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Short communication

Quasilineability and topological properties of the set of fuzzy numbers

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Abstract

In this paper we show that the cardinality of the set of fuzzy numbers coincides with that of the real numbers. We also show that the set of triangular fuzzy numbers is nowhere dense within the set of fuzzy numbers (with a suitable distance) and that the set of real numbers is also nowhere dense within the set of triangular fuzzy numbers. In addition, we introduce the concept of quasilineability and study the set of bounded fuzzy number sequences that do not have a lower limit and that of monotonic decreasing, bounded with respect a partial ordering and not convergent.

Keywords: Cardinality; Fuzzy number; Quasilineability; Sequence of fuzzy numbers; Triangular fuzzy number

1. Introduction

A fuzzy set is a set whose elements have degrees of membership. Fuzzy sets were introduced by Zadeh [25] in 1965 with a potential applicability in Information Processing, and since then they have been used in a wide range of fields, such as Decision Analysis, Operation Research and Statistics, Psychology, Bioinformatics, etc.

Fuzzy numbers are a special case of a convex, normalized fuzzy set of the real line. They were also introduced by Zadeh in 1975 (see [26]) in order to analyze approximated numeric values.

Bounded and convergent sequences of fuzzy numbers were introduced by Matloka [17], where it is proved that every convergent sequence of fuzzy numbers is bounded; and in [18], Nanda studies the spaces of bounded and convergent sequences of fuzzy numbers and shows that they are complete metric spaces. For a review of sequences of fuzzy numbers, see e.g. [5].

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Aseev [2] introduced the concept of quasilinear space defined by the partial order relation generalizing classical linear spaces. In [24], the authors have proved that classical sets —consisting of the bounded, convergent, null and absolutely p-summable— sequences of fuzzy numbers are normed quasilinear spaces.

On the other hand, it is common to try to decide if given a set Ω and a subset of it Σ, the second is "large" inside the first. When we have a metric d defined in Ω we can use topological category to analyze the size of subsets. In particular, a subset Σ of a metric space (Ω, d) is called *nowhere dense* if it is not dense in any open ball $B(x, r)$ of radius $r > 0$, i.e., its closure has empty interior, denoted by $\text{int}(\overline{\Sigma}) = \emptyset$, (see [19]). The subset Σ is called *meager* or of *first category* in (Ω, d) if it can be covered by a countable union of nowhere dense sets. Σ is called of *second category* if it is not meager. Finally, Σ is called *co-meager* if $\Sigma^c = \Omega \setminus \Sigma$ is meager. Following [9], in complete metric spaces, first category sets are the "small" sets, co-meager sets are the "large" sets and second-category sets are merely "not small". We will use this technique to study the size of the triangular fuzzy numbers and $\mathbb{R}$ inside the set of fuzzy numbers.

When Ω is a vectorial space over a field $\mathbb{K}$ (usually $\mathbb{R}$ or $\mathbb{C}$) and A is not, the search of a vectorial space V (as well as other algebraic structures) contained in A is known as *lineability*. To the extent that the dimension of V is large tells us whether A is within Ω. The term lineability was coined by V. I. Gurariy ([14]) and is introduced in the Ph. D. Dissertation of Seoane-Sepúlveda ([22]) where a systematic study of this notion in various types of problems and vectorial spaces is made for the first time. The study of lineability appears in several different areas such as Real and Complex Analysis ([7,8,11]), Measure Theory ([6]), Set Theory ([12]), etc. See [15] for an example in which the space used is ultrametric. We refer to [1] as the main reference of this field of study.

The main point in the theory of lineability is the search for large algebraic structures composed of mathematical objects enjoying certain special (usually, pathological) properties. This theory cannot be applied directly in the case of fuzzy numbers since we do not work with vectorial spaces. However, we can adapt it since we are in quasilinear spaces.

The paper is organized as follows. After some preliminaries concerning quasilinear spaces and triangular fuzzy numbers (Section 2), in Section 3 we prove that the set of the triangular fuzzy numbers is a quasilinear space, show that this set is nowhere dense in the set of all fuzzy numbers on $\mathbb{R}$, and study the set of sequences of bounded fuzzy numbers that have no lower limit and that of fuzzy numbers monotonic decreasing, bounded with respect a partial ordering and not convergent. Finally, Section 4 is devoted to conclusions.

2. Preliminaries

First, we recall some basic notation, definitions and results in the theory of fuzzy numbers (see, e.g., [5] and the references therein for more details).

Definition 1. A *fuzzy number* is a fuzzy set on the real axis, i.e., a mapping $u : \mathbb{R} \to [0, 1]$ which satisfies the following four conditions:

(i) u is normal, i.e., there exists an $\alpha_0 \in \mathbb{R}$ such that $u(\alpha_0) = 1$.
(ii) u is fuzzy convex, i.e., $u(\alpha s + (1-\alpha)t) \geq \min\{u(s), u(t)\}$ for all $s, t \in \mathbb{R}$ and for all $\alpha \in [0, 1]$.
(iii) u is upper semi-continuous, i.e., $\limsup_{x \to x_0} u(x) \leq u(x_0)$.
(iv) The set $[u]_0 = \overline{\{t \in \mathbb{R} : u(t) > 0\}}$ is compact, where $\overline{\{t \in \mathbb{R} : u(t) > 0\}}$ denotes the closure of the set $\{t \in \mathbb{R} : u(t) > 0\}$ in the usual topology of $\mathbb{R}$.

