LOCAL TRIPLE DERIVATIONS ON REAL C*-ALGEBRAS AND JB*-TRIPLES

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ABSTRACT. We study when a local triple derivation on a real JB*-triple is a triple derivation. We find an example of a (real linear) local triple derivation on a rank-one Cartan factor of type I which is not a triple derivation. On the other hand, we find sufficient conditions on a real JB*-triple E to guarantee that every local triple derivation on E is a triple derivation.

1. INTRODUCTION

Derivations on Banach algebras, C*-algebras, and more recently on real and complex JB*-triples, constitute a field of interest for the researcher in Functional Analysis and operator algebras. Derivations on C*-algebras constitute one of the most studied subclasses of the class of linear operators and they have been considered since the very earliest stages of development of the theory of Banach algebras. One of the reasons for this interest is that each *-derivation on a C*-algebra A (respectively, each triple derivation on a JB*-triple E) generates a one parameter group of *-automorphisms on A (respectively, a one parameter group of *-triple-automorphisms on E). We recall that an (associative) *derivation* on a Banach algebra A is a linear mapping $D : A \to A$ satisfying $D(ab) = D(a)b + aD(b)$, for every a, b in A. It is due to S. Sakai [31] that every (associative) derivation on a C*-algebra is automatically continuous; actually, J.R. Ringrose proved that the same statement holds for every (associative) derivation from a C*-algebra A to a Banach A-bimodule M (compare [30]).

In 1990, R.V. Kadison introduced the concept of local derivation. A linear mapping T on a Banach algebra A is a *local derivation* if for each a in A there is a derivation D_a on A with $D_a(a) = T(a)$. Kadison's contribution on local derivations determines that every bounded local derivation on a von Neumann algebra (i.e. a C*-algebra which is also a dual Banach space) is a derivation (cf. [21]). Concerning local derivations on C*-algebras, the culminating result is due to B.E. Johnson [20], who proved that every bounded local derivation from a C*-algebra A into a Banach A-bimodule is a derivation. Furthermore, local derivations on C*-algebras are continuous even if not assumed a priori to be so.

In recent years, the research on local derivation gave rise to the study of local triple derivations on the wider class of JB*-triples. A simple motivation relies in the fact that every C*-algebra, when equipped with the ternary product $\{x, y, z\} = \frac{1}{2}(xy^*z + zy^*x)$, lies in the wider category of JB*-triples. A JB*-triple is essentially a complex Banach space E equipped with a triple product $\{.,.,.\} : E \times E \times E \to E$ which is conjugate linear in the middle variable and symmetric and linear in the outer variables satisfying certain algebraic-analytic axioms (see Subsection 1.1 for more details). A *triple derivation* on a JB*-triple E is a linear mapping $\delta : E \to E$ satisfying Leibnitz's rule for the triple product, that is,

$$\delta \{a, b, c\} = \{\delta(a), b, c\} + \{a, \delta(b), c\} + \{a, b, \delta(c)\}, \tag{1}$$

for every $a, b, c \in E$. Inspired by the research line opened by Ringrose and Sakai, T. Barton and Y. Friedman proved that triple derivations on JB*-triples are automatically continuous (cf. [3]).

Local theory for triple derivations was initiated by M. Mackey in [26]. A *local triple derivation* on a JB*-triple E is a linear map $T : E \to E$ such that for each a in E there exists a triple derivation δ_a on E satisfying $T(a) = \delta_a(a)$. In the just mentioned paper, Mackey proves an analogous result to that of Kadison for local triple derivations, and shows that every bounded local triple derivation on a JBW*-triple (i.e. a JB*-triple which is also a dual Banach space) is a triple derivation. A similar result for bounded local triple derivations on unital C*-algebras was established by M. Burgos, J.J. Garcés and the first and third authors of this note (see [7]). The problems and questions about triple derivations have been completely solved in the recent note [8], where M. Burgos and the first and third authors of this note establish that every continuous local triple derivation on a JB*-triple is a triple derivation; moreover, local triple derivations on a JB*-triple are continuous even if not assumed a priori to be so. In the same paper, the authors open the scope to the study of real-linear local triple derivations on JB*-triples and to the study of local triple derivations in the wider class of real JB*-triples (i.e. norm-closed real subtriples of JB*-triples). In the real setting it is proved that every local triple derivation T on a real JB*-triple E is continuous and for every x in E we have:

$$T \{x, x, x\} = 2 \{T(x), x, x\} + \{x, T(x), x\}.$$

When E is a (complex) JB*-triple and T is complex linear, the polarisation formula

$$8\{x, y, x\} = \sum_{k=0}^{3} \sum_{j=1}^{2} i^k (-1)^j \left\{x + i^k y + (-1)^j z, x + i^k y + (-1)^j z, x + i^k y + (-1)^j z\right\}$$

gives

$$T \{x, y, z\} = \{T(x), y, z\} + \{x, T(y), z\} + \{x, y, T(z)\},$$

and hence T is a triple derivation. In the real setting, the lacking of a polarisation formula in the terms given above, makes invalid the above argument. Similar difficulties appear in the studies of (surjective) real linear isometries between JB*-triples and real JB*-triples developed by Ch.-H. Chu, T. Dang, B. Russo, and B. Ventura [10], T. Dang [11], J.M. Isidro, W. Kaup and A. Rodríguez [19], F.J. Fernández-Polo, J. Martínez and A.M. Peralta [14] and M. Apazoglou and A.M. Peralta [2]. The current knowledge concerning local triple derivations on real JB*-triples leads us to the following problem:

Problem 1.1. [8, Problem 2.6] *Is every (bounded) local triple derivation on a real JB*-triple a triple derivation?*

In this paper we give a complete answer to the above question. Unfortunately, the real setting provides some difficulties and obstacles, and the answer to Problem 1.1 is, in general, negative (compare Example 2.4). However, in our main result (Theorem 3.4) we shall determine sufficient conditions on a real JB*-triple E to guarantee that every local triple derivation on E is a triple derivation. Among the consequences it follows that every local triple derivation on a real C*-algebra is a triple derivation. It is superfluous to say that the techniques needed in the real setting are completely different and independent from those employed in the case of complex JB*-triples.

