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A new approach on Lipschitz compact operators

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The notions of Lipschitz-free compact and Lipschitz-free weakly compact operators between metric spaces are introduced. Some nonlinear versions of Schauder's theorem and Gantmacher's theorem on compact and weakly compact linear operators are proved. The Davis–Figiel–Johnson–Pełczyński factorization theorem is stated for Lipschitz-free weakly compact operators.

0. Introduction

In this note we deal with Lipschitz-free compact and Lipschitz-free weakly compact operators between metric spaces. They are nonlinear versions of the notions of compact and weakly compact linear operators between Banach spaces. We give several characterizations of Lipschitz-free compact and Lipschitz-free weakly compact operators. These results are nonlinear versions of the classical theorems due to Schauder and Gantmacher on compact and weakly compact linear operators, respectively. We also obtain a version for Lipschitz-free weakly compact operators of the factorization theorem of W.J. Davis et al. [6]. Similar versions of these results were stated for Banach-valued Lipschitz operators in [10]. The relationships between different classes of Lipschitz operators are studied. The key tool to obtain our results is a process of linearization of Lipschitz mappings provided by the Lipschitz-free space over a pointed metric space. We dedicate the following section to recall this process and present some known classes of Lipschitz operators.

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1. Preliminaries

A pointed metric space X is a metric space with a base point that we always will represent by 0. If X is a normed space, 0 will be its origin. We denote by d the distance in any metric space.

Let X and Y be pointed metric spaces. Let us recall that a map $f: X \to Y$ is Lipschitz if there exists a real constant $C \geq 0$ such that $d(f(x), f(y)) \leq Cd(x, y)$ for all $x, y \in X$. The infimum of such constants is denoted by $\mathrm{Lip}(f)$. In other words,

$$\mathrm{Lip}(f) = \sup\left\{\frac{d(f(x), f(y))}{d(x, y)} : x, y \in X,\ x \neq y\right\}.$$

We denote by $\mathrm{Lip}_0(X, Y)$ the set of all Lipschitz maps f from X into Y such that $f(0) = 0$. The elements of $\mathrm{Lip}_0(X, Y)$ are also referred to as Lipschitz operators. If E is a Banach space over the field $\mathbb{K}$ of real or complex numbers, $\mathrm{Lip}_0(X, E)$ is a Banach space with the Lipschitz norm Lip. The space $\mathrm{Lip}_0(X, \mathbb{K})$ is known as the Lipschitz dual of X and denoted frequently by $X^\#$.

The Lipschitz-free Banach space $\mathcal{F}(X)$ over a pointed metric space X is the closed linear span in $(X^\#)^*$ of the evaluation functionals $\delta_x: X^\# \to \mathbb{K}$ with $x \in X$, where

$$\delta_x(f) = f(x) \qquad \left(f \in X^\#\right).$$

This space was called and denoted so by G. Godefroy and N.J. Kalton in [9]. We refer to Weaver's book [16] for a complete study about spaces of Lipschitz functions.

Notation. Let E and F be Banach spaces. We denote by $\mathcal{L}(E, F)$ the Banach space of all bounded linear operators from E into F with the usual norm. $\mathcal{K}(E, F)$ and $\mathcal{W}(E, F)$ stand for the spaces of compact and weakly compact linear operators from E into F, respectively. As is customary, E^* stands for the dual space of E, B_E for the closed unit ball of E and J_E for the canonical isometric embedding from E into E^{**}. Given $M \subset E$, we denote by $\overline{\Gamma}(M)$ the closed, convex, balanced hull of M in E. For any $T \in \mathcal{L}(E, F)$, T^* denotes the adjoint operator of T from F^* into E^*.

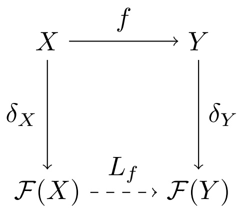
We gather in the next theorem some properties of the Lipschitz-free space over a pointed metric space.

Theorem 1.1. *Let X and Y be pointed metric spaces.*

(i). *The Dirac map $\delta_X: X \to \mathcal{F}(X)$ given by $\delta_X(x) = \delta_x$ is a (nonlinear) isometry.*
(ii). *$\mathcal{F}(X)^*$ is isometrically isomorphic to $X^\#$ via the evaluation map $Q_X: X^\# \to \mathcal{F}(X)^*$ given by $Q_X(g)(\gamma) = \gamma(g)$ for all $g \in X^\#$ and $\gamma \in \mathcal{F}(X)$.*
(iii). *The closed unit ball of $\mathcal{F}(X)$ is the closed, convex, balanced hull in $(X^\#)^*$ of the set*

