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Nice operators and surjective isometries ☆

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ABSTRACT

We show the existence of infinite-dimensional Banach spaces in which every nice operator is an isometric isomorphism. Moreover, the results contain, as particular cases, certain previously known facts about the description of the isometric isomorphisms between spaces of continuously differentiable functions.

1. Introduction

Throughout this note it will be assumed that $\mathbb{K}$ is either the real field $\mathbb{R}$ or the complex field $\mathbb{C}$. As usual, we denote

$$\mathbb{T} = \{\alpha \in \mathbb{K} : |\alpha| = 1\}.$$

Given a Banach space X, the symbols B_X and S_X will stand for the closed unit ball and the unit sphere of X, respectively:

$$B_X = \{x \in X : \|x\| \le 1\}, \qquad S_X = \{x \in X : \|x\| = 1\}.$$

Furthermore, E_X will be the set of extreme points of B_X. Notice that E_X can be empty if X is infinite-dimensional. Nevertheless, if X^* denotes the dual Banach space of X, it is well known that $E_{X^*} \neq \emptyset$. On the other hand, for any $A \subset X$, $\operatorname{span} A$ denotes the linear span of the set A.

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Let Y be another Banach space over the same field $\mathbb{K}$ and let $L(X,Y)$ denote the space of linear and bounded mappings from X into Y equipped with the operator norm. According to the custom, we will write $L(X)$ instead of $L(X,X)$. The adjoint of T will be represented by T^*.

Assume for a moment that X and Y are isometrically isomorphic and let T be a linear isometry from X onto Y. If $T_1, T_2 \in B_{L(X,Y)}$ and $T = \frac{1}{2}(T_1 + T_2)$, it is clear that $T^* = \frac{1}{2}(T_1^* + T_2^*)$. Since T^* is an isometric isomorphism, $T^*(E_{Y^*}) = E_{X^*}$ and consequently T^*, T_1^* and T_2^* coincide on E_{Y^*}. Moreover, B_{Y^*} is the closure in the weak-$*$ topology of the convex hull of E_{Y^*} (a well known consequence of Krein–Milman and Banach–Alaoglú theorems). The linearity and weak-$*$ continuity of T^*, T_1^* and T_2^* ensure that $T^* = T_1^* = T_2^*$ and hence $T = T_1 = T_2$. This proves that T is an extreme point of the closed unit ball of $L(X,Y)$.

Extreme points of $B_{L(X,Y)}$ (for arbitrary Banach spaces X and Y) are known in the literature as extreme operators or extreme contractions. In accordance with the previous comments, the identity mapping from X onto itself is an extreme operator for any Banach space X.

One can easily note that the only essential condition on T we used is $T^*(E_{Y^*}) \subset E_{X^*}$. The following concept appears for the first time in [8] and it is motivated precisely by such an inclusion:

An operator $T \in L(X,Y)$ is said to be **nice** if $T^*(E_{Y^*}) \subset E_{X^*}$.

The connection of such operators with extreme contractions was initially studied by Blumenthal, Lindenstrauss and Phelps [3] in the context of continuous function spaces and subsequently by Sharir [10–13] in the same setting and also within the context of L_1-spaces.

As we have already noted, isometric isomorphisms are nice operators and each operator of this last class is an extreme contraction. The overlap between nice operators and extreme contractions does occur in certain significant families of Banach spaces. This is the case, for example, if X and Y are L_1-spaces. On the other hand, if X and Y are continuous function spaces, the coincidence between both types of operators frequently (but not always) takes place. An excellent source of information is the paper by Roy [9]. It contains a complete discussion of extreme points that pays special attention to the particular case of extreme contractions. More recent results can be seen in [1,2,6,7].

Nice operators between continuous function spaces are weighted composition operators. Specifically, if K_1 and K_2 are compact Hausdorff spaces, an operator $T \in L(C(K_1), C(K_2))$ is nice if, and only if, there exists an extreme point e of the closed unit ball of $C(K_2)$ (namely, a continuous function $e : K_2 \to \mathbb{K}$ with $|e(t)| = 1$, $\forall t \in K_2$) and a continuous mapping $\varphi : K_2 \to K_1$ such that

$$Tf = e(f \circ \varphi), \quad \forall f \in C(K_1).$$

Furthermore, T is indeed an isometric isomorphism if and only if φ is a homeomorphism (an influential result universally known as the Banach–Stone theorem).

It is, therefore, clear that even in the case $X = Y$ a nice operator $T : X \to Y$ need not be a surjective isometry. To narrow this comment further, you may think, for example, of the operator $T : (\mathbb{R}^2, \|\cdot\|_\infty) \to (\mathbb{R}^2, \|\cdot\|_\infty)$ given by

$$T(x,y) = (x,x), \quad \forall (x,y) \in \mathbb{R}^2,$$

whose adjoint can be identified with the operator $S : (\mathbb{R}^2, \|\cdot\|_1) \to (\mathbb{R}^2, \|\cdot\|_1)$ defined by

$$S(x,y) = (x+y, 0), \quad \forall (x,y) \in \mathbb{R}^2.$$

On the other hand, if X and Y are finite-dimensional, it is easy to find situations in which each nice operator is an isometric isomorphism. Consider, for example, the case $X = Y = H$, where H is a (finite-dimensional) Hilbert space.

In the third section of this paper we will show the existence of infinite dimensional Banach spaces X and Y with the same property (the coincidence between nice operators and surjective linear isometries). The geometric nature of this fact will become apparent in the fourth section.

