Shannon entropy of symmetric Pollaczek polynomials

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Abstract

We discuss the asymptotic behavior (as $n \to \infty$) of the entropic integrals

$$E_n = -\int_{-1}^{1} \log \left(p_n^2(x)\right) p_n^2(x) w(x)\, dx\,,$$

and

$$F_n = -\int_{-1}^{1} \log \left(p_n^2(x) w(x)\right) p_n^2(x) w(x)\, dx,$$

when w is the symmetric Pollaczek weight on $[-1, 1]$ with main parameter $\lambda \geq 1$, and p_n is the corresponding orthonormal polynomial of degree n. It is well known that w does not belong to the Szegő class, which implies in particular that $E_n \to -\infty$. For this sequence we find the first two terms of the asymptotic expansion. Furthermore, we show that $F_n \to \log(\pi) - 1$, proving that this "universal behavior" extends beyond the Szegő class. The asymptotics of E_n has also a curious interpretation in terms of the mutual energy of two relevant sequences of measures associated with p_n's.

1 Introduction and statement of results

Different information measures, and in particular, the Shannon entropy, has found application in many branches of science. In quantum mechanics, the uncertainty in the localization of a particle in ordinary space is quantitatively measured by the so-called position information entropy

$$S_\rho = -\int \rho(\vec{r}) \log \rho(\vec{r}) d\vec{r}\,,$$

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of the probability density $\rho(\vec{r}) = |\psi(\vec{r})|^2$, where $\psi(\vec{r})$ is the wavefunction of its dynamical state. This functional leads, for instance, to a stronger version of the celebrated Heisenberg's uncertainty principle, a fundamental law of nature [4]. This fact and the effective implementation of the density functional theory of complex many-electron systems [17], which uses the single-particle density as the basic variable, are responsible for the fact that the study of the entropy has become a standard tool in atomic and molecular physics, and in condensed matter theories. The exact or explicit determination of the information entropies of complex many-particle systems is an extremely difficult problem. Only recently a small progress has been achieved yielding in some cases closed formulas for the information entropies of the simplest 1-dimensional single-particle systems and the three-dimensional systems of particles moving in a central or spherically symmetric potential. For these systems the wavefunctions are expressible in terms of some special functions, and the determination of the corresponding information entropies boils down naturally to the computation of entropic functionals for sequences of orthogonal polynomials (cf. [22, 23]; a state-of-the art of this topic up to 2001 is given in [10]). In particular, given a positive unit itegrable weight w on $[-1, 1]$, and the sequence of corresponding orthonormal polynomials $\{p_n\}_{n\geq 0}$, we may define the Shannon entropy of these polynomials either as

$$E_n = E_n(w) \stackrel{\text{def}}{=} -\int_{-1}^{1} \log\left(p_n^2(x)\right) p_n^2(x) w(x)\, dx\,, \tag{1}$$

or as

$$F_n = F_n(w) \stackrel{\text{def}}{=} -\int_{-1}^{1} \log\left(p_n^2(x) w(x)\right) p_n^2(x) w(x)\, dx. \tag{2}$$

They are obviously related by $E_n(w) - F_n(w) = G_n(w)$, where

$$G_n = G_n(w) \stackrel{\text{def}}{=} \int_{-1}^{1} \log(w(x)) p_n^2(x) w(x)\, dx. \tag{3}$$

Hence, we are faced with two different problems. One is the explicit computation of (1)–(2) for fixed n's (either as a closed formula or numerically). Observe that a naive evaluation of these functionals by means of quadratures encounters the difficulty of the zeros of p_n, that belong to the interval of orthogonality. Some of the contributions in this sense are [5, 7, 8, 9, 22]. A second problem is the study of the asymptotic behavior of $\{E_n\}$, $\{F_n\}$, and $\{G_n\}$ when $n \to \infty$, which has a special interest in the analysis of the highly-excited (Rydberg) states of numerous quantum-mechanical systems

[22]. In this sense there have been important contributions in the last few years [2, 6, 7, 10, 11, 14, 20]. In a recent paper [3] the authors have studied the asymptotic behavior of these functionals under the assumption that the weight of orthogonality satisfies the Szegő condition,

$$\int_{-1}^{1} \frac{\log(w(x))}{\sqrt{1-x^2}}\, dx > -\infty\,. \tag{4}$$

Under an additional assumption on the growth of the polynomials on the interval of orthogonality they proved that both E_n and F_n (and in consequence, also G_n) converge, and

$$\lim_n F_n(w) = \log(\pi) - 1 \tag{5}$$

(notice that F_n is taken here with a slightly different normalization than in [3]). The authors of [3] conjectured that the limit in (5) is valid for a larger class of weights; from their work it follows also that if $w > 0$ on $(-1,1)$ does not satisfy (4), then $E_n(w)$ and $G_n(w)$ diverge to $-\infty$. The Pollaczek polynomials constitute the first and the best known example of a family of orthogonal polynomials on $[-1,1]$ with respect to a weight *not satisfying* the Szegő condition (4). In this paper we deal with the *symmetric* Pollaczek polynomials, $p_n^\lambda(x;a)$, that depend on two real parameters, $\lambda > 0$, $a \geq 0$, and that may be defined by the recurrence relation

$$x p_n^\lambda(x;a) = a_{n+1}\, p_{n+1}^\lambda(x;a) + a_n\, p_{n-1}^\lambda(x;a), \quad p_{-1}^\lambda(x;a) = 0\,, \quad p_0^\lambda(x;a) = 1\,, \tag{6}$$

with the coefficients

$$a_n = \frac{1}{2}\sqrt{\frac{n(n+2\lambda-1)}{(n+\lambda+a)(n+\lambda+a-1)}}\,. \tag{7}$$

It is known (see [21, Appendix]) that these polynomials are orthonormal on $[-1,1]$ with respect to the unitary weight function

$$w_\lambda(x;a) = \frac{2^{2\lambda}(\lambda+a)}{2\pi\,\Gamma(2\lambda)}\,(1-x^2)^{\lambda-1/2}\, e^{(2\arccos x-\pi)\frac{ax}{\sqrt{1-x^2}}} \left|\Gamma\left(\lambda + i\frac{ax}{\sqrt{1-x^2}}\right)\right|^2, \tag{8}$$

where $\Gamma(x)$ denotes the gamma function. From (8) it is clear that Pollaczek polynomials $p_n^\lambda(x;0)$ (that is, for $a = 0$) reduce to orthonormal Gegenbauer polynomials with parameter λ. In the sequel, whenever it cannot lead us into confusion, we omit the explicit reference to the parameters λ and a from the notation of the polynomials.

Our main goal is to study the asymptotic behavior of the sequences $E_n(w)$, $F_n(w)$ and $G_n(w)$ as $n \to \infty$, when w is the symmetric Pollaczek weight. We can summarize our main results saying that limit (5) is proved to be valid also for $w = w_\lambda(\cdot; a)$, with the restriction $\lambda \geq 1$ (so, this fact extends beyond the Szegő class, as conjectured), and we find the main part of the asymptotic expansion of $E_n(w)$ and $G_n(w)$, up to the $o(1)$ terms. Namely, we establish the following: for F_n we prove that indeed, also for the symmetric Pollaczek weight, limit (5) is still valid:

Theorem 1.1 *For the symmetric Pollaczek weight $w = w_\lambda(\cdot; a)$, with $a \geq 0$ and $\lambda \geq 1$,*

$$F_n(w) = \log(\pi) - 1 + o(1), \qquad n \to \infty.$$

Remark 1.2 The restriction $\lambda \geq 1$ comes from the method of proof; we believe that Theorem 1.1 is valid for the whole range of the parameter λ, that is, for $\lambda > 0$.

For the divergent sequence $\{G_n\}$ we find the first two terms of its asymptotic expansion:

Theorem 1.3 *For the symmetric Pollaczek weight $w = w_\lambda(\cdot; a)$, with $a \geq 0$ and $\lambda > 0$,*

$$G_n(w) = -2a \log(n) + 2a + \log\left(\frac{\Gamma(\lambda + a)\Gamma(\lambda + a + 1)}{\pi\ \Gamma(2\lambda)}\right) + o(1), \quad n \to \infty\,.$$

As a straightforward corollary we obtain

Corollary 1.4 *For the symmetric Pollaczek weight $w = w_\lambda(\cdot; a)$, with $a \geq 0$ and $\lambda \geq 1$,*

$$E_n(w) = -2a \log(n) + \tau(\lambda; a) + o(1), \qquad n \to \infty, \tag{9}$$

where

$$\tau(\lambda; a) \stackrel{def}{=} 2a - 1 + \log\left(\frac{\Gamma(\lambda + a)\Gamma(\lambda + a + 1)}{\Gamma(2\lambda)}\right). \tag{10}$$

Remark 1.5 The value $\tau(\lambda; 0) = -1 + \log(\Gamma(\lambda)\Gamma(\lambda + 1)/\Gamma(2\lambda))$ matches $\lim_n E_n$ for orthonormal Gegenbauer polynomials, found in [2].

For illustration, we have computed the entropy $E_n(w_\lambda(\cdot; a))$ for $n = 1, 2, \ldots, 500$, $\lambda = 5, 15$, and $a = 5, 10, 15$ (Fig. 1), using the numerical algorithm from [8], which admits as the only input data the expression of the recurrence coefficients a_n in (7). For comparison, in Fig. 2 we plot the difference $E_n(w_\lambda(\cdot;a)) - \widetilde{E}_n(w_\lambda(\cdot;a))$, where

$$\widetilde{E}_n(w_\lambda(\cdot;a)) \stackrel{\text{def}}{=} -2a\log(n) + \tau(\lambda;a)\,.$$

In [14], a rather general result about the leading term of the asymptotics of E_n has been established, that we state here for the symmetric case: for even weights functions w on $[-1,1]$ which belong to the class $\mathcal{F}(C^2+)$ introduced in [13] (and whose definition we recall in Section 2), if an additional assumption on the behavior of w at ± 1 (see Eq. (1) in [14]) is satisfied, then

$$E_n(w) = -\frac{2}{\pi}\int_{-\alpha_n}^{\alpha_n} \frac{Q(x)}{\sqrt{\alpha_n^2 - x^2}}\,dx\,\big(1+o(1)\big), \quad n\to\infty\,, \tag{11}$$

where

$$Q(x) \stackrel{\text{def}}{=} -\frac{1}{2}\log\frac{w(x)}{w(0)} \tag{12}$$

is the "external field", associated with the weight w, and α_n is the *Mahskar-Rakhmanov-Saff number* (or MRS number), defined as the unique solution of the integral equation

$$\frac{1}{\pi}\int_{-\alpha_n}^{\alpha_n} \frac{x\,Q'(x)}{\sqrt{\alpha_n^2 - x^2}}\,dx = n \tag{13}$$

(see e.g. [19] for details). Unfortunately, the assumptions from [14] on the behavior of w at ± 1 are not fulfilled by the symmetric Pollaczek weights. Nevertheless, the result of Corollary 1.4 above shows that the assertion in [14] is still valid:

Corollary 1.6 *For the symmetric Pollaczek weight $w = w_\lambda(\cdot;a)$, with $a \ge 0$ and $\lambda \ge 1$, formula* (11) *holds.*

Remark 1.7 We should observe however that the result in [14] is sharp: there the $o(1)$ term of (11) has a power decay, fact which is not true for the Pollaczek weight, where the decay is logarithmic.