1.1. **Notation and preliminaries.** A complex Banach space, E, together with a triple product $\{.,.,.\} : E \times E \times E \to E$, which is continuous, symmetric and linear in the outer variables and conjugate linear in the inner one, satisfying:

a) *Jordan Identity*:

$$L(a,b)\{x,y,z\} = \{L(a,b)x,y,z\} - \{x, L(b,a)y, z\} + \{x, y, L(a,b)z\},$$

for all $a, b, x, y, z \in E$, where $L(a,b)x := \{a,b,x\}$;

b) For each $a \in E$, the operator $L(a,a)$ is hermitian with non-negative spectrum and $\| \{a,a,a\} \| = \|a\|^3$,

is said to be a (complex) *JB*-triple.*

Every C*-algebra is a complex JB*-triple with respect to the triple product $\{x,y,z\} = \frac{1}{2}(xy^*z + zy^*x)$, and in the same way every JB*-algebra with respect to $\{a,b,c\} = (a \circ b^*) \circ c + (c \circ b^*) \circ a - (a \circ c) \circ b^*$. Other examples of JB*-triples are given by the so-called building blocks of JB*-triples. We refer to the (complex) *Cartan factors* of type 1 to 6 defined as follows: the Cartan factor of type 1 (also denoted by $I^{\mathbb{C}}$) is the Banach space $L(H,K)$ of bounded linear operators between two complex Hilbert spaces, H and K, where the triple product is defined by $\{x,y,z\} = 2^{-1}(xy^*z + zy^*x)$. Cartan factors of types 2 and 3 are the subtriples of $L(H)$ defined by $II^{\mathbb{C}} = \{x \in L(H) : x = -jx^*j\}$ and $III^{\mathbb{C}} = \{x \in L(H) : x = jx^*j\}$, respectively, where j is a conjugation on H. A Cartan factor of type 4 or IV is a complex spin factor, that is, a complex Hilbert space provided with a conjugation $x \mapsto \overline{x}$, triple product

$$\{x,y,z\} = \langle x/y \rangle z + \langle z/y \rangle x - \langle x/\bar{z} \rangle \bar{y},$$

and norm given by $\|x\|^2 = \langle x/x \rangle + \sqrt{\langle x/x \rangle^2 - | \langle x/\overline{x} \rangle |^2}$. The Cartan factors of types 5 and 6 consist of matrices over the eight dimensional complex Cayley division algebra $\mathbb{O}$; the type VI is the space of all hermitian 3x3 matrices over $\mathbb{O}$, while the type V is the subtriple of 1x2 matrices with entries in $\mathbb{O}$ (compare [25], [15], [12] and [23]).

In 1995, J.M. Isidro, W. Kaup and A. Rodríguez introduce a class of real Banach spaces, called real JB*-triples, containing all real and complex C*-algebras and all (complex) JB*-triples. A *real JB*-triple* is a norm-closed real subtriple of a (complex) JB*-triple (cf. [19]). Every real JB*-triple E can be also regarded as a *real form* of a complex JB*-triple, that is, there exist a (complex) JB*-triple E_c and a conjugate linear isometry $\tau : E_c \to E_c$ of period 2 such that $E = \{b \in E_c : \tau(b) = b\}$. We can actually identify E_c with the complexification of E.

Real forms of Cartan factors are called *real Cartan factors.* W. Kaup classified all real Cartan factors in [23, Corollary 4.4] (the classification in the finite dimensional case is due to O. Loos [25, pages 11.5-11.7]). Real Cartan factors as classified, up to triple isomorphisms, as follows: Let X and Y be two real Hilbert spaces, let P and Q be two Hilbert spaces over the quaternion field $\mathbb{H}$, and finally, let H be a complex Hilbert.

(1) $I^{\mathbb{R}} := \mathcal{L}(X, Y)$

(5) $II^{\mathbb{H}} := \{w \in \mathcal{L}(P) : w^* = w\}$

(2) $I^{\mathbb{H}} := \mathcal{L}(P, Q)$

(6) $III^{\mathbb{R}} := \{x \in \mathcal{L}(X) : x^* = x\}$

(3) $II^{\mathbb{C}} := \{z \in \mathcal{L}(H) : z^* = z\}$

(7) $III^{\mathbb{H}} := \{w \in \mathcal{L}(P) : w^* = -w\}$

(4) $II^{\mathbb{R}} := \{x \in \mathcal{L}(X) : x^* = -x\}$

(8) $IV^{r,s} := E$, where $E = X_1 \oplus^{\ell_1} X_2$ and X_1, X_2 are closed linear subspaces, of dimensions r and s, of a real Hilbert space, X, of dimension greater or equal to three, so that $X_2 = X_1^{\perp}$, with triple product

$$\{x, y, z\} = \langle x/y \rangle z + \langle z/y \rangle x - \langle x/\bar{z} \rangle \bar{y},$$

where $\langle ./. \rangle$ is the inner product in X and the involution $x \to \bar{x}$ on E is defined by $\bar{x} = (x_1, -x_2)$ for every $x = (x_1, x_2)$. This factor is known as a *real spin factor.*

(9) $V^{\mathbb{O}_0} := M_{1,2}(\mathbb{O}_0)$

(11) $VI^{\mathbb{O}_0} := H_3(\mathbb{O}_0)$

(10) $V^{\mathbb{O}} := M_{1,2}(\mathbb{O})$

(12) $VI^{\mathbb{O}} := H_3(\mathbb{O})$

where $\mathbb{O}_0$ is the real split Cayley algebra over the field of the real numbers and $\mathbb{O}$ is the real division Cayley algebra (known also as the algebra of real division octonions). The real Cartan factors $(ix) - (xii)$ are called *exceptional real Cartan factors.*

By a *generalized real Cartan factor* we shall mean a real Cartan factor or a complex Cartan factor regarded as a real JB*-triple.