$$\left\{\frac{\delta_x - \delta_y}{d(x, y)} : x, y \in X,\ x \neq y\right\}.$$

(iv). *For each $f \in \mathrm{Lip}_0(X, Y)$, the Lipschitz adjoint map $f^\#: Y^\# \to X^\#$, given by $f^\#(g) = gf$ for all $g \in Y^\#$, is a continuous linear operator and $\left\|f^\#\right\| = \mathrm{Lip}(f)$.*
(v). *For each $f \in \mathrm{Lip}_0(X, Y)$, there exists a unique operator $L_f \in \mathcal{L}(\mathcal{F}(X), \mathcal{F}(Y))$ such that $(L_f)^* = Q_X f^\#(Q_Y)^{-1}$. Furthermore, $\|L_f\| = \mathrm{Lip}(f)$.*
(vi). *For each $f \in \mathrm{Lip}_0(X, Y)$, there exists a unique operator $L_f \in \mathcal{L}(\mathcal{F}(X), \mathcal{F}(Y))$ such that $L_f\delta_X = \delta_Y f$, that is, the following diagram commutes:*



(vii). *If X, Y and Z are pointed metric spaces, $f \in \mathrm{Lip}_0(X,Y)$ and $g \in \mathrm{Lip}_0(Y,Z)$, then $L_{gf} = L_g L_f$.*

Proof. The statement (i) was proved by R.F. Arens and J. Eells Jr. in [1] (see also the paper [14] by E. Michael). The assertions (ii) and (iii) were obtained in [10, Lemma 1.1]. See [11, Corollary 4.2] for another proof of (ii). The statement (iv) was stated by J.G. Peng and Z.B. Xu in [15, Proposition 1].

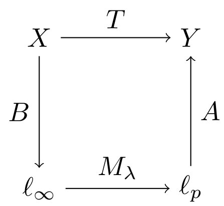
In order to show (v), let $f \in \mathrm{Lip}_0(X,Y)$. By (ii) and (iv), $Q_X f^{\#}(Q_Y)^{-1}$ is a continuous linear operator from $(\mathcal{F}(Y)^*, w^*)$ into $(\mathcal{F}(X)^*, w^*)$, where w^* denotes the weak∗ topology. Hence there is a unique operator $L_f \in \mathcal{L}(\mathcal{F}(X), \mathcal{F}(Y))$ such that $(L_f)^* = Q_X f^{\#}(Q_Y)^{-1}$. Clearly, $\|L_f\| = \|(L_f)^*\| = \left\|Q_X f^{\#}(Q_Y)^{-1}\right\| = \left\|f^{\#}\right\| = \mathrm{Lip}(f)$.

The statement (vi), which can be deduced readily from (v), is Lemma 3.1 in [12]. See [9, Lemma 2.2] for a particular case.

To prove (vii), the assertion (vi) provides operators $L_f \in \mathcal{L}(\mathcal{F}(X), \mathcal{F}(Y))$ and $L_g \in \mathcal{L}(\mathcal{F}(Y), \mathcal{F}(Z))$ such that $L_f \delta_X = \delta_Y f$ and $L_g \delta_Y = \delta_Z g$. Clearly, $gf \in \mathrm{Lip}_0(X,Z)$. Since $\delta_Z gf = L_g \delta_Y f = L_g L_f \delta_X$ and $L_g L_f \in \mathcal{L}(\mathcal{F}(X), \mathcal{F}(Y))$, we infer that $L_{gf} = L_g L_f$ by the uniqueness given in (vi). □

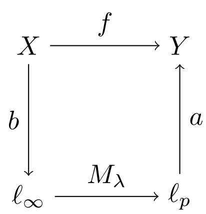
We devote the rest of this section to recall and relate some known classes of Lipschitz operators. In what follows, p will be in $[1, \infty)$.

An operator between Banach spaces $T \in \mathcal{L}(X,Y)$ is p-nuclear if there are operators $A \in \mathcal{L}(\ell_p, Y)$, $B \in \mathcal{L}(X, \ell_\infty)$ and a diagonal operator $M_\lambda \in \mathcal{L}(\ell_\infty, \ell_p)$ induced by a sequence $\lambda \in \ell_p$ such that the following diagram commutes:



Define $\nu_p(T) = \inf \|A\| \cdot \|M_\lambda\| \cdot \|B\|$, the infimum being taken over all factorizations as above.