A specific construction is not necessary because, as we shall see, there are classical Banach spaces satisfying what is expressed in the preceding paragraph. They are spaces of continuously differentiable functions on compact intervals of $\mathbb{R}$. Therefore we will make a detailed study of nice operators between them. We will consider two natural norms in such spaces and in general neither injectivity nor surjectivity of the operators will be assumed. Our results include, as a particular case, the description of surjective linear isometries, whose precedents can be found for complex scalars in [4] and for both scalar fields in [5]. The latter reference contains very general and relevant results that are applicable to various function spaces with different norms.

2. Continuously differentiable function spaces

In this section we will complete the notation to be used and discuss some basic facts about spaces of differentiable functions with continuous derivative. The proofs are elementary and have been incorporated in view of the preliminary nature of this subject.

Let K be a compact interval of $\mathbb{R}$ and $C^1(K)$ the vector space of scalar valued continuously differentiable functions defined on K. The symbol $l(K)$ will denote the length of K. Furthermore, u_K and i_K will be the elements in $C^1(K)$ given by

$$u_K(t) = 1, \qquad i_K(t) = t, \quad \text{for every } t \in K.$$

If $t \in K$, we will consider the functionals $\delta_t, \delta'_t : C^1(K) \to \mathbb{K}$, defined by

$$\delta_t(x) = x(t), \qquad \delta'_t(x) = x'(t), \quad \text{for each } x \in C^1(K).$$

The following notations will also be used:

$$\nabla_K = \{\delta_t : t \in K\} \quad \text{and} \quad \nabla'_K = \{\delta'_t : t \in K\}.$$

We shall often need the next algebraic property of the newly considered functionals:

Proposition 1. *Let $n \in \mathbb{N}$, $t_1, \dots, t_n \in K$ and $\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n \in \mathbb{K}$. Assume that $t_i \neq t_j$, for any $i, j \in \{1, \dots, n\}$ with $i \neq j$ and $|\alpha_i| + |\beta_i| \neq 0$, for every $i \in \{1, \dots, n\}$. Then the functionals $\alpha_1\delta_{t_1} + \beta_1\delta'_{t_1}$, $\dots, \alpha_n\delta_{t_n} + \beta_n\delta'_{t_n}$ are linearly independent.*

Proof. Let $\lambda_1, \dots, \lambda_n$ be scalars such that $\sum_{i=1}^n \lambda_i(\alpha_i\delta_{t_i} + \beta_i\delta'_{t_i}) = 0$ and define $x^* = \sum_{i=1}^n \lambda_i(\alpha_i\delta_{t_i} + \beta_i\delta'_{t_i})$. For each $i_0 \in \{1, \dots, n\}$ consider the functions $x_{i_0}, y_{i_0} \in C^1(K)$ given by

$$x_{i_0}(t) = \prod_{i\in\{1,\dots,n\}\setminus\{i_0\}} (t - t_i)^2(t - t_{i_0}), \qquad y_{i_0}(t) = \prod_{i\in\{1,\dots,n\}\setminus\{i_0\}} (t - t_i), \quad \forall t \in K.$$

Obviously $x_{i_0}(t_i) = 0$ for any $i \in \{1, \dots, n\}$, $x'_{i_0}(t_i) = y_{i_0}(t_i) = 0$, for any $i \in \{1, \dots, n\}\setminus\{i_0\}$ and $x'_{i_0}(t_{i_0}) = \prod_{i\in\{1,\dots,n\}\setminus\{i_0\}}(t_{i_0} - t_i)^2 \neq 0$. It is therefore clear that $0 = x^*(x_{i_0}) = \lambda_{i_0}\beta_{i_0}x'_{i_0}(t_{i_0})$ and consequently $\lambda_{i_0}\beta_{i_0} = 0$. Since i_0 is arbitrary $x^* = \sum_{i=1}^n \lambda_i\alpha_i\delta_{t_i}$. Thus $0 = x^*(y_{i_0}) = \lambda_{i_0}\alpha_{i_0}y_{i_0}(t_{i_0})$ and the inequality $y_{i_0}(t_{i_0}) \neq 0$ gives us that $\lambda_{i_0}\alpha_{i_0} = 0$.

Let $i \in \{1, \dots, n\}$. According to the above note that $\lambda_i\alpha_i = \lambda_i\beta_i = 0$. Then

$$0 = |\lambda_i||\alpha_i| + |\lambda_i||\beta_i| = |\lambda_i|\big(|\alpha_i| + |\beta_i|\big)$$

and it can be concluded that $\lambda_i = 0$. $\square$

It is convenient to isolate the following consequence of the previous result.

Corollary 2. *Consider $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{K}$ and $t, s \in K$ such that*

$$\alpha_1 \delta_t + \beta_1 \delta_t' = \alpha_2 \delta_s + \beta_2 \delta_s'.$$

Then $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 0$ or $\alpha_1 = \alpha_2$, $\beta_1 = \beta_2$ and $t = s$.

Proof. If $t \neq s$, necessarily $|\alpha_i| + |\beta_i| = 0$ for some $i \in \{1, 2\}$. Hence

$$\alpha_1 \delta_t + \beta_1 \delta_t' = \alpha_2 \delta_s + \beta_2 \delta_s' = 0$$

and evaluating such functionals at u_K and i_K, it is obtained that $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 0$.

If $t = s$ the above mentioned functionals can be used to conclude that $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$. □

The next observation completes our presentation of preliminary results.

