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Differential operator for discrete Gegenbauer–Sobolev orthogonal polynomials: eigenvalues and asymptotics

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Abstract

We consider the following discrete Sobolev inner product involving the Gegenbauer weight

$$(f, g)_S := \int_{-1}^1 f(x)g(x)(1-x^2)^\alpha dx + M[f^{(j)}(-1)g^{(j)}(-1) + f^{(j)}(1)g^{(j)}(1)],$$

where $\alpha > -1$, $j \in \mathbb{N} \cup \{0\}$, and $M > 0$. Our main objective is to calculate the exact value

$$r_0 = \lim_{n \rightarrow \infty} \frac{\log \left(\max_{x \in [-1,1]} |\tilde{Q}_n^{(\alpha, M, j)}(x)| \right)}{\log \tilde{\lambda}_n}, \quad \alpha \geq -1/2,$$

where $\{\tilde{Q}_n^{(\alpha, M, j)}\}_{n \geq 0}$ is the sequence of orthonormal polynomials with respect to this Sobolev inner product. These polynomials are eigenfunctions of a differential operator and the obtaining of the asymptotic behavior of the corresponding eigenvalues, $\tilde{\lambda}_n$, is the principal key to get the result. This value r_0 is related to the convergence of a series in a left-definite space. In addition, to complete the asymptotic study of this family of nonstandard polynomials we give the Mehler–Heine formulae for the corresponding orthogonal polynomials.

Keywords: Sobolev orthogonality · differential operators · asymptotics.

Mathematics Subject Classification (2010): 33C47 · 42C05

1 Introduction

In the framework of Sobolev orthogonality, we consider the nonstandard inner product

$$(f, g)_S := \int_{-1}^1 f(x)g(x)(1-x^2)^\alpha dx + M[f^{(j)}(-1)g^{(j)}(-1) + f^{(j)}(1)g^{(j)}(1)], \quad (1)$$

where $\alpha > -1$, $j \in \mathbb{N} \cup \{0\}$, and $M > 0$. It is usually known as a Gegenbauer–Sobolev inner product because its absolutely continuous part involves the classical Gegenbauer weight. We denote by $\{Q_n^{(\alpha, M, j)}\}_{n \geq 0}$ the sequence of orthogonal polynomials with respect to (1). Along this paper we also use the Sobolev orthonormal polynomials, denoted by $\{\tilde{Q}_n^{(\alpha, M, j)}\}_{n \geq 0} = \left\{ \frac{Q_n^{(\alpha, M, j)}}{\sqrt{(Q_n^{(\alpha, M, j)}, Q_n^{(\alpha, M, j)})_S}} \right\}_{n \geq 0}$.

Consider a sequence of polynomials, $\{P_n\}_{n \geq 0}$, orthogonal with respect to a symmetric inner-product ϕ that also satisfy a (possibly infinite order) spectral differential equation. In [10] the authors give conditions for polynomials orthogonal with respect to a related discrete Sobolev inner product of the form $(f, g)_S = \phi(f, g) + Mf^{(j)}(c)g^{(j)}(c)$ to also satisfy a (possibly infinite order) spectral differential equation. H. Bavinck in [6] extended this result to polynomials $\{Q_n\}_{n \geq 0}$ orthogonal with respect to discrete Sobolev inner products of the form $(f, g)_S = \phi(f, g) + Mf^{(j)}(c_1)g^{(j)}(c_1) + Nf^{(k)}(c_2)g^{(k)}(c_2)$. Specifically Bavinck constructs a differential equation ℓ and eigenvalues $\tilde{\lambda}_n$ such that

$$\ell[Q_n](x) = \sum_{i=1}^{\infty} a_i(x)Q_n^{(i)}(x) = \tilde{\lambda}_n Q_n(x).$$

Central to the construction of this differential equation is the reproducing polynomial kernel $K_n(x, y) := \sum_{i=0}^n \frac{Q_i^{(\alpha, M, j)}(x)Q_i^{(\alpha, M, j)}(y)}{(Q_i^{(\alpha, M, j)}, Q_i^{(\alpha, M, j)})_S}$.

Here we consider the Gegenbauer–Sobolev polynomials $Q_n^{(\alpha, M, j)}$ orthogonal with respect to the discrete Sobolev–type inner product (1). We are interested in properties of the related polynomial kernel scaled by the eigenvalues:

$$K(x, y; r) = \sum_{i=0}^{\infty} \tilde{\lambda}_i^{-r} \frac{Q_i^{(\alpha, M, j)}(x)Q_i^{(\alpha, M, j)}(y)}{(Q_i^{(\alpha, M, j)}, Q_i^{(\alpha, M, j)})_S}.$$

They are useful for some applications that two of the authors [12] are developing. Since that work is unfinished we only give a brief motivation without details. The results on this paper are self-contained and do not depend on this motivation.

Let $(H, \langle \cdot, \cdot \rangle_H)$ be the completion space of polynomials under the inner product $(f, g)_S$. Let \mathbf{T} be the self-adjoint operator in H generated by the differential expression $\mathbf{L} + M\mathbf{A}$, where \mathbf{L} is the linear differential operator associated with the Gegenbauer polynomials and \mathbf{A} is an operator that we will define later. \mathbf{T} exists as an unbounded operator in H since $\tilde{\lambda}_n \rightarrow \infty$. In this case the *left-definite* space $H_r(\mathbf{T})$ with inner-product

$$\langle f, g \rangle_r := \langle \mathbf{T}^r f, g \rangle_H$$

on the linear manifold $\mathcal{D}(\mathbf{T}^{r/2})$ yields a Hilbert space. Furthermore we may take the power \mathbf{T}^r as a self-adjoint operator in a left-definite space (see [11] for details).

Now take r_0 to be the least number such that for each $r > r_0$ the kernel $K(x, y; r)$ converges both absolutely and in the left-definite space $H_r(\mathbf{T})$. Then for $r > r_0$ the sequence $\left\{ \frac{Q_n^{(\alpha, M, j)}(x)}{\sqrt{\tilde{\lambda}_n(Q_n^{(\alpha, M, j)}, Q_n^{(\alpha, M, j)})_S}} \right\}_{n \geq 0}$ forms a complete, orthonormal basis for $H_r(\mathbf{T})$ and the reproducing property follows from the Parseval identity:

$$\langle \mathbf{T}^r K(x, \cdot; r_0^+ + r), f \rangle_{H_{r_0^+}} = \langle K(x, \cdot; r_0^+), f \rangle_{H_{r_0^+}} = f(x).$$

Notice this gives $K(x, y; r)$ as the reproducing kernel for the $r - r_0^+$ left-definite space of the left-definite operator acting in the r_0^+ left-definite space generated by \mathbf{T} .

It can be shown [12] that

$$r_0 = \lim_{n \rightarrow \infty} \frac{\log \left(\sup_{x \in [-1, 1]} |\tilde{Q}_n^{(\alpha, M, j)}(x)| \right)}{\log \tilde{\lambda}_n}. \quad (2)$$

In this paper we find a value for r_0 valid for $\alpha \geq -1/2$, $M > 0$ and $j \in \mathbb{N}$. The case $\alpha \in (-1, -1/2)$ remains as an open problem since our technique has not worked.

In addition, to complete the study of the polynomials $Q_n^{(\alpha, M, j)}$ we focus our attention on the Mehler–Heine asymptotics. These Mehler–Heine type formulae are interesting twofold: they provide the scaled asymptotics for $Q_n^{(\alpha, M, j)}$ on compact sets of the complex plane and they supply us with asymptotic information about the location of the zeros of these polynomials in terms of the zeros of other known special functions. This result is independent of the main result obtained in Section 4 but it can be deduced straightforwardly from the results in Section 3 and moreover it has its own interest.

