A SINGULAR ELLIPTIC EQUATION WITH NATURAL GROWTH IN THE GRADIENT AND A VARIABLE EXPONENT

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Abstract. In this paper we consider singular quasilinear elliptic equations with quadratic gradient and a singular term with a variable exponent

$$\begin{cases} -\Delta u + \dfrac{|\nabla u|^2}{u^{\gamma(x)}} = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Here Ω is an open bounded set of $\mathbb{R}^N$, $\gamma(x)$ is a positive continuous function and f is positive function that belongs to a certain Lebesgue space.

We show, among other results, that there exists a solution in the natural energy space $H_0^1(\Omega)$ to this problem when $\gamma(x)$ is strictly less than 2 in a strip around the boundary; while there is no solution in the energy space when there exists $\Gamma \subset \partial\Omega$ with $|\Gamma|_{N-1} > 0$ such that $\gamma(x) > 2$ on Γ.

Moreover, since we work by approximation we can analyze the behavior of the approximated solutions u_n in the case in which there is no solution in $H_0^1(\Omega)$.

1 Introduction

In the framework of quasilinear elliptic equations with quadratic growth in the gradient, here we are concerned with the existence of solutions for the following boundary value problem:

$$\begin{cases} -\Delta u + \dfrac{|\nabla u|^2}{u^{\gamma(x)}} = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is an open, bounded subset of $\mathbb{R}^N$ $(N \geq 3)$, $0 \leq f \in L^q(\Omega)$ with $q \geq \frac{N}{2}$ satisfying

$$m_\omega(f) \stackrel{\text{def}}{=} \text{ess inf}\{f(x) : x \in \omega\} > 0, \quad \forall \omega \subset\subset \Omega_\delta \tag{1.2}$$

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where $\Omega_\delta := \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) < \delta\}$, for $\delta > 0$ fixed, and $\gamma(x) \in C^1(\overline{\Omega})$ is a positive function.

If the lower order term is nonsingular, namely

$$\begin{cases} -\Delta u + g(x,u)|\nabla u|^2 = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

with g a Carathéodory function in $\Omega \times [0, \infty)$, problem (**??**) has been exhaustively studied in [**?**, **?**, **?**] with data f in suitable Lebesgue spaces.

In the case in which the lower order term is singular, there are several papers that deal with existence and nonexistence of solutions when γ is a positive constant, namely with the model problem

$$\begin{cases} -\Delta u + \dfrac{|\nabla u|^2}{u^\gamma} = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.4}$$

First, existence of solutions for (**??**) was proved in [**?**, **?**, **?**] for $0 < \gamma \leq 1$ and the uniqueness of solution for $0 < \gamma < 1$ in [**?**]. We also quote the paper [**?**]. Specifically, the existence of positive solutions of (**??**) is proved in [**?**] for $\gamma \leq 1$ provided $0 \not\equiv f \in L^q(\Omega)$ ($q > \frac{2N}{N+2}$) with $f \geq 0$. In [**?**] it is proved the existence of solution if $\gamma < 2$ when a strong condition on f is assumed (see [**?**] for the parabolic case). More precisely, it is imposed condition (**??**) in the whole Ω. Moreover nonexistence is proved if $\gamma > 2$ or if $\gamma = 2$ and $\lambda_1(f) > 1$, where $\lambda_1(f)$ denotes the first positive eigenvalue of the laplacian operator $-\Delta$ with zero Dirichlet boundary conditions and weight $f \in L^q(\Omega)$, ($q > N/2$). In [**?**] the author prove the same result as in [**?**] avoiding, in the case $0 < \gamma < 1$, the assumption that f must be strictly positive in compact subsets of Ω (see also [**?**]). Later, in [**?**] it is proved the nonexistence of solution assuming only that $\gamma \geq 2$.

In the present paper, we deal with a variable exponent and we analyze how the behavior of $\gamma(x)$ influences the existence and nonexistence of solutions. We may have a region inside Ω where $\gamma(x) < 2$ and another region where $\gamma(x) \geq 2$.

The main goal here is to explain that what matters for existence of solutions is the behaviour of $\gamma(x)$ near the boundary.

The idea to prove the existence result consists in approximating the singular term $s^{-\gamma(x)}$ continuously, such that the non singular approximated problems fall into the framework in [**?**] and therefore they have finite energy solution u_n, for every $n \in \mathbb{N}$. We will prove that, for $\gamma(x) < 2$ near the boundary, the approximating solutions u_n converge to a positive solution of (**??**). As $f \in L^q(\Omega)$ with $q \geq \frac{N}{2}$ it is easy to prove ([**?**]) that exist a priori estimates of the solutions u_n in $H_0^1(\Omega)$. Observe, that due to singularity of the lower order term, the approximated lower order term blow up as $u_n(x)$

is converging to zero. This is the reason why it is not possible to apply the ideas of [?, ?, ?] to show the strong convergence of ∇u_n in $L^2(\Omega)$ (and thus the strong convergence of the approximated solutions u_n in $H_0^1(\Omega)$ to a solution of (??)). The keypoint to overcome this difficulty consists in proving that u_n are uniformly away from zero in every compact set inside Ω. We show here that $\gamma(x)$ must be less than 2 only near the boundary for obtaining this kind of estimate. This principle allows us to prove that the sequence of approximating solutions converges locally to a solution of (??).

