A PhD thesis in Mathematics

MEASURING PROJECTIVITY IN ABELIAN CATEGORIES. APPLICATIONS TO COMPLEXES

July 2022

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Measuring projectivity in abelian categories. Applications to complexes

(Medida de la proyectividad en categorías abelianas. Aplicaciones a complejos)

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Program: Doctorate Program in Mathematics

Memory carried out at the Mathematics departments
of the Mohammed V University
and of the University of Almería
under the direction of professors
Dr. Driss Bennis
and Dr. Luis Oyonarte
to aspire to the degree
Doctor of Mathematical Sciences.

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To my beloved Malak

Acknowledgments

I wouldn't have achieved this thesis without the wise counsel of Prof. Driss Bennis and Prof. Luis Oyonarte. I am indebted to them for all my advanced mathematical knowledge and experience. In addition to the priceless guidance in the investigation world, they have always been worthful and supporting friends with whom I enjoy talking with. Their insight and patience are always appreciated.

I would like to express my deepest gratitude to Prof. J. R. García Rozas with who I had the opportunity to discuss and realize the articles involved in this thesis. It has been my privilege to work closely with him and learn from his knowledge and experience.

My stays at the University of Almería have been supported by the frame of the European exchange program Erasmus + Ka1 Mobility between Program and Partner Countries. I would like to thank the staff of the international offices of the university Mohammed V and the university of Almería, who has been so helpful in every bureaucratic stuff during these four years.

Many thanks to the amazing friends who have spent this period of my life with me and made Rabat and Almeria my home. I am grateful for every moment that we have shared together.

Words cannot express my gratitude to my family for their endless support in any moment of my life, especially, my mother and my grandparents, who have been the best encouragement for me.

Contents

Resumen

En los últimos años, López-Permouth y varios colaboradores han introducido un nuevo enfoque en el estudio de la proyectividad, inyectividad y planitud clásicas de los módulos (mira por ejemplo [28, 6, 14]). De esta manera, introdujeron los dominios de subproyectividad de módulos como una herramienta para medir, de alguna manera, el nivel de proyectividad de dichos módulos (y no solo para determinar si el módulo es proyectivo o no). En esta memoria desarrollamos un nuevo tratamiento de la subproyectividad en cualquier categoría abeliana que arroja más luz sobre algunos de sus diversos aspectos importantes. Es decir, en términos de subproyectividad, se unifican algunos resultados clásicos y se caracterizan algunos anillos clásicos. También se muestra que, en algunas categorías, la subproyectividad mide nociones distintas a la proyectividad. Además, este nuevo enfoque permite, además de establecer generalizaciones de resultados conocidos, construir nuevos ejemplos como el dominio de subproyectividad de la clase de objetos Gorenstein proyectivos, la clase de complejos DG-proyectivos y tipos particulares de representaciones lineales de quivers finitos.

Asimismo, en esta memoria ampliamos nuestro estudio a la categoría de complejos sobre una categoría abeliana. Probamos que la noción de subproyectividad proporciona una nueva visión de los morfismos homotópicamente nulos en la categoría de complejos y damos varios resultados que enfatizan la importancia de la subproyectividad en la categoría de complejos; damos algunas aplicaciones caracterizando algunos anillos clásicos y establecemos varios ejemplos que nos permiten reflejar el alcance y los límites de nuestros resultados.

Résumé

Au cours des dernières années, López-Permouth et plusieurs collaborateurs ont introduit une nouvelle approche dans l'étude de la projectivité, de l'injectivité et de la planéité classiques des modules (voir par exemple [28, 6, 14]). De cette façon, ils ont introduit les domaines de sous-projectivité des modules comme un outil pour mesurer, en quelque sorte, le niveau de projectivité d'un tel module (donc pas seulement pour déterminer si le module est projectif ou non). Dans ce mémoire, nous développons un nouveau traitement de la sous-projectivité dans toute catégorie abélienne qui éclaire davantage certains de ses divers aspects importants. A savoir, en termes de sous-projectivité, certains résultats classiques sont unifiés et certains anneaux classiques sont caractérisés. On montre aussi que, dans certaines catégories, la sous-projectivité mesure des notions autres que la projectivité. De plus, cette nouvelle approche permet, en plus d'établir des généralisations de résultats connus, de construire de nouveaux exemples tels que le domaine de sous-projectivité de la classe des objets projectifs de Gorenstein, la classe des complexes DG-projectifs et des types particuliers de représentations d'un carquois linéaire fini.

Aussi, dans cette thèse, nous étudions profondément la sous-projectivité dans la catégorie des complexes sur une categorie abélienne. Nous prouvons que la notion de sous-projectivité fournit une nouvelle vision des morphismes nuls-homotopiques dans la catégorie des complexes et nous montrons à travers plusieurs resultats l'importance de la sous-projectivité dans la categorie des complexes. Notamment, la sous-projectivité offre une nouvelle vision aux quelques notions classiques des anneaux. En plus, divers exemples sont donnés afin de supporter la nouvelle vision et aussi de discuter les limites de quelques résultats.

Abstract

In the last few years, López-Permouth and several collaborators have introduced a new approach in the study of the classical projectivity, injectivity and flatness of modules (see for instance, [28, 6, 14]). This way, they introduced subprojectivity domains of modules as a tool to measure, somehow, the projectivity level of such a module (so not just to determine whether or not the module is projective). In this memory, we develop a new treatment of the subprojectivity in any abelian category which shed more light on some of its various important aspects. Namely, in terms of subprojectivity, some classical results are unified and some classical rings are characterized. It is also shown that, in some categories, the subprojectivity measures notions other than the projectivity. Furthermore, this new approach allows, in addition to establishing nice generalizations of known results, to construct various new examples such as the subprojectivity domain of the class of Gorenstein projective objects, the class of DG-projective complexes and particular types of representations of a finite linear quiver.

Also, in this memory we extend our study to the category of complexes over an abelian category. We prove that the subprojectivity notion provides a new sight of null-homotopic morphisms in the category of complexes and we give various results which emphasize the importance of subprojectivity in the category of complexes; we give some applications by characterizing some classical rings and establish various examples that allow us to reflect the scope and limits of our results.

[0]**Key Words.** subprojectivity; subprojectivity domain; abelian category; complex; homotopy; projective; flat

Author's papers involved in this thesis :

1. H. Amzil, D. Bennis, J.R. García Rozas, H. Ouberka and L. Oyonarte, *Subprojectivity in abelian categories*, Appl Categor Struct **29** (2021), 889–913.

2. D. Bennis, J.R. García Rozas, H. Ouberka and L. Oyonarte, *A new approach to projectivity in the categories of complexes*, Annali di Matematica (2022). https://doi.org/10.1007/s10231-022-01223-9

3. D. Bennis, J.R. García Rozas, H. Ouberka and L. Oyonarte, *A new approach to projectivity in the categories of complexes, II*, to appear in Bull. Malays. Math. Sci. Soc. Available at arXiv:2203.01067

Introduction

In this memory we will work mainly on an abelian category with enough projectives, although we will find the biggest applications and examples on the category of modules over an associative ring with unit. It is worth mentioning the existing connection between abelian and module categories through the well-known Gabriel's theorem: an abelian category is equivalent to a module category if and only if it is cocomplete and has a finitely generated projective generator (see for instance [35, page 211]).

Throughout the thesis, $\mathscr{A}$ will denote an abelian category with enough projectives, $\mathscr{C}(\mathscr{A})$ the category of complexes over $\mathscr{A}$, $\mathscr{K}(\mathscr{A})$ the homotopy category of $\mathscr{C}(\mathscr{A})$ and R will denote an associative (non necessarily commutative) ring with a unit element. The category of left R-modules will be denoted by R-Mod, the category of complexes of left R-modules will be denoted by $\mathscr{C}(R)$ and the homotopy category of $\mathscr{C}(R)$ will be denoted by $\mathscr{K}(R)$. Modules are, unless otherwise explicitly stated, left R-modules.

To any given class of objects $\mathscr{C}$ of $\mathscr{A}$ we associate its right Ext-orthogonal class,

$$\mathscr{C}^{\perp} = \{X \in \mathscr{A} \mid \mathrm{Ext}^1(C,X) = 0, C \in \mathscr{C}\},$$

and its left Ext-orthogonal class,

$${}^{\perp}\mathscr{C} = \{X \in \mathscr{A} \mid \mathrm{Ext}^1(X,C) = 0, C \in \mathscr{C}\}.$$

In particular, if $\mathscr{C} = \{M\}$ then we simply write ${}^{\perp}\mathscr{C} = {}^{\perp}M$ and $\mathscr{C}^{\perp} = M^{\perp}$.

Many studies are done every year on projective, injective and flat modules. Many of them involve concepts derived from relative projectivity, injectivity and flatness. Rather than saying whether a module has a certain property or not, each module is assigned a relative domain that, somehow, measures to which extent it has this particular property. For instance, the study of flatness was accessed in [14, 18] from two slightly similar alternative perspectives as both use the tensor product. Then, another perspective on the flatness of modules was introduced in [4] where the authors use flat precovers to define and study flat-precover completing domains.

On the other hand, relative injectivity, injectivity domains and the notion of a poor module (modules with the smallest possible injectivity domain) have been studied in [2, 24, 31]. Dually, relative projectivity, projectivity domains and the notion of a p-poor modules have been studied in [29, 31]. In contrast to the notion of relative injectivity, Aydğdu and López-Permouth introduced in [6] the notion of subinjectivity. Then, Holston et al. introduced in [28] the projective analog of subinjectivity and called it subprojectivity. The purpose of [28] was to introduce a new approach on the analysis of the projectivity of a module. However, the study of the subprojectivity goes beyond that aim and, indeed provides, among other things, an interesting new side on some other known notions. This opens a new important area of research which attracts many authors.

In this thesis, we extend the notion of subprojectivity to an abelian category $\mathscr{A}$ with enough projectives and we show that subprojectivity domains may not be restricted to a single object. On the contrary, the subprojectivity domains of a whole class of objects can be computed, giving rise to the characterization of the subprojectivity domain of several homologically interesting classes of objects. We also show that subprojectivity can be used to measure characteristics different from the projectivity in some particular categories such as the category of complexes and the one or representations of quivers (see Example 2.1.3). Then, we go deeper in the investigation of subprojectivity in the category of complexes of $\mathscr{A}$ (which has enough projectives since $\mathscr{A}$ is supposed to have enough projectives) and we show that subprojectivity of complexes is relatively linked to that of null-homotopy of morphisms.

Recall from [28] that for two objects M and N of $\mathscr{A}$, M is said to be N-subprojective if for every epimorphism $g : B \to N$ and every morphism $f : M \to N$, there exists a morphism $h : M \to B$ such that $gh = f$. The subprojectivity domain of any object M, denoted $\underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(M)$, is defined as the class of all objects N such that M is N-subprojective, and the subprojectivity domain of a whole class $\mathfrak{C}$ of $\mathscr{A}$, $\underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(\mathfrak{C})$, is defined as the class of objects N such that every C of $\mathfrak{C}$ is N-subprojective.

Now, we summarize the contents of this memory.

Chapter I: Preliminaries

This chapter is the preliminaries parts of the memory. We recall some basic terminologies and results which will be used in the rest of the thesis.

Chapter II: Measuring projectivity in abelian categories

In this chapter, we develop a new treatment of the subprojectivity in the categorical context. This study provides new interesting tools to develop this area of research. Indeed, we obtain, for instance, generalizations of several results using new methods which give a different light to the way they are seen now, which in addition, gives new perspectives. The current study provides also new powerful tools in constructing various interesting examples. For instance, we know and it is easy to show that the subprojectivity domain of a projective object P is the whole category $\mathscr{A}$, which is exactly the right

Ext-orthogonal class of P. So we can write $\underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(P) = P^{\perp}$. So it is natural to ask how far we can go by extending this equality. We will show that, at least, it is possible to extend it to objects which are embedded in projective ones (see Proposition 2.1.6). As a consequence, we deduce that, if M is a Gorenstein projective object, then there is an object N such that $\underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(M) = N^{\perp}$.

We also introduce subprojectivity domains of classes as a natural extension of the subprojectivity domains of objects and we show, among several other things, that the fact that a subprojectivity domain of a class coincides with its first right Ext-orthogonal class can be characterized in terms of preenvelopes and precovers.

The chapter is organized as follows:

In section 2.1, we investigate subprojectivity domains of objects. We start by giving examples in the category of complexes and the category of representations of a quiver which show that the role of subprojectivity could go beyond the measure of the projectivity, and that indeed it can be effectively used to measure other properties such as the exactness of complexes or determine when a morphism is monic (see Example 2.1.3). The main contribution of this section is the elaboration of two new ways to treat the subprojectivity of objects. The first one is a functorial characterization of the subprojectivity of objects (see 2. and 3. of Proposition 2.1.4 and Proposition 2.1.23) and the second one characterizes the subprojectivity of objects in terms of factorizations of morphisms (see 4. and 5. of Proposition 2.1.4). This contribution allows to easily establish new and interesting results and examples throughout the paper. For instance, Corollary 2.1.7 shows that if M is a strongly Gorenstein projective object then $\underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(M) = M^{\perp}$. And Corollaries 2.1.9 and 2.1.10 give, in terms of subprojectivity, a new way to see how an object can be embedded in a projective object.

We also introduce and investigate subprojectivity domains of classes as a natural extension of subprojectivity domains of objects. This notion leads, among other things, to a unification of several well-known results (see Corollaries 2.1.27, 2.1.28, 2.1.29 and 2.1.30). We determine subprojectivity domains of various classes such as the one of DG-projective complexes (Proposition 2.1.15), the one of strongly Gorenstein projective objects (Proposition 2.1.16), the one of finitely presented objects (Proposition 2.1.19), the one of finitely generated modules (Proposition 2.1.24), and the one of simple modules (Proposition 2.1.25). We show in Proposition 2.1.14 that the subprojectivity domain of a class $\mathscr{L}$ does not change even if we modify this class to $Add(\mathscr{L})$ (i.e. the class of all objects which are isomorphic to direct summands of direct sums of copies of objects of the class $\mathscr{L}$). As consequences, the subprojectivity domains of the classes of all Gorenstein projective objects, of all pure-projective objects and of all semisimple modules are determined (see Corollaries 2.1.17, 2.1.20 and 2.1.26).

Section 2.2 is devoted to the study of some closure properties of subprojectivity domains. We extend the study done in [28] and we give new results. In Proposition 2.2.1 we show that the subprojectivity domain of any class is closed under extensions, finite direct sums and direct summands. In Proposition 2.2.4 we show that the subprojectivity domain of any finitely generated object is closed under pure-subobjects (if the category is locally finitely presented Grothendieck). Then, we characterize when are the subprojectivity domains closed under kernels of epimorphisms (Proposition 2.2.5 and Example 2.2.6). In Proposition 2.2.7 we show that the subprojectivity domain of a class $\mathscr{L}$ is closed under subobjects if and only if the subprojectivity domain of any object of $\mathscr{L}$ is closed under subobjects. This leads to new characterizations of known notions. For instance, in Corollary 2.2.8, we show that, for any ring R, the weak global dimension of R is at most 1 if and only if the subprojectivity domain of each finitely presented module is closed under submodules. In Corollary 2.2.9 we prove that a left coherent ring R is left semihereditary if and only if the subprojectivity domain of each of its finitely generated modules is closed under submodules. Similarly, in Proposition 2.2.10, we generalize [28, Proposition 2.14] by showing that the subprojectivity domain of a class $\mathscr{L}$ is closed under arbitrary direct products if and only if the subprojectivity domain of any of its objects is closed under arbitrary direct products. This result allows us to give a much direct proof (see Corollary 2.2.11) of a characterization of coherent rings established by Durğun in [20, Proposition 2.3]. Then, we discuss the closeness under direct sums of the subprojectivity domains of classes. In [28, Proposition 2.13], it was shown that the subprojectivity domain of any finitely generated module is closed under arbitrary direct sums. Here, using the functorial characterization of the subprojectivity domains, we show that this also holds for small objects (see Proposition 2.2.12). We end Section 2.2 with a discussion on whether or not the subprojectivity domains are closed under direct limits and we show that this holds for finitely presented objects (see Proposition 2.2.14).

Finally, in Section 2.3 we relate subprojectivity domains with right Ext-orthogonal of classes and the existence of precovers and preenvelopes. The main results (Theorems 2.3.1 and 2.3.10) state that, under some conditions on the category $\mathscr{A}$ and on the class $\mathscr{L}$, the following conditions are equivalent.

1. $\mathscr{L}^{\perp} = \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(\mathscr{L})$.

2. $\mathscr{L}^{\perp}$ is closed under kernels of epimorphism and cokernls of monomorphisms and contains $Proj_{\mathscr{A}}$.

3. $\mathscr{L} \bigcap \mathscr{L}^{\perp} = Proj_{\mathscr{A}}$ and every object in $\mathscr{L}^{\perp}$ has a special $\mathscr{L}$-precover.

4. $Proj_{\mathscr{A}} \subseteq \mathscr{L}^{\perp}$, $\underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(\mathscr{L})$ is closed under cokernels of monomorphisms and every $M \in \mathscr{L}$ has an $\mathscr{L}^{\perp}$-preenvelope which is projective.

Inspired by the work of Parra and Rada ([37]), we show that, if we assume further conditions on $\mathscr{A}$, then the closure under direct products of the subprojectivity domains of classes can be characterized in terms of preenvelopes (see Proposition 2.3.9). Also, in this section we give two consequences of Theorem 2.3.1; we characterize quasi-Fröbenius rings in the meaning of subprojectivity (see Corollary 2.3.2); and we show that every object in $\mathscr{G}\mathscr{P}_{\mathscr{A}}^{\perp}$ has a special $\mathscr{G}\mathscr{P}_{\mathscr{A}}$-precover (see Corollary 2.3.3).

Chapter III: Measuring projectivity of complexes

In this chapter we go deeper in the investigation of subprojectivity in $\mathscr{C}(\mathscr{A})$, the category of complexes of $\mathscr{A}$, which has enough projectives since $\mathscr{A}$ is supposed to have enough projectives. In this sense, when studying subprojectivity of complexes, it is observed that the concept of subprojectivity is relatively closely linked to that of null-homotopy of morphisms.

The chapter is organized as follows:

In Section 3.1, we investigate the relationship between the subprojectivity of complexes and the null-homotopy of morphisms. Namely, in Theorem 3.1.4, we prove that if $N_n \in \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(M_n)$ for every $n \in \mathbb{Z}$, then we get that $N \in \underline{\mathfrak{Pr}}_{\mathscr{C}(\mathscr{A})}^{-1}(M)$ if and only if $\mathrm{Hom}_{\mathscr{K}(\mathscr{A})}(M[-1],K) = 0$ (where $M[-1]$ denotes the -1-shift of M) for every short exact sequence of complexes $0 \to K \to P \to N \to 0$ with P projective. The proof of this theorem is based on a new characterization of the subprojectivity of an object in any abelian category with enough projectives in terms of the splitting of some particular type of short exact sequences (Proposition 3.1.1).

The second main result of the section (Theorem 3.1.12) assures that if M and N are two complexes with $N_{n+1} \in \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(M_n)$ for every $n \in \mathbb{Z}$, then the conditions $N \in \underline{\mathfrak{Pr}}_{\mathscr{C}(\mathscr{A})}^{-1}(M)$ and $\mathrm{Hom}_{\mathscr{K}(\mathscr{A})}(M,N) = 0$ are equivalent. This time, the idea is based on a new characterization of subprojectivity in terms of factorizations through contractible complexes (Proposition 3.1.10).

Theorem 3.1.12 allows us to determine exactly when a complex N is in the subprojectivity domain of all the shifts $M[n]$ of a given complex M (Proposition 3.1.15), which, at the same time, helps in characterizing subprojectivity domains of complexes of the form $\oplus_{n\in\mathbb{Z}}\overline{M}[n]$ (Proposition 3.1.16) and of the form $\oplus_{n\in\mathbb{Z}}\underline{M}[n]$ (Proposition 3.1.17) for a given object M in $\mathscr{A}$. A particular case of Proposition 3.1.17 typifies exact complexes in terms of subprojectivity in the following sense: if $\mathscr{A}$ has a projective generator P, then N is exact if and only if $N \in \underline{\mathfrak{Pr}}_{\mathscr{C}(\mathscr{A})}^{-1}(\underline{P}[n])$ for every $n \in \mathbb{Z}$ (see Corollary 3.1.18). Motivated by this result, we asked whether subprojectivity can measure the exactness of a complex N at each N_i. In fact, we prove that, for any complex N and any $n \in \mathbb{Z}$, $N \in \underline{\mathfrak{Pr}}_{\mathscr{C}(\mathscr{A})}^{-1}(\underline{P}[n])$ if and only if $H_n(N) = 0$ (see Proposition 3.1.19). This result allows us to answer two interesting questions. Namely, we provide an example showing that the subprojectivity domains are not closed under kernel of epimorphisms (see Example 3.1.20), and we give an example showing that the equivalence of Theorem 3.1.4

mentioned above does not hold in general if we replace the condition "P is projective" by "$P \in \underline{\mathfrak{Pr}}_{\mathscr{C}(\mathscr{A})}^{-1}(M)$" (see Remark 3.1.5 and Example 3.1.21). It is worth mentioning that the necessity and the importance of the conditions given in the main Theorems 3.1.4 and 3.1.12 are deeply discussed in Propositions 3.1.6 and Example 3.1.13, and that semisimple categories (in the sense that every object is projective) are also characterized in terms of subprojectivity. In fact, this was a consequence of the study of the condition "$N_{n+1} \in \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(M_n)$ for every $n \in \mathbb{Z}$" assumed in Theorem 3.1.12; we prove that the category $\mathscr{A}$ must be semisimple when this condition implies the condition $N \in \underline{\mathfrak{Pr}}_{\mathscr{C}(\mathscr{A})}^{-1}(M)$ for every two complexes M and N (Proposition 3.1.14).

In Section 3.2, we study the relationship between the subprojectivity of a complex and the subprojectivity of its cycles. As a natural question, inspired by some classical facts, we ask whether for two complexes M and N, $N \in \underline{\mathfrak{Pr}}_{\mathscr{C}(\mathscr{A})}^{-1}(M)$ under the condition "$Z_n(N) \in \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(M_n)$ for every $n \in \mathbb{Z}$" (Theorem 3.2.2) and under the condition "$N_n \in \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(Z_{n-1}(M))$ for every $n \in \mathbb{Z}$" (Theorem 3.2.7). We show that this holds for every exact complex N and every bounded below complex M (see Theorem 3.2.2). In Theorem 3.2.7, we show that this holds for every exact complex M and every bounded above complex N only if every projective object is injective.

The relations between the conditions given in the main Theorems 3.2.2 and 3.2.7 are also deeply discussed. Indeed, we show that the condition "M is bounded below" in Theorem 3.2.2 cannot be dropped (see Example 3.2.4). Also, we give an example showing that the reverse implication of Theorem 3.2.2 does not hold true in general (see Example 3.2.5) and another example showing that the reverse implication of the first assertion of Theorem 3.2.7 does not hold true in general, that is, given two complexes M and N, the condition " $N \in \underline{\mathfrak{Pr}}_{\mathscr{C}(\mathscr{A})}^{-1}(M)$", is not sufficient to assure that $N_n \in \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(Z_{n-1}(M))$ for every $n \in \mathbb{Z}$.

To finish this section, we study the case of contractible complexes which satisfies that $N \in \underline{\mathfrak{Pr}}_{\mathscr{C}(\mathscr{A})}^{-1}(M)$ if and only if $N_n \in \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(Z_{n-1}(M))$ for every $n \in \mathbb{Z}$ (Proposition 3.2.8). This can be seen as a new extension of the known fact that the projective complexes are exactly the contractible ones with projective cycles.

Section 3.3 is devoted to the study of subprojectivity domains of classes of complexes. Inspired by some classical facts, we focus our study on the classes of complexes constructed from a class of objects $\mathscr{L}$. We will use the following terminology:

- The class of complexes X such that every $X_n \in \mathscr{L}$ will be denoted by $\#\mathscr{L}$.
- The class of bounded complexes (resp., bounded below complexes) X such that every non zero $X_n \in \mathscr{L}$ will be denoted by $\mathscr{C}^b(\mathscr{L})$ (resp., $\mathscr{C}^-(\mathscr{L})$).

- The class of exact complexes X such that every $Z(X)_n \in \mathscr{L}$ will be denoted by $\widetilde{\mathscr{L}}$.

For some particular cases of classes $\mathscr{L}$ of objects in $\mathscr{A}$, the classes of complexes $\#\mathscr{L}$, $\mathscr{C}^b(\mathscr{L})$ and $\widetilde{\mathscr{L}}$ are usual. For instance, we have $\mathscr{C}(\mathscr{A}) = \#\mathscr{A}$, $\mathscr{P}_{\mathscr{C}(\mathscr{A})} = \widetilde{\mathscr{P}_{\mathscr{A}}}$ (i.e., the class of projective complexes (Proposition 1.2.2)) and $\mathscr{F}\mathscr{P}_{\mathscr{C}(R)} = \mathscr{C}^b(\mathscr{F}\mathscr{P}_{R\text{-Mod}})$ (i.e., the class of finitely presented complexes of modules (see [26, Lemma 4.1.1])). Thus, the following questions arise naturally: let $\mathscr{L}$ and $\mathscr{G}$ be two classes such that $\underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(\mathscr{L}) = \mathscr{G}$.

1. When do we have $\underline{\mathfrak{Pr}}_{\mathscr{C}(\mathscr{A})}^{-1}(\widetilde{\mathscr{L}}) = \#\mathscr{G}$?

2. When do we have $\underline{\mathfrak{Pr}}_{\mathscr{C}(\mathscr{A})}^{-1}(\#\mathscr{L}) = \widetilde{\mathscr{G}}$?

3. When do we have $\underline{\mathfrak{Pr}}_{\mathscr{C}(\mathscr{A})}^{-1}(\mathscr{C}^b(\mathscr{L})) = \widetilde{\mathscr{G}}$?

We will show in Theorem 3.3.1, that if $\mathscr{L}$ and $\mathscr{G}$ contain the zero object and $\mathscr{L}$ is closed under extensions, then $\underline{\mathfrak{Pr}}_{\mathscr{C}(\mathscr{A})}^{-1}(\widetilde{\mathscr{L}}) = \#\mathscr{G}$ if and only if $\underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(\mathscr{L}) = \mathscr{G}$ and $\mathrm{Hom}_{\mathscr{K}(\mathscr{A})}(M,N) = 0$ for any $M \in \widetilde{\mathscr{L}}$ and $N \in \#\mathscr{G}$.

To answer the second question, we will show in Theorem 3.3.2, that if $\mathscr{A}$ has a projective generator P with $0, P \in \mathscr{L}$. Then, $\underline{\mathfrak{Pr}}_{\mathscr{C}(\mathscr{A})}^{-1}(\#\mathscr{L}) = \widetilde{\mathscr{G}}$ if and only if $\underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(\mathscr{L}) = \mathscr{G}$ and $\mathrm{Hom}_{\mathscr{K}(\mathscr{A})}(M,N) = 0$ for any $M \in \#\mathscr{L}$ and $N \in \widetilde{\mathscr{G}}$.

Finally in Theorem 3.3.6, we answer Question 3 as follows: if $\mathscr{A}$ has a projective generator P with $0, P \in \mathscr{L}$, then we get that $\underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(\mathscr{L}) = \mathscr{G}$ if and only if $\underline{\mathfrak{Pr}}_{\mathscr{C}(\mathscr{A})}^{-1}(\mathscr{C}^b(\mathscr{L})) = \widetilde{\mathscr{G}}$.

We end this section by giving some consequences of the main results already established (see Propositions 3.3.3 and 3.3.7 and Corollaries 3.3.8, 3.3.9, 3.3.10, 3.3.11, 3.3.12 and 3.3.13).

Finally, Section 3.4 is devoted to some applications to the category of complexes of modules. As consequences of the results of the above sections, we give some new characterizations of some classical rings. We give, in Proposition 3.4.1, a characterization of quasi-Fröbenius rings. In Proposition 3.4.2 we characterize left hereditary rings in terms of subprojectivity as those rings for which every subcomplex of a DG-projective complex is DG-projective. Furthermore, we do it without the condition "Every exact complex of projective modules is projective" needed in [46, Proposition 2.3].

Following the same context, subprojectivity also makes it possible to characterize rings of weak global dimension at most 1, and using subprojectivity domains we prove that these rings are the ones over which subcomplexes of DG-flat complexes are always also DG-flat (Proposition 3.4.3). As a consequence, left semi-hereditary rings are also characterized in terms of subprojectivity (Corollary 3.4.4).

PRELIMINARIES

This chapter is the introductory part of the memory. We recall some basic terminology and results (without proofs) which will be used in the next two chapters. Also, in this chapter we extend and prove some well known results from the category of modules to any abelian category. The reader is supposed to be familiar with the language of categories.

1.1 Finitely generated, finitely presented and pure-projective objects

The categorical setting for this section is that of Grothendieck categories for which our main reference is [43].

Recall that a Grothendieck category is a cocomplete abelian category with a generating set, and with exact direct limits. Throughout this section, $\mathscr{G}$ will denote a Grothendieck category.

To orient the reader, we summarize some standard facts about Grothendieck categories. First, a Grothendieck category is always complete and every object $B \in \mathscr{G}$ has an injective envelope. In particular, $\mathscr{G}$ has enough injectives. A useful fact is that any Grothendieck category is locally small, meaning the lattice of subobjects of any given object A (that is, the class of subobjects of A), denoted by $\mathscr{L}(A)$, is in fact a set (see [43, Proposition IV.6.6]). Another useful fact in Grothendieck categories is the following: if $\{A_i;\ i \in I\}$ is a family of subobjects of an object A (which can be seen as a direct family with the order given by the inclusions $A_i \to A_j$ for every $i < j$), then the direct limit $\varinjlim A_i$ coicides with the sum $\sum A_i$. In this case, we call it the direct union of the family, and it is usually represented by either $\cup A_i$ or $\sum A_i$.

In this section we recall finiteness conditions on Grothendieck categories: we state the definition of finitely generated, finitely presented and pure-projective objects and we give some known properties. In particular, the definitions of finitely generated objects given in [42] and [35] are not the one we adopt here, but have not found any proof of this equivalence.

Finitely generated objects appear quite often in the study of the categorical homology theory. There are many different equivalent definitions of this type of objects, each one useful in different contexts. In our case, we say that an object A of $\mathcal{G}$ is finitely generated if for any family of objects $\{X_i; i \in I\}$ and any epimorphism $\varphi : \oplus_I X_i \to A$, there exists a finite subset $J \subseteq I$ such that the restriction of φ to $\oplus_J X_i$ is an epimorphism. We denote the class of all finitely generated objects of $\mathcal{G}$ by $\mathcal{FG}_{\mathcal{G}}$.

We start with a result, useful to prove the equivalence mentioned above between our definition of finitely generated objects and the ones given in [42] and [35].

Proposition 1.1.1. *The following conditions are equivalent for any object A of* $\mathcal{G}$.

1. *A is finitely generated.*
2. *If* $\{A_i;\ i \in I\}$ *is a directed family of subobjects of A with* $\sum_I A_i = A$*, then there is* $j \in I$ *such that* $A_j = A$.
3. *If* $\{A_i;\ i \in I\}$ *is an ascending chain of subobjects of A with* $\sum_I A_i = A$*, then there is* $j \in I$ *such that* $A_j = A$.
4. *If* $\{A_i;\ i \in I\}$ *is any family of subobjects of A with* $\sum_I A_i = A$*, then there is a finite subset* $F \subseteq I$ *such that* $\sum_F A_i = A$.

Proof. 1. $\Rightarrow$ 2. Let $\lambda_i : A_i \to A$, $k_i : A_i \to \oplus_I A_i$ and $\mu : \sum_I A_i \to A$ denote the inclusion monomorphisms. The family $\{\lambda_i;\ i \in I\}$ induce a unique $\lambda : \oplus_I A_i \to A$ such that $\lambda k_i = \lambda_i$, and we know the ker-coker factorization of λ is

$$\begin{tikzcd} \oplus_I A_i \arrow[rr, "\lambda"] \arrow[dr, "\overline{\lambda}"'] & & A \\ & \sum_I A_i \arrow[ur, "\mu"'] & \end{tikzcd}$$

Now, if $\sum_I A_i = A$ then μ is the identity and then λ is an epimorphism.

Being A finitely generated ensures we can find a finite subset $F \subseteq I$ such that the restriction of λ, $\lambda' : \oplus_F A_i \to A$, is an epimorphism. But the ker-coker factorization of λ' is

$$\begin{tikzcd} \oplus_F A_i \arrow[rr, "\lambda'"] \arrow[dr, "\overline{\lambda'}"'] & & A \\ & \sum_F A_i \arrow[ur, "\mu'"'] & \end{tikzcd}$$

so the inclusion μ' is an isomorphism and then it is the identity.

In other words, $A = \sum_F A_i$, but F is finite and $\{A_i;\ i \in I\}$ is a directed family, so there is some $j \in I$ such that $A_i \subseteq A_j$ for every $i \in F$. Then, $A = \sum_F A_i \subseteq A_j$ and then $A = A_j$.
2. $\Rightarrow$ 3. Clear since any ascending chain of subobjects is a directed family.
3. $\Leftrightarrow$ 4. This is due to [35, Theorem 1, page 204].
4. $\Rightarrow$ 1. Suppose we have an epimorphism $\varphi : \oplus_I C_i \to A$. Then, for any $i \in I$ let $k_i : C_i \to \oplus_I C_i$ be the canonical injection and call $A_i = \mathrm{Im}\ (\varphi k_i)$. The ker-coker factorization of each φk_i is then

$$\begin{tikzcd} C_i \arrow[r, "k_i"] \arrow[dr, "\overline{\varphi k_i}"'] & \oplus_I C_i \arrow[r, "\varphi"] & A \\ & A_i \arrow[ur, "\lambda_i"'] & \end{tikzcd}$$

where λ_i is the inclusion monomorphism.

If we call now $k'_i : A_i \to \oplus_I A_i$ the canonical injections, the family of inclusions $\{\lambda_i;\ i \in I\}$ induce a unique $\xi : \oplus_I A_i \to A$ such that $\xi k'_i = \lambda_i\ \forall i \in I$.

On the other hand, if we let $\psi = \oplus_I \overline{\varphi k_i}$ then we know that the diagram

$$\begin{tikzcd} C_i \arrow[r, "k_i"] \arrow[d, "\overline{\varphi k_i}"'] & \oplus_I C_i \arrow[d, "\psi"] \\ A_i \arrow[r, "k'_i"'] & \oplus_I A_i \end{tikzcd}$$

is commutative for every $i \in I$. Thus, the diagram

$$\begin{tikzcd} \oplus_I C_i \arrow[r, "\varphi"] \arrow[d, "\psi"'] & A \arrow[d, equal] \\ \oplus_I A_i \arrow[r, "\xi"'] & A \end{tikzcd}$$

is commutative because $\xi \psi k_i = \xi k'_i \overline{\varphi k_i} = \lambda_i \overline{\varphi k_i} = \varphi k_i\ \forall i \in I$, so indeed $\xi \psi = \varphi$. Since φ is an epimorphism we get that ξ is an epimorphism too. Now, from the ker-coker factorization of ξ:

$$\begin{tikzcd} \oplus_I A_i \arrow[rr, "\xi"] \arrow[dr, "\overline{\xi}"'] & & A \\ & \sum_I A_i \arrow[ur, "\lambda"'] & \end{tikzcd}$$

(λ is the inclusion monomorphism) we see that λ is an isomorphism since ξ is an epimorphism, so λ is actually the identity, that is, $\sum_I A_i = A$. Then, by the hypotheses there is a finite subset $F \subseteq I$ such that $A = \sum_F A_i$. But then, if $\xi' : \oplus_F A_i \to A$ is the restriction of ξ to $\oplus_F A_i$, the ker-coker factorization of ξ' is

$$\begin{tikzcd} \oplus_F A_i \arrow[rr, "\xi'"] \arrow[dr, "\overline{\xi'}"'] & & A \arrow[dl, equal] \\ & \sum_F A_i & \end{tikzcd}$$

which means that ξ' is an epimorphism.