The structure of the paper is the following: in Section 2 we give a background about Gegenbauer orthogonal polynomials, $C_n^{(\alpha)}$, introducing the properties of these polynomials that will be used along the paper. In Section 3 we establish connection formulae between the polynomials $C_n^{(\alpha)}$ and $Q_n^{(\alpha, M, j)}$ which are useful to state an upper bound for $\|Q_n^{(\alpha, M, j)}\|_\infty = \max_{x \in [-1, 1]} |Q_n^{(\alpha, M, j)}(x)|$. The results in this section are the keys to obtain the new results in the next sections. In Section 4 we tackle our main objective, so we calculate the exact value of r_0 in Theorem 2. For this purpose it is essential to obtain the asymptotic behavior of the eigenvalues associated with the linear differential operator $\mathbf{T} = \mathbf{L} + M\mathbf{A}$, i.e.

$$\mathbf{T}Q_n^{(\alpha, M, j)}(x) = (\mathbf{L} + M\mathbf{A})Q_n^{(\alpha, M, j)}(x) = \tilde{\lambda}_n Q_n^{(\alpha, M, j)}(x).$$

As far as we know it is the first time in the framework of Sobolev orthogonality that the asymptotic behavior of the eigenvalues is studied. Finally, as we have commented previously, in Section 5 we complete the study of this family of Sobolev polynomials $Q_n^{(\alpha, M, j)}$ establishing the corresponding Mehler–Heine type asymptotics and giving additional information about the asymptotic behavior of the zeros of these polynomials.

Along the text we will use the following notation: if a_n and b_n are two sequences of real numbers, then $a_n \approx b_n$ means $\lim_{n \rightarrow +\infty} a_n/b_n = 1$.

2 Gegenbauer orthogonal polynomials: a background

In [20] Gegenbauer polynomials are considered as those polynomials which are orthogonal with respect to the inner product

$$(f, g) = \int_{-1}^1 f(x)g(x)(1-x^2)^{\lambda-1/2} dx, \quad \lambda > -1/2.$$

We denote these polynomials by $P_n^{(\lambda)}$ with the normalization convention

$$P_n^{(\lambda)}(1) = \frac{\Gamma(n+2\lambda)}{\Gamma(n+1)\Gamma(2\lambda)}.$$

We take $\alpha := \lambda - 1/2$ and consider the polynomials $\frac{P_n^{(\lambda)}(x)}{P_n^{(\lambda)}(1)}$. We denote by $\{C_n^{(\alpha)}\}_{n \geq 0}$ this sequence of Gegenbauer orthogonal polynomials (this normalization is also used in [9]). In this way, it is obvious that the polynomials $C_n^{(\alpha)}$ are orthogonal with respect to

$$(f, g)_\alpha := \int_{-1}^1 f(x)g(x)(1-x^2)^\alpha dx,$$

with $C_n^{(\alpha)}(1) = 1$, and using the symmetry of these polynomials we have $C_n^{(\alpha)}(-1) = (-1)^n$. So, the inner product (1) can be rewritten as

$$(f, g)_S = (f, g)_\alpha + M[f^{(j)}(-1)g^{(j)}(-1) + f^{(j)}(1)g^{(j)}(1)].$$

Now, we recall some properties of Gegenbauer orthogonal polynomials. These properties can be found in [9] or [20] among others.

- Derivatives:

$$\left(C_n^{(\alpha)}\right)^{(k)}(x) := \frac{d^k C_n^{(\alpha)}(x)}{dx^k} = \frac{(-1)^k (n+2\alpha+1)_k (-n)_k}{2^k (\alpha+1)_k} C_{n-k}^{(\alpha+k)}(x), \quad (3)$$

with $k = 0, 1, \dots$, where $(a)_k$ denotes the Pochhammer's symbol, i.e., $(a)_k = a(a+1) \cdots (a+k-1) = \frac{\Gamma(a+k)}{\Gamma(a)}$, $k \geq 1$, $(a)_0 = 1$.

- Differential equation and eigenvalues:

$$(x^2 - 1) \left(C_n^{(\alpha)} \right)''(x) + 2(\alpha + 1)x \left(C_n^{(\alpha)} \right)'(x) = \lambda_n C_n^{(\alpha)}(x),$$

$$\lambda_n = n(n + 2\alpha + 1). \quad (4)$$

- Leading coefficient:

$$k_n(\alpha) := \frac{(n + 2\alpha + 1)_n}{2^n (\alpha + 1)_n} = \frac{\Gamma(2n + 2\alpha + 1)\Gamma(\alpha + 1)}{2^n \Gamma(n + \alpha + 1)\Gamma(n + 2\alpha + 1)}. \quad (5)$$

- Squared norm:

$$\|C_n^{(\alpha)}\|_\alpha^2 := \int_{-1}^1 (C_n^{(\alpha)}(x))^2 (1 - x^2)^\alpha dx = \frac{2^{2\alpha+1}\Gamma^2(\alpha + 1)\Gamma(n + 1)}{(2n + 2\alpha + 1)\Gamma(n + 2\alpha + 1)}. \quad (6)$$

Next we compute certain limits to be used later. To do this, we take into account (see, for example, [4, f. (5.11.13)] or [13, f. (7)])

$$\lim_{n \rightarrow +\infty} \frac{n^{b-a}\Gamma(n + a)}{\Gamma(n + b)} = 1. \quad (7)$$

In the next lemma we provide some useful asymptotic behaviors of Gegenbauer polynomials.

Lemma 1 For $k \in \mathbb{N} \cup \{0\}$, we have

$$\lim_{n \rightarrow +\infty} \frac{\left(C_n^{(\alpha)} \right)^{(k)}(1)}{n^{2k}} = \frac{1}{2^k (\alpha + 1)_k}. \quad (8)$$

Furthermore,

$$\lim_{n \rightarrow +\infty} \|C_n^{(\alpha)}\|_\alpha^2 n^{2\alpha+1} = 2^{2\alpha}\Gamma^2(\alpha + 1). \quad (9)$$

Proof: Using (3), (7) and the fact that $C_n^\alpha(1) = 1$, we obtain

$$\begin{aligned} \lim_{n \rightarrow +\infty} \frac{\left(C_n^{(\alpha)} \right)^{(k)}(1)}{n^{2k}} &= \frac{(-1)^k}{2^k (\alpha + 1)_k} \lim_{n \rightarrow \infty} \frac{(n + 2\alpha + 1)_k (-n)_k}{n^{2k}} \\ &= \frac{(-1)^k}{2^k (\alpha + 1)_k} (-1)^k = \frac{1}{2^k (\alpha + 1)_k}. \end{aligned}$$

Formula (9) is deduced in a straightforward way from (6) using (7). \square

We will use the following notation:

$$K_n^{(j,k)}(x,y) = \sum_{i=0}^n \frac{\left(C_i^{(\alpha)}\right)^{(j)}(x) \left(C_i^{(\alpha)}\right)^{(k)}(y)}{\|C_i^{(\alpha)}\|_\alpha^2},$$

$$\kappa_{2n}^{(j,k)}(x,y) = \sum_{i=0}^n \frac{\left(C_{2i}^{(\alpha)}\right)^{(j)}(x) \left(C_{2i}^{(\alpha)}\right)^{(k)}(y)}{\|C_{2i}^{(\alpha)}\|_\alpha^2}, \quad (10)$$

$$\tilde{\kappa}_{2n}^{(j,k)}(x,y) = \sum_{i=0}^n \frac{\left(C_{2i+1}^{(\alpha)}\right)^{(j)}(x) \left(C_{2i+1}^{(\alpha)}\right)^{(k)}(y)}{\|C_{2i+1}^{(\alpha)}\|_\alpha^2}. \quad (11)$$

Notice that $K_n^{(0,0)}(x,y) = K_n(x,y)$ are the usual kernel polynomials associated with Gegenbauer polynomials.