A real or complex JBW*-triple is a JB*-triple which is also a dual Banach space. The second dual of a real or complex JB*-triple is a JBW*-triple (see [13], [19]). Every real or complex JBW*-triple admits a unique (isometric) predual and its product is separately weak*-continuous (compare [4] and [27]).

For each element a in a real or complex JB*-triple, E, the symbol $Q(a)$ will denote the mapping on E defined by $Q(a)(x) = \{a, x, a\}$.

An element e in a real or complex JB*-triple, E, is called a tripotent whenever $\{e, e, e\} = e$. Every tripotent $e \in E$ induces a decomposition of E,

$$E = E_0(e) \oplus E_1(e) \oplus E_2(e),$$

where $E_k(e) := \{x \in E : L(e, e)x = \frac{k}{2}x\}$ is a subtriple of E (compare [25, Theorem 3.13]). The natural projection of E onto $E_k(e)$ will be denoted by $P_k(e)$. This

decomposition is called the Peirce decomposition associated with the tripotent e. The following *Peirce arithmetic* is satisfied:

$$\{E_0(e), E_2(e), E\} = \{E_2(e), E_0(e), E\} = 0,$$

and

$$\{E_i(e), E_j(e), E_k(e)\} \subseteq E_{i-j+k}(e),$$

where $E_l(e) = 0$ for every $l \neq 0, 1, 2$. A tripotent e in E is called *minimal* when $E^1(e) := \{x \in E : Q(e)(x) = x\} = \mathbb{R}e$.

Two elements a, b in a real or complex JB*-triple E are said to be *orthogonal* (written $a \perp b$) if $L(a, b) = 0$. It is known that $a \perp b \Leftrightarrow \{a, a, b\} = 0 \Leftrightarrow \{b, b, a\} = 0$ (cf. [6, Lemma 1]). We shall say that two sets $A, B \subseteq E$ are orthogonal if $a \perp b$, for every $a \in A$ and $b \in B$.

The rank, $r(E)$, of a real or complex JB*-triple E, is the minimal cardinal number r satisfying $card(S) \leq r$ whenever S is an orthogonal subset of E, i.e. $0 \notin S$ and $x \perp y$ for every $x \neq y$ in S.

Let E be a real JB*-triple. It is known that the JB*-subtriple E_a generated by a single element a in E is JB*-triple isomorphic (and hence isometric) to $C_0(L, \mathbb{R})$ for some locally compact Hausdorff space $L \subseteq (0, \|a\|]$, such that $L \cup \{0\}$ is compact. It is also known the existence of a triple isomorphism Ψ from E_a onto $C_0(L, \mathbb{R})$, satisfying $\Psi(a)(t) = t$ ($t \in L$) (compare [22, Lemma 1.14] and [9, §3, page 81]). It follows that in a real JB*-triple of rank one every norm-one element is a tripotent.

Given two elements a, b in a real or complex JB*-triple, E, it follows from the Jordan identity that $\delta(a, b) := L(a, b) - L(b, a)$ is a triple derivation on E. An *inner triple derivation* on E is a triple derivation δ which can be written as finite sum of derivations of the form $\delta(a, b)$ with $a, b \in E$, i.e.,

$$\delta = \sum_{j=1}^{n} \delta(a_j, b_j).$$

We say that E satisfies the *inner derivation property* (IDP) when every triple derivation on E is inner. Unfortunately, there exist examples of real and complex JB*-triples which do not satisfy the IDP (compare [17]). When the space of all inner triple derivations on E is dense in the space of all triple derivations on E, with respect to the strong operator topology of $L(E)$, we shall say that E has the *inner approximation property* (IAP for short). The advantage of the latter property being that every real or complex JB*-triple has the IAP (see [3, Theorem 4.6] and [17, Theorem 5]).

2. Properties of local triple derivations on real Cartan factors

In this section we shall determine those generalized real Cartan factors C such that every local triple derivation on C is a triple derivation.

Let $T : E \to E$ be a local triple derivation on a real JB*-triple. Corollary 2.5 in [8] assures that $T\{a, a, a\} = 2\{T(a), a, a\} + \{a, T(a), a\}$, for every $a \in E$. If we consider the symmetrized triple product $<a, b, c> := \frac{1}{3}(\{a, b, c\} + \{c, a, b\} + \{b, c, a\})$, which is trilinear and symmetric, a real polarisation formula gives that T is a triple derivation of the symmetrized Jordan triple product $< ., ., . >$. In other words, [8, Corollary 2.5] shows that every local triple derivation on a real JB*-triple is a derivation of the symmetrized triple product. We shall see in this section that the

reciprocal statement is true whenever E is a generalized real Cartan factor. In a first step we consider real JB*-triples of rank one.