Now, if we let φ' be the restriction of φ to $\oplus_F C_i$ and $\psi' = \oplus_F \overline{\varphi k_i}$ (the corresponding restriction of ψ), we have a commutative diagram

$$\begin{tikzcd} \oplus_F C_i \arrow[r, "\varphi'"] \arrow[d, "\psi'"'] & A \arrow[d, equal] \\ \oplus_F A_i \arrow[r, "\xi'"'] & A \end{tikzcd}$$

All the morphisms $\overline{\varphi k_i}$ are epimorphisms so ψ' is an epimorphism, and we have already seen that ξ' is an epimorphism, so we get that φ' is an epimorphism. □

Recall that the class of all finitely generated objects is closed under quotients and extensions (see for instance [43, page 121, Lemma 3.1]). In particular, it is closed under direct summands and finite direct sums.

The existence of finitely generated objects in Grothendieck categories is not guaranteed at all, but there is a type of (Grothendieck) categories in which every object is a quotient of a direct sum of finitely generated objects. All module categories are Grothendieck categories of this type.

Definition 1.1.2. *The category $\mathscr{G}$ is said to be locally finitely generated provided that it has a generating set of finitely generated objects.*

Let us now define finitely presented objects.

Definition 1.1.3. *An object A of $\mathscr{G}$ is said to be finitely presented if for every exact sequence $0 \to K \to L \to A \to 0$ with L finitely generated, K is finitely generated. We denote the class of all finitely presented objects of $\mathscr{G}$ by $\mathscr{F}\mathscr{P}_{\mathscr{G}}$.*

The category $\mathscr{G}$ is said to be locally finitely presented provided that it has a generating set of finitely presented objects.

We recall the following characterizations of finitely presented objects.

Proposition 1.1.4. *If $\mathscr{G}$ is a locally finitely generated category, then for an object A the following conditions are equivalent.*

1. *A is finitely presented.*

2. *There exists a short exact sequence $0 \to K \to L \to A \to 0$ with L finitely presented and K finitely generated.*

3. $\mathrm{Hom}_{\mathscr{G}}(A,-)$ *preserves direct limits.*

Proof. 1. $\Leftrightarrow$ 3. is due to [43, page 122, Proposition 3.4]
1. $\Rightarrow$ 2. If A is finitely presented, then the short exact sequence $0 \to 0 \to A \to A \to 0$ verifies 2.
2. $\Rightarrow$ 1. Let $0 \to C \to B \to A \to 0$ be any short exact sequence with B finitely generated and let us prove that C is finitely generated. Consider the following pullback diagram

$$\begin{tikzcd} & & 0 \arrow[d] & 0 \arrow[d] & \\ & & K \arrow[r, equal] \arrow[d] & K \arrow[d] & \\ 0 \arrow[r] & C \arrow[r] \arrow[d, equal] & D \arrow[r] \arrow[d] & L \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & C \arrow[r] & B \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & 0 \\ & & 0 & 0 & \end{tikzcd}$$

with L finitely presented and K finitely generated. Then, D is finitely generated since K and B are, but, L is finitely presented, thus C is finitely generated. $\square$

Now, we turn our attention to pure-projective objects. Recall that a short exact sequence

$$0 \longrightarrow A \xrightarrow{k} B \xrightarrow{p} C \longrightarrow 0$$

in $\mathscr{G}$ is said pure if $\mathrm{Hom}_{\mathscr{G}}(F,p)$ is epic for every F finitely presented. In this case, $p : B \to C$ is called pure epimorphism, $k : A \to B$ pure monomorphism and A pure subobject of B. An object of $\mathscr{G}$ is said to be pure-projective if it is projective with respect to every pure short exact sequence in $\mathscr{G}$. The class of all pure projective objects of $\mathscr{G}$ will be denoted by $\mathscr{P}\mathscr{P}_{\mathscr{G}}$.

Using the definition, we immediately get that every finitely presented object is pure-projective, but the reverse is not true in general. In fact, it is well known that a module is pure projective if and only if it is a direct summand of a direct sum of finitely presented modules. In Proposition 1.1.6, we will prove the categorical version of this fact. But first, we will prove the following result which is well known in the category of modules.

Proposition 1.1.5. *We suppose that the category $\mathscr{G}$ is locally finitely presented. Then, for every object M there exists a pure epimorphism $\oplus F_i \to M$ such that each F_i is finitely presented.*

Proof. Let $\mathscr{S}_{\mathscr{G}}$ be a representative set of finitely presented objects in $\mathscr{G}$. For $F \in \mathscr{S}_{\mathscr{G}}$, put $\Delta_F = \mathrm{Hom}_{\mathscr{G}}(F,M)$ and consider the morphism $\phi_F : F^{(\Delta_F)} \to M$ such that for every $f \in \Delta_F$, $\phi_F k_f = f$ where $k_f : F \to F^{(\Delta_F)}$ is the canonical injection. These can be extended to $\phi : \oplus_{F\in\mathscr{S}_{\mathscr{G}}} F^{(\Delta_F)} \to M$. Then, for every $F \in \mathscr{S}_{\mathscr{G}}$ and every $f \in \Delta_F$, $\phi\mu_F k_f = f$ ($\mu_F : F^{(\Delta_F)} \to \oplus_{F\in\mathscr{S}_{\mathscr{G}}} F^{(\Delta_F)}$ is the canonical injection). Thus, for every $F \in \mathscr{S}_{\mathscr{G}}$, $\mathrm{Hom}_{\mathscr{G}}(F,\phi)$ is epic.

Now, it remains to prove that $\phi : \oplus_{F\in\mathscr{S}_{\mathscr{G}}} F^{(\Delta_F)} \to M$ is epic. For let $g : M \to X$ be a morphim such that $g\phi = 0$. Now, since the category is locally finitely presented there exists an epimorphism $\psi : \oplus_{F\in\mathscr{X}} F \to M$ for some set $\mathscr{X} \subseteq \mathscr{S}_{\mathscr{G}}$. Then, for every $P \in \mathscr{X}$, $\phi\mu_P k_{\psi\eta_P} = \psi\eta_P$ where $\eta_P : P \to \oplus_{F\in\mathscr{X}} F$ is the canonical injection. Thus, $g\psi\eta_P = 0$ for every $P \in \mathscr{X}$ and then $g\psi = 0$. But ψ is epic, so $g = 0$. Thus $\phi : \oplus_{F\in\mathscr{S}_{\mathscr{G}}} F^{(\Delta_F)} \to M$ is a pure-epimorphism. □

Proposition 1.1.6. *We suppose that the category is locally finitely presented. An object is pure-projective if and only if it is a direct summand of a direct sum of finitely presented objects.*

Proof. Let P be a pure-projective object and $g : \oplus F \to P$ be a pure epimorphism such that each F is finitely presented which exists by Proposition 1.1.5. Thus, $g : \oplus F \to P$ splits. That is, P is a direct summand of $\oplus F$.
The converse holds true since every finitely presented object is pure-projective, a direct sum of pure projective objects is pure-projective and a direct summand of a pure projective object is again pure projective. □

1.2 Category of complexes

In this section we fix some notations from [15] and recall some definitions and basic results on the category of complexes that will be used throughout this memory.

By a complex X of objects of $\mathscr{A}$ we mean a sequence of objects and morphisms in $\mathscr{A}$

$$\cdots \longrightarrow X_2 \xrightarrow{d_2} X_1 \xrightarrow{d_1} X_0 \xrightarrow{d_0} X_{-1} \xrightarrow{d_{-1}} X_{-2} \longrightarrow \cdots$$

such that $d_n d_{n+1} = 0$ for all $n \in \mathbb{Z}$. If $\mathrm{Im}\, d_{n+1} = \mathrm{Ker}\ d_n$ for all $n \in \mathbb{Z}$ then we say that X is exact. We denote by $\varepsilon_n^X : X_n \to \mathrm{Im}\, d_n$ the canonical epimorphism and by $\mu_n^X : \mathrm{Ker}(d_{n-1}) \to X_{n-1}$ the canonical monomorphism.

The n^{th} boundary (respectively, cycle, homology) of a complex X is defined as $\mathrm{Im}\ d^X_{n+1}$ (respectively, $\mathrm{Ker}\, d^X_n$, $\mathrm{Ker}\, d^X_n/\mathrm{Im}\ d^X_{n+1}$) and it is denoted by $B_n(X)$ (respectively, $Z_n(X)$, $H_n(X)$). The elements in $B_n(X)$ are called boundaries and the elements in $Z_n(X)$ are called cycles.

Given a class $\mathscr{L}$ of objects in $\mathscr{A}$, a complex

$$X:\ \cdots \longrightarrow X_{i+1} \xrightarrow{d_{i+1}} X_i \xrightarrow{d_i} X_{i-1} \longrightarrow \cdots$$

is said to be $\mathrm{Hom}_{\mathscr{A}}(\mathscr{L},-)$-exact (resp., $\mathrm{Hom}_{\mathscr{A}}(-,\mathscr{L})$-exact) if it becomes exact after applying $\mathrm{Hom}_{\mathscr{A}}(L,-)$ (resp., $\mathrm{Hom}_{\mathscr{A}}(-,L)$) for every $L \in \mathscr{L}$.

Throughout the thesis, we use the following particular kind of complexes:

Disc complex. Given an object M, we denote by $\overline{M}$ the complex

$$\cdots \longrightarrow 0 \longrightarrow M \xrightarrow{\mathrm{id}_M} M \longrightarrow 0 \longrightarrow \cdots$$

with all terms 0 except M in the degrees 1 and 0.

Sphere complex. Also, for an object M, we denote by $\underline{M}$ the complex

$$\cdots \longrightarrow 0 \longrightarrow M \longrightarrow 0 \longrightarrow \cdots$$

with all terms 0 except M in the degree 0.

Shift complex. Let X be a complex with differential d^X and fix an integer n. We denote by $X[n]$ the complex consisting of X_{i-n} in degree i with differential $(-1)^n d^X_{i-n}$.

Now, by a morphism of complexes $f: X \to Y$ we mean a family of morphisms $f_i : X_i \to Y_i$ such that $d^Y_i f_i = f_{i-1} d^X_i$ for all $i \in \mathbb{Z}$. That is, the following diagram commutes

$$\begin{tikzcd}
\cdots \arrow[r] & X_{i+1} \arrow[r, "d^X_{i+1}"] \arrow[d, "f_{i+1}"] & X_i \arrow[r, "d^X_i"] \arrow[d, "f_i"] & X_{i-1} \arrow[r] \arrow[d, "f_{i-1}"] & \cdots \\
\cdots \arrow[r] & Y_{i+1} \arrow[r, "d^Y_{i+1}"'] & Y_i \arrow[r, "d^Y_i"'] & Y_{i-1} \arrow[r] & \cdots
\end{tikzcd}$$

The category of complexes of $\mathscr{A}$ will be denoted by $\mathscr{C}(\mathscr{A})$. The category of complexes of modules over the ring R will be denoted by $\mathscr{C}(R)$.

It follows straight from the definition of a morphism of complexes $f: X \to Y$ that it maps boundaries to boundaries and cycles to cycles. Thus, it induces a family of morphisms $H_n(f)$ in homology

$$\begin{tikzcd}
0 \arrow[r] & B_n(X) \arrow[r] \arrow[d] & Z_n(X) \arrow[r] \arrow[d] & H_n(X) \arrow[r] \arrow[d, "H_n(f)"] & 0 . \\
0 \arrow[r] & B_n(Y) \arrow[r] & Z_n(Y) \arrow[r] & H_n(Y) \arrow[r] & 0
\end{tikzcd} \tag{1.1}$$

A morphism $f : X \to Y$ of complexes is called a quasi-isomorphism if the induced morphisms $H_n(f) : H_n(X) \to H_n(N)$ are all isomorphisms. A quasi-isomorphism is marked by a $\backsimeq$ next to the arrow.

Remark 1.2.1. *It is immediate from the definition of a morphism of complexes that an epic morphism is epic on boundaries. An application of the Snake Lemma to the diagram 1.1 shows that an epic quasi-isomorphism is epic on cycles as well. On the other hand, a quasi-isomorphism that is epic on cycles is also epic on boundaries (apply Snake lemma to the diagram 1.1) and hence epic as morphism of complexes as we can deduce from the diagrams*

$$\begin{tikzcd}
0 \arrow[r] & Z_n(X) \arrow[r] \arrow[d] & X_n \arrow[r] \arrow[d] & B_{n-1}(X) \arrow[r] \arrow[d] & 0 . \\
0 \arrow[r] & Z_n(Y) \arrow[r] & Y_n \arrow[r] & B_{n-1}(Y) \arrow[r] & 0
\end{tikzcd}$$

A morphism of complexes $f : X \to Y$ is said to be null-homotopic if, for all $n \in \mathbb{Z}$, there exist morphisms $s_n : X_n \to Y_{n+1}$ such that for any n we have $f_n = d^Y_{n+1} s_n + s_{n-1} d^X_n$, and then we say that f is null-homotopic by s. For a complex X, id_X is null-homotopic if and only if X is of the form $\oplus_{n\in\mathbb{Z}} \overline{M_n}[n]$ for some family of objects M_n. A complex of this special type is called contractible.

Two morphisms of complexes f and g are homotopic, $f \sim g$ in symbols, if $f - g$ is null-homotopic. The relation $f \sim g$ is an equivalence relation. The homotopy category $\mathscr{K}(\mathscr{A})$ is defined as the one having the same objects as $\mathscr{C}(\mathscr{A})$, and which morphisms are homotopy equivalence classes of morphisms in $\mathscr{C}(\mathscr{A})$.

For complexes X and Y, we let $\mathrm{Hom}^\bullet(X,Y)$ denote the complex of abelian groups with

$$\mathrm{Hom}^\bullet(X,Y)_n = \prod_{i\in\mathbb{Z}} \mathrm{Hom}_{\mathscr{A}}(X_i, Y_{i+n})$$

and

$$d_n^{\mathrm{Hom}^\bullet(X,Y)}(\psi) = (d^Y_{i+n}\psi_i - (-1)^n \psi_{i-1} d^X_i)_{i\in\mathbb{Z}}.$$

Note that for every $n \in \mathbb{Z}$,

$$Z_n(\mathrm{Hom}^\bullet(X,Y)) = \mathrm{Hom}_{\mathscr{C}(\mathscr{A})}(X[n],Y) = \mathrm{Hom}_{\mathscr{C}(\mathscr{A})}(X,Y[-n])$$

and

$$H_n(\mathrm{Hom}^\bullet(X,Y)) = \mathrm{Hom}_{\mathscr{K}(\mathscr{A})}(X[n],Y) = \mathrm{Hom}_{\mathscr{K}(\mathscr{A})}(X,Y[-n]).$$

For every complex X, $\mathrm{Hom}^\bullet(X,-)$ is a left exact functor from the category of complexes of $\mathscr{A}$ to the category of complexes of abelian groups.

The following characterization of projective complexes is well known in the case of complexes of modules. We recall its extension to complexes in an abelian category from [34, Proposition 2.3.6].

Proposition 1.2.2. *For a complex P of $\mathscr{C}(\mathscr{A})$, the following conditions are equivalent.*

1. *P is projective as an object of $\mathscr{C}(\mathscr{A})$.*
2. *P is contractible and the components P_i are all projective in $\mathscr{A}$.*
3. *P is exact and the cycles $Z_i(P)$ are all projective in $\mathscr{A}$.*

Recall that a complex P in $\mathscr{C}(\mathscr{A})$ is said to be DG-projective if its components are projective and $\mathrm{Hom}^{\bullet}(P,E)$ is exact for every exact complex E. The class of DG-projective complexes will be denoted by $\mathscr{DGP}_{\mathscr{C}(\mathscr{A})}$.

We end this setion by proving the categorical version of some properties of DG-projective complexes in order to use them later.

Recentely, a study of DG-projective complexes over some general categories was done in [39]. From this study we can deduce the following result which is well known for DG-projective complexes of modules.

Proposition 1.2.3. *If $\mathscr{A}$ is locally finitely presented, then for every complex N there exists a DG-projective complex X and an epic quasi-isomorphism $f : X \to N$.*

Proof. From [39, Theorem 6.6] there exists a special DG-projective precover $f : X \to N$, that is, f is epic and $\mathrm{Ker}f$ holds in $\mathscr{DGP}^{\perp}_{\mathscr{C}(\mathscr{A})}$, which is the class of exact complexes. Thus, $f : X \to N$ is a quasi-isomorphism. □

The following characterization of DG-projective complexes is well known in the case of complexes of modules (see for instance [17, (3.2.5) Theorem]).

Proposition 1.2.4. *A complex P is DG-projective if and only if* $\mathrm{Hom}^{\bullet}(P,-)$ *preserves epic quasi-isomorphisms.*

Proof. Suppose that P is DG-projective and let $g : A \to B$ be an epic quasi-isomorphism. If we consider the long exact sequence of homology of $0 \to K \to A \xrightarrow{g} B \to 0$, we get that K is exact. Then, if we consider the long exact sequence of homology of $0 \longrightarrow \mathrm{Hom}^{\bullet}(P,K) \longrightarrow \mathrm{Hom}^{\bullet}(P,A) \xrightarrow{\mathrm{Hom}^{\bullet}(P,g)} \mathrm{Hom}^{\bullet}(P,B) \longrightarrow 0$ we get that $\mathrm{Hom}^{\bullet}(P,g)$ is a quasi-isomorphism since $\mathrm{Hom}^{\bullet}(P,K)$ is exact (K is exact and P is supposed to be DG-projective). On the other hand, $\mathrm{Hom}_{\mathscr{A}}(P_m, g_n)$ is epic for every $m,n \in \mathbb{Z}$ since each P_m is projective. Thus, $\mathrm{Hom}^{\bullet}(P,g)$ is epic.

Conversely, let $g : Q \to P$ be an epic quasi-isomorphism with Q is DG-projective (see Proposition 1.2.3). Then, $\mathrm{Hom}^\bullet(P,g)$ is an epic quasi-ismorphism, hence, we get $\mathrm{Hom}_{\mathscr{C}(\mathscr{A})}(P,g) = Z_0(\mathrm{Hom}^\bullet(P,g))$ is epic by Remarak 1.2.1. Thus, P is a direct summand of Q which is DG-projective. Therefore, P is DG-projective. □

Proposition 1.2.5. *A complex is DG-projective and exact if and only if it is projective.*

Proof. Let P be an exact DG-projective complex and $g : Q \to P$ an epimorphism with Q projective. The epimorphism $g : Q \to P$ is a quasi-isomorphism since Q is also exact. Then, $\mathrm{Hom}^\bullet(P,g)$ is an epic quasi-isomorphism since P is DG-projective. Thus, we get by Remarak 1.2.1 that $\mathrm{Hom}^\bullet(P,g)$ is epic on cycles. In particular, we get that $\mathrm{Hom}_{\mathscr{C}(\mathscr{A})}(P,g) = Z_0(\mathrm{Hom}^\bullet(P,g))$ is epic. Thus, P is a direct summand of Q, hence it is projective.

Conversely, let P be a projective complex. Then, P is contractible with projective components. Thus, for every exact complex E and every $n \in \mathbb{Z}$, $H_n(\mathrm{Hom}^\bullet(P,E)) = \mathrm{Hom}_{\mathscr{K}(\mathscr{A})}(P,E[n]) = 0$ since P is contractible (that is, id_P is null-homotopic). Thus, $\mathrm{Hom}^\bullet(P,E)$ is exact whenever E is exact. □

1.3 Gorenstein projective objects

A complete projective resolution of an object M is an exact sequence of projective objects
$$\mathbf{P} : \cdots \to P_1 \to P_0 \to P_{-1} \to \cdots$$
which is $\mathrm{Hom}_{\mathscr{A}}(-,Proj_{\mathscr{A}})$-exact with, $M = Z_0(\mathbf{P})$.

An object G is called Gorenstein projective, if it has a complete projective resolution. We use $\mathscr{G}\mathscr{P}_{\mathscr{A}}$ to denote the class of all Gorenstein projective objects.

If $\mathbf{P}$ is a complete projective resolution of an object, then by symmetry, all the kernels of $\mathbf{P}$ are Gorenstein projective objects. It is clear that every projective objects is Gorenstein projective.

Using the definition, we immediately get the following characterization of Gorenstein projective objects which is well known in the case of category of modules and the one of complexes of modules.

Proposition 1.3.1. *An object G is Gorenstein projective if and only if there exists an exact complex of projectives P_i:*
$$0 \to G \to P_0 \to P_{-1} \to \cdots$$
which is $\mathrm{Hom}_{\mathscr{A}}(-,Proj_{\mathscr{A}})$*-exact, and* $\mathrm{Ext}^i_{\mathscr{A}}(G,Q) = 0$ *for all $i \geqslant 1$ and all projective objects Q.*

In [10], a particular case of Gorenstein projective modules was introduced and these modules were called strongly Gorenstein projectives. The reason behind the introduction of this class of modules was giving a nice characterization for Gorenstein projective modules. Namely, the authors proved that a module is Gorenstein projective if and only if it is a direct summand of a strongly Gorenstein projective module. Here, we extend this fact to abelian categories.

Definition 1.3.2. *An object M of $\mathscr{A}$ is said to be strongly Gorenstein projective if it has a complete projective resolution of the form*

$$\cdots \longrightarrow P \xrightarrow{f} P \xrightarrow{f} P \xrightarrow{f} \cdots$$

We use $\mathscr{SGP}_{\mathscr{A}}$ to denote the class of all strongly Gorenstein projective objects of $\mathscr{A}$.

We prove the following characterization of strongly Gorenstein projective objects, which was given in [10, Proposition 2.9] for modules.

Proposition 1.3.3. *The following conditions are equivalent for an object M.*

1. *M is strongly Gorenstein projective.*
2. *There is a short exact sequence $0 \to M \to P \to M \to 0$ which is $\mathrm{Hom}_{\mathscr{A}}(-, Proj_{\mathscr{A}})$-exact and with P projective.*
3. *There is a short exact sequence $0 \to M \to P \to M \to 0$ with P projective and $\mathrm{Ext}^1_{\mathscr{A}}(M,Q) = 0$ for every $Q \in Proj_{\mathscr{A}}$*

Proof. 1. $\Leftrightarrow$ 2. is clear.

For 2. $\Leftrightarrow$ 3. we consider the exact sequence $\cdots \to \mathrm{Hom}_{\mathscr{A}}(P,Q) \to \mathrm{Hom}_{\mathscr{A}}(M,Q) \to \mathrm{Ext}^1_{\mathscr{A}}(M,Q) \to \mathrm{Ext}^1_{\mathscr{A}}(P,Q) = 0 \to \cdots$ for every $Q \in Proj_{\mathscr{A}}$. Then, $0 \to M \to P \to M \to 0$ is $\mathrm{Hom}_{\mathscr{A}}(-,Q)$-exact if and only if $\mathrm{Ext}^1_{\mathscr{A}}(M,Q) = 0$ for every projective object Q. $\square$

Proposition 1.3.4. *If $\mathscr{A}$ has direct sums and they are exact, then an object is Gorenstein projective if and only if it is a direct summand of a strongly Gorenstein projective one.*

Proof. Let G be a Gorenstein projective object. Then, G has a complete projective resolution

$$\mathbf{P}: \ \cdots \to P_1 \to P_0 \to P_{-1} \to \cdots$$

Consider the exact sequences

$$0 \to Z_n(\mathbf{P}) \to P_n \to Z_{n-1}(\mathbf{P}) \to 0$$

and let

$$0 \to \oplus_n Z_n(\mathbf{P}) \to \oplus_n P_n \to \oplus_n Z_{n-1}(\mathbf{P}) \to 0$$

be their direct sum which is exact since direct sums are exact. Now, let Q be a projective object in $\mathscr{A}$ and consider the following commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathscr{A}}(\oplus_n P_n, Q) \arrow[r] \arrow[d, "\cong"'] & \mathrm{Hom}_{\mathscr{A}}(\oplus_n Z_n(\mathbf{P}), Q) \arrow[d, "\cong"] \\
\prod_n \mathrm{Hom}_{\mathscr{A}}(P_n, Q) \arrow[r] & \prod_n \mathrm{Hom}_{\mathscr{A}}(Z_n(\mathbf{P}), Q)
\end{tikzcd}$$

The second row is epic since $0 \to Z_n(\mathbf{P}) \to P_n \to Z_{n-1}(\mathbf{P}) \to 0$ is $\mathrm{Hom}_{\mathscr{A}}(-,Q)$-exact for every n. Then, the first one is also epic, thus $0 \to \oplus_n Z_n(\mathbf{P}) \to \oplus_n P_n \to \oplus_n Z_{n-1}(\mathbf{P}) \to 0$ is also $\mathrm{Hom}_{\mathscr{A}}(-,Q)$-exact. Therefore, $\oplus_n Z_n(\mathbf{P})$ is strongly Gorenstein projective.

The converse holds since the class of Gorenstein projective objects is closed under direct summands (see the proof of [30, Theorem 2.5]) and every strongly Gorenstein projective object is Gorenstein projective. □

In [47, Theorem 2.2], it was proven that a complex of modules is Gorenstein projective if and only if its components are Gorenstein projective. Using the same arguments one can extend this fact to complexes on $\mathscr{A}$.

Proposition 1.3.5. *A complex is Gorenstein projective in $\mathscr{C}(\mathscr{A})$ if and only if its components are Gorenstein projective in $\mathscr{A}$.*

MEASURING PROJECTIVITY IN ABELIAN CATEGORIES

In this chapter we develop a new treatment of the subprojectivity in any abelian category $\mathscr{A}$ with enough projectives. Namely, in terms of subprojectivity, some classical results are unified and some classical rings are characterized. It is also shown that, in some categories, the subprojectivity measures notions other than the projectivity. Furthermore, this new approach allows, in addition to establishing nice generalizations of known results, to construct various new examples such as the subprojectivity domain of the class of Gorenstein projective objects, the class of DG-projective complexes and particular types of representations of a finite linear quiver.

The chapter ends with a study showing that the fact that the subprojectivity domain of a class coincides with its first right Ext-orthogonal class can be characterized in terms of the existence of precovers and preenvelopes.

2.1 Subprojectivity domains

Subprojectivity of objects is a notion studied up to a certain level of deepness in categories of modules. However, it is a categorical type concept which has not even been considered in this general setting. The aim of this section is thus to explore the meaning of subprojectivity domains of objects and classes in nice categories from the homological point of view: abelian categories.

We start by recalling what subprojectivity means.

Definition 2.1.1 ([28])**.** *Given two objects M and N in $\mathscr{A}$, M is said to be N-subprojective if for every morphism $f : M \to N$ and every epimorphism $g : K \to N$, there exists a morphism $h : M \to K$ such that $gh = f$.*

The subprojectivity domain, or domain of subprojectivity, of M is defined as the class

$$\underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(M) := \{N \in \mathscr{A} : M \text{ is } N\text{-subprojective}\}.$$

As mentioned in the introduction, subprojectivity domains were introduced in [28] to, somehow, measure the projectivity of modules. So for instance it is clear that a module is projective precisely when its subprojectivity domain is the whole category R-Mod. Of course, one immediately sees that this is not a situation which holds just in module categories. On the opposite, it does in every abelian category with enough projectives. We state it as a proposition.

Proposition 2.1.2. *Let M be an object of $\mathscr{A}$. Then the following statements are equivalent.*

1. $\underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(M)$ *is the whole abelian category* $\mathscr{A}$.
2. *M is projective.*
3. $M \in \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(M)$.

Proof. (1) $\Rightarrow$ (3) is clear.

For (3) $\Rightarrow$ (2) let $g : P \to M$ be an epimorphism with P projective. Then, $\mathrm{Hom}_{\mathscr{A}}(M, g)$ is epic since $M \in \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(M)$ and then M is a direct summand of P, that is, M is projective.

To prove (2) $\Rightarrow$ (1) let A be an object of $\mathscr{A}$ and $g : B \to A$ be an epimorphism. Then, $\mathrm{Hom}_{\mathscr{A}}(M, g)$ is epic since M is projective. Thus, $A \in \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(M)$. □

But in some cases subprojectivity can measure notions other than that of projectivity. We give two examples showing this fact.

Example 2.1.3.

- *Sometimes, the exactness of a complex can be guaranteed whenever it is known its membership to the subprojectivity domain of a DG-projective complex. Namely, if $\mathscr{A}$ is locally finitely presented, N is any complex in $\mathscr{C}(\mathscr{A})$ and $f : X \to N$ is an epic quasi-isomorphism where X is DG-projective, then $N \in \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(X)$ if and only if N is exact. Indeed, suppose that $N \in \underline{\mathfrak{Pr}}_{\mathscr{C}(\mathscr{A})}^{-1}(X)$ and consider an epimorphism $g : P \to N$ with P projective. Then, there exists a morphism $h : X \to P$ such that $f = gh$, so $H_n(f) = H_n(g)H_n(h)$ for every $n \in \mathbb{Z}$, and since f is a quasi-isomorphism, all the morphisms $H_n(g) : H_n(P) \to H_n(N)$ are epimorphisms. But P is exact, so N is also exact.*

  Conversely, if N is exact then X is also exact since they are quasi-isomorphic, so X is projective by Proposition 1.2.5. Therefore, $N \in \underline{\mathfrak{Pr}}_{\mathscr{C}(\mathscr{A})}^{-1}(X)$ by Proposition 2.1.2.

- *In some cases, subprojectivity in the category of representations by modules of the linear quiver with two vertices and one arrow, denoted by* A_2*, characterizes monomorphisms in R-Mod. Namely, if* $g : N_1 \to N_2$ *is a morphism of modules then* $g : N_1 \to N_2$ *is monic if and only if* $N_1 \xrightarrow{g} N_2 \in \underline{\mathfrak{Pr}}^{-1}_{Rep(A_2)}(R \to 0)$ *where* $Rep(A_2)$ *denote the category of reperesentations of* A_2*. To prove this we first state and prove the following equivalence for any module M*

$$N_1 \xrightarrow{g} N_2 \in \underline{\mathfrak{Pr}}^{-1}_{Rep(A_2)}(M \to 0) \Leftrightarrow \mathrm{Hom}_R(M, \mathrm{Ker}g) = 0.$$

  For the first implication, we suppose that $N_1 \xrightarrow{g} N_2 \in \underline{\mathfrak{Pr}}^{-1}_{Rep(A_2)}(M \to 0)$*, let* $f : M \to \mathrm{Ker}g$ *be a morphism of modules and* $i : \mathrm{Ker}g \to N_1$ *be the canonical injection. We get the following commutative diagram*

$$\begin{tikzcd}
& & & M \arrow[rr] \arrow[dlll, dashed, "h"'] \arrow[ddll, "if"] & & 0 \arrow[dlll, dashed] \arrow[ddll] \\
P_1 \arrow[rr, "\pi"] \arrow[dr, "\beta"'] & & P_2 \arrow[dr, "\alpha"] & & & \\
& N_1 \arrow[rr, "g"'] & & N_2 & &
\end{tikzcd}$$

  where $P_1 \xrightarrow{\pi} P_2$ *is a projective representation and* (β, α) *is an epimorphism in the category* $Rep(A_2)$ *which exist by [23, Theorem 5.1.3]. Therefore,* $\pi h = 0$ *so* $h = 0$ *since* π *is monic (see [21, Theorem 4.1]). Then,* $if = \beta h = 0$ *so* $f = 0$*.*

  Conversely, for every $(f, 0) \in \mathrm{Hom}_{Rep(A_2)}(M \to 0, N_1 \to N_2)$*,* $gf = 0$ *so there exists a morphism* $t : M \to \mathrm{Ker}g$ *such that* $f = it$*. Then* $\mathrm{Hom}_{Rep(A_2)}(M \to 0, N_1 \to N_2) = 0$ *since* $\mathrm{Hom}_R(M, \mathrm{Ker}g) = 0$*. So clearly* $N_1 \xrightarrow{g} N_2 \in \underline{\mathfrak{Pr}}^{-1}_{Rep(A_2)}(M \to 0)$*.*

  Now, if we apply the above equivalence to R we get that $N_1 \xrightarrow{g} N_2 \in \underline{\mathfrak{Pr}}^{-1}_{Rep(A_2)}(R \to 0)$ *if and only if* $\mathrm{Hom}_R(R, \mathrm{Ker}g) = 0$ *which is equivalent to* $\mathrm{Ker}g = 0$*.*

The following result provides new ways to treat and use subprojectivity.

Proposition 2.1.4. *Let M and N be two objects of* $\mathscr{A}$*. Then the following conditions are equivalent.*

1. *M is N-subprojective.*

2. *There exists a morphism* $g : P \to N$ *with P projective and* $\mathrm{Hom}_{\mathscr{A}}(M, g)$ *an epimorphism.*

3. *There exists a morphism* $g : P \to N$ *with* $P \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{A}}(M)$ *and* $\mathrm{Hom}_{\mathscr{A}}(M, g)$ *an epimorphism.*

4. *Every morphism $M \to N$ factors through a projective object.*

5. *Every morphism $M \to N$ factors through an object in $\underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(M)$.*

Proof. (1) $\Rightarrow$ (2) Suppose that M is N-subprojective and let $g : P \to N$ be an epimorphism with P projective. Thus, $\mathrm{Hom}_{\mathscr{A}}(M,g)$ is an epimorphism since M is N-subprojective.
(2) $\Rightarrow$ (4) is clear
(2) $\Rightarrow$ (3) and (4) $\Rightarrow$ (5) are clear since every projective object holds in $\underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(M)$.
(3) $\Rightarrow$ (1) Assume that there exists such a morphism $g : P \to N$ and let $K \to N$ be an epimorphism. Then apply $\mathrm{Hom}_{\mathscr{A}}(M,-)$ to the pullback diagram

$$\begin{tikzcd} D \arrow[r] \arrow[d] & P \arrow[r] \arrow[d, "g"] & 0 \\ K \arrow[r] & N \arrow[r] & 0 \end{tikzcd}$$

to get

$$\begin{tikzcd} \mathrm{Hom}_{\mathscr{A}}(M,D) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathscr{A}}(M,P) \arrow[d, "\mathrm{Hom}_{\mathscr{A}}(M,g)"] \\ \mathrm{Hom}_{\mathscr{A}}(M,K) \arrow[r] & \mathrm{Hom}_{\mathscr{A}}(M,N). \end{tikzcd}$$

Thus, $\mathrm{Hom}_{\mathscr{A}}(M,K) \to \mathrm{Hom}_{\mathscr{A}}(M,N)$ is epic since $\mathrm{Hom}_{\mathscr{A}}(M,D) \to \mathrm{Hom}_{\mathscr{A}}(M,P)$ and $\mathrm{Hom}_{\mathscr{A}}(M,g)$ are epimorphisms by assumption.
(5) $\Rightarrow$ (1) Consider any morphism $f : M \to N$ and any epimorphism $g : K \to N$. Then, by the assumption, there exists an object $L \in \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(M)$ and a commutative diagram

$$\begin{tikzcd} & M \arrow[dl, "\alpha"'] \arrow[dd, "f"] & \\ L \arrow[dr, "\beta"] & & \\ K \arrow[r, "g"'] & N \arrow[r] & 0. \end{tikzcd}$$

Now let $k : P \to L$ be an epimorphism with P projective. Then, by the projectivity of P and the fact that $L \in \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(M)$, the diagram

$$\begin{tikzcd} & & M \arrow[dll, dashed, "\gamma"'] \arrow[dl, "\alpha"'] \arrow[dd, "f"] & \\ P \arrow[r, "k"] \arrow[dr, dashed, "h"'] & L \arrow[dr, "\beta"] & & \\ & K \arrow[r, "g"'] & N \arrow[r] & 0 \end{tikzcd}$$

can be completed commutatively. Thus, $N \in \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(M)$ $\square$

We now give an example, in the context of representations of quivers by modules, of the usefulness of the above fact. But first we recall that the linear quiver

$$v_n \to v_{n-1} \to \cdots \to v_2 \to v_1$$

is denoted by A_n and the category of representations of A_n is denoted by $Rep(A_n)$. As in [21], we use $\overline{M}[i]$, for a module M, to denote the representation

$$0 \to 0 \to \cdots \to 0 \to M \xrightarrow{id} \cdots \xrightarrow{id} M \xrightarrow{id} M$$

where the last M is in the i'th place.
Following [36, Section 2], we know that a representation

$$M_n \xrightarrow{f_{n-1}} M_{n-1} \longrightarrow \cdots \longrightarrow M_2 \xrightarrow{f_1} M_1$$

of A_n is projective if and only if it is a direct sum of the following projective representations:

$$\overline{P_1}[1]: \ 0 \to 0 \to \cdots \to 0 \to 0 \to P_1,$$

$$\overline{P_2}[2]: \ 0 \to 0 \to \cdots \to 0 \to P_2 \xrightarrow{id} P_2,$$

$$\vdots$$

$$\overline{P_n}[n]: \ P_n \xrightarrow{id} P_n \xrightarrow{id} \cdots \xrightarrow{id} P_n \xrightarrow{id} P_n \xrightarrow{id} P_n,$$

where the P_i's are all projective modules. Thus, for a module M, the representation $\overline{M}[i]$ is projective if and only if M is projective. We generalize this fact to the case of subprojectivity.