The notion of Lipschitz p-nuclear operator was introduced by D. Chen and B. Zheng in [5] for mappings from a metric space into a Banach space. Given pointed metric spaces X and Y, we say that $f \in \mathrm{Lip}_0(X,Y)$ is Lipschitz p-nuclear if there exist $a \in \mathrm{Lip}_0(\ell_p, Y)$, $b \in \mathrm{Lip}_0(X, \ell_\infty)$ and a diagonal operator $M_\lambda \in \mathcal{L}(\ell_\infty, \ell_p)$ with $\lambda \in \ell_p$ giving rise to the following commutative diagram:



Define $\mathrm{Lip}_{pN}(f) = \inf \mathrm{Lip}(a) \cdot \|M_\lambda\| \cdot \mathrm{Lip}(b)$, the infimum being taken over all factorizations as above, and denote by $\mathrm{Lip}_{0pN}(X,Y)$ the set of all Lipschitz p-nuclear operators from X into Y.

Let us recall now that an operator between Banach spaces $T \in \mathcal{L}(X,Y)$ is p-integral if there are a probability measure μ and two operators $A \in \mathcal{L}(L_p(\mu), Y)$ and $B \in \mathcal{L}(X, L_\infty(\mu))$ such that the following diagram commutes:

$$\begin{array}{ccccc} X & \xrightarrow{T} & Y & \xrightarrow{J_Y} & Y^{**} \\ {\scriptstyle B}\downarrow & & & & \uparrow{\scriptstyle A} \\ L_\infty(\mu) & & \xrightarrow{I_{p,\infty}} & & L_p(\mu) \end{array}$$

where $I_{p,\infty}: L_\infty(\mu) \to L_p(\mu)$ is the formal inclusion operator. Denote $\iota_p(T) = \inf \|A\| \cdot \|B\|$, where the infimum is extended over all operators A and B as above.

J.D. Farmer and W.B. Johnson introduced the following generalization in [8]. Given pointed metric spaces X and Y, a Lipschitz operator $f \in \mathrm{Lip}_0(X,Y)$ is Lipschitz p-integral if there exist a probability measure μ and two Lipschitz operators $a \in \mathrm{Lip}_0(L_p(\mu), (Y^\#)^*)$ and $b \in \mathrm{Lip}_0(X, L_\infty(\mu))$ so that the following diagram commutes:

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & \xrightarrow{J_Y^L} & (Y^\#)^* \\ {\scriptstyle b}\downarrow & & & & \uparrow{\scriptstyle a} \\ L_\infty(\mu) & & \xrightarrow{I_{\infty,p}} & & L_p(\mu) \end{array}$$

where J_Y^L is the canonical isometry from Y into $(Y^\#)^*$. The triple (a, b, μ) is called a Lipschitz p-integral factorization of f. We denote by $\mathrm{Lip}_{0pI}(X,Y)$ the set of all Lipschitz p-integral operators from X into Y. With each Lipschitz p-integral operator f, we associate $\mathrm{Lip}_{pI}(f) = \inf \mathrm{Lip}(a) \cdot \mathrm{Lip}(b)$, where the infimum is taken over all Lipschitz p-integral factorizations of f.

We now introduce a smaller class of Lipschitz p-integral operators.

Definition 1.2. Let X and Y be pointed metric spaces. A Lipschitz operator $f \in \mathrm{Lip}_0(X,Y)$ is called a strongly Lipschitz p-integral operator if there exist a probability measure μ, a bounded linear operator $A \in \mathcal{L}(L_p(\mu), (Y^\#)^*)$ and a Lipschitz operator $b \in \mathrm{Lip}_0(X, L_\infty(\mu))$ such that the following diagram commutes:

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & \xrightarrow{J_Y^L} & (Y^\#)^* \\ {\scriptstyle b}\downarrow & & & & \uparrow{\scriptstyle A} \\ L_\infty(\mu) & & \xrightarrow{I_{\infty,p}} & & L_p(\mu) \end{array}$$

The triple (A, b, μ) is called a strongly Lipschitz p-integral factorization of f. Define

$$\mathrm{Lip}_{pSI}(f) = \inf \|A\| \cdot \mathrm{Lip}(b),$$

the infimum being taken over all such factorizations. We denote by $\mathrm{Lip}_{0pSI}(X,Y)$ the set of all strongly Lipschitz p-integral operators from X into Y.

Clearly, $\mathrm{Lip}_{0pSI}(X,Y) \subset \mathrm{Lip}_{0pI}(X,Y)$ and $\mathrm{Lip}_{pI}(f) \leq \mathrm{Lip}_{pSI}(f)$ for every $f \in \mathrm{Lip}_{0pSI}(X,Y)$.

The preceding definition extends that of Banach-valued strongly Lipschitz p-integral operator in [10]. Indeed, when Y is a Banach space, the mapping $J_Y^L\colon Y \to (Y^{\#})^*$ can be replaced in Definition 1.2 by the canonical injection $J_Y\colon Y \to Y^{**}$ since $PJ_Y^L = J_Y$ where $P\colon (Y^{\#})^* \to Y^{**}$ is a linear projection of one-norm (see [13, Theorem 2]).