Proposition 3. *Let I and J be two intervals of $\mathbb{R}$ with nonempty interior and $\varphi : J \to I$ a function such that $x' \circ \varphi$ belongs to $C^1(J)$ for any $x \in C^1(I)$. Then φ is a constant function.*

Proof. The function φ is differentiable with continuous derivative because, in fact, $\varphi = x' \circ \varphi$ for any primitive x of the identity in I. The problem reduces to proving that if φ is non-constant there exists $x \in C^1(I)$ such that the function $x' \circ \varphi$ does not belong to $C^1(J)$. Indeed, if φ is not constant, there is at least one point t_0 in the interior of the interval J such that $\varphi'(t_0) \neq 0$. Assume for example that $\varphi'(t_0) > 0$. Then φ is strictly increasing in a certain neighborhood of t_0. Now consider the function $x : I \to \mathbb{R}$ given by

$$x(s) = \begin{cases} (s - \varphi(t_0))^2 & \text{if } s \geq \varphi(t_0) \\ 0 & \text{if } s < \varphi(t_0). \end{cases}$$

Obviously $x \in C^1(I)$ and

$$x'(s) = \begin{cases} 2(s - \varphi(t_0)) & \text{if } s \geq \varphi(t_0) \\ 0 & \text{if } s < \varphi(t_0). \end{cases}$$

Let $\{t_n\}$ be a sequence of elements in the aforementioned neighborhood of t_0 such that $\{t_n\} \to t_0$. Then $\{\varphi(t_n)\}$ converges to $\varphi(t_0)$. If $\{t_n\}$ is strictly increasing, the same happens with the sequence $\{\varphi(t_n)\}$ and therefore $x'(\varphi(t_n)) = 0$, for every $n \in \mathbb{N}$. In consequence,

$$\left\{ \frac{x'(\varphi(t_n)) - x'(\varphi(t_0))}{t_n - t_0} \right\} = \{0\} \to 0.$$

On the other hand, if $\{t_n\}$ is strictly decreasing, so does the sequence $\{\varphi(t_n)\}$ and thus $x'(\varphi(t_n)) = 2(\varphi(t_n) - \varphi(t_0))$, for each $n \in \mathbb{N}$. In this way,

$$\left\{ \frac{x'(\varphi(t_n)) - x'(\varphi(t_0))}{t_n - t_0} \right\} = \left\{ \frac{2(\varphi(t_n) - \varphi(t_0))}{t_n - t_0} \right\} \to 2\varphi'(t_0) > 0.$$

Hence, the function $x' \circ \varphi$ is not differentiable at t_0. □

Before concluding this section consider two compact intervals of $\mathbb{R}$, $K_1 = [a_1, b_1]$ and $K_2 = [a_2, b_2]$, and let $\varphi : K_2 \to K_1$ be an isometry. Evidently the existence of isometries from K_2 into K_1 is equivalent to the condition $l(K_2) \leq l(K_1)$. It is well known and easy to check that there exists $c \in \mathbb{R}$ such that either

$$\varphi(t) = t + c, \quad \text{for every } t \in K_2 \tag{1}$$

or

$$\varphi(t) = -t + c, \quad \text{for every } t \in K_2. \tag{2}$$

To be more precise, given a real number c, the equality (1) defines an isometry from K_2 into K_1 if, and only if, $a_1 - a_2 \leq c \leq b_1 - b_2$. Similarly, the equality (2) provides an isometry from K_2 into K_1 if, and only if, $a_1 + b_2 \leq c \leq a_2 + b_1$. In particular, if $l(K_2) = l(K_1)$, there are exactly two isometries (in this case surjectives) from K_2 onto K_1. One of them is given by (1) with $c = a_1 - a_2 = b_1 - b_2$ and the other by (2) with $c = a_1 + b_2 = a_2 + b_1$. Thus, for example, the isometries from the interval $[0, 1]$ to itself are the identity mapping and the function $t \mapsto 1 - t$. It is also clear that a mapping $\varphi : K_2 \to K_1$ is an isometry if, and only if, φ is differentiable and $|\varphi'(t)| = 1$, for every $t \in K_2$.

3. On the coincidence between nice operators and onto isometries

In the context of continuously differentiable function spaces, different norms, equivalent and complete, are commonly used. All of them give rise to the adequate convergence notion for these spaces. The norm we are going to consider in this section is particularly frequent, easy to use and mainly will allow us to illustrate the existence of infinite-dimensional Banach spaces where the class consisting of the nice operators matches the one formed by the isometric isomorphisms.

Let K be a compact interval of $\mathbb{R}$ and denote by X the Banach space $(C^1(K), \|\cdot\|)$, where

$$\|x\| = \max\left\{\|x\|_\infty, \left\|x'\right\|_\infty\right\}, \quad \text{for every } x \in C^1(K). \tag{3}$$

Throughout this section, any reference to the maximum norm should be interpreted in terms of the previous equality.

It is known (see [5]) that the set of the extreme points of the closed unit ball of X^* is given by

$$E_{X^*} = \mathbb{T}\nabla_K \cup \mathbb{T}\nabla'_K.$$

Obviously, the sets

$$\left\{x^* \in X^* : \left|x^*(u_K)\right| < \frac{1}{2}\right\} \quad \text{and} \quad \left\{x^* \in X^* : \left|x^*(u_K)\right| > \frac{1}{2}\right\}$$

are disjoint and w^*-open. Furthermore, $\mathbb{T}\nabla'_K$ is contained in the first one and $\mathbb{T}\nabla_K$ is included in the second of them. Indeed, $x^*(u_K) = 0$, for every $x^* \in \mathbb{T}\nabla'_K$, while $|x^*(u_K)| = 1$, for every $x^* \in \mathbb{T}\nabla_K$. Moreover, it is clear that the sets ∇_K and ∇'_K are w^*-connected. The just stated facts will be useful to characterize nice operators between continuously differentiable function spaces under the maximum norm.