Proposition 2.1.5. *If M is a module and $(N,\delta) = N_n \overset{\delta_n}{\to} N_{n-1} \overset{\delta_{n-1}}{\to} \cdots \overset{\delta_3}{\to} N_2 \overset{\delta_2}{\to} N_1$ is a representation of A_n ($n \geq 2$) in R-*Mod*. Then, for an integer $1 \leq i \leq n$, $N \in \underline{\mathfrak{Pr}}_{Rep(A_n)}^{-1}(\overline{M}[i])$ if and only if $N_i \in \underline{\mathfrak{Pr}}_{R\text{-Mod}}^{-1}(M)$.*

Proof. For simplicity in notation we only prove the case of A_2 (A_n follows by the same arguments). Thus, we just need to discuss two cases: $0 \to M$ and $M \xrightarrow{id} M$.

1. Choose any representation $\mathbf{N} : N_2 \to N_1$, and any epimorphism $(\alpha,\beta) : \mathbf{P} \to \mathbf{N}$ from a projective representation $\mathbf{P} : P_2 \to P_1 \in Rep(A_2)$. Suppose that $\mathbf{N} \in \underline{\mathfrak{Pr}}_{Rep(A_2)}^{-1}(0 \to$

M) and let $f : M \to N_1$ be any morphism of modules. Then, $(0, f) : (0 \to M) \to \mathbf{N}$ is a morphism of representations and the diagram

$$\begin{tikzcd}
 & 0 \arrow[rr] \arrow[dl, dashed] \arrow[dd] & & M \arrow[dl, dashed, "h"'] \arrow[dd, "f"] \\
P_2 \arrow[rr] \arrow[dr, "\alpha"'] & & P_1 \arrow[dr, "\beta"'] & \\
 & N_2 \arrow[rr] & & N_1
\end{tikzcd}$$

can be completed commutatively. Therefore, $f = \beta h$ and, by Proposition 2.1.4, $N_1 \in \underline{\mathfrak{Pr}}^{-1}_{R\text{-Mod}}(M)$.

Conversely, suppose that $N_1 \in \underline{\mathfrak{Pr}}^{-1}_{R\text{-Mod}}(M)$ and let $(0, f) : (0 \to M) \to \mathbf{N}$ be a morphism of representations. Then, there exists a morphism $h : M \to P_1$ such that $f = \beta h$. Therefore, $(0, f) = (\alpha, \beta)(0, h)$ and, by Proposition 2.1.4, $\mathbf{N} \in \underline{\mathfrak{Pr}}^{-1}_{Rep(A_2)}(0 \to M)$.

2. To prove the necessary condition we choose any representation $N_2 \to N_1 \in \underline{\mathfrak{Pr}}^{-1}_{Rep(A_2)}(M \xrightarrow{id} M)$ and any morphism of modules $f : M \to N_2$. Then, there exists a morphism of representations (k, h) completing commutatively the diagram

$$\begin{tikzcd}
 & M \arrow[rr, equal] \arrow[dl, dashed, "k"'] \arrow[dd, "f"] & & M \arrow[dl, dashed, "h"'] \arrow[dd, "gf"] \\
P_2 \arrow[rr] \arrow[dr, "\alpha"'] & & P_1 \arrow[dr, "\beta"'] & \\
 & N_2 \arrow[rr, "g"] & & N_1
\end{tikzcd}$$

where $P_2 \to P_1$ is a projective representation and (α, β) an epimorphism in the category $Rep(A_2)$. Therefore $f = \alpha k$ and then, again by Proposition 2.1.4, $N_2 \in \underline{\mathfrak{Pr}}^{-1}_{R\text{-Mod}}(M)$.

Conversely, let $N_2 \to N_1$ be a representation in $Rep(A_2)$. Suppose that N_2 is in $\underline{\mathfrak{Pr}}^{-1}_{R\text{-Mod}}(M)$ and consider a projective representation $P_2 \to P_1$, an epimorphism (α, β) from $P_2 \to P_1$ onto $N_2 \to N_1$, and a morphism of representations (f_2, f_1) from $M \xrightarrow{id} M$ to $N_2 \to N_1$. Then, there exists $h : M \to P_2$ such that $f_2 = \alpha h$. Therefore, we get the following commutative diagram

$$\begin{tikzcd}
 & M \arrow[rr, equal] \arrow[dl, "h"'] \arrow[dd, "f_2"] & & M \arrow[dl, "\pi h"'] \arrow[dd, "f_1"] \\
P_2 \arrow[rr, "\pi"] \arrow[dr, "\alpha"'] & & P_1 \arrow[dr, "\beta"'] & \\
 & N_2 \arrow[rr, "g"] & & N_1
\end{tikzcd}$$

so $f_1 = gf_2 = g\alpha h = \beta\pi h$ and hence $(f_2, f_1) = (\alpha,\beta)(h,\pi h)$. This means by Proposition 2.1.4 that $N_2 \to N_1 \in \underline{\mathfrak{Pr}}^{-1}_{Rep(A_2)}(M \xrightarrow{id} M)$. □

Last result leads to some interesting consequences. We start by the following.

Proposition 2.1.6. *Let* $0 \to M \to Q \to M' \to 0$ *be a short exact sequence with* Q *projective. Then* $M'^{\perp} \subseteq \underline{\mathfrak{Pr}}^{-1}_{\mathscr{A}}(M)$*. If moreover* $Proj_{\mathscr{A}} \subseteq M'^{\perp}$ *then* $\underline{\mathfrak{Pr}}^{-1}_{\mathscr{A}}(M) = M'^{\perp}$.

Proof. Let N be any object in $\mathscr{A}$ and consider the long exact sequence

$$\longrightarrow \mathrm{Hom}_{\mathscr{A}}(Q,N) \longrightarrow \mathrm{Hom}_{\mathscr{A}}(M,N) \longrightarrow \mathrm{Ext}^1_{\mathscr{A}}(M',N) \longrightarrow \mathrm{Ext}^1_{\mathscr{A}}(Q,N) \longrightarrow$$

If $N \in M'^{\perp}$ then $\mathrm{Ext}^1_{\mathscr{A}}(M',N) = 0$ so $\mathrm{Hom}_{\mathscr{A}}(Q,N) \to \mathrm{Hom}_{\mathscr{A}}(M,N)$ is epic, that is, any morphism $M \to N$ factors through the projective object Q. Then, by Proposition 2.1.4 we deduce that $N \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{A}}(M)$.

Suppose in addition that $Proj(\mathscr{A}) \subseteq M'^{\perp}$. Let $N \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{A}}(M)$ and let $P \to N$ be an epimorphism. We apply the functors $\mathrm{Hom}_{\mathscr{A}}(-,P)$ and $\mathrm{Hom}_{\mathscr{A}}(-,N)$ to $M \to Q$ to get the following commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathscr{A}}(Q,P) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathscr{A}}(M,P) \arrow[r] \arrow[d] & \mathrm{Ext}^1_{\mathscr{A}}(M',P) \arrow[r] & \mathrm{Ext}^1_{\mathscr{A}}(Q,P) \\
\mathrm{Hom}_{\mathscr{A}}(Q,N) \arrow[r] & \mathrm{Hom}_{\mathscr{A}}(M,N) \arrow[r] & \mathrm{Ext}^1_{\mathscr{A}}(M',N) \arrow[r] & \mathrm{Ext}^1_{\mathscr{A}}(Q,N)
\end{tikzcd}$$

Since Q is projective, $\mathrm{Ext}^1_{\mathscr{A}}(Q,N) = 0$. Then, to prove that $\mathrm{Ext}^1_{\mathscr{A}}(M',N) = 0$ it is sufficient to prove that $\mathrm{Hom}_{\mathscr{A}}(Q,N) \to \mathrm{Hom}_{\mathscr{A}}(M,N)$ is epic. But $N \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{A}}(M)$ so $\mathrm{Hom}_{\mathscr{A}}(M,P) \to \mathrm{Hom}_{\mathscr{A}}(M,N)$ is an epimorphism, and of course $\mathrm{Hom}_{\mathscr{A}}(Q,P) \to \mathrm{Hom}_{\mathscr{A}}(M,P)$ is epic ($Ext^1(M',P) = 0$ by assumption), so we get that $\mathrm{Hom}_{\mathscr{A}}(Q,N) \to \mathrm{Hom}_{\mathscr{A}}(M,N)$ is epic. □

An example of an object satisfying the condition of Proposition 2.1.6 can be found among strongly Gorenstein projective objects.

Corollary 2.1.7. *If* M *is a strongly Gorenstein projective object then* $\underline{\mathfrak{Pr}}^{-1}_{\mathscr{A}}(M) = M^{\perp}$.

Proof. Let $0 \to M \to Q \to M \to 0$ be a short exact sequence with Q projective. Then by Proposition 2.1.6 $\underline{\mathfrak{Pr}}^{-1}_{\mathscr{A}}(M) = M^{\perp}$ since $Proj(\mathscr{A}) \subseteq M^{\perp}$. □

The converse of Corollary 2.1.7 does not hold in general. A counterexample can be found in commutative local artinian principal ideal rings: in [9] it is proved that over one such a ring every module is 2-strongly Gorenstein projective, that is, there exists an exact sequence $0 \to M \to P_2 \to P_1 \to M \to 0$ with P_1 and P_2 projective and $M \in {}^{\perp}Proj(\mathscr{A})$.

Thus, using [12, Theorem 3.7] one can prove that if the ring admits more than two proper ideals then the maximal ideal cannot be strongly Gorenstein projective. So for instance, the ideal $(2+8\mathbb{Z})$ of the ring $\mathbb{Z}/8\mathbb{Z}$ is not strongly Gorenstein projective. However, we do have $\underline{\mathfrak{Pr}}^{-1}_{(\mathbb{Z}/8\mathbb{Z})-\mathrm{Mod}}(2+8\mathbb{Z}) = (2+8\mathbb{Z})^{\perp}$ by the following result.

Proposition 2.1.8. *If R is a commutative local artinian principal ideal ring, then for every module M, $\underline{\mathfrak{Pr}}^{-1}_{R\text{-Mod}}(M) = M^{\perp}$.*

Proof. We can assume that M is a non-projective module.

By [28, Proposition 4.5] we know that $\underline{\mathfrak{Pr}}^{-1}_{R\text{-Mod}}(M) = Proj_{R\text{-Mod}}$ so if we prove that $M^{\perp} = Proj_{R\text{-Mod}}$ we will be done.

Now, since R is a commutative local artinian principal ideal ring, every module is a direct sum of cyclic modules, and the only composition series of the ring is

$$0 = x^m R \subseteq x^{m-1}R \subseteq \cdots \subseteq xR = Rad(R) \subseteq R,$$

where x is a generator of $Rad(R)$ (the Jacobson radical of R). Therefore, the result will follow if we show that $Proj_{R\text{-Mod}} = (R/x^iR)^{\perp}$ for every $0 < i < m$ ($\underline{\mathfrak{Pr}}^{-1}_{R\text{-Mod}}(\oplus M_i) = \bigcap \underline{\mathfrak{Pr}}^{-1}_{R\text{-Mod}}(M_i)$ by [28, Proposition 2.10]).

Of course we have $Proj_{R\text{-Mod}} \subseteq (R/x^iR)^{\perp}$ since $Proj_{R\text{-Mod}} = Inj_{R\text{-Mod}}$ (R is a QF-ring). And on the other hand, if N is a non projective module then there is an i such that R/x^iR is a direct summand of N. This means $\mathrm{Ext}_R(R/x^iR, R/x^jR)$ is a direct summand of $\mathrm{Ext}_R(N, R/x^jR)$ for every j. But $\mathrm{Ext}_R(R/x^iR, R/x^jR) \neq 0$ for all $0 < i, j < m$ by [44, Example 4.5] so we are done. □

Proposition 2.1.6 says that if an object M can be embedded in a projective object then there exists an object M' such that $(M')^{\perp} \subseteq \underline{\mathfrak{Pr}}^{-1}_{\mathscr{A}}(M)$. Therefore, $\underline{\mathfrak{Pr}}^{-1}_{\mathscr{A}}(M)$ contains the class of injective objects. This fact was proved using different arguments in [19, Lemma 2.2] by giving a list of equivalences. The following result extends such a list of equivalent conditions.

Corollary 2.1.9. *Assume that $\mathscr{A}$ has enough injectives and let M be an object of $\mathscr{A}$. The following conditions are equivalent.*

1. *M can be embedded in a projective object P.*
2. *There exists an object M' such that $(M')^{\perp} \subseteq \underline{\mathfrak{Pr}}^{-1}_{\mathscr{A}}(M)$.*
3. *$Inj_{\mathscr{A}} \subseteq \underline{\mathfrak{Pr}}^{-1}_{\mathscr{A}}(M)$.*

Proof. (1) $\Rightarrow$ (2) If $j : M \to P$ is a monomorphism, then $(\mathrm{Coker}\, j)^{\perp} \subseteq \underline{\mathfrak{Pr}}^{-1}_{\mathscr{A}}(M)$ by Proposition 2.1.6.

(2) $\Rightarrow$ (3) Clear since $Inj_{\mathscr{A}} \subseteq M'^{\perp}$.

$(3) \Rightarrow (1)$ Let $f : M \to E$ be a monomorphism with E injective (f exists since $\mathscr{A}$ is assumed to have enough injectives). Then by (3) there exists two morphisms $\beta : M \to P$ and $\alpha : P \to E$ such that P is projective and $f = \alpha\beta$. Thus, $\beta : M \to P$ is a monomorphism since $f : M \to E$ is. $\square$

Now, we prove that when the object is finitely generated (and can be embedded in a projective object) then its subprojectivity domain contains a larger class than that of the injectives. Recall that an object A is said FP-injective if $\mathrm{Ext}^1_{\mathscr{A}}(F,A) = 0$ for every finitely presented object F, that is, $A \in \mathscr{F}\mathscr{P}^{\perp}_{\mathscr{A}}$. Recall that every object embeds in an FP-injective object (see [13, Corollary 3.7]).

Corollary 2.1.10. *Suppose that $\mathscr{A}$ is Grothendieck with a system of finitely generated projective generators and let M be a finitely generated object. The following conditions are equivalent.*

1. *M can be embedded in a projective object.*
2. *There exists a finitely presented object M' such that $(M')^{\perp} \subseteq \underline{\mathfrak{Pr}}^{-1}_{\mathscr{A}}(M)$.*
3. *$\mathscr{F}\mathscr{P}^{\perp}_{\mathscr{A}} \subseteq \underline{\mathfrak{Pr}}^{-1}_{\mathscr{A}}(M)$.*
4. *For an FP-injective preenvelope $i : M \hookrightarrow E$ of M, $E \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{A}}(M)$.*

Proof. $(1) \Rightarrow (2)$. Let P be a projective object such that M is embedded in P and $g : \oplus_{i\in I}F_i \to P$ be an epimorphism (which splits since P is projective) such that each F_i is finitely generated projective. Thus, there is a monomorphism $h : P \to \oplus_{i\in I}F_i$ such that $gh = \mathrm{id}_P$. By [35, Page 206] there is a finite subset J of I such that $h : P \to \oplus_{i\in I}F_i$ factors through $\oplus_{i\in J}F_i$. Then, P and so M can be embedded in the finitely generated projective object $\oplus_{i\in J}F_i$. Thus, $\oplus_{i\in J}F_i/M$ is finitely presented (see Proposition 1.1.4). Then, we apply Proposition 2.1.6 to $0 \to M \to \oplus_{i\in J}F_i \to \oplus_{i\in J}F_i/M \to 0$.

$(2) \Rightarrow (3)$ Holds since $\mathscr{F}\mathscr{P}^{\perp}_{\mathscr{A}} \subseteq (M')^{\perp}$.

$(3) \Rightarrow (4)$ Clear.

$(4) \Rightarrow (1)$. Let $g : P \to E$ be an epimorphism with P projective. Since $E \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{A}}(M)$, the diagram

$$\begin{tikzcd} & M \arrow[dl, dashed, "h"'] \arrow[d, "i"] & \\ P \arrow[r, "g"'] & E \arrow[r] & 0 \end{tikzcd}$$

can be completed commutatively by h. Thus, h must be injective. $\square$

We now fix our attention on classes of objects: we introduce and investigate subprojectivity domains of classes instead of just single objects. The subprojectivity domain of a class $\mathscr{X}$ is defined as the class of all objects holding in the subprojectivity domain of each object of $\mathscr{X}$.

Definition 2.1.11. *The subprojectivity domain, or domain of subprojectivity, of a class of objects $\mathscr{M}$ of $\mathscr{A}$ is defined as*

$$\underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(\mathscr{M}) := \{N \in \mathscr{A} : M \text{ is } N\text{-subprojective for every } M \in \mathscr{M}\}.$$

Therefore, if $\mathscr{M} := \{M\}$ then $\underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(\mathscr{M}) = \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(M)$.

Proposition 2.1.2 characterizes when the subprojectivity domain of an object is the whole abelian category $\mathscr{A}$. The following extension to classes of such a proposition can be used to unify various classical results.

Proposition 2.1.12. *Let $\mathscr{L}$ be a class of objects of $\mathscr{A}$. The following conditions are equivalent.*

1. *$\underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(\mathscr{L})$ is the whole abelian category $\mathscr{A}$.*
2. *Every object of $\mathscr{L}$ is projective.*
3. *$\mathscr{L} \subseteq \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(\mathscr{L})$.*

Proof. To prove $(1) \Rightarrow (2)$, let L be in $\mathscr{L}$ and $P \to L$ be an epimorphism with P projective. Since $L \in \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(L)$, $P \to L$ splits. Hence L is projective.

The implication $(2) \Rightarrow (3)$ is clear since $\underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(\mathscr{L})$ contains the class of projectives.

To prove $(3) \Rightarrow (1)$, consider an object L in $\mathscr{L}$. By assumption $L \in \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(L)$, hence L is projective (Proposition 2.1.2). So $\underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(L)$ coincide with $\mathscr{A}$ for any L in $\mathscr{L}$. Therefore, $\underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(\mathscr{L})$ is $\mathscr{A}$. $\square$

Recall that if $\mathscr{A}$ is Grothendieck, then an object F in $\mathscr{A}$ is said to be flat if every short exact sequence $0 \to A \to B \to F \to 0$ is pure (see [42]). Thus, if $\mathscr{S}_{\mathscr{A}}$ is the representative set of finitely presented objects, then F is flat if and only if $\mathrm{Hom}_{\mathscr{A}}(\oplus_{M \in \mathscr{S}_{\mathscr{A}}} M, -)$ makes exact every short exact sequence of the form $0 \to A \to B \to F \to 0$. This means that $\underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(\oplus_{M \in \mathscr{S}_{\mathscr{A}}} M)$ is the class of flat objects (as proved in [19, Proposition 2.1] for modules). On the other hand, $\underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(\mathscr{S}_{\mathscr{A}})$ is the class of flat objects (see Proposition 2.1.19). Thus, $\underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(\mathscr{S}_{\mathscr{A}}) = \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(\oplus_{M \in \mathscr{S}_{\mathscr{A}}} M)$. The following result, which was already proven for the category of modules in [28, Proposition 2.10], is a generalization of this fact.

Proposition 2.1.13. *Suppose that $\mathscr{A}$ has direct sums and let $\{M_i\}_{i \in I}$ be a set of objects of $\mathscr{A}$. Then, $\underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(\oplus_{i \in I} M_i) = \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(\{M_i\}_{i \in I})$.*

Proof. Let $g : K \to N$ be an epimorphism. The following diagram is commutative

$$\begin{tikzcd}
\mathrm{Hom}_{\mathscr{A}}(\oplus_{i\in I}M_i, K) \arrow[r, "\mathrm{Hom}_{\mathscr{A}}(\oplus_{i\in I}M_i, g)"] \arrow[d, "\psi^K"'] & \mathrm{Hom}_{\mathscr{A}}(\oplus_{i\in I}M_i, N) \arrow[d, "\psi^N"] \\
\prod_{i\in I}\mathrm{Hom}_{\mathscr{A}}(M_i, K) \arrow[r, "\prod_{i\in I}\mathrm{Hom}_{\mathscr{A}}(M_i, g)"'] & \prod_{i\in I}\mathrm{Hom}_{\mathscr{A}}(M_i, N)
\end{tikzcd}$$

where ψ^K and ψ^N are isomorphisms. Hence, the morphism $\mathrm{Hom}_{\mathscr{A}}(\oplus_{i\in I}M_i, g)$ is epic if and only if $\prod_{i\in I}\mathrm{Hom}_{\mathscr{A}}(M_i, g)$ is epic. Then the morphism $\mathrm{Hom}_{\mathscr{A}}(\oplus_{i\in I}M_i, g)$ is epic if and only if $\mathrm{Hom}_{\mathscr{A}}(M_i, g)$ is epic for every $i \in I$. Therefore, $N \in \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(\oplus_{i\in I}M_i)$ if and only if $N \in \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(M_i)$ for every $i \in I$. □

Proposition 2.1.13 shows that reducing sets to a singleton while preserving the same subprojectivity domain is possible. The following result shows how far we can modify classes while preserving the same subprojectivity domain. For we will use the following known terminology: if $\mathscr{L}$ is a class of objects of $\mathscr{A}$, we denote by $Sum(\mathscr{L})$ the class of all objects which are isomorphic to direct sums of objects of $\mathscr{L}$, by $Summ(\mathscr{L})$ the class of all objects which are isomorphic to direct summands of objects of $\mathscr{L}$, and by $Add(\mathscr{L})$ the class $Summ(Sum(\mathscr{L}))$.

Proposition 2.1.14. *Let $\mathscr{L}$ be a class of objects of $\mathscr{A}$. Then*

$$\underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(Add(\mathscr{L})) = \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(Sum(\mathscr{L})) = \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(Summ(\mathscr{L})) = \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(\mathscr{L}).$$

If $\mathscr{L}$ is a set, then all these classes coincide with the class $\underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(\oplus_{L\in\mathscr{L}}L)$.

Proof. It is clear that $\underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(Add(\mathscr{L}))$ holds inside $\underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(\mathscr{L})$ since $\mathscr{L}$ holds inside $Add(\mathscr{L})$.

Conversely, let N be in $\underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(\mathscr{L})$ and M in $Add(\mathscr{L})$. Then, there exist M' in $Add(\mathscr{L})$ and a family $\{L_i\}$ in $\mathscr{L}$ such that $M \oplus M' = \oplus_i L_i$. By Proposition 2.1.13, $N \in \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(M)$ so $N \in \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(Add(\mathscr{L}))$. Therefore, $\underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(Add(\mathscr{L})) = \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(\mathscr{L})$.

Now, it is clear that $\mathscr{L} \subseteq Summ(\mathscr{L}) \subseteq Add(\mathscr{L})$ and that $\mathscr{L} \subseteq Sum(\mathscr{L}) \subseteq Add(\mathscr{L})$, so we get $\underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(Add(\mathscr{L})) \subseteq \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(Summ(\mathscr{L})) \subseteq \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(\mathscr{L})$ and $\underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(Add(\mathscr{L})) \subseteq \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(Sum(\mathscr{L})) \subseteq \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(\mathscr{L})$. Therefore,

$$\underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(Add(\mathscr{L})) = \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(Sum(\mathscr{L})) = \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(Summ(\mathscr{L})) = \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(\mathscr{L}).$$

If $\mathscr{L}$ is a set then, by Proposition 2.1.13, $\underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(\mathscr{L}) = \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(\oplus_{L\in\mathscr{L}}L)$. □

2.1.1 Examples of homologically interesting classes

In this subsection, we investigate subprojectivity domains of homologically important classes in a general abelian category (with some conditions), the category of complexes and the category of modules. And then, we apply Proposition 2.1.12 to unify various known results.

We start by noticing that Example 2.1.3 helps us deduce that the subprojectivity domain of the class of all DG-projective complexes is a subclass of the class of exact complexes. The next proposition shows that we have an equality.

Proposition 2.1.15. *If $\mathscr{A}$ is locally finitely presented, then the subprojectivity domain of the class of DG-projective complexes is the class of all exact complexes.*

Proof. Let E be a complex. By Proposition 1.2.3, there exists an epic quasi-isomorphism $g : P \to E$ where P is DG-projective.
If we suppose that E holds in the subprojectivity domain of the class of DG-projectives, then $E \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(\mathscr{A})}(P)$. Thus, we get by Example 2.1.3 that E is exact.
Conversely, suppose that E is exact and let M be a DG-projective complex. Then, $\mathrm{Hom}^{\bullet}(M,g)$ is an epic quasi-isomorphism, in particular, epic on the cycles (see Remark 1.2.1), thus $\mathrm{Hom}_{\mathscr{C}(\mathscr{A})}(M,g)$ is epic. Then, every morphism $M \to E$ factors through P which is exact (since E is) and then, projective by Proposition 1.2.5. Therefore, E holds in the subprojectivity domain of the class of DG-projectives by Proposition 2.1.4. □

The case of the class of strongly Gorenstein projective objects can be deduced directly from Corollary 2.1.7.

Proposition 2.1.16. *The subprojectivity domain of the class of strongly Gorenstein projective objects is the class $\mathscr{SGP}_{\mathscr{A}}^{\perp}$.*

Proof. Let N be an object of $\mathscr{A}$. $N \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{A}}(\mathscr{SGP}_{\mathscr{A}})$ means that $N \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{A}}(M)$ for every $M \in \mathscr{SGP}_{\mathscr{A}}$. By Corollary 2.1.7 we have that $\underline{\mathfrak{Pr}}^{-1}_{\mathscr{A}}(M) = M^{\perp}$ for every $M \in \mathscr{SGP}_{\mathscr{A}}$. Thus, $N \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{A}}(\mathscr{SGP}_{\mathscr{A}})$ is equivalent to $N \in M^{\perp}$ for every $M \in \mathscr{SGP}_{\mathscr{A}}$, which is equivalent to $N \in \mathscr{SGP}_{\mathscr{A}}^{\perp}$. □

Using Proposition 2.1.14 we determine the subprojectivity domain of the class of Gorenstein projective objects. If direct sums exist and they are exact then an object is Gorenstein projective if and only if it is a direct summand of a strongly Gorenstein projective one (see Proposition 1.3.4), so clearly $\mathscr{SGP}_{\mathscr{A}}^{\perp} = \mathscr{GP}_{\mathscr{A}}^{\perp}$. Thus, we have the following result.

Corollary 2.1.17. *If direct sums exist and they are exact then, the subprojectivity domain of the class of Gorenstein projective objects is the class $\mathscr{GP}_{\mathscr{A}}^{\perp}$.*

Example 2.1.18. *In the category of modules* R-Mod*, if* R *is a ring with finite Gorenstein global dimension (see [11]), then the subprojectivity domain of the class of Gorenstein projective modules is the class of all modules with finite projective dimension. Indeed, over a ring* R *with finite Gorenstein global dimension, the class* $\mathscr{G}\mathscr{P}^{\perp}_{R\text{-Mod}}$ *coincides with the class of all modules with finite projective dimension. We do not have a precise reference but one can see that it is a simple consequence of [16, Lemma 2.17].*

The case of the class of finitely presented objects can be deduced directly from the categorical definition of flat objects.

Proposition 2.1.19. *The subprojectivity domain of the class of finitely presented objects in a Grothendieck category is the class of flat objects.*

Recall now that an object is pure-projective if and only if it is a direct summand of a direct sum of finitely presented objects (see Proposition 1.1.6). As a direct consequence of Proposition 2.1.14 and Proposition 2.1.19, we get the following result.

Corollary 2.1.20. *If* $\mathscr{A}$ *is a locally finitely presented Grothendieck category then the subprojectivity domain of the class of all pure-projective objects is precisely the class of all flat objects.*

Now, we turn our attention to some important classes of modules defined in terms of factorization of morphisms. Recall for a class of finitely generated modules $\mathscr{S}$, the class of $\mathscr{S}$-proj modules was defined in [37] as the class of modules N such that every morphism $f : S \to N$, where $S \in \mathscr{S}$, factors through a free module. Proposition 2.1.22 shows that $\mathscr{S}$-proj is a subprojectivity domain.

First we give the following lemma which will be useful in the proof of Proposition 2.1.22.

Lemma 2.1.21. *If* $f : M \to P$ *is a morphism of modules such that* M *is finitely generated and* P *is projective, then* $f : M \to P$ *factors through a finitely generated free module.*

Proof. Let $g : R^{(I)} \to P$ be an epimorphism and $h : M \to R^{(I)}$ be a morphism such that $gh = f$. Now consider the ker-coker factorization of h

$$\begin{tikzcd} \operatorname{Im} h \arrow[d, "\mu"'] & M \arrow[l, "\varepsilon"'] \arrow[dl, "h"'] \arrow[d, "f"] \\ R^{(I)} \arrow[r, "g"'] & P \end{tikzcd}$$

Since $\operatorname{Im} h$ is finitely generated, there is a finite subset J of I such that $\mu : \operatorname{Im} h \to R^{(I)}$ factors through $R^{(J)}$. Thus, $f : M \to P$ factors through $R^{(J)}$. $\square$

Proposition 2.1.22. *The subprojectivity domain of a class of finitely generated modules $\mathscr{S}$ is precisely the class of $\mathscr{S}$-proj modules.*

Proof. Let $N \in \underline{\mathfrak{Pr}}_{R\text{-Mod}}^{-1}(\mathscr{S})$ and $f : M \to N$ be a morphism with $M \in \mathscr{S}$. Then, there exist two morphisms $\beta : M \to P$ and $\alpha : P \to N$ with P projective and $f = \alpha\beta$. We get by Lemma 2.1.21 that $\beta : M \to P$ factors through a finitely generated free module F which implies that $f : M \to N$ factors through F. Thus, N is $\mathscr{S}$-projective. The other inclusion holds true by Proposition 2.1.4. □

In the light of Proposition 2.1.4, now it is natural to ask whether or not the membership in the subprojectivity domain of a class of finitely generated objects can be characterized by the factorization through finitely generated projective objects. The following proposition shows that this is possible in a Grothendieck category with a family of finitely generated projective objects.

Proposition 2.1.23. *Suppose that $\mathscr{A}$ is Grothendieck with a system of finitely generated projective generators $\mathscr{G}$ and let $\mathscr{S}$ be a class of finitely generated objects. Then, the following conditions are equivalent.*

1. *$N \in \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(\mathscr{S})$.*
2. *Every morphism $M \to N$ with $M \in \mathscr{S}$ factors through a finite direct sum of objects of $\mathscr{G}$.*
3. *Every monomorphism $M \to N$ with $M \in \mathscr{S}$ factors through a finite direct sum of objects of $\mathscr{G}$.*
4. *Every morphism $M \to N$ with $M \in \mathscr{S}$ factors through a finitely generated projective object.*
5. *Every monomorphism $M \to N$ with $M \in \mathscr{S}$ factors through a finitely generated projective object.*

Proof. (2) $\Leftrightarrow$ (3) and (4) $\Leftrightarrow$ (5) are clear since every morphism $f : X \to Y$ factors through $\operatorname{Im} f$ which is finitely generated whenever X is finitely generated. To prove (1) $\Rightarrow$ (2) let $f : M \to N$ be a morphism with $M \in \mathscr{S}$. Then, we get by Proposition 2.1.4 two morphisms $\beta : M \to P$ and $\alpha : P \to N$ with P projective and $f = \alpha\beta$. Let $g : \oplus_{i\in I} Q_i \to P$ be an epimorphism such that each Q_i holds in $\mathscr{G}$. Since P is projective, there exists a morphism $h : P \to \oplus_{i\in I} Q_i$ such that $gh = \mathrm{id}_P$. We get by [35, Page 206] two morphisms $\gamma : \oplus_{i\in J} Q_i \to \oplus_{i\in I} Q_i$ and $\delta : M \to \oplus_{i\in J} Q_i$ such that J is a finite subset of I and $h\beta = \gamma\delta$. Then, $f = \alpha\beta = \alpha g h\beta = \alpha g\gamma\delta$, that is, $f : M \to N$ factors through $\oplus_{i\in J} Q_i$.
(2) $\Rightarrow$ (4) is clear since the finite direct sum of finitely generated projective objects is again a finitely generated projective object.
For (4) $\Rightarrow$ (1) we apply Proposition 2.1.4. □

Now, we apply Proposition 2.1.23 to charaterize some known classes of modules defined by means of factorizations.

Recall that a module M is said to be f-projective if for every finitely generated submodule C of M, the inclusion map $C \to M$ factors through a finitely generated free module. Then, we have the following result.

Proposition 2.1.24. *The subprojectivity domain of the class of all finitely generated modules is the class of f-projective modules.*

Proof. Apply 1. $\Leftrightarrow$ 3. of Proposition 2.1.23. $\square$

In a similar way to Proposition 2.1.24, we can determine the subprojectivity domain of the class of simple modules. Recall that a module N is called simple-projective if, for any simple module M, every morphism $f : M \to N$ factors through a finitely generated free module (see [32, Definition 2.1]).

Proposition 2.1.25. *The subprojectivity domain of the class of simple modules is the class of simple-projective modules.*

Proof. Apply 1. $\Leftrightarrow$ 2. of Proposition 2.1.23. $\square$

Another interesting example can be found in the class of semisimple modules, whose subprojectivity domain can also be found by simply applying Proposition 2.1.14 and Proposition 2.1.25.

Corollary 2.1.26. *The subprojectivity domain of the class of semisimple modules is the class of simple-projective modules.*

As mentioned before, Proposition 2.1.12 can be used to unify various known results: applying it to the class of Gorenstein projective objects and to the class of strongly Gorenstein projective objects we get Corollary 2.1.27 (see [25]); Applying it to the class of finitely presented objects and to the class of pure-projective objects we get Corollary 2.1.28 (see [25]); Applying it to the class of finitely generated modules we get Corollary 2.1.29; and finally, applying it to the class of simple modules and to the one of semisimple modules we get Corollary 2.1.30.

Corollary 2.1.27. *The following conditions are equivalent.*

1. *Every object of $\mathscr{A}$ holds in $\mathscr{G}\mathscr{P}^{\perp}_{\mathscr{A}}$.*
2. *Every Gorenstein projective object is projective.*
3. *Every Gorenstein projective object holds in $\mathscr{G}\mathscr{P}^{\perp}_{\mathscr{A}}$.*
4. *Every strongly Gorenstein projective is projective.*

5. *Every strongly Gorenstein projective holds in* $\mathscr{G}\mathscr{P}_{\mathscr{A}}^{\perp}$.