Finally, we recall that an operator $T \in \mathcal{L}(X,Y)$ between Banach spaces is p-summing if there exists a constant $C \geq 0$ such that regardless of the natural number n and regardless of the choice of vectors $\{v_i\}_{i=1}^n$ in X, we have

$$\left(\sum_{i=1}^{n} \|T(v_i)\|^p\right)^{\frac{1}{p}} \leq C \sup_{v^* \in B_{X^*}} \left(\sum_{i=1}^{n} |v^*(v_i)|^p\right)^{\frac{1}{p}}.$$

The infimum of such constants C is denoted by $\pi_p(T)$.

A generalization of this concept was given by J.D. Farmer and W.B. Johnson in [8]. For pointed metric spaces X and Y, it is said that $f \in \mathrm{Lip}_0(X,Y)$ is Lipschitz p-summing if there exists a constant $C \geq 0$ such that regardless of the natural number n and regardless of the choices of points $\{x_i\}_{i=1}^n$ and $\{y_i\}_{i=1}^n$ in X, we have

$$\left(\sum_{i=1}^{n} d(f(x_i), f(y_i))^p\right)^{\frac{1}{p}} \leq C \sup_{g \in B_{X^{\#}}} \left(\sum_{i=1}^{n} |g(x_i) - g(y_i)|^p\right)^{\frac{1}{p}}.$$

We denote the infimum of such constants by $\mathrm{Lip}_{pS}(f)$ and the set of all Lipschitz p-summing operators from X into Y by $\mathrm{Lip}_{0pS}(X,Y)$. Lipschitz p-summing operators have been studied by J.A. Chávez-Domínguez in [2] and D. Chen and B. Zheng in [4].

Next, we study the relationships between the aforementioned classes of Lipschitz operators. Some properties of Lipschitz p-summing operators are needed.

It is known that $\mathrm{Lip}_{0pS}(X,Y)$ enjoys the ideal property (see [3,8]). More concretely, if X, Y, Z and V are pointed metric spaces, $f \in \mathrm{Lip}_0(X,Y)$, $g \in \mathrm{Lip}_{0pS}(Y,Z)$ and $h \in \mathrm{Lip}_0(Z,V)$, then $hgf \in \mathrm{Lip}_{0pS}(X,V)$ and $\mathrm{Lip}_{pS}(hgf) \leq \mathrm{Lip}(h)\mathrm{Lip}_{pS}(g)\mathrm{Lip}(f)$.

By the very definition, it is clear that $\mathrm{Lip}_{0pS}(X,Y)$ also enjoys the injectivity property. More specifically, if $i\colon Y \to Z$ is an isometry, then $f \in \mathrm{Lip}_{0pS}(X,Y)$ if and only if $if \in \mathrm{Lip}_{0pS}(X,Z)$. In this case, $\mathrm{Lip}_{pS}(if) = \mathrm{Lip}_{pS}(f)$.

In a clear parallel to the linear case, we have the following.

Proposition 1.3. *Let X and Y be pointed metric spaces and let $1 \leq p < \infty$.*

(i). $\mathrm{Lip}_{0pN}(X,Y) \subset \mathrm{Lip}_{0pI}(X,Y)$ *and* $\mathrm{Lip}_{pI}(f) \leq \mathrm{Lip}_{pN}(f)$ *for each* $f \in \mathrm{Lip}_{0pN}(X,Y)$.
(ii). $\mathrm{Lip}_{0pI}(X,Y) \subset \mathrm{Lip}_{0pS}(X,Y)$ *and* $\mathrm{Lip}_{pS}(f) \leq \mathrm{Lip}_{pI}(f)$ *for each* $f \in \mathrm{Lip}_{0pI}(X,Y)$.

Proof. (i) Let $f \in \mathrm{Lip}_{0pN}(X,Y)$ and consider a typical Lipschitz p-nuclear factorization for f:

$$f = aM_\lambda b\colon X \xrightarrow{b} \ell_\infty \xrightarrow{M_\lambda} \ell_p \xrightarrow{a} Y.$$

It is known (see, for example, [7, p. 111]) that the linear operator M_λ is strictly p-integral, that is, it admits a factorization

$$M_\lambda = AI_{\infty,p}B\colon \ell_\infty \xrightarrow{B} L_\infty(\mu) \xrightarrow{I_{\infty,p}} L_p(\mu) \xrightarrow{A} \ell_p,$$

where μ is a probability measure, $A \in \mathcal{L}(L_p(\mu), \ell_p)$ and $B \in \mathcal{L}(\ell_\infty, L_\infty(\mu))$. Moreover, the strictly p-integral norm of M_λ, defined by $\inf \|A\| \cdot \|B\|$ where the infimum is taken over all such factorizations, is $\|M_\lambda\|$. So we arrive at a Lipschitz p-integral factorization for f:

$$J_Y^L f = J_Y^L a A I_{\infty,p} B b \colon X \xrightarrow{b} \ell_\infty \xrightarrow{B} L_\infty(\mu) \xrightarrow{I_{\infty,p}} L_p(\mu) \xrightarrow{A} \ell_p \xrightarrow{a} Y \xrightarrow{J_Y^L} (Y^\#)^*.$$