Finally, asymptotic formula (9) has a curious interpretation in terms of the behavior of the mutual energy of two relevant sequences of probability measures on $[-1,1]$ associated with p_n's:

$$\rho_n = \frac{1}{n}\sum_{j=1}^{n} \delta_{\zeta_j^{(n)}} \qquad \text{and} \qquad d\nu_n(x) = p_n^2(x) w(x)\,dx\,, \tag{14}$$

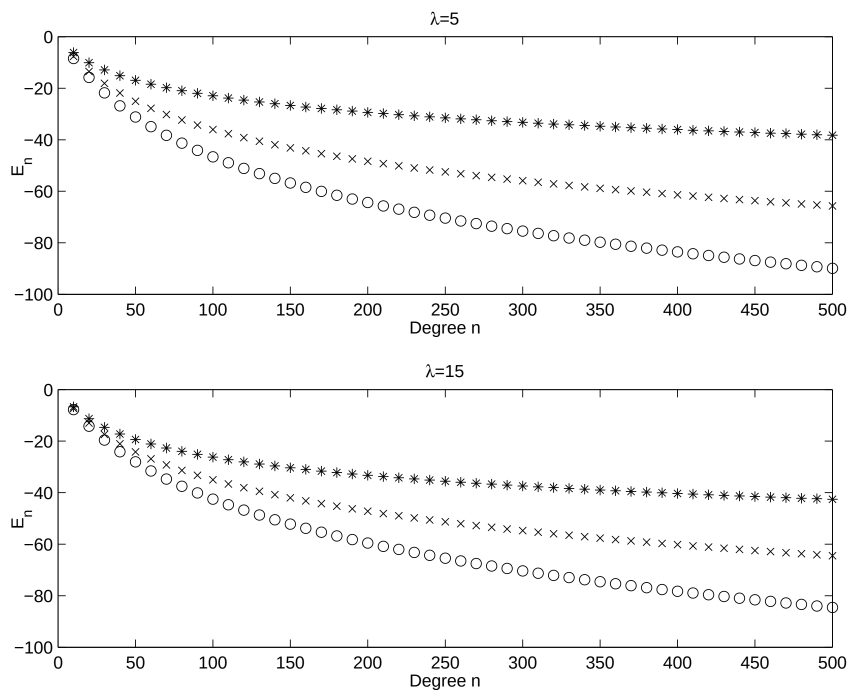

Figure 1: Entropy $E_n(w_\lambda(\cdot;a))$ for $n = 1, 2, \ldots, 500$, with $\lambda = 5$ (upper) and $\lambda = 15$ (lower). We use the values $a = 5$ ('*'), $a = 10$ ('x') and $a = 15$ ('o').

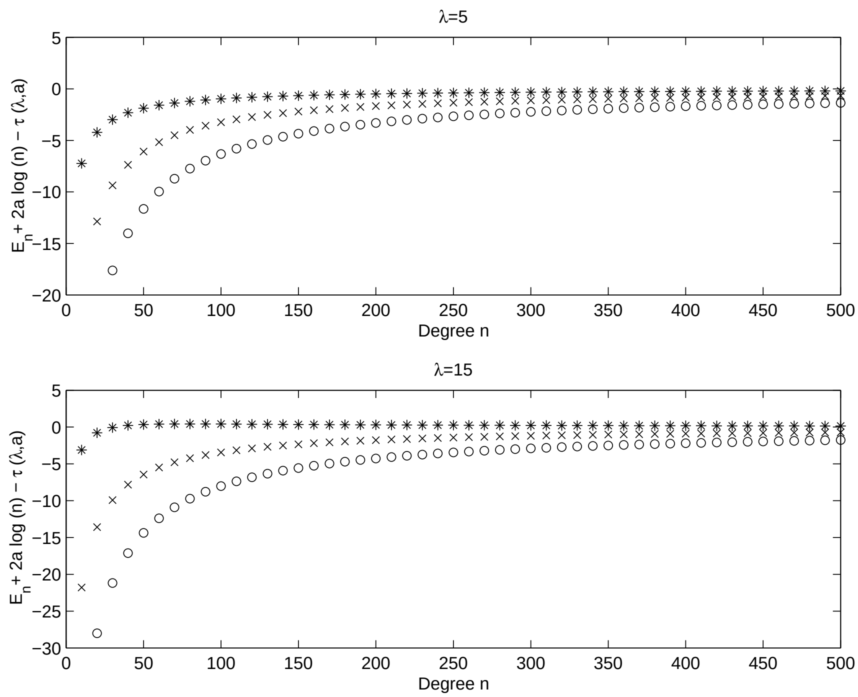

Figure 2: $E_n(w_\lambda(\cdot;a)) - \widetilde{E}_n(w_\lambda(\cdot;a))$ for $n = 1, 2, \ldots, 500$, with $\lambda = 5$ (upper) and $\lambda = 15$ (lower). We use the values $a = 5$ ('*'), $a = 10$ ('x') and $a = 15$ ('o').

where $-1 < \zeta_1^{(n)} < \cdots < \zeta_n^{(n)} < 1$ are the zeros of the polynomial p_n. Both measures are standard objects of study in the analytic theory of orthogonal polynomials. For instance, the normalized zero counting measure ρ_n is closely connected with the n-th root asymptotics of p_n, while ν_n is associated with the behavior of the ratio p_{n+1}/p_n as $n \to \infty$ (see [16, 18]).

If ρ and ν are Borel (generally speaking, real signed) measures on $\mathbb{C}$, we denote by

$$V^{\rho}(z) \stackrel{\text{def}}{=} -\int \log|z-t|\,d\rho(t) \tag{15}$$

the logarithmic potential of ρ, and by

$$I[\nu,\rho] \stackrel{\text{def}}{=} \int V^{\nu}(z)\,d\rho(z) = -\iint \log|z-t|\,d\nu(t)\,d\rho(z)\,,$$

the mutual energy of ν and ρ. The latter is connected with the entropy (1) by the formula

$$E_n = -2\log\gamma_n + 2\sum_{j=1}^{n} V^{\nu_n}(\zeta_j^{(n)}) = -2\log\gamma_n + 2n\,I[\rho_n,\nu_n]\,, \tag{16}$$

where

$$\gamma_n = 2^n \left(\frac{(\lambda+a+1)_n\ (\lambda+a)_n}{n!\ (2\lambda)_n} \right)^{1/2} > 0 \tag{17}$$

is the leading coefficient of p_n, and $(z)_n = \Gamma(z+n)/\Gamma(z)$ denotes as usual the Pochhammer's symbol. It is well known that as long as the orthogonality weight $w > 0$ a.e. on $[-1,1]$, both ρ_n and ν_n tend (as $n \to \infty$) in the weak-* sense to the Chebyshev (equilibrium) distribution of the interval, which implies that $\lim_{n\to\infty} I[\rho_n,\nu_n] = \log(2)$. In [3] a rather surprising "universal" behavior of the next term of the asymptotic expansion of $I[\rho_n,\nu_n]$ was observed. Namely, if the orthogonality weight satisfies the Szegő condition (4) with an additional assumption on the growth of the sequence of $\{p_n\}$ on $[-1,1]$, then $I[\rho_n,\nu_n] = \log(2) - 1/(2n) + o(1/n)$, $n \to \infty$. However, this result is no longer valid for the Pollaczek polynomials, as it follows from formula (9):

Corollary 1.8 *For the symmetric Pollaczek weight $w = w_\lambda(\cdot; a)$, with $a \geq 0$ and $\lambda \geq 1$,*

$$I[\rho_n,\nu_n] = \log(2) - \frac{1-2a}{2n} + o\left(\frac{1}{n}\right)\,, \quad n \to \infty\,.$$

Observe that the second term of asymptotics is still independent of the main parameter λ, and matches the result in [3] for $a = 0$.

The structure of this paper is as follows. In Section 2 we gather some technical facts about the weight function $w_\lambda(\cdot; a)$; in particular, we show that this weight does not satisfy (4), and for $\lambda \geq 1$, it belongs to the class $\mathcal{F}(C^2+)$. Section 3 contains some results about the equilibrium measure of total mass n in the external field Q; it is needed for the proof of Theorem 1.1 and Corollary 1.4 (Section 4). We defer the proof of the asymptotics of the sequence $\{G_n\}$ (Theorem 1.3) to Section 5. Finally, corollaries 1.6 and 1.8 are established in Section 6.