Theorem 4. *Let K_1 and K_2 be compact intervals of $\mathbb{R}$, $X = C^1(K_1)$ and $Y = C^1(K_2)$, provided with their respective maximum norms.*

i) *If* $\mathbb{K}=\mathbb{R}$*, the existence of nice operators from* X *into* Y *is equivalent to the condition* $l(K_1)\geq l(K_2)$*. Furthermore, in such a case, an operator* $T: X\to Y$ *is nice if, and only if, there is a scalar* $\alpha_0\in\mathbb{T}$ *and an isometry* $\varphi: K_2\to K_1$ *such that*

$$(Tx)(t)=\alpha_0 x\big(\varphi(t)\big),\quad \textit{for any } t\in K_2 \textit{ and } x\in X. \tag{4}$$

ii) *Suppose* $\mathbb{K}=\mathbb{C}$*. If* $l(K_1)<l(K_2)$*, there are no more nice operators than those of the form*

$$(Tx)(t)=\alpha(t)x(s_0),\quad \textit{for any } t\in K_2 \textit{ and } x\in X, \tag{5}$$

or

$$(Tx)(t)=\alpha(t)x'(s_0),\quad \textit{for any } t\in K_2 \textit{ and } x\in X, \tag{6}$$

where s_0 *is a point of* K_1 *and* α *is a differentiable function from* K_2 *into* $\mathbb{T}$ *whose derivative is continuous and* $|\alpha'(t)|=1$*, for every* $t\in K_2$*.*
On the other hand, if $l(K_1)\geq l(K_2)$*, an operator* $T: X\to Y$ *is nice if, and only if, it is of the form (4), (5) or (6).*

Proof. The operators described in the statements are obviously nice. Therefore, everything is reduced to proving that any nice operator is given by one of these descriptions. Suppose, then, that $T: X\to Y$ is a nice operator. Since the set $T^*(\nabla_{K_2})$ is w^*-connected and it is contained in $\mathbb{T}\nabla_{K_1}\cup\mathbb{T}\nabla'_{K_1}$ one and only one of the following two assertions is satisfied:

A.1. $T^*(\nabla_{K_2})\subset\mathbb{T}\nabla_{K_1}$ or
A.2. $T^*(\nabla_{K_2})\subset\mathbb{T}\nabla'_{K_1}$.

Likewise, one of the two following conditions holds:

B.1. $T^*(\nabla'_{K_2})\subset\mathbb{T}\nabla'_{K_1}$ or
B.2. $T^*(\nabla'_{K_2})\subset\mathbb{T}\nabla_{K_1}$.

First, assume that the assertion $A.1$ is true. Then, for any $t\in K_2$, there are $\alpha(t)\in\mathbb{T}$ and $\varphi(t)\in K_1$ such that

$$T^*(\delta_t)=\alpha(t)\delta_{\varphi(t)}. \tag{7}$$

The mappings α and φ are continuously differentiable because $\alpha=Tu_{K_1}$ and $\varphi=\frac{Ti_{K_1}}{\alpha}$. According to the equality (7)

$$(Tx)(t)=\alpha(t)x\big(\varphi(t)\big),\quad \text{for any } x\in X \text{ and } t\in K_2. \tag{8}$$

In consequence, given $x\in X$,

$$(Tx)'(t)=\alpha'(t)x\big(\varphi(t)\big)+\alpha(t)\varphi'(t)x'\big(\varphi(t)\big),\quad \text{for every } t\in K_2,$$

that is to say,

$$T^*\big(\delta'_t\big)=\alpha'(t)\delta_{\varphi(t)}+\alpha(t)\varphi'(t)\delta'_{\varphi(t)},\quad \text{for every } t\in K_2. \tag{9}$$

If $B.1$ holds and $t \in K_2$, there are $\gamma(t) \in \mathbb{T}$ and $\psi(t) \in K_1$ such that $T^*(\delta'_t) = \gamma(t)\delta'_{\psi(t)}$. Therefore,

$$\alpha'(t)\delta_{\varphi(t)} + \alpha(t)\varphi'(t)\delta'_{\varphi(t)} = \gamma(t)\delta'_{\psi(t)}.$$

It follows by applying Corollary 2 (or directly by evaluating such functional at u_{K_1}, i_{K_1} and $(i_{K_1})^2$) that $\alpha'(t) = 0$, $\alpha(t)\varphi'(t) = \gamma(t)$ and $\varphi(t) = \psi(t)$, for each $t \in K_2$. Hence, α is constant and $|\varphi'(t)| = 1$, for any $t \in K_2$. This implies that φ is an isometry from K_2 into K_1 and in particular $l(K_1) \geq l(K_2)$. There is, furthermore, $\alpha_0 \in \mathbb{T}$ such that

$$(Tx)(t) = \alpha_0 x\big(\varphi(t)\big), \quad \text{for any } x \in X \text{ and } t \in K_2. \tag{10}$$

Now assume that, in addition to $A.1$, $B.2$ is satisfied. Then, given $t \in K_2$, there are $\beta(t) \in \mathbb{T}$ and $\eta(t) \in K_1$ such that $T^*(\delta'_t) = \beta(t)\delta_{\eta(t)}$. Taking into account (9), one infers that

$$\alpha'(t)\delta_{\varphi(t)} + \alpha(t)\varphi'(t)\delta'_{\varphi(t)} = \beta(t)\delta_{\eta(t)}$$

and in accordance with Corollary 2, $\alpha'(t) = \beta(t)$ and $\alpha(t)\varphi'(t) = 0$, for every $t \in K_2$. Thus

$$\big|\alpha(t)\big| = \big|\alpha'(t)\big| = 1 \quad \text{and} \quad \varphi'(t) = 0, \quad \text{for every } t \in K_2. \tag{11}$$

The first two equalities of the previous line (and therefore the conditions $A.1$ and $B.2$) are manifestly incompatible if $\mathbb{K} = \mathbb{R}$. However, in the complex case there is no problem and, as it can be deduced from (8) and (11), there is $s_0 \in K_1$ such that

$$(Tx)(t) = \alpha(t)x(s_0), \quad \text{for any } x \in X \text{ and } t \in K_2.$$