Proposition 1 *Let k and s be nonnegative integer numbers. Then,*

$$\lim_{n \rightarrow +\infty} \frac{K_{n-1}^{(k,s)}(1,1)}{n^{2k+2s+2\alpha+2}} = \frac{1}{2^{2\alpha+k+s+1}} C_{k,s}, \quad (12)$$

$$\lim_{n \rightarrow +\infty} \frac{\kappa_{2(n-1)}^{(k,s)}(1,1)}{n^{2k+2s+2\alpha+2}} = \lim_{n \rightarrow +\infty} \frac{\tilde{\kappa}_{2(n-1)}^{(k,s)}(1,1)}{n^{2k+2s+2\alpha+2}} = 2^{k+s} C_{k,s}, \quad (13)$$

where

$$C_{k,s} = \frac{1}{(k+s+\alpha+1)\Gamma(\alpha+k+1)\Gamma(\alpha+s+1)}.$$

Proof: First, we observe that

$$n^{2\alpha+2k+2s+2} - (n-1)^{2\alpha+2k+2s+2} \approx (2\alpha+2k+2s+2)n^{2\alpha+2k+2s+1}.$$

Using Stolz's criterion, (8), and (9) we get

$$\begin{aligned} \lim_{n \rightarrow +\infty} \frac{K_{n-1}^{(k,s)}(1,1)}{n^{2k+2s+2\alpha+2}} &= \lim_{n \rightarrow +\infty} \frac{K_{n-1}^{(k,s)}(1,1) - K_{n-2}^{(k,s)}(1,1)}{n^{2k+2s+2\alpha+2} - (n-1)^{2k+2s+2\alpha+2}} \\ &= \lim_{n \rightarrow +\infty} \frac{\frac{\left(C_{n-1}^{(\alpha)}\right)^{(k)}(1) \left(C_{n-1}^{(\alpha)}\right)^{(s)}(1)}{\|C_{n-1}^{(\alpha)}\|_\alpha^2}}{2(\alpha+k+s+1)n^{2\alpha+2k+2s+1}} \\ &= \frac{1}{2(\alpha+k+s+1)} \lim_{n \rightarrow +\infty} \frac{\left(C_{n-1}^{(\alpha)}\right)^{(k)}(1)}{n^{2k}} \frac{\left(C_{n-1}^{(\alpha)}\right)^{(s)}(1)}{n^{2s}} \frac{1}{\|C_{n-1}^{(\alpha)}\|_\alpha^2 n^{2\alpha+1}} \\ &= \frac{1}{2(\alpha+k+s+1)} \frac{1}{2^k(\alpha+1)_k} \frac{1}{2^s(\alpha+1)_s} \frac{1}{2^{2\alpha}\Gamma^2(\alpha+1)}. \end{aligned}$$

Finally, using $(\alpha + 1)_k = \frac{\Gamma(\alpha + k + 1)}{\Gamma(\alpha + 1)}$ we obtain (12). To establish (13) we can proceed in the same way. \square

All the results introduced in this section about the well-known family of Gegenbauer orthogonal polynomials are necessary and used in the next sections.

3 Connection formulae and some asymptotic behaviors

It is well known that $\{C_i^{(\alpha)}\}_{i=0}^m$ constitute a basis of the linear space $\mathbb{P}_m[x]$ of polynomials with real coefficients and degree at most m . Therefore, the Gegenbauer–Sobolev polynomials orthogonal with respect to (1), with leading coefficient $k_n(\alpha)$ given in (5), can be expressed as

$$\begin{aligned} Q_{2n}^{(\alpha, M, j)}(x) &= C_{2n}^{(\alpha)}(x) + \sum_{i=0}^{n-1} a_{2n, 2i} C_{2i}^{(\alpha)}(x), \\ Q_{2n+1}^{(\alpha, M, j)}(x) &= C_{2n+1}^{(\alpha)}(x) + \sum_{i=0}^{n-1} a_{2n+1, 2i+1} C_{2i+1}^{(\alpha)}(x). \end{aligned}$$

Thus, applying a well-established procedure (see, for example, [16, Sect. 2] among others), we deduce the following connection formulae.

Proposition 2 *We have,*

$$Q_{2n}^{(\alpha, M, j)}(x) = C_{2n}^{(\alpha)}(x) - \frac{2M \left(C_{2n}^{(\alpha)}\right)^{(j)}(1) \kappa_{2(n-1)}^{(j, 0)}(1, x)}{1 + 2M \kappa_{2(n-1)}^{(j, j)}(1, 1)}, \quad (14)$$

$$Q_{2n+1}^{(\alpha, M, j)}(x) = C_{2n+1}^{(\alpha)}(x) - \frac{2M \left(C_{2n+1}^{(\alpha)}\right)^{(j)}(1) \tilde{\kappa}_{2(n-1)}^{(j, 0)}(1, x)}{1 + 2M \tilde{\kappa}_{2(n-1)}^{(j, j)}(1, 1)}. \quad (15)$$

Proof: For $i = 0, \dots, n-1$ fixed, we have

$$\begin{aligned} 0 &= (Q_{2n}^{(\alpha, M, j)}(x), C_{2i}^{(\alpha)}(x))_S = \left(C_{2n}^{(\alpha)}(x) + \sum_{k=0}^{n-1} a_{2n, 2k} C_{2k}^{(\alpha)}(x), C_{2i}^{(\alpha)}(x) \right)_S \\ &= \left(C_{2n}^{(\alpha)}(x), C_{2i}^{(\alpha)}(x) \right)_\alpha + \sum_{k=0}^{n-1} a_{2n, 2k} \left(C_{2k}^{(\alpha)}(x), C_{2i}^{(\alpha)}(x) \right)_\alpha \\ &+ M \left[\left(Q_{2n}^{(\alpha, M, j)} \right)^{(j)}(-1) \left(C_{2i}^{(\alpha)} \right)^{(j)}(-1) + \left(Q_{2n}^{(\alpha, M, j)} \right)^{(j)}(1) \left(C_{2i}^{(\alpha)} \right)^{(j)}(1) \right] \\ &= a_{2n, 2i} \|C_{2i}^{(\alpha)}\|_\alpha^2 + 2M \left(Q_{2n}^{(\alpha, M, j)} \right)^{(j)}(1) \left(C_{2i}^{(\alpha)} \right)^{(j)}(1), \end{aligned}$$

thus,

$$a_{2n,2i} = \frac{-2M \left(Q_{2n}^{(\alpha,M,j)}\right)^{(j)}(1) \left(C_{2i}^{(\alpha)}\right)^{(j)}(1)}{\|C_{2i}^{(\alpha)}\|_{\alpha}^2},$$

and

$$\begin{aligned} Q_{2n}^{(\alpha,M,j)}(x) &= C_{2n}^{(\alpha)}(x) + \sum_{i=0}^{n-1} \frac{-2M \left(Q_{2n}^{(\alpha,M,j)}\right)^{(j)}(1) \left(C_{2i}^{(\alpha)}\right)^{(j)}(1)}{\|C_{2i}^{(\alpha)}\|_{\alpha}^2} C_{2i}^{(\alpha)}(x) \\ &= C_{2n}^{(\alpha)}(x) - 2M \left(Q_{2n}^{(\alpha,M,j)}\right)^{(j)}(1) \sum_{i=0}^{n-1} \frac{\left(C_{2i}^{(\alpha)}\right)^{(j)}(1) C_{2i}^{(\alpha)}(x)}{\|C_{2i}^{(\alpha)}\|_{\alpha}^2} \\ &= C_{2n}^{(\alpha)}(x) - 2M \left(Q_{2n}^{(\alpha,M,j)}\right)^{(j)}(1) \kappa_{2(n-1)}^{(j,0)}(1, x). \end{aligned}$$

Differentiating the above expression j times and evaluating at $x = 1$ yields

$$\left(Q_{2n}^{(\alpha,M,j)}\right)^{(j)}(1) = \frac{\left(C_{2n}^{(\alpha)}\right)^{(j)}(1)}{1 + 2M \kappa_{2(n-1)}^{(j,j)}(1, 1)}, \quad (16)$$

which proves (14). For the odd case, relation (15) is established in the same way. \square

Proposition 2 is very useful to obtain the following relative asymptotics at the point $x = 1$.