In order to prove our nonexistence result we follow the ideas in [?] adapted for Sobolev functions vanishing only in a part of the boundary.

Our main results are the following (it is assumed that $\partial\Omega$ is Lipschitz and we denote by n_e the exterior normal vector to $\partial\Omega$, see the comments before the statement of the main results).

Theorem 1.1 (Existence). *Let $f \in L^q(\Omega)$ with $q \geq \frac{N}{2}$ satisfying (??) and $\gamma(x) < 2$ on $\partial\Omega$ or $\gamma(x) \leq 2$ on $\partial\Omega$ with $\dfrac{\partial\gamma(x)}{\partial n_e} > 0$, then there exists $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ a solution to problem (??).*

Theorem 1.2 (Nonexistence). *If there exists $\Gamma \subset \partial\Omega$ with $|\Gamma|_{N-1} > 0$ such that $\gamma(x) > 2$ on Γ or $\gamma(x) = 2$ on Γ with $\dfrac{\partial\gamma(x)}{\partial n_e} \leq 0$ there then (??) admits no solution $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$.*

We remark that what we will use to show existence of a solution is that $\gamma(x) < 2$ for every x in a strip around $\partial\Omega$ inside Ω. Our hypothesis on $\gamma(x)$ in Theorem **??** guarantee this fact. Note that we can extend the existence result to functions $\gamma(x)$ such that $\gamma(x) < 2$ on $A \subset \partial\Omega$ and $\gamma(x) = 2$ on $\partial\Omega \setminus A$ with $\dfrac{\partial\gamma(x)}{\partial n_e} > 0$ there.

For the nonexistence part we use that there is an open set $D \subset \Omega$ such that $\gamma(x) \geq 2$ in D and $|\partial D \cap \partial\Omega|_{N-1} > 0$. Remark that the conditions on $\gamma(x)$ assumed in Theorem **??** imply the existence of such set D.

The paper is organized as follows. Section 2 is devoted to describe the approximated problems and we prove some properties that we need in the proof of our main results. In Section 3 we prove the main results. We analyze the behavior of the solutions to the approximated problems in Section 4.

Notations. As usual, for every $s \in \mathbb{R}$ we consider the positive and negative parts given by $s^+ = \max\{s, 0\}$ and $s^- = \min\{s, 0\}$. For any $k > 0$ we set $T_k(s) = \min(k, \max(s, -k))$ and $G_k(s) = s - T_k(s)$. We denote by $|E|$ the Lebesgue measure of a measurable set E in $\mathbb{R}^N$ and by $|\Gamma|_{N-1}$ the $(N-1)$-dimensional surface measure of Γ. For $1 \leq p \leq +\infty$, $\|u\|_p$ is the usual norm of a function $u \in L^p(E)$. We equipped the standard Sobolev space $H_0^1(E)$ with the usual norm $\|u\| = \left(\int_E |\nabla u|^2\right)^{1/2}$.

Moreover, for any $q > 1$, $q' = \frac{q}{q-1}$ will be the Hölder conjugate exponent of q, while for any $1 < p < N$, $p^* = \frac{Np}{N-p}$ is the Sobolev conjugate exponent of p. As usual, $\mathcal{S}$ denotes the best Sobolev constant, i.e.,

$$\mathcal{S} = \sup_{\|u\|_{H_0^1(\Omega)}=1} \|u\|_{L^{2^*}(\Omega)}.$$

Following [**?**], we set $\varphi_\lambda(s) = se^{\lambda s^2}$, $\lambda > 0$; we will use here that for every a, $b > 0$ we have

$$a\varphi_\lambda'(s) - b|\varphi_\lambda(s)| \geq \frac{a}{2}, \tag{1.5}$$

if $\lambda > \frac{b^2}{4a^2}$. We will also denote by $\varepsilon(n)$ any quantity that goes to 0 as n goes to infinity.

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2 Preliminary results

Let us start giving our definition of solution to problem (**??**).

Definition 2.1. We say that $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ is a positive solution for (**??**) if $u > 0$ a.e. $x \in \Omega$,

$$\frac{|\nabla u|^2}{u^{\gamma(x)}} \in L^1(\Omega)$$

and

$$\int_\Omega \nabla u \nabla \varphi + \int_\Omega \frac{|\nabla u|^2}{u^{\gamma(x)}}\varphi = \int_\Omega f(x)\varphi,$$

for every $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$ with $\varphi \geq 0$.