We recall now the behavior of a local triple derivation on a tripotent element. Let $T : E \to E$ be a local triple derivation on a real JB*-triple and let e be a tripotent in E. Then, by [8, (6), page 6],

$$(2) \qquad P_0(e)(T(e)) = 0 \text{ and } P_2(e)(T(e)) = -Q(e)(T(e)).$$

Proposition 2.1. *Let E be a real JB*-triple of rank one. Every (real linear) triple derivation of the symmetrized Jordan triple product, $T : E \to E$, is a local triple derivation.*

Proof. We have to show that, for every x in E, there exists a triple derivation $\delta : E \to E$ such that $T(x) = \delta(x)$. To this end, let x be a non-zero element in E and set $u := \frac{x}{\|x\|}$. We notice that in a real JB*-triple of rank one every norm-one element is a minimal tripotent (compare subsection 1.1). Finally, it is easily verified, via Peirce arithmetic and (2), that the (inner) triple derivation

$$\delta = \frac{1}{2\|x\|} \delta(T(x) + 3P_1(u)T(x), u),$$

satisfies

$$\delta(x) = \frac{1}{2} \Big(\{T(x), u, u\} + 3\{P_1(u)T(x), u, u\} - \{u, T(x), u\} - 3\{u, P_1(u)T(x), u\} \Big)$$

$$= \frac{1}{2}\|x\| \Big(\{T(u), u, u\} + 3\{P_1(u)T(u), u, u\} - \{u, T(u), u\} \Big)$$

$$= P_2(u)T(x) + P_1(u)T(x) = T(x).$$

$\square$

Haplessly, we do not know if the statement in the above Proposition 2.1 remains valid for every real JB*-triple. That is, we do not know whether every triple derivation of the symmetrized triple product on a real JB*-triple is a local triple derivation. However, Propositions 2.1 and 2.6 guarantee that local triple derivations and triple derivation of the symmetrized triple product on a generalized real Cartan factor are the same notions.

We shall exhibit, in the sequel, a series of results guaranteing that in some particular generalized real Cartan factor of rank one (concretely, real Hilbert spaces regarded as real Cartan factor of type $I^{\mathbb{R}}$ and real spin factors), every local triple derivation is a triple derivation.

Lemma 2.2. *Let E be a generalized real Cartan factor of type $I^{\mathbb{R}}$, and let $T : E \to E$ be a real linear mapping. The following statements are equivalent:*

(a) T is a local triple derivation;
(b) T is a triple derivation for the symmetrized triple product;
(c) T is a bounded skew-symmetric operator (i.e. $T^* = -T$);
(d) T is a triple derivation.

Proof. We recall that E is a real Hilbert space and the triple product of E is given by

$$\{x, y, z\} = \frac{1}{2}(<x/y>z + <z/y>x),$$

where $< \cdot / \cdot >$ is the inner product on E.

The implication $(a) \Rightarrow (b)$ is proved in [8, Corollary 2.5]. The equivalence $(c) \Leftrightarrow (d)$ was established in [17, Lemma 3, Section 3.3], while $(d) \Rightarrow (a)$ is obvious.

We shall prove $(b) \Rightarrow (c)$. Let T be a derivation for the symmetrized triple product $< \cdot, \cdot, \cdot >$. For each $x \in E$, we have that $T\{x, x, x\} = 2\{Tx, x, x\} + \{x, Tx, x\}$ and hence

$$\|x\|^2 T(x) = < T(x)/x > x + \|x\|^2 T(x) + < x/T(x) > x,$$

which gives,

$$< T(x)/x >= - < x/T(x) >, \qquad \forall x \in E,$$

and hence $T^* = -T$. $\square$

We deal now with rank-one real spin factors.

Lemma 2.3. *Let E be a real spin factor of rank one and let $T : E \to E$ be a (real) linear mapping. The following statements are equivalent:*

(a) T is a local triple derivation;
(b) T is a triple derivation for the symmetrized triple product;
(c) T is a bounded skew-symmetric operator ($T^* = -T$);
(d) T is a triple derivation.

Proof. We recall that E is a real Hilbert space with inner product $< \cdot/\cdot >$, whose triple product is defined by $\{x, y, z\} = < x/y > z + < z/y > x - < x/z > y$ (compare [23, Theorem 4.1 and Proposition 5.4]).

The arguments given in the proof of Lemma 2.2 remain valid to prove the implications $(a) \Rightarrow (b)$, $(d) \Rightarrow (a)$, while $(c) \Leftrightarrow (d)$ was essentially obtained in [17, Section 3.2].

$(b) \Rightarrow (c)$ If T is a triple derivation for the symmetrized triple product $< \cdot, \cdot, \cdot >$, then

$$T\{x, x, x\} = 2\{T(x), x, x\} + \{x, T(x), x\},$$

$$\|x\|^2 T(x) = 2 < T(x)/x > x + 2\|x\|^2 T(x) - 2 < T(x)/x > x$$

$$+ 2 < x/T(x) > x - \|x\|^2 T(x),$$

and hence $< x/T(x) >= 0$, for all $x \in E$, which concludes the proof. $\square$

Generalized real Cartan factors of rank one treated in the above Lemmas 2.3 and 2.2 are essentially real Hilbert spaces and every local triple derivation on them is a triple derivation. Unfortunately, there exist other examples of rank-one real JB*-triples having an essentially "complex" or "quaternionic" nature. The next example shows the existence of rank-one Cartan factors C having a complex structure such that there exists a local triple derivation on C which is not a triple derivation.