We will now analyze the case $A.2$. In this situation, given $t \in K_2$, there are $\alpha(t) \in \mathbb{T}$ and $\varphi(t) \in K_1$ such that $T^*(\delta_t) = \alpha(t)\delta'_{\varphi(t)}$. Therefore

$$(Tx)(t) = \alpha(t)x'\big(\varphi(t)\big), \quad \text{for every } x \in X \tag{12}$$

and obviously $\alpha = T(i_{K_1})$, so that the function α is continuously differentiable. In view of (12), $x' \circ \varphi \in Y$, for every $x \in X$ and by virtue of Proposition 3, φ is a constant function. Consequently, there is $s_0 \in K_1$ such that

$$(Tx)(t) = \alpha(t)x'(s_0). \tag{13}$$

This equality and ultimately the possibility $A.2$ cannot occur in the real case. In fact, under such a condition, the function α is constant and one can deduce from (13) that $(Tx)'(t) = 0$, for any $t \in K_2$ and $x \in X$, that is, $T^*(\delta'_t) = 0$, for every $t \in K_2$, which is not possible when T is a nice operator. Thus, in the real case there are no more nice operators than those described by (10) and only if the inequality $l(K_1) \geq l(K_2)$ holds. Moreover, if $\mathbb{K} = \mathbb{C}$, Eq. (13) allows to deduce that $T^*(\delta'_t) = \alpha'(t)\delta'_{s_0}$ and since T is nice, $|\alpha'(t)| = 1$, for every $t \in K_2$. $\square$

We conclude this section with two immediate consequences of the preceding theorem.

Corollary 5. *Let X and Y be as in Theorem 4.*

i) *X and Y are isometrically isomorphic if, and only if, $l(K_1) = l(K_2)$, in which case an operator $T : X \to Y$ is an isometric isomorphism if, and only if, there is $\alpha_0 \in \mathbb{T}$ and an isometric bijection $\varphi : K_2 \to K_1$ such that*

$$(Tx)(t) = \alpha_0 x\big(\varphi(t)\big), \quad \textit{for any } t \in K_2 \textit{ and } x \in X.$$

ii) *If $\mathbb{K} = \mathbb{R}$ and $l(K_1) = l(K_2)$, every nice operator from X into Y is an isometric isomorphism.*

In [5] more information about the first paragraph of the previous result can be obtained.

4. A change in the norm

As we said at the beginning, in this section we will show the geometric nature of the problem we are analyzing. If one replaces the previously considered norm on $C^1(K)$ by an equivalent norm, the set of nice operators may be different of the set of isometric isomorphisms (both for $\mathbb{K} = \mathbb{R}$ and for $\mathbb{K} = \mathbb{C}$).

Let K be a compact interval of $\mathbb{R}$ and consider the norm on $C^1(K)$ defined by

$$|||x||| = \max\big\{\big|x(t)\big| + \big|x'(t)\big| : t \in K\big\}, \quad \text{for every } x \in C^1(K). \tag{14}$$

If $X = (C^1(K), |||\cdot|||)$, the set of extreme points of the closed unit ball of X^* is given, as may be seen in [4] and [5], by

$$E_{X^*} = \big\{\alpha\delta_t + \beta\delta'_t : \alpha, \beta \in \mathbb{T},\ t \in K\big\}.$$

The symbol $|||\cdot|||$ will be also used for the corresponding dual norm on X^*. Evidently $|||\delta_t - \delta_s||| \le |t - s|$, for any $t, s \in K$. Moreover, if $n \in \mathbb{N}$, $\alpha_1, \ldots, \alpha_n \in \mathbb{T}$ and $t_1, \ldots, t_n$ are pairwise different elements of K

$$\big|\big|\big|\alpha_1\delta'_{t_1} + \cdots + \alpha_n\delta'_{t_n}\big|\big|\big| = n. \tag{15}$$

Below we will describe the nice operators between continuously differentiable function spaces with respect to the norm just introduced. The results we get are new and have an independent interest to the problem we are addressing. Notice that the description of surjective isometries between such spaces can be easily obtained as a particular case.

Let thus K_1 and K_2 be compact intervals of $\mathbb{R}$. From now on, X and Y will stand for the spaces $C^1(K_1)$ and $C^1(K_2)$ endowed with their respective norms (14).

Lemma 6. *Assume that $T : X \to Y$ is a nice operator. Then Tu_{K_1} is a constant function. Namely, there exists $\alpha_0 \in \mathbb{T}$ such that*

$$(Tu_{K_1})(t) = \alpha_0, \quad \textit{for every } t \in K_2.$$

Proof. Define $e = Tu_{K_1}$ and consider $t \in K_2$ and $\beta \in \mathbb{T}$. Since T is nice, there are $\alpha_1, \beta_1 \in \mathbb{T}$ and $t_1 \in K_1$ such that $T^*(\delta_t + \beta\delta'_t) = \alpha_1\delta_{t_1} + \beta_1\delta'_{t_1}$. As a consequence,

$$\big|e(t) + \beta e'(t)\big| = \big|\alpha_1 u_{K_1}(t_1) + \beta_1 u'_{K_1}(t_1)\big| = |\alpha_1| = 1. \tag{16}$$

Now suppose $\mathbb{K} = \mathbb{R}$. Then $|e(t) \pm e'(t)| = 1$ and therefore $2e(t) \in \{-2, 0, 2\}$, that is, $e(t) \in \{-1, 0, 1\}$, for every $t \in K_2$. Thus e is a constant function and, in view of the equality (16), either $e(t) = 1$, for every $t \in K_2$, or $e(t) = -1$, for every $t \in K_2$.