In order to prove our results the approach is to consider the following approximating problems. For every $n \in \mathbb{N}$ let u_n be the solution to

$$\begin{cases} -\Delta u_n + \dfrac{u_n^+ |\nabla u_n|^2}{\left(u_n^+ + \frac{1}{n}\right)^{\gamma(x)+1}\left(1 + \frac{1}{n}|\nabla u_n|^2\right)} = f & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

Now, we prove some estimates that we will need in what follows.

Proposition 2.2. *There exists at least one positive solution* $0 < u_n \in H_0^1(\Omega) \cap L^\infty(\Omega)$ *of the approximating problem* (**??**). *In addition, the sequence* $\{u_n\}$ *is bounded in* $H_0^1(\Omega)$ *and in* $L^\infty(\Omega)$, *i.e. there exists* $C > 0$ *independent of* n *with*

$$\|u_n\|_{H_0^1(\Omega)} \leq C, \ \|u_n\|_{L^\infty(\Omega)} \leq C, \qquad \forall n \in \mathbb{N}.$$

Remark 2.3. Standard regularity arguments imply that u_n is Hölder continuous.

Proof. Classical results allow us to deduce that the problem (**??**) has a solution u_n that belongs to $H_0^1(\Omega)$ (see [**?**]) and to $L^\infty(\Omega)$ (see [**?**]).

To prove the a priori estimate in $L^\infty(\Omega)$ we take $\varphi = G_k(u_n)$ as test function in (**??**) to obtain

$$\int_\Omega |\nabla G_k(u_n)|^2 + \int_\Omega \frac{u_n^+ |\nabla u_n|^2}{\left(u_n^+ + \frac{1}{n}\right)^{\gamma(x)+1} \left(1 + \frac{1}{n}|\nabla u_n|^2\right)} G_k(u_n) = \int_\Omega f(x) G_k(u_n).$$

Using the positivity of the lower order term we deduce that

$$\int_\Omega |\nabla G_k(u_n)|^2 \leq \int_\Omega f(x) G_k(u_n).$$

Now, by Stampacchia's method, see [**?**], it follows from this inequality the existence of $C > 0$ such that that

$$\|u_n\|_{L^\infty(\Omega)} \leq C.$$

Now, we prove the a priori estimate in the Sobolev space. Taking u_n as test function in (**??**) and using Hölder and Sobolev inequalities we arrive to

$$\begin{aligned}\int_\Omega |\nabla u_n|^2 + \int_\Omega \frac{u_n^+ |\nabla u_n|^2}{\left(u_n^+ + \frac{1}{n}\right)^{\gamma(x)+1} \left(1 + \frac{1}{n}|\nabla u_n|^2\right)} u_n \leq & \|f\|_q \|u_n\|_{q'} \\ \leq & C\,\mathcal{S} \|f\|_q \|u_n\|.\end{aligned}$$

Using the positivity of the lower order term and that q' is the conjugate exponent of q (note that for $q > N/2$ we have $q' < 2^*$) we conclude that the sequence u_n is bounded in $H_0^1(\Omega)$. Therefore, up to a subsequence, $u_n \rightharpoonup u$ for some $u \in H_0^1(\Omega)$.

On the other hand, taking u_n^- as a test function in (**??**) we obtain

$$\int_\Omega |\nabla u_n^-|^2 + \int_\Omega \frac{u_n^+ |\nabla u_n|^2}{\left(u_n^+ + \frac{1}{n}\right)^{\gamma(x)+1} \left(1 + \frac{1}{n}|\nabla u_n|^2\right)} u_n^- = \int_\Omega f u_n^-$$

and as f is nonnegative we get

$$\int_\Omega |\nabla u_n^-|^2 = \int_\Omega f u_n^- \leq 0.$$

Therefore, we deduce that $u_n \geq 0$. Moreover, since

$$-\Delta u_n + n^{\|\gamma\|_{L^\infty(\Omega)}+2} u_n \geq f$$

the strong maximum principle assures that $u_n > 0$. □

Now we prove that the solutions of the approximated problems u_n are away from zero in every compact subset of Ω. In this proof is where we appreciate that $\gamma(x)$ must be less than or equal to 2 only near of the boundary in order to obtain our existence result.

Proposition 2.4. *Let $f \in L^q(\Omega)$ with $q \geq \frac{N}{2}$ satisfying (**??**) and $\gamma(x) < 2$ on $\partial\Omega$ or $\gamma(x) \leq 2$ on $\partial\Omega$ with $\dfrac{\partial\gamma(x)}{\partial n_e} > 0$ then there exists $c_\omega > 0$ such that $u_n \geq c_\omega$ for every $\omega \subset\subset \Omega$.*

Proof. Let us consider

$$\Omega_\eta = \{x \in \Omega \ : \ \mathrm{dist}(x, \partial\Omega) < \eta\}.$$

Given $\omega \subset\subset \Omega$ there exists $\eta > 0$ such that $\omega \subset \Omega \setminus \overline{\Omega}_\eta$. The conclusion follows from the fact that there exists $c > 0$ such that $u_n(x) \geq c$ a.e. $x \in \Omega \setminus \overline{\Omega}_\eta$. Note that it is enough to show this for η small.