We denote by $\mathcal{F}$ the set of all fuzzy numbers on $\mathbb{R}$. The set $\mathbb{R}$ can be embedded in $\mathcal{F}$, since each $r \in \mathbb{R}$ can be regarded as a fuzzy number $\overline{r}$ defined by

$$\overline{r}(t) := \begin{cases} 1, & t = r, \\ 0, & t \neq r \end{cases}$$

(in order to simplify the notation, we identify r with $\overline{r}$ and we will always write r). Furthermore, fuzzy numbers generalize closed intervals.

The α-*level set* (or α-*cut*) $[u]_\alpha$ of $u \in \mathcal{F}$ is defined by

$$[u]_\alpha := \begin{cases} \overline{\{t \in \mathbb{R} : u(t) > 0\}}, & \alpha = 0. \\ \{t \in \mathbb{R} : u(t) \geq \alpha\}, & 0 < \alpha \leq 1. \end{cases}$$

The set $[u]_\alpha$ is a closed, bounded and non empty interval for each $\alpha \in [0,1]$, which is defined by $[u]_\alpha := \left[u^-(\alpha), u^+(\alpha)\right]$.

The Representation theorem [13] expresses a fuzzy number as two real-valued functions, and has been proved useful in solving many problems concerning fuzzy numbers (see, e.g. [16]).

Theorem 1. *[Representation theorem] For $u \in \mathcal{F}$ and for each $\alpha \in [0,1]$, let $[u]_\alpha = \left[u^-(\alpha), u^+(\alpha)\right]$. Then the following statements hold:*

(i) *u^- is a bounded and non-decreasing left-continuous function on* $(0,1]$.
(ii) *u^+ is a bounded and non-increasing left-continuous function on* $(0,1]$.
(iii) *The functions u^- and u^+ are right-continuous at* $\alpha = 0$.
(iv) $u^-(1) \leq u^+(1)$.

Conversely, if the pair of functions u^- and u^+ satisfies the conditions (i)–(iv), then there exists a unique $u \in \mathcal{F}$ such that $[u]_\alpha = \left[u^-(\alpha), u^+(\alpha)\right]$ for each $\alpha \in [0,1]$.

A partial ordering, denoted by $\preceq$, on $\mathcal{F}$ can be defined by saying $u \preceq v$ if, and only if, $[u]_\alpha \subseteq [v]_\alpha$ for all $\alpha \in [0,1]$, i.e., $v^-(\alpha) \leq u^-(\alpha)$ and $u^+(\alpha) \leq v^+(\alpha)$ for all $\alpha \in [0,1]$ (see [21]).

The metric D on $\mathcal{F}$, by means of the *Hausdorff metric*, is defined by

$$D(u,v) := \sup_{\alpha \in [0,1]} \max\left\{\left|u^-(\alpha) - v^-(\alpha)\right|, \left|u^+(\alpha) - v^+(\alpha)\right|\right\}.$$

$(\mathcal{F}, D)$ is a complete metric space and the restriction of D to $\mathbb{R}$ is the usual distance (see [20] for details).

Several types of fuzzy numbers are often used in applications (see [10]). One of them is the *triangular fuzzy number* (tfn, for short) which is characterized by $[u]_\alpha = [a + \alpha(b-a), c - \alpha(c-b)]$ for all $\alpha \in [0,1]$. Note that, for a tfn u, we have the membership function with the triangular form

$$u(x) = \begin{cases} \dfrac{x-a}{b-a}, & a \leq x < b, \\ \dfrac{c-x}{c-b}, & b \leq x \leq c, \\ 0, & \text{elsewhere}, \end{cases}$$

when $a < b < c$;

$$u(x) = \begin{cases} \dfrac{c-x}{c-b}, & b \leq x \leq c, \\ 0, & \text{elsewhere}, \end{cases}$$

when $a = b < c$; and similarly for the cases $a < b = c$ and $a = b = c$. We denote by $\mathcal{T}$ the set of all triangular fuzzy numbers.

We will use the following operations for any pair u, v of (triangular) fuzzy numbers and any real number λ in terms of α-level sets:

i. $u + v$: $[u+v]_\alpha = [u]_\alpha + [v]_\alpha = \left[u^-(\alpha) + v^-(\alpha), u^+(\alpha) + v^+(\alpha)\right]$; and
ii. λu:

$$\lambda[u]_\alpha = \begin{cases} \left[\lambda u^-(\alpha), \lambda u^+(\alpha)\right], & \lambda \geq 0, \\ \left[\lambda u^+(\alpha), \lambda u^-(\alpha)\right], & \lambda < 0, \end{cases}$$

for all $\alpha \in [0, 1]$.

A *sequence* $u = (u_n)$ of fuzzy numbers is a function u from the set $\mathbb{N}$ into $\mathcal{F}$. The set of all sequences of fuzzy numbers is denoted by $w(\mathcal{F})$, and a sequence $(u_n) \in w(\mathcal{F})$ is called *convergent* with limit $u \in \mathcal{F}$ if, and only if, for every $\varepsilon > 0$ there exists an $n(\varepsilon) \in \mathbb{N}$ such that $D(u_k, u) < \varepsilon$ for all $k > n(\varepsilon)$.