In the complex case, β can be replaced by ± 1 and $\pm i$ in order to obtain that

$$\left|e(t)\right|^2 + \left|e'(t)\right|^2 \pm 2\mathrm{Re}\big(e(t)\overline{e'(t)}\big) = 1,$$
$$\left|e(t)\right|^2 + \left|e'(t)\right|^2 \mp 2\,\mathrm{Im}\big(e(t)\overline{e'(t)}\big) = 1.$$

It follows that $e(t)\overline{e'(t)} = 0$ or, equivalently, $e(t)e'(t) = 0$. In this way, the images of the functions e and e' are contained in $\{0\} \cup \mathbb{T}$. According to (16) it is clear that $e(K_2) \subset \mathbb{T}$ and since $ee' = 0$ necessarily $e'(K_2) \subset \{0\}$. □

Theorem 7. *Consider an operator $T \in L(X, Y)$.*

i) *Assume that $l(K_1) < l(K_2)$. Then T is nice if, and only if, there are $t_0 \in K_1$ and $\alpha_0, \beta_0 \in \mathbb{T}$ such that*

$$(Tx)(t) = \alpha_0 x(t_0) + \beta_0 x'(t_0), \quad \text{for any } x \in X \text{ and } t \in K_2. \tag{17}$$

ii) *Suppose now that $l(K_1) \geq l(K_2)$. Then T is nice if, and only if, one of the following two statements is true:*
   a) *T is of the form (17).*
   b) *There is a scalar $\alpha_0 \in \mathbb{T}$ and an isometry $\varphi : K_2 \to K_1$ such that*

$$(Tx)(t) = \alpha_0 x\big(\varphi(t)\big), \quad \text{for any } x \in X \text{ and } t \in K_2. \tag{18}$$

Proof. The operators described by Eqs. (17) and (18) are clearly nice. It is, therefore, sufficient to show that any nice operator from X into Y can be expressed in one of those ways.

Assume then that T is a nice operator. By the preceding lemma there is a scalar $\alpha_0 \in \mathbb{T}$ such that $(Tu_{K_1})(t) = \alpha_0$, for every $t \in K_2$.

Since $T = \alpha_0(\frac{1}{\alpha_0}T)$ and $\frac{1}{\alpha_0}T$ is a nice operator which maps u_{K_1} to u_{K_2}, also assume from this point that T itself associates u_{K_2} with u_{K_1}. To reach the general description of the nice operators it suffices to multiply by α_0 the expression of T we are going to obtain under the assumed hypothesis.

With $\alpha_0 = 1$, fix a point $t \in K_2$ and a scalar $\beta \in \mathbb{T}$. Obviously there are $\varphi(t, \beta) \in K_1$ and $\eta(t, \beta) \in \mathbb{T}$ such that

$$T^*\big(\delta_t + \beta\delta_t'\big) = \delta_{\varphi(t,\beta)} + \eta(t, \beta)\delta'_{\varphi(t,\beta)} \tag{19}$$

and, as we see next, the constructed mappings φ and η are continuous. First observe that the equality (19) can be expressed equivalently as follows:

$$(Tx)(t) + \beta(Tx)'(t) = x\big(\varphi(t, \beta)\big) + \eta(t, \beta)x'\big(\varphi(t, \beta)\big),$$

for any $x \in X$, $t \in K_2$ and $\beta \in \mathbb{T}$. In particular, given $t \in K_2$ and $\beta \in \mathbb{T}$,

$$(Ti_{K_1})(t) + \beta(Ti_{K_1})'(t) = \varphi(t, \beta) + \eta(t, \beta)$$
$$\big(T\big(i_{K_1}^2\big)\big)(t) + \beta\big(T\big(i_{K_1}^2\big)\big)'(t) = \varphi(t, \beta)^2 + 2\eta(t, \beta)\varphi(t, \beta)$$
$$\big(T\big(i_{K_1}^3\big)\big)(t) + \beta\big(T\big(i_{K_1}^3\big)\big)'(t) = \varphi(t, \beta)^3 + 3\eta(t, \beta)\varphi(t, \beta)^2.$$

Therefore, the mappings $\varphi+\eta$, $\varphi^2+2\eta\varphi$ and $\varphi^3+3\eta\varphi^2$ are continuous. It follows that so are the functions $(\varphi+\eta)^2$ and $(\varphi+\eta)^3$. Observing that

$$(\varphi+\eta)^2-\left(\varphi^2+2\eta\varphi\right)=\eta^2$$
$$(\varphi+\eta)^3-\left(\varphi^3+3\eta\varphi^2\right)=3\varphi\eta^2+\eta^3$$

we can say that η^2 and $3\varphi\eta^2+\eta^3$ are continuous. Consequently so is $3\varphi+\eta$. Finally, the equalities

$$\varphi=\frac{1}{2}\big((3\varphi+\eta)-(\varphi+\eta)\big)$$
$$\eta=-\frac{1}{2}\big((3\varphi+\eta)-3(\varphi+\eta)\big)$$

allow to conclude that φ and η are continuous.

Now put

$$Q=K_2\times\mathbb{T}.$$

The above arguments are valid for real or complex scalars. From this moment we will reason separately in each of such situations.

Complex case: The first step is to prove that the function φ does not depend on β. To this end, consider $t\in K_2$. If the mapping $\beta\mapsto\varphi(t,\beta)$, from $\mathbb{T}$ into K_1, were non-constant, by Proposition 1, infinite linearly independent vectors of X^* would appear in the right side of equality (19). This is not possible since the vectors of the left side of such equality are contained in a two-dimensional subspace of X^*. Therefore, equality (19) can be written in the form

$$T^*(\delta_t)+\beta T^*\left(\delta'_t\right)=\delta_{\varphi(t)}+\eta(t,\beta)\delta'_{\varphi(t)},\quad\text{for every } (t,\beta)\in Q. \tag{20}$$

Let $y=Ti_{K_1}$. In accordance with the above equalities,

$$y(t)+\beta y'(t)=\varphi(t)+\eta(t,\beta).$$

It follows readily that

1. $y'(t)=0$, for every $t\in K_2$ or
2. $|y'(t)|=1$, for every $t\in K_2$.