2 The weight function

As a first step in our analysis we study the behavior of the symmetric Pollaczek weight function $w_\lambda(\cdot; a)$ defined in (8), which we denote simply by w whenever it cannot lead us into confusion. Using the notation

$$t \stackrel{\text{def}}{=} \frac{ax}{\sqrt{1-x^2}}, \quad x \in \Delta \stackrel{\text{def}}{=} [-1, 1],$$

we rewrite its definition as

$$w(x) = \frac{2^{2\lambda}\,(\lambda + a)}{2\pi\,\Gamma(2\lambda)}(1-x^2)^{\lambda-1/2}\, e^{(2\arccos x - \pi)t}\, |\Gamma(\lambda + it)|^2, \quad x \in \Delta. \tag{18}$$

This is an even function on Δ, strictly positive in $(-1, 1)$, but vanishing at the end points. The fast (exponential) decay at ± 1 is precisely the reason why w does not satisfy (4). Indeed, using the asymptotic formula (6.1.40) from [1] (see also [15, §2.11]), we can easily obtain that

$$2\log|\Gamma(\lambda + ti)| = -\pi t + 2(\lambda - 1/2)\log t + 2\log\sqrt{2\pi} + \frac{\lambda(\lambda-1)(2\lambda-1)}{6t^2} + \mathcal{O}(t^{-4}),$$

when $x \to 1^-$ (which denotes in what follows the one-sided limit from the left). Thus, (18) yields that

$$\begin{aligned}\log w(x) &= \log\left(\frac{2^{2\lambda}\,(\lambda + a)}{2\pi\,\Gamma(2\lambda)}\right) + (\lambda - 1/2)\log(1-x^2) + (2\arccos x - \pi)t \\ &\quad + \log|\Gamma(\lambda + it)|^2 \\ &= -2\pi t + (\lambda - 1/2)\log(x^2) + 2t\arccos x + \log\left(\frac{2^{2\lambda}\,(\lambda + a)a^{2\lambda-1}}{2\pi\,\Gamma(2\lambda)}\right) \\ &\quad + 2\log\sqrt{2\pi} + \frac{\lambda(\lambda-1)(2\lambda-1)}{6t^2} + \mathcal{O}(t^{-4}), \quad x \to 1^-. \end{aligned} \tag{19}$$

Since

$$2t \arccos x = 2a - \frac{2a}{3}(1-x^2) + \mathcal{O}(1-x^2)^2, \qquad x \to 1^-, \tag{20}$$

we obtain that

$$\log w(x) = -2\pi t + \mathcal{O}(1), \qquad x \to 1^-.$$

In consequence,

$$w(x) = \exp(-2\pi|t| + \mathcal{O}(1)) = \exp\left(\frac{-2\pi a|x|}{\sqrt{1-x^2}} + \mathcal{O}(1)\right), \quad |x| \to 1^-, \tag{21}$$

showing that for this weight the integral in (4) is divergent. The previous analysis motivates the introduction of functions w_0 and s on Δ, such that

$$w_0(x) \stackrel{\text{def}}{=} e^{-2\pi|t|} = \exp\left(-\frac{2\pi a|x|}{\sqrt{1-x^2}}\right), \quad \text{and} \quad w(x) = w_0(x)e^{s(x)}. \tag{22}$$

Lemma 2.1 *Function $s \in C^\infty(-1,1)$ is even and continuous in $[-1,1]$.*

Proof. We need to check only the existence of finite limits of s at ± 1 (the rest is trivial). From (19) it is clear that

$$s(x) = \log\frac{w(x)}{w_0(x)} = \log\left(\frac{2^{2\lambda}\ (\lambda+a)}{2\pi\ \Gamma(2\lambda)}\ a^{2\lambda-1}\right) + (\lambda - 1/2)\log(x^2)$$
$$+ 2t\ \arccos x + \frac{\lambda(\lambda-1)(2\lambda-1)}{6t^2} + \mathcal{O}(t^{-4}),$$

and using (20) we get

$$s(x) = \log\left(\frac{2^{2\lambda}(\lambda+a)}{\Gamma(2\lambda)}\ a^{2\lambda-1}e^{2a}\right) + \mathcal{O}(x-1), \quad x \to 1^-,$$

which concludes the proof. □

Proposition 2.2 *For $\lambda \ge 1$, the weight $w = w_\lambda(\cdot;a)$ belongs to the class $\mathcal{F}(C^2+)$.*

Recall that $w \in \mathcal{F}(C^2+)$ (see [13]) if the corresponding external field Q introduced in (12) is positive and verifies the following conditions:

a) Q' is continuous in Δ.

b) Q'' exists and is positive in $\Delta \setminus \{0\}$.

c) $\lim_{|x|\to 1^-} Q(x) = \infty$.

d) Function

$$T(x) \stackrel{\text{def}}{=} \frac{xQ'(x)}{Q(x)},$$

is cuasi-increasing in $(0,1)$ cuasi-decreasing in $(-1,0)$, and

$$T(x) \geq \Lambda > 1, \quad x \in (-1,1). \tag{23}$$

e) There exists $C_1 > 0$ such that

$$\frac{Q''(x)Q(x)}{(Q'(x))^2} \leq C_1, \quad x \in (-1,1).$$

f) There exist a compact subinterval J, contained in $(-1,1)$, and $C_2 > 0$ such that

$$\frac{Q''(x)Q(x)}{(Q'(x))^2} \geq C_2,$$

for all $x \in (-1,1) \setminus J$, except a subset with zero measure.

A function $f : I \to [0,+\infty)$ is cuasi-increasing if $\forall x < y \in I,\ \exists C > 0$, such that $f(x) < C\ f(y)$.

We prove Proposition 2.2 in several steps.

Lemma 2.3 *If a function $f \in C[0,+\infty)$ is positive and decreasing, then for $t > 0$,*

$$I(t) \stackrel{def}{=} \int_0^{+\infty} f(u)\sin(ut)\,du \in (0,+\infty]\,. \tag{24}$$

Proof. With the change of variable $v = ut$ we can write

$$I(t) = \frac{1}{t}\int_0^{\infty} f(v/t)\ \sin(v)\ dv.$$

We denote $g(v) = f(v/t)$; then

$$\begin{aligned}
tI(t) &= \sum_{k=0}^{\infty} \int_{k\pi}^{(k+1)\pi} g(v) \ \sin v \ dv = \sum_{k=0}^{\infty} \int_0^{\pi} g(k\pi + v) \ \sin(k\pi + v) \ dv \\
&= \sum_{k=0}^{\infty} \int_0^{\pi} g(2k\pi + v) \sin(2k\pi + v) dv \\
&\quad + \sum_{k=0}^{\infty} \int_0^{\pi} g((2k+1)\pi + v) \sin((2k+1)\pi + v) dv \\
&= \sum_{k=0}^{\infty} \int_0^{\pi} g(2k\pi + v) \ \sin(v) \ dv - \sum_{k=0}^{\infty} \int_0^{\pi} g((2k+1)\pi + v) \ \sin(v) \ dv \\
&= \sum_{k=0}^{\infty} \int_0^{\pi} \Big(g(2k\pi + v) - g((2k+1)\pi + v) \Big) \ \sin(v) \ dv.
\end{aligned}$$

Since each integral in the series is strictly positive, it proves (24). □

Now we gather some properties of the digamma and trigamma functions in the following technical lemma:

Lemma 2.4 *For the digamma function $\psi(x) = \Gamma'(x)/\Gamma(x)$ the following statements hold: for $\lambda \geq 1$,*

i) $\operatorname{Re} \psi'(\lambda + it)$ *is a strictly positive even function of $t \in \mathbb{R}$, and*

$$\lim_{t \to \pm\infty} t \operatorname{Re} \psi'(\lambda + it) = 0. \tag{25}$$

ii) $\operatorname{Im} \psi(\lambda + it)$ *is an odd function of $t \in \mathbb{R}$, strictly positive in* $(0, +\infty)$*, and*

$$\operatorname{Im} \psi(\lambda \pm it) = \pm \frac{\pi}{2} + \mathcal{O}(t^{-1}), \quad t \to +\infty. \tag{26}$$

Remark 2.5 Limits (25)–(26) are valid in fact for $\lambda > 0$.

Proof. The symmetry of both the real and the imaginary parts of $\psi(\lambda + it)$ follows from the well known property

$$\psi(\overline{z}) = \overline{\psi(z)};$$

so, we restrict our attention to $t > 0$.

For *i)* we consider the integral representation of the trigamma function,

$$\psi'(z) = \int_0^{+\infty} e^{-uz} \frac{u}{1-e^{-u}} du, \tag{27}$$

(see e.g. [12, formula 3.41.371.6]), from where

$$\operatorname{Re}\psi'(\lambda+it) = \int_0^{\infty} e^{-u\lambda} \frac{u}{1-e^{-u}} \cos(ut)\, du.$$

Integrating by parts it can be reduced to

$$\operatorname{Re}\psi'(\lambda+it) = \frac{1}{t}\int_0^{\infty} e^{-\lambda u}\, \frac{ue^{-u}-(1-\lambda u)(1-e^{-u})}{(1-e^{-u})^2} \sin(ut)\, du.$$

It is easy to check that for $\lambda \geq 1$, function

$$f(u) = e^{-\lambda u}\, \frac{ue^{-u}-(1-\lambda u)(1-e^{-u})}{(1-e^{-u})^2},$$

is positive and decreasing on $(0,+\infty)$. Hence, the first part of *i)* follows from Lemma 2.3. On the other hand, by [1, formula 6.4.12],

$$\psi'(z) = \frac{1}{z} + \mathcal{O}\left(\frac{1}{z^2}\right), \quad z \to \infty, \quad |\arg z| < \pi,$$

so that

$$(\lambda+it)\psi'(\lambda+it) = 1 + \mathcal{O}\left(\frac{1}{t}\right), \quad t \to \pm\infty.$$

In particular,

$$\begin{aligned} 0 &= \lim_{t\to\pm\infty} \operatorname{Im}\big((\lambda+it)\psi'(\lambda+it)\big) = \lambda \lim_{t\to\pm\infty} \operatorname{Im}\big(\psi'(\lambda+it)\big) + \lim_{t\to\pm\infty} t\operatorname{Re}\psi'(\lambda+it) \\ &= \lim_{t\to\pm\infty} t\operatorname{Re}\psi'(\lambda+it), \end{aligned}$$

which proves (25). For *ii)* we may use the series expansion [1, formula 6.3.16],

$$\psi(1+z) = -\gamma + \sum_{k=1}^{\infty} \frac{z}{k(k+z)}, \quad -z \notin \mathbb{N},$$

according to which

$$\operatorname{Im}\psi(1+z) = -\sum_{k=1}^{\infty} \operatorname{Im}\left(\frac{1}{k+z}\right) = \sum_{k=1}^{\infty} \frac{\operatorname{Im}(z)}{|k+z|^2} > 0 \quad \text{if } \operatorname{Im}(z) > 0.$$

Finally, by the asymptotic formula [1, formula 6.3.18],

$$\psi(z) = \log z + \mathcal{O}\left(\frac{1}{z}\right), \quad z \to \infty, \quad |\arg z| < \pi,$$

from which (26) is immediate. □

Now we are ready to analyze whether the weight belongs to $\mathcal{F}(C^2+)$. Conditions a) and b) are a straightforward consequence of the following statement:

Lemma 2.6 *The even function $Q \in C^\infty(-1,1)$ satisfies*

$$\frac{d^k\, Q(x)}{dx^k} > 0 \quad \textit{for } x \in (0,1) \textit{ and } k = 0,1,2.$$

Proof. By definition, $Q(0) = 0$, and by symmetry, $Q'(0) = 0$, so

$$Q''(x) > 0 \quad \Rightarrow \quad Q'(x) > 0 \quad \Rightarrow \quad Q(x) > 0, \quad x \in (0,1).$$

But for $x \in (0,1)$ we have $t > 0$, and

$$\begin{aligned} Q''(x) = {} & \frac{(\lambda - 1/2)(1+x^2)}{(1-x^2)^2} + \frac{ax^2 + 2a}{(1-x^2)^2} + \frac{3ax}{(1-x^2)^{5/2}}\, \frac{1}{2}(\pi - 2\arccos x) \\ & + \frac{3ax}{(1-x^2)^{5/2}} \operatorname{Im}\psi(\lambda + it) + \frac{a^2}{(1-x^2)^3} \operatorname{Re}\psi'(\lambda + it) > 0, \end{aligned} \tag{28}$$

where we have used Lemma 2.4. □

Since by (21),

$$Q(x) = -\frac{1}{2}\log w(x) + \frac{1}{2}\log w(0) = \frac{1}{2}\,\frac{2\pi a|x|}{\sqrt{1-x^2}} + \mathcal{O}(1), \quad |x| \to 1^-, \tag{29}$$

condition c) also trivially holds.