Corollary 2.1.28. *Suppose that* $\mathscr{A}$ *is Grothendieck. Then the following conditions are equivalent.*

1. *Every object of* $\mathscr{A}$ *is flat.*
2. *Every finitely presented object is projective.*
3. *Every finitely presented object is flat.*
4. *Every pure-projective object is projective.*
5. *Every pure-projective object is flat.*

Corollary 2.1.29. *The following conditions are equivalent.*

1. *Every module is f-projective.*
2. *Every finitely generated module is projective, that is R is a semisimple artinian ring.*
3. *Every finitely generated module is f-projective.*

Corollary 2.1.30. *The following conditions are equivalent.*

1. *Every module is simple-projective.*
2. *Every simple module is projective.*
3. *Every simple module is simple-projective.*
4. *Every semisimple module is projective.*
5. *Every semisimple module is simple-projective.*

2.2 Closure properties of subprojectivity domains

The aim of this section is to investigate the closure properties of subprojectivity domains. This study leads to some new characterizations of known notions.

We start with Proposition 2.2.1 which is a generalization of [1, Proposition 3], [28, Proposition 2.11] and [28, Proposition 2.12]. Though it can be proved by using similar arguments to those of the results it generalizes, we give an alternative proof since we think it provides new and useful ideas.

Proposition 2.2.1. *The subprojectivity domain of any class in $\mathscr{A}$ is closed under extensions, finite direct sums and direct summands.*

Proof. Clearly it suffices to prove the result for subprojectivity domain of objects so let us consider a single object M of $\mathscr{A}$ and study its subprojectivity domain.

For let $0 \to A \to B \to C \to 0$ be a short exact sequence of objects and suppose that A and C are in $\underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(M)$. Consider then two epimorphisms $P_A \to A$ and $P_C \to C$ with P_A and P_C projective. By Horseshoe Lemma we get the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P_A & \longrightarrow & P_B & \longrightarrow & P_C & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \end{array}$$

with P_B projective. Apply then $\mathrm{Hom}_{\mathscr{A}}(M,-)$ to get the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Hom}_{\mathscr{A}}(M,P_A) & \longrightarrow & \mathrm{Hom}_{\mathscr{A}}(M,P_B) & \longrightarrow & \mathrm{Hom}_{\mathscr{A}}(M,P_C) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{Hom}_{\mathscr{A}}(M,A) & \longrightarrow & \mathrm{Hom}_{\mathscr{A}}(M,B) & \longrightarrow & \mathrm{Hom}_{\mathscr{A}}(M,C) & \longrightarrow & 0 \end{array}$$

with exact rows (P_C is projective and C holds in $\underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(M)$).

Since A and C hold in $\underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(M)$, the two morphisms $\mathrm{Hom}_{\mathscr{A}}(M,P_A) \to \mathrm{Hom}_{\mathscr{A}}(M,A)$ and $\mathrm{Hom}_{\mathscr{A}}(M,P_C) \to \mathrm{Hom}_{\mathscr{A}}(M,C)$ are epic. Then, $\mathrm{Hom}_{\mathscr{A}}(M,P_B) \to \mathrm{Hom}_{\mathscr{A}}(M,B)$ is also epic and then we get $B \in \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(M)$ (by Proposition 2.1.4).

Now, the closure under extensions of $\underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(M)$ proves its closure under finite direct sums.

And finally, let $N \in \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(M)$ and A be a direct summand of N. If $p : N \to A$ is the canonical projection then $\mathrm{Hom}_{\mathscr{A}}(M,p)$ is epic and then, by Proposition 2.1.4, we get that $A \in \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(M)$. □

Now, we prove that the subprojectivity domains of finitely generated objects in a locally finitely presented Grothendieck category are closed under pure monomorphisms. First we prove the following two results which will be useful to prove Proposition 2.2.4.

Lemma 2.2.2. *Let (D,g',f') be a pullback of two morphisms $g : E \to B$ and $f : A \to B$. If g is a pure epimorphism and f is a pure monomorphism, then f' is a pure monomorphism.*

Proof. We have the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & D & \xrightarrow{f'} & E & \xrightarrow{h'} & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} & & \| & & \\ 0 & \longrightarrow & A & \xrightarrow[f]{} & B & \xrightarrow[h]{} & C & \longrightarrow & 0 \end{array}$$

We get that h' is a pure epimorphism since g and h are. Thus, f' is a pure monomorphism. $\square$

Lemma 2.2.3. *Suppose that $\mathscr{A}$ is a locally finitely presented Grothendieck category and let $k : K \to \oplus_{i\in I}F_i$ be a pure monomorphism such that each F_i is finitely presented. Then, for every morphism $f : M \to K$ with M finitely generated there exists a morphism $j : \oplus F_i \to K$ such that $jkf = f$.*

Proof. Since M is finitely generated, there exists a finite subset J of I, together with a morphism $h : M \to \oplus_{i\in J}F_i$, such that $kf = gh$ where $g : \oplus_{i\in J}F_i \to \oplus_{i\in I}F_i$ is the canonical injection (see [35, page 206, Lemma 3]). Then, we have a commutative diagram with exact rows

$$\begin{tikzcd}
 & M \arrow[r, "h"] \arrow[d, "f"] & \oplus_{i\in J}F_i \arrow[r, "\gamma"] \arrow[d, "g"] & C \arrow[r] \arrow[d, "l"] & 0 \\
0 \arrow[r] & K \arrow[r, "k"'] & \oplus_{i\in I}F_i \arrow[r, "\gamma'"'] & C' \arrow[r] & 0
\end{tikzcd}$$

Since C is finitely presented and $k : K \to \oplus_{i\in I}F_i$ is a pure monomorphism, there exists a morphism $t : C \to \oplus_{i\in I}F_i$ such that $\gamma' t = l$. Then, $\gamma' t\gamma = l\gamma = \gamma' g$, hence there exists a morphism $s : \oplus_{i\in J}F_i \to K$ such that $ks = g - t\gamma$ and then $ksh = gh = kf$. But k is a monomorphism, then $sh = f$.

Now, the morphism $g : \oplus_{i\in J}F_i \to \oplus_{i\in I}F_i$ is a split monomorphism, so there exists a morphism $g' : \oplus_{i\in I}F_i \to \oplus_{i\in J}F_i$ such that $g'g = \mathrm{id}$. Now, put $j = sg'$, then $kjkf = ksg'kf = ksg'gh = ksh = kf$. Since k is a monomorphism, we get $jkf = f$. $\square$

Proposition 2.2.4. *If $\mathscr{A}$ is a locally finitey presented Grothendieck category, then the subprojectivity domain of any finitely generated object is closed under pure subobjects.*

Proof. Let M be a finitely generated object, $N \in \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(M)$, and $i : K \to N$ be a pure monomorphism. Let us prove that $K \in \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(M)$. For let $f : M \to K$ be a morphism, $\alpha : P \to N$ and $\beta : M \to P$ be two morphisms such that P is projective and $if = \alpha\beta$ ($N \in \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(M)$). Now, let $g : \oplus F_i \to N$ be a pure epimorphism with each F_i finitely presented (g exists by Proposition 1.1.5). Then, there exists a morphism $\gamma : P \to \oplus F_i$ such that $\alpha = g\gamma$. We get the commutative diagram

$$\begin{tikzcd}
M \arrow[dr, dotted, "\lambda"] \arrow[drr, bend left, "\gamma\beta"] \arrow[ddr, bend right, "f"'] & & \\
 & D \arrow[r, "k"] \arrow[d, "h"] & \oplus F_i \arrow[d, "g"] \\
 & K \arrow[r, "i"'] & N
\end{tikzcd}$$

in which D is a pullback.

By Lemma 2.2.2, $k : D \to \oplus F_i$ is a pure-monomorphism. Then, there exists a morphism $j : \oplus F_i \to D$ such that $jk\lambda = \lambda$ (see Lemma 2.2.3). Then, $if = \alpha\beta = g\gamma\beta = gk\lambda = ih\lambda = ihjk\lambda = ihj\gamma\beta$, then $f = hj\gamma\beta$ since i is monic. Thus, $f : M \to K$ factors through P. Therefore, $k \in \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(M)$ by Proposition 2.1.4. □

One can easily deduce from Proposition 2.2.4 that in a locally finitely presented Grothendieck category, the subprojectivity domain of the class of finitely generated objects is closed under kernels of epimorphisms. Indeed, consider a short exact sequence $0 \to A \to B \to C \to 0$ such that B and C hold in the subprojectivity domain of the class of finitely generated objects. Then, for every finitely presented object M, $0 \to \mathrm{Hom}_{\mathscr{A}}(M,A) \to \mathrm{Hom}_{\mathscr{A}}(M,B) \to \mathrm{Hom}_{\mathscr{A}}(M,C) \to 0$ is exact since $C \in \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(M)$. That is, A is a pure subobject of B. We conclude by Proposition 2.2.4 that A holds in the subprojectivity domain of the class of finitely generated objects. Therefore, it is natural to ask whether or not subprojectivity domains are closed under kernels of epimorphisms. In fact, we will see in Example 3.1.20 that this is not true in general. Here, we characterize when subprojectivity domains are closed under kernels of epimorphisms.

Proposition 2.2.5. *Let $\mathscr{L}$ be a class of objects of $\mathscr{A}$. Then, the following conditions are equivalent.*

1. *$\underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(\mathscr{L})$ is closed under kernels of epimorphisms.*

2. *For every short exact sequence $0 \to C \to P \to A \to 0$ where P is projective, if $A \in \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(\mathscr{L})$ then $C \in \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(\mathscr{L})$.*

3. *For every epimorphism $P \to A$ with P projective and $A \in \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(\mathscr{L})$, the pullback object of P over A holds in $\underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(\mathscr{L})$.*

Proof. (1) ⇒ (2) is clear since every projective holds in $\underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(\mathscr{L})$. To prove (2) ⇒ (1) consider an exact sequence

$$0 \to C \to B \to A \to 0$$

with B, $A \in \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(\mathscr{L})$ and the pullback diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & K \arrow[r, equal] \arrow[d] & K \arrow[d] & \\
0 \arrow[r] & C \arrow[r] \arrow[d, equal] & D \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C \arrow[r] & B \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

where P is a projective object. $A \in \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(\mathscr{L})$, so by assumption $K \in \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(\mathscr{L})$. Then, by Proposition 2.2.1, $D \in \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(\mathscr{L})$, and since C is a direct summand of D, we deduce using again Proposition 2.2.1 that $C \in \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(\mathscr{L})$.

To prove (2) $\Leftrightarrow$ (3), let $P \to A$ be an epimorphism with P projective and consider the following diagram where D is the pullback of P over A

$$\begin{tikzcd}
0 \arrow[r] & C \arrow[r] \arrow[d, equal] & D \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & 0\ . \\
0 \arrow[r] & C \arrow[r] & P \arrow[r] & A \arrow[r] & 0
\end{tikzcd}$$

Suppose that $A \in \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(\mathscr{L})$. By Proposition 2.2.1, we have $D \in \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(\mathscr{L})$ if and only if $C \in \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(\mathscr{L})$. $\square$

As examples of classes satisfying the conditions of Proposition 2.2.5, we give the following.

Example 2.2.6. *1. Let M be a strongly Gorenstein projective object. Then $\underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(M)$ is closed under kernels of epimorphisms.*

Indeed, let $0 \to C \to P \to A \to 0$ be a short exact sequence with P projective and $A \in \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(M)$. If we consider the long exact sequence

$$\cdots \to \mathrm{Hom}_{\mathscr{A}}(M,P) \to \mathrm{Hom}_{\mathscr{A}}(M,A) \to \mathrm{Ext}^1_{\mathscr{A}}(M,C) \to \mathrm{Ext}^1_{\mathscr{A}}(M,P) \to \cdots,$$

then $\mathrm{Ext}^1_{\mathscr{A}}(M,C) = 0$ so $C \in \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(M)$ (see Corollary 2.1.7). Therefore, by Proposition 2.2.5, $\underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(M)$ is closed under kernels of epimorphisms.

2. *Suppose that $\mathscr{A}$ is a locally finitely presented Grothendieck category and let $\mathscr{L}$ be any class of finitely generated objects containing all finitely presented objects. Then, the subprojectivity domain of $\mathscr{L}$ is closed under kernels of epimorphisms. In particular, the class of f-projective modules (that is, the subprojectivity domain of the class of all finitely gnerated modules) is closed under kernels of epimorphisms.*

   To show this, let $0 \to A \to B \to C \to 0$ be a short exact sequence with $B,\ C \in \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(\mathscr{L})$. Since $\mathscr{L}$ contains all finitely presented objects the sequence $0 \to A \to B \to C \to 0$ is pure, so by Proposition 2.2.4 we get that $A \in \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(\mathscr{L})$.

In [28, Proposition 2.15] it is proved that a ring R is right hereditary if and only if the subprojectivity domain of any right R-module is closed under submodules. Since $\underline{\mathfrak{Pr}}_{\text{Mod-}R}^{-1}(\text{Mod-}R)$ is the class of projective right R-modules where Mod-R denotes the category of right R-modules, one could replace the statement "R is right hereditary" by "$\underline{\mathfrak{Pr}}_{\text{Mod-}R}^{-1}(\text{Mod-}R)$ is closed under submodules", getting then that $\underline{\mathfrak{Pr}}_{\text{Mod-}R}^{-1}(\text{Mod-}R)$ is closed under submodules if and only if $\underline{\mathfrak{Pr}}_{Mod-R}^{-1}(M)$ is closed under submodules for every right R-module M. Thus, the next proposition gives an extension of this result to an arbitrary class $\mathscr{L}$ of objects of $\mathscr{A}$.

Proposition 2.2.7. *Let $\mathscr{L}$ be a class of objects of $\mathscr{A}$. Then, the following two conditions are equivalent.*

1. *The subprojectivity domain of $\mathscr{L}$ is closed under subobjects.*

2. *The subprojectivity domain of any object of $\mathscr{L}$ is closed under subobjects.*

Proof. $(2) \Rightarrow (1)$ is immediate.

To prove $(1) \Rightarrow (2)$ let $M \in \mathscr{L}$ and suppose that $B \in \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(M)$. Now let A be a subobject of B. We get the following pullback diagram

$$\begin{tikzcd} 0 \arrow[r] & D \arrow[r] \arrow[d, "g"'] & P \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] & 0 \end{tikzcd}$$

where $P \to B$ is an epimorphism and P is projective. Now, apply $\mathrm{Hom}_{\mathscr{A}}(M,-)$ to the previous diagram getting the following commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Hom}_{\mathscr{A}}(M,D) \arrow[r] \arrow[d, "\mathrm{Hom}_{\mathscr{A}}(M,g)"'] & \mathrm{Hom}_{\mathscr{A}}(M,P) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathscr{A}}(M,C) \arrow[d, equal] \\ 0 \arrow[r] & \mathrm{Hom}_{\mathscr{A}}(M,A) \arrow[r] & \mathrm{Hom}_{\mathscr{A}}(M,B) \arrow[r] & \mathrm{Hom}_{\mathscr{A}}(M,C) \end{tikzcd}$$

Since $B \in \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(M)$, we conclude that $\mathrm{Hom}_{\mathscr{A}}(M,g)$ is epic. Now, since P is projective, we have that $P \in \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(\mathscr{L})$, then, by (1), $D \in \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(\mathscr{L})$. Therefore, using Proposition 2.1.4, we get that $A \in \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(M)$. □

As a consequence, we get the following result, established first in [20, Proposition 2.4].

Corollary 2.2.8. *The weak global dimension of R is at most* 1 *if and only if the subprojectivity domain of each finitely presented module is closed under submodules.*

Proof. We know that the weak global dimension of R is at most 1 if and only if the class of all flat modules is closed under submodules. But the subprojectivity domain of the class of finitely presented modules is precisely the class of flat modules. Then, we just have to apply Proposition 2.2.7. □

Recall that R is left coherent if and only if the category R-mod of finitely generated (left) R-modules is abelian. Recall also that R is left semihereditary if the class of all finitely generated projective (left) R-modules is closed under submodules. Then, applying Proposition 2.2.7 to the class of finitely generated modules in the category R-mod we get the following.

Corollary 2.2.9. *Let R be a left coherent ring. Then, R is left semihereditary if and only if* $\underline{\mathfrak{Pr}}_{R\text{-}mod}^{-1}(M)$ *is closed under submodules for each finitely generated module M.*

In [28, Proposition 2.14] it is studied when the subprojectivity domain of any module is closed under arbitrary direct products. This can be extended to the categorical setting provided (of course) that $\mathscr{A}$ has direct products.

Proposition 2.2.10. *Suppose that $\mathscr{A}$ has direct products and let $\mathscr{L}$ be a class of objects of $\mathscr{A}$. Then the following conditions are equivalent.*

1. *The subprojectivity domain of $\mathscr{L}$ is closed under arbitrary direct products.*
2. *The subprojectivity domain of any object of $\mathscr{L}$ is closed under arbitrary direct products.*

Proof. (2) ⇒ (1) is immediate.

For (1) ⇒ (2) let M be an object of $\mathscr{L}$, $\{N_i\}_{i\in I}$ be a family of objects in $\underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(M)$ and $\{g_i : P_i \to N_i\}_{i\in I}$ be a family of epimorphisms where each P_i is projective. Consider the following commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathscr{A}}(M,\prod_{i\in I}P_i) \arrow[r, "\mathrm{Hom}_{\mathscr{A}}(M,\prod_{i\in I}g_i)"] \arrow[d, "\psi^P"'] & \mathrm{Hom}_{\mathscr{A}}(M,\prod_{i\in I}N_i) \arrow[d, "\psi^N"] \\
\prod_{i\in I}\mathrm{Hom}_{\mathscr{A}}(M,P_i) \arrow[r, "\prod_{i\in I}\mathrm{Hom}_{\mathscr{A}}(M,g_i)"] & \prod_{i\in I}\mathrm{Hom}_{\mathscr{A}}(M,N_i)
\end{tikzcd}$$

where ψ^N and ψ^P are the natural isomorphisms. The commutativity of the above diagram gives that $\mathrm{Hom}_{\mathscr{A}}(M,\prod_{i\in I} g_i)$ is epic. Since each P_i is in $\underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(\mathscr{L})$, $\prod_{i\in I} P_i$ is, by assumption, in $\underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(\mathscr{L})$. Then $\prod_{i\in I} P_i$ is in $\underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(M)$. By Proposition 2.1.4, $\prod_{i\in I} N_i \in \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(M)$ as desired. □

The result [28, Proposition 2.14] shows that a ring R is a right perfect and left coherent ring if and only if the subprojectivity domain of any right module is closed under arbitrary direct products. This holds since $\underline{\mathfrak{Pr}}_{R\text{-Mod}}^{-1}(R\text{-Mod})$ is the class of projective modules. Here we can give a much direct proof of a characterization of coherent rings given by Durğun in [20, Proposition 2.3] using also the same property applied to a different class.

Corollary 2.2.11. *Let R be a ring. Then R is right coherent if and only if the subprojectivity domain of any finitely presented left module is closed under direct products.*

It is a natural question at this point to ask about the closure of subprojectivity domains under arbitrary direct sums. There is not a clear answer to this. In [28, Proposition 2.13] it is shown that the subprojectivity domain of any finitely generated module is closed under arbitrary direct sums. Now, we will see that the class for which subprojectivity domains are closed under direct sums is larger than that of finitely generated modules, since it contains that of small modules. Whether or not this is the largest class with this property we don't know, but it would be of a great interest to know to what point this class can be enlarged.

So suppose that $\mathscr{A}$ is an abelian category with direct sums and let M be an object in $\mathscr{A}$, $\{N_i\}_{i\in I}$ be a family of objects in $\underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(M)$ and consider a family of epimorphisms $P_i \to N_i$, where each P_i is projective. We have the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \oplus_{i\in I}\mathrm{Hom}_{\mathscr{A}}(M,P_i) \arrow[r, "\phi_P"] \arrow[d, "\alpha"'] & \mathrm{Hom}_{\mathscr{A}}(M,\oplus_{i\in I}P_i) \arrow[r] \arrow[d, "\beta"'] & \mathrm{Coker}\phi_P \arrow[r] \arrow[d, "\gamma"'] & 0 \\
0 \arrow[r] & \oplus_{i\in I}\mathrm{Hom}_{\mathscr{A}}(M,N_i) \arrow[r, "\phi_N"'] & \mathrm{Hom}_{\mathscr{A}}(M,\oplus_{i\in I}N_i) \arrow[r] & \mathrm{Coker}\phi_N \arrow[r] & 0
\end{tikzcd}$$

Clearly, β is epic if and only if γ is epic since α is an epimorphism. Consequently, if M is a small object, that is, $\mathrm{Hom}_{\mathscr{A}}(M,-)$ preserves direct sums, then $\mathrm{Coker}\phi_P = \mathrm{Coker}\phi_N = 0$. Thus, the subprojectivity domain of M is closed under direct sums. We state this as a proposition.

Proposition 2.2.12. *The subprojectivity domain of any small object is closed under direct sums.*

Recall that in a locally finitely generated Grothendieck category an object M is finitely presented if and only $\mathrm{Hom}_{\mathscr{A}}(M,-)$ preserves direct limits. In particular, finitely presented objects are small. Thus, we get by Proposition 2.2.12 that the subprojectivity dmain of any finitely presented object is closed under direct sums. But, is the subprojectivity domain of any finitely presented object closed under direct limits? In Proposition 2.2.14 we will give a positive anwzer to this question.

First we prove the following result.

Lemma 2.2.13. *For any direct system $(N_i)_{i\in I}$, the natural morphism $\lambda : \oplus_{i\in I}N_i \to \varinjlim_{i\in I} N_i$ is a pure-epimorphism.*

Proof. First note that there exists $\lambda : \oplus_{i\in I}N_i \to \varinjlim_{i\in I} N_i$ such that for every $j \in I$, $\lambda k_j = \lambda_j$ where $k_j : N_j \to \oplus_{i\in I}N_i$ and $\lambda_j : N_j \to \varinjlim_{i\in I} N_i$ are the canonical morphisms. To prove that $\lambda : \oplus_{i\in I}N_i \to \varinjlim_{i\in I} N_i$ is epic let $f : \varinjlim_{i\in I} N_i \to X$ be a morphism such that $f\lambda = 0$. Then, for every $j \in I$, $f\lambda_j = f\lambda k_j = 0$. Thus $f = 0$. Now, to prove that $\lambda : \oplus_{i\in I}N_i \to \varinjlim_{i\in I} N_i$ is a pure-epimorphism let M be a finitely presented object and consider the following commutative diagram

$$\begin{tikzcd} \oplus_{i\in I}\mathrm{Hom}_{\mathscr{A}}(M,N_i) \arrow[r] \arrow[d, "\cong"'] & \varinjlim_{i\in I} \mathrm{Hom}_{\mathscr{A}}(M,N_i) \arrow[d, "\cong"'] \\ \mathrm{Hom}_{\mathscr{A}}(M,\oplus_{i\in I}N_i) \arrow[r] & \mathrm{Hom}_{\mathscr{A}}(M,\varinjlim_{i\in I} N_i) \end{tikzcd}$$

The upper morphism is epic (using the same arguments we used to prove that λ is epic). Then the lower morphism is so. $\square$

Proposition 2.2.14. *The subprojectivity domain of any finitely presented object in a locally finitely generated Grothendieck category is closed under direct limits.*

Proof. Let M be a finitely presented object and $(N_i)_{i\in I}$ be a direct system in $\mathscr{A}$ such that $N_i \in \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(M)$, and consider the natural morphism $\oplus_{i\in I}N_i \to \varinjlim_{i\in I}$ which is a pure epimorphism by Lemma 2.2.13. Thus, every morphism $M \to \varinjlim_{i\in I} N_i$ factors through $\oplus_{i\in I}N_i$ which is an object of $\underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(M)$ by Proposition 2.2.12. Therefore, $\varinjlim_{i\in I} N_i \in \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(M)$, by Proposition 2.1.4. $\square$

Notice that subprojectivity domains are not closed under direct limits in general. Indeed, if the subprojectivity domain of any object of $\mathscr{A}$ is closed under direct limits then $\underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(\mathscr{A})$, which is the class of projective objects, is closed under direct limits, which is not the case.

2.3 Ext-orthogonal classes, precovers and preenvelopes

The aim of this section is to establish the relation between the subprojectivity domains and the Ext-orthogonal classes. The idea behind this is inspired by the following discussion: fix a class $\mathscr{L}$ and consider a short exact sequence $0 \to K \to P \to N \to 0$ with P projective. For every $M \in \mathscr{L}$ we have the exact sequence

$$\cdots \to \mathrm{Hom}_{\mathscr{A}}(M,P) \to \mathrm{Hom}_{\mathscr{A}}(M,N) \to \mathrm{Ext}^1_{\mathscr{A}}(M,K) \to \mathrm{Ext}^1_{\mathscr{A}}(M,P) \to \cdots.$$

So if we assume that $\mathscr{L}^{\perp}$ contains all projective objects, we get the following equivalence: $N \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{A}}(\mathscr{L})$ if and only if $K \in \mathscr{L}^{\perp}$. However, it does not seem clear how to get new results if we relate the subprojectivity domains with a property on kernels of epimorphisms. But, if we suppose moreover that $\mathscr{L}$ contains all projective objects and that it is closed under kernels of epimorphisms, then $\mathscr{L}^{\perp}$ will be closed under cokernels of monomorphisms (see the proof of [26, Lemma 1.2.8]). So by the above equivalence we get the following implication: if $N \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{A}}(\mathscr{L})$ then $N \in \mathscr{L}^{\perp}$, that is, $\underline{\mathfrak{Pr}}^{-1}_{\mathscr{A}}(\mathscr{L}) \subseteq \mathscr{L}^{\perp}$. In Theorem 2.3.1 and Theorem 2.3.10 we provide a necessary and sufficient condition to have the equality $\underline{\mathfrak{Pr}}^{-1}_{\mathscr{A}}(\mathscr{L}) = \mathscr{L}^{\perp}$.

Recall that, given a class of objects $\mathscr{F}$ in $\mathscr{A}$, an $\mathscr{F}$-precover of an object M is a morphism $F \to M$ with $F \in \mathscr{F}$, such that $\mathrm{Hom}_{\mathscr{A}}(F',F) \to \mathrm{Hom}_{\mathscr{A}}(F',M) \to 0$ is exact for any $F' \in \mathscr{F}$. An $\mathscr{F}$-precover is said to be special provided that it is an epimorphism with kernel in the class $\mathscr{F}^{\perp}$. $\mathscr{F}$-preenvelopes and special $\mathscr{F}$-preenvelopes are defined dually.

Theorem 2.3.1. *Suppose that $\mathscr{A}$ has enough injectives and let $\mathscr{L}$ be a class of objects of $\mathscr{A}$ which is closed under kernels of epimorphisms and which contains the class $Proj_{\mathscr{A}}$. Then, the following conditions are equivalent.*

1. *$\mathscr{L}^{\perp} = \underline{\mathfrak{Pr}}^{-1}_{\mathscr{A}}(\mathscr{L})$.*

2. *$\mathscr{L}^{\perp}$ is closed under kernels of epimorphism, cokernels of monomorphisms and contains $Proj_{\mathscr{A}}$.*

3. *$\mathscr{L} \bigcap \mathscr{L}^{\perp} = Proj_{\mathscr{A}}$ and every object in $\mathscr{L}^{\perp}$ has a special $\mathscr{L}$-precover.*

Proof. 1. $\Rightarrow$ 2. Clearly $Proj_{\mathscr{A}} \subseteq \underline{\mathfrak{Pr}}^{-1}_{\mathscr{A}}(\mathscr{L}) = \mathscr{L}^{\perp}$. Now, let us prove that $\mathscr{L}^{\perp}$ is closed under kernels of epimorphisms.

Let $0 \to K \to L \to N \to 0$ be a short exact sequence such that $N, L \in \mathscr{L}^{\perp}$ and let us prove that $K \in \mathscr{L}^{\perp}$. To do so, consider the following long exact sequence for some M in $\mathscr{L}$

$$\cdots \longrightarrow \mathrm{Hom}_{\mathscr{A}}(M,L) \longrightarrow \mathrm{Hom}_{\mathscr{A}}(M,N) \longrightarrow \mathrm{Ext}^1_{\mathscr{A}}(M,K) \longrightarrow \mathrm{Ext}^1_{\mathscr{A}}(M,L) = 0$$

Since N is taken in $\mathscr{L}^{\perp}$ and $\mathscr{L}^{\perp}$ coincide with $\underline{\mathfrak{Pr}}^{-1}_{\mathscr{A}}(\mathscr{L})$ by assumption, $\mathrm{Hom}_{\mathscr{A}}(M,L) \to \mathrm{Hom}_{\mathscr{A}}(M,N)$ is epic, hence $K \in \mathscr{L}^{\perp}$.

Now, let us prove that $\mathscr{L}^{\perp}$ is closed under cokernels of monomorphisms. Let $0 \to K \to L \to N \to 0$ be a short exact sequence such that $K, L \in \mathscr{L}^{\perp}$, and consider the following pullback diagram

$$\begin{tikzcd} & & 0 \arrow[d] & 0 \arrow[d] & \\ & & H \arrow[r, equal] \arrow[d] & H \arrow[d] & \\ 0 \arrow[r] & K \arrow[r] \arrow[d, equal] & D \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & K \arrow[r] & L \arrow[r] \arrow[d] & N \arrow[r] \arrow[d] & 0 \\ & & 0 & 0 & \end{tikzcd}$$

where P is a projective object. We proved that $Proj_{\mathscr{A}} \subseteq \mathscr{L}^{\perp}$ and $\mathscr{L}^{\perp}$ is always closed under extensions and kernels of monomorphisms, then P, D and H are in $\mathscr{L}^{\perp}$. So $\mathrm{Hom}_{\mathscr{A}}(M,P) \to \mathrm{Hom}_{\mathscr{A}}(M,N)$ is epic, for every object M of $\mathscr{L}$, hence $N \in \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(\mathscr{L})$ (see Proposition 2.1.4). By assumption we get that $N \in \mathscr{L}^{\perp}$, as desired.

2. $\Rightarrow$ 1. Let N be an object of $\mathscr{A}$, $0 \to K \to P \to N \to 0$ be a short exact sequence with P projective, and consider the following long exact sequence for some M in $\mathscr{L}$

$$\cdots \longrightarrow \mathrm{Hom}_{\mathscr{A}}(M,P) \longrightarrow \mathrm{Hom}_{\mathscr{A}}(M,N) \longrightarrow \mathrm{Ext}^1_{\mathscr{A}}(M,K) \longrightarrow \mathrm{Ext}^1_{\mathscr{A}}(M,P)$$

Since $\mathscr{L}^{\perp}$ is assumed to contain all projective objects, $\mathrm{Hom}_{\mathscr{A}}(M,P) \to \mathrm{Hom}_{\mathscr{A}}(M,N)$ is epic if and only if $K \in M^{\perp}$, hence, by Proposition 2.1.4, $N \in \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(\mathscr{L})$ if and only if $K \in \mathscr{L}^{\perp}$. But $K \in \mathscr{L}^{\perp}$ if and only if $N \in \mathscr{L}^{\perp}$ by assumption. Therefore $N \in \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(\mathscr{L})$ if and only if $N \in \mathscr{L}^{\perp}$.

1. $\Rightarrow$ 3. If $M \in \mathscr{L} \bigcap \mathscr{L}^{\perp}$ then $M \in \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(\mathscr{L})$ by condition 1., and then $M \in \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(M)$. Hence, M is projective by Proposition 2.1.2.

Conversely, any projective P holds in $\mathscr{L}$ by the hypotheses, and of course $P \in \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(\mathscr{L})$. But $\underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(\mathscr{L}) = \mathscr{L}^{\perp}$, so indeed $P \in \mathscr{L} \bigcap \mathscr{L}^{\perp}$.

To prove the second assertion let $N \in \mathscr{L}^{\perp}$ and let us show that any epimorphism $g : P \to N$ with P projective is indeed a special $\mathscr{L}$-precover.

That g is an $\mathscr{L}$-precover is clear since, by assumption, $P \in \mathscr{L}$ and $N \in \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(\mathscr{L})$.

Now, for an object $L \in \mathscr{L}$, being $\mathrm{Hom}_{\mathscr{A}}(L,P) \to \mathrm{Hom}_{\mathscr{A}}(L,N) \to 0$ exact and being $\mathrm{Ext}^1_{\mathscr{A}}(L,P) = 0$ implies that $\mathrm{Ext}^1_{\mathscr{A}}(L, \mathrm{Ker} g) = 0$, that is, $\mathrm{Ker} g \in \mathscr{L}^{\perp}$.

3. $\Rightarrow$ 1. Let $N \in \mathscr{L}^{\perp}$, $M \in \mathscr{L}$ and consider a special $\mathscr{L}$-precover

$$0 \to K \to L \to N \to 0$$

of N.

Since N and K are in $\mathscr{L}^{\perp}$, L does too and then we get $L \in \mathscr{L} \bigcap \mathscr{L}^{\perp}$. But $\mathscr{L} \bigcap \mathscr{L}^{\perp} = Proj_{\mathscr{A}}$ so by Proposition 2.1.4, $N \in \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(M)$.

Conversely, let $N \in \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(\mathscr{L})$ and let

$$0 \to K \to P \to N \to 0$$

be a short exact sequence with P projective. For any $L \in \mathscr{L}$ the associated long exact sequence looks like

$$\cdots \to \mathrm{Ext}^1_{\mathscr{A}}(L,P) \to \mathrm{Ext}^1_{\mathscr{A}}(L,N) \to \mathrm{Ext}^2_{\mathscr{A}}(L,K) \to \cdots .$$

But $\mathrm{Ext}^1_{\mathscr{A}}(L,P) = 0$ since P is projective (so $P \in \mathscr{L}^{\perp}$ by the hypothesis), so proving that $\mathrm{Ext}^2_{\mathscr{A}}(L,K) = 0$ for every $L \in \mathscr{L}$ will give $N \in \mathscr{L}^{\perp}$.

Let $0 \to C \to Q \to L \to 0$ be a short exact sequence with Q projective and $L \in \mathscr{L}$. Since $\mathscr{L}$ is closed under kernels of epimorphisms, $Q,\ L \in \mathscr{L}$ implies $C \in \mathscr{L}$. Now, in the long exact sequence

$$\cdots \to \mathrm{Ext}^1_{\mathscr{A}}(C,K) \to \mathrm{Ext}^2_{\mathscr{A}}(L,K) \to \mathrm{Ext}^2_{\mathscr{A}}(Q,K) \to \cdots$$

we have $\mathrm{Ext}^1_{\mathscr{A}}(C,K) = 0$: indeed, in the long exact sequence

$$\mathrm{Hom}_{\mathscr{A}}(C,P) \to \mathrm{Hom}_{\mathscr{A}}(C,N) \to \mathrm{Ext}^1_{\mathscr{A}}(C,K) \to \mathrm{Ext}^1_{\mathscr{A}}(C,P) \to \cdots$$

$\mathrm{Ext}^1_{\mathscr{A}}(C,P) = 0$ (since $P \in \mathscr{L}^{\perp}$ by the hypothesis) and the first morphism is epic since $N \in \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(\mathscr{L})$ and $C \in \mathscr{L}$.

On the other hand, $\mathrm{Ext}^2_{\mathscr{A}}(Q,K) = 0$, so indeed $\mathrm{Ext}^2_{\mathscr{A}}(L,K) = 0$. □

In the category of complexes of modules $\mathscr{C}(R)$, Theorem 2.3.1 can be used to characterize the subprojectivity domain of the class $\mathscr{E}$ of exact complexes. We recall that every projective complex is exact, that the class $\mathscr{E}$ is closed under kernels of epimorphisms and that it is special precovering in the whole category of complexes (see [26, Theorem 2.3.17]). It is also known that $\mathscr{E} \bigcap \mathscr{E}^{\perp}$ is the class of injective complexes (see [26, Proposition 2.3.7]). So, by Theorem 2.3.1, we get the following result.