Proposition 3 *Let k be a nonnegative integer. Then, we have*

$$\lim_{n \rightarrow +\infty} \frac{\left(Q_n^{(\alpha,M,j)}\right)^{(k)}(1)}{\left(C_n^{(\alpha)}\right)^{(k)}(1)} = \frac{k - j}{j + k + \alpha + 1}.$$

Proof: We only prove the even case since the proof for the odd case is similar. We differentiate the expression (14) k times and evaluate at $x = 1$. Then, we divide by $\left(C_{2n}^{(\alpha)}\right)^{(k)}(1)$ and use the

limit relations (8) and (13) to obtain

$$\begin{aligned}
\lim_{n \rightarrow +\infty} \frac{\left(Q_{2n}^{(\alpha, M, j)}\right)^{(k)}(1)}{\left(C_{2n}^{(\alpha)}\right)^{(k)}(1)} &= 1 - \lim_{n \rightarrow +\infty} \frac{2M \left(C_{2n}^{(\alpha)}\right)^{(j)}(1) \kappa_{2(n-1)}^{(j, k)}(1, 1)}{\left(1 + 2M \kappa_{2(n-1)}^{(j, j)}(1, 1)\right) \left(C_{2n}^{(\alpha)}\right)^{(k)}(1)} \\
&= 1 - \lim_{n \rightarrow +\infty} \frac{2M \frac{\left(C_{2n}^{(\alpha)}\right)^{(j)}(1) \kappa_{2(n-1)}^{(j, k)}(1, 1)}{n^{2j}}}{\frac{\left(C_{2n}^{(\alpha)}\right)^{(k)}(1)}{n^{4j+2k+2\alpha+2}} + 2M \frac{\kappa_{2(n-1)}^{(j, j)}(1, 1)}{n^{4j+2\alpha+2}} \frac{\left(C_{2n}^{(\alpha)}\right)^{(k)}(1)}{n^{2k}}} \\
&= 1 - \frac{(2j + \alpha + 1)(\alpha + 1)_k \Gamma(\alpha + j + 1)}{(\alpha + 1)_j (j + k + \alpha + 1) \Gamma(\alpha + k + 1)} \\
&= 1 - \frac{2j + \alpha + 1}{j + k + \alpha + 1} = \frac{k - j}{j + k + \alpha + 1}. \quad \square
\end{aligned}$$

In the following proposition we show that the norm of the Gegenbauer–Sobolev orthogonal polynomials, induced by the nonstandard inner product (1), behaves like the norm of classical Gegenbauer polynomials.

Proposition 4 *We have,*

$$\lim_{n \rightarrow \infty} \frac{\|Q_n^{(\alpha, M, j)}\|_S}{\|C_n^{(\alpha)}\|_\alpha} = 1.$$

Proof: Again, we only prove the even case.

$$\left(Q_{2n}^{(\alpha, M, j)}, Q_{2n}^{(\alpha, M, j)}\right)_S = \left(Q_{2n}^{(\alpha, M, j)}, C_{2n}^{(\alpha)}\right)_S = \|C_{2n}^{(\alpha)}\|_\alpha^2 + 2M \left(Q_{2n}^{(\alpha, M, j)}\right)^{(j)}(1) \left(C_{2n}^{(\alpha)}\right)^{(j)}(1).$$

Then, applying (16) we get

$$\begin{aligned}
\frac{\|Q_{2n}^{(\alpha, M, j)}\|_S^2}{\|C_{2n}^{(\alpha)}\|_\alpha^2} &= 1 + \frac{2M \left(Q_{2n}^{(\alpha, M, j)}\right)^{(j)}(1) \left(C_{2n}^{(\alpha)}\right)^{(j)}(1)}{\|C_{2n}^{(\alpha)}\|_\alpha^2} \\
&= 1 + \frac{2M \left(\left(C_{2n}^{(\alpha)}\right)^{(j)}(1)\right)^2}{\left(1 + 2M \kappa_{2(n-1)}^{(j, j)}(1, 1)\right) \|C_{2n}^{(\alpha)}\|_\alpha^2} \\
&= 1 + \frac{2M \frac{\left(\left(C_{2n}^{(\alpha)}\right)^{(j)}(1)\right)^2}{n^{4j}}}{\frac{1 + 2M \kappa_{2(n-1)}^{(j, j)}(1, 1)}{n^{4j+2\alpha+2}} \|C_{2n}^{(\alpha)}\|_\alpha^2 n^{2\alpha+1}}.
\end{aligned}$$

It is enough to observe that taking limits in the above expression, and applying (8)–(13), we get the result. \square

The number of terms of the connection formula given in Proposition 2 depends on n , so this number increases when n grows. To avoid this, we can give another *connection* formula in which the polynomials $Q_n^{(\alpha, M, j)}$ can be expressed as a finite linear combination of polynomials not depending on n .

Proposition 5 *There exists a family of real numbers $\{\gamma_{n,i}\}_{i=0}^{j+1}$, not identically zero, such that the following connection formula holds*

$$Q_n^{(\alpha, M, j)}(x) = \sum_{i=0}^{j+1} \gamma_{n,i} (1-x^2)^i \left(C_{n-i}^{(\alpha+i)} \right)^{(i)}(x), \quad n \geq 2j+2. \quad (17)$$

Proof: An analogous result was established in a similar framework in [19, Th. 1], although in that paper the discrete part of the Sobolev inner product is located at only one point c . The proof in this context is essentially the same and we omit the details. \square

Proposition 6 *Let $\{\gamma_{n,i}\}_{i=0}^{j+1}$ be the coefficients given in (17). Then,*

$$\lim_{n \rightarrow +\infty} \gamma_{n,i} = \gamma_i \in \mathbb{R}, \quad 0 \leq i \leq j+1,$$

where

$$\gamma_i = \begin{cases} -j, & \text{if } i = 0; \\ (-1)^i \frac{\frac{i-j}{j+i+\alpha+1} - \sum_{k=0}^{i-1} \gamma_k \binom{i}{k} (-2)^k k! \frac{(\alpha+1)_i}{(\alpha+k+1)_i}}{i! \frac{2^i (\alpha+1)_i}{(\alpha+i+1)_i}}, & \text{if } 1 \leq i \leq j+1. \end{cases} \quad (18)$$

Proof: As we have commented in the previous proposition, this result was also established in a similar context in [19, Th. 1] (see also [13, Th. 2]). But, now the discrete part of our inner product (1) is concentrated in two points, not only in one like the references cited. Anyway, the technique is the same. However, we include the main lines of the proof because in this concrete case we can establish the exact value of γ_i which has interest by itself.