A set $E \subseteq \mathcal{F}$ is said to be *bounded from above* (*below*) if there exists $u \in \mathcal{F}$, called an *upper* (a *lower*) *bound* of E, such that $v \preceq u$ ($u \preceq v$) for every $v \in \mathcal{F}$. u is called the *supremum* (*infimum*) of E if u is an upper (lower) bound and $u \preceq u'$ ($u' \preceq u$) for all upper (lower) bounds u'. E is said to be *bounded* if it is bounded from above and below. See [3,18,23] for more details.

We recall the basic notion of lineability (see [1]).

Definition 2. Let κ be a cardinal number, V a vector space and $A \subset V$. We say that A is κ-lineable if there is a vector space $L \subset V$ such that L has dimension κ and $L \setminus \{0\} \subset A$.

As we have mentioned, lineability cannot be applied directly in the case of fuzzy numbers, but we can adapt it. Aseev [2] introduced new results in functional analysis with the concept of quasilinear space, which we recall now.

Definition 3. A set X is called a *quasilinear space* (qls, for short) if a partial order relation ($\preceq$), an algebraic sum operation ($+$) and an operation of multiplication by real numbers ($\cdot$) are defined on it and satisfy the following properties for any elements $x, y, z, v \in X$ and any real numbers α, β:

(q1) $x \preceq x$.
(q2) $x \preceq z$ if $x \preceq y$ and $y \preceq z$.
(q3) $x = y$ if $x \preceq y$ and $y \preceq x$.
(q4) $x + y = y + x$.
(q5) $x + (y + z) = (x + y) + z$.
(q6) There exists an element $\theta \in X$, called neutral element, such that $\theta + x = x$.
(q7) $\alpha \cdot (\beta \cdot x) = (\alpha\beta) \cdot x$.
(q8) $\alpha \cdot (x + y) = \alpha \cdot x + \alpha \cdot y$.
(q9) $1 \cdot x = x$.
(q10) $0 \cdot x = \theta$.
(q11) $(\alpha + \beta) \cdot x \preceq \alpha \cdot x + \beta \cdot x$.
(q12) $x + z \preceq y + v$ if $x \preceq y$ and $z \preceq v$.
(q13) $\alpha \cdot x \preceq \alpha \cdot y$ if $x \preceq y$.

The quasilinear space in Definition 3 will be denoted by $(X, \preceq, +, \cdot)$. Note that a linear space is a qls with the partial order relation "=".

Theorem 2 *([21]). $(\mathcal{F}, \preceq, +, \cdot)$ is a qls with $\theta = 0$.*

Let X be a qls and $Y \subseteq X$. Then Y is called a *subspace* of X whenever Y is a qls with the same partial ordering and the same operations on X to Y. Furthermore, $Y \subset X$ is a *cone* of X if it is closed for the sum and the product by non-negative scalars.

The elements of a subset Λ of a qls X are said to be *quasilinear independent* (*ql-independent*, for short) if there doesn't exist any quasilinear combination of the form $\sum_{i=1}^{n} \alpha_i x_i$ with $x_i \in \Lambda$ and $\alpha_i \in \mathbb{R}$, $i = 1, \ldots, n$, such that $\theta \preceq \sum_{i=1}^{n} \alpha_i x_i$ (see [4]).

Given a qls X and a cardinal number ν, a subset $M \subset X$ is called $\nu-$*quasilineable* if it contains ν ql-independent elements and the subspace generated by them is contained in M. A subset $N \subset X$ is $\nu-$*coneable* if it has ν ql-independent elements and the cone generated by them is contained in N. The search for these sets is called *quasi-lineability*.

3. Main results

In this section we will study, on the one hand, the cardinality of the set of fuzzy numbers and its size in topological terms; and, on the other, the quasilineability of sequences of fuzzy numbers.

3.1. The cardinality of the set of fuzzy numbers

By using Theorem 1 we can know how many fuzzy numbers exist, that is, we will establish the cardinality of the set $\mathcal{F}$; specifically, it is the same as that of $\mathbb{R}$, as the following result shows, where $\mathfrak{c}$ represents the cardinality of the set of real numbers.

Theorem 3. *The cardinality of the sets $\mathcal{F}$ and $\mathcal{T}$ is $\mathfrak{c}$.*

Proof. On the one hand, since $\mathbb{R} \subset \mathcal{F}$, we have that the cardinality of $\mathcal{F}$ is greater than or equal to $\mathfrak{c}$. On the other hand, we check how many functions from $[0, 1]$ to $\mathbb{R}$ satisfy that they are left-continuous at $]0, 1]$ and right-continuous at the point 0. This set of functions is denoted by $\mathcal{S}$. Since the set $\mathbb{Q} \cap]0, 1[$ is dense in $[0, 1]$, where $\mathbb{Q}$ represents the set of rational numbers, every function $f \in \mathcal{S}$ is uniquely determined by the values of this function in $\mathbb{Q} \cap]0, 1]$. Therefore, the cardinality of $\mathcal{S}$ is smaller than or equal to that of $\mathbb{R}^{\mathbb{N}}$ —i.e., the set of the sequences of real numbers—, which is $\mathfrak{c}$. Since $\mathcal{S}$ contains the constant functions, its cardinal is $\mathfrak{c}$. Similarly, the set $\mathcal{D}$ of the functions from $[0, 1]$ to $\mathbb{R}$ satisfying that they are right-continuous in $[0, 1[$ and left-continuous at the point 1 has cardinality $\mathfrak{c}$. Since Theorem 1 provides an injective application from $\mathcal{F}$ to $\mathcal{S} \times \mathcal{D}$, we have that the cardinality of $\mathcal{F}$ is less than or equal to $\mathfrak{c}$. We conclude that the cardinality of $\mathcal{F}$ is $\mathfrak{c}$.