Hence $f \in \mathrm{Lip}_{0pI}(X,Y)$ and $\mathrm{Lip}_{pI}(f) \leq \mathrm{Lip}(aA)\mathrm{Lip}(Bb) \leq \mathrm{Lip}(a)\,\|A\|\,\|B\|\,\mathrm{Lip}(b)$. Passing to the infimum twice, we obtain $\mathrm{Lip}_{pI}(f) \leq \mathrm{Lip}(a)\,\|M_\lambda\|\,\mathrm{Lip}(b)$ and $\mathrm{Lip}_{pI}(f) \leq \mathrm{Lip}_{pN}(f)$.

(ii) Let $f \in \mathrm{Lip}_{0pI}(X,Y)$ and take a Lipschitz p-integral factorization as

$$J_Y^L f = a I_{\infty,p} b \colon X \xrightarrow{b} L_\infty(\mu) \xrightarrow{I_{\infty,p}} L_p(\mu) \xrightarrow{a} (Y^\#)^*.$$

Notice that $I_{\infty,p}$ is p-summing with $\pi_p(I_{\infty,p}) = 1$ by [7, 2.9 (d)]. Hence $I_{\infty,p} \in \mathrm{Lip}_{0pS}(L_\infty(\mu), L_p(\mu))$ and $\mathrm{Lip}_{pS}(I_{\infty,p}) = 1$ by [8, Theorem 2]. Then $J_Y^L f$ is in $\mathrm{Lip}_{0pS}(X,(Y^\#)^*)$ by its ideal property. By its injectivity property, it follows that $f \in \mathrm{Lip}_{0pS}(X,Y)$ and

$$\mathrm{Lip}_{pS}(f) = \mathrm{Lip}_{pS}(J_Y^L f) = \mathrm{Lip}_{pS}(a I_{\infty,p} b) \leq \mathrm{Lip}(a)\mathrm{Lip}_{pS}(I_{\infty,p})\mathrm{Lip}(b) = \mathrm{Lip}(a)\mathrm{Lip}(b).$$

Then the relation $\mathrm{Lip}_{pS}(f) \leq \mathrm{Lip}_{pI}(f)$ follows readily. □

2. The results

Following [10, Definition 2.1], let us recall that a base-point preserving map f from a pointed metric space X to a Banach space Y is Lipschitz compact (Lipschitz weakly compact) if

$$\left\{ \frac{f(x) - f(y)}{d(x,y)} \colon x, y \in X,\ x \neq y \right\}$$

is a relatively compact (respectively, relatively weakly compact) subset of Y. Our aim now is to introduce this property for maps taking values in a pointed metric space Y. The problem which would raise the possible lack of linear structure in such a space Y can be avoided if one observes in the light of Theorem 1.1 (i) that $f \colon X \to Y$ is Lipschitz if and only if

$$\left\{ \frac{\delta_{f(x)} - \delta_{f(y)}}{d(x,y)} \colon x, y \in X,\ x \neq y \right\}$$

is a bounded subset of $\mathcal{F}(Y) \subset (Y^\#)^*$. This motivates the following concepts.

Definition 2.1. Let X and Y be pointed metric spaces. A base-point preserving map $f \colon X \to Y$ is Lipschitz-free compact (Lipschitz-free weakly compact) if

$$\left\{ \frac{\delta_{f(x)} - \delta_{f(y)}}{d(x,y)} \colon x, y \in X,\ x \neq y \right\}$$

is a relatively compact (respectively, relatively weakly compact) subset of $\mathcal{F}(Y)$.

We denote by $\mathrm{Lip}_{0FK}(X,Y)$ and $\mathrm{Lip}_{0FW}(X,Y)$ the sets of all Lipschitz-free compact and Lipschitz-free weakly compact operators from X to Y, respectively. Notice that

$$\mathrm{Lip}_{0FK}(X,Y) \subset \mathrm{Lip}_{0FW}(X,Y) \subset \mathrm{Lip}_0(X,Y).$$

In the case that Y is a Banach space, we have the following fact.