To begin we differentiate the formula (17) k times and evaluate at $x = 1$. Then, for $0 \leq k \leq j+1$, we find

$$\left(Q_n^{(\alpha, M, j)} \right)^{(k)}(1) = \sum_{i=0}^k \gamma_{n,i} \binom{k}{i} (-1)^i i! \left(\sum_{l=0}^{k-i} \frac{i!}{(i-l)!} 2^{i-l} \left(C_{n-i}^{(\alpha+i)} \right)^{(k-l)}(1) \right). \quad (19)$$

Taking into account Proposition 3, we divide (19) by $\left(C_n^{(\alpha)}\right)^{(k)}(1)$ to see the result if and only if

$\lim_{n \rightarrow +\infty} \frac{\left(C_{n-i}^{(\alpha+i)}\right)^{(k-l)}(1)}{\left(C_n^{(\alpha)}\right)^{(k)}(1)} \in \mathbb{R}$, with $0 \leq l \leq k - i$. But this is true by Lemma 1. In fact,

$$\lim_{n \rightarrow +\infty} \frac{\left(C_{n-i}^{(\alpha+i)}\right)^{(k-l)}(1)}{\left(C_n^{(\alpha)}\right)^{(k)}(1)} = \begin{cases} \frac{(\alpha+1)_k}{(\alpha+i+1)_k}, & \text{if } l = 0; \\ 0, & \text{if } 1 \leq l \leq k - i. \end{cases} \quad (20)$$

We have proved that the sequences $\{\gamma_{n,i}\}_n$ are convergent with $i \in \{0, \dots, j+1\}$ when $n \rightarrow \infty$. Now, we want to compute explicitly the corresponding limits γ_i with $0 \leq i \leq j+1$. Taking $k = 0$ in (19) and using Proposition 3, we get

$$\lim_{n \rightarrow +\infty} \frac{Q_n^{(\alpha, M, j)}(1)}{C_n^{(\alpha)}(1)} = \lim_{n \rightarrow +\infty} Q_n^{(\alpha, M, j)}(1) = \lim_{n \rightarrow +\infty} \gamma_{n,0} = \frac{-j}{j + \alpha + 1}. \quad (21)$$

Thus, we can construct a recursive algorithm based on (19) and, paying attention to (20), we deduce easily (18). \square

Finally, we give an upper bound of the uniform norm of the Sobolev polynomials. This result will be useful to establish one of our main target in Section 4.

Theorem 1 *Let $Q_n^{(\alpha, M, j)}(x)$ be the orthogonal polynomials with respect to (1), then*

$$\|Q_n^{(\alpha, M, j)}\|_\infty := \max_{x \in [-1, 1]} \left| Q_n^{(\alpha, M, j)}(x) \right| \leq \begin{cases} \frac{3j+2\alpha+2}{j+\alpha+1} + D, & \text{if } \alpha \geq -1/2; \\ F n^{-\alpha-1/2}, & \text{if } -1 < \alpha < -1/2, \end{cases}$$

when $n \rightarrow +\infty$, being D and F positive constants independent of n .

Proof: Taking $\alpha = \lambda - 1/2$ and considering the expression (4.7.1) in [20], we have

$$C_n^{(\alpha)}(x) = \frac{P_n^{(\lambda)}(x)}{P_n^{(\lambda)}(1)} = \frac{\Gamma(n+1)\Gamma(2\alpha+1)}{\Gamma(n+2\alpha+1)} P_n^{(\lambda)}(x) = \frac{\Gamma(n+1)\Gamma(\alpha+1)}{\Gamma(n+\alpha+1)} P_n^{(\alpha, \alpha)}(x), \quad (22)$$

where $P_n^{(\alpha, \beta)}$ are the classical Jacobi polynomials orthogonal with respect to the weight function $(1-x)^\alpha(1+x)^\beta$, $\alpha, \beta > -1$.

Now, we use a uniform bound of $|P_n^{(\alpha, \alpha)}|$ given in [1, f. (22.14.1)] and with more detail in [20, f. (7.33.2)–(7.33.3)] to deduce, via (22), that we have for $-1 \leq x \leq 1$,

$$|P_n^{(\alpha, \alpha)}(x)| \leq \begin{cases} P_n^{(\alpha, \alpha)}(1) = \binom{n+\alpha}{n} \approx \frac{n^\alpha}{\Gamma(\alpha+1)}, & \text{if } \alpha \geq -1/2; \\ C n^{-1/2}, & \text{if } -1 < \alpha < -1/2, \end{cases} \quad (23)$$

where C is a constant. To prove the result we use different approaches according to each case.

- Case $\alpha \geq -1/2$. From (22) and (23), it is clear that $\max_{x \in [-1,1]} |C_n^{(\alpha)}(x)| = 1$, and this maximum is reached at $x = 1$. We only prove the even case since the proof of the odd case is totally similar. First, we have

$$\begin{aligned}
\max_{x \in [-1,1]} |\kappa_{2(n-1)}^{(j,0)}(1, x)| &= \max_{x \in [-1,1]} \left| \sum_{i=0}^{n-1} \frac{\left(C_{2i}^{(\alpha)}\right)^{(j)}(1) C_{2i}^{(\alpha)}(x)}{\|C_{2i}^{(\alpha)}\|_{\alpha}^2} \right| \\
&\leq \sum_{i=0}^{n-1} \frac{\left(C_{2i}^{(\alpha)}\right)^{(j)}(1) \max_{x \in [-1,1]} |C_{2i}^{(\alpha)}(x)|}{\|C_{2i}^{(\alpha)}\|_{\alpha}^2} \\
&= \sum_{i=0}^{n-1} \frac{\left(C_{2i}^{(\alpha)}\right)^{(j)}(1) C_{2i}^{(\alpha)}(1)}{\|C_{2i}^{(\alpha)}\|_{\alpha}^2} = \kappa_{2(n-1)}^{(j,0)}(1, 1).
\end{aligned}$$

Therefore, using (14) and the previous bound we get,

$$\begin{aligned}
\max_{x \in [-1,1]} |Q_{2n}^{(\alpha, M, j)}(x)| &\leq \max_{x \in [-1,1]} |C_{2n}^{(\alpha)}(x)| + \max_{x \in [-1,1]} \left| \frac{2M \left(C_{2n}^{(\alpha)}\right)^{(j)}(1) \kappa_{2(n-1)}^{(j,0)}(1, x)}{1 + 2M \kappa_{2(n-1)}^{(j,j)}(1, 1)} \right| \\
&= 1 + \frac{2M \left(C_{2n}^{(\alpha)}\right)^{(j)}(1) \kappa_{2(n-1)}^{(j,0)}(1, 1)}{1 + 2M \kappa_{2(n-1)}^{(j,j)}(1, 1)}.
\end{aligned}$$

On the other hand, it was established in the proof of Proposition 3 for $k = 0$ that

$$\lim_{n \rightarrow \infty} \frac{2M \left(C_{2n}^{(\alpha)}\right)^{(j)}(1) \kappa_{2(n-1)}^{(j,0)}(1, 1)}{1 + 2M \kappa_{2(n-1)}^{(j,j)}(1, 1)} = \frac{2j + \alpha + 1}{j + \alpha + 1}.$$

Thus, we claim that for any positive constant D and n large enough we have

$$\max_{x \in [-1,1]} |Q_{2n}^{(\alpha, M, j)}(x)| \leq 1 + \frac{2j + \alpha + 1}{j + \alpha + 1} + D = \frac{3j + 2\alpha + 2}{j + \alpha + 1} + D.$$

In fact, numerical experiments indicate that the sequence $\frac{2M \left(C_{2n}^{(\alpha)}\right)^{(j)}(1) \kappa_{2(n-1)}^{(j,0)}(1, 1)}{1 + 2M \kappa_{2(n-1)}^{(j,j)}(1, 1)}$ is decreasing, so D cannot be removed.

- Case $-1 < \alpha < -1/2$. For our purpose it is easier to take into account (17). In this way, we

get

$$\begin{aligned} \max_{x \in [-1,1]} \left| Q_n^{(\alpha, M, j)}(x) \right| &\leq \sum_{i=0}^{j+1} \max_{x \in [-1,1]} \left| \gamma_{n,i} (1-x^2)^i \left(C_{n-i}^{(\alpha+i)} \right)^{(i)}(x) \right| \\ &\leq (j+2) \max_{i \in \{0, \dots, j+1\}} \max_{x \in [-1,1]} \left| \gamma_{n,i} (1-x^2)^i \left(C_{n-i}^{(\alpha+i)} \right)^{(i)}(x) \right|. \end{aligned}$$