On the other hand, since $\mathbb{R} \subset \mathcal{T} \subset \mathcal{F}$, it is immediately obtained that the cardinality of $\mathcal{T}$ is also $\mathfrak{c}$. □

3.2. Topological size of $\mathcal{T}$

In this subsection we prove that the set $\mathcal{T}$ is a qls, show that this set is nowhere dense in the set $\mathcal{F}$, and provide several topological properties. Among the results that we obtain, we highlight that, from a topological point of view, $\mathcal{T}$ is a very small subset of $\mathcal{F}$, and something similar happens with $\mathbb{R}$ as a subset of $\mathcal{T}$.

Since the set $\mathcal{T}$ is (arithmetically) closed for the sum and the product by scalars, then it trivially satisfies the conditions (q1)–(q13) in Definition 3; thus we have the following result.

Theorem 4. *The set $\mathcal{T}$ is a qls.*

The next three results show the topological relations, in terms of nowhere density, among the sets $\mathcal{T}$, $\mathcal{F}$ and $\mathbb{R}$.

Theorem 5. *The set $\mathcal{T}$ is (topologically) closed and nowhere dense in $\mathcal{F}$.*

Proof. To check that the set $\mathcal{T}$ is (topologically) closed, let us consider a sequence of triangular fuzzy numbers (t_n) that is convergent to a fuzzy number u. The fuzzy number t_n is characterized by three real numbers, (a_n, b_n, c_n), where b_n is the value that satisfies $t_n(b_n) = 1$, a_n is the supremum of the values x less than b_n for which $t_n(x) = 0$, and c_n is the infimum of those values x greater or equal than b_n that satisfy $t_n(x) = 0$. Since (t_n) is convergent, it is also bounded; this tells us that the sequence (a_n, b_n, c_n) is bounded. We take a subsequence $\left(t_{\sigma(n)}\right) = \left(a_{\sigma(n)}, b_{\sigma(n)}, c_{\sigma(n)}\right)$ of (t_n) such that it is convergent to $t = (a, b, c)$. Then, the number t is the limit of $\left(t_{\sigma(n)}\right)$. Therefore, u must be equal to t, that is, u is a triangular fuzzy number.

In order to prove that $\mathcal{T}$ is nowhere dense in $\mathcal{F}$, since $\mathcal{T}$ is closed in $\mathcal{F}$, we only need to check that $\operatorname{int}(\mathcal{T}) = \emptyset$. Suppose $u \in \operatorname{int}(\mathcal{T})$, and let $\varepsilon > 0$ such that $B(u, \varepsilon) \subset \mathcal{T}$. For $n \in \mathbb{N}$, let $v_{1/n}$ be the fuzzy numbers given by

$$v_{1/n}(t) = \begin{cases} 1, & -\dfrac{1}{n} < t < \dfrac{1}{n}, \\ 0, & \text{otherwise.} \end{cases}$$

The sequence $\left(u + v_{1/n}\right)$ converges to u, as $n \to \infty$, but $u + v_{1/n} \notin \mathcal{T}$, so that $B(u, \varepsilon) \not\subset \mathcal{T}$, whence we have a contradiction, and we conclude that $\mathcal{T}$ is nowhere dense in $\mathcal{F}$. □

Theorem 6. *$\mathbb{R}$ is nowhere dense in $\mathcal{T}$.*

Proof. First, note that $\mathbb{R}$ is closed in $\mathcal{T}$ since the restriction of D to $\mathbb{R}$ is the usual distance, so we need to check that $\mathrm{int}(\mathbb{R}) = \emptyset$. Assume $u \in \mathbb{R}$, and consider the sequence of triangular fuzzy numbers (u_n) given by

$$u_n(t) = \begin{cases} \dfrac{t-u+n}{n}, & u-n \le t < u, \\ \dfrac{u-t+n}{n}, & u < t \le u+n, \\ 0, & \text{otherwise.} \end{cases}$$

It is clear that (u_n) converges to u, as $n \to \infty$; therefore, we have $\mathrm{int}(\mathbb{R}) = \emptyset$, and hence $\mathbb{R}$ is nowhere dense in $\mathcal{T}$. □

As an immediate consequence of Theorems 5 and 6, we have the following result.

Corollary 7. *$\mathbb{R}$ is nowhere dense in $\mathcal{F}$.*

3.3. Quasilineability of subsets of sequences of fuzzy numbers

The partial order $\preceq$ on $\mathcal{F}$ is used to show that $(\mathcal{F}, \preceq, +, \cdot)$ is a qls —recall Theorem 2—, but it is not suitable for other purposes: for example, it does not allow to compare two real numbers with each other. However, it is common to use the order $\lesssim$ —i.e., for any pair of fuzzy numbers (u, v), $u \lesssim v$ denotes $u^-(\alpha) \le v^-(\alpha)$ and $u^+(\alpha) \le v^+(\alpha)$ for all $\alpha \in [0, 1]$. This order extends the usual order of the real numbers to $\mathcal{F}$. In certain contexts this extension behaves very differently from the usual order of the real numbers. Furthermore, it occurs on large sets, in the sense of quasilineability.