We turn now to the even function

$$T(x) = \frac{xQ'(x)}{Q(x)};$$

let us show that it is cuasi-increasing in $(0,1)$. Since T is continuous and positive on the bounded interval $(0,1)$, it is sufficient to show that it does not blow up at the left end point, nor it vanishes at the right one. Recall that

$$Q(0) = Q'(0) = 0, \quad Q''(0) = (\lambda - 1/2) + 2a + a^2\psi'(\lambda) > 0\,;$$

in particular,

$$Q(x) = \frac{(\lambda - 1/2) + 2a + a^2\psi'(\lambda)}{2}\, x^2 + \mathcal{O}(x^3)\,, \quad x \to 0\,. \tag{30}$$

Hence,

$$\lim_{x\to 0} T(x) = \lim_{x\to 0} \frac{xQ'(x)}{Q(x)} = \lim_{x\to 0} \frac{Q'(x)/x}{Q(x)/x^2} = \frac{Q''(0)}{\frac{1}{2}Q''(0)} = 2 > 0\,. \tag{31}$$

On the other hand,

$$Q'(x) = \frac{(\lambda - 1/2)x}{1-x^2} + \frac{ax}{1-x^2} + \frac{1}{2}\,\frac{(\pi - 2\arccos x)\; a}{(1-x^2)^{3/2}} + \frac{a}{(1-x^2)^{3/2}}\,\mathrm{Im}\,\psi(\lambda + it), \tag{32}$$

and by Lemma 2.4,

$$\lim_{x\to 1^-} (1-x^2)^{3/2}\; Q'(x) = a\pi\,. \tag{33}$$

Together with (29) it shows that

$$\lim_{x\to 1^-} T(x) = \lim_{x\to 1^-} \frac{xQ'(x)}{Q(x)} = \lim_{x\to 1^-} \frac{1}{1-x^2}\;\frac{\pi a}{\pi a} = +\infty. \tag{34}$$

In conclusion, T is cuasi-increasing in $(0,1)$.

On the other hand, if $\zeta \in (0,1)$ is a local minimum of T, then

$$\left.\frac{d\log(T(x))}{dx}\right|_{x=\zeta} = 0 \quad \Rightarrow \quad \frac{1}{\zeta} + \frac{Q''}{Q'}(\zeta) - \frac{Q'}{Q}(\zeta) = 0\,,$$

or equivalently,

$$T(\zeta) = 1 + \frac{\zeta Q''(\zeta)}{Q'(\zeta)} > 1\,.$$

Taking into account also the behavior at $x = 0$ and $x = 1$ (see (31) and (34)), we obtain (23).

Finally, let us check conditions e) and f). Denote

$$H(x) \stackrel{\text{def}}{=} \frac{Q(x)Q''(x)}{(Q'(x))^2}\,.$$

This is an even, continuous and positive function on $(-1,1)$, with $H(0) = 1/2$, where we have used (30). On the other hand,

$$\lim_{x\to 1^-}(1-x^2)^{1/2}Q(x) = \lim_{x\to 1^-}(1-x^2)^{3/2}Q'(x) = a\pi\,,$$

where we have used (29) and (33). Also from (28) it follows that

$$\lim_{x\to 1^-}(1-x^2)^{5/2}Q''(x) = \frac{3}{2}\,a\pi + \frac{3}{2}\,a\pi + a\lim_{x\to 1^-} t\operatorname{Re}\psi'(\lambda+it),$$

and by (25), $\lim_{x\to 1^-}(1-x^2)^{5/2}\ Q''(x) = 3a\pi$. Gathering these identities we obtain that

$$\lim_{x\to 1^-} H(x) = \lim_{x\to 1^-}\frac{(1-x^2)^{1/2}Q(x)\,(1-x^2)^{5/2}Q''(x)}{(1-x^2)^3(Q'(x))^2} = 3\,.$$

Hence, H can be extended as a strictly positive and continuous (and thus, uniformly continuous) function on $[-1,1]$; from this fact conditions e) and f) follow automatically. This concludes the proof of Proposition 2.2.

3 Equilibrium measure

The equilibrium measure μ_n on $[-1,1]$ of total weight n in the external field Q plays a prominent role in the asymptotics of the orthogonal polynomials, and we gather in this section some of its properties needed further. By Lemma 2.6, function $Q \in C^\infty(-1,1)$ is strictly convex, and $Q(-1^+) = Q(1^-) = +\infty$. In consequence, (see e.g. [19]), μ_n is absolutely continuous and supported on the interval $[-\alpha_n,\alpha_n]$, where α_n is the MRS number, defined by (13). If we denote by $\sigma_n(x)$ the density (μ_n') of μ_n, then

$$\int_{-\alpha_n}^{\alpha_n}\sigma_n(x)\,dx = n\,,$$

and the characterizing property of the equilibrium is

$$V^{\mu_n}(x) + Q(x)\begin{cases} = b_n(=\text{const}), & x\in[-\alpha_n,\alpha_n],\\ > b_n, & \alpha_n < |x| \le 1\,,\end{cases}$$

where V^{μ_n} is the logarithmic potential of μ_n (cf. (15)). We analyze first the asymptotic behavior of $\{\alpha_n\}$, needed in the proof of Corollary 1.6. It is known that $\alpha_n \to 1^-$, and even more, that $1-\alpha_n = \mathcal{O}(1/n)$ (see [13, §1.6]), but we are looking for a more precise information. The following technical lemma is useful for the estimation of the behavior of the integral in (13):

Lemma 3.1 *Let $f(u,x)$ be defined for $u,x \in [0,1]$ with f and $\partial f/\partial x$ continuous in $[0,1]^2$, $f(1,1) \neq 0$; then when $u \to 1^-$,*

$$\int_0^1 \frac{f(u,x)}{\sqrt{1-u^2x^2}} \frac{dx}{\sqrt{1-x^2}} = -\frac{f(1,1)}{2} \log(1-u) \big(1+o(1)\big), \tag{35}$$

$$\int_0^1 \frac{f(u,x)}{(1-u^2x^2)} \frac{dx}{\sqrt{1-x^2}} = \frac{\pi f(1,1)}{2\sqrt{2}} \frac{1}{\sqrt{1-u}} \big(1+o(1)\big), \tag{36}$$

$$\int_0^1 \frac{f(u,x)}{(1-u^2x^2)^{3/2}} \frac{dx}{\sqrt{1-x^2}} = \frac{f(1,1)}{2} \frac{1}{1-u} \big(1+o(1)\big). \tag{37}$$

Formula (35) appears in [12, formula 8.113.3] for $f(u,x)=1$; the proof is standard, and we omit it here for the sake of brevity.

Now we can obtain the first two terms of the asymptotics of α_n:

Proposition 3.2 *The MRS numbers α_n satisfy*

$$\alpha_n = 1 - \frac{a}{n} + o\left(\frac{1}{n}\right), \quad n \to \infty.$$

Proof. Formula (13) defining the MRS numbers may be rewritten as

$$\frac{2}{\pi} \int_0^1 \frac{\alpha_n x \; Q'(\alpha_n x)}{\sqrt{1-x^2}} \, dx = n, \tag{38}$$

which motivates the study of the asymptotics (as $u \to 1^-$) of the integral

$$\begin{aligned} &\frac{2}{\pi} \int_0^1 \frac{ux \; Q'(ux)}{\sqrt{1-x^2}} \, dx \\ &\quad = \frac{1}{\pi} \int_0^1 \frac{ux \; ((\lambda - 1/2)2ux + 2aux)}{(1-u^2x^2) \sqrt{1-x^2}} \, dx + \frac{1}{\pi} \int_0^1 \frac{ux \; (-2)a \; \arccos(ux)}{(1-u^2x^2)^{3/2} \sqrt{1-x^2}} \, dx \\ &\quad + \frac{1}{\pi} \int_0^1 \frac{ux \left(\pi a + 2a \operatorname{Im} \psi \left(\lambda + i \frac{aux}{\sqrt{1-u^2x^2}}\right)\right)}{(1-u^2x^2)^{3/2} \sqrt{1-x^2}} \, dx = I_1(u) + I_2(u) + I_3(u). \end{aligned}$$

Using Lemma 3.1 we have that for $u \to 1^-$,

$$I_1(u) = \frac{1}{\pi}((2\lambda - 1) + 2a) \frac{\pi}{2\sqrt{2}} \frac{1}{\sqrt{1-u}} \big(1+o(1)\big) = o\left(\frac{1}{1-u}\right),$$

$$I_3(u) = \frac{1}{\pi} \; (\pi a + 2a\pi/2) \frac{1}{2} \; \frac{1}{1-u} \big(1+o(1)\big) = \frac{a}{1-u} \big(1+o(1)\big).$$

Moreover, since

$$f(u,x) = \frac{\arccos(ux)}{\sqrt{1-u^2x^2}}\,,$$

satisfies the conditions of Lemma 3.1, we have that

$$I_2(u) = -\frac{1}{\pi}\int_0^1 \frac{ux2af(u,x)}{(1-u^2x^2)\sqrt{1-x^2}} = o\left(\frac{1}{1-u}\right), \quad u \to 1^-.$$

Summarizing,

$$\frac{2}{\pi}\int_0^1 \frac{ux\,Q'(ux)}{\sqrt{1-x^2}}\,dx = \frac{a}{1-u}\big(1+o(1)\big),$$

and equation (38) for the MRS numbers can be rewritten as

$$\frac{a}{1-\alpha_n}\big(1+o(1)\big) = n,$$

which proves the statement. □

We turn now to the analysis of the density σ_n. It is convenient to introduce the normalized translation of σ_n to Δ,

$$\sigma_n^*(u) = \frac{\alpha_n}{n}\,\sigma_n(\alpha_n u)\,, \quad u \in \Delta\,, \tag{39}$$

as well as the cumulative distribution

$$\Phi_n(\theta) \stackrel{\text{def}}{=} \pi\int_{\cos\theta}^1 \sigma_n^*(t)dt\,, \tag{40}$$

which is obviously a smooth and strictly increasing function on $[0,\pi]$; moreover, $\Phi_n : [0,\pi] \to [0,\pi]$ is a bijection, and the inverse function $\Phi_n^{[-1]}$ exists (observe that our definition differs in normalization from that used in [13, 14]). We summarize some properties of Φ_n in the following lemma.