Example 2.4. *Let $H = \mathbb{C}^2$ be the 2-dimensional complex Hilbert space equipped with its natural inner product $< (\lambda_1, \lambda_2)|(\mu_1, \mu_2) >= \lambda_1 \overline{\mu_1} + \lambda_2 \overline{\mu_2}$. We equip H with its structure of (rank-one) complex Cartan factor of type $I^{\mathbb{C}}$, that is,*

$$2\{\lambda, \mu, \nu\} = < \lambda|\mu > \nu + < \nu|\mu > \lambda.$$

Of course, we consider the real JB-triple obtained from H by restricting scalar multiplication to the real numbers.*

Let $T : H = \mathbb{R}^4 \to H = \mathbb{R}^4$ be the real linear mapping whose associate matrix,

in the canonical basis, is given by $\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, *equivalently,*

$$T(\lambda_1, \lambda_2) = (\Re e(\lambda_2), -\Re e(\lambda_1)).$$

Clearly, T is only $\mathbb{R}$-linear. It is not hard to check that $\Re e < T(\lambda_1, \lambda_2)|(\lambda_1, \lambda_2) >= 0$, and thus,

$$2\{T(\lambda_1, \lambda_2), (\lambda_1, \lambda_2), (\lambda_1, \lambda_2)\} + \{(\lambda_1, \lambda_2), T(\lambda_1, \lambda_2), (\lambda_1, \lambda_2)\}$$

$$=< T(\lambda_1, \lambda_2)|(\lambda_1, \lambda_2) > (\lambda_1, \lambda_2) + (|\lambda_1|^2 + |\lambda_2|^2)T(\lambda_1, \lambda_2)$$

$$+ < (\lambda_1, \lambda_2)|T(\lambda_1, \lambda_2) > (\lambda_1, \lambda_2)$$

$$= (|\lambda_1|^2 + |\lambda_2|^2)T(\lambda_1, \lambda_2) + 2\Re e < T(\lambda_1, \lambda_2)|(\lambda_1, \lambda_2) > (\lambda_1, \lambda_2)$$

$$= (|\lambda_1|^2 + |\lambda_2|^2)T(\lambda_1, \lambda_2) = T\{(\lambda_1, \lambda_2), (\lambda_1, \lambda_2), (\lambda_1, \lambda_2)\},$$

which shows that $T\{x, x, x\} = 2\{T(x), x, x\} + \{x, T(x), x\}$, for every $x \in H$. A priori, this is not enough to guarantee that T is a local derivation. However, Proposition 2.1 assures that T is a local triple derivation.

On the other hand, since

$$T\{(1, 0), (i, 0), (1, 0)\} = (0, 0)$$

and

$$2\{T(1, 0), (i, 0), (1, 0)\} + \{(1, 0), T(i, 0), (1, 0)\} = (0, i),$$

we deduce that T is not a triple derivation.

Motivated by an example given by R.V. Kadison in [21], M. Mackey provides an example of a local triple derivation on the *-algebra $\mathbb{C}(x)$, of rational functions in the real variable x over $\mathbb{C}$, which is not a triple derivation. The particularity of the real setting provides, in Example 2.4, a new example of a real linear local triple derivation on a finite dimensional (complex) JB*-triple which is not a triple derivation.

We shall give a shorter argument to justify the statements in the above Example 2.4. The following observation will be very useful.

Proposition 2.5. *Let E be a complex JB*-triple. Every real linear triple derivation $\delta : E \to E$ is complex linear.*

Proof. Having in mind that, for each a, b in E, $L(a, b) : E \to E$ is $\mathbb{C}-$linear, every (real linear) inner derivation on E is $\mathbb{C}-$linear.

Suppose now that $\delta : E \to E$ is a real linear derivation. Since every real JB*-triple satisfies IAP, given $\varepsilon > 0$ and $x \in E$, there exists a inner derivation $\widehat{\delta}$ such that $\|\delta(x) - \widehat{\delta}(x)\| < \frac{\varepsilon}{2}$ and $\|\delta(ix) - \widehat{\delta}(ix)\| < \frac{\varepsilon}{2}$. Therefore

$$\|i\delta(x) - \delta(ix)\| \le \|i\delta(x) - i\widehat{\delta}(x)\| + \|\widehat{\delta}(ix) - \delta(ix)\| < \varepsilon.$$

The arbitrariness of ε and x guarantee the desired statement. $\square$

Since the mapping T defined in Example 2.4 is defined on a complex JB*-triple but it is not complex linear, it follows from the above proposition that T is not triple derivation.

After dealing with generalized real Cartan factors of rank one, we shall show that local triple derivations on generalized real Cartan factors of rank > 1 are triple derivations.

Proposition 2.6. *Let C be a generalized real Cartan factor of rank > 1 and let $T : C \to C$ be a linear map. The following are equivalent:*

(a) *T is a triple derivation;*
(b) *T is a local triple derivation;*
(c) *T is a triple derivation of the symmetrized triple product.*

Proof. The implication $(b) \Rightarrow (c)$ is a consequence of [8, Corollary 2.5], while $(a) \Rightarrow (b)$ is clear. To prove $(c) \Rightarrow (a)$, let $T : C \to C$ be a triple derivation of the symmetrized triple product $< . , . , . >$. It is well known in real JB*-triple theory that $\{\exp(tT) : C \to C\}_{t \in \mathbb{R}}$ is a one-parameter group of automorphisms of the symmetrized triple product. By [19, Theorem 4.8], $\exp(tT)$ is a surjective isometry for every real t. Since C is of rank > 1, Corollary 2.15 in [14] assures that $\exp(tT)$ is a triple automorphism of the original triple product and hence

$$\exp(tT) \{x, y, z\} = \{\exp(tT)(x), \exp(tT)(y), \exp(tT)(z)\},$$

for every $x, y, z \in C$ and $t \in \mathbb{R}$. Finally, the identity

$$\frac{\partial}{\partial t}|_{t=0} (\exp(tT) \{x, y, z\}) = \frac{\partial}{\partial t}|_{t=0} (\{\exp(tT)(x), \exp(tT)(y), \exp(tT)(z)\}),$$

gives $T \{x, y, z\} = \{T(x), y, z\} + \{x, T(y), z\} + \{x, y, T(z)\}$. $\qquad\square$

Combining Propositions 2.1 and 2.6 we can also deduce that local triple derivations and triple derivations of the symmetrized triple product on a generalized real Cartan factor define the same objects.