In the first case, y is a constant function and clearly

$$y(t)=\varphi(t)+\eta(t,\beta). \tag{21}$$

If Im stands for imaginary, there is a real number c such that

$$c=\operatorname{Im}\big(y(t)\big)=\operatorname{Im}\big(\eta(t,\beta)\big).$$

Thus the function η is constant and by (21) so is φ. Hence, there are $t_0\in K_1$ and $\lambda_0\in\mathbb{T}$ such that

$$T^*(\delta_t)+\beta T^*\left(\delta'_t\right)=\delta_{t_0}+\lambda_0\delta'_{t_0},\quad\text{for every } (t,\beta)\in Q.$$

From the previous equality it follows that $T^*(\delta'_t)=0$, for every $t\in K_2$. In consequence,

$$(Tx)(t) = x(t_0) + \lambda_0 x'(t_0), \quad \text{for any } t \in K_2 \text{ and } x \in X.$$

Suppose now that $|y'(t)| = 1$, for every $t \in K_2$. In such a case, $y(t) = \varphi(t)$ and $\beta y'(t) = \eta(t,\beta)$, for every $(t,\beta) \in Q$. Therefore φ is an isometry from K_2 into K_1 and, in particular, $l(K_2) \leq l(K_1)$. The equality (20) ensures that

$$(Tx)(t) + \beta(Tx)'(t) = x\big(\varphi(t)\big) + \beta\varphi'(t)x'\big(\varphi(t)\big), \quad \text{for any } x \in X \text{ and } (t,\beta) \in Q.$$

By summing the resulting equalities for $\beta = 1$ and $\beta = -1$ one concludes that

$$(Tx)(t) = x\big(\varphi(t)\big), \quad \text{for any } x \in X \text{ and } t \in K_2.$$

Real case: Given $\beta \in \mathbb{T}$, the function $t \mapsto \eta(t,\beta)$, from K_2 into $\mathbb{T}$, is constant. Thereupon, η only depends on β. We thus write $\eta(\beta)$ instead of $\eta(t,\beta)$.

Fix $t \in K_2$. In accordance with equality (19),

$$T^*\delta_t + T^*\delta'_t = \delta_{\varphi(t,1)} + \eta(1)\delta'_{\varphi(t,1)} \quad \text{and} \quad T^*\delta_t - T^*\delta'_t = \delta_{\varphi(t,-1)} + \eta(-1)\delta'_{\varphi(t,-1)}. \tag{22}$$

Consequently,

$$T^*\delta_t = \frac{\delta_{\varphi(t,1)} + \delta_{\varphi(t,-1)}}{2} + \frac{\eta(1)\delta'_{\varphi(t,1)} + \eta(-1)\delta'_{\varphi(t,-1)}}{2}. \tag{23}$$

Now, consider the sets $A = \{t \in K_2 : \varphi(t,1) = \varphi(t,-1)\}$ and $B = K_2 \setminus A$. Evidently, A is closed and we are going to see immediately that so is B. For this purpose let $\{t_n\}$ be a convergent sequence of elements in B, $t = \lim t_n$ and suppose to reach a contradiction that $t \in A$. By virtue of (23),

$$T^*\delta_{t_n} = \frac{\delta_{\varphi(t_n,1)} + \delta_{\varphi(t_n,-1)}}{2} + \frac{\eta(1)\delta'_{\varphi(t_n,1)} + \eta(-1)\delta'_{\varphi(t_n,-1)}}{2}, \quad \text{for every } n \in \mathbb{N}$$

and $T^*\delta_t = \delta_{\varphi(t,1)} + \frac{\eta(1)+\eta(-1)}{2}\delta'_{\varphi(t,1)}$. It is clear that the sequences $\{T^*\delta_{t_n}\}$ and $\{\delta_{\varphi(t_n,1)} + \delta_{\varphi(t_n,-1)}\}$ converge in the norm topology to $T^*\delta_t$ and $\delta_{\varphi(t,1)} + \delta_{\varphi(t,-1)}$, respectively. Therefore, the sequence $\{\eta(1)\delta'_{\varphi(t_n,1)} + \eta(-1)\delta'_{\varphi(t_n,-1)}\}$ also converges in norm to $(\eta(1) + \eta(-1))\delta'_{\varphi(t,1)}$. This is a contradiction because according to (15)

$$\big|\big|\big|\eta(1)\delta'_{\varphi(t_n,1)} + \eta(-1)\delta'_{\varphi(t_n,-1)} - \big(\eta(1) + \eta(-1)\big)\delta'_{\varphi(t,1)}\big|\big|\big| \geq 2, \quad \text{for every } n \in \mathbb{N}.$$

In order to prove that $A = K_2$, assume first that there exists an open interval $V \subset K_2$ and a point $t_0 \in K_1$ such that $\varphi(t,1) = t_0$, for every $t \in V$. Define $y = Ti_{K_1}$. The equality (23) ensures that

$$y(t) = \frac{t_0 + \varphi(t,-1)}{2} + \frac{\eta(1) + \eta(-1)}{2}, \quad \text{for each } t \in V$$

and hence the mapping $t \mapsto \varphi(t,-1)$ is continuously differentiable on V. Likewise, given $x \in X$,

$$(Tx)(t) = \frac{x(t_0) + x(\varphi(t,-1))}{2} + \frac{\eta(1)x'(t_0) + \eta(-1)x'(\varphi(t,-1))}{2}, \quad \text{for every } t \in V.$$