Next, we need to define an order on the set $w(\mathcal{F})$ and some operations that endow it with a quasilinear space structure (see [5] for more details). The addition $(+)$ and scalar multiplication $(\cdot)$ are defined on $\mathcal{F}$ by $(u_k) + (v_k) = (u_k + v_k)$ and $\lambda\,(u_k) = (\lambda u_k)$, for $(u_k), (v_k) \in w(\mathcal{F})$ and $\lambda \in \mathbb{R}$. The unit element of $w(\mathcal{F})$ with respect to addition is $\theta = (0)$. The partial ordering "$\ll$" on $w(\mathcal{F})$ is defined as follows: for $u = (v_k), v = (v_k) \in w(\mathcal{F})$, we have $u \ll v$ if, and only if, $u_k \preceq v_k$ for all $k \in \mathbb{N}$. Thus, we have the following result.

Theorem 8 *([24]). $(w(\mathcal{F}), \ll, +, \cdot)$ is a qls.*

We say that a sequence $(u_k) \in w(\mathcal{F})$ is monotonic decreasing if $u_{k+1} \preceq u_k$ for all $k \in \mathbb{N}$.

The next results show that the behavior of sequences of fuzzy numbers can be different from that of sequences in $\mathbb{R}$. Specifically, it is known that there are increasing and upper bounded sequences with respect to $\lesssim$ which are not convergent —an example can be found in [5, p. 322]. We may wonder if this example responds to an exceptional situation or has a large size from the point of view of quasilineability. Since when multiplying by a negative number the trend of a sequence is reversed, we must use positive numbers. Therefore, we must study the coneability of $w(\mathcal{F})$. Since the cardinality of $\mathcal{F}$ is $\mathfrak{c}$ —recall Theorem 3—, we have that the cardinality of $w(\mathcal{F})$ is $\mathfrak{c}$ too. Therefore, any cone or subspace of $w(\mathcal{F})$ has dimension less than or equal to $\mathfrak{c}$, being this dimension the maximum that could be obtained.

The following theorem shows that finding bounded and monotonic decreasing sequences that do not converge is not an exceptional situation from the point of view of quasilineability since the subset of $w(\mathcal{F})$ formed by these sequences is coneable and the cardinality of ql-independent sequences in that cone is the maximum possible, that is, $\mathfrak{c}$.

Theorem 9. *The set of monotonic decreasing, bounded with respect to $\lesssim$ and not convergent sequences of $\omega(\mathcal{F})$ is $\mathfrak{c}$-coneable.*

Proof. Given $\beta \in]0, 1[$, we define the real number $\beta_n := e^{-n\beta}$ and the fuzzy number

$$u_{\beta,n}(t) := \begin{cases} 1 - t^{(n\beta)^{-1}}, & 0 \le t \le 1, \\ 0, & \text{otherwise,} \end{cases}$$

and consider the sequence of fuzzy numbers $\beta_* = \left(\beta_n + u_{\beta,n}\right)$.

We check that the sequence β_* is monotonic decreasing. Since $\left[\beta_n + u_{\beta,n}\right]_\alpha = \left[\beta_n, \beta_n + (1-\alpha)^{n\beta}\right]$, we have $\beta_n > \beta_{n+1}$ and $\beta_n + (1-\alpha)^{n\beta} > \beta_{n+1} + (1-\alpha)^{(n+1)\beta}$ for all $n \in \mathbb{N}$, i.e., β_* is monotonic decreasing.

On the other hand, we have $\beta_n > 0$ for all $n \in \mathbb{N}$, whence the sequence β_* is bounded below by 0.

In order to prove that the sequence β_* is not convergent, we check that it is not a Cauchy sequence. Assume β_* is a Cauchy sequence, then given $0 < \varepsilon < 1/2$, there exists $n_\varepsilon := n(\varepsilon) \in \mathbb{N}$ such that $D\left[\beta_n + u_{\beta,n}, \beta_{n_\varepsilon} + u_{\beta,n_\varepsilon}\right] < \varepsilon$ for all $n > n_\varepsilon$. By taking $\alpha = 1/\sqrt{n}$, we have

$$\left[\beta_n + u_{\beta,n}\right]_{1/\sqrt{n}} = \left[e^{-n\beta}, e^{-n\beta} + \left(1 - \frac{1}{\sqrt{n}}\right)^{n\beta}\right],$$

$$\left[\beta_{n_\varepsilon} + u_{\beta,n_\varepsilon}\right]_{1/\sqrt{n}} = \left[e^{-n_\varepsilon\beta}, e^{-n_\varepsilon\beta} + \left(1 - \frac{1}{\sqrt{n}}\right)^{n_\varepsilon\beta}\right].$$