We will prove this fact in two steps. In the first one we prove that there exists $c > 0$ such that $u_n(x) > c$ for every $x \in \partial(\Omega \setminus \overline{\Omega}_\eta)$. Then, in the second step, we will use this inequality to prove the claim in the whole $\Omega \setminus \overline{\Omega}_\eta$.

Step 1. We may assume that $\eta < \delta$, where δ is given by (**??**). Since $\gamma(x) < 2$ on $\partial\Omega$ or $\gamma(x) \leq 2$ on $\partial\Omega$ with $\dfrac{\partial\gamma(x)}{\partial n_e} > 0$ then there exists $\eta_1 \in (0, \delta)$ such that, for every $\eta < \eta_1$ there exists $\gamma^*_\eta < 2$ with

$$0 \leq \gamma(x) \leq \gamma^*_\eta < 2$$

for every $x \in \Omega_\eta \setminus \overline{\Omega}_{\frac{\eta}{4}}$. Thus we will assume that $0 < 2\eta < \eta_1 < \delta$ and we also have that $\partial(\Omega \setminus \overline{\Omega}_\eta) \subset \omega_1$ with

$$\omega_1 := \left\{ x \in \Omega : \frac{3\eta}{4} < \mathrm{dist}(x, \partial\Omega) < \frac{5\eta}{4} \right\}.$$

Observe that $\omega_1 \subset\subset W$ where

$$W := \left\{ x \in \Omega : \frac{\eta}{2} < \mathrm{dist}(x, \partial\Omega) < 2\eta \right\} \subset \Omega_{2\eta} \setminus \overline{\Omega}_{\frac{\eta}{2}}.$$

For every $0 < s < C$, with C given by Proposition **??**, and $x \in W$ we have that

$$\frac{s}{(s + \frac{1}{n})^{\gamma(x)+1}} \leq \frac{(C+1)^{\gamma^*_{2\eta}}}{s^{\gamma^*_{2\eta}}}.$$

Taking

$$h(s) = \frac{(C+1)^{\gamma^*_{2\eta}}}{s^{\gamma^*_{2\eta}}}$$

we have that $0 < u_n \in H^1(W) \cap C(W)$ is a supersolution to the equation

$$-\Delta z + h(z)|\nabla z|^2 = T_1(f) \qquad \text{in } W.$$

Therefore, we can use Proposition 2.3 in [?] (note that condition (**??**) implies that $T_1(f)$ satisfies (1.4) of that paper in W and, since $\gamma^*_{2\eta} < 2$, the function h satisfies (1.7) of [?]). We deduce the existence of $c_{\omega_1} > 0$ that $u_n(x) \geq c_{\omega_1}$ for every $x \in \omega_1$, $n \in \mathbb{N}$.

Step 2. Using that, from Step 1, $u_n(x) \geq c_{\omega_1}$ in $\partial(\Omega \setminus \overline{\Omega}_\eta)$ we prove now that $u_n \geq c_{\omega_1}$ in $D := \Omega \setminus \overline{\Omega}_\eta$.

We take $\phi_k \in C^1_0(\Omega)$, with $\phi_k \geq 0$ and $\mathrm{supp}(\phi_k) \subset\subset D$, as test function in (**??**) and we obtain

$$\int_D \nabla u_n \nabla \phi_k + \int_D \frac{u_n |\nabla u_n|^2}{\left(u_n + \frac{1}{n}\right)^{\gamma(x)+1} \left(1 + \frac{1}{n}|\nabla u_n|^2\right)} \phi_k = \int_D f \phi_k.$$

Thus, by density, for every nonnegative $\phi \in H^1_0(D) \cap L^\infty(D)$ we have

$$\int_D \nabla u_n \nabla \phi + \int_D \frac{u_n |\nabla u_n|^2}{\left(u_n + \frac{1}{n}\right)^{\gamma(x)+1} \left(1 + \frac{1}{n}|\nabla u_n|^2\right)} \phi = \int_D f \phi.$$

Using that

$$\frac{u_n |\nabla u_n|^2}{\left(u_n + \frac{1}{n}\right)^{\gamma(x)+1} \left(1 + \frac{1}{n}|\nabla u_n|^2\right)} \leq (C+1)^{\|\gamma\|_{L^\infty(\Omega)}} \frac{|\nabla u_n|^2}{\left(u_n + \frac{1}{n}\right)^{\|\gamma\|_{L^\infty(\Omega)}}}$$

we obtain, with $c = (C+1)^{\|\gamma\|_{L^\infty(\Omega)}}$, that

$$\int_D \nabla u_n \nabla \phi + \int_D c \frac{|\nabla u_n|^2}{\left(u_n + \frac{1}{n}\right)^{\|\gamma\|_{L^\infty(\Omega)}}} \phi \geq \int_D f \phi,$$

for every $0 \leq \phi \in H^1_0(D) \cap L^\infty(D)$.