3. Local triple derivations on general real JB*-triples

In this section we shall find sufficient conditions on a general real JB*-triple to guarantee that every local triple derivation on it is a triple derivation. We begin stating several properties of local triple derivations.

We recall that a subspace I of a real JB*-triple E is a *triple ideal* if $\{E, E, I\} + \{E, I, E\} \subseteq I$. It is known that a subtriple I of E is a triple ideal if and only if $\{E, E, I\} \subseteq I$ or $\{E, I, E\} \subseteq I$ or $\{E, I, I\} \subseteq I$ (compare [5]).

Lemma 3.1. *Let $T : E \to E$ be a triple derivation of the symmetrized triple product on a real JB*-triple and let I be a triple ideal of E. Then $T(I) \subseteq I$.*

Proof. Given $a \in I$, we can find b in the JB*-subtriple of E generated by a such that $\{b, b, b\} = a$ (notice that, in particular, b lies in I). Since $T(a) = T \{b, b, b\} = 2 \{T(b), b, b\} + \{b, T(b), b\} \in I$, we get the desired statement. $\qquad\square$

Since the bidual, E^{**}, of a real JB*-triple E is a JBW*-triple (see [19]), the following lemma, which is a variant of [18, Proposition 2.1], follows from the separate weak*-continuity of the triple product of E^{**} (cf. [27]) and the weak*-density of E in E^{**}.

Lemma 3.2. *Let E be a real JB*-triple and let $T : E \to E$ be a triple derivation of the symmetrized triple product. Then $T^{**} : E^{**} \to E^{**}$ is a weak*-continuous triple derivation of the symmetrized triple product.* $\square$

The atomic decomposition established in [29, Theorem 3.6] assures that every JBW*-triple W decomposes as an orthogonal sum

$$W = A \oplus^{\infty} N,$$

where A and N are weak*-closed ideals of W, A being the weak*-closed real linear span of all minimal tripotents in W, N containing no minimal tripotents and $A \perp N$. It is also proved in [29, Theorem 3.6] that A is an orthogonal sum of weak*-closed triple ideals which are generalized real Cartan factors.

The tools required in our arguments will need the following Gelfand-Naimark type theorem for real JB*-triples, which has been borrowed from [14, Proposition 3.1].

Proposition 3.3. *Let E be a real JB*-triple. Let A denote the atomic part of E^{**}, $j : E \hookrightarrow E^{**}$ the canonical embedding, and $\pi : E^{**} \to A$ the canonical projection of E^{**} onto A. Then A writes as an orthogonal sum of weak*-closed triple ideals which are generalized real Cartan factors, and the mapping $\pi \circ j : E \to A$ is an isometric triple embedding.* $\square$

We are now in a position to give sufficient conditions on a real JB*-triple E to guarantee that every local triple derivation on E is a triple derivation. We recall first, that, by [23, Proposition 5.4], every real JB*-triple of rank one is precisely one of the following rank-one generalized real Cartan factors: a rank-one type $I^{\mathbb{R}}$, $I^{\mathbb{C}}$, $I^{\mathbb{H}}$, a rank-one real spin factor $IV^{n,0}$, and $V^{\mathbb{O}} := M_{1,2}(\mathbb{O})$.

Theorem 3.4. *Let E be a real JB*-triple whose second dual doesn't contain rank-one generalized real Cartan factors of types $I^{\mathbb{C}}$, $I^{\mathbb{H}}$ nor $V^{\mathbb{O}} := M_{1,2}(\mathbb{O})$. Then every local triple derivation on E is a triple derivation.*

Proof. Let $T : E \to E$ be a local triple derivation on E. We have already commented that Corollary 2.5 in [8] assures that T is a triple derivation of the symmetrized triple product. Lemma 3.2 implies that $T^{**} : E^{**} \to E^{**}$ is a triple derivation of the symmetrized triple product. We have also argued that, by the atomic decomposition of E^{**}, there exists a family of mutually orthogonal, weak*-closed triple ideals $\{C_i : i \in \Lambda\} \cup \{N\}$ of E^{**} such that

$$E^{**} = \left(\bigoplus_i^{\ell_\infty} C_i \right) \bigoplus^{\ell_\infty} N.$$

Lemma 3.1 shows that $T^{**}(N) \subseteq N$ and $T^{**}(C_i) \subseteq C_i$, for every $i \in \Lambda$. By hypothesis, each C_i is a generalized real Cartan factor of rank > 2 or a rank-one generalized real Cartan factor of type $I^{\mathbb{R}}$ (i.e. $L(H, \mathbb{R})$, for a real Hilbert space H), or a real spin factor of rank one. Now, Lemmas 2.2 and 2.3 and Proposition 2.6 imply that $T^{**}|_{C_i} : C_i \to C_i$ is a triple derivation for every i, and hence $T^{**}|_{\bigoplus_i^{\ell_\infty} C_i} : \bigoplus_i^{\ell_\infty} C_i \to \bigoplus_i^{\ell_\infty} C_i$ also is a triple derivation.