Note that

$$\max\left\{\left|e^{-n\beta} - e^{-n_\varepsilon\beta}\right|, \left|e^{-n\beta} + \left(1 - \frac{1}{\sqrt{n}}\right)^{n\beta} - e^{-n_\varepsilon\beta} - \left(1 - \frac{1}{\sqrt{n}}\right)^{n_\varepsilon\beta}\right|\right\}$$
$$= e^{-n_\varepsilon\beta} - e^{-n\beta} + \left(1 - \frac{1}{\sqrt{n}}\right)^{n_\varepsilon\beta} - \left(1 - \frac{1}{\sqrt{n}}\right)^{n\beta} > \left(1 - \frac{1}{\sqrt{n}}\right)^{n_\varepsilon\beta} - \left(1 - \frac{1}{\sqrt{n}}\right)^{n\beta},$$

and since $n_\varepsilon\beta$ is fixed, then

$$\left(1 - \frac{1}{\sqrt{n}}\right)^{n_\varepsilon\beta} \to 1 \quad \text{and} \quad \left(1 - \frac{1}{\sqrt{n}}\right)^{n\beta} = \left[\left(1 - \frac{1}{\sqrt{n}}\right)^{\sqrt{n}}\right]^{\sqrt{n}\beta} \to 0$$

as $n \to +\infty$, so that, from a certain value, we have

$$\left(1 - \frac{1}{\sqrt{n}}\right)^{n_\varepsilon\beta} - \left(1 - \frac{1}{\sqrt{n}}\right)^{n\beta} > z,$$

where z is any real number in $]0,1[$, in particular, $z = 1/2$; therefore, we have a contradiction, whence the sequence $\left(\beta_n + u_{\beta,n}\right)$ is not convergent.

Given m real numbers $\beta^{(i)} \in]0,1[$, $i = 1, \ldots, m$, we define the real numbers $\beta_n^{(i)} := e^{-n\beta^{(i)}}$ and the fuzzy numbers

$$u_{\beta^{(i)},n}(t) := \begin{cases} 1 - t^{\left(n\beta^{(i)}\right)^{-1}}, & 0 \le t \le 1, \\ 0, & \text{otherwise,} \end{cases}$$

and consider the sequence of fuzzy numbers $\beta_*^{(i)} = \left(\beta_n^{(i)} + u_{\beta^{(i)},n}\right)$ for $i = 1, \ldots, n$. We construct a combination of the form

$$\delta := \sum_{i=1}^{m} \rho_i \beta_*^{(i)}$$

with $\rho_i > 0$ for all $i = 1, \ldots, m$. We now prove that δ is a monotonic decreasing, bounded and not convergent sequence of fuzzy numbers.

The α-level set of δ_n is given by

$$[\delta_n]_\alpha = \left[\sum_{i=1}^{m} \rho_i \beta_n^{(i)}, \sum_{i=1}^{m} \rho_i \beta_n^{(i)} + \sum_{i=1}^{m} \rho_i\,(1-\alpha)^{n\beta^{(i)}}\right].$$

Since $\rho_i > 0$ and $\beta_n^{(i)} > \beta_{n+1}^{(i)}$ for every $i = 1, \ldots, m$ and for all $n \in \mathbb{N}$, we have

$$\sum_{i=1}^{m} \rho_i \beta_n^{(i)} > \sum_{i=1}^{n} \rho_i \beta_{n+1}^{(i)}$$

and

$$\sum_{i=1}^{m} \rho_i \beta_n^{(i)} + \sum_{i=1}^{m} \rho_i \,(1-\alpha)^{n\beta^{(i)}} > \sum_{i=1}^{m} \rho_i \beta_{n+1}^{(i)} + \sum_{i=1}^{m} \rho_i \,(1-\alpha)^{(n+1)\beta^{(i)}} .$$

This guarantees that the sequence δ is monotonic decreasing.

On the other hand, we have $\beta_n^{(i)} > 0$ for all $n \in \mathbb{N}$ and for every $i = 1, \ldots, m$, whence the sequences $\beta_*^{(i)}$ are bounded below by 0, and since $\rho_i > 0$, we have that δ is bounded below 0.

In order to prove that the sequence δ is not convergent, we check that it is not a Cauchy sequence. Assume δ is Cauchy sequence, then given $0 < \varepsilon < \rho_1/2$, there exists $n_\varepsilon := n(\varepsilon)$ such that $D\left[\delta_n, \delta_{n_\varepsilon}\right] < \varepsilon$ for all $n > n_\varepsilon$. By taking $\alpha = 1/\sqrt{n}$, we have

$$[\delta_n]_{1/\sqrt{n}} = \left[\sum_{i=1}^{m} \rho_i e^{-n\beta^{(i)}}, \sum_{i=1}^{m} \left(\rho_i e^{-n\beta^{(i)}} + \left(1-\frac{1}{\sqrt{n}}\right)^{n\beta^{(i)}}\right)\right],$$
$$\left[\delta_{n_\varepsilon}\right]_{1/\sqrt{n}} = \left[\sum_{i=1}^{m} \rho_i e^{-n_\varepsilon\beta^{(i)}}, \sum_{i=1}^{m} \left(\rho_i e^{-n_\varepsilon\beta^{(i)}} + \left(1-\frac{1}{\sqrt{n}}\right)^{n_\varepsilon\beta^{(i)}}\right)\right].$$