Proposition 2.2. *Let X be a pointed metric space and let Y be a Banach space. Then every Lipschitz-free (weakly) compact operator from X to Y is Lipschitz (weakly) compact.*

Proof. If $f: X \to Y$ is a Lipschitz-free (weakly) compact operator, then

$$\left\{ \frac{\delta_{f(x)} - \delta_{f(y)}}{d(x,y)} : x, y \in X,\ x \neq y \right\}$$

is relatively (weakly) compact in $\mathcal{F}(Y)$. By [9, Lemma 2.4], there is a bounded linear operator β_Y from $\mathcal{F}(Y)$ into Y (the barycentric map) so that $\beta_Y \circ \delta_Y(y) = y$ for $y \in Y$. It follows that

$$\left\{ \frac{f(x) - f(y)}{d(x,y)} : x, y \in X,\ x \neq y \right\} = \left\{ \beta_Y \left(\frac{\delta_{f(x)} - \delta_{f(y)}}{d(x,y)} \right) : x, y \in X,\ x \neq y \right\}$$

is relatively (weakly) compact in Y, and so f is Lipschitz (weakly) compact. □

We study now the relation between the Lipschitz-free compactness of a Lipschitz operator f in $\mathrm{Lip}_0(X,Y)$ and the compactness of its linearization L_f in $\mathcal{L}(\mathcal{F}(X), \mathcal{F}(Y))$. By the way, we obtain a nonlinear version of Schauder's theorem on the compactness of the adjoint operator of a compact linear operator between Banach spaces.

Theorem 2.3. *Let X and Y be pointed metric spaces and $f \in \mathrm{Lip}_0(X,Y)$. The following are equivalent:*

(i). *f is Lipschitz-free compact.*
(ii). *L_f is compact in $\mathcal{L}(\mathcal{F}(X), \mathcal{F}(Y))$.*
(iii). *$f^{\#}$ is compact in $\mathcal{L}(Y^{\#}, X^{\#})$.*

Proof. Put $\widetilde{X} = \{(x,y) \in X^2 : x \neq y\}$, consider $\delta_{\widetilde{X}} : (x,y) \mapsto (\delta_x - \delta_y)/d(x,y)$ from $\widetilde{X}$ to $\mathcal{F}(X)$, take its image $\delta_{\widetilde{X}}(\widetilde{X})$ and notice that

$$L_f(\delta_{\widetilde{X}}(\widetilde{X})) = \left\{ \frac{\delta_{f(x)} - \delta_{f(y)}}{d(x,y)} : x, y \in X,\ x \neq y \right\}.$$

Since $B_{\mathcal{F}(X)} = \overline{\Gamma}(\delta_{\widetilde{X}}(\widetilde{X}))$ by Theorem 1.1 (iii), the equivalence between (i) and (ii) follows from the inclusions

$$L_f(\delta_{\widetilde{X}}(\widetilde{X})) \subset L_f(\overline{\Gamma}(\delta_{\widetilde{X}}(\widetilde{X}))) \subset \overline{\Gamma}(L_f(\delta_{\widetilde{X}}(\widetilde{X}))).$$

The equivalence between (ii) and (iii) is deduced from the equality $(L_f)^* = Q_X f^{\#}(Q_Y)^{-1}$, stated in Theorem 1.1 (v), by applying Schauder's theorem and the ideal property of compact linear operators between Banach spaces. □

We now state a similar characterization for Lipschitz-free weakly compact operators and also show that those Lipschitz-free operators factor through reflexive spaces. So we give both nonlinear versions of Gantmacher's theorem on the weak compactness of the adjoint operator of a weakly compact linear operator between Banach spaces and Davis–Figiel–Johnson–Pełczyński theorem on the factorization of a weakly compact linear operator between Banach spaces through reflexive spaces.

Theorem 2.4. *Let X and Y be pointed metric spaces and $f \in \mathrm{Lip}_0(X,Y)$. The following are equivalent:*

(i). *f is Lipschitz-free weakly compact.*
(ii). *L_f is weakly compact in $\mathcal{L}(\mathcal{F}(X), \mathcal{F}(Y))$.*
(iii). *$f^\#$ is weakly compact in $\mathcal{L}(Y^\#, X^\#)$.*
(iv). *There exist a reflexive Banach space F, a Lipschitz operator $b \in \mathrm{Lip}_0(X, F)$ and a bounded linear operator $T \in \mathcal{L}(F, \mathcal{F}(Y))$ such that $\delta_Y f = Tb$.*