Corollary 2.3.2. *R is quasi-Fröbenius if and only if the subprojectivity domain of the class of exact complexes of modules is $\mathscr{E}^{\perp}$.*

The question of whether or not any object has a special $\mathscr{GP}_{\mathscr{A}}$-precover has been a subject of many papers. Here, as a consequence of Corollary 2.1.17 and Theorem 2.3.1 (since it is known that the class $\mathscr{GP}_{\mathscr{A}}$ is closed under kernels of epimorphisms) we immediately get a partial answer which has been recently known following different methods (see [48, Proposition 4.1]).

Corollary 2.3.3. *If $\mathscr{A}$ has enough injectives and direct sums which are exact then, every object in $\mathscr{GP}_{\mathscr{A}}^{\perp}$ has a special $\mathscr{GP}_{\mathscr{A}}$-precover.*

We have seen in Theorem 2.3.1 that the fact that the subprojectivity domain of a class $\mathscr{L}$ (under some conditions on $\mathscr{L}$) coincides with its first right orthogonal class is equivalent to the existence of special precovers plus another condition. We will see in Theorem 2.3.10 that this is also equivalent to existence of preenveloppes (under some conditions on the category $\mathscr{A}$ and the class $\mathscr{L}$). But first we give some conditions for the class of projectives to be locally initially small and then we show when any object of $\mathscr{L}$ has a $\underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(\mathscr{L})$-preenvelope.

Recall that a class $\mathscr{F}$ of objects is locally initially small if for every object M of $\mathscr{A}$ there exists a set $\mathscr{F}_M \subseteq \mathscr{F}$ such that every morphism $M \to F$ with $F \in \mathscr{F}$ factors through a direct product of objects in $\mathscr{F}_M$ (see [40, Definition 2.1]). In [40, Proposition 2.9] it is established that the class of projective modules is always locally initially small. The argument consists in proving that for any set X, the class $Summ(X)$ is locally initially small, and following the arguments given in [40] one can see that this holds in any Grothendieck category with enough projectives. However the arguments in the categorical sething are rather cumbersome, so, for the reader's convenience, we think it is necessary to set them out here, even though they are nothing more than a translation into categorical language of what is done in [40, Proposition 2.9]. We split the proof into several results, and we start with the following purely categorical fact.

Given two families of objects of $\mathscr{A}$ $\{A_i; i \in I\}$ and $\{B_i; i \in I\}$, and a set of morphisms $\{f_i : A_i \to B_i; i \in I\}$, we always have two induced morphisms $f : \oplus A_i \to \oplus B_i$ and $f' : \prod A_i \to \prod B_i$ such that

1. f is the unique morphism satisfying $k_i' f_i = f k_i$ for every i, where $k_i : A_i \to \oplus A_i$ and $k_i' : B_i \to \oplus B_i$ are the canonical injections.
2. f' is the unique morphism satisfying $\pi_i' f' = f_i \pi_i$ for every i, where $\pi_i : \prod A_i \to A_i$ and $\pi_i' : \prod B_i \to B_i$ are the canonical projections.

Now, we know there is a unique morphism (acually monic since $\mathscr{A}$ is Grothendieck, see for instance [35, Corollary 2, page 188]) $\lambda : \oplus A_i \to \prod A_i$ satisfying $\pi_i \lambda k_j = \delta_j^i$ for every $i, j \in I$, and similarly a unique $\lambda' : \oplus B_i \to \prod B_i$ such that $\pi_i' \lambda' k_j' = \delta_j^i$ for every $i, j \in I$ (where δ_j^i is the Kroneker delta).

We have the following.

Lemma 2.3.4. *The diagram*

$$\begin{tikzcd} \oplus A_i \arrow[r, "\lambda"] \arrow[d, "f"'] & \prod A_i \arrow[d, "f'"] \\ \oplus B_i' \arrow[r, "\lambda"] & \prod B_i \end{tikzcd}$$

is commutative.

Proof. The morphism $f'\lambda - \lambda' f : \oplus A_i \to \prod B_i$ is such that $\pi_i'(f'\lambda - \lambda' f)k_j = \pi_i' f'\lambda k_j - \pi_i'\lambda' f k_j = f_i \pi_i \lambda k_j - \pi_i' \lambda' k_j' f_j = f_i \delta_j^i - \delta_j^i f_j$ for every $i, j \in I$.

If $i \neq j$ then $\delta_j^i = 0$ and then $\pi_i'(f'\lambda - \lambda' f)k_j = 0$.

If $i = j$ then $\pi_i'(f'\lambda - \lambda' f)k_j = f_i - f_i = 0$.

Thus, $\pi_i'(f'\lambda - \lambda' f)k_j = 0\ \forall i \in I, \forall j \in I$, so, by the properties of the product $(f'\lambda - \lambda' f)k_j = 0\ \forall j \in I$, and then, by the properties of the coproduct $f'\lambda - \lambda' f = 0$.

Therefore, $f'\lambda = \lambda' f$ and then the diagram is commutative. □

Proposition 2.3.5. *Let X be a finitely generated object of $\mathscr{A}$ and $\{A_i\}_{i\in I}$ be a family of objects of $\mathscr{A}$. Then, for every morphism $g : X \to \oplus_{i\in I} A_i$, the set $\{i \in I / \pi_i g \neq 0\}$ is finite, where $\pi_j : \oplus_{i\in I} A_i \to A_j$ is the canonical projection for every $j \in I$.*

Proof. Let $M = \mathrm{Im}\ g$ and consider the ker-coker decomposition of g,

$$\begin{tikzcd}
X \arrow[rr, "g"] \arrow[dr, "\bar{g}"'] & & \oplus_{i\in I} A_i \\
& M \arrow[ur, "f"'] &
\end{tikzcd}$$

If we let $\mathscr{F}$ be the set of all finite subsets of I and define, for every $F \in \mathscr{F}$, A_F to be the image of the canonical morphism $\oplus_{i\in F} A_i \to \oplus_{i\in I} A_i$, condition (5) of [38, Chapter 4, Theorem 4.6] says that $M = \sum_{F\in\mathscr{F}} (M \cap A_F)$.

Now, for every $F \in \mathscr{F}$, we define $(M \cap A_F, \gamma_F, \eta_F)$ as the pullback of $\alpha_F : A_F \to \oplus_{i\in I} A_i$ and $f : M \to \oplus_{i\in I} A_i$. Now, for every two sets F, F' in $\mathscr{F}$ such that $F \subset F'$, we have taht $A_F \subseteq A_{F'}$, so there is a morphism $\alpha_F^{F'} : A_F \to A_{F'}$ such that the diagram

$$\begin{tikzcd}
& A_F \arrow[dl, "\alpha_F^{F'}"'] \arrow[d, "\alpha_F"] \\
A_{F'} \arrow[r, "\alpha_{F'}"'] & \oplus_{i\in I} A_i
\end{tikzcd}$$

commutes. Then, the universal property of the pullback guarantees the existence of a family of morphisms $\beta_F^{F'}$ such that the diagrams

$$\begin{tikzcd}
M \cap A_F \arrow[rr, "\gamma_F"] \arrow[dr, dashed, "\beta_F^{F'}"] \arrow[ddr, "\eta_F"'] & & A_F \arrow[dr, "\alpha_F^{F'}"] \arrow[ddr, "\alpha_F"'] & \\
& M \cap A_{F'} \arrow[rr, "\gamma_{F'}"] \arrow[d, "\eta_{F'}"] & & A_{F'} \arrow[d, "\alpha_{F'}"] \\
& M \arrow[rr, "f"'] & & \oplus_{i\in I} A_i
\end{tikzcd}$$

commute. Therefore, the family $\{(M\cap A_F)_{F\in\mathscr{F}},(\beta_F^{F'})_{F\subset F'}\}$ is a directed family of subobjects of M.

But M is finitely generated so there exists $F_0\in\mathscr{F}$ such that $M=M\cap A_{F_0}$ (see Proposition 1.1.1) and then, η_{F_0} is an isomorphism. Let $j\in I\setminus F_0$. We have $\pi_j f\eta_{F_0}=\pi_j\alpha_{F_0}\gamma_{F_0}=0$. Hence $\pi_j f=0$, for every $j\in I\setminus F_0$. Then $\{i\in I/\pi_i f\neq 0\}\subset F_0$. Therefore the set $\{i\in I/\pi_i f\neq 0\}$ is finite. Since $g=f\bar{g}$ and $\bar{g}$ epic, the set $\{i\in I/\pi_i g\neq 0\}$ is also finite. $\square$

Corollary 2.3.6. *Let $\mathscr{A}$ be a locally finitely generated Grothendieck category, A be an object of $\mathscr{A}$ and I be any index set. For any morphism $f:M\to A^{(I)}$ and any element $i\in I$, the set $[i]=\{j\in I/\pi_i f=\pi_j f\}$ is finite, where π_j is the canonical projection to the j-th component.*

Proof. We assume, without loss of generality, that $\pi_i f\neq 0$ for every $i\in I$.

Since $\mathscr{A}$ is locally finitely generated, there exists an epimorphism $g:\oplus_{\alpha\in F}X_\alpha\to M$ with all X_α finitely generated objects.

Call $k_\alpha:X_\alpha\to\oplus_{i\in F}X_i$ the canonical monomorphism for any $\alpha\in F$. We claim that for any $j\in I$ there exists some $\alpha_j\in F$ such that $\pi_j fgk_{\alpha_j}\neq 0$. Indeed, if $\pi_j fgk_\alpha=0$ for every $\alpha\in F$ then $\pi_j fg=0$, and since g is epic, $\pi_j f=0$, a contradiction. Therefore, $[j]\subseteq\{i\in I/\pi_i fgk_{\alpha_j}\neq 0\}$, and Proposition 2.3.5 says that this set is finite. $\square$

Proposition 2.3.7. *Let $\mathscr{F}$ be a set of objects of $\mathscr{A}$. Then, Sum$(\mathscr{F})$ is locally initially small.*

Proof. Let M be any object of $\mathscr{A}$ and $f:M\to\oplus_{F\in\mathscr{F}}F^{(X_F)}$ be any morphism.

Let $p_F:\oplus_{F\in\mathscr{F}}F^{(X_F)}\to F^{(X_F)}$ and $p_x:F^{(X_F)}\to F$ denote the canonical epimorphisms for any $F\in\mathscr{F}$ and any $x\in X_F$. Then, call $f_x=p_xp_Ff$.

Now, the equivalence relation in each X_F given by

$$x\sim y\Leftrightarrow f_x=f_y$$

provides a number of equivalence classes $[x]$, each of which having finite cardinality by Corollary 2.3.6. Let us denote $\overline{X}_F$ the quotient set $X_F/\sim$.

Now, for every equivalence class $[x]\in\overline{X}_F$ let us denote by $\Delta_{F,[x]}:F\to F^{[x]}$ the unique morphism induced by the identities $\mathrm{id}_F:F\to F$. That is, $\Delta_{F,[x]}$ is the unique morphism making commutative the diagrams

$$\begin{tikzcd} F^{[x]} \arrow[r, "\pi'_y"] & F \\ & F \arrow[ul, "\Delta_{F,[x]}"] \arrow[u, equal] \end{tikzcd}$$

for every $y \in [x]$, where π'_y are the canonical projections. The family $\{\Delta_{F,[x]}; [x] \in \overline{X}_F\}$ induces a unique morphism $\Delta'_F : F^{\overline{X}_F} \to \prod_{[x]\in\overline{X}_F} F^{[x]}$ such that the diagram

$$\begin{tikzcd}
F^{\overline{X}_F} \arrow[r, "\Delta'_F"] \arrow[d, "\overline{\pi}_{[x]}"'] & \prod_{[x]\in\overline{X}_F} F^{[x]} \cong F^{X_F} \arrow[d, "\pi_{[x]}"] \\
F \arrow[r, "\Delta_{F,[x]}"'] & F^{[x]}
\end{tikzcd}$$

(where $\overline{\pi}_{[x]}$ and $\pi_{[x]}$ are the canonical projections) commutes for every $[x] \in \overline{X}_F$. But then the diagram

$$\begin{tikzcd}
F^{\overline{X}_F} \arrow[r, "\Delta'_F"] \arrow[d, "\overline{\pi}_{[x]}"'] & F^{X_F} \arrow[d, "\pi_{[x]}"] \\
F \arrow[r, "\Delta_{F,[x]}"'] \arrow[dr, equal] & F^{[x]} \arrow[d, "\pi'_y"] \\
& F
\end{tikzcd}$$

commutes, and indeed, Δ'_F is the unique morphism making commutative the outer diagram, for if $\phi : F^{\overline{X}_F} \to \prod_{[x]\in\overline{X}_F} F^{[x]}$ is such that $\pi'_y\pi_{[x]}\phi = \overline{\pi}_{[x]}$, then $\pi'_y\Delta_{F,[x]}\overline{\pi}_{[x]} = \overline{\pi}_{[x]} = \pi'_y\pi_{[x]}\phi$ $\forall y \in [x]$ and then $\pi_{[x]}\Delta'_F = \Delta_{F,[x]}\overline{\pi}_{[x]} = \pi_{[x]}\phi$. But the unicity of Δ'_F implies that $\phi = \Delta'_F$.

Notice that $\pi'_y\pi_{[x]} = \pi_y$ $\forall y \in [x]$ and $\forall [x]$, so we actually have that Δ'_F is the unique morphism such that $\pi_x\Delta'_F = \overline{\pi}_{[x]}$ $\forall x \in X_F$, $\forall F \in \mathscr{F}$.

We can repeat the same argument with the family $\Delta'_F; F \in \mathscr{F}$ getting a unique morphism $\Delta' = \prod_{F\in\mathscr{F}} F^{\overline{X}_F} \to \prod_{F\in\mathscr{F}} F^{X_F}$ that makes the diagram commutes

$$\begin{tikzcd}
\prod_{F\in\mathscr{F}} F^{\overline{X}_F} \arrow[r, "\Delta'"] \arrow[d, "\overline{\pi}_F"'] & \prod_{F\in\mathscr{F}} F^{X_F} \arrow[d, "\pi_F"] \\
F^{\overline{X}_F} \arrow[r, "\Delta'_F"'] & F^{X_F}
\end{tikzcd}$$

for every $F \in \mathscr{F}$ (where $\overline{\pi}_F$ always denote the canonical projection).

Again, we see that Δ' is the unique morphism verifiying that $\pi_x\pi_F\Delta' = \overline{\pi}_{[x]}\overline{\pi}_F$ $\forall x \in X_F$, $\forall F \in \mathscr{F}$.

But the family $\{\Delta_{F,[x]}; [x] \in \overline{X}_F\}$ induces a unique morphism $\Delta_F : F^{(\overline{X}_F)} \to \oplus_{[x]\in\overline{X}_F} F^{[x]}$ such that the diagram

$$\begin{tikzcd} F \arrow[r, "\Delta_{F,[x]}"] \arrow[d, "\overline{k}_{[x]}"'] & F^{[x]} \arrow[d, "k_{[x]}"] \\ F^{(\overline{X}_F)} \arrow[r, "\Delta_F"'] & \oplus_{[x]\in\overline{X}_F} F^{[x]} \end{tikzcd}$$

commutes $\forall [x] \in \overline{X}_F$.

And the family $\{\Delta_F; F \in \mathscr{F}\}$ induces a unique morphism

$$\Delta : \oplus_{F\in\mathscr{F}} F^{(\overline{X}_F)} \to \oplus_{F\in\mathscr{F}} \oplus_{[x]\in\overline{X}_F} F^{[x]}$$

such that the diagram

$$\begin{tikzcd} F^{(\overline{X}_F)} \arrow[r, "\Delta_F"] \arrow[d, "\overline{k}_F"'] & \oplus_{[x]\in\overline{X}_F} F^{[x]} \arrow[d, "k_F"] \\ \oplus_{F\in\mathscr{F}} F^{(\overline{X}_F)} \arrow[r, "\Delta"'] & \oplus_{F\in\mathscr{F}} \oplus_{[x]\in\overline{X}_F} F^{[x]} \end{tikzcd}$$

commutes for all $F \in \mathscr{F}$.

But $|[x]| < \infty \ \forall x$ so $\oplus_{F\in\mathscr{F}} \oplus_{[x]\in\overline{X}_F} F^{[x]} \cong \oplus_{F\in\mathscr{F}} F^{(X_F)}$. If we let $\lambda : \oplus_{F\in\mathscr{F}} F^{(X_F)} \to \prod_{F\in\mathscr{F}} F^{X_F}$ and $\overline{\lambda} : \oplus_{F\in\mathscr{F}} F^{(\overline{X}_F)} \to \prod_{F\in\mathscr{F}} F^{\overline{X}_F}$ be the canonical morphisms, we see that the diagram

$$\begin{tikzcd} \oplus_{F\in\mathscr{F}} F^{(\overline{X}_F)} \arrow[r, "\overline{\lambda}"] \arrow[d, "\Delta"'] & \prod_{F\in\mathscr{F}} F^{\overline{X}_F} \arrow[d, "\Delta'"] \\ \oplus_{F\in\mathscr{F}} F^{(X_F)} \arrow[r, "\lambda"'] & \prod_{F\in\mathscr{F}} F^{X_F} \end{tikzcd}$$

is commutative by Lemma 2.3.4.

If for any $F \in \mathscr{F}$ we consider the family of morphisms $\{f_x; [x] \in \overline{X}_F\}$, we get a unique morphism $h_F : M \to F^{\overline{X}_F}$ (for each F) such that

$$\begin{tikzcd} F^{\overline{X}_F} \arrow[rr, "\overline{\pi}_{[x]}"] & & F \\ & M \arrow[ul, dashed, "h_F"] \arrow[ur, "f_x"'] & \end{tikzcd}$$

commutes for every $[x] \in \overline{X}_F$, and this new family $\{h_F; F \in \mathscr{F}\}$ induces a unique morphism $h : M \to \prod_{F\in\mathscr{F}} F^{\overline{X}_F}$ such that

$$\begin{tikzcd} \prod_{F\in\mathscr{F}} F^{\overline{X}_F} \arrow[rr, "\pi_F"] & & F^{\overline{X}_F} \\ & M \arrow[ul, dashed, "h"] \arrow[ur, "h_F"'] & \end{tikzcd}$$

commutes $\forall F \in \mathscr{F}$.

Thus, h is the unique morphism satisfying $\overline{\pi}_{[x]}\overline{\pi}_F h = f_x$ for every $[x] \in \overline{X}_F$ and every $F \in \mathscr{F}$.

Of course, we also have the morphism

$$M \xrightarrow{\quad f \quad} \oplus_{F\in\mathscr{F}} F^{(X_F)} \xrightarrow{\quad \lambda \quad} \prod_{F\in\mathscr{F}} F^{X_F}$$

and $\pi_x \pi_F \lambda f = p_x p_F f \ \forall x \in X_F, \forall f \in \mathscr{F}$, so indeed $\lambda f : M \to \prod_{F\in\mathscr{F}} F^{X_F}$ is the unique morphism such that $\pi_x \pi_F \lambda f = f_x \ \forall x \in X_F, \forall F \in \mathscr{F}$.

Recall (see for instance [35, Corollary 2, page 188]) that λ is a monomorphism, so $\mathrm{Im}\,(\lambda f) \cong \mathrm{Im}\,(f)$ and then we see that the morphism λf has a factorization

$$\begin{tikzcd}
& M \arrow[rr, "\lambda f"] \arrow[dl, "\overline{f}"'] \arrow[dr, "f"] & & \prod_{F\in\mathscr{F}} F^{X_F} \\
\mathrm{Im}\,(\lambda f) \arrow[rr, "k"'] & & \oplus_{F\in\mathscr{F}} F^{(X_F)} \arrow[ur, "\lambda"'] &
\end{tikzcd}$$

and the morphism h has a factorization

$$\begin{tikzcd}
M \arrow[rr, "h"] \arrow[dr, "\overline{h}"'] & & \prod_{F\in\mathscr{F}} F^{(X_F)} \\
& \mathrm{Im}\,(h) \arrow[ur, "\beta"'] &
\end{tikzcd}$$

We claim that $\mathrm{Im}\,(h) \subseteq \oplus_{F\in\mathscr{F}} F^{(X_F)}$.

Let us prove that indeed $\mathrm{Im}\,(\beta) \subseteq \oplus_{F\in\mathscr{F}} F^{(X_F)}$. We have the diagram

$$\begin{tikzcd}
M \arrow[rrr, "h"] \arrow[dr, "\overline{h}"'] \arrow[dddd, equal] & & & \prod_{F\in\mathscr{F}} F^{\overline{X}_F} \arrow[dddd, "\Delta'"] \\
& \mathrm{Im}\,(h) \arrow[urr, "\beta"] & & \\
& & \oplus_{F\in\mathscr{F}} F^{\overline{X}_F} \arrow[uur, "\overline{\lambda}"] \arrow[d, "\Delta"] & \\
& \mathrm{Im}\,(\lambda f) \arrow[r, "k"] & \oplus_{F\in\mathscr{F}} F^{X_F} \arrow[dr, "\lambda"] & \\
M \arrow[ur, "\overline{f}"] \arrow[urr, "f"'] \arrow[rrr, "\lambda f"'] & & & \prod_{F\in\mathscr{F}} F^{X_F}
\end{tikzcd}$$

in which all possible subdiagrams are commutative (including the outer square since we have already seen that $\pi_x \pi_F \Delta' h = \overline{\pi}_{[x]} \overline{\pi}_F h = f_x$ $\forall x \in X_F$, $\forall F \in \mathscr{F}$ and $\pi_x \pi_F \lambda f = f_x$ $\forall x \in X_F$). Moreover, λk is a monomorphism so $\lambda k \overline{f}$ is the epic-monic factorization of λf and so $\lambda k = ker(coker(\lambda f))$. Let us call $c = coker(\lambda f)$. We have $c\Delta' h = c\lambda f = 0$, but $c\Delta' h = c\Delta' \beta \overline{h}$ so $c\Delta' \beta = 0$ and then there is a unique $\alpha : \text{Im }(h) \to \text{Im }(\lambda f)$ such that $\Delta' \beta = \lambda k \alpha$.

Each of the $\Delta_{[x],F}$ is a splitting monomorphism, so the induces Δ_F is a splitting monomorphism and then Δ is a splitting monomorphism too. This means we have a morphism $\Phi : \oplus_{F \in \mathscr{F}} F^{(X_F)} \to \oplus_{F \in \mathscr{F}} F^{(\overline{X}_F)}$ such that $\Phi\Delta = \text{id}$.

The same argument, reasoning with the product, gives $\Psi : \prod_{F \in \mathscr{F}} F^{(X_F)} \to \prod_{F \in \mathscr{F}} F^{(\overline{X}_F)}$ such that $\Psi\Delta' = \text{id}$.

Moreover Lemma 2.3.4 says that $\Psi\lambda = \overline{\lambda}\Phi$. We then consider the morphism

$$\phi : \text{Im }(h) \to \oplus_{F \in \mathscr{F}} F^{(\overline{X}_F)}$$

given by $\phi = \Phi k \alpha$.

We have

$$\overline{\lambda}\phi = \overline{\lambda}\Phi k \alpha = \psi \lambda k \alpha = \psi \Delta' \beta = \beta.$$

Therefore, we see that indeed $\text{Im }(\beta) \subseteq \oplus_{F \in \mathscr{F}} F^{(\overline{X}_F)}$ (β is a monomorphism so ϕ is a monomorphism too) and so that the morphism h factors through $\oplus_{F \in \mathscr{F}} F^{(\overline{X}_F)}$ ($\text{Im }(h) \subseteq \oplus_{F \in \mathscr{F}} F^{(\overline{X}_F)}$).

This means we can assume that the original morphisms f is such that $f_x \neq f_y$, $\forall x; y \in F$ with $x \neq y$. In this case, the map

$$\begin{aligned} X_F &\to \text{Hom}_{\mathscr{A}}(M, F) \\ x &\mapsto f_x \end{aligned}$$

is an injection, so we assume $X_F \subseteq \text{Hom}_{\mathscr{A}}(M, F)$ and then $\oplus_{F \in \mathscr{F}} F^{(X_F)}$ is a direct summand of $F_M = \oplus_{F \in \mathscr{F}} F^{(\text{Hom}(M,F))}$ for every $f \in \mathscr{F}$.

Since F_M is totally independent of the morphism f, we see that the set $\{F_M\}$ makes $Sum(\mathscr{F})$ to be locally initially small. □

Corollary 2.3.8. *If $\mathscr{A}$ is a locally finitely generated Grothendieck category with enough projectives, then $Proj_{\mathscr{A}}$ is locally initially small.*

Proof. If $\mathscr{G}$ is a system of projective generators of $\mathscr{A}$, then $Proj_{\mathscr{A}} = Add(\mathscr{G})$. But $Sum(\mathscr{G})$ is locally initially small Proposition 2.3.7, and then trivially $Summ(Sum(\mathscr{G})) = Add(\mathscr{G})$ is locally initially small. □

As mentioned before, with Corollary 2.3.6, the proof of [40, Proposition 2.9] follows in any locally finitely generated Grothendieck category with enough projectives $\mathscr{A}$, and as a consequence, if $\mathscr{A}$ has a system of projective generators $\mathscr{G}$, then we see that the class $Proj_{\mathscr{A}}$ is locally initially small. Indeed, it is easy to check that $Proj_{\mathscr{A}} = Add(\mathscr{G})$. But $Sum(\mathscr{G})$ is a locally initially small class, so $Add(\mathscr{G})$ is locally initially small too.

With the use of this fact, we can prove the following.

Proposition 2.3.9. *Let $\mathscr{L}$ be a class of objects of $\mathscr{A}$. Then, the following conditions are equivalent.*

1. *Every object M of $\mathscr{L}$ has a projective preenvelope.*

2. *Every object M of $\mathscr{L}$ has a $\underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(\mathscr{L})$-preenvelope.*

If, in addition, $\mathscr{A}$ is a locally finitely generated Grothendieck category with a system of projective generators then 1. and 2. above are equivalent to

3. *$\underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(\mathscr{L})$ is closed under direct products.*

Proof. (1) $\Rightarrow$ (2). Let $M \to Q$ be a projective preenvelope of M. Let us prove that $M \to Q$ is a $\underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(\mathscr{L})$-preenvelope. For let $N \in \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(\mathscr{L})$ and let $P \to N$ be an epimorphism with P projective. Apply the functors $\mathrm{Hom}_{\mathscr{A}}(M,-)$ and $\mathrm{Hom}_{\mathscr{A}}(Q,-)$ to $P \to N$ to get the following commutative diagram with exact rows

$$\begin{array}{ccccc}
\mathrm{Hom}_{\mathscr{A}}(Q,P) & \longrightarrow & \mathrm{Hom}_{\mathscr{A}}(Q,N) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \\
\mathrm{Hom}_{\mathscr{A}}(M,P) & \longrightarrow & \mathrm{Hom}_{\mathscr{A}}(M,N) & \longrightarrow & 0
\end{array}$$

with $\mathrm{Hom}_{\mathscr{A}}(Q,P) \to \mathrm{Hom}_{\mathscr{A}}(M,P)$ an epimorphism ($M \to Q$ is a projective preenvelope). Therefore, $\mathrm{Hom}_{\mathscr{A}}(Q,N) \to \mathrm{Hom}_{\mathscr{A}}(M,N)$ is also an epimorphism.

(2) $\Rightarrow$ (1). Let $f : M \to N$ be a $\underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(\mathscr{L})$-preenvelope of M and $g : P \to N$ be an epimorphism with P projective. Since $N \in \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(\mathscr{L})$ there exists a morphism $h : M \to P$ such that $f = gh$. Let us prove that $h : M \to P$ is a projective preenvelope.

For let $h' : M \to P'$ be a morphism with P' projective. Since $f : M \to N$ is a $\underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(\mathscr{L})$-preenvelope there exists a morphism $g' : N \to P'$ such that $h' = g'f$. Hence $h' = g'gh$ and $g'g : P \to P'$ is the morphism we were looking for.

(2) $\Rightarrow$ (3). Let $\{N_i\}_{i\in I}$ be a family of objects such that each $N_i \in \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(\mathscr{L})$, $f : M \to \prod_{i\in I} N_i$ be any morphism such that $M \in \mathscr{L}$ and $g : M \to N$ be a $\underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(\mathscr{L})$-preenvelope of M. Then, for every $j \in I$, there exists a morphism $\gamma_j : N \to N_j$ such that $\gamma_j g = \pi_j f$ where $\pi_j : \prod_{i\in I} N_i \to N_j$ is the canonical projection. Now, let $\alpha : P \to N$ be an epimorphism

with P projective. Then, we get the following diagram

$$\begin{tikzcd}
M \arrow[dr, dotted, "\beta"] \arrow[drr, bend left, "f"] \arrow[ddr, bend right, "g"'] & & \\
& P \arrow[r, dotted, "\gamma"] \arrow[d, "\alpha"'] & \prod N_i \arrow[d, "\pi_j"] \\
& N \arrow[r, "\gamma_j"'] & N_j
\end{tikzcd}$$

in which β exists such that $g = \alpha\beta$ since $N \in \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(M)$ and γ exists such that for every $j \in I$, $\pi_j\gamma = \gamma_j\alpha$ by the universal property of direct products. Then, we get that for every $j \in I$, $\pi_j f = \gamma_j g = \gamma_j\alpha\beta = \pi_j\gamma\beta$. Thus, $f = \gamma\beta$. Then, we conclude by Proposition 2.1.4 that $\prod_{i\in I} N_i \in \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(\mathscr{L})$

$(3) \Rightarrow (2)$. Let $M \in \mathscr{L}$. Given $N \in \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(\mathscr{L})$ we fix an epimorphism $P \to N$ from a projective P. Since the class of projective objects is locally initially small, there exists a set $\mathscr{X}$ of projective objects such that any morphism $M \to P$ factors through a product of objects in the set $\mathscr{X}$. But every morphism $M \to N$ factors through P, and such factorization $M \to P$ factors through a product of objects in the set $\mathscr{X}$, so we have just seen that every morphism $M \to N$ with $N \in \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(\mathscr{L})$ factors through a product of elements of $\mathscr{X}$.

Call now $K = \prod_{P\in\mathscr{X}} P^{\mathrm{Hom}_{\mathscr{A}}(M,P)}$. Since $\underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(\mathscr{L})$ is supposed to be closed under direct products, we see that $K \in \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(\mathscr{L})$.

Now, for each $P \in \mathscr{X}$ there exists a canonical morphism $\lambda_P : M \to P^{\mathrm{Hom}_{\mathscr{A}}(M,P)}$, so there is a unique $\lambda : M \to K$ such that $\pi_P\lambda = \lambda_P$ for every $P \in \mathscr{X}$, where π_P are the canonical projections. We claim that $\lambda : M \to K$ is a $\underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(\mathscr{L})$-preenvelope of M.

To show this, take any morphism $f : M \to N$ with $N \in \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(\mathscr{L})$, so there exist $h : M \to \prod_{X\in\mathscr{X}} X$ and $g : \prod_{X\in\mathscr{X}} X \to N$ such that $f = gh$. Consider the projections $\pi_X : K \to X^{\mathrm{Hom}_{\mathscr{A}}(M,X)}$ and $\pi_{p_X h} : X^{\mathrm{Hom}_{\mathscr{A}}(M,X)} \to X$ (the projection to the component $p_X h$ where $p_X : \prod_{X\in\mathscr{X}} X \to X$ is the canonical projection). By the universal property of the direct product there exists a unique morphism $\gamma : K \to \prod_{X\in\mathscr{X}} X$ such that $p_X\gamma = \pi_{p_X h}\pi_X$. Therefore, $p_X\gamma\lambda = \pi_{p_X h}\pi_X\lambda = \pi_{p_X h}\lambda_X = p_X h$ for all $X \in \mathscr{X}$, so $\gamma\lambda = h$ and hence $g\gamma\lambda = gh = f$. We then get that $\lambda : M \to K$ is a $\underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(\mathscr{L})$-preenvelope of M. $\square$

Now, we are in position to prove Theorem 2.3.10.

Theorem 2.3.10. *Suppose that $\mathscr{A}$ is a locally finitely generated Grothendieck category with a system of projective generators and let $\mathscr{L}$ be a class of objects which contains the class $Proj_{\mathscr{A}}$. Then, the following conditions are equivalent.*

1. $\mathscr{L}^{\perp} = \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(\mathscr{L})$.

2. *$Proj_{\mathscr{A}} \subseteq \mathscr{L}^{\perp}$, $\underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(\mathscr{L})$ is closed under cokernels of monomorphisms and every $M \in \mathscr{L}$ has an $\mathscr{L}^{\perp}$-preenvelope which is projective.*

Proof. 1. $\Rightarrow$ 2. Clearly $Proj_{\mathscr{A}} \subseteq \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(\mathscr{L}) = \mathscr{L}^{\perp}$.

Now, let

$$0 \to A \to B \to C \to 0$$

be exact with $A,\ B \in \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(\mathscr{L})$. To prove that $C \in \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(\mathscr{L})$ choose any $L \in \mathscr{L}$ and apply $\mathrm{Hom}_{\mathscr{A}}(L,-)$ to the exact sequence. We get a long exact sequence

$$0 \to \mathrm{Hom}_{\mathscr{A}}(L,A) \to \mathrm{Hom}_{\mathscr{A}}(L,B) \to \mathrm{Hom}_{\mathscr{A}}(L,C) \to \mathrm{Ext}^1_{\mathscr{A}}(L,A) \to \cdots$$

with $\mathrm{Ext}^1_{\mathscr{A}}(L,A) = 0$ since $L \in \mathscr{L}$ and $A \in \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(\mathscr{L}) = \mathscr{L}^{\perp}$. Then, $\mathrm{Hom}_{\mathscr{A}}(L,B) \to \mathrm{Hom}_{\mathscr{A}}(L,C)$ is epic. Thus, Proposition 2.1.4 immediately gives that $C \in \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(\mathscr{L})$.

Finally, if $M \in \mathscr{L}$ let $f : M \to N$ be a $\underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(\mathscr{L})$-preenvelope, which exists by Proposition 2.3.9 since $\mathscr{L}^{\perp}$ is always closed under direct products.

Now find an epimorphism $g : P \to N$ from a projective P. Since $N \in \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(\mathscr{L})$ there exists a morphism $h : M \to P$ such that $f = gh$. We claim that $h : M \to P$ is a $\underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(\mathscr{L})$-preenvelope. Indeed, let $k : M \to N'$ be a morphism with $N' \in \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(\mathscr{L})$. Since $f : M \to N$ is a $\underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(\mathscr{L})$-preenvelope, there exists $l : N \to N'$ such that $k = lf$, hence $k = lgh$. Therefore, $h : M \to P$ is a $\underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(\mathscr{L})$-preenvelope.

2. $\Rightarrow$ 1. Let $N \in \mathscr{L}^{\perp}$, choose any $M \in \mathscr{L}$, any morphism $f : M \to N$ and a $\mathscr{L}^{\perp}$-preenvelope $g : M \to Q$ of M with Q projective. Then, there exists $h : Q \to N$ such that $f = hg$, so $N \in \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(\mathscr{L})$ by Proposition 2.1.4.

Conversely, let $N \in \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(\mathscr{L})$, choose any $M \in \mathscr{L}$ and take an $\mathscr{L}^{\perp}$-preenvelope $g : M \to Q$, where Q is projective. Of course every $\mathscr{L}^{\perp}$-preenvelope is injective since $\mathscr{L}^{\perp}$ contains the class of injectives, so if C is the cokernel of g we get a long exact sequence

$$\cdots \longrightarrow \mathrm{Ext}^1_{\mathscr{A}}(Q,N) \longrightarrow \mathrm{Ext}^1_{\mathscr{A}}(M,N) \longrightarrow \mathrm{Ext}^2_{\mathscr{A}}(C,N) \longrightarrow \cdots$$

Since Q is projective, showing that $\mathrm{Ext}^2_{\mathscr{A}}(C,N) = 0$ would give that $\mathrm{Ext}^1_{\mathscr{A}}(M,N) = 0$, so let's prove that $\mathrm{Ext}^i_{\mathscr{A}}(C,N) = 0,\ i = 1,2$.