Since

$$\max\left\{\left|\sum_{i=1}^{m} \rho_i e^{-n\beta^{(i)}} - \sum_{i=1}^{m} \rho_i e^{-n_\varepsilon\beta^{(i)}}\right|,\right.$$
$$\left.\left|\sum_{i=1}^{m} \rho_i e^{-n\beta^{(i)}} + \left(1-\frac{1}{\sqrt{n}}\right)^{n\beta^{(i)}} - \sum_{i=1}^{m} \left(\rho_i e^{-n_\varepsilon\beta^{(i)}} + \left(1-\frac{1}{\sqrt{n}}\right)^{n_\varepsilon\beta^{(i)}}\right)\right|\right\}$$
$$= \sum_{i=1}^{m} \rho_i \left(e^{-n_\varepsilon\beta^{(i)}} - e^{-n\beta^{(i)}}\right) + \sum_{i=1}^{m} \rho_i \left(\left(1-\frac{1}{\sqrt{n}}\right)^{n_\varepsilon\beta^{(i)}} - \left(1-\frac{1}{\sqrt{n}}\right)^{n\beta^{(i)}}\right)$$
$$> \rho_1 \left(\left(1-\frac{1}{\sqrt{n}}\right)^{n_\varepsilon\beta^{(1)}} - \left(1-\frac{1}{\sqrt{n}}\right)^{n\beta^{(1)}}\right) > \frac{\rho_1}{2}$$

(in the last inequality we have applied a reasoning similar to that described above), which is a contradiction, whence the sequence δ is not convergent.

Now we check that the sequences $\beta_*^{(i)}$ are ql-independent.

Assume

$$\mathbf{0} \ll \delta = \sum_{i=1}^{m} \gamma_i \beta_*^{(i)},$$

with $\gamma_i \in \mathbb{R}$ for $i = 1, \ldots, m$, where $\mathbf{0}$ denotes the sequence with all its elements equal to 0. The fuzzy numbers $\beta_n^{(i)} + u_{\beta^{(i)},n}$ satisfy $\left[\beta_n^{(i)} + u_{\beta^{(i)},n}\right]_1 = \beta_n^{(i)}$, so that

$$[\delta_n]_1 = \sum_{i=1}^{m} \rho_i \beta_n^{(i)} = \sum_{i=1}^{m} \rho_i e^{-n\beta^{(i)}}.$$

In order to obtain $\mathbf{0} \ll \delta$, it would be necessary

$$\sum_{i=1}^{m} \rho_i e^{-n\beta^{(i)}} = 0 \quad \text{for all } n \in \mathbb{N}. \tag{1}$$

Suppose, without loss of generality, $\beta^{(1)} = \min\left\{\beta^{(1)}, \ldots, \beta^{(m)}\right\}$. When n tends to $+\infty$ we have

$$\sum_{i=1}^{m} \rho_i e^{-n\beta^{(i)}} > \frac{\rho_1 e^{-n\beta^{(1)}}}{2};$$

therefore, (1) cannot happen. We conclude that the sequences β_* are ql-independent, whence the result follows. □

Given a bounded sequence (x_n) of real numbers, the sequence $\left(\inf_{k\geq n} x_k\right)$ is a monotonic increasing. Therefore, there exists the limit $\lim_{n\to+\infty}\left(\inf_{k\geq n} x_k\right)$, known as the lower limit of (x_n) and noted by $\liminf x_n$. However, this is not the case for bounded sequences of fuzzy numbers. Specifically, we say that the bounded sequence $(u_n)\in w(\mathcal{F})$ has a lower limit, denoted by $\liminf u_n$, when the limit $\lim_{n\to+\infty}\left(\inf_{k\geq n} u_k\right)$ exists.

It is known that there are bounded sequences that have no lower limit —an example can be found in [5, p. 326]. Again, we wonder if this example responds to an exceptional situation or has a large size from the point of view of quasilineability. Unlike the case studied earlier, now we do not have to limit ourselves to positive numbers and we can study quasilinearability. The following theorem shows again that the best possible result is obtained and that from the quasilianeability the set is large and the result is optima, that is, quasilineability is obtained and the maximum dimension is $\mathfrak{c}$. Furthermore, the set of bounded sequences of fuzzy numbers that do not have a lower limit is “large” from the point of view of quasilineability.

Theorem 10. *The set of bounded sequences of $\omega(\mathcal{F})$ for which the lower limit does not exist is $\mathfrak{c}$-quasilineable.*

Proof. Given $\beta\in]0,1[$, we define the sequence $\beta_* = \left(u_{\beta,n}\right)$, with

$$u_{\beta,n}(t) := \begin{cases} e^{\beta|t|/n}, & -1\leq t\leq 1,\\ 0, & \text{otherwise.}\end{cases}$$

Define $\beta_{**} = \left(u_{\beta,n}+e^{-\beta n}\right)$. We now check that $\liminf\beta_{**}$ does not exist. We have

$$u^*_{\beta,n}(t) = \inf_{k\geq n} u_{\beta,k}(t) + e^{-\beta n} = \begin{cases} 1, & -1\leq t\leq e^{-\beta n},\\ e^{\beta(t-e^{-\beta n})/n}, & e^{-\beta n}<t\leq 1+e^{-\beta n},\\ 0, & \text{otherwise.}\end{cases}$$

If the sequence $\left(u^*_{\beta,n}\right)_n$ is convergent, then it must be a Cauchy sequence. Consider

$$\left[u^*_{\beta,n}\right]_{1-1/n^2} = \left[-1, e^{-n\beta}+\frac{n\ln\left(1-n^{-2}\right)}{\beta}\right].$$

Note that, for m big enough, we have $\left[u^*_{\beta,m}\right]_{1-1/n^2}\supset[-1,1]$. Thus, the Hausdorff distance between $\left[u^*_{\beta,n}\right]_{1-1/n^2}$ and $\left[u^*_{\beta,m}\right]_{1-1/n^2}$ satisfies

$$D\left(\left[u^*_{\beta,n}\right]_{1-1/n^2},\left[u^*_{\beta,m}\right]_{1-1/n^2}\right)\geq 1-e^{-n\beta}-\frac{n\ln\left(1-n^{-2}\right)}{\beta}>\frac{1}{2}.$$

Therefore, $\left(u^*_{\beta,n}\right)_n$ is not a Cauchy sequence and is not convergent, that is, the sequences β_{**} do not have lower limit.