Proof. Similarly to the proof of Theorem 2.3, we can prove that (i), (ii) and (iii) are equivalent. Now, if (ii) holds, then there exist a reflexive Banach space F and operators $T \in \mathcal{L}(F, \mathcal{F}(Y))$ and $S \in \mathcal{L}(\mathcal{F}(X), F)$ such that $L_f = TS$ by applying Davis–Figiel–Johnson–Pełczyński factorization theorem [6]. Define $b = S\delta_X$. Clearly, $b \in \mathrm{Lip}_0(X, F)$ and $\delta_Y f = L_f \delta_X = TS\delta_X = Tb$. Then (iv) follows. Finally, assume that (iv) is true. We have

$$\left\{\frac{\delta_{f(x)} - \delta_{f(y)}}{d(x,y)} : x, y \in X,\ x \neq y\right\} = \left\{T\left(\frac{b(x) - b(y)}{d(x,y)}\right) : x, y \in X,\ x \neq y\right\}.$$

Notice that

$$\left\{\frac{b(x) - b(y)}{d(x,y)} : x, y \in X,\ x \neq y\right\}$$

is a bounded subset of the reflexive Banach space F and therefore it is relatively weakly compact. Since a linear operator T between normed spaces is norm-to-norm continuous if and only if it is weak-to-weak continuous, it follows that

$$\left\{T\left(\frac{b(x) - b(y)}{d(x,y)}\right) : x, y \in X,\ x \neq y\right\}$$

is a relatively weakly compact subset of $\mathcal{F}(Y)$, and we obtain (i). □

Proposition 2.5. *Let X and Y be pointed metric spaces and $1 \leq p < \infty$. Every strongly Lipschitz p-integral operator from X to Y is Lipschitz-free weakly compact.*

Proof. Let $f \in \mathrm{Lip}_{0pSI}(X, Y)$ and take a strongly Lipschitz p-integral factorization

$$J_Y^L f = AI_{\infty,p} b : X \xrightarrow{b} L_\infty(\mu) \xrightarrow{I_{\infty,p}} L_p(\mu) \xrightarrow{A} (Y^\#)^*.$$

If $p > 1$, then $L_p(\mu)$ is reflexive and hence f is Lipschitz-free weakly compact by Theorem 2.4. For $p = 1$, take $q > 1$ and factor $I_{\infty,1} : L_\infty(\mu) \to L_1(\mu)$ through the space $L_q(\mu)$ in the form

$$I_{\infty,1} = I_{q,1} I_{\infty,q} : L_\infty(\mu) \xrightarrow{I_{\infty,q}} L_q(\mu) \xrightarrow{I_{q,1}} L_1(\mu).$$

Then we obtain the same conclusion. □

We now study the ideal property of the aforementioned classes of Lipschitz-free operators.

Proposition 2.6. *Let X, Y, Z and V be pointed metric spaces, $f \in \mathrm{Lip}_0(X, Y)$ and $h \in \mathrm{Lip}_0(Z, V)$. If $g \in \mathrm{Lip}_{0FK}(Y, Z)$ ($\mathrm{Lip}_{0FW}(Y, Z)$), then $hgf \in \mathrm{Lip}_{0FK}(X, V)$ (respectively, $\mathrm{Lip}_{0FW}(X, V)$).*

Proof. Assume that $g \in \mathrm{Lip}_{0FK}(Y, Z)$. Then $L_g \in \mathcal{K}(\mathcal{F}(Y), \mathcal{F}(Z))$ by Theorem 2.3. Since $\mathcal{K}(\mathcal{F}(Y), \mathcal{F}(Z))$ is a Banach operator ideal, then $L_h L_g L_f \in \mathcal{K}(\mathcal{F}(X), \mathcal{F}(V))$. By Theorem 1.1 (vii), it follows that

$L_{hgf} \in \mathcal{K}(\mathcal{F}(X), \mathcal{F}(V))$. This means that $hgf \in \mathrm{Lip}_{0FK}(X, V)$ by Theorem 2.3. The case Lip_{0FW} is proved similarly. □

We study the relationships of a Lipschitz p-summing (p-integral, p-nuclear) operator f in $\mathrm{Lip}_0(X, Y)$ and its linearization L_f in $\mathcal{L}(\mathcal{F}(X), \mathcal{F}(Y))$.

Theorem 2.7. *Let X, Y be pointed metric spaces, $f \in \mathrm{Lip}_0(X, Y)$ and $p \in [1, \infty)$.*

(i). *If L_f is p-summing, then f is Lipschitz p-summing and $\mathrm{Lip}_{pS}(f) \leq \pi_p(L_f)$.*
(ii). *If L_f is p-integral, then f is Lipschitz p-integral and $\mathrm{Lip}_{pI}(f) \leq \iota_p(L_f)$.*
(iii). *If L_f is p-nuclear, then f is Lipschitz p-nuclear and $\mathrm{Lip}_{pN}(f) \leq \nu_p(L_f)$.*