Choose then any morphism $f : M \to N$ and find an epimorphism $h : P \to N$ from a projective P. Then, by the N-subprojectivity of M, the diagram

$$\begin{tikzcd}
 & M \arrow[r, "g"] \arrow[dl, dashed, "k"'] \arrow[d, "f"] & Q \\
P \arrow[r, "h"'] & N \arrow[r] & 0
\end{tikzcd}$$

can be completed commutatively by k. But P is projective (so it holds in $\mathscr{L}^{\perp}$ by the hypotheses), and g is a $\mathscr{L}^{\perp}$-preenvelope, so there is a morphism $l : Q \to P$ such that $lg = k$. Therefore, $f = hk = hlg$ and then $\mathrm{Hom}_{\mathscr{A}}(g,N)$ is an epimorphism, so from the long exact sequence

$$\cdots \longrightarrow \mathrm{Hom}_{\mathscr{A}}(Q,N) \longrightarrow \mathrm{Hom}_{\mathscr{A}}(M,N) \longrightarrow \mathrm{Ext}^1_{\mathscr{A}}(C,N) \longrightarrow \mathrm{Ext}^1_{\mathscr{A}}(Q,N) = 0$$

we see that $\mathrm{Ext}^1_{\mathscr{A}}(C,N) = 0$.

Now, if $0 \to N \to E \to D \to 0$ is exact and E is injective, we get an associated long exact sequence

$$\cdots \longrightarrow \mathrm{Ext}^1_{\mathscr{A}}(C,D) \longrightarrow \mathrm{Ext}^2_{\mathscr{A}}(C,N) \longrightarrow \mathrm{Ext}^2_{\mathscr{A}}(C,E) = 0 \longrightarrow \cdots$$

But we have already proved that $\mathscr{L}^{\perp} \subseteq \underline{\mathfrak{Pr}}^{-1}_{\mathscr{A}}(\mathscr{L})$, so $E \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{A}}(\mathscr{L})$, and N does too, so since $\underline{\mathfrak{Pr}}^{-1}_{\mathscr{A}}(\mathscr{L})$ is closed under cokernels of monomorphisms we get that D is also in $\underline{\mathfrak{Pr}}^{-1}_{\mathscr{A}}(\mathscr{L})$. Hence, by the same arguments as before, $\mathrm{Ext}^1_{\mathscr{A}}(C,D) = 0$ and then $\mathrm{Ext}^2_{\mathscr{A}}(C,N) = 0$. $\square$

MEASURING PROJECTIVITY OF COMPLEXES

In this chapter, we extend the study of the subprojectivity domains from an abelian category $\mathscr{A}$ with enough projectives to the category of complexes $\mathscr{C}(\mathscr{A})$ (which has enough projectives since $\mathscr{A}$ has). Namely, we study the relationship between the subprojectivity domains of complexes in $\mathscr{C}(\mathscr{A})$ and the subprojectivity domains of their components and cycles in $\mathscr{A}$. This study shows that the subprojectivity notion provides a new sight of null-homotopic morphisms in the category of complexes and gives various results which emphasize the importance of subprojectivity in the category of complexes. Namely, we give some applications by characterizing some classical rings and establish various examples that allow us to reflect the scope and limits of our results.

3.1 Subprojectivity and null-homotopy

In this section, a first treatment of the subprojectivity in the category of complexes will be done. We will prove among several things that the concept of subprojectivity in the category of complexes is closely linked to that of null-homotopy of morphisms.

We start with a new characterization of subprojectivity in terms of splitting short exact sequences which will be considered somehow as the subprojectivity analogue of the classical characterization of projectivity.

Proposition 3.1.1. *Let M and N be two objects of $\mathscr{A}$. Then the following conditions are equivalent.*

1. $N \in \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(M)$.
2. *For every epimorphism $g : K \to N$ and every morphism $f : M \to N$, the epimorphism $g' : D \to M$ given by the pullback (D, g', f') of g and f, splits.*

3. *There exists an epimorphism $g : P \to N$ with P projective such that for every morphism $f : M \to N$, the epimorphism $g' : D \to M$ given by the pullback (D, g', f') of g and f, splits.*

4. *There exists an epimorphism $g : P \to N$ with $P \in \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(M)$ such that for every morphism $f : M \to N$, the epimorphism $g' : D \to M$ given by the pullback (D, g', f') of g and f, splits.*

Proof. 1. $\Rightarrow$ 2. Let $g : K \to N$ be an epimorphism, $f : M \to N$ be a morphism and (D, g', f') be their pullback. Since $N \in \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(M)$, there exists a morphism $h : M \to K$ such that the following diagram commutes

$$\begin{tikzcd} M \arrow[drr, "\mathrm{id}_M"] \arrow[ddr, "h"'] & & \\ & D \arrow[r, "g'"'] \arrow[d, "f'"] & M \arrow[d, "f"] \\ & K \arrow[r, "g"'] & N \end{tikzcd}$$

Then, by the universal property of pullbacks, there exists a morphism $k : M \to D$ such that $g'k = id_M$. Hence g' splits, as desired.

2. $\Rightarrow$ 3. This is clear since the category $\mathscr{A}$ is supposed to have enough projectives.

3. $\Rightarrow$ 4. This is clear since every projective object belongs to $\underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(M)$.

4. $\Rightarrow$ 1. Let $g : P \to N$ be the epimorphism of statement 4., $f : M \to N$ be a morphism and (D, g', f') their pullback

$$\begin{tikzcd} D \arrow[r, "g'"] \arrow[d, "f'"'] & M \arrow[d, "f"] \\ P \arrow[r, "g"] & N \end{tikzcd}$$

Then, by assumption, there exists a morphism $h : M \to D$ such that $g'h = id_M$, hence $f = fg'h = gf'h$. Therefore, $N \in \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(M)$ (see Proposition Lemma-subproj).

□

The following two lemmas will be useful in the proof of Theorem 3.1.4.

Lemma 3.1.2. *For two complexes M and N with $N \in \underline{\mathfrak{Pr}}_{\mathscr{C}(\mathscr{A})}^{-1}(M)$, $\mathrm{Hom}_{\mathscr{K}(\mathscr{A})}(M, N) = 0$.*

Proof. Let $f \in \mathrm{Hom}_{\mathscr{C}(\mathscr{A})}(M, N)$, then there exist two morphisms $\alpha : P \to N$ and $\beta : M \to P$ such that P is projective and $f = \alpha\beta$ (see Proposition 2.1.4). Now, id_P is null-homotopic since P is contractible, thus, the composition $\alpha \mathrm{id}_P \beta$ is null-homotopic. Therefore, $\mathrm{Hom}_{\mathscr{K}(\mathscr{A})}(M, N) = 0$.

□

Lemma 3.1.3. *If (D,g',f') is the pullback of two morphisms of complexes $g: C \to B$ and $f: A \to B$, then (D_n, g'_n, f'_n) is the pullback of $g_n: C_n \to B_n$ and $f_n: A_n \to B_n$ for every $n \in \mathbb{Z}$.*

Proof. Let $\alpha: X \to A_n$ and $\beta: X \to C_n$ be two morphisms of $\mathscr{A}$ such that $f_n\alpha = g_n\beta$ and consider the two morphisms of complexes $\overline{\alpha}: \overline{X}[n-1] \to A$ and $\overline{\beta}: \overline{X}[n-1] \to C$ induced by α and β, respectively. It is straightforward to verify that $f\overline{\alpha} = g\overline{\beta}$, so there exists a unique morphism of complexes $h: \overline{X}[n-1] \to D$ such that $g'h = \overline{\alpha}$ and $f'h = \overline{\beta}$. Then, $g'_n h_n = \alpha$ and $f'_n h_n = \beta$.

The unicity of $h_n: X \to D_n$ comes from the unicity of h. $\square$

Now, we give the first main result of this section.

Theorem 3.1.4. *Let M and N be two complexes such that $N_n \in \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(M_n)$ for every $n \in \mathbb{Z}$. Then, the following statements are equivalent.*

1. *$N \in \underline{\mathfrak{Pr}}_{\mathscr{C}(\mathscr{A})}^{-1}(M)$.*
2. *For every short exact sequence $0 \to K \to P \to N \to 0$ with P projective, the equation $\operatorname{Hom}_{\mathscr{K}(\mathscr{A})}(M[-1],K) = 0$ holds.*
3. *There exists a short exact sequence $0 \to K \to P \to N \to 0$ with P projective such that $\operatorname{Hom}_{\mathscr{K}(\mathscr{A})}(M[-1],K) = 0$.*
4. *There exists a short exact sequence $0 \to K \to P \to N \to 0$ with $P \in \underline{\mathfrak{Pr}}_{\mathscr{C}(\mathscr{A})}^{-1}(M)$ such that $\operatorname{Hom}_{\mathscr{K}(\mathscr{A})}(M[-1],K) = 0$.*

Proof. 1. $\Rightarrow$ 2. Let $0 \to K \to P \to N \to 0$ be a short exact sequence with P projective and consider the following commutative diagram with exact rows

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Z_0(\operatorname{Hom}^\bullet(M,K)) & \longrightarrow & \operatorname{Hom}^\bullet(M,K)_0 & \longrightarrow & B_{-1}(\operatorname{Hom}^\bullet(M,K)) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Z_0(\operatorname{Hom}^\bullet(M,P)) & \longrightarrow & \operatorname{Hom}^\bullet(M,P)_0 & \longrightarrow & B_{-1}(\operatorname{Hom}^\bullet(M,P)) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Z_0(\operatorname{Hom}^\bullet(M,N)) & \longrightarrow & \operatorname{Hom}^\bullet(M,N)_0 & \longrightarrow & B_{-1}(\operatorname{Hom}^\bullet(M,N)) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

The first and second columns are exact since $N \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(\mathscr{A})}(M)$ and $N_n \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{A}}(M_n)$ for every $n \in \mathbb{Z}$, respectively. Hence, the third column is also exact.

Now, applying the Snake Lemma to the following commutative diagram with exact rows and columns

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 \to & B_{-1}(\mathrm{Hom}^\bullet(M,K)) & \to & B_{-1}(\mathrm{Hom}^\bullet(M,P)) & \to & B_{-1}(\mathrm{Hom}^\bullet(M,N)) & \to 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
0 \to & Z_{-1}(\mathrm{Hom}^\bullet(M,K)) & \to & Z_{-1}(\mathrm{Hom}^\bullet(M,P)) & \to & Z_{-1}(\mathrm{Hom}^\bullet(M,N)) &
\end{array}$$

we get the exact sequence

$$0 \to H_{-1}(\mathrm{Hom}^\bullet(M,K)) \to H_{-1}(\mathrm{Hom}^\bullet(M,P)) \to H_{-1}(\mathrm{Hom}^\bullet(M,N))$$

but $H_{-1}(\mathrm{Hom}^\bullet(M,P)) = \mathrm{Hom}_{\mathscr{K}(\mathscr{A})}(M[-1],P) = 0$ by Lemma 3.1.2. Thus,

$$\mathrm{Hom}_{\mathscr{K}(\mathscr{A})}(M[-1],K) = H_{-1}(\mathrm{Hom}^\bullet(M,K)) = 0.$$

2. $\Rightarrow$ 3. Clear since the category of complexes $\mathscr{C}(\mathscr{A})$ has enough projectives.

3. $\Rightarrow$ 4. This is clear since every projective complex belongs to $\underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(\mathscr{A})}(M)$.

4. $\Rightarrow$ 1. Let $0 \to K \to P \to N \to 0$ be the short exact sequence of statement 4., $f : M \to N$ be any morphism of complexes and consider the following pullback diagram

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r] \arrow[d, equal] & D \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, "f"] & 0 \\
0 \arrow[r] & K \arrow[r] & P \arrow[r] & N \arrow[r] & 0
\end{tikzcd}$$

For every $n \in \mathbb{Z}$, D_n is a pullback by Lemma 3.1.3, so by assumption and Proposition 3.1.1 the short exact sequence $0 \to K \to D \to M \to 0$ is degreewise splits. Then, this sequence is equivalent to a short exact sequence $0 \to K \to M(g) \to M \to 0$ being $M(g)$ the mapping cone of a morphism $g : M[-1] \to K$, but $g : M[-1] \to K$ is null-homotopic by assumption so $0 \to K \to M(g) \to M \to 0$ splits (see [22, Proposition 3.3.2]). Therefore, the sequence $0 \to K \to D \to M \to 0$ splits too and then $N \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(\mathscr{A})}(M)$ by Proposition 3.1.1.

□

Remark 3.1.5. *It is natural to ask whether, as in the case of exact sequences* $0 \to K \to P \to N \to 0$ *with P projective, the statements of Theorem 3.1.4 are equivalent to the following: "for every short exact sequence* $0 \to K \to P \to N \to 0$ *with* $P \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(\mathscr{A})}(M)$, *the equation* $\mathrm{Hom}_{\mathscr{K}(\mathscr{A})}(M[-1],K) = 0$ *holds". We will see in Example 3.1.21 that they are not equivalent.*

Given two complexes M and N, it is natural to ask if $N \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(\mathscr{A})}(M)$ is sufficient to get that, for every $n \in \mathbb{Z}$, $N_n \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{A}}(M_n)$. This is not true in general. Indeed, we can always consider, over a non semisimple ring R, two modules X and Y with $Y \notin \underline{\mathfrak{Pr}}^{-1}_{R\text{-Mod}}(X)$, while it is clear that we always have $\underline{Y} \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(R)}(\overline{X})$ since every morphism $\overline{X} \to \underline{Y}$ is zero. Nevertheless, the answer to the question would be positive if we assume, furthermore, that N belongs to $\underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(\mathscr{A})}(M[-1])$.

Proposition 3.1.6. *Let M and N be two complexes such that*

$$N \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(\mathscr{A})}(M[-1]) \bigcap \underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(\mathscr{A})}(M).$$

Then, $N_n \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{A}}(M_n)$ for every $n \in \mathbb{Z}$.

Proof. Let P be a projective complex and $P \to N$ be an epimorphism of complexes. Since $N, P \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(\mathscr{A})}(M[-1])$, $\mathrm{Hom}_{\mathscr{K}(\mathscr{A})}(M[-1], P) = \mathrm{Hom}_{\mathscr{K}(\mathscr{A})}(M[-1], N) = 0$ by Lemma 3.1.2. So, the horizontal maps of the following commutative diagram are isomorphisms

$$\begin{array}{ccc} B_{-1}(\mathrm{Hom}^\bullet(M,P)) & \longrightarrow & Z_{-1}(\mathrm{Hom}^\bullet(M,P)) \\ \downarrow & & \downarrow \\ B_{-1}(\mathrm{Hom}^\bullet(M,N)) & \longrightarrow & Z_{-1}(\mathrm{Hom}^\bullet(M,N)) \end{array}$$

The morphism

$$Z_{-1}(\mathrm{Hom}^\bullet(M,P)) \to Z_{-1}(\mathrm{Hom}^\bullet(M,N))$$

coincides with $\mathrm{Hom}_{\mathscr{C}(\mathscr{A})}(M[-1], P) \to \mathrm{Hom}_{\mathscr{C}(\mathscr{A})}(M[-1], N)$, and it is epic since $N \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(\mathscr{A})}(M[-1])$, so the map $B_{-1}(\mathrm{Hom}^\bullet(M,P)) \to B_{-1}(\mathrm{Hom}^\bullet(M,N))$ must also be epic.

Now, consider the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Z_0(\mathrm{Hom}^\bullet(M,P)) & \longrightarrow & \mathrm{Hom}^\bullet(M,P)_0 & \longrightarrow & B_{-1}(\mathrm{Hom}^\bullet(M,P)) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & Z_0(\mathrm{Hom}^\bullet(M,N)) & \longrightarrow & \mathrm{Hom}^\bullet(M,N)_0 & \longrightarrow & B_{-1}(\mathrm{Hom}^\bullet(M,N)) & \longrightarrow & 0 \end{array}$$

The map $Z_0(\mathrm{Hom}^\bullet(M,P)) \to Z_0(\mathrm{Hom}^\bullet(M,N))$ is epic since $N \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(\mathscr{A})}(M)$, so again $\mathrm{Hom}^\bullet(M,P)_0 \to \mathrm{Hom}^\bullet(M,N)_0$ is epic so we see that every morphism $M_n \to N_n$ factors through P_n for every $n \in \mathbb{Z}$.

□

Though the fact that a complex N belongs to the subprojectivity domain of another complex M does not imply that the components of N necessarily belong to the subprojectivity domains of the components of M, the answer is completely different if we ask about cycles of N instead of components of N. We can see this in the following result.

Lemma 3.1.7. *Let N be a complex and M be an object in $\mathscr{A}$. If $N \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(\mathscr{A})}(\underline{M}[n])$ for some $n \in \mathbb{Z}$, then $Z_n(N) \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{A}}(M)$.*

Proof. Let $f : M \to Z_n(N)$ be any morphism of $\mathscr{A}$ and $\underline{f} : \underline{M}[n] \to N$ be the induced morphism of complexes. By assumption $\underline{f}$ factors as

$$\begin{tikzcd} \underline{M}[n] \arrow[r, "\alpha"'] \arrow[rr, bend left, "\underline{f}"] & P \arrow[r, "\beta"'] & N \end{tikzcd}$$

for some projective complex P. Then, $d^P_n \alpha_n = 0$, so there exists a morphism $h : M \to Z_n(P)$ such that $\mu^P_n h = \alpha_n$.

On the other side, the morphism β induces a morphism $g : Z_n(P) \to Z_n(N)$ such that $\mu^N_n g = \beta_n \mu^P_n$. Then, we have

$$\mu^N_n gh = \beta_n \mu^P_n h = \beta_n \alpha_n = \underline{f}_n = \mu^N_n f,$$

that is, $f = gh$, so f factors through the projective object $Z_n(P)$. □

Another natural question at this point is whether the inverse implication of Proposition 3.1.6 is true or not. Namely, given two complexes M and N, is the condition "$N_n \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{A}}(M_n)$ for every $n \in \mathbb{Z}$", sufficient to assure that $N \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(\mathscr{A})}(M)$? Again, this is not true in general since, for instance, for exact complexes of modules it only holds over left hereditary rings (see Proposition 3.4.2).

We have studied so far the relation between subprojectivity and null-homotopic morphisms involving kernels of epimorphisms. We will now see that this relation can also be described without considering such kernels (Theorem 3.1.12).

We start by characterizing when a contractible complex holds in the subprojectivity domain of another complex. We need the following lemma.

Lemma 3.1.8. *Let M be a complex, N be an object of $\mathscr{A}$ and $n \in \mathbb{Z}$. Then, $\overline{N}[n] \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(\mathscr{A})}(M)$ if and only if $N \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{A}}(M_n)$.*

Proof. Suppose that $\overline{N}[n] \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(\mathscr{A})}(M)$ and let $f : M_n \to N$ be a morphism in $\mathscr{A}$. The induced morphism $\overline{f} : M \to \overline{N}[n]$ (that is, $\overline{f}_n = f$) factors through a projective complex P by the hypothesis, so f factors through the projective object P_n.

Conversely, let $f : M \to \overline{N}[n]$ be a morphism of complexes. Since $N \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{A}}(M_n)$, the morphism f_n factors as

$$\begin{tikzcd} M_n \arrow[r, "\alpha"'] \arrow[rr, bend left, "f_n"] & P \arrow[r, "\beta"'] & N \end{tikzcd}$$

for some projective object P of $\mathscr{A}$. Then, if we let $g : M \to \overline{P}[n]$ be the morphism of complexes with $g_n = \alpha$ and $g_{n+1} = \alpha d^M_{n+1}$, and $h : \overline{P}[n] \to \overline{N}[n]$ be the morphism of complexes with $h_n = h_{n+1} = \beta$, we clearly get that $f = hg$, hence $\overline{N}[n] \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(\mathscr{A})}(M)$.

□

Proposition 3.1.9. *Let M be a complex and $(N_n)_{n\in\mathbb{Z}}$ be a family of objects of $\mathscr{A}$. Then, $\oplus_{n\in\mathbb{Z}}\overline{N_n}[n] \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(\mathscr{A})}(M)$ if and only if $N_n \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{A}}(M_n)$ for every $n \in \mathbb{Z}$.*

Proof. If $\oplus_{n\in\mathbb{Z}}\overline{N_n}[n] \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(\mathscr{A})}(M)$ then $\overline{N_n}[n] \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(\mathscr{A})}(M)$ for every $n \in \mathbb{Z}$ since $\underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(\mathscr{A})}(M)$ is closed under direct summands (see Proposition 2.2.1). Then, by Lemma 3.1.8 we get that for every $n \in \mathbb{Z}$, $N_n \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{A}}(M_n)$.

Conversely, if $N_n \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{A}}(M_n)$ for every $n \in \mathbb{Z}$ then $\overline{N_n}[n] \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(\mathscr{A})}(M)$ for every $n \in \mathbb{Z}$ again by Lemma 3.1.8.

Now, let $f : M \to \oplus_{n\in\mathbb{Z}}\overline{N_n}[n]$ be a morphism of complexes and, for every m, choose an epimorphism $g^m : \overline{P_m}[m] \to \overline{N_m}[m]$ with P_m a projective object of $\mathscr{A}$.

If we let

$$\pi^m : \oplus_{n\in\mathbb{Z}}\overline{N_n}[n] \to \overline{N_m}[m]$$

be the projection morphism, for any m there exists a morphism $h^m : M \to \overline{P_m}[m]$ such that $\pi^m f = g^m h^m$.

But $\oplus_{n\in\mathbb{Z}}\overline{P_n}[n]$ coincides with $\prod_{n\in\mathbb{Z}} \overline{P_n}[n]$, so if we call

$$\pi'^m : \oplus_{n\in\mathbb{Z}}\overline{P_n}[n] \to \overline{P_m}[m]$$

the projection morphism, we get a morphism $h : M \to \oplus_{n\in\mathbb{Z}}\overline{P_n}[n]$ such that $\pi'^m h = h^m$ for every m.

Therefore, for every $m \in \mathbb{Z}$ we have

$$\pi^m f = g^m h^m = g^m \pi'^m h = \pi^m (\oplus g^n) h$$

so we see that $f = (\oplus g^n)h$. This means that f factors through the projective complex $\oplus_{n\in\mathbb{Z}}\overline{P_n}[n]$ and so that $\oplus_{n\in\mathbb{Z}}\overline{N_n}[n] \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(\mathscr{A})}(M)$.

□

The following result characterizes subprojectivity in terms of factorization of morphisms through contractible complexes.

Proposition 3.1.10. *Let M and N be two complexes. The following conditions are equivalent.*

1. $N \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(\mathscr{A})}(M)$.

2. *Every morphism $M \to N$ factors through a contractible complex $\oplus_{n\in\mathbb{Z}}\overline{X_n}[n]$ such that $X_n \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{A}}(M_n)$ for every $n \in \mathbb{Z}$.*

Proof. 1. $\Rightarrow$ 2. Clear since every morphism $M \to N$ factors through a projective complex by Proposition 2.1.4 and every projective complex is a contractible complex of projective objects of $\mathscr{A}$. 2. $\Rightarrow$ 1. Apply Proposition 3.1.9 to get that $\oplus_{n\in\mathbb{Z}}\overline{X_n}[n] \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(\mathscr{A})}(M)$ and conclude by Proposition 2.1.4.

□

Lemma 3.1.11. *Let $f : X \to Y$ be a null-homotopic morphism of complexes by a morphism s. If every morphism $s_n : X_n \to Y_{n+1}$ of $\mathscr{A}$ factors through an object L_{n+1}, then $f : X \to Y$ factors through the contractible complex $\oplus_{n\in\mathbb{Z}}\overline{L_{n+1}}[n]$. In particular, $f : X \to Y$ factors through the contractible complex $\oplus_{n\in\mathbb{Z}}\overline{Y_{n+1}}[n]$.*

Proof. Suppose that for any n there exist two morphisms $\alpha_n : X_n \to L_{n+1}$ and $\beta_n : L_{n+1} \to Y_{n+1}$ such that $s_n = \beta_n\alpha_n$. Then, we have the situation

$$\begin{tikzcd}
X_{n+1} \arrow[rr, "d^X_{n+1}"] \arrow[ddr, "f_{n+1}"'] & & X_n \arrow[rr, "d^X_n"] \arrow[dl, "\alpha_n"'] \arrow[ddl, "s_n"] \arrow[ddr, "f_n"'] & & X_{n-1} \arrow[dl, "\alpha_{n-1}"'] \arrow[ddl, "s_{n-1}"] \arrow[ddr, "f_{n-1}"] & \\
& L_{n+1} \arrow[d, "\beta_n"] & & L_n \arrow[d, "\beta_{n-1}"] & & \\
& Y_{n+1} \arrow[rr, "d^Y_{n+1}"] & & Y_n \arrow[rr, "d^Y_n"] & & Y_{n-1}
\end{tikzcd}$$

For every $n \in \mathbb{Z}$, let $p^1_{n+1} : L_{n+1} \oplus L_n \to L_{n+1}$ and $p^2_n : L_{n+1} \oplus L_n \to L_n$ be the canonical projections, and $k^1_{n+1} : L_{n+1} \to L_{n+1} \oplus L_n$ and $k^2_n : L_n \to L_{n+1} \oplus L_n$ be the canonical injections. Now, call Z the complex $\oplus_{n\in\mathbb{Z}}\overline{L_{n+1}}[n]$ and consider, for every $n \in \mathbb{Z}$, the two morphisms of $\mathscr{A}$ $h_n : L_{n+1} \oplus L_n \to Y_n$ given by $h_n = d^Y_{n+1}\beta_n p^1_{n+1} + \beta_{n-1}p^2_n$, and $g_n : X_n \to L_{n+1} \oplus L_n$ given by $g_n = (\alpha_n, \alpha_{n-1}d^X_n)$. We claim that both $h : Z \to Y$ and $g : X \to Z$ are morphisms of complexes.

For any $n \in \mathbb{Z}$, we have $d^Y_n h_n = d^Y_n(d^Y_{n+1}\beta_n p^1_{n+1} + \beta_{n-1}p^2_n) = d^Y_n\beta_{n-1}p^2_n$, and $h_{n-1}d^Z_n = (d^Y_n\beta_{n-1}p^1_n + \beta_{n-2}p^2_{n-1})k^1_n p^2_n = d^Y_n\beta_{n-1}P^1_n k^1_n p^2_n = d^Y_n\beta_{n-1}p^2_n$, so h is a morphism of complexes, and for any $n \in \mathbb{Z}$ we have

$$g_{n-1}d^X_n = (\alpha_{n-1}, \alpha_{n-2}d^X_{n-1})d^X_n = (\alpha_{n-1}d^X_n, 0) = d^Z_n(\alpha_n, \alpha_{n-1}d^X_n) = d^Z_n g_n,$$

so g is also a morphism of complexes.

Now we see that $f = hg$ since for any $n \in \mathbb{Z}$ we have

$$h_n g_n = d^Y_{n+1}\beta_n\alpha_n + \beta_{n-1}\alpha_{n-1}d^X_n = d^Y_{n+1}s_n + s_{n-1}d^X_n = f_n.$$

Therefore, $f : X \to Y$ factors through the contractible complex $Z = \oplus_{n\in\mathbb{Z}}\overline{L_{n+1}}[n]$.

□

Theorem 3.1.12. *Let M and N be two complexes such that $N_{n+1} \in \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(M_n)$ for every $n \in \mathbb{Z}$. Then, $N \in \underline{\mathfrak{Pr}}_{\mathscr{C}(\mathscr{A})}^{-1}(M)$ if and only if $\mathrm{Hom}_{\mathscr{K}(\mathscr{A})}(M,N) = 0$.*

Proof. If $N \in \underline{\mathfrak{Pr}}_{\mathscr{C}(\mathscr{A})}^{-1}(M)$ then it is clear by Lemma 3.1.2 that $\mathrm{Hom}_{\mathscr{K}(\mathscr{A})}(M,N) = 0$.

Conversely, if $\mathrm{Hom}_{\mathscr{K}(\mathscr{A})}(M,N) = 0$ then, by Lemma 3.1.11, every morphism $M \to N$ factors through the contractible complex $\oplus_{n\in\mathbb{Z}}\overline{N_{n+1}}[n]$, so $N \in \underline{\mathfrak{Pr}}_{\mathscr{C}(\mathscr{A})}^{-1}(M)$ by Proposition 3.1.10.

□

The following example shows that the condition $N_{n+1} \in \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(M_n)$ for every $n \in \mathbb{Z}$ in Theorem 3.1.12 cannot be removed in general.

Example 3.1.13. *Let X be any non-projective module and choose any other module Y out of the subprojectivity domain of X (such modules exist over any non semisimple ring). It is clear that $\mathrm{Hom}_{\mathscr{K}(R)}(\overline{X},\overline{Y}) = 0$ and, by Lemma 3.1.8, that $\overline{Y} \notin \underline{\mathfrak{Pr}}_{\mathscr{C}(R)}^{-1}(\overline{X})$.*

Given two complexes M and N, it is clear that the condition "$N_{n+1} \in \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(M_n)$ for every $n \in \mathbb{Z}$" is not enough in general to get $N \in \underline{\mathfrak{Pr}}_{\mathscr{C}(\mathscr{A})}^{-1}(M)$. For instance, if $\mathscr{A}$ is semisimple (in the sense that every object is projective) and M is not exact (so M is not a projective complex), then for sure we can find complexes not in $\underline{\mathfrak{Pr}}_{\mathscr{C}(\mathscr{A})}^{-1}(M)$.

In the following result we prove that this condition suffices for exact complexes if and only if $\mathscr{A}$ is semisimple.

Proposition 3.1.14. *The following conditions are equivalent.*

1. *$\mathscr{A}$ is semisimple.*
2. *For every complex M and every exact complex N, if $N_{n+1} \in \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(M_n)$ for every $n \in \mathbb{Z}$, then $N \in \underline{\mathfrak{Pr}}_{\mathscr{C}(\mathscr{A})}^{-1}(M)$.*
3. *For every object M of $\mathscr{A}$ and every exact complex N, if there exists $n \in \mathbb{Z}$ such that $N_{n+1} \in \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(M)$, then $N \in \underline{\mathfrak{Pr}}_{\mathscr{C}(\mathscr{A})}^{-1}(\underline{M}[n])$.*

Proof. 1. $\Rightarrow$ 2. Every exact complex N is projective so $N \in \underline{\mathfrak{Pr}}_{\mathscr{C}(\mathscr{A})}^{-1}(M)$ for every complex M.

2. $\Rightarrow$ 3. Clear.

3. $\Rightarrow$ 1. Let M be an object of $\mathscr{A}$ and $\mathscr{P}$ be a projective resolution of M. Then, $\mathscr{P}_1 \in \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(\underline{M})$ and so $\mathscr{P} \in \underline{\mathfrak{Pr}}_{\mathscr{C}(\mathscr{A})}^{-1}(\underline{M})$ by assumption. Then, by Lemma 3.1.7, $M = Z_0(\mathscr{P}) \in \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(M)$. This means that M is projective and therefore that $\mathscr{A}$ is semisimple.

□

Given two complexes M and N, it is natural to ask whether $N \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(\mathscr{A})}(M)$ implies that $N_{n+1} \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{A}}(M_n)$ for every $n \in \mathbb{Z}$. This is not true in general. For intance, In the category of R-modules, if we take any non-projective module X and choose any other module Y out of the the subprojectivity domain of X (such modules exist over any non semisimple ring). Then, the complex $\underline{Y}[2]$ belongs to $\underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(R)}(\overline{X})$ since $\mathrm{Hom}_{\mathscr{C}(R)}(\overline{X}, \underline{Y}[2]) = 0$, but $\underline{Y}[2]_2 = Y \notin \underline{\mathfrak{Pr}}^{-1}_{R\text{-Mod}}(X)$.

However, if we add the condition "$N \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(\mathscr{A})}(M[1])$", then Proposition 3.1.6 says that $N_{n+1} \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{A}}(M_n)$ for every $n \in \mathbb{Z}$.

Inspired by Proposition 3.1.6, we give the following result.

Proposition 3.1.15. *Let M and N be two complexes. The following statements are equivalent.*

1. *$N \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(\mathscr{A})}(M[n])$ for every $n \in \mathbb{Z}$.*

2. *For every $i, j \in \mathbb{Z}$, $N_i \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{A}}(M_j)$, and $\mathrm{Hom}_{\mathscr{K}(\mathscr{A})}(M[n], N) = 0$ for every $n \in \mathbb{Z}$.*

Proof. Apply Proposition 3.1.6 and Theorem 3.1.12. □

Now, we give some applications of Proposition 3.1.15. Namely, given any object M of $\mathscr{A}$, Proposition 3.1.15 can be used to study the subprojectivity domain of the complexes $\oplus_{n\in\mathbb{Z}}\overline{M}[n]$ (Proposition 3.1.16) and $\oplus_{n\in\mathbb{Z}}\underline{M}[n]$ (Proposition 3.1.17).

Proposition 3.1.16. *Let N be a complex and M be an object of $\mathscr{A}$. The following statements are equivalent.*

1. *$N \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(\mathscr{A})}(\oplus_{n\in\mathbb{Z}}\overline{M}[n])$.*

2. *$N \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(\mathscr{A})}(\overline{M}[n])$ for every $n \in \mathbb{Z}$.*

3. *$N_n \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{A}}(M)$ for every $n \in \mathbb{Z}$.*

Proof. 1. ⇔ 2. Clear by Proposition 2.1.14.

2. ⇔ 3. Clear by Proposition 3.1.15 since $\mathrm{Hom}_{\mathscr{K}(\mathscr{A})}(\overline{M}[n], N) = 0$ for every $n \in \mathbb{Z}$. □

Proposition 3.1.17. *Let N be a complex and M be an object of $\mathscr{A}$. The following statements are equivalent.*

1. *$N \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(\mathscr{A})}(\oplus_{n\in\mathbb{Z}}\underline{M}[n])$.*

2. *$N \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(\mathscr{A})}(\underline{M}[n])$ for every $n \in \mathbb{Z}$.*

3. *N is $\mathrm{Hom}_{\mathscr{A}}(M,-)$-exact and $N_n \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{A}}(M)$ for every $n \in \mathbb{Z}$.*

Proof. 1. $\Leftrightarrow$ 2. Clear by Proposition 2.1.14.

2. $\Rightarrow$ 3. By Proposition 3.1.15 we know that $N_n \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{A}}(M)$ for every $n \in \mathbb{Z}$ and that $H_n(\mathrm{Hom}^\bullet(\underline{M},N)) = \mathrm{Hom}_{\mathscr{K}(\mathscr{A})}(\underline{M}[n],N) = 0$. But $\mathrm{Hom}^\bullet(\underline{M},N)$ is nothing but the complex

$$\cdots \to \mathrm{Hom}_{\mathscr{A}}(M,N_{n+1}) \to \mathrm{Hom}_{\mathscr{A}}(M,N_n) \to \mathrm{Hom}_{\mathscr{A}}(M,N_{n-1}) \to \cdots$$

Thus, N is $\mathrm{Hom}_{\mathscr{A}}(M,-)$-exact

3. $\Rightarrow$ 2. Let $n \in \mathbb{Z}$ and $f : \underline{M}[n] \to N$ be a morphism of complexes. Since $d^N_n f_n = 0$ we get that $f_n \in \mathrm{Ker}(\mathrm{Hom}_{\mathscr{A}}(M,d^N_n)) = \mathrm{Im}\ (\mathrm{Hom}_{\mathscr{A}}(M,d^N_{n+1}))$, so there exists a morphism $g : M \to N_{n+1}$ such that $d^N_{n+1} g = f_n$. Thus, f is null-homotopic and $\mathrm{Hom}_{\mathscr{K}(\mathscr{A})}(\underline{M}[n],N) = 0$ for every $n \in \mathbb{Z}$. Proposition 3.1.15 says then that $N \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(\mathscr{A})}(\underline{M}[n])$ for every $n \in \mathbb{Z}$. $\square$

From now on we will assume in this section that $\mathscr{A}$ has a projective generator P.