We are going to compute $\max_{x \in [-1,1]} \left| \gamma_{n,i} (1-x^2)^i \left(C_{n-i}^{(\alpha+i)} \right)^{(i)}(x) \right|$. First, we observe that using (3) we get

$$\begin{aligned} (1-x^2)^i \left(C_{n-i}^{(\alpha+i)} \right)^{(i)}(x) &= (1-x^2)^i \rho_{n,i} C_{n-2i}^{(\alpha+2i)}(x) \\ &= (1-x^2)^i \rho_{n,i} \frac{\Gamma(n-2i+1)\Gamma(\alpha+2i+1)}{\Gamma(n+\alpha+1)} P_{n-2i}^{(\alpha+2i, \alpha+2i)}(x), \end{aligned}$$

where

$$\rho_{n,i} = \frac{(-1)^i (n+2\alpha+i+1)_i (-n+i)_i}{2^i (\alpha+i+1)_i} \approx \frac{n^{2i}}{2^i (\alpha+i+1)_i}. \quad (24)$$

Using (23), (24) and Proposition 6, we get for n large enough,

$$\begin{aligned} \max_{x \in [-1,1]} \left| \gamma_{n,i} (1-x^2)^i \rho_{n,i} \frac{\Gamma(n-2i+1)\Gamma(\alpha+2i+1)}{\Gamma(n+\alpha+1)} P_{n-2i}^{(\alpha+2i, \alpha+2i)}(x) \right| \\ \leq E n^{2i} n^{-\alpha-2i} n^{-1/2} = E n^{-\alpha-1/2}, \end{aligned}$$

where E is a positive constant, which proves the result for this case. \square

4 Asymptotics behavior of the eigenvalues of Gegenbauer-Sobolev orthogonal polynomials and the exact value of r_0

In Section 3 we have established some of the necessary results to attack the problem of computing the value of r_0 given in (2). But we need a bit more: the asymptotic behavior of the eigenvalues of Gegenbauer-Sobolev polynomials.

In [7] the authors claim that there exists a linear differential operator of the form $\mathbf{T} = \mathbf{L} + M\mathbf{A}$ for discrete Sobolev orthogonal polynomials with respect to an inner product such as

$$(f, g) = \int_I f(x)g(x)d\mu + M[f^{(j)}(-c)g^{(j)}(-c) + f^{(j)}(c)g^{(j)}(c)], \quad c > 0,$$

where μ is a finite symmetric Borel measure supported on the interval I . \mathbf{L} is the linear differential operator associated with the standard polynomials orthogonal with respect to μ . This operator $\mathbf{L} + M\mathbf{A}$ can have infinite order. Obviously, the inner product (1) here considered lies in this framework.

In addition, the authors give expressions for the eigenvalues associated with $\mathbf{L} + M\mathbf{A}$. Then, if we particularize this for the Gegenbauer–Sobolev orthogonal polynomials, we have

$$(\mathbf{L} + M\mathbf{A})Q_n^{(\alpha, M, j)}(x) = \tilde{\lambda}_n Q_n^{(\alpha, M, j)}(x).$$

We are looking for the asymptotic behavior of $\tilde{\lambda}_n$ which is the key to establish one of our main goals in this work.

Following [7], we get that

$$\tilde{\lambda}_n = \lambda_n + M\mu_n, \tag{25}$$

where λ_n are as in (4), $\mu_0 = 0$ and the numbers $\{\mu_m\}_{m=0}^{j+1}$ can be chosen arbitrarily. Then, $\{\mu_m\}_{m=j+2}^{\infty}$ and the operator \mathbf{A} are uniquely determined once the choice of these arbitrary numbers has been done (see [7, Sec. 2.2] or with more detail Theorem 2.1 in [5] where this statement is established). In fact, they obtain

$$\begin{aligned} \mu_{j+2t} &= \mu_j + \sum_{i=1}^t (\lambda_{j+2i} - \lambda_{j+2i-2}) q_{j+2i, j+2i}, \quad t \in \mathbb{N}, j \in \mathbb{N} \cup \{0\}, \\ \mu_{j+2t+1} &= \mu_{j+1} + \sum_{i=1}^t (\lambda_{j+2i+1} - \lambda_{j+2i-1}) q_{j+2i+1, j+2i+1}, \quad t \in \mathbb{N}, j \in \mathbb{N} \cup \{0\}, \end{aligned}$$

where

$$q_{n,n} = K_{n-1}^{(j,j)}(1, 1) + (-1)^{n+j} K_{n-1}^{(j,j)}(1, -1).$$

Since $\{\mu_m\}_{m=0}^{j+1}$ can be chosen arbitrarily, for simplicity we take $\mu_0 = \dots = \mu_{j+1} = 0$. Using (4), we obtain

$$\mu_{j+2t} = 2 \sum_{i=1}^t (2j + 4i + 2\alpha - 1) q_{j+2i, j+2i}, \quad t \in \mathbb{N}, j \in \mathbb{N} \cup \{0\}, \tag{26}$$

$$\mu_{j+2t+1} = 2 \sum_{i=1}^t (2j + 4i + 2\alpha + 1) q_{j+2i+1, j+2i+1}, \quad t \in \mathbb{N}, j \in \mathbb{N} \cup \{0\}. \tag{27}$$

We are going to establish the asymptotic behavior of the sequence $\{\mu_n\}_n$ given by (26)–(27) when $n \rightarrow \infty$. First, we need a technical result.

Proposition 7 *We have,*

$$q_{s+2i,s+2i} = \begin{cases} 2\kappa_{s+2i-2}^{(j,j)}(1,1), & \text{if } s \text{ is even;} \\ 2\tilde{\kappa}_{s+2i-3}^{(j,j)}(1,1), & \text{if } s \text{ is odd,} \end{cases} \quad (28)$$

where $\kappa_{2m}^{(j,k)}(x,y)$ and $\tilde{\kappa}_{2m}^{(j,k)}(x,y)$ are given in (10) and (11), respectively.

Proof: We use the definition of $q_{s,s}$.

$$\begin{aligned} q_{s+2i,s+2i} &= K_{s+2i-1}^{(j,j)}(1,1) + (-1)^{s+j} K_{s+2i-1}^{(j,j)}(1,-1) \\ &= \sum_{m=0}^{s+2i-1} \frac{\left((C_m^{(\alpha)})^{(j)}(1) \right)^2}{\|C_m^{(\alpha)}\|_{\alpha}^2} + (-1)^{s+j} \sum_{m=0}^{s+2i-1} \frac{(C_m^{(\alpha)})^{(j)}(1) (C_m^{(\alpha)})^{(j)}(-1)}{\|C_m^{(\alpha)}\|_{\alpha}^2} \\ &= \sum_{m=0}^{s+2i-1} \frac{\left((C_m^{(\alpha)})^{(j)}(1) \right)^2 (1 + (-1)^{s+m+2j})}{\|C_m^{(\alpha)}\|_{\alpha}^2}. \end{aligned}$$

Then, if s is even we get

$$q_{s+2i,s+2i} = 2 \sum_{m=0, m \text{ even}}^{s+2i-1} \frac{\left((C_m^{(\alpha)})^{(j)}(1) \right)^2}{\|C_m^{(\alpha)}\|_{\alpha}^2} = 2\kappa_{s+2i-2}^{(j,j)}(1,1).$$

The odd case is established in the same way. \square

It is easy to observe from (25) that if we want to obtain the asymptotic behavior of $\tilde{\lambda}_n$ we must know the one of μ_n . Thus, we establish the following result.