Now, consider

$$H_n(s) = \int_1^s \frac{c}{(s + \frac{1}{n})^{\|\gamma\|_{L^\infty(\Omega)}}} dt.$$

If we take in the previous inequality $e^{-H_n(u_n)}(c_{\omega_1} - u_n)^+ \in H^1_0(D) \cap L^\infty(D)$ as test function it follows that

$$-\int_{D \cap \{c_{\omega_1} \geq u_n\}} |\nabla u_n|^2 e^{-H_n(u_n)} \geq \int_D f e^{-H_n(u_n)} (c_{\omega_1} - u_n)^+ \geq 0.$$

Then, $(c_{\omega_1} - u_n)^+ \equiv 0$ and therefore $u_n \geq c_{w_1}$ in D. □

3 Proofs of the main results

Proof of Theorem **??**. The result follows from the following steps. First we prove that $u_n \to u$ strongly in $H^1_{\mathrm{loc}}(\Omega)$ and next that we can pass to the limit in (**??**).

Step 1. $u_n \to u$ strongly in $H^1_{\rm loc}(\Omega)$. Here we prove that

$$\lim_{n\to+\infty}\int_\Omega |\nabla(u_n-u)|^2\phi \,=0, \quad \forall \phi\in C_0^\infty(\Omega) \text{ with } \phi\geq 0. \tag{3.1}$$

Reasoning as in [?], we consider the function $\varphi_\lambda(s)$ defined in (**??**) and we choose $\varphi_\lambda(u_n-u)\phi$ as test function in (**??**), we have

$$\begin{aligned}\int_\Omega \nabla u_n\cdot\nabla(u_n-u)\varphi_\lambda'(u_n-u)\phi+\int_\Omega \nabla u_n\cdot\nabla\phi\,\varphi_\lambda(u_n-u)\\ +\int_\Omega \frac{u_n|\nabla u_n|^2}{\left(u_n+\frac1n\right)^{\gamma(x)+1}\left(1+\frac1n|\nabla u_n|^2\right)}\varphi_\lambda(u_n-u)\phi\\ =\int_\Omega f\,\varphi_\lambda(u_n-u)\phi.\end{aligned}$$

Since, up to a subsequence, $u_n\to u$ weakly in $H^1_0(\Omega)$ and strongly in $L^2(\Omega)$, we note that

$$\int_\Omega f\,\varphi_\lambda(u_n-u)\phi-\int_\Omega \nabla u_n\cdot\nabla\phi\,\varphi_\lambda(u_n-u)\,=\varepsilon(n).$$

Moreover, choosing $\omega_\phi\subset\subset\Omega$ with $\operatorname{supp}\phi\subset\omega_\phi$, from Proposition **??**, Proposition **??** and the fact that $\gamma(x)\in C(\overline{\Omega})$, we deduce that

$$\begin{aligned}\int_\Omega \frac{u_n|\nabla u_n|^2}{\left(u_n+\frac1n\right)^{\gamma(x)+1}\left(1+\frac1n|\nabla u_n|^2\right)}\varphi_\lambda(u_n-u)\phi\\ \geq -c(\omega_\phi)\int_\Omega |\nabla u_n|^2|\varphi_\lambda(u_n-u)|\phi\,.\end{aligned}$$

Thus, it follows that

$$\begin{aligned}\int_\Omega \nabla u_n\cdot\nabla(u_n-u)\varphi_\lambda'(u_n-u)\phi-\\ -c(\omega_\phi)\int_\Omega |\nabla u_n|^2|\varphi_\lambda(u_n-u)|\phi\,\leq\varepsilon(n).\end{aligned}\tag{3.2}$$

Adding

$$-\int_\Omega \nabla u\cdot\nabla(u_n-u)\varphi_\lambda'(u_n-u)\phi=\varepsilon(n)$$

in both sides of (**??**) and since

$$\begin{aligned}\int_\Omega |\nabla u_n|^2|\varphi_\lambda(u_n-u)|\phi \leq& 2\int_\Omega |\nabla(u_n-u)|^2|\varphi_\lambda(u_n-u)|\phi\\ &+2\int_\Omega |\nabla u|^2|\varphi_\lambda(u_n-u)|\phi\\ =&2\int_\Omega |\nabla(u_n-u)|^2|\varphi_\lambda(u_n-u)|\phi\,+\varepsilon(n),\end{aligned}$$

we find

$$\int_\Omega |\nabla(u_n-u)|^2\Big[\varphi_\lambda'(u_n-u)-2c(\omega_\phi)|\varphi_\lambda(u_n-u)|\Big]\phi\,\leq\varepsilon(n).$$

Choosing λ such that (**??**) holds with $a = 1$ and $b = 2c(\omega_\phi)$, we conclude that (**??**) is satisfied.