Lemma 3.3 *For Φ_n defined in* (40)*,*

i) $\Phi_n'(\theta) \to 1$ *pointwise in* $(0,\pi)$.

ii)

$$\int_0^\pi \left|\frac{1}{\Phi_n'(\Phi_n^{[-1]}(\eta))} - 1\right|\,d\eta = o(1), \qquad n\to\infty.$$

iii) $\Phi_n(\theta) \to \theta$ *and* $\Phi_n^{[-1]}(\theta) \to \theta$ *as* $n \to \infty$ *uniformly in* $[0,\pi]$.

Proof. Let $0 < \varepsilon < 1/2$, and $x = \cos\theta \in (-1+2\varepsilon, 1-2\varepsilon)$. By Lemma 6.5 of [13],

$$\Phi_n'(\theta) - 1 = \pi\sigma_n^*(\cos\theta)\sin\theta - 1 = \frac{\alpha_n}{\pi n}\, PV \int_{-1}^{1} \frac{Q'(\alpha_n u)\sqrt{1-u^2}}{u-x}\, du\,,$$

where PV means the principal value of integral. Hence,

$$\begin{aligned}\Phi_n'(\theta) - 1 = \frac{\alpha_n}{\pi n}\Bigg(&\int_{-1+\varepsilon}^{1-\varepsilon} \frac{Q'(\alpha_n u)\sqrt{1-u^2} - Q'(\alpha_n x)\sqrt{1-x^2}}{u-x}\, du \\ &+ Q'(\alpha_n x)\sqrt{1-x^2}PV\int_{-1+\varepsilon}^{1-\varepsilon}\frac{du}{u-x} + \int_{-1}^{-1+\varepsilon}\frac{Q'(\alpha_n u)\sqrt{1-u^2}}{u-x}\, du \\ &+ \int_{1-\varepsilon}^{1}\frac{Q'(\alpha_n u)\sqrt{1-u^2}}{u-x}\, du\Bigg)\,.\end{aligned} \tag{41}$$

The first two terms within parentheses in the right hand side of (41) are uniformly bounded. Let us estimate

$$\left|\int_{1-\varepsilon}^{1}\frac{Q'(\alpha_n u)\sqrt{1-u^2}}{u-x}\, du\right| \le \frac{1}{\varepsilon}\int_{1-\varepsilon}^{1} Q'(\alpha_n u)\sqrt{1-u^2}\, du$$

(the remaining term is analyzed in a similar fashion). Integrating by parts,

$$\int_{1-\varepsilon}^{1} Q'(\alpha_n u)\sqrt{1-u^2}\, du = -\frac{1}{\alpha_n}\, Q(\alpha_n(1-\varepsilon))\sqrt{\varepsilon(2-\varepsilon)} + \frac{1}{\alpha_n}\int_{1-\varepsilon}^{1}\frac{uQ(\alpha_n u)}{\sqrt{1-u^2}}\, du\,,$$

and using (35) we get that

$$\left|\Phi_n'(\theta) - 1\right| = \mathcal{O}\left(\frac{\log(n)}{n}\right)\,,$$

which proves *i)*.

On the other hand, by [14, lemma 4.2 a)], the sequence $|\Phi_n'(\theta)|$ is uniformly bounded on $[0,\pi]$. Thus, by the dominated convergence theorem,

$$\begin{aligned}\int_0^{\pi}\left|\frac{1}{\Phi_n'(\Phi_n^{[-1]}(\eta))} - 1\right|\, d\eta &= \int_0^{\pi}\left|\frac{1}{\Phi_n'(\theta)} - 1\right|\, \Phi_n'(\theta)d\theta \\ &= \int_0^{\pi}\left|1 - \Phi_n'(\theta)\right|\, d\theta \to 0\,, \quad n\to\infty\,.\end{aligned}$$

Furthermore, given $\theta \in [0,\pi]$,

$$|\Phi_n(\theta)-\theta| = \left|\int_0^\theta (\Phi_n'(\eta)-1)d\eta\right| \le \int_0^\pi \left|\Phi_n'(\eta)-1\right| d\eta = o(1),$$

and so the uniform convergence of Φ_n on $[0,\pi]$ follows again by the dominated convergence theorem. Finally, if $\theta = \Phi_n(\eta)$,

$$\left|\Phi_n^{[-1]}(\theta)-\theta\right| = \left|\Phi_n^{[-1]}(\Phi_n(\eta)) - \Phi_n(\eta)\right| = |\eta - \Phi_n(\eta)|\,,$$

showing that $\Phi_n^{[-1]}(\theta)$ converges uniformly to θ on $[0,\pi]$. This concludes the proof. $\square$

4 Asymptotics of F_n: proof of Theorem 1.1

We follow the scheme of proof of [3]. We have established already that the weight $w \in \mathcal{F}(C^2+)$; furthermore, $w(x) > 0$ on $(-1,1)$, and in consequence, it is an Erdős-Turán weight. One of the most relevant facts about these weights is that the sequence $p_n^2(x)w(x)dx$ converges in the weak-$*$ topology to the equilibrium (Robin) measure μ of $[-1,1]$ (see [16, 18]). In other words, for any $f \in C[-1,1]$,

$$\lim_{n\to\infty} \int_{-1}^1 f(x)p_n^2(x)w(x)\ dx = \frac{1}{\pi}\int_{-1}^1 f(x)\frac{dx}{\sqrt{1-x^2}}. \tag{42}$$

We make use also of the following technical lemmas. In the sequel we write that $x_n \sim y_n$ if the ratios x_n/y_n and y_n/x_n are uniformly bounded in n.

Lemma 4.1

$$\int_{\Delta\setminus[-\alpha_n,\alpha_n]} \log(p_n^2(x)\ w(x))\ p_n^2(x)\ w(x)\ dx = o(1)\,, \quad n\to\infty\,.$$

Proof. Since $w \in \mathcal{F}(C^2+)$, by [13, Theorem 1.18],

$$\sup_{x\in\Delta}\left|p_n(x)\ \sqrt{w(x)}\right| \sim n^{1/6}\ (\alpha_n)^{-1/3}\ \left(\frac{T(\alpha_n)}{\alpha_n}\right)^{1/6} \sim n^{1/3},$$

where we have taken into account Proposition 3.2. Thus, for $\varepsilon > 0$, there exists $C_1 > 0$ such that

$$\left(p_n^2(x)\ w(x)\right)^{1+\varepsilon} \le C_1\, n^{(2+2\varepsilon)/3}$$

and so, if $\varepsilon < 1/2$, then

$$\int_{\Delta\setminus[-\alpha_n,\alpha_n]} \log(p_n^2(x)\ w(x))\ p_n^2(x)\ w(x)\ dx \le \int_{\Delta\setminus[-\alpha_n,\alpha_n]} (p_n^2(x)\ w(x))^{1+\varepsilon}\ dx$$

$$\le C_1\ n^{(2+2\varepsilon)/3} \int_{\Delta\setminus[-\alpha_n,\alpha_n]} dx \le C_2\ n^{\frac{2+2\varepsilon}{3}-1} = C_2\ n^{\frac{-1+2\varepsilon}{3}} = o(1).$$

For a lower bound it is sufficient to take into account that function

$$\mathcal{R}(y) = y^2 \log(y^2) \tag{43}$$

is bounded from below on $[0,+\infty)$, and thus

$$\int_{\Delta\setminus[-\alpha_n,\alpha_n]} \log(p_n^2(x)\,w(x))\,p_n^2(x)\,w(x)\,dx \ge C \int_{\Delta\setminus[-\alpha_n,\alpha_n]} dx = o(1)\,, \quad n\to\infty\,,$$

which concludes the proof. □

Lemma 4.2

$$\lim_{n\to\infty} \int_{-\alpha_n}^{\alpha_n} \log(\sqrt{\alpha_n^2 - x^2})\ p_n^2(x)\ w(x)\ dx = -\log(2). \tag{44}$$

Proof. Let

$$\ell_n(x) \stackrel{\text{def}}{=} \begin{cases} \log(\sqrt{\alpha_n^2 - x^2}), & x \in (-\alpha_n, \alpha_n), \\ 0, & x \in [-1,1] \setminus (-\alpha_n, \alpha_n)\,. \end{cases}$$

Obviously, $\ell_n(x) \to \log(\sqrt{1-x^2})$ pointwise for $x \in (-1,1)$. Furthermore, there exists $C_1 > 0$ such that for $t > 1$ it holds that $(\log(t))^3 < C_1 t$. Then, by Lebesgue dominated convergence theorem,

$$\left\| \ell_n(x) - \log(\sqrt{1-x^2}) \right\|_{L^3} = o(1)\,, \quad n \to \infty$$

(here and in the sequel $\|\cdot\|_{L^p}$ denotes the p-norm with respect to the Lebesgue measure on Δ).