Proposition 8 *It holds*

$$\lim_{n \rightarrow +\infty} \frac{\mu_{2n}}{n^{4j+2\alpha+4}} = \lim_{n \rightarrow +\infty} \frac{\mu_{2n+1}}{n^{4j+2\alpha+4}} = \frac{2^{2j+3}}{(2j+\alpha+2)(2j+\alpha+1)\Gamma^2(\alpha+j+1)}.$$

Proof: For n large enough we can write $2n = 2m + j$, and so j is even. To establish this result we

are going to use the Stolz's criterium and formulae (13) and (28).

$$\begin{aligned}
& \lim_{n \rightarrow +\infty} \frac{\mu_{2n}}{n^{4j+2\alpha+4}} = \lim_{m \rightarrow +\infty} \frac{\mu_{j+2m}}{m^{4j+2\alpha+4}} \\
&= \lim_{m \rightarrow +\infty} \frac{2 \sum_{i=1}^m (2j+4i+2\alpha-1)q_{j+2i,j+2i} - 2 \sum_{i=1}^{m-1} (2j+4i+2\alpha-1)q_{j+2i,j+2i}}{m^{4j+2\alpha+4} - (m-1)^{4j+2\alpha+4}} \\
&= \frac{1}{(2j+\alpha+2)} \lim_{m \rightarrow +\infty} \frac{(2j+4m+2\alpha-1)q_{j+2m,j+2m}}{m^{4j+2\alpha+3}} \\
&= \frac{1}{2j+\alpha+2} \lim_{m \rightarrow +\infty} \frac{(2j+4m+2\alpha+1) 2\kappa_{2(m-1+j/2)}^{(j,j)}(1,1)}{m^{4j+2\alpha+2}} \\
&= \frac{1}{2^{2j+3}} \\
&= \frac{1}{(2j+\alpha+2)(2j+\alpha+1)\Gamma^2(\alpha+j+1)}.
\end{aligned}$$

Analogously, for n large enough $2n+1 = 2m+j+1$, so $j+1$ is odd. Then, to prove the other limit we can use (28) with $s = j+1$. \square

Finally, we are ready to establish the asymptotic behavior of the eigenvalues $\tilde{\lambda}_n$.

Proposition 9 *Let $\tilde{\lambda}_n$ be the eigenvalues associated with the linear differential operator $\mathbf{T} = \mathbf{L} + M\mathbf{A}$. Then,*

$$\lim_{n \rightarrow +\infty} \frac{\tilde{\lambda}_n}{n^{4j+2\alpha+4}} = \frac{M}{2^{2j+2\alpha+1}(2j+\alpha+2)(2j+\alpha+1)\Gamma^2(\alpha+j+1)}.$$

Proof: Applying (4) and Proposition 8

$$\begin{aligned}
\lim_{n \rightarrow +\infty} \frac{\tilde{\lambda}_n}{n^{4j+2\alpha+4}} &= \lim_{n \rightarrow +\infty} \frac{\lambda_n + M\mu_n}{n^{4j+2\alpha+4}} = \lim_{n \rightarrow +\infty} \frac{M\mu_n}{(n/2)^{4j+2\alpha+4} 2^{4j+2\alpha+4}} \\
&= \frac{M}{(2j+\alpha+2)(2j+\alpha+1)\Gamma^2(\alpha+j+1)2^{2j+2\alpha+1}}. \quad \square
\end{aligned}$$

To conclude this section, we establish the main goal of this paper.

Theorem 2 *Let $\tilde{Q}_n^{(\alpha, M, j)}(x)$ be orthonormal polynomials with respect to (1), $j > 0$, and $\tilde{\lambda}_n$ the eigenvalues associated with the linear differential operator $\mathbf{T} = \mathbf{L} + M\mathbf{A}$. Then, for $\alpha \geq -1/2$,*

$$r_0 = \lim_{n \rightarrow +\infty} \frac{\log(\max_{x \in [-1,1]} |\tilde{Q}_n^{(\alpha, M, j)}(x)|)}{\log(\tilde{\lambda}_n)} = \frac{\alpha + 1/2}{4j + 2\alpha + 4}.$$

Proof: We are going to use Theorem 1, Proposition 4, Proposition 9, and (9). We have

$$\begin{aligned}
& \frac{\log(\max_{x \in [-1,1]} |\tilde{Q}_n^{(\alpha, M, j)}(x)|)}{\log(\tilde{\lambda}_n)} = \frac{\log(\max_{x \in [-1,1]} |Q_n^{(\alpha, M, j)}(x)| / \|Q_n^{(\alpha, M, j)}\|_S)}{\log(\tilde{\lambda}_n)} \\
& = \frac{\log\left(\max_{x \in [-1,1]} |Q_n^{(\alpha, M, j)}(x)| / (\|Q_n^{(\alpha, M, j)}\|_S n^{\alpha+1/2} n^{-\alpha-1/2})\right)}{\log\left(\frac{\tilde{\lambda}_n}{n^{4j+2\alpha+4}} n^{4j+2\alpha+4}\right)} \\
& = \frac{\log(\max_{x \in [-1,1]} |Q_n^{(\alpha, M, j)}(x)|) - \log\left(\|Q_n^{(\alpha, M, j)}\|_S n^{\alpha+1/2}\right) - (-\alpha - 1/2) \log(n)}{(4j + 2\alpha + 4) \log(n) + \log\left(\frac{\tilde{\lambda}_n}{n^{4j+2\alpha+4}}\right)}.
\end{aligned}$$

Taking limits, we obtain

$$\lim_{n \rightarrow +\infty} \frac{\log(\max_{x \in [-1,1]} |\tilde{Q}_n^{(\alpha, M, j)}(x)|)}{\log(\tilde{\lambda}_n)} = \frac{\alpha + 1/2}{4j + 2\alpha + 4}.$$

It is enough to take limits to prove the result.

Note that we need to have a lower and an upper bound for $\max_{x \in [-1,1]} |Q_n^{(\alpha, M, j)}(x)|$. The upper bound is guaranteed by Theorem 1. We also need that the lower bound is greater than zero. This is deduced from the fact that $\max_{x \in [-1,1]} |Q_n^{(\alpha, M, j)}(x)| \geq |Q_n^{(\alpha, M, j)}(1)|$ and by (21) $\lim_{n \rightarrow \infty} |Q_n^{(\alpha, M, j)}(1)| = \frac{j}{j+\alpha+1}$. However, this argument does not work when $j = 0$. \square

Remark.

- When $\alpha \geq -1/2$ and $j = 0$ we cannot assure that r_0 exists. In the case it exists, using some inequalities, it would be easy to establish that $r_0 \in \left[\frac{-\alpha-3/2}{2\alpha+4}, \frac{\alpha+1/2}{2\alpha+4}\right]$.
- When $\alpha \in (-1, -1/2)$ we could proceed in a similar way as in the proof of Theorem 2, but again we have the same problem with the lower bound as in the case $\alpha \geq -1/2$ and $j = 0$. Thus, this result remains as an open problem.
- It is worth paying attention to the fact that the value of r_0 for Gegenbauer polynomials is very different of the one for discrete Gegenbauer–Sobolev polynomials. For Gegenbauer orthogonal polynomials and $\alpha \geq -1/2$, we get

$$r_0 = \frac{\alpha + 1/2}{2},$$

and for discrete Gegenbauer-Sobolev orthogonal polynomials with $\alpha \geq -1/2$ and $j \in \mathbb{N}$, we have

$$r_0 = \frac{\alpha + 1/2}{4j + 2\alpha + 4}.$$

Essentially, this is due to the different asymptotic behavior of the eigenvalues $\lambda_n \approx n^2$ and $\tilde{\lambda}_n \approx G n^{4j+2\alpha+4}$, when $n \rightarrow +\infty$. The constant G is computed in Proposition 9.

5 Mehler–Heine asymptotics for Gegenbauer–Sobolev orthogonal polynomials

To complete this study we address the issue of Mehler-Heine formulae which are very relevant because they describe in detail the asymptotic behavior around the point $x = 1$ where we have located the perturbation (using the symmetry of these polynomials we also have the information around the point $x = -1$). This type of asymptotics has been considered in several frameworks. In the context of Sobolev orthogonality there is a wide literature, we can cite the surveys [15] and [17], and the references therein. Even more recently and conceptually closer to the inner product (1) we can point out [13, 14, 19] among others.