Step 2. We pass to the limit in (**??**). Choosing $\frac{1}{\varepsilon}T_\varepsilon(u_n)$ as test function in (**??**), we obtain

$$\int_\Omega \frac{T_\varepsilon(u_n)}{\varepsilon} \frac{u_n|\nabla u_n|^2}{\left(u_n + \frac{1}{n}\right)^{\gamma(x)+1}\left(1 + \frac{1}{n}|\nabla u_n|^2\right)} \le \int_\Omega f\,.$$

If we take the limit as ε tends to zero, and we use that $u_n > 0$ in Ω, we get

$$\int_\Omega \frac{u_n|\nabla u_n|^2}{\left(u_n + \frac{1}{n}\right)^{\gamma(x)+1}\left(1 + \frac{1}{n}|\nabla u_n|^2\right)} \le \int_\Omega f\,. \tag{3.3}$$

Since

$$-\Delta u_n = f - \frac{u_n|\nabla u_n|^2}{\left(u_n + \frac{1}{n}\right)^{\gamma(x)+1}\left(1 + \frac{1}{n}|\nabla u_n|^2\right)}\,,$$

and the right hand side is bounded in $L^1(\Omega)$ by the assumptions on f and by (**??**). Then we can apply Lemma 1 of [**?**] (see also [**?**]) to deduce that, up to (not relabeled) subsequences, ∇u_n converges to ∇u a.e. in Ω.

Using Fatou lemma in (**??**), we get

$$\int_\Omega \frac{|\nabla u|^2}{u^{\gamma(x)}} \le \int_\Omega f.$$

Therefore, to conclude the proof we only have to show that u is a distributional solution of the problem (**??**). We begin by passing to the limit as $n \to \infty$ in the equation satisfied by u_n, that is, in

$$\int_\Omega \nabla u_n \cdot \nabla\phi + \int_\Omega \frac{u_n|\nabla u_n|^2}{\left(u_n + \frac{1}{n}\right)^{\gamma(x)+1}\left(1 + \frac{1}{n}|\nabla u_n|^2\right)}\phi = \int_\Omega f\phi\,,\ \forall\phi \in C_0^\infty(\Omega).$$

First of all, the weak convergence of u_n to u implies that

$$\lim_{n\to+\infty}\int_\Omega \nabla u_n \nabla\phi = \int_\Omega \nabla u \nabla \phi\,, \quad \forall\phi \in C_0^\infty(\Omega). \tag{3.4}$$

On the other hand, if we fix $\omega \subset\subset \Omega$, then, by Proposition **??**, Proposition **??** and since $\gamma(x) \in C(\overline{\Omega})$, we get

$$\frac{u_n}{\left(u_n + \frac{1}{n}\right)^{\gamma(x)+1}} \le c(\omega), \qquad \forall n >> 1, \text{ and } \forall x \in \omega.$$

Consequently, if $E \subset \omega$ it follows that

$$\int_E \frac{u_n|\nabla u_n|^2}{\left(u_n + \frac{1}{n}\right)^{\gamma(x)+1}\left(1 + \frac{1}{n}|\nabla u_n|^2\right)} \le c(\omega)\int_E |\nabla u_n|^2. \tag{3.5}$$

Let $\varepsilon > 0$ be fixed. Since u_n is strongly compact in $H^1_{\rm loc}(\Omega)$ and there exist n_ε, δ_ε such that for every $E \subset \omega \subset\subset \Omega$ with meas$(E) < \delta_\varepsilon$, we have

$$\int_E |\nabla u_n|^2 < \frac{\varepsilon}{c(\omega)}, \quad \forall n \geq n_\varepsilon.$$

In conclusion, by (**??**), we see that meas$\,(E) < \delta_\varepsilon$ implies

$$\int_E \frac{u_n |\nabla u_n|^2}{\left(u_n + \frac{1}{n}\right)^{\gamma(x)+1} \left(1 + \frac{1}{n}|\nabla u_n|^2\right)} \leq \varepsilon, \quad \forall n \geq n_\varepsilon,$$

i.e., the sequence

$$\frac{u_n |\nabla u_n|^2}{\left(u_n + \frac{1}{n}\right)^{\gamma(x)+1} \left(1 + \frac{1}{n}|\nabla u_n|^2\right)}$$

is equiintegrable. This, together with its a.e. convergence to $\frac{|\nabla u|^2}{u^{\gamma(x)}}$, implies by Vitali's theorem that

$$\lim_{n\to+\infty} \int_\Omega \frac{u_n |\nabla u_n|^2}{\left(u_n + \frac{1}{n}\right)^{\gamma(x)+1} \left(1 + \frac{1}{n}|\nabla u_n|^2\right)} \phi = \int_\Omega \frac{|\nabla u|^2}{u^{\gamma(x)}} \phi, \quad \forall \phi \in C_0^\infty(\Omega).$$

Therefore, using the above limit and (**??**) we conclude that

$$\int_\Omega \nabla u \nabla \phi + \int_\Omega \frac{|\nabla u|^2}{u^{\gamma(x)}} \phi = \int_\Omega f \phi\,, \qquad \forall \phi \in C_0^\infty(\Omega),$$

as we wanted to show. □

Now we prove our nonexistence result.