Let us denote $A = \bigoplus_i^{\ell_\infty} C_i$ and keep the notation employed in Proposition 3.3. Since the mapping $\Phi = \pi \circ j : E \to A$ is an isometric triple embedding,

$T^{**}|_A : A \to A$ is a triple derivation, and $\Phi T = T^{**}\Phi$, we have

$$\Phi T(\{x, y, z\}) = T^{**}|_A \{\Phi(x), \Phi(y), \Phi(z)\}$$

$$= \{T^{**}|_A\Phi(x), \Phi(y), \Phi(z)\} + \{\Phi(x), T^{**}|_A\Phi(y), \Phi(z)\} + \{\Phi(x), \Phi(y), T^{**}|_A\Phi(z)\}$$

$$= \Phi\left(\{T(x), y, z\} + \{x, T(y), z\} + \{x, y, T(z)\}\right),$$

witnessing that T is a triple derivation. $\qquad\square$

Remark 3.5. *The rank-one, generalized real Cartan factors avoided in Theorem 3.4 appear also in the problem of determining those real JB*-triples E satisfying that every surjective isometry on E is a triple isomorphism. More precisely, W. Kaup established, in [23, Lemma 5.12], that a real JB*-triple of rank one, E, satisfies that every surjective linear isometry on E is triple isomorphism if and only if E is isomorphic to a rank-one real Cartan factor of type $I^{\mathbb{R}}$ or a real spin factor of rank one, equivalently, E is not isomorphic to one of the rank-one generalized real Cartan factors we have avoided in the statement of Theorem 3.4.*

In order to get some applications of our main result, we refresh some definitions. A real JB*-algebra is a norm closed self-adjoint real Jordan subalgebra of a (complex) JB*-algebra. The class of J*B-algebras introduced by K. Alvermann in [1] coincides with the class of unital real JB*-algebras. As in the complex setting, every real C*-algebra is a real JB*-algebra with respect to the Jordan product and every real JB*-algebra is a real JB*-triple with product

$$\{a, b, c\} = (a \circ b^*) \circ c + (c \circ b^*) \circ a - (a \circ c) \circ b^*$$

(see [16] and [24] for the basic background on JB- and JB*-algebras and real C*-algebras, respectively).

The proof of Corollary 4.4 in [14] shows that the bidual of a real JB*-algebra can only contain rank-one generalized real Cartan factors of the form $\mathbb{R}$, $\mathbb{C}$ or $IV^{n,0}$. So, the following result follows a direct consequence of Theorem 3.4.

Corollary 3.6. *Every local triple derivation on a real JB*-algebra or a real C*-algebra is a triple derivation.* $\qquad\square$

The counter-example exhibited in Example 2.4 and the consequent restrictions appearing in the real setting are inviting us to explore the connections with 2-local triple derivations. In the associative setting, 2-local derivations and 2-local homomorphisms were introduced by P. Šemrl in [32]. Given a associative algebra A, a mapping $T : A \to A$ is called a *2-local derivation* if for every $a, b \in A$ there is an (associative) derivation $D_{a,b} : A \to A$, depending on a and b, such that $D_{a,b}(a) = T(a)$ and $D_{a,b}(b) = T(b)$. In the just quoted paper, Šemrl shows that if H is an infinite-dimensional separable Hilbert space, then every 2-local derivation on $L(H)$ is a derivation of $L(H)$. Šemrl's paper has motivated over forty publications during the last years. Here we extend the notion to the triple setting. Let $T : E \to E$ be a mapping on a real or complex JB*-triple. We shall say that T is a *2-local triple derivation* when given elements $a, b \in A$ there is a triple derivation $\delta_{a,b} : E \to E$, depending on a and b, such that $\delta_{a,b}(a) = T(a)$ and $\delta_{a,b}(b) = T(b)$.

Every 2-local triple derivation $T : E \to E$ is 1-homogeneous, that is, $T(\lambda a) = \lambda T(a)$, for every a in E and every scalar λ. Indeed, let us take a triple derivation $\delta_{a,\lambda a} : E \to E$ such that $T(\lambda a) = \delta_{a,\lambda a}(\lambda a) = \lambda\delta_{a,\lambda a}(a) = \lambda T(a)$. The above Example 2.4 provides an example of a local triple derivation on a rank-one real

JB*-triple which is not a triple derivation. This kind of counterexample is not valid for additive 2-local triple derivation. Indeed, let $T : E \to E$ be an additive 2-local triple derivation on a real JB*-triple. We have already seen that, in this case, T is linear. For each derivation $\delta : E \to E$ the mapping $\widetilde{\delta} : \widetilde{E} \to \widetilde{E}$, $\widetilde{\delta}(x+iy) := \delta(x)+i\delta(y)$ defines a $\mathbb{C}$-linear triple derivation on the complexification of E (compare [17, Remark 1]). We also consider a $\mathbb{C}$-linear mapping on $\widetilde{E}$ defined by $\widetilde{T}(x+iy) := T(x)+iT(y)$. Let us take $a, b \in E$ and a triple derivation $\delta_{a,b} : E \to E$ satisfying $\delta_{a,b}(a) = T(a)$ and $\delta_{a,b}(b) = T(b)$. In this case, the mapping $\widetilde{\delta}_{a,b}$ is a derivation on $\widetilde{E}$ and $\widetilde{T}(a+ib) = T(a)+iT(b) = \delta_{a,b}(a)+i\delta_{a,b}(b) = \widetilde{\delta}_{a,b}(a+ib)$. Therefore, $\widetilde{T}$ is a local triple derivation on the complex JB*-triple $\widetilde{E}$, and hence Theorems 2.4 and 2.8 in [8] assure that $\widetilde{T}$ (and hence T) is a (continuous) triple derivation.

Corollary 3.7. *Let $T : E \to E$ be a (not necessarily continuous) additive 2-local triple derivation on a real JB*-triple. Then T is a continuous triple derivation.* $\square$

Problem 3.8. *Is every 2-local triple derivation on a real JB*-triple additive? Equivalently, is every 2-local triple derivation on a real JB*-triple a triple derivation?*

References

[1] K. Alvermann, Real normed Jordan Banach algebras with an involution, *Arch. Math. (Basel)* **47**, no. 2, 135-150 (1986).