Proof. (i) Notice first that, by Theorem 1.1 (i)–(vi),

$$d(f(x), f(y)) = \left\|\delta_{f(x)} - \delta_{f(y)}\right\| = \|L_f\delta_x - L_f\delta_y\| = \|L_f(\delta_x - \delta_y)\|$$

for all $x, y \in X$. If L_f is p-summing, we have

$$\left(\sum_{i=1}^{n} \|L_f(\gamma_i)\|^p\right)^{\frac{1}{p}} \leq \pi_p(L_f) \sup_{F \in B_{\mathcal{F}(X)^*}} \left(\sum_{i=1}^{n} |F(\gamma_i)|^p\right)^{\frac{1}{p}}$$

for any finite set of vectors $\{\gamma_i\}_{i=1}^n$ in $\mathcal{F}(X)$. Then

$$\begin{aligned}\left(\sum_{i=1}^{n} d(f(x_i), f(y_i))^p\right)^{\frac{1}{p}} &\leq \pi_p(L_f) \sup_{F \in B_{\mathcal{F}(X)^*}} \left(\sum_{i=1}^{n} |F(\delta_{x_i} - \delta_{y_i})|^p\right)^{\frac{1}{p}} \\ &= \pi_p(L_f) \sup_{g \in B_{X^\#}} \left(\sum_{i=1}^{n} |g(x_i) - g(y_i)|^p\right)^{\frac{1}{p}}.\end{aligned}$$

Consequently, $f \in \mathrm{Lip}_{0pS}(X, Y)$ and $\mathrm{Lip}_{pS}(f) \leq \pi_p(L_f)$.

(ii) If L_f is p-integral, take a p-integral factorization in the form

$$J_{\mathcal{F}(Y)}L_f = AI_{\infty,p}B\colon \mathcal{F}(X) \overset{B}{\to} L_\infty(\mu) \overset{I_{\infty,p}}{\to} L_p(\mu) \overset{A}{\to} \mathcal{F}(Y)^{**}.$$

Let $I_{\mathcal{F}(Y),(Y^\#)^*}$ be the inclusion operator from $\mathcal{F}(Y)$ into $(Y^\#)^*$. Using that $(Q_Y)^*J_{\mathcal{F}(Y)} = I_{\mathcal{F}(Y),(Y^\#)^*}$, $L_f\delta_X = \delta_Y f$ and $I_{\mathcal{F}(Y),(Y^\#)^*}\delta_Y = J_Y^L$, we obtain the following Lipschitz p-integral factorization for f:

$$J_Y^L f = (Q_Y)^*AI_{\infty,p}B\delta_X\colon X \overset{\delta_X}{\to} \mathcal{F}(X) \overset{B}{\to} L_\infty(\mu) \overset{I_{\infty,p}}{\to} L_p(\mu) \overset{A}{\to} \mathcal{F}(Y)^{**} \overset{(Q_Y)^*}{\to} (Y^\#)^*.$$

Hence $f \in \mathrm{Lip}_{0pI}(X, Y)$ and $\mathrm{Lip}_{pI}(f) \leq \mathrm{Lip}((Q_Y)^*A)\mathrm{Lip}(B\delta_X) \leq \|A\|\,\|B\|$. Taking infimum, we infer that $\mathrm{Lip}_{pI}(f) \leq \iota_p(L_f)$.

(iii) Take a p-nuclear factorization for L_f:

$$L_f = AM_\lambda B\colon \mathcal{F}(X) \overset{B}{\to} \ell_\infty \overset{M_\lambda}{\to} \ell_p \overset{A}{\to} \mathcal{F}(Y).$$

We deduce a Lipschitz p-nuclear factorization for f:

$$f = \delta_Y^{-1}AM_\lambda B\delta_X\colon X \overset{\delta_X}{\to} \mathcal{F}(X) \overset{B}{\to} \ell_\infty \overset{M_\lambda}{\to} \ell_p \overset{A}{\to} \delta_Y(Y) \overset{\delta_Y^{-1}}{\to} Y.$$

Then $f \in \mathrm{Lip}_{0pN}(X,Y)$ and $\mathrm{Lip}_{pN}(f) \le \mathrm{Lip}(\delta_Y^{-1}A)\,\|M_\lambda\|\,\mathrm{Lip}(B\delta_X) \le \|A\|\,\|M_\lambda\|\,\|B\|$. Hence $\mathrm{Lip}_{pN}(f) \le \nu_p(L_f)$. □

Remark 2.8. The converses in Theorem 2.7 do not always hold. Notice that the identity operator I on $\mathbb{R}$ is p-nuclear and hence Lipschitz p-nuclear, but its linearization L_I is the identity map on the infinite-dimensional Banach space $\mathcal{F}(\mathbb{R})$ and thus cannot be p-summing by the weak Dvoretzky–Rogers theorem [7, 2.18].

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