Furthermore, from [13, Theorem 13.6] it follows that the sequence $\|p_n\sqrt{w}\|_{L^p}$ is uniformly bounded as long as $p < 4$ (and in particular, for $p = 3$). Thus, by Hölder inequality,

$$\left| \int_{-1}^{1} \left(\ell_n(x) - \log(\sqrt{1-x^2}) \right) p_n^2(x)\ w(x)dx \right|$$
$$\le \left\| \ell_n(x) - \log(\sqrt{1-x^2}) \right\|_{L^3} \left\| p_n^2\ w \right\|_{L^{3/2}}\ ,$$

which implies that

$$\begin{aligned}
&\int_{-\alpha_n}^{\alpha_n} \log(\sqrt{\alpha_n^2 - x^2})\ p_n^2(x)\ w(x)\ dx \\
&= \int_{-1}^{1} \log(\sqrt{1-x^2})\ p_n^2(x)\ w(x)\ dx + o(1)\,, \quad n \to \infty\,.
\end{aligned} \tag{45}$$

On the other hand, if for $\varepsilon \in (0,1)$ we denote

$$\log_\varepsilon(x) \stackrel{\text{def}}{=} \max\{\log(\varepsilon), \log(\sqrt{1-x^2})\} \in C[-1,1]\,,$$

then

$$\begin{aligned}
&\left| \int_{-1}^{1} \log(\sqrt{1-x^2})\ p_n^2(x)\ w(x)\ dx - \frac{1}{\pi} \int_{-1}^{1} \log(\sqrt{1-x^2})\ \frac{dx}{\sqrt{1-x^2}} \right| \\
&\le \left| \int_{-1}^{1} \Big(\log(\sqrt{1-x^2}) - \log_\varepsilon(x) \Big)\ p_n^2(x)\ w(x)\ dx \right| \\
&+ \left| \int_{-1}^{1} \log_\varepsilon(x)\ \left(p_n^2(x)\ w(x) - \frac{1}{\pi\sqrt{1-x^2}} \right)\ dx \right| \\
&+ \left| \int_{-1}^{1} \Big(\log_\varepsilon(x) - \log(\sqrt{1-x^2}) \Big)\ \frac{1}{\pi\sqrt{1-x^2}}\, dx \right| = I_1 + I_2 + I_3\,.
\end{aligned}$$

By (42), $I_2 = o(1)$, as $n \to \infty$, while

$$I_3 = -\int_{\sqrt{1-x^2}<\varepsilon} \log(\sqrt{1-x^2})\ \frac{1}{\pi\sqrt{1-x^2}}\, dx = \mathcal{O}(\varepsilon)\,.$$

Using the same arguments as for (45) we find also that $I_1 = \mathcal{O}(\varepsilon)$. Taking into account that $\varepsilon > 0$ is arbitrary and using (45) we obtain that

$$\begin{aligned}
&\int_{-\alpha_n}^{\alpha_n} \log(\sqrt{\alpha_n^2 - x^2})\ p_n^2(x)\ w(x)\ dx \\
&= \frac{1}{\pi} \int_{-1}^{1} \log(\sqrt{1-x^2})\ \frac{dx}{\sqrt{1-x^2}} + o(1)\,, \quad n \to \infty\,.
\end{aligned}$$

The identity

$$\frac{1}{\pi} \int_{-1}^{1} \log(\sqrt{1-x^2})\ \frac{dx}{\sqrt{1-x^2}} = -\log(2)$$

is straightforward, which concludes the proof. □

Remark 4.3 This result is not surprising: if we denote by $\widehat{\nu}_n$ the absolutely continuous measure on $[-\alpha_n, \alpha_n]$ with $\widehat{\nu}_n'(x) = p_n^2(x)w(x)$, then $\widehat{\nu}_n \to \mu$ in the weak-$*$ topology, where μ is the Robin measure of Δ. The integral in the left hand side of (44) can be rewritten as

$$-\frac{1}{2}\left(V^{\widehat{\nu}_n}(-\alpha_n) + V^{\widehat{\nu}_n}(\alpha_n)\right) \longrightarrow -\frac{1}{2}\left(V^{\mu}(-1) + V^{\mu}(1)\right) = -\log(2)\,.$$

Now we turn to the proof of Theorem 1.1. Using function Φ_n introduced in (40), let us denote

$$\begin{aligned} f_n(\alpha_n \cos\theta) &\stackrel{\text{def}}{=} \sqrt{\alpha_n}\; p_n(\alpha_n\cos\theta)\;\sqrt{w(\alpha_n\cos\theta)}\;\sqrt{\sin\theta}, \\ g_n(\alpha_n\cos\theta) &\stackrel{\text{def}}{=} \sqrt{\frac{2}{\pi}}\cos\left(\frac{\theta}{2} - \frac{\pi}{4} + n\Phi_n(\theta)\right). \end{aligned} \tag{46}$$

Since $w \in \mathcal{F}(C^2+)$, by [13, Theorem 15.1 and Lemma 15.4],

$$\int_0^{\pi} \left| f_n(\alpha_n\cos\theta) - g_n(\alpha_n\cos\theta)\right| d\theta = o(1)\,, \quad n\to\infty\,, \tag{47}$$

(where we have used that on the bounded interval convergence in L^2 is stronger than in L^1). Let us rewrite the definition of F_n as

$$\begin{aligned} F_n = &-\int_{-\alpha_n}^{\alpha_n} \log(p_n^2(x)\; w(x)) p_n^2(x) w(x)\; dx \\ &-\int_{\Delta\setminus[-\alpha_n,\alpha_n]} \log(p_n^2(x)\; w(x))\; p_n^2(x)\; w(x)\; dx \\ = &-\int_{-\alpha_n}^{\alpha_n} \log(p_n^2(x)\; w(x))\; p_n^2(x)\; w(x)\; dx + o(1)\,, \end{aligned}$$

where we have used Lemma 4.1. With the notation (43) and (46) it is equivalent to

$$\begin{aligned} F_n &= -\int_{-\alpha_n}^{\alpha_n} \mathcal{R}(f_n(x))\;\frac{dx}{\sqrt{\alpha_n^2 - x^2}} + \int_{-\alpha_n}^{\alpha_n} \log(\sqrt{\alpha_n^2 - x^2})\; p_n^2(x)\; w(x)\; dx + o(1) \\ &= -\int_0^{\pi} \mathcal{R}(f_n(\alpha_n\cos\theta))\; d\theta - \log 2 + o(1)\,, \quad n\to\infty\,, \end{aligned} \tag{48}$$

(see Lemma 4.2). Since by [13, Theorem 1.17] there exists a constant $M > \sqrt{2/\pi}$ such that for all $n \in \mathbb{N}$, $\left|f_n(\alpha_n\cos\theta)\right| \le M$, for $\theta \in [0,\pi]$, we get by

(47)

$$\int_0^\pi |\mathcal{R}(f_n(\alpha_n \cos\theta)) - \mathcal{R}(g_n(\alpha_n \cos\theta))|\, d\theta$$
$$\leq \max_{y\in[0,M]} \mathcal{R}'(y) \int_0^\pi |f_n(\alpha_n \cos\theta) - g_n(\alpha_n \cos\theta)|\, d\theta = o(1)\,,$$

which yields

$$F_n = -\int_0^\pi \mathcal{R}\left(g_n(\alpha_n \cos\theta)\right)\, d\theta - \log 2 + o(1)\,, \quad n\to\infty\,.$$

With the change of variable $\eta = \Phi_n(\theta)$ we rewrite

$$\begin{aligned} F_n &= -\int_0^\pi \mathcal{R}\left(\sqrt{\frac{2}{\pi}}\cos\left(\Phi_n^{[-1]}(\eta) - \frac{\pi}{4} + n\eta\right)\right)\ \frac{d\eta}{\Phi_n'(\Phi_n^{[-1]}(\eta))} - \log 2 + o(1) \\ &= -\int_0^\pi \mathcal{R}\left(\sqrt{\frac{2}{\pi}}\cos\left(\Phi_n^{[-1]}(\eta) - \frac{\pi}{4} + n\eta\right)\right)\ d\eta - \log 2 + o(1) \\ &= -\int_0^\pi \mathcal{R}\left(\sqrt{\frac{2}{\pi}}\cos\left(\eta - \frac{\pi}{4} + n\eta\right)\right)\ d\eta - \log 2 + o(1), \end{aligned} \tag{49}$$

where we have used Lemma 3.3. It remains to use the following analogue of the Lebesgue lemma, proved in [2] under weaker conditions:

Lemma 4.4 *Let g be a π-periodic continuous function on $[0,+\infty)$, and $h \in C[0,\pi]$. Then*

$$\int_0^\pi f(n\theta + h(\theta))\ d\theta = \int_0^\pi f(\theta) d\theta + o(1).$$

Applying this lemma to (49), we obtain finally

$$F_n = -1 + \log(2) + \log(\pi) - \log(2) + o(1) = \log(\pi) - 1 + o(1)\,, \quad n\to\infty\,.$$

5 Asymptotics of G_n: proof of Theorem 1.3

We start again with some technical results:

Lemma 5.1 *When $n\to\infty$,*

$$\int_0^1 x(1-x^2)\ s'(x)\ p_n^2(x)\ w(x) dx = B_2 + o(1),$$

where

$$B_2 = \int_0^1 x(1-x^2)\ s'(x) \frac{1}{\pi\sqrt{1-x^2}}\ dx\,. \tag{50}$$

Proof. By (42), it is sufficient to show that $(1-x^2)s'(x)$ can be extended as a continuous function to the whole interval Δ; for this purpose we only need to show that the limit

$$\lim_{x\to 1^-} (1-x^2)\ s'(x)$$

exists. From the explicit expression for s it is easy to find that for $x \in (0,1)$,

$$\begin{aligned}(1-x^2)s'(x) =& (1-x^2)\,\frac{w'(x)}{w(x)} - (1-x^2)\,\frac{w_0'(x)}{w_0(x)} = (1-x^2)\ \frac{w'(x)}{w(x)} + \frac{2\pi a}{\sqrt{1-x^2}} \\ =& -2x(\lambda-1/2) - 2ax + 2\arccos x\ \frac{a}{\sqrt{1-x^2}} \\ & -2\frac{t}{x}\left(\operatorname{Im}\psi(\lambda+it) - \frac{\pi}{2}\right).\end{aligned}$$

It remains to use (26), and the statement follows. □

Let us denote

$$p_n(x) = \gamma_n x^n + \beta_n x^{n-2} + \text{lower degree terms}; \tag{51}$$

the explicit expression for γ_n was given in (17).