[2] M. Apazoglou and A. M. Peralta, Linear isometries between real JB*-triples and C*-algebras, *Quart. J. Math. Oxford*, 2013; doi: 10.1093/qmath/hat033

[3] T.J. Barton, Y. Friedman, Bounded derivations of JB*-triples, *Quart. J. Math. Oxford* **41**, 255-268 (1990).

[4] T.J. Barton, R.M. Timoney, Weak*-continuity of Jordan triple products and its applications. Math. Scand. **59**, 177-191 (1986).

[5] L.J. Bunce, Structure of representations and ideals of homogeneous type in Jordan algebras, *Quart. J. Math. Oxford* (2) **37**, no. 145, 1-10 (1986).

[6] M. Burgos, F.J. Fernández-Polo, J. Garcés, J. Martínez, A.M. Peralta, Orthogonality preservers in C*-algebras, JB*-algebras and JB*-triples, *J. Math. Anal. Appl.* **348**, 220-233 (2008).

[7] M. Burgos, F.J. Fernández-Polo, J.J. Garcés, A.M. Peralta, Local triple derivations on C*-algebras, to appear in *Communications in Algebra*.

[8] M. Burgos, F.J. Fernández-Polo, A.M. Peralta, Local triple derivations on C*-algebras and JB*-triples, preprint 2013. arXiv:1303.4569v2

[9] M. Burgos, A.M. Peralta, M. Ramírez, M.E. Ruiz Morillas, von Neumann regularity in Jordan-Banach triples, to appear in *Proceedings of Jordan structures in Algebra and Analysis Meeting. Tribute to El Amin Kaidi for his 60th birthday. Almería, 20, 21 y 22 de Mayo de 2009*, Edited by J. Carmona et al., Universidad de Almería, Ed. Círculo Rojo 2010.

[10] Ch.-H. Chu, T. Dang, B. Russo, B. Ventura, Surjective isometries of real C*-algebras, *J. London Math. Soc.* **47**, 97-118 (1993).

[11] T. Dang, Real isometries between JB*-triples, *Proc. Amer. Math. Soc.* **114**, 971-980 (1992).

[12] T. Dang, Y. Friedman, Classification of JB*-triple factors and applications, *Math. Scand.* **61**, 292-330 (1987).

[13] S. Dineen, The second dual of a JB*-triple system, in: J. Mujica (Ed.), Complex analysis, Functional Analysis and Approximation Theory, North-Holland, Amsterdam, 1986.

[14] F.J. Fernández-Polo, J. Martínez Moreno, and A.M. Peralta, Surjective isometries between real JB*-triples, *Math. Proc. Cambridge Phil. Soc.*, **137** 709-723 (2004).

[15] Y. Friedman, B. Russo, A Gelfand-Naimark theorem for JB*-triples, *Duke Math. J.*, **53**, 139-148 (1986).

[16] H. Hanche-Olsen, E. Størmer, *Jordan operator algebras, Monographs and Studies in Mathematics 21*, Pitman, London-Boston-Melbourne 1984.

[17] T. Ho, J. Martinez-Moreno, A.M. Peralta, B. Russo, Derivations on real and complex JB*-triples, *J. London Math. Soc.* (2) **65**, no. 1, 85-102 (2002).

[18] T. Ho, A.M. Peralta, B. Russo, Ternary Weakly Amenable C*-algebras and JB*-triples, to appear in *Quart. J. Math. Oxford* 00 (2012), 1-31; doi:10.1093/qmath/has032.

[19] J.M. Isidro, W. Kaup, A. Rodríguez, On real forms of JB*-triples, *Manuscripta Math.* **86**, 311-335 (1995).

[20] B.E. Johnson, Local derivations on C*-algebras are derivations, *Trans. Amer. Math. Soc.* **353**, 313-325 (2001).

[21] R. Kadison, Local derivations, *J. Algebra* **130**, no. 2, 494-509 (1990).

[22] W. Kaup, A Riemann Mapping Theorem for bounded symmetric domains in complex Banach spaces, *Math. Z.* **183**, 503-529 (1983).

[23] W. Kaup, On real Cartan factors, *Manuscripta Math.* **92**, 191-222 (1997).

[24] B.R. Li, *Real Operator Algebras*, World Scientific Publishing (Singapore), 2003.

[25] O. Loos, *Bounded symmetric domains and Jordan pairs*, Math. Lectures, University of California, Irvine 1977.

[26] M. Mackey, Local derivations on Jordan triples, to appear in *Bull. London Math. Soc.* 2013. doi: 10.1112/blms/bdt007 (arXiv:1207.5394v1).

[27] J. Martínez, A.M. Peralta, Separate weak*-continuity of the triple product in dual real JB*-triples, *Math. Z.* **234**, 635-646 (2000).

[28] A.M. Peralta, B. Russo, Automatic continuity of derivations on C*-algebras and JB*-triples, preprint 2010.

[29] A.M. Peralta, L.L. Stachó, Atomic decomposition of real JBW*-triples, *Quart. J. Math. Oxford* **52**, no. 1, 79-87 (2001).

[30] J.R. Ringrose, Automatic continuity of derivations of operator algebras, *J. London Math. Soc.* (2) **5** , 432-438 (1972).

[31] S. Sakai, On a conjecture of Kaplansky, *Tohoku Math. J.*, **12**, 31-33 (1960).

[32] P. Sˇemrl, Local automorphisms and derivations on $B(H)$, *Proc. Amer. Math. Soc.* **125**, no. 9, 2677-2680 (1997).

[33] V. Shul'man, Operators preserving ideals in C*-algebras, *Studia Math.* **109**, 67-72 (1994).

[34] L.L. Stacho, A projection principle concerning biholomorphic automorphisms, *Acta Sci. Math.* **44**, 99-124 (1982).

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