Proof of Theorem **??**. From our hypothesis, we may assume that $\Gamma = \partial D \cap \partial\Omega$ with $D \subset \Omega$ open such that $\gamma(x) \geq 2$ for every $x \in D$.

We prove the result using the ideas of [**?**]. Assume on the contrary that there exists some $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ solution of (**??**) with $u > 0$ a.e. in Ω such that

$$\int_\Omega \frac{|\nabla u|^2}{u^{\gamma(x)}} dx < +\infty.$$

Since $\gamma(x) \geq 2$, we know in D that

$$\int_D \frac{|\nabla u|^2}{(u+\varepsilon)^2} \leq \int_D \frac{|\nabla u|^2}{u^2} \leq c \int_D \frac{|\nabla u|^2}{u^{\gamma(x)}} < +\infty, \forall \varepsilon > 0,$$

i.e.,

$$\int_D |\nabla(\ln(u+\varepsilon) - \ln(\varepsilon))|^2 \leq C_3, \quad \forall \varepsilon > 0.$$

Denoting $z_\varepsilon = |\ln(u+\varepsilon) - \ln(\varepsilon)|$, we have that $z_\varepsilon \in H^1(D)$ with $z_\varepsilon = 0$ on $\partial D \cap \partial\Omega$. Now we observe that there exists a constant C_4 such that

$$\int_D g^2 \leq C_4 \int_D |\nabla g|^2 \tag{3.6}$$

for any function $g \in H^1(D)$ with $g = 0$ on Γ. To see this fact, we argue by contradiction. Assume that there is a sequence g_n such that $\int_D |\nabla g_n|^2 \to 0$ and $\int_D g_n^2 = 1$. Then g_n converges strongly in $H^1(D)$ to a function g_0 that verifies $\int_D |\nabla g_0|^2 = 0$ (hence, $g_0 = cte$) $\int_D g_0^2 = 1$ and $g_0 = 0$ on Γ, a contradiction. Thus, using the generalized Poincare's inequality (**??**) we get

$$\int_D z_\varepsilon^2 \leq C_4 \int_D |\nabla z_\varepsilon|^2 \leq C_4 C_3 := C_5, \quad \forall \varepsilon > 0.$$

Denote $E_n = \{x \in D : u(x) > \frac{1}{n}\}$ for every $n \in \mathbb{N}$. Then we have

$$\{x \in D; u(x) > 0\} = \bigcup_{n=1}^{\infty} E_n,$$

which implies that

$$0 < |D| \leq \sum_{n=1}^{\infty} |E_n|$$

and then there exists $n_0 \in \mathbb{N}$ such that $|E_{n_0}| > 0$. We deduce

$$\left| \ln\left(\frac{1}{n_0} + \varepsilon\right) - \ln(\varepsilon) \right|^2 \cdot |E_{n_0}| \leq \int_{E_{n_0}} |\ln(u+\varepsilon) - \ln(\varepsilon)|^2 \leq C_5,$$

for every $\varepsilon > 0$, therefore

$$\left| \ln\left(\frac{1}{n_0} + \varepsilon\right) - \ln(\varepsilon) \right|^2 \leq \frac{C_5}{|E_{n_0}|} < +\infty, \qquad \forall \varepsilon > 0.$$

Now, as ε goes to zero, we obtain a contradiction. □

4 Behavior of the approximating solutions u_n

In this section we analyze the behavior of the solutions of the approximating problems (**??**) in the case in which there is no solution in the Sobolev space $H_0^1(\Omega)$.

We consider here the case $\overline{\Omega} = \overline{D}_1 \cup \overline{D}_2$, where $D_1, D_2 \subset \Omega$ are open sets with $|\partial D_2 \cap \partial\Omega|_{N-1} > 0$ and

$$\gamma(x) < 2 \text{ for every } x \in D_1,$$

$$\gamma(x) \geq 2 \text{ for every } x \in \overline{D}_2.$$

This will be referred as condition (H).

In this case Theorem **??** assures that there is no solution $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ of (**??**). We explain what occurs with the approximations u_n in the following result.