Thus the mapping $t \mapsto x'(\varphi(t,-1))$ is continuously differentiable on V, for any $x \in X$. Pursuant to Proposition 3, there is $t_1 \in K_1$ such that $\varphi(t,-1) = t_1$, for every $t \in V$. Clearly then

$$(Tx)(t) = \frac{x(t_0) + x(t_1)}{2} + \frac{\eta(1)x'(t_0) + \eta(-1)x'(t_1)}{2}, \quad \text{for any } x \in X \text{ and } t \in V.$$

In consequence, $(Tx)'(t) = 0$, for any $x \in X$ and $t \in V$. That is to say $T^*\delta'_t = 0$, for every $t \in V$. The equalities (22) allow to deduce that

$$T^*\delta_t = \delta_{t_0} + \eta(1)\delta'_{t_0} = \delta_{t_1} + \eta(-1)\delta'_{t_1}, \quad \text{for every } t \in V.$$

In particular $t_0 = t_1$ and thus $\varphi(t,1) = \varphi(t,-1)$, for each $t \in V$. Therefore the set A is nonempty and necessarily $A = K_2$.

Suppose now that the mapping $t \mapsto \varphi(t,1)$ is not constant on any open interval contained in K_2. Then, given $s \in K_2$, there is a sequence $\{s_n\}$ in K_2 such that $\{s_n\} \to s$ and $\{\varphi(s_n,1)\}$ is strictly monotonic. Suppose, to arrive at a contradiction, that $A = \emptyset$. Equivalently $\varphi(t,1) \neq \varphi(t,-1)$, for every $t \in K_2$. In this way $\varphi(s,1) \neq \varphi(s,-1)$ and since $\{\varphi(s_n,-1)\} \to \varphi(s,-1)$ it can be assumed that $\varphi(s_n,-1) \neq \varphi(s,1)$, for every $n \in \mathbb{N}$. According to (23)

$$T^*\delta_{s_n} = \frac{\delta_{\varphi(s_n,1)} + \delta_{\varphi(s_n,-1)}}{2} + \frac{\eta(1)\delta'_{\varphi(s_n,1)} + \eta(-1)\delta'_{\varphi(s_n,-1)}}{2}, \quad \text{for every } n \in \mathbb{N}.$$

It follows, as we saw above, that the sequence $\{\eta(1)\delta'_{\varphi(s_n,1)} + \eta(-1)\delta'_{\varphi(s_n,-1)}\}$ converges in norm to $\eta(1)\delta'_{\varphi(s,1)} + \eta(-1)\delta'_{\varphi(s,-1)}$. But this is a contradiction because, for any $n \in \mathbb{N}$, the points $\varphi(s_n,1)$, $\varphi(s_n,-1)$ and $\varphi(s,1)$ are pairwise different and by virtue of (15)

$$\left|\!\left|\!\left|\eta(1)\delta'_{\varphi(s_n,1)} + \eta(-1)\delta'_{\varphi(s_n,-1)} - \eta(1)\delta'_{\varphi(s,1)} - \eta(-1)\delta'_{\varphi(s,-1)}\right|\!\right|\!\right| \geq 2,$$

for every $n \in \mathbb{N}$. We thus conclude that $\varphi(t,1) = \varphi(t,-1)$ for any $t \in K_2$. From now on we will write $\varphi(t)$ to represent the common value of both sides of the previous equality.

Making use once again of (23)

$$T^*\delta_t = \delta_{\varphi(t)} + \frac{\eta(1) + \eta(-1)}{2}\delta'_{\varphi(t)}, \quad \text{for every } t \in K_2. \tag{24}$$

Select $t \in K_2$ and define as before $y = Ti_{K_1}$. Then $y(t) = \varphi(t) + \frac{\eta(1)+\eta(-1)}{2}$ and hence φ is continuously differentiable on K_2. Moreover, from the equalities (22) it follows that

$$y(t) + y'(t) = \varphi(t) + \eta(1), \qquad y(t) - y'(t) = \varphi(t) + \eta(-1).$$

Consequently, $y'(t) = \frac{\eta(1)-\eta(-1)}{2}$. As a result of that, if $\eta(1) = \eta(-1)$ the function y, and hence φ, is constant. In this manner there is $t_0 \in K_2$ such that

$$T^*\delta_t = \delta_{t_0} + \eta(1)\delta'_{t_0}, \quad \text{for every } t \in K_2,$$

that is, $(Tx)(t) = x(t_0) + \eta(1)x'(t_0)$, for every $t \in K_2$.

Finally, if $\eta(1) = -\eta(-1)$, then $|\varphi'(t)| = |y'(t)| = |\eta(1)| = 1$, for any $t \in K_2$. Therefore φ is an isometry and, according to (24),

$$(Tx)(t) = x\big(\varphi(t)\big), \quad \text{for every } t \in K_2. \qquad \square$$

As an immediate consequence we obtain the following description of the isometries from X onto Y.

Corollary 8. *The spaces X and Y are isometrically isomorphic if, and only if, $l(K_1) = l(K_2)$. Furthermore, in such a case, an operator $T : X \to Y$ is an isometric isomorphism if, and only if, there are $\alpha_0 \in \mathbb{T}$ and an isometric bijection $\varphi : K_2 \to K_1$ such that*

$$(Tx)(t) = \alpha_0 x\big(\varphi(t)\big), \quad \textit{for any } t \in K_2 \textit{ and } x \in X.$$

This last result was already known as can be seen in [4] and [5].

Thus, regardless of the scalar field and even assuming that $l(K_1) = l(K_2)$ Theorem 7 shows in particular that nice operators and surjective isometries are different sets under the norm (14).

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