Lemma 5.2 *The followings identities hold:*

$$\int_{-1}^1 x\ p_n(x)\ p_n'(x)\ w(x)\ dx = n\,, \tag{52}$$

and

$$\int_{-1}^1 x^3 p_n(x)\ p_n'(x)\ w(x)\ dx = n(a_{n+1}^2 + a_n^2) - 2a_n a_{n-1}\frac{\beta_n}{\gamma_{n-2}}, \tag{53}$$

where a_n are the coefficients of the recurrence relation (6), and γ_n, β_n are the coefficients of p_n defined in (51).

Proof. By the recurrence relation (6),

$$xp_n(x)p_n'(x) = a_{n+1}p_{n+1}(x)p_n'(x) + a_n p_{n-1}(x)p_n'(x),$$

so that

$$\int_{-1}^{1} x p_n(x) p_n'(x)\ w(x)\ dx = a_n \int_{-1}^{1} p_{n-1}(x) p_n'(x)\ w(x)\ dx$$
$$= a_n \frac{n\gamma_n}{\gamma_{n-1}} \int_{-1}^{1} p_{n-1}^2(x)\ w(x)\ dx = n,$$

where we have used the well known fact that $a_n = \gamma_{n-1}/\gamma_n$. This proves (52). Again, from (6) it is easy to find that

$$x^2 p_n(x) = a_{n+2} a_{n+1} p_{n+2}(x) + (a_{n+1}^2 + a_n^2) p_n(x) + a_n a_{n-1} p_{n-2},$$

and we get

$$\int_{-1}^{1} x^3 p_n(x) p_n'(x)\ w(x)\ dx = (a_{n+1}^2 + a_n^2) \int_{-1}^{1} x p_n(x) p_n'(x)\ w(x)\ dx$$
$$+ a_n a_{n-1} \int_{-1}^{1} x p_{n-2} p_n'(x)\ w(x)\ dx. \tag{54}$$

First integral in the right hand side was computed in (52), and it remains to concentrate our attention on the second one. Since

$$x p_n'(x) = n\gamma_n x^n + (n-2)\beta_n x^{n-2} + \text{ lower degree terms}$$
$$= n p_n(x) - 2\frac{\beta_n}{\gamma_{n-2}} p_{n-2}(x) + \text{ lower degree terms},$$

we have

$$\int_{-1}^{1} x p_{n-2}(x) p_n'(x)\ w(x)\ dx = -2\frac{\beta_n}{\gamma_{n-2}}.$$

Substituting it in (54), we obtain (53). □

We find next an expression for the ratio β_n/γ_n in terms of the coefficients of the recurrence relation:

Lemma 5.3 *With the notations introduced in (6) and (51),*

$$\frac{\beta_{n+1}}{\gamma_{n+1}} = \frac{\beta_n}{\gamma_n} - a_n^2, \qquad n \in \mathbb{N}; \tag{55}$$

in particular

$$\frac{\beta_{n+1}}{\gamma_{n+1}} = -\sum_{k=1}^{n} a_k^2. \tag{56}$$

Proof. Comparing the coefficients of x^{n-1} in both sides of (6) we obtain that

$$\beta_n = a_{n+1}\beta_{n+1} + a_n\gamma_{n-1},$$

so that

$$\frac{\beta_{n+1}}{\gamma_{n+1}} = \frac{1}{a_{n+1}}\frac{\gamma_n}{\gamma_{n+1}}\frac{\beta_n}{\gamma_n} - \frac{a_n}{a_{n+1}}\frac{\gamma_{n-1}}{\gamma_n}\frac{\gamma_n}{\gamma_{n+1}},$$

and so, using the identity

$$a_n = \frac{\gamma_{n-1}}{\gamma_n},$$

(55) holds. Formula (56) follows from (55) and the fact that $\beta_1 = 0$. □

Remark 5.4 Observe that we have used only the symmetry of the recurrence relation, so these formulas are valid for any even weight function on $[-1, 1]$.

Corollary 5.5 *For the symmetric Pollaczek polynomials the following asymptotic formula is valid:*

$$\frac{\beta_n}{\gamma_n} = -\frac{n}{4} + \frac{a}{2}\log n - \frac{a-\lambda}{4} - \frac{a}{2}\psi(a+\lambda) + \mathcal{O}\left(\frac{1}{n}\right), \qquad n \to \infty. \tag{57}$$

Proof. From (7) it is easy to obtain that

$$a_k^2 = 1 + \frac{a^2 - \lambda^2 - a + \lambda}{k + \lambda + a - 1} + \frac{\lambda^2 - a^2 - \lambda - a}{k + \lambda + a},$$

and (56) lead us to

$$\frac{\beta_n}{\gamma_n} = -\frac{n-1}{4} + \frac{1}{4}\sum_{k=1}^{n-1}\frac{2a}{k+\lambda+a-1} + \frac{1}{4}\frac{-\lambda^2 + a^2 + \lambda + a}{n + \lambda + a - 1} - \frac{1}{4}\frac{-\lambda^2 + a^2 + \lambda + a}{\lambda + a}.$$

Using that

$$\sum_{k=1}^{n-1}\frac{1}{k+\lambda+a-1} = \psi(n+\lambda+a-1) - \psi(\lambda+a) = \log(n) - \psi(\lambda+a) + \mathcal{O}\left(\frac{1}{n}\right),$$

when $n \to \infty$, we finally get

$$\frac{\beta_n}{\gamma_n} = -\frac{n-1}{4} + \frac{2a}{4}(\log(n) - \psi(\lambda+a)) + \frac{1}{4}(\lambda - a - 1) + \mathcal{O}\left(\frac{1}{n}\right),$$

which is equivalent to the statement of the Lemma. □

Now we can prove Theorem 1.3. Remember that the technique of [14] is not valid here because an additional assumption on w from [14] is not satisfied. The central idea in our proof is to take advantage of the fact that the main contribution to the asymptotics of G_n comes from the behavior of the weight w at the endpoints of Δ (see Section 2). Using functions w_0 and s introduced in (22), we write G_n in the form

$$G_n = \int_{-1}^{1} \log(w_0(x))\ p_n^2(x)\ e^{s(x)}\ w_0(x)\ dx + \int_{-1}^{1} s(x)\ p_n^2(x)\ w(x)\ dx.$$

In particular, since $s \in C[-1,1]$ (see Lemma 2.1), applying (42) in the second integral we have

$$G_n = \int_{-1}^{1} \log(w_0(x))\ p_n^2(x)\ e^{s(x)}\ w_0(x)\ dx + B_1 + o(1), \tag{58}$$

where

$$B_1 = \lim_{n\to\infty} \int_{-1}^{1} s(x)\ p_n^2(x)\ w(x)\ dx = \int_{-1}^{1} \frac{s(x)}{\pi\ \sqrt{1-x^2}}\ dx.$$

If we denote

$$g(x) = \log(w_0(x)) = -\ \frac{2\pi a|x|}{\sqrt{1-x^2}},$$

taking into account the symmetry, we can rewrite the integral in the right hand side of (58) as

$$\begin{aligned}\int_{-1}^{1} g(x)\ p_n^2(x)\ e^{s(x)}\ e^{g(x)}\ dx &= 2\int_0^1 \frac{g(x)}{g'(x)}\ p_n^2(x)\ e^{s(x)}\ e^{g(x)} g'(x)\ dx \\ &= 2\int_0^1 \left(\frac{g(x)}{g'(x)}\ p_n^2(x)\ e^{s(x)}\right)\ de^{g(x)}.\end{aligned}$$

Observe that for $x \in [0,1]$, $g(x)/g'(x) = x(1-x^2)$, so integrating by parts,

$$\begin{aligned}&\int_{-1}^{1} g(x)\ p_n^2(x)\ e^{s(x)}\ e^{g(x)}\ dx \\ &\quad = 2\left[x(1-x^2)\ p_n^2(x)\ w(x)\right]_{x=0}^{x=1} - 2\int_0^1 \left(x(1-x^2)\ p_n^2(x)\ e^{s(x)}\right)'\ w_0(x)\ dx \\ &\quad = -\int_{-1}^{1} (1-3x^2) p_n^2(x)\ w(x)\ dx - \int_{-1}^{1} x(1-x^2)\ \left(p_n^2(x)\right)'\ w(x)\ dx \\ &\quad\quad - 2\int_0^1 x(1-x^2)\ p_n^2(x)\ s'(x)\ w(x)\ dx\,.\end{aligned}$$

The asymptotics of each of these three integrals can be computed by means of (42), Lemma 5.2, and Lemma 5.1, respectively, obtaining that

$$\begin{aligned} G_n &= 2n(a_{n+1}^2 + a_n^2 - 1) - 4a_n a_{n-1} \frac{\beta_n}{\gamma_{n-2}} + B_1 - 2B_2 + \frac{1}{2} + o(1) \\ &= 2n(a_{n+1}^2 + a_n^2 - 1) - 4\frac{\beta_n}{\gamma_n} + B_1 - 2B_2 + \frac{1}{2} + o(1), \quad n \to \infty . \end{aligned}$$

Using that $a_n \to 1/2$ (see (7)) and (57), we get that

$$G_n = -2a\log(n) + B_1 - 2B_2 + \frac{1}{2} + B_3 + o(1), \tag{59}$$

where

$$B_3 = -2a + (a-\lambda) + 2a\psi(a+\lambda) = -a - \lambda + 2a\,\psi(a+\lambda). \tag{60}$$

Let us simplify the expression of the constant term of this asymptotics. First,

$$B_1 - 2B_2 = \int_{-1}^{1} \frac{s(x)}{\pi\,\sqrt{1-x^2}}\,dx - 2\int_0^1 x\sqrt{1-x^2}\;s'(x)\frac{1}{\pi}\,dx,$$

and integrating by parts the second integral,

$$\begin{aligned} B_1 - 2B_2 &= 2\int_0^1 \frac{s(x)}{\pi\,\sqrt{1-x^2}} - \frac{2}{\pi}\left[x\sqrt{1-x^2}s(x)\right]_{x=0}^{x=1} + 2\int_0^1 \frac{1-2x^2}{\pi\sqrt{1-x^2}}\;s(x)\,dx \\ &= \frac{4}{\pi}\int_0^1 s(x)\;\sqrt{1-x^2}\,dx. \end{aligned} \tag{61}$$