Theorem 4.1. *Assume (**??**) for every $\delta > 0$ and that condition (H) is satisfied. Then the weak limit u of the sequence u_n satisfies that $0 < u$ in D_1, $u \equiv 0$ in D_2, $u \in H_0^1(D_1) \cap L^\infty(\Omega)$ and u satisfies*

$$\begin{cases} -\Delta u + \dfrac{|\nabla u|^2}{u^{\gamma(x)}} = f & \text{in } D_1, \\ u = 0 & \text{on } \partial D_1. \end{cases} \tag{4.1}$$

Moreover, there exists a Radon measure $\nu_0 \in \mathcal{M}(\Omega)$ supported in D_2 such that, in the sense of distributions,

$$\begin{cases} -\Delta u + \dfrac{|\nabla u|^2}{u^{\gamma(x)}} \chi_{D_1} = f - \nu_0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.2}$$

Proof. Observe that the sequence u_n of solutions of (**??**) weakly converges in $H_0^1(\Omega)$ to $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ (using the Sobolev's estimate proved in Proposition **??**). Moreover, Proposition **??** is valid for $\omega \subset\subset D_1$ (observe that what we use in that result is that $\gamma(x) < 2$ for every x in a strip around $\partial\Omega$ inside Ω) and, in particular, $u > 0$ in D_1. Even more, as in (**??**) we have that

$$\nu_n = \frac{u_n |\nabla u_n|^2}{(u_n + \frac{1}{n})^{\gamma(x)+1}(1 + \frac{1}{n}|\nabla u_n|^2)} \text{ is bounded in } L^1(\Omega),$$

therefore, the result of [**?**] yields that (up to subsequences) ∇u_n converges to ∇u almost everywhere in Ω. Thus there exists a positive Radon measure $\nu \in \mathcal{M}(\Omega)$ such that, up to a subsequence, $\nu_n \to \nu$ in the weak-$*$ topology of measures. Since we can use Fatou lemma to obtain that $\frac{|\nabla u|^2}{u^{\gamma(x)}} \in L^1(D_1)$ we can even assume that $\nu = \frac{|\nabla u|^2}{u^{\gamma(x)}} \chi_{D_1}(x) + \nu_0$, where ν_0 is a nonnegative bounded Radon measure on Ω.

Now we claim that $u = 0$ in $\overline{D}_2$. Indeed, if $D = \{x \in \overline{D}_2 : u(x) > 0\}$ and $|D| > 0$ then, since $\gamma(x) \geq 2$ in $\overline{D}_2$ we can argue as in the proof of Theorem **??** (observe that $u \in H^1(D)$ and $u = 0$ on a subset of ∂D of positive measure). For example, if $D = D_2$ then $u = 0$ on $\partial D_2 \cap \partial\Omega$. As another example, we mention that if $\overline{D} \subset D_2$ then $u = 0$ on ∂D. Thus we reach a contradiction and the claim is proved.

As a consequence $u \in H_0^1(D_1)$ and, as in the proof of Theorem **??**, we can pass to the limit in the approximating problems to prove (**??**). In addition, (**??**) follow from the weak-$*$ convergence of ν_n. Finally, in order to prove that ν_0 is supported in D_2 we observe that, taking $\phi \in C_0^\infty(D_1)$ as test function in (**??**) and (**??**) and substracting we obtain that

$$\int_\Omega \phi \, d\nu_0 = 0.$$

On the other hand, taking $\phi \in C_0^\infty(D_2)$ as test function in (**??**) and using that $u = 0$ in D_2 we get that

$$\int_\Omega \phi \, d\nu_0 = \int_\Omega f\phi. \qquad \square$$

Remark 4.2. Now we just remark that when we consider (H) in the case $\partial D_2 \cap \partial\Omega = \emptyset$ we have proved that the weak limit u of the sequence u_n satisfies that $0 < u$ in Ω and it is a solution to (**??**). This is a consequence of the fact that we have $\gamma(x) < 2$ in a strip near the boundary of Ω, and hence the approximations converge to a solution to (**??**) as was proved in Theorem **??**.

The case in which $\partial D_2 \cap \partial\Omega \neq \emptyset$ with $|\partial D_2 \cap \partial\Omega|_{N-1} = 0$ is left open.

Remark 4.3. Finally we point out that, as in [**?**] or [**?**], the above results can be generalized to a more general class of differential operators. More precisely we can consider

$$\begin{cases} -\mathrm{div}(M(x,u)\nabla u) + Q(x,u)\dfrac{|\nabla u|^2}{u^{\gamma(x)}} = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

with $M(x,s)$ a matrix with coefficients $m_{i,j}(x,s)$, such that Q and $m_{i,j}$ are Carathéodory functions, $i,j = 1,\dots,N$ and for some positive constants a, b, α, β it is satisfied that

$$0 < a \le Q(x,s) \le b, \quad s > 0,$$
$$0 < \alpha|\xi|^2 \le M(x,s)\xi\cdot\xi, \quad |M(x,s)| \le \beta, \quad s > 0,\ x \in \Omega,\ \xi \in \mathbb{R}^N.$$

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