To establish Mehler–Heine formula for the discrete Gegenbauer–Sobolev orthogonal polynomials considered in this work, we need the corresponding formula for classical Jacobi orthogonal polynomials. For α, β real numbers and s an integer number, it holds (see [20, Th. 8.1.1]):

$$\lim_{n \rightarrow \infty} n^{-\alpha} P_n^{(\alpha, \beta)} \left(\cos \left(\frac{x}{n+s} \right) \right) = \lim_{n \rightarrow \infty} \frac{1}{n^\alpha} P_n^{(\alpha, \beta)} \left(1 - \frac{x^2}{2(n+s)^2} \right) = (x/2)^{-\alpha} J_\alpha(x), \quad (29)$$

uniformly on compact subsets of \mathbb{C} , where $J_\alpha(x)$ denotes the Bessel function of the first kind, i.e.,

$$J_\alpha(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \Gamma(k + \alpha + 1)} \left(\frac{x}{2} \right)^{2k + \alpha}.$$

The integer number s will play an important role in the proof of Theorem 3. The original statement of Mehler–Heine formula for classical Jacobi polynomials was made with $s = 0$, but it can be extended for every integer number s as it was established in the proof of Corollary 1 in [3]. In that paper it was proved a more general result: if $(f_n)_n$ is a sequence of holomorphic functions on \mathbb{C} and $(b_n)_n$ is a sequence of complex numbers satisfying $\lim_{n \rightarrow \infty} \frac{b_n}{b_{n+s}} = 1$ for every integer number s such that $(f_n(z/b_n))_n$ converges to a function f uniformly on compact subsets of \mathbb{C} , then

$$\lim_{n \rightarrow \infty} f_n \left(\frac{z}{b_{n+s}} \right) = f(z),$$

uniformly on compact subsets of \mathbb{C} for every integer s .

Thus, we can claim:

Theorem 3 *For the sequence $\{Q_n^{(\alpha, M, j)}\}_{n \geq 0}$ the following Mehler–Heine formula holds*

$$\lim_{n \rightarrow \infty} Q_n^{(\alpha, M, j)} \left(\cos \left(\frac{x}{n} \right) \right) = \lim_{n \rightarrow +\infty} Q_n^{(\alpha, M, j)} \left(1 - \frac{x^2}{2n^2} \right) = \varphi_{\alpha, j}(x), \quad (30)$$

uniformly on compact subsets of \mathbb{C} , where

$$\varphi_{\alpha, j}(x) = \sum_{i=0}^{j+1} 2^i \gamma_i \Gamma(\alpha + i + 1) (x/2)^{-\alpha} J_{\alpha+2i}(x), \quad (31)$$

with the coefficients γ_i given in (18).

Proof: Scaling adequately in (17) and using (3), (22) and (24), we get

$$\begin{aligned} & \lim_{n \rightarrow +\infty} Q_n^{(\alpha, M, j)} \left(1 - \frac{x^2}{2n^2} \right) = \\ & \lim_{n \rightarrow +\infty} \sum_{i=0}^{j+1} \gamma_{n,i} \rho_{n,i} \frac{x^{2i}}{n^{2i}} \left(1 - \frac{x^2}{4n^2} \right)^i \frac{\Gamma(n-2i+1)\Gamma(\alpha+2i+1)}{\Gamma(n+\alpha+1)} P_{n-2i}^{(\alpha+2i, \alpha+2i)} \left(1 - \frac{x^2}{2n^2} \right) = \\ & \lim_{n \rightarrow +\infty} \sum_{i=0}^{j+1} \gamma_{n,i} x^{2i} \frac{\rho_{n,i}}{n^{2i}} \left(1 - \frac{x^2}{4n^2} \right)^i \frac{n^{\alpha+2i}\Gamma(n-2i+1)\Gamma(\alpha+2i+1)}{\Gamma(n+\alpha+1)} \frac{P_{n-2i}^{(\alpha+2i, \alpha+2i)} \left(1 - \frac{x^2}{2n^2} \right)}{n^{\alpha+2i}}. \end{aligned}$$

It only remains to apply the asymptotic behaviors given by (7), Proposition 6, (24) and (29) to obtain the result. \square

Next, we consider the zeros of the polynomials $Q_n^{(\alpha, M, j)}$. When $j = 0$ the inner product (1) is standard, i.e, it is related to the measure μ given by $d\mu = (1-x)^\alpha(1+x)^\beta dx + M(\delta(x+1) + \delta(x-1))$ where $\delta(x)$ is the Dirac's delta function. Thus, all the zeros of $Q_n^{(\alpha, M, 0)}$ are real and are contained within $(-1, 1)$.

However, when $j > 0$ the situation changes. It was proved by H. G. Meijer in [18, Th. 4.1] (see also [2, Lemma 2]) in a more general framework that the polynomial $Q_n^{(\alpha, M, j)}(x)$, $n \geq 1$, has n real and simple zeros and at most two of them are located outside $(-1, 1)$. However, on the one hand, using Proposition 3 we have that $\lim_{n \rightarrow +\infty} Q_n^{(\alpha, M, j)}(1) = \frac{-j}{j+\alpha+1} < 0$, but on the other hand, the leading coefficient of $Q_n^{(\alpha, M, j)}(x)$ is $k_n(\alpha) > 0$ given by (5), so we have $\lim_{x \rightarrow +\infty} Q_n^{(\alpha, M, j)}(x) = +\infty$. Thus, we deduce that there exists a zero within $(1, +\infty)$ and, by the symmetry of the polynomials, another one within $(-\infty, -1)$. Therefore, we can summarize it in the following result.

Proposition 10 *If $j > 0$ the polynomial $Q_n^{(\alpha, M, j)}(x)$, $n \geq 1$, has n real and simple zeros and, for n large enough, exactly two of them are located outside $(-1, 1)$. If $j = 0$, then all the zeros are within $(-1, 1)$.*

Finally, we establish the asymptotic behavior of the zeros as a consequence of Theorem 3. It is only necessary to apply Hurwitz's theorem (see [20, Th. 1.91.3]) to (30). We denote by $[a]$ the integer part of a . Thus, we have

Proposition 11 *For $j > 0$, we denote by $s_{n,i}$, $i = 1, \dots, [n/2] - 1$, the $[n/2] - 1$ positive zeros of $Q_n^{(\alpha, M, j)}$ within $(0, 1)$ in a decreasing order, i.e., $s_{n, [n/2]-1} < s_{n, [n/2]-2} < \dots < s_{n,1}$. Then,*

$$\lim_{n \rightarrow \infty} n \arccos(s_{n,i}) = y_i, \quad i = 1, \dots, [n/2] - 1,$$

where $0 < y_1 < \dots < y_{[n/2]-1}$ denote the first $[n/2] - 1$ positive real zeros of the function $\varphi_{\alpha,j}$ given in (31).

For $j = 0$ the inner product (1) appears in [8] as a very particular case in a context which involves continuous Sobolev polynomials. Notice that in this situation using (18) we deduce that the limit function (31) is

$$\varphi_{\alpha,0}(x) = -\Gamma(\alpha + 1)(x/2)^{-\alpha} J_{\alpha+2}(x).$$

The above result was obtained in [8, Proposition 2] but in that paper the inner product considered was

$$(f, g)_S := \int_{-1}^1 f(x)g(x)(1-x^2)^{\alpha-1/2} dx + M(f(-1)g(-1) + f(1)g(1)),$$

where the corresponding Sobolev orthogonal polynomials were monic. Then, to compare both results we must take into account these facts. Anyway, the zeros of $Q_n^{(\alpha, M, 0)}$ behave asymptotically like the zeros of $x^{-\alpha} J_{\alpha+2}(x)$ (obviously in [8, Proposition 2] α must be changed by $\alpha + 1/2$). Thus, following the above notation, in the case $j = 0$ we get

$$\lim_{n \rightarrow \infty} s_{n,1} = 1, \quad \lim_{n \rightarrow \infty} n \arccos(s_{n,i}) = j_i^{(\alpha+2)}, \quad i = 2, \dots, [n/2],$$

where $0 < j_1^{(\alpha+2)} < \dots < j_{[n/2]}^{(\alpha+2)}$ denote the first $[n/2]$ positive zeros of the Bessel function of the first kind $J_{\alpha+2}$.

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