If we let $M = P$ in Proposition 3.1.17, then the condition "N is $\mathrm{Hom}_{\mathscr{A}}(P,-)$-exact" means that N is exact (since P preserves and reflects exactness by it's definition). This leads to the following characterization of exact complexes in terms of subprojectivity.

Corollary 3.1.18. *Let P be a projective generator of $\mathscr{A}$ and N be a complex. The following assertions are equivalent.*

1. *N is exact.*

2. *$N \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(\mathscr{A})}(\oplus_{n\in\mathbb{Z}}\underline{P}[n])$.*

3. *$N \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(\mathscr{A})}(\underline{P}[n])$ for every $n \in \mathbb{Z}$.*

There is now a natural question which comes to mind after Corollary 3.1.18: we have described, for the projective generator P, how the subprojectivity domain of the set of complexes $\{\underline{P}[n], n \in \mathbb{Z}\}$ is, so, what about the subprojectivity domain of each of the complexes $\underline{P}[n]$? Can we describe them as well?

Given a complex N, we know, by Theorem 3.1.12, that $N \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(\mathscr{A})}(\underline{P}[n])$ if and only if $\mathrm{Hom}_{\mathscr{K}(\mathscr{A})}(\underline{P}[n],N) = 0$. But,

$$\mathrm{Hom}_{\mathscr{K}(\mathscr{A})}(\underline{P}[n],N) = H_n(\mathrm{Hom}^\bullet(\underline{P},N)).$$

So, the condition $\mathrm{Hom}_{\mathscr{K}(\mathscr{A})}(\underline{P}[n],N) = 0$ is equivalent to $H_n(N) = 0$ since P is a projective generator of $\mathscr{A}$. We state this fact in the following proposition.

Proposition 3.1.19. *Let P be a projective generator of $\mathscr{A}$, N be a complex, and $n \in \mathbb{Z}$. The following assertions are equivalent.*

1. $N \in \underline{\mathfrak{Pr}}_{\mathscr{C}(\mathscr{A})}^{-1}(\underline{P}[n])$.
2. $\mathrm{Hom}_{\mathscr{K}(\mathscr{A})}(\underline{P}[n], N) = 0$.
3. $H_n(N) = 0$.

Now, with Proposition 3.1.19 in hand, it is easy to see that subprojectivity domains are not closed under kernels of epimorphisms in general.

Example 3.1.20. *Consider the short exact sequence of complexes*

$$0 \to \underline{P} \to \overline{P} \to \underline{P}[1] \to 0,$$

where P is the projective generator of $\mathscr{A}$. It is clear by Proposition 3.1.19 that $\underline{P}[1]$ and $\overline{P}$ both hold in $\underline{\mathfrak{Pr}}_{\mathscr{C}(\mathscr{A})}^{-1}(\underline{P})$, but $\underline{P}$ does not. Therefore, the subprojectivity domain of $\underline{P}$ is not closed under kernels of epimorphisms.

Moreover, Proposition 3.1.19 helps us to answer a question raised in Remark 3.1.5. Precisely, it is understood by the equivalence $(1 \Leftrightarrow 4)$ in Theorem 3.1.4 that the second assertion remains equivalent to the first assertion even if we replace the condition "Q is projective" with $Q \in \underline{\mathfrak{Pr}}_{\mathscr{C}(\mathscr{A})}^{-1}(M)$. However, this fact does not hold true. Namely, the following example shows that if we replace "Q is projective" with $Q \in \underline{\mathfrak{Pr}}_{\mathscr{C}(\mathscr{A})}^{-1}(M)$ in assertion 2, the equivalent does not hold.

Example 3.1.21. *Let $0 \to N_3 \to N_2 \to N_1 \to 0$ be a short exact sequence in $\mathscr{A}$ such that $N_3 \neq 0$ and let $X_i := \overline{N_i} \oplus \underline{N_i}[-1]$ for $i \in \{1,2,3\}$. Then, we have an induced exact sequence of complexes $0 \to X_3 \to X_2 \to X_1 \to 0$.*

Moreover, we see that for $i \in \{1,2,3\}$ it holds that $H_0(X_i) = H_0(\overline{N_i}) \oplus H_0(\underline{N_i}[-1]) = 0$ and that $H_{-1}(X_i) = H_{-1}(\overline{N_i}) \oplus H_{-1}(\underline{N_i}[-1]) = H_{-1}(\underline{N_i}[-1]) = N_i$. Thus, we can assert that $N_1, N_2 \in \underline{\mathfrak{Pr}}_{\mathscr{C}(\mathscr{A})}^{-1}(\underline{P})$ and that $\mathrm{Hom}_{\mathscr{K}(\mathscr{A})}(\underline{P}[-1], X_3) \neq 0$ where P is the projective generator of $\mathscr{A}$ (see Proposition 3.1.19).

3.2 Characterizing complexes through their cycles

The purpose of this section is to study the relationship between the subprojectivity of complexes and the subprojectivity of their cycles. Namely, for two complexes M and N, we investigate whether $N \in \underline{\mathfrak{Pr}}_{\mathscr{C}(\mathscr{A})}^{-1}(M)$ under the condition "$Z_n(N) \in \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(M_n)$ for every $n \in \mathbb{Z}$" (in the case of Theorem 3.2.2) and under the condition "$N_n \in \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(Z_{n-1}(M))$ for every $n \in \mathbb{Z}$"(in the case of Theorem 3.2.7).

We start with the following result which will be useful later.

Lemma 3.2.1. *Let $\mathscr{L}$ be a class of objects in $\mathscr{A}$, M a bounded below complex and N an exact complex which is* $\operatorname{Hom}_{\mathscr{A}}(\mathscr{L},-)$*-exact. If every morphism $M_n \to Z_n(N)$ factors through an object in $\mathscr{L}$ then every morphism of complexes $M \to N$ is null-homotopic by a morphism s such that every $s_n : M_n \to N_{n+1}$ factors through an object in $\mathscr{L}$.*

Proof. Let $f : M \to N$ be a morphism of complexes. We are going to construct a family of morphisms $s_n : M_n \to N_{n+1}$, such that $f_n = d^N_{n+1}s_n + s_{n-1}d^M_n$. We suppose that $M_n = 0$ for every $n < 0$, then $d^N_0 f_0 = 0$, so there exists a morphism $t_0 : M_0 \to Z_0(N)$ such that $\mu^N_1 t_0 = f_0$. By assumption, there exist two morphisms $\beta_0 : M_0 \to L_0$ and $\alpha_0 : L_0 \to Z_0(N)$ with $L_0 \in \mathscr{L}$ and $t_0 = \alpha_0\beta_0$. Since N is $\operatorname{Hom}_{\mathscr{A}}(\mathscr{L},-)$-exact, there exists a morphism $\gamma_0 : L_0 \to N_1$ such that $\varepsilon^N_1\gamma_0 = \alpha_0$.

$$\begin{tikzcd}
\cdots \arrow[r, "d_2^M"] & M_1 \arrow[rr, "d_1^M"] \arrow[dd, "f_1"] & & M_0 \arrow[r] \arrow[dl, "\beta_0"'] \arrow[dd, "f_0"] \arrow[ddl, "t_0"] & 0 \arrow[r] \arrow[dd] & \cdots \\
& & L_0 \arrow[dl, "\gamma_0"'] \arrow[dd, "\alpha_0"] & & & \\
\cdots \arrow[r, "d_1^N"] & N_1 \arrow[rr, "d_1^N"] \arrow[dr, "\varepsilon_1^N"'] & & N_0 \arrow[r, "d_0^N"] & N_{-1} \arrow[r] & \cdots \\
& & Z_0(N) \arrow[ur, "\mu_1^N"'] \arrow[dr] & & & \\
& 0 \arrow[ur] & & 0 & &
\end{tikzcd}$$

Let $s_0 = \gamma_0\beta_0$. One can check that $d^N_1 s_0 = f_0$, hence $d^N_1 s_0 d^M_1 = f_0 d^M_1 = d^N_1 f_1$. Thus, there exists a morphism $t_1 : M_1 \to Z_1(N)$ such that $\mu^N_2 t_1 = f_1 - s_0 d^M_1$. By assumption, there exist two morphisms $\beta_1 : M_1 \to L_1$ and $\alpha_1 : L_1 \to Z_1(N)$ with $L_1 \in \mathscr{L}$ and $t_1 = \alpha_1\beta_1$. Since N is $\operatorname{Hom}_{\mathscr{A}}(\mathscr{L},-)$-exact, there exists a morphism $\gamma_1 : L_1 \to N_2$ such that $\varepsilon^N_2\gamma_1 = \alpha_1$. Let $s_1 = \gamma_1\beta_1$, then $d^N_2 s_1 = \mu^N_2\varepsilon^N_2\gamma_1\beta_1 = \mu^N_2\alpha_1\beta_1 = \mu^N_2 t_1 = f_1 - s_0 d^M_1$. Using the same arguments we construct $s_n : M_n \to N_{n+1}$, such that $f_n = d^N_{n+1}s_n + s_{n-1}d^M_n$, for any $n \geqslant 0$. For $n < 0$, we take $s_n = 0$. $\square$

Now, we are in position to prove the first main result of this section.

Theorem 3.2.2. *Let N be an exact complex and M a bounded below complex. Then, if every $Z_n(N) \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{A}}(M_n)$, then $N \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(\mathscr{A})}(M)$.*

Proof. Suppose that $Z_n(N) \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{A}}(M_n)$ for every $n \in \mathbb{Z}$. Then, every morphism $M_n \to Z_n(N)$ factors through a projective object. Thus, by Lemma 3.2.1, every morphism $f : M \to N$ is null-homotopic by a morphism s such that each s_n factors through a projective object. Then, every $f : M \to N$ factors through a projective complex, by Lemma 3.1.11. Thus, $N \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(\mathscr{A})}(M)$. $\square$

Example 3.2.3. *The subprojectivity domain of a finitely presented objects in $\mathscr{A}$ contains the class of all flat objects (see Proposition 2.1.19). By Theorem 3.2.2, we get that the subprojectivity domain of a bounded below complex of finitely presented objects contains the class of all flat complexes (of course if $\mathscr{A}$ is Grothendieck).*

The following example shows that Theorem 3.2.2 fails without assuming the condition "M is bounded below".

The example will be given in a category which has enough injectives, has objects which are not projectives, and every injective object is projective. An example of such a category is the category of modules over a quasi-Fröbenius ring which is not semisimple (for instance, $\mathbb{Z}/4\mathbb{Z}$ is a quasi-Fröbenius ring which is not semisimple, see [33] for more details and examples about quasi-Fröbenius rings).

Example 3.2.4. *Suppose that $\mathscr{A}$ has enough injectives, has objects which are not projectives, and every injective object is projective. Then, there exist a non bounded complex P and an exact complex E such that E does not hold in $\underline{\mathfrak{Pr}}_{\mathscr{C}(\mathscr{A})}^{-1}(P)$ and $Z_n(E) \in \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(P_n)$ for every $n \in \mathbb{Z}$.*

Proof. Let M be a non projective object, N an object such that N does not hold in $\underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(M)$, E an exact complex with $E_1 = N$, $E_n = 0$ for every $n > 1$ and E_n is injective for every $n < 1$ (we can construct such a complex since $\mathscr{A}$ has enough injectives) and P an other exact complex with projective components and $Z_0(P) = M$ (we can construct the componenets P_n for $n > 0$ since $\mathscr{A}$ has enough projectives, and the components P_n for $n \leqslant 0$ can be constructed since $\mathscr{A}$ has enough injectives and every injective object is projective). Since the components of P are projectives, it is clear that for every $n \in \mathbb{Z}$, $Z_n(E) \in \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(P_n)$. Now, suppose that $E \in \underline{\mathfrak{Pr}}_{\mathscr{C}(\mathscr{A})}^{-1}(P)$ and let $f : M \to N$ be a morphism in $\mathscr{A}$. We construct a morphism of complexes $g : P \to E$ as follows

$$\begin{tikzcd}
\cdots \arrow[r] & P_2 \arrow[r, "d_2^P"] \arrow[dd, "g_2=0"'] & P_1 \arrow[rr, "d_1^P"] \arrow[dr, "\varepsilon_1^P"] \arrow[dd, "g_1=f\varepsilon_1^P"'] & & P_0 \arrow[rr, "d_0^P"] \arrow[dr, "\varepsilon_0^P"] \arrow[dd, "g_0"'] & & P_{-1} \arrow[r] \arrow[dd, "g_{-1}"] & \cdots \\
& & & M \arrow[ur, "\mu_1^P"'] \arrow[dl, "f"] & & Z_1(P) \arrow[ur, "\mu_0^P"'] \arrow[dd, "z_1"] & & \\
\cdots \arrow[r] & 0 \arrow[r] & N \arrow[rr, "d_1^E"'] & & E_0 \arrow[rr, "d_0^E"] \arrow[dr, "\varepsilon_0^E"'] & & E_{-1} \arrow[r] & \cdots \\
& & & & & Z_1(E) \arrow[ur, "\mu_0^E"'] & &
\end{tikzcd}$$

where $g_0 : P_0 \to E_0$ exists such that $g_0\mu_1^P = d_1^E f$ since E_0 is injective (one can verifies that $g_0 d_1^P = d_1^E g_1$), $z_1 : Z_1(P) \to Z_1(E)$ exists such that $z_1\varepsilon_0^P = \varepsilon_0^E g_0$ by the universal property of cokernels, $g_{-1} : P_{-1} \to E_{-1}$ exists such that $g_{-1}\mu_0^P = \mu_0^E z_1$ since E_{-1}

is injective. It is clear that $g_{-1}d_0^P = d_0^E g_0$. Using the same arguments we construct $g_n : P_n \to E_n$ such that $g_n d_{n+1}^P = d_{n+1}^E g_{n+1}$ for every $n < 0$. For every $n > 1$ we take $g_n = 0$. Now, the morphism of complexes $g : P \to E$ factors through a projective complex Q since we supposed that $E \in \underline{\mathfrak{Pr}}_{\mathscr{C}(\mathscr{A})}^{-1}(P)$. Let $\beta : P \to Q$ and $\alpha : Q \to E$ be two morphisms of complexes such that $f = \alpha\beta$. Consider the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "\mu_1^P"] \arrow[d, "\gamma"'] & P_0 \arrow[r, "d_0^P"] \arrow[d, "\beta_0"'] & P_{-1} \arrow[r] \arrow[d, "\beta_{-1}"'] & \cdots \\
0 \arrow[r] & Z_0(Q) \arrow[r, "\mu_1^Q"] \arrow[d, "\delta"'] & Q_0 \arrow[r, "d_0^Q"] \arrow[d, "\alpha_0"'] & Q_{-1} \arrow[r] \arrow[d, "\alpha_{-1}"'] & \cdots \\
0 \arrow[r] & N \arrow[r, "d_1^E"] & E_0 \arrow[r, "d_0^E"] & E_{-1} \arrow[r] & \cdots
\end{tikzcd}$$

The morphisms $\gamma : M \to Z_0(Q)$ and $\delta : Z_0(Q) \to N$ exist and make the diagram commute, by the universal property of kernels. We claim that $f = \delta\gamma$. Indeed, $d_1^E f \varepsilon_1^P = d_1^E g_1 = g_0 d_1^P = \alpha_0\beta_0\mu_1^P\varepsilon_1^P = \alpha_0\mu_1^Q\gamma\varepsilon_1^P = d_1^E\delta\gamma\varepsilon_1^P$, then $f = \delta\gamma$. Thus, any morphism $f : M \to N$ factors through a projective object in $\mathscr{A}$. Then, $N \in \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(M)$ which is not the case. Therefore, E does not hold in $\underline{\mathfrak{Pr}}_{\mathscr{C}(\mathscr{A})}^{-1}(P)$ even if $Z_n(E) \in \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(P_n)$ for every $n \in \mathbb{Z}$. $\square$

The following example in the category of modules shows that the reverse implication of Theorem 3.2.2 does not hold true in general.

Recall that a ring R is said to be a left IF-ring if every injective R-module is flat. Obviously, every von Neumann regular ring is a left IF-ring but the converse does not hold. In fact, an example of a quasi-Fröbenius ring which is not semisimple can be seen as an example of a left IF-ring which is not a von Neauman regular ring since quasi-Fröbenius rings are If and Noetherian, and von Neauman regular rings which are Noetherian are semisimple (see for instance [**?**, **?**, **?**] for more details and examples about IF-rings).

Example 3.2.5. *Let R be an IF ring which is not von Neumann regular, then there exists a finitely presented module which is not projective, so there exists a module K which does not belong to the subprojectivity domain of M. Consider an exact complex $N : \cdots \to 0 \to K \to E \to C \to 0 \to \cdots$ such that K is in position 2 and E is an injective module. Since M is finitely presented (then every flat module holds in its subprojectivity domain) and R is an IF-ring, $E \in \underline{\mathfrak{Pr}}_{R\text{-Mod}}^{-1}(M)$. Then, $N \in \underline{\mathfrak{Pr}}_{\mathscr{C}(R)}^{-1}(\overline{M})$ (as we will see in Lemma 3.2.8). However, $Z_1(N) = K$ does not belong to $\underline{\mathfrak{Pr}}_{R\text{-Mod}}^{-1}(M)$.*

Now, we turn our attention to the second aim of this section which is to investigate, for two complexes M and N, whether $N \in \underline{\mathfrak{Pr}}_{\mathscr{C}(\mathscr{A})}^{-1}(M)$ if $N_n \in \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(Z_{n-1}(M))$ for every $n \in \mathbb{Z}$. We will see in Theorem 3.2.7 that this holds true for M exact and N bounded above only if every projective object is injective. But first, we prove the following lemma which will be useful in the proof of Theorem 3.2.7.

Lemma 3.2.6. *Let $\mathscr{L}$ be a class of objects in $\mathscr{A}$, M be an exact complex which is $\operatorname{Hom}_{\mathscr{A}}(-,\mathscr{L})$-exact and N a bounded above complex. If every morphism $Z_{n-1}(M) \to N_n$ factors through an object in $\mathscr{L}$, then every morphism of complexes $M \to N$ is null-homotopic by a morphism s such that every s_n factors through an object in $\mathscr{L}$.*

Proof. Let $f : M \to N$ be a morphism of complexes. We are going to construct a family of morphisms $s_n : M_{n-1} \to N_n$, such that $f_n = d^N_{n+1}s_{n+1} + s_n d^M_n$. We suppose that $N_n = 0$ for every $n > 0$. Then $f_0 d^M_1 = 0$, so there exists a morphism $t_0 : Z_{-1}(M) \to N_0$ such that $t_0 \varepsilon^M_0 = f_0$. Then, there exist two morphisms $\beta_0 : Z_{-1}(M) \to L_0$ and $\alpha_0 : L_0 \to N_0$ with $L_0 \in \mathscr{L}$ and $t_0 = \alpha_0\beta_0$. Since M is $\operatorname{Hom}_{\mathscr{A}}(-,\mathscr{L})$-exact, there exists a morphism $\gamma_0 : M_{-1} \to L_0$ such that $\gamma_0 \mu^M_0 = \beta_0$.

$$\begin{tikzcd}
\cdots \arrow[r] & M_0 \arrow[rrr, "d^M_0"] \arrow[dr, "\varepsilon^M_0"] \arrow[dddr, "f_0"'] & & & M_{-1} \arrow[r, "d^M_{-1}"] \arrow[ddl, "\gamma_0"'] \arrow[ddd, "f_{-1}"] & \cdots \\
& & Z_{-1}(M) \arrow[urr, "\mu^M_0"] \arrow[dd, "t_0"'] \arrow[dr, "\beta_0"] & & & \\
& & & L_0 \arrow[dl, "\alpha_0"] & & \\
0 \arrow[rr] & & N_0 \arrow[rr, "d^N_0"'] & & N_{-1} \arrow[r, "d^M_{-1}"'] & \cdots
\end{tikzcd}$$

Let $s_0 = \alpha_0\gamma_0$. One can check that $s_0 d^M_0 = f_0$, hence $f_{-1}d^M_0 = d^N_0 f_0 = d^N_0 s_0 d^M_0$, so there exists a morphism $t_{-1} : Z_{-2}(M) \to N_{-1}$ such that $t_{-1}\varepsilon^M_{-1} = f_{-1} - d^N_0 s_0$. Then, there exist two morphisms $\beta_{-1} : Z_{-2}(M) \to L_{-1}$ and $\alpha_{-1} : L_{-1} \to N_{-1}$ with $L_{-1} \in \mathscr{L}$ and $t_{-1} = \alpha_{-1}\beta_{-1}$. Since M is $\operatorname{Hom}_{\mathscr{A}}(-,\mathscr{L})$-exact there exists a morphism $\gamma_{-1} : M_{-2} \to L_{-1}$ such that $\gamma_{-1}\mu^M_{-1} = \beta_{-1}$. Let $s_{-1} = \alpha_{-1}\gamma_{-1}$. Then, $f_{-1} - d^N_0 s_0 = t_{-1}\varepsilon^M_{-1} = \alpha_{-1}\beta_{-1}\varepsilon^M_{-1} = \alpha_{-1}\gamma_{-1}\mu^M_{-1}\varepsilon^M_{-1} = \alpha_{-1}\gamma_{-1}d^M_{-1} = s_1 d^M_{-1}$, hence $f_{-1} = d^N_0 s_0 + s_{-1}d^M_{-1}$. Using the same arguments we construct, and for any $n \leqslant 0$, $s_n : M_{n-1} \to N_n$, such that $f_n = d^N_{n+1}s_{n+1} + s_n d^M_n$, for $n > 0$, we take $s_n = 0$. Therefore, $f : M \to N$ is null homotopic by the morphism s such that every s_n factors through an object L_n in $\mathscr{L}$. $\square$

Theorem 3.2.7. *Suppose that $\mathscr{A}$ has enough injectives. Then, the following conditions are equivalent.*

1. *For every exact complex M and every bounded above complex N, if for every $n \in \mathbb{Z}$, $N_n \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{A}}(Z_{n-1}(M))$, then $N \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(\mathscr{A})}(M)$.*

2. *For every exact complex M and every module N, if there exists $n \in \mathbb{Z}$ such that $N \in \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(Z_{n-1}(M))$, then $\underline{N}[n] \in \underline{\mathfrak{Pr}}_{\mathscr{C}(\mathscr{A})}^{-1}(M)$.*

3. *Every projective object is injective.*

Proof. 1. $\Rightarrow$ 2. is clear.
2. $\Rightarrow$ 3. Let P be a projective object and $i : P \to E$ be a monomorphism with E is injective. Let us prove that $\mathrm{Hom}_{\mathscr{A}}(i,M) : \mathrm{Hom}_{\mathscr{A}}(E,M) \to \mathrm{Hom}_{\mathscr{A}}(P,M)$ is epic for every object M. For let $f : P \to M$ be a morphism and consider the exact complex $X : \cdots \to 0 \to P \to E \to C \to 0 \to \cdots$ with P is in the 0-position. Then, $\underline{M}$ holds in the subprojectivity domain of X by assumption. Thus, there exist two morphisms of complexes $\beta : X \to Q$ and $\alpha : Q \to \underline{M}$ such that Q is projective and $\alpha\beta = \phi$ where $\phi_0 = f$ and $\phi_i = 0$ otherwise. We have $\alpha_0 d_1^Q = 0$, hence there exists a morphism $h : Z_{-1}(Q) \to M$ such that $h\varepsilon_0^Q = \alpha_0$. Since Q is projective, the morphism $\mu_0^Q : Z_{-1}(Q) \to Q_{-1}$ splits, that is, there exists a morphism $\nu_0^Q : Q_{-1} \to Z_{-1}(Q)$ such that $\nu_0^Q\mu_0^Q = \mathrm{id}$. Then,

$$h\nu_0^Q\beta_{-1}i = h\nu_0^Q d_0^Q\beta_0 = h\nu_0^Q\mu_0^Q\varepsilon_0^Q\beta_0 = h\varepsilon_0^Q\beta_0 = \alpha_0\beta_0 = \phi_0 = f.$$

Thus, $\mathrm{Hom}_{\mathscr{A}}(i,M) : \mathrm{Hom}_{\mathscr{A}}(E,M) \to \mathrm{Hom}_{\mathscr{A}}(P,M)$ is epic for every object M. Therefore, P is injective.
3. $\Rightarrow$ 1. Let M be an exact complex and N a bounded above complex such that, for every $n \in \mathbb{Z}$, $N_n \in \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(Z_{n-1}(M))$. Hence, every morphism $Z_{n-1}(M) \to N_n$ factors through a projective object. Then, by Lemma 3.2.6, every morphism of complexes $M \to N$ is null-homotopic by a morphism s such that every $s_n : M_n \to N_{n+1}$ factors through a projective object (M is $\mathrm{Hom}_{\mathscr{A}}(-,\mathscr{P}_{\mathscr{A}})$-exact since every projective object is injective). Then, by Lemma 3.1.11, every morphism of complexes $M \to N$ factors through a projective complex. Therefore, $N \in \underline{\mathfrak{Pr}}_{\mathscr{C}(\mathscr{A})}^{-1}(M)$. □

Another natural question at this point is whether the inverse implication of the first assertion of Theorem 3.2.7 is true or not. Namely, given two complexes M and N, is the condition " $N \in \underline{\mathfrak{Pr}}_{\mathscr{C}(\mathscr{A})}^{-1}(M)$" sufficient to assure that $N_n \in \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(Z_{n-1}(M))$ for every $n \in \mathbb{Z}$? Again, this is not true in general since, for instance, if X and Y are two objects with $Y \notin \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(X)$, we have $\underline{Y} \in \underline{\mathfrak{Pr}}_{\mathscr{C}(\mathscr{A})}^{-1}(\underline{X}[-1])$ since every morphism $\underline{X}[-1] \to \underline{Y}$ is zero while $\underline{Y}_0 \in \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(Z_{-1}(\underline{X}[-1]))$. Nevertheless, the answer to the question would be positive if we assume that M is contractible. Namely, we will see in Proposition 3.2.9 that if M is contractible, then $N \in \underline{\mathfrak{Pr}}_{\mathscr{C}(\mathscr{A})}^{-1}(M)$ if and only if $N_n \in \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(Z_{n-1}(M))$ for every $n \in \mathbb{Z}$. The following result treats a particular case which will be useful.

Lemma 3.2.8. *Let N be a complex and M an object of $\mathscr{A}$. Then, for every $n \in \mathbb{Z}$, $N \in \underline{\mathfrak{Pr}}_{\mathscr{C}(\mathscr{A})}^{-1}(\overline{M}[n])$ if and only if $N_{n+1} \in \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(M)$.*

Proof. Suppose that $N \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(\mathscr{A})}(\overline{M}[n])$ and let $f : M \to N_{n+1}$ be a morphism in $\mathscr{A}$. Then, by assumption, there exists a morphism of complexes $\overline{f} : \overline{M}[n] \to N$ such that $\overline{f}_{n+1} = f$. $\overline{f} : \overline{M}[n] \to N$ factors through a projective complex P. Hence, $f : M \to N_{n+1}$ factors through the projective object P_{n+1}.
Conversely, Suppose that $N_{n+1} \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{A}}(M)$ and let $f : \overline{M}[n] \to N$ be a morphism of complexes. Then, by assumption, there exists two morphisms $\alpha : P \to N_{n+1}$ and $\beta : M \to P$ such that $f_{n+1} = \alpha\beta$. We define two morphisms of complexes $g : \overline{P}[n] \to N$ and $h : \overline{M}[n] \to \overline{P}[n]$ such that $g_{n+1} = \alpha$, $g_n = d^N_{n+1}\alpha$, $h_{n+1} = h_n = \beta$, and $g_m = h_m = 0$ otherwise. It is clear that $f = gh$. Therefore, $N \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(\mathscr{A})}(\overline{M}[n])$. □

Proposition 3.2.9. *Let N be a complex and $\{M_n\}_{n\in\mathbb{Z}}$ be a family of objects in $\mathscr{A}$. Then, we get that $N \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(\mathscr{A})}(\oplus_{n\in\mathbb{Z}}\overline{M_n}[n])$ if and only if $N_{n+1} \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{A}}(M_n)$ for every $n \in \mathbb{Z}$.*

Proof. We have $\underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(\mathscr{A})}(\oplus_{n\in\mathbb{Z}}\overline{M_n}[n]) = \underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(\mathscr{A})}(\{\overline{M_n}[n]\ n \in \mathbb{Z}\})$, by Proposition 2.1.14, and we conclude by Lemma 3.2.8. □

3.3 Application to some well-behaved classes of complexes

In this section, we go further in the study of subprojectivity in the category of complexes and we will prove that the subprojectivity in the category of complexes provides also a new interesting unified framework of the classical projectivity and flatness from which arise the following natural questions that we stated in the introduction.

1. When do we have $\underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(\mathscr{A})}(\widetilde{\mathscr{L}}) = \#\mathscr{G}$?

2. When do we have $\underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(\mathscr{A})}(\#\mathscr{L}) = \widetilde{\mathscr{G}}$?

3. When do we have $\underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(\mathscr{A})}(\mathscr{C}^b(\mathscr{L})) = \widetilde{\mathscr{G}}$?

The first main result, which answers the first question, is given as follows.

Theorem 3.3.1. *Let $\mathscr{L}$ and $\mathscr{G}$ be two classes of objects in $\mathscr{A}$ which contain the zero object and such that $\mathscr{L}$ is closed under extensions. Then, the following conditions are equivalent.*

1. *$\underline{\mathfrak{Pr}}^{-1}_{\mathscr{A}}(\mathscr{L}) = \mathscr{G}$ and $\mathrm{Hom}_{\mathscr{K}(\mathscr{A})}(M,N) = 0$ for any $N \in \#\mathscr{G}$ and $M \in \widetilde{\mathscr{L}}$.*

2. *$\underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(\mathscr{A})}(\widetilde{\mathscr{L}}) = \#\mathscr{G}$.*

Proof. 1. $\Rightarrow$ 2. Let $N \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(\mathscr{A})}(\widetilde{\mathscr{L}})$ and $M \in \mathscr{L}$. Then, $\oplus_{n\in\mathbb{Z}}\overline{M}[n] \in \widetilde{\mathscr{L}}$. Hence, $N \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(\mathscr{A})}(\oplus_{n\in\mathbb{Z}}\overline{M}[n])$. Then, by Proposition 3.1.16, for every $n \in \mathbb{Z}$, $N_n \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{A}}(M)$. Thus, for every $n \in \mathbb{Z}$, $N_n \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{A}}(\mathscr{L}) = \mathscr{G}$. Then, $\underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(\mathscr{A})}(\widetilde{\mathscr{L}}) \subseteq \#\mathscr{G}$. Conversely, let $N \in \#\mathscr{G}$ and $M \in \widetilde{\mathscr{L}}$. Then, for every $n \in \mathbb{Z}$, $N_n \in \mathscr{G}$ and $M_n \in \mathscr{L}$ since $\mathscr{L}$ is closed under extensions. Then, for every $n \in \mathbb{Z}$, $N_{n+1} \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{A}}(M_n)$. Then, $N \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(\mathscr{A})}(M)$, by Theorem 3.1.12, since $\mathrm{Hom}_{\mathscr{K}(\mathscr{A})}(M,N) = 0$. Therefore, $N \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(\mathscr{A})}(\widetilde{\mathscr{L}})$.

2. $\Rightarrow$ 1. Let $N \in \#\mathscr{G}$ and $M \in \widetilde{\mathscr{L}}$, then $N \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(\mathscr{A})}(M)$. Thus, by Lemma 3.1.2, $\mathrm{Hom}_{\mathscr{K}(\mathscr{A})}(M,N) = 0$. Now, let us prove that $\underline{\mathfrak{Pr}}^{-1}_{\mathscr{A}}(\mathscr{L}) = \mathscr{G}$. For let $N \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{A}}(\mathscr{L})$ and $M \in \widetilde{\mathscr{L}}$, then $N \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{A}}(M_0)$ since $M_0 \in \mathscr{L}$ ($\mathscr{L}$ is closed under extensions). Then, $\overline{N} \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(\mathscr{A})}(M)$ by Lemma 3.1.8. Then, $\overline{N} \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(\mathscr{A})}(\widetilde{\mathscr{L}}) = \#\mathscr{G}$. Thus, $N \in \mathscr{G}$. Therefore, $\underline{\mathfrak{Pr}}^{-1}_{\mathscr{A}}(\mathscr{L}) \subseteq \mathscr{G}$. Conversely, let $N \in \mathscr{G}$ and $M \in \mathscr{L}$, hence $\overline{N} \in \#\mathscr{G}$ and $\overline{M} \in \widetilde{\mathscr{L}}$. Then, $\overline{N} \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(\mathscr{A})}(\overline{M})$ by assumption. Then, by Lemma 3.1.8, $N \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{A}}(M)$, then $N \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{A}}(\mathscr{L})$. Therefore, $\underline{\mathfrak{Pr}}^{-1}_{\mathscr{A}}(\mathscr{L}) = \mathscr{G}$. □

Now, the second main result of this section is as follows :

Theorem 3.3.2. *Suppose that $\mathscr{A}$ has a projective generator P and let $\mathscr{L}$ and $\mathscr{G}$ be two classes of objects in $\mathscr{A}$ such that $0, P \in \mathscr{L}$. Then, the following conditions are equivalent.*

1. $\underline{\mathfrak{Pr}}^{-1}_{\mathscr{A}}(\mathscr{L}) = \mathscr{G}$ *and* $\mathrm{Hom}_{\mathscr{K}(\mathscr{A})}(M,N) = 0$ *for any* $M \in \#\mathscr{L}$ *and* $N \in \widetilde{\mathscr{G}}$.

2. $\underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(\mathscr{A})}(\#\mathscr{L}) = \widetilde{\mathscr{G}}$.

Proof. 1. $\Rightarrow$ 2. Let $N \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(\mathscr{A})}(\#\mathscr{L})$, then $N \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(\mathscr{A})}(\underline{P}[n])$ for every $n \in \mathbb{Z}$, so N is exact by Corollary 3.1.18. Now, let $L \in \mathscr{L}$, then $N \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(\mathscr{A})}(\underline{L}[n])$ for every $n \in \mathbb{Z}$. Then, by Lemma 3.1.7, $Z_n(N) \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{A}}(L)$ for every $L \in \mathscr{L}$ and $n \in \mathbb{Z}$. Then, $Z_n(N) \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{A}}(\mathscr{L}) = \mathscr{G}$ for every $n \in \mathbb{Z}$. Thus, $N \in \widetilde{\mathscr{G}}$. Conversely, let $N \in \widetilde{\mathscr{G}}$ and $L \in \#\mathscr{L}$. For every $n \in \mathbb{Z}$, $Z_n(N) \in \mathscr{G} = \underline{\mathfrak{Pr}}^{-1}_{\mathscr{A}}(\mathscr{L})$, hence $N_n \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{A}}(\mathscr{L})$ since $\underline{\mathfrak{Pr}}^{-1}_{\mathscr{A}}(\mathscr{L})$ is closed under extensions. Then, $N \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(\mathscr{A})}(L)$ by Theorem 3.1.12 since $\mathrm{Hom}_{\mathscr{K}(\mathscr{A})}(M,N) = 0$ by assumption. Thus, $N \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(\mathscr{A})}(\#\mathscr{L})$.