Now we check that the sequences β_{**}, with $\beta\in]0,1[$, are ql-independent.

Assume

$$\mathbf{0}\ll\delta=\sum_{i=1}^m \rho_i\beta^{(i)}_{**},$$

with $\rho_i\in\mathbb{R}$. The fuzzy numbers $\beta^{(i)}_{**n}$ satisfy $\left[\beta^{(i)}_{**n}\right]_1 = e^{-\beta n}$, so that

$$[\delta_n]_1 = \sum_{i=1}^m \rho_i e^{-\beta^{(i)}n}.$$

To have $\mathbf{0} \ll \delta$, it would be necessary

$$\sum_{i=1}^{m} \rho_i e^{-\beta^{(i)} n} = 0 \quad \text{for all } n \in \mathbb{N}. \tag{2}$$

Suppose, without loss of generality, $\beta^{(1)} = \min\left\{\beta^{(1)}, \ldots, \beta^{(m)}\right\}$. When n tends to $+\infty$, we have

$$\sum_{i=1}^{m} \rho_i e^{-n\beta^{(i)}} > \frac{\rho_1 e^{-n\beta^{(1)}}}{2},$$

so that (2) cannot happen; therefore, the sequences β_{**} are ql-independent.

Finally, we check that the limit

$$\liminf \sum_{i=1}^{m} \rho_i \beta_{**}^{(i)}$$

does not exist. Note

$$\rho\beta_{**}(t) = \begin{cases} e^{\frac{\beta\left|t-\rho e^{-\beta n}\right|}{n}}, & t \in \left[-|\rho| + \rho e^{-\beta n}, |\rho| + \rho e^{-\beta n}\right], \\ 0, & \text{otherwise.} \end{cases}$$

Then we have

$$\left[\sum_{i=1}^{m} \rho_i \beta_{**}^{(i)}\right]_{\frac{n^2-1}{n^2}} = \left[\sum_{i=1}^{m} \frac{n \ln \frac{n^2-1}{n^2}}{\beta^{(i)}} + \rho_i e^{-n\beta^{(i)}}, \sum_{i=1}^{m} \frac{-n \ln \frac{n^2-1}{n^2}}{\beta^{(i)}} + \rho_i e^{-n\beta^{(i)}}\right]$$

and

$$\left[\inf_{k\geq n} \sum_{i=1}^{m} \rho_i \beta_{**,k}^{(i)}(t)\right]_{\frac{n^2-1}{n^2}} = \left[-\sum_{i=1}^{m} |\rho_i|, \sum_{i=1}^{m} \frac{-n \ln \frac{n^2-1}{n^2}}{\beta^{(i)}} + \rho_i e^{-n\beta^{(i)}}\right]$$

If the limit

$$\lim \left(\inf_{k\geq n} \sum_{i=1}^{m} \rho_i \beta_{**,k}^{(i)}(t)\right)$$

exists, then the sequence must be a Cauchy sequence, and from certain n_ε, we have that the Hausdorff distance of

$$\left[-\sum_{i=1}^{m} |\rho_i|, \sum_{i=1}^{m} \frac{-n \ln \frac{n^2-1}{n^2}}{\beta^{(i)}} + \rho_i e^{-n\beta^{(i)}}\right]$$

and

$$\left[\inf_{k\geq r} \sum_{i=1}^{m} \rho_i \beta_{**,k}^{(i)}(t)\right]_{\frac{n^2-1}{n^2}}$$

must be less than ε. Since for r big enough, we have

$$\left[\inf_{k\geq r} \sum_{i=1}^{m} \rho_i \beta_{**,k}^{(i)}(t)\right]_{\frac{n^2-1}{n^2}} \supset \left[-\sum_{i=1}^{m} |\rho_i|, \sum_{i=1}^{m} |\rho_i|\right],$$

the distance is greater than $\sum_{i=1}^{m} |\rho_i|/2$, so that it cannot be a Cauchy sequence, whence the result easily follows. □

4. Conclusions

In this paper, we have checked the cardinality of fuzzy numbers, proved that the set of the triangular fuzzy numbers is a quasilinear space, showed that this set is nowhere dense in the set of all fuzzy numbers on $\mathbb{R}$, and provided several topological properties. Moreover, we have studied the set of sequences of bounded fuzzy numbers that have no lower limit and that of fuzzy numbers monotonic decreasing, bounded with respect a partial ordering and not convergent. It is known that these two subsets of $w(\mathcal{F})$ are not empty and there are some examples of elements of these subsets in the literature —e.g., we refer to [5] and the references therein. We have showed that, from the point of quasilineability, these are not exceptional situations, obtaining optimal results with maximum dimensions which, in both cases, is $\mathfrak{c}$.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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