2. $\Rightarrow$ 1. It is clear that $\mathrm{Hom}_{\mathscr{K}(\mathscr{A})}(M,N) = 0$ for any $M \in \#\mathscr{L}$ and $N \in \widetilde{\mathscr{G}}$ by Lemma 3.1.2. To prove that $\underline{\mathfrak{Pr}}^{-1}_{\mathscr{A}}(\mathscr{L}) = \mathscr{G}$, let $N \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{A}}(\mathscr{L})$, then for every $M \in \#\mathscr{L}$, $N \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{A}}(M_0)$, hence $\overline{N} \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(\mathscr{A})}(M)$ (see Lemma 3.1.8). Therefore, $\overline{N} \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(\mathscr{A})}(\#\mathscr{L}) = \widetilde{\mathscr{G}}$, hence $N \in \mathscr{G}$. Conversely, let $N \in \mathscr{G}$ and $M \in \mathscr{L}$ then, $\overline{N} \in \widetilde{\mathscr{G}}$ and $\overline{M} \in \#\mathscr{L}$. Then,

$\overline{N} \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(\mathscr{A})}(\overline{M})$ by assumption. Thus, $N \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{A}}(M)$ by Lemma 3.1.8. Therefore, $N \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{A}}(\mathscr{L})$. $\square$

It was proven in [45] that the class $\mathscr{GP}^{\perp}_{\mathscr{C}(R)}$ is inside $\widetilde{\mathscr{GP}^{\perp}_{R\text{-Mod}}}$, that is, the class of exact complexes with cycles in $\mathscr{GP}^{\perp}_{R\text{-Mod}}$ (see [45, Proposition 3.4 and Remark 3.3]). Here, we show when these two classes coincide in $\mathscr{C}(\mathscr{A})$.

Proposition 3.3.3. *If direct sums exist and they are exact, then* $\mathscr{GP}^{\perp}_{\mathscr{C}(\mathscr{A})} = \widetilde{\mathscr{GP}^{\perp}_{\mathscr{A}}}$ *if and only if* $\operatorname{Hom}_{\mathscr{K}(\mathscr{A})}(M,N) = 0$ *for any Gorenstein projective complex* M *and any complex* $N \in \widetilde{\mathscr{GP}^{\perp}_{\mathscr{A}}}$.

Proof. We have $\underline{\mathfrak{Pr}}^{-1}_{\mathscr{A}}(\mathscr{GP}_{\mathscr{A}}) = \mathscr{GP}^{\perp}_{\mathscr{A}}$ by Corollary 2.1.17, then, by Theorem 3.3.2, $\underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(\mathscr{A})}(\#\mathscr{GP}_{\mathscr{A}}) = \widetilde{\mathscr{GP}^{\perp}_{\mathscr{A}}}$ if and only if $\operatorname{Hom}_{\mathscr{K}(R)}(M,N) = 0$ for any $M \in \#\mathscr{GP}_{\mathscr{A}}$ and $N \in \widetilde{\mathscr{GP}^{\perp}_{\mathscr{A}}}$. But $\#\mathscr{GP}_{\mathscr{A}} = \mathscr{GP}_{\mathscr{C}(\mathscr{A})}$, by Proposition 1.3.5. Then, we get that $\underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(\mathscr{A})}(\#\mathscr{GP}_{\mathscr{A}}) = \underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(\mathscr{A})}(\mathscr{GP}_{\mathscr{C}(\mathscr{A})}) = \mathscr{GP}^{\perp}_{\mathscr{C}(\mathscr{A})}$ (see Corollary 2.1.17). Therefore, $\mathscr{GP}^{\perp}_{\mathscr{C}(\mathscr{A})} = \widetilde{\mathscr{GP}^{\perp}_{\mathscr{A}}}$ if and only if $\operatorname{Hom}_{\mathscr{K}(\mathscr{A})}(M,N) = 0$ for any M Gorenstein projective complex and any $N \in \widetilde{\mathscr{GP}^{\perp}_{\mathscr{A}}}$. $\square$

Now, we turn our attention to the third question. In the study of this question, a new type of classes appears naturally which are defined as follows.

Definition 3.3.4. *Given* $\mathscr{L}$ *a class of objects of* $\mathscr{A}$, *a complex* X *is said to be a* $dg\mathscr{L}$ *complex, if* $X_n \in \mathscr{L}$, *for each* $n \in \mathbb{Z}$, *and* $\operatorname{Hom}^{\bullet}(X,G)$ *is exact whenever* G *is an exact complex with cycles in* $\underline{\mathfrak{Pr}}^{-1}_{\mathscr{A}}(\mathscr{L})$. *We denote the class of* $dg\mathscr{L}$ *complexes by* $dg\widetilde{\mathscr{L}}$.

The terminology used in this definition is inspired from $dg\mathscr{L}$ complexes introduced by Gillespie [27] based on the fact that, when $(\mathscr{L},\mathscr{G})$ is the classical cotorsion pair $(Proj_{\mathscr{A}},\mathscr{A})$, $dg\widetilde{\mathscr{L}}$ is nothing but the classical DG-projective complexes.

Lemma 3.3.5. *If* $\mathscr{L}$ *is a class which contains* 0, *then* $\mathscr{C}^{-}(\mathscr{L}) \subseteq dg\widetilde{\mathscr{L}}$.

Proof. Set $\mathscr{G} = \underline{\mathfrak{Pr}}^{-1}_{\mathscr{A}}(\mathscr{L})$ and let $X \in \mathscr{C}^{-}(\mathscr{L})$ and $f : X \to N$ be a morphism of complexes with $N \in \widetilde{\mathscr{G}}$. N is exact and $Z_n(N) \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{A}}(\mathscr{L})$, for each $n \in \mathbb{Z}$, hence N is $\operatorname{Hom}_{\mathscr{A}}(Proj_{\mathscr{A}},-)$-exact and every morphism $X_n \to Z_n(N)$ factors through a projective in $\mathscr{A}$. Therefore $f : X \to N$ is null-homotopic, by Lemma 3.2.1. Thus, $\operatorname{Hom}^{\bullet}(X,N)$ is exact whenever N is a $\widetilde{\mathscr{G}}$ complex. $\square$

Now, our third main result of this section is given as follows.

Theorem 3.3.6. *Suppose that $\mathscr{A}$ has a projective generator P and let $\mathscr{L}$ and $\mathscr{G}$ be two classes of modules such that $0, P \in \mathscr{L}$. Then, the following conditions are equivalent.*

1. $\underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(\mathscr{L}) = \mathscr{G}$.
2. $\underline{\mathfrak{Pr}}_{\mathscr{C}(\mathscr{A})}^{-1}(\mathscr{C}^b(\mathscr{L})) = \widetilde{\mathscr{G}}$.
3. $\underline{\mathfrak{Pr}}_{\mathscr{C}(\mathscr{A})}^{-1}(\mathscr{C}^-(\mathscr{L})) = \widetilde{\mathscr{G}}$.
4. $\underline{\mathfrak{Pr}}_{\mathscr{C}(\mathscr{A})}^{-1}(dg\widetilde{\mathscr{L}}) = \widetilde{\mathscr{G}}$.

Proof. 1. $\Rightarrow$ 4. Let us prove that $\underline{\mathfrak{Pr}}_{\mathscr{C}(\mathscr{A})}^{-1}(dg\widetilde{\mathscr{L}}) \subseteq \widetilde{\mathscr{G}}$. For let $N \in \underline{\mathfrak{Pr}}_{\mathscr{C}(\mathscr{A})}^{-1}(dg\widetilde{\mathscr{L}})$, then $N \in \underline{\mathfrak{Pr}}_{\mathscr{C}(\mathscr{A})}^{-1}(\underline{P}[n])$ for every $n \in \mathbb{Z}$, by Lemma 3.3.5, so N is exact by Corollary 3.1.18. Now, let $n \in \mathbb{Z}$ and $L \in \mathscr{L}$, then $N \in \underline{\mathfrak{Pr}}_{\mathscr{C}(\mathscr{A})}^{-1}(\underline{L}[n])$ for every $n \in \mathbb{Z}$, by Lemma 3.3.5. Then, by Lemma 3.1.7, $Z_n(N) \in \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(L)$ for every $L \in \mathscr{L}$. Then, $Z_n(N) \in \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(\mathscr{L}) = \mathscr{G}$ for every $n \in \mathbb{Z}$. Thus, $N \in \widetilde{\mathscr{G}}$. Now, let us prove that $\widetilde{\mathscr{G}} \subseteq \underline{\mathfrak{Pr}}_{\mathscr{C}(\mathscr{A})}^{-1}(dg\widetilde{\mathscr{L}})$. Let $N \in \widetilde{\mathscr{G}}$ and $M \in dg\widetilde{\mathscr{L}}$, then $\mathrm{Hom}_{\mathscr{K}(\mathscr{A})}(M,N) = 0$ since $\mathrm{Hom}^\bullet(X,N)$ is exact. For every $n \in \mathbb{Z}$, $Z_n(N) \in \mathscr{G} = \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(\mathscr{L})$, hence $N_n \in \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(\mathscr{L})$ since $\underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(\mathscr{L})$ is closed under extensions. Then, $N \in \underline{\mathfrak{Pr}}_{\mathscr{C}(\mathscr{A})}^{-1}(dg\widetilde{\mathscr{L}})$ by Theorem 3.1.12. Thus, $N \in \underline{\mathfrak{Pr}}_{\mathscr{C}(\mathscr{A})}^{-1}(dg\widetilde{\mathscr{L}})$.
4. $\Rightarrow$ 1. Let $N \in \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(\mathscr{L})$, then for every $M \in dg\widetilde{\mathscr{L}}$, $N \in \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(M_0)$ since $M_0 \in \mathscr{L}$. Then, for every $M \in dg\widetilde{\mathscr{L}}$, $\overline{N} \in \underline{\mathfrak{Pr}}_{\mathscr{C}(\mathscr{A})}^{-1}(M)$ (see Lemma 3.1.8). Therefore, $\overline{N} \in \underline{\mathfrak{Pr}}_{\mathscr{C}(\mathscr{A})}^{-1}(dg\widetilde{\mathscr{L}}) = \widetilde{\mathscr{G}}$, hence $N \in \mathscr{G}$. Conversely, let $N \in \mathscr{G}$ and $M \in \mathscr{L}$ then, $\overline{N} \in \widetilde{\mathscr{L}}$ and $\overline{M} \in dg\widetilde{\mathscr{L}}$ (see Lemma 3.3.5). Then, $\overline{N} \in \underline{\mathfrak{Pr}}_{\mathscr{C}(\mathscr{A})}^{-1}(\overline{M})$ by assumption. Then, $N \in \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(M)$ by Lemma 3.1.8. Then, $N \in \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(\mathscr{L})$.
(4. $\Leftrightarrow$ 1.) $\Rightarrow$ 2. It is clear that $\widetilde{\mathscr{G}} \subseteq \underline{\mathfrak{Pr}}_{\mathscr{C}(\mathscr{A})}^{-1}(\mathscr{C}^b(\mathscr{L}))$ since $\mathscr{C}^-(\mathscr{L}) \subseteq dg\widetilde{\mathscr{L}}$ by Lemma 3.3.5. Conversely, let $N \in \underline{\mathfrak{Pr}}_{\mathscr{C}(\mathscr{A})}^{-1}(\mathscr{C}^b(\mathscr{L}))$. Then, $N \in \underline{\mathfrak{Pr}}_{\mathscr{C}(\mathscr{A})}^{-1}(\underline{P}[n])$ for every $n \in \mathbb{Z}$, so N is exact by Corollary 3.1.18. Now, let $n \in \mathbb{Z}$ and $L \in \mathscr{L}$, then $N \in \underline{\mathfrak{Pr}}_{\mathscr{C}(\mathscr{A})}^{-1}(\underline{L}[n])$ for every $n \in \mathbb{Z}$. Then, by Lemma 3.1.7, $Z_n(N) \in \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(L)$ for every $L \in \mathscr{L}$. Then, $Z_n(N) \in \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(\mathscr{L}) = \mathscr{G}$ for every $n \in \mathbb{Z}$. Thus, $N \in \widetilde{\mathscr{G}}$.
For 2. $\Rightarrow$ 1. and 3. $\Rightarrow$ 1. we use the same arguments of 4. $\Rightarrow$ 1.
(2. $\Leftrightarrow$ 4.) $\Rightarrow$ 3. is clear since $\mathscr{C}^b(\mathscr{L}) \subseteq \mathscr{C}^-(\mathscr{L}) \subseteq dg\widetilde{\mathscr{L}}$ (by Lemma 3.3.5).

□

The following result is a direct consequence of Theorem 3.3.6 and it will be useful to unify various known results and characterize important classes of complexes.

Proposition 3.3.7. *Suppose that $\mathscr{A}$ has a projective generator P and let $\mathscr{L}$ and $\mathscr{G}$ be two classes in $\mathscr{A}$ such that $0, P \in \mathscr{L}$ and $\underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(\mathscr{L}) = \mathscr{G}$. Then, the following conditions are equivalent for any complex F.*

1. *$F \in \widetilde{\mathscr{G}}$.*
2. *Every morphism $X \to F$, with $X \in \mathscr{C}^b(\mathscr{L})$, factors through a projective complex.*
3. *Every morphism $X \to F$, with $X \in \mathscr{C}^-(\mathscr{L})$, factors through a projective complex.*
4. *Every morphism $X \to F$, with $X \in dg\widetilde{\mathscr{L}}$, factors through a projective complex.*

Proof. 1. $\Leftrightarrow$ 2. Applying Theorem 3.3.6 we get that $\underline{\mathfrak{Pr}}_{\mathscr{C}(\mathscr{A})}^{-1}(\mathscr{C}^b(\mathscr{L})) = \widetilde{\mathscr{G}}$. Then, we conclude by Proposition 2.1.4.
1. $\Leftrightarrow$ 3. and 1. $\Leftrightarrow$ 4. hold using the same arguments. □

In Section 2.1, we characterized the subprojectivity domains of several homologically interesting classes in $\mathscr{A}$. Now, we will apply Proposition 3.3.7 to get some interesting characterizations of objects in $\mathscr{C}(\mathscr{A})$.

We start with the following characterization of exact complexes.

Corollary 3.3.8. *If $\mathscr{A}$ has a projective generator, then the following conditions are equivalent for a complex F in $\mathscr{C}(\mathscr{A})$.*

1. *F is exact.*
2. *Every morphism $X \to F$, where X is a bounded complex of projectives, factors through a projective complex.*
3. *Every morphism $X \to F$, where X is a bounded below complex of projectives, factors through a projective complex.*
4. *Every morphism $X \to F$,where X is a DG-projective complex, factors through a projective complex.*

Proof. Following the notations of Proposition 3.3.7, let $\mathscr{L}$ be the class of projective objects and $\mathscr{G} = \mathscr{A}$. Then, $\widetilde{\mathscr{G}}$ is the class of exact complexes, and $dg\widetilde{\mathscr{L}}$ is the class of DG-projective complexes. □

Then, we give a characterization of projective complexes.

Corollary 3.3.9. *If $\mathscr{A}$ has a projective generator, then the following conditions are equivalent for a complex F in $\mathscr{C}(\mathscr{A})$.*

1. *F is exact and every cycle is projective.*

2. *Every morphism $X \to F$, where X is bounded, factors through a projective complex.*

3. *Every morphism $X \to F$, where X is bounded below, factors through a projective complex.*

4. *Every morphism $X \to F$, factors through a projective complex.*

Proof. Using the notations of Proposition 3.3.7, let $\mathscr{L} = \mathscr{A}$ and $\mathscr{G}$ be the class of projective objects, $Proj_{\mathscr{A}}$. Then, $\widetilde{\mathscr{G}} = Proj_{\mathscr{C}(\mathscr{A})}$ (see Proposition 1.2.2) and $dg\widetilde{\mathscr{L}} = \mathscr{C}(\mathscr{A})$ since for every $X \in \mathscr{C}(\mathscr{A})$ and $G \in Porj_{\mathscr{C}(\mathscr{A})}$, $H_n(\mathrm{Hom}^\bullet(X,G)) = \mathrm{Hom}_{\mathscr{K}(\mathscr{A})}(X[n],G) = 0$ for every $n \in \mathbb{Z}$ (id_G is null homotopic since G is contractible). $\square$

In the case of the class of Gorenstein projective objects and the class of strongly Gorenstein projective objects we have the following.

Corollary 3.3.10. *If $\mathscr{A}$ has a projective generator, then the following conditions are equivalent for a complex F in $\mathscr{C}(\mathscr{A})$.*

1. *F is exact and every cycle holds in $\widetilde{\mathscr{G}\mathscr{P}^{\perp}_{\mathscr{A}}}$.*

2. *Every morphism $X \to F$, where X is a bounded complex of Gorenstein projective components, factors through a projective complex.*

3. *Every morphism $X \to F$, where X is a bounded below complex of Gorenstein projective components, factors through a projective complex.*

4. *Every morphism $X \to F$, where X is a $dg\widetilde{\mathscr{G}\mathscr{P}_{\mathscr{A}}}$ complex, factors through a projective complex.*

5. *Every morphism $X \to F$, where X is a bounded complex of strongly Gorenstein projective components, factors through a projective complex.*

6. *Every morphism $X \to F$, where X is a bounded below complex of strongly Gorenstein projective components, factors through a projective complex.*

7. *Every morphism $X \to F$, where X is a $dg\widetilde{\mathscr{S}\mathscr{G}\mathscr{P}_{\mathscr{A}}}$ complex, factors through a projective complex.*

Proof. For 1. $\Leftrightarrow$ 2. $\Leftrightarrow$ 3. $\Leftrightarrow$ 4., we apply Proposition 3.3.7 to $\mathscr{L} = \mathscr{G}\mathscr{P}_{\mathscr{A}}$ and $\mathscr{G} = \widetilde{\mathscr{G}\mathscr{P}^{\perp}_{\mathscr{A}}}$ (see Corollary 2.1.17). And for 1. $\Leftrightarrow$ 5. $\Leftrightarrow$ 6. $\Leftrightarrow$ 7., we apply it to $\mathscr{L} = \mathscr{S}\mathscr{G}\mathscr{P}_{\mathscr{A}}$ and $\mathscr{G} = \widetilde{\mathscr{S}\mathscr{G}\mathscr{P}^{\perp}_{\mathscr{A}}} = \widetilde{\mathscr{G}\mathscr{P}^{\perp}_{\mathscr{A}}}$ (see Proposition 2.1.16). $\square$

In the case of a class of finitely generated objects, to apply Proposition 3.3.7, we have to assume that $\mathscr{A}$ has a projective generator which is finitely generated and since we are studying finitely generated objects in Grothendieck categories, then $\mathscr{A}$ should be a Grothendieck category with a finitely generated projective generator. But in that case, $\mathscr{A}$ will be equivalent to the category of modules. For this reason, we give Corollaries 3.3.11 3.3.12 and 3.3.13 in R-Mod. However, it is an interesting open question to prove these results for a general locally finitely generated Grothendieck category with enough projectives (which include the case of representations of infinite quivers and the one of graded modules over a graded ring over an infinite group)

We start with the case of finitely presented modules which gives the following characterization of flat complexes of modules. First, recall that a complex of modules is finitely presented if and only if it is bounded of finitely presented components, that is $\mathscr{FG}_{\mathscr{C}(\mathscr{A})} = \mathscr{C}^b(\mathscr{FG}_{R\text{-Mod}})$ (see [26, Lemma 4.1.1]).

Corollary 3.3.11. *The following conditions are equivalent for a complex of modules F.*

1. *F is exact and every cycle is flat.*
2. *Every morphism $X \to F$, where X is a finitely presented complex, factors through a projective complex.*
3. *Every morphism $X \to F$, where X is a bounded complex below of finitely presented components, factors through a projective complex.*
4. *Every morphism $X \to F$, where X is a $dg\widetilde{\mathscr{F}\mathscr{P}}$ complex, factors through a projective complex.*
5. *Every morphism $X \to F$, where X is a bounded below complex of pure projective components, factors through a projective complex.*
6. *Every morphism $X \to F$, where X is a bounded below complex of pure projective components, factors through a projective complex.*
7. *Every morphism $X \to F$, where X is a $dg\widetilde{\mathscr{P}\mathscr{P}_{\mathscr{A}}}$ complex, factors through a projective complex.*

Proof. We know from Proposition 2.1.19 and Corollary 2.1.20 that the subprojectivity domain of the class of finitely presented modules and the one of pure projective modules coincide with the class of flat modules. Then, to show 1. $\Leftrightarrow$ 2. $\Leftrightarrow$ 3. $\Leftrightarrow$ 4., we take $\mathscr{L}$ to be the class of finitely presented modules and $\mathscr{G}$ to be the class of flat modules and we apply Proposition 3.3.7. For 1. $\Leftrightarrow$ 5. $\Leftrightarrow$ 6. $\Leftrightarrow$ 7., we take $\mathscr{L}$ to be the class of pure porjective modules and $\mathscr{G}$ to be the class of flat modules and we apply Proposition 3.3.7. $\square$

Now, we give the case of finitely generated modules.

Corollary 3.3.12. *the following conditions are equivalent for a complex of modules F.*

1. *F is exact and every cycle is f-projective.*
2. *Every morphism $X \to F$, where X is a finitely generated complex, factors through a projective complex.*
3. *Every morphism $X \to F$, where X is a bounded below complex of finitely generated components, factors through a projective complex.*
4. *Every morphism $X \to F$, where X is a $dg\widetilde{\mathscr{F}\mathscr{G}_{R\text{-Mod}}}$ complex, factors through a projective complex.*

Proof. Following the notations of Proposition 3.3.7, let $\mathscr{L}$ be the class of finitely generated modules and $\mathscr{G}$ the one of f-projective modules (see Proposition 2.1.24). Then, $\mathscr{C}^b(\mathscr{L})$ is the class of finitely generated complexes (see [26, Lemma 4.1.1]). $\square$

Finally, we apply Proposition 3.3.7 to class of simple modules and the one of semisimple modules.

Corollary 3.3.13. *The following conditions are equivalent for a complex of modules F.*

1. *F is exact and every cycle is simple projective.*
2. *Every morphism $X \to F$, where X is a bounded complex of simple components, factors through a projective complex.*
3. *Every morphism $X \to F$, where X is a bounded below complex of simple components, factors through a projective complex.*
4. *Every morphism $X \to F$, where X is a $dg\widetilde{\mathscr{S}_{R\text{-Mod}}}$ complex, factors through a projective complex.*
5. *Every morphism $X \to F$, where X is a bounded complex of semisimple components, factors through a projective complex.*
6. *Every morphism $X \to F$, where X is bounded below complex of semisimple components, factors through a projective complex.*
7. *Every morphism $X \to F$, where X is a $dg\widetilde{\mathscr{S}\mathscr{S}_{R\text{-Mod}}}$ complex, factors through a projective complex.*

Proof. We know from Proposition 2.1.25 and Corollary 2.1.26 that the subprojectivity domain of the class of simple modules and the one of semisimple modules coincide with the class of simple projective modules. Then, to show 1. $\Leftrightarrow$ 2. $\Leftrightarrow$ 3. $\Leftrightarrow$ 4., we take $\mathscr{L}$ to be the class of simple modules and $\mathscr{G}$ to be the class of simple projectives and we apply Proposition 3.3.7. For 1. $\Leftrightarrow$ 5. $\Leftrightarrow$ 6. $\Leftrightarrow$ 7., we take $\mathscr{L}$ to be the class of semisimple modules and $\mathscr{G}$ to be the class of simple projectives and we apply Proposition 3.3.7. $\square$

3.4 Characterization of rings by subprojectivity of complexes

This section is devoted to some applications to complexes of modules. Namely, we give new characterizations of some classical rings by the meaning of subprojectivity.

We start with the following characterization of quasi-Fröbenius rings.

Proposition 3.4.1. *The following conditions are equivalent.*

1. *R is quasi-Fröbenius.*

2. *For every exact complex M and every bounded above complex N, if for every $n \in \mathbb{Z}$, $N_n \in \underline{\mathfrak{Pr}}^{-1}_{R\text{-Mod}}(Z_{n-1}(M))$, then $N \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(R)}(M)$.*

3. *For every exact complex M and every module N, if there exists $n \in \mathbb{Z}$ such that $N \in \underline{\mathfrak{Pr}}^{-1}_{R\text{-Mod}}(Z_{n-1}(M))$, then $\underline{N}[n] \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(R)}(M)$.*

Proof. Recall that R is quasi-Fröbenius if and only if every projective module is injective and apply Theorem 3.2.7 to get the desired result. $\square$

Recall that the ring R is said to be left hereditary if any left R-submodule of a projective left R-module is projective. Recall also that a complex P is said to be DG-projective if its components are projective and $\mathrm{Hom}^\bullet(P,E)$ is exact for every exact complex E. In [46, Proposition 2.3] it is proved that, under certain conditions, a ring is left hereditary if and only if every subcomplex of a DG-projective complex is DG-projective. Among these conditions, the authors included: "Every exact complex of projective modules is projective". In this section, using the properties of subprojectivity domains, we will show that the latter equivalence holds without the mentioned assumption.

Proposition 3.4.2. *For any ring R, the following statements are equivalent.*

1. *R is left hereditary.*

2. *For every complex M and every exact complex N, if $N_n \in \underline{\mathfrak{Pr}}^{-1}_{R\text{-Mod}}(M_n)$ for every $n \in \mathbb{Z}$, then $N \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(R)}(M)$.*

3. *For every module M and every exact complex N, if there exists $n \in \mathbb{Z}$ such that $N_n \in \underline{\mathfrak{Pr}}_{R\text{-Mod}}^{-1}(M)$, then $N \in \underline{\mathfrak{Pr}}_{\mathscr{C}(R)}^{-1}(\underline{M}[n])$.*

4. *Every subcomplex of a DG-projective complex is DG-projective.*

Proof. 1. $\Rightarrow$ 2. If $0 \to K \to P \to N \to 0$ is a short exact sequence of complexes with P projective then K is exact (P and N are exact) and all cycles $Z_n(K)$ are projective by 1. Therefore, K is projective and then, by Lemma 3.1.2, $\mathrm{Hom}_{\mathscr{K}(R)}(M[-1], K) = 0$, so $N \in \underline{\mathfrak{Pr}}_{\mathscr{C}(R)}^{-1}(M)$ by Theorem 3.1.4.

2. $\Rightarrow$ 3. Clear.

3. $\Rightarrow$ 1. Let Q be a projective module and Y be any submodule of Q. Let us prove that $\underline{\mathfrak{Pr}}_{R\text{-Mod}}^{-1}(Y) = R$-Mod. For let X be a module and consider the exact complex

$$\mathscr{C} : \cdots \to 0 \to X \to E(X) \to C \to 0 \to \cdots$$

($E(X)$ in the 0-position). By [19, Lemma 2.2] $E(X) \in \underline{\mathfrak{Pr}}_{R\text{-Mod}}^{-1}(Y)$. So we get that $\mathscr{C} \in \underline{\mathfrak{Pr}}_{\mathscr{C}(R)}^{-1}(\underline{Y})$. Then, $X \in \underline{\mathfrak{Pr}}_{R\text{-Mod}}^{-1}(Y)$ by Lemma 3.1.7.

1. $\Rightarrow$ 4. Let P be a DG-projective complex and Q a subcomplex of P. Then, every module Q_n is projective by condition 1.

Now, let E be an exact complex and let us prove that $\mathrm{Hom}^\bullet(Q, E)$ is exact.

Let $0 \to E \to I \to C \to 0$ be a short exact sequence of complexes with I injective. Since every module Q_n is projective we get that for every $n \in \mathbb{Z}$, $\mathrm{Hom}^\bullet(Q, I)_n \to \mathrm{Hom}^\bullet(Q, C)_n$ is epic, and for every $i, j \in \mathbb{Z}$, $C_i \in \underline{\mathfrak{Pr}}_{R\text{-Mod}}^{-1}(Q_j)$. Then, by condition 2. we get that $C \in \underline{\mathfrak{Pr}}_{\mathscr{C}(R)}^{-1}(Q[n])$ for every $n \in \mathbb{Z}$ (C is exact since I and E are exact), so for every $n \in \mathbb{Z}$, $Z_n(\mathrm{Hom}^\bullet(Q, I)) \to Z_n(\mathrm{Hom}^\bullet(Q, C))$ is epic. Therefore, for every $n \in \mathbb{Z}$ the two first columns of the commutative diagram with exact rows

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & Z_n(\mathrm{Hom}^\bullet(Q,E)) & \to & \mathrm{Hom}^\bullet(Q,E)_n & \to & B_{n-1}(\mathrm{Hom}^\bullet(Q,E)) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & Z_n(\mathrm{Hom}^\bullet(Q,I)) & \to & \mathrm{Hom}^\bullet(Q,I)_n & \to & B_{n-1}(\mathrm{Hom}^\bullet(Q,I)) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & Z_n(\mathrm{Hom}^\bullet(Q,C)) & \to & \mathrm{Hom}^\bullet(Q,C)_n & \to & B_{n-1}(\mathrm{Hom}^\bullet(Q,C)) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

are exact, so the third is also exact.

Now consider, for every $n \in \mathbb{Z}$, the commutative diagram with exact rows

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & B_n(\mathrm{Hom}^\bullet(Q,E)) & \longrightarrow & Z_n(\mathrm{Hom}^\bullet(Q,E)) & \longrightarrow & H_n(\mathrm{Hom}^\bullet(Q,E)) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & B_n(\mathrm{Hom}^\bullet(Q,I)) & \longrightarrow & Z_n(\mathrm{Hom}^\bullet(Q,I)) & \longrightarrow & H_n(\mathrm{Hom}^\bullet(Q,I)) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & B_n(\mathrm{Hom}^\bullet(Q,C)) & \longrightarrow & Z_n(\mathrm{Hom}^\bullet(Q,C)) & \longrightarrow & H_n(\mathrm{Hom}^\bullet(Q,C)) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

The first and second columns are exact, so the third one is also exact. But, for every $n \in \mathbb{Z}$, $H_n(\mathrm{Hom}^\bullet(Q,I)) = \mathrm{Hom}_{\mathscr{K}(R)}(Q[n],I) = 0$ since I is contractible.

4. $\Rightarrow$ 1. Let Q be a projective module and Y a submodule of Q. Since $\underline{Y}$ is a subcomplex of the DG-projective complex $\underline{Q}$, $\underline{Y}$ must be DG-projective by assumption, so Y is projective.

□

It is a well-known fact that a ring is left semi-hereditary if and only if it is left coherent and every submodule of a flat module is flat (i.e., the weak global dimension of the ring is at most 1). Using subprojectivity we can prove a similar result in the categories of complexes. Namely, a ring is left semi-hereditary if and only if it is left coherent and every subcomplex of a DG-flat complex is DG-flat (Corollary 3.4.4). This is so because rings for which subcomplexes of DG-flat complexes are DG-flat are precisely those of weak global dimension at most 1 (Proposition 3.4.3).

Recall that the subprojectivity domain of the class of all finitely presented complexes (respectively, modules) is the class of all flat complexes (respectively, modules) (see Proposition 2.1.19). Recall also that a complex F is said to be DG-flat if F_n is flat for every $n \in \mathbb{Z}$ and the complex $E \otimes^\bullet F$ is exact for any exact complex E of right R-modules (see [5]).

Proposition 3.4.3. *For any ring R, the following assertions are equivalent.*

1. *The weak global dimension of R is at most* 1.

2. *For every finitely presented complex M and every exact complex N such that* $N_n \in \underline{\mathfrak{Pr}}^{-1}_{R\text{-Mod}}(M_n)$ *for every* $n \in \mathbb{Z}$, *then* $N \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(R)}(M)$.

3. *For every finitely presented module M and every exact complex N, if there exists $n \in \mathbb{Z}$ such that $N_n \in \underline{\mathfrak{Pr}}^{-1}_{R\text{-Mod}}(M)$, then $N \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(R)}(\underline{M}[n])$.*

4. *Every subcomplex of a DG-flat complex is DG-flat.*

Proof. 1. $\Rightarrow$ 2. Consider a short exact sequence of complexes $0 \to K \to P \to N \to 0$ with P projective. Since all cycles $Z_n(P)$ are projective, every cycle $Z_n(K)$ is flat by assumption. Then, K is flat (K is exact since P and N are), so $K \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(R)}(M[-1])$ and hence $\mathrm{Hom}_{\mathscr{K}(R)}(M[-1], K) = 0$ by Lemma 3.1.2. Therefore, $N \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(R)}(M)$ by Theorem 3.1.4.

2. $\Rightarrow$ 3. Clear.

3. $\Rightarrow$ 1. Let X be a submodule of a flat module F. Let us prove that $X \in \underline{\mathfrak{Pr}}^{-1}_{R\text{-Mod}}(M)$ for every finitely presented module M. For let M be a finitely presented module and consider the exact complex

$$\mathscr{F} : \cdots \to 0 \to X \to F \to C \to 0 \to \cdots$$

with F in the 0-position.

Since $F \in \underline{\mathfrak{Pr}}^{-1}_{R\text{-Mod}}(M)$ we have that $\mathscr{F} \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(R)}(\underline{M})$ by assumption, and then $X \in \underline{\mathfrak{Pr}}^{-1}_{R\text{-Mod}}(M)$ by Lemma 3.1.7.

1. $\Rightarrow$ 4. Let F be a DG-flat complex, N be a subcomplex of F and $P \to N$ be an epic quasi-isomorphism with P DG-projective. To prove that N is DG-flat it is sufficient to prove that for every finitely presented complex M, $\mathrm{Hom}_{\mathscr{C}(R)}(M, P) \to \mathrm{Hom}_{\mathscr{C}(R)}(M, N)$ is epic (see [17, Proposition 6.2]). For let $f : M \to N$ be a morphism of complexes with M finitely presented and consider the following pullback diagram

$$\begin{tikzcd}
0 \arrow[r] & E \arrow[r] \arrow[d, equal] & D \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, "f"] & 0 \\
0 \arrow[r] & E \arrow[r] & P \arrow[r] & N \arrow[r] & 0
\end{tikzcd}$$

Every module N_n is flat by 1, so $N_n \in \underline{\mathfrak{Pr}}^{-1}_{R\text{-Mod}}(M_n)$ for every $n \in \mathbb{Z}$, and hence the short exact sequence $0 \to E \to D \to M \to 0$ splits at the module level by Proposition 3.1.1 since for every $n \in \mathbb{Z}$, D_n is a pullback (see Lemma 3.1.3). Then, the sequence $0 \to E \to D \to M \to 0$ is equivalent to a short exact sequence $0 \to E \to M(g) \to M \to 0$ where $M(g)$ is the mapping cone of a morphism $g : M[-1] \to E$ (see [22, Section 3.3]).

Now, every module E_n is flat by condition 1. So, $E_n \in \underline{\mathfrak{Pr}}^{-1}_{R\text{-Mod}}(M_{n+1})$ for every $n \in \mathbb{Z}$. Thus, $E \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(R)}(M[-1])$ by condition 2 and then by Lemma 3.1.2 we get that $\mathrm{Hom}_{\mathscr{K}(R)}(M[-1], E) = 0$.

In particular, $g : M[-1] \to E$ is null-homotopic so the sequence $0 \to E \to M(g) \to M \to 0$ splits (see [22, Proposition 3.3.2])and then the sequence $0 \to E \to D \to M \to 0$ splits. Therefore, f clearly factors through $P \to N$.

4. $\Rightarrow$ 1. Let F be a flat module and Y a submodule of F. Then $\underline{Y}$ is a subcomplex of the DG-flat complex $\underline{F}$, so $\underline{Y}$ is also DG-flat by assumption and therefore Y is flat. $\square$

Corollary 3.4.4. *For any ring R the following statements are equivalent.*

1. *R is left semi-hereditary.*
2. *R is left coherent and for every finitely presented complex M and every exact complex N, if $N_n \in \underline{\mathfrak{Pr}}^{-1}_{R\text{-Mod}}(M_n)$ for every $n \in \mathbb{Z}$, then $N \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(R)}(M)$.*
3. *R is left coherent and for every finitely presented module M and every exact complex N, if there exists $n \in \mathbb{Z}$ such that $N_n \in \underline{\mathfrak{Pr}}^{-1}_{R\text{-Mod}}(M)$, then $N \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(R)}(\underline{M}[n])$.*
4. *R is left coherent and every subcomplex of a DG-flat complex is DG-flat.*

Proof. Since R is left semi-hereditary if and only if it is left coherent and has the weak global dimension at most 1, we apply Proposition 3.4.3 the get the result. $\square$

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