Using the explicit expression for s on $[0,1]$, the right hand side in (61) is reduced to

$$\begin{aligned} &\log\left(\frac{2^{2\lambda}\;(\lambda+a)}{2\pi\;\Gamma(2\lambda)}\right)\;\frac{4}{\pi}\int_0^1 \sqrt{1-x^2}dx + \frac{4(\lambda-1/2)}{\pi}\int_0^1 \log(1-x^2)\;\sqrt{1-x^2}\,dx \\ &\quad + \frac{4}{\pi}\int_0^1 2ax\arccos x\,dx + \frac{4}{\pi}\int_0^1 \log|\Gamma(\lambda+it)|^2\;\sqrt{1-x^2}\,dx + 4\int_0^1 ax\,dx \\ &= \log\left(\frac{2^{2\lambda}\;(\lambda+a)}{2\pi\;\Gamma(2\lambda)}\right) + (1-2\log(2))(\lambda-1/2) \\ &\quad + a + \frac{4}{\pi}\int_0^1 \log|\Gamma(\lambda+it)|^2\;\sqrt{1-x^2}\,dx + 2a \\ &= \log\left(\frac{(\lambda+a)}{\pi\;\Gamma(2\lambda)}\right) + \lambda + 3a + \frac{4}{\pi}\int_0^1 \log\left|\Gamma\left(\lambda + i\frac{ax}{\sqrt{1-x^2}}\right)\right|^2\;\sqrt{1-x^2}\,dx. \end{aligned} \tag{62}$$

Let us compute now the value of this last integral. With the change of variables $u = x/\sqrt{1-x^2}$ we obtain that

$$\int_0^1 \log \left| \Gamma\left(\lambda + i\frac{ax}{\sqrt{1-x^2}}\right)\right|^2 \sqrt{1-x^2}\, dx = \int_0^{+\infty} \log |\Gamma(\lambda + iau)|^2 \frac{du}{(1+u^2)^2}$$
$$= \int_{-\infty}^{+\infty} \log |\Gamma(\lambda + iau)| \frac{du}{(1+u^2)^2} .$$

With $\lambda > 0$, $a \geq 0$, function

$$f(u) \stackrel{\text{def}}{=} \frac{\log(\Gamma(\lambda + iau))}{(1+u^2)^2}$$

is meromorphic and single valued in the lower half plane $\{\mathrm{Im}(u) < 0\}$, with a double pole at $u = -i$. Taking into account that by Stirling formula,

$$\log(\Gamma(\lambda + iau)) \sim (\lambda + iau)|u| \log(\lambda + iau) \quad \text{as } u \to \infty,\ \mathrm{Im}(u) < 0\,,$$

we may apply the residue calculus to establish that

$$\int_{-\infty}^{+\infty} \log(\Gamma(\lambda + iau)) \frac{du}{(1+u^2)^2} = -2\pi i \operatorname*{res}_{u=-i} f(u) = \frac{\pi}{2}\left(a\psi(\lambda + a) - \log(\Gamma(\lambda + a))\right) .$$

Taking the real part, we get that

$$\frac{4}{\pi}\int_0^1 \log \left| \Gamma\left(\lambda + i\frac{ax}{\sqrt{1-x^2}}\right)\right|^2 \sqrt{1-x^2}\, dx = 2a\psi(\lambda + a) - 2\log(\Gamma(\lambda + a)) . \tag{63}$$

Gathering (60)–(63) in (59) we conclude the proof of Theorem 1.3.

Remark 5.6 The idea of this proof can be applied also to the case of a non-symmetric weight of the form

$$w(x) = \exp\left\{-\frac{4c}{(1-x)^\alpha} - \frac{4d}{(1+x)^\alpha} + s(x)\right\},$$

where $s \in C^1[-1,1]$, and $\alpha \in [1/2, 1]$.

6 Proof of corollaries 1.6 and 1.8

Proof of Corollary 1.6: Consider the integral

$$I_n = -\frac{2}{\pi}\int_{\alpha_{-n}}^{\alpha_n} \frac{Q(x)}{\sqrt{(\alpha_n - x)(x - \alpha_{-n})}}\, dx = -\frac{4}{\pi}\int_0^{\alpha_n} \frac{Q(x)}{\sqrt{(\alpha_n - x)(x - \alpha_{-n})}}\, dx.$$

By (22),

$$
\begin{aligned}
I_n &= -\frac{4}{\pi}\int_0^{\alpha_n}\left(\frac{\pi a|x|}{\sqrt{1-x^2}}+\frac{1}{2}\log(w(0))-\frac{s(x)}{2}\right)\,\frac{1}{\sqrt{\alpha_n^2-x^2}}\,dx\\
&= -\frac{4}{\pi}\int_0^{1}\left(\frac{\pi a|\alpha_n x|}{\sqrt{1-\alpha_n^2x^2}}+\frac{1}{2}\log(w(0))-\frac{s(\alpha_n x)}{2}\right)\,\frac{1}{\sqrt{1-x^2}}\,dx\\
&= -\frac{4}{\pi}\int_0^{1}\frac{\pi a|\alpha_n x|}{\sqrt{1-\alpha_n^2x^2}}\,\frac{1}{\sqrt{1-x^2}}\,dx\\
&\quad -\frac{4}{\pi}\int_0^{1}\left(\frac{1}{2}\log(w(0))-\frac{s(\alpha_n x)}{2}\right)\,\frac{1}{\sqrt{1-x^2}}\,dx.
\end{aligned}
$$

Since $s \in C[-1,1]$, the second integral is bounded; hence, by (35),

$$
I_n = -4a\alpha_n\,\frac{-1}{2}\,\log(1-\alpha_n)+\mathcal{O}(1) = 2a\log(1-\alpha_n)+\mathcal{O}(1)\,,\quad n\to\infty\,.
$$

Finally, from the asymptotics of α_n found in Proposition 3.2 we obtain

$$
I_n = 2a\log(a/n)+\mathcal{O}(1) = -2a\log(n)+\mathcal{O}(1)\,,\quad n\to\infty\,,
$$

and comparing this expression with the result of Corollary 1.4, the statement follows.

Proof of Corollary 1.8: Taking into account the relation between the entropy and the mutual energy (16) we obtain that

$$
I[\rho_n,\nu_n] = \frac{E_n+2\log(\gamma_n)}{2n}\,.
$$

By (17),

$$
2\log(\gamma_n) = 2n\log(2)+2a\log(n)-\log\left(\frac{\Gamma(\lambda+a+1)\;\Gamma(\lambda+a)}{\Gamma(2\lambda)}\right)+o(1),
$$

so that by (9),

$$
I[\nu_n,\lambda_n] = \log(2)+\frac{1}{2n}\left(\tau(\lambda,a)-\log\left(\frac{\Gamma(\lambda+a+1)\;\Gamma(\lambda+a)}{\Gamma(2\lambda)}\right)\right)+o\left(\frac{1}{n}\right).
$$

The use of the explicit expression for $\tau(\lambda,a)$ in (10) concludes the proof of the Corollary.

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References

[1] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions*, Dover Publ., New York, 1972.

[2] A. I. Aptekarev, V. S. Buyarov, and J. S. Dehesa, *Asymptotic behavior of the L^p-norms and the entropy for general orthogonal polynomials*, Russian Acad. Sci. Sb. Math., 82 (1995), pp. 373–395.

[3] B. Beckermann, A. Martínez-Finkelshtein, E. A. Rakhmanov, and F. Wielonsky, *Asymptotic upper bounds for the entropy of orthogonal polynomials in the Szegő class*, J. Math. Physics, 45 (2004), pp. 4239–4254.

[4] I. Bialynicki-Birula and J. Mycielsky, *Uncertainty relations for information entropy in wave mechanics*, Commun. Math. Phys., 44 (1975), pp. 129–132.

[5] V. S. Buyarov, *On information entropy of Gengenbauer polynomials*, Vesntik Moskow Univ., Ser. **1** 6 (1997), pp. 8–11 (in russian).

[6] V. S. Buyarov, J. S. Dehesa, A. Martínez-Finkelshtein, and E. B. Saff, *Asymptotics of the information entropy for Jacobi and Laguerre polynomials with varying weights*, J. Approx. Theory, 99 (1999), pp. 153–166.

[7] V. S. Buyarov, P. López-Artés, A. Martínez-Finkelshtein, and W. Van Assche, *Information entropy of Gegenbauer polynomials*, J. Phys. A, 33 (2000), pp. 6549–6560.

[8] V. Buyarov, J. S. Dehesa, A. Martínez-Finkelshtein, and J. Sánchez-Lara, *Computation of the entropy of polynomials orthogonal on an interval*, SIAM J. Sci. Comp., 26 (2004), pp. 488–509.

[9] J. S. Dehesa, W. Van Assche, R. J. Yáñez, *Information entropy of classical orthogonal polynomials and their application to the harmonic oscillator and Coulomb potentials*, Methods Appl. Anal. **4** (1997), pp. 91–110.

[10] J. S. Dehesa, A. Martínez-Finkelshtein, and J. Sánchez-Ruiz, *Quantum information entropies and orthogonal polynomials*, J. Comput. Appl. Math., 133 (2001), pp. 23–46.

[11] J. S. Dehesa, R. J. Yañez, A. I. Aptekarev, and V. S. Buyarov, *Strong asymptotics of Laguerre polynomials and information entropies of 2D harmonic oscillator and 1D Coulomb potentials*, J. Math. Physics, 39 (1998), pp. 3050–3060.

[12] I. S. Gradshtein and I. M. Ryzhik, *Table of Integrals, Series and Products*, Academic Press, San Diego, CA, fifth ed., 1995.

[13] E. Levin and D. S. Lubinsky, *Orthogonal Polynomials for Exponential Weights*, vol. 4 of CMS Books in Mathematics, Springer Verlag, 2001.

[14] E. Levin and D. S. Lubinsky, *Asymptotics for entropy integrals associated with exponential weights*, J. Comput. Appl. Math., 156 (2003), pp. 265–283.

[15] Y. L. Luke, *Mathematical Functions and their Approximations*, Academic Press, New York, 1975.

[16] A. Máté, P. Nevai, V. Totik, *Strong and weak convergence of orthogonal polynomials*, Am. J. Math. **109** (1987), pp. 239–282.

[17] R. G. Parr and W. Yang, *Density Functional Theory of Atoms and Molecules*, Oxford University Press, New York, 1989.

[18] E. A. Rakhmanov, *On the asymptotics of the ratio of orthogonal polynomials*, Math. USSR Sb., 32 (1977), pp. 199–213.

[19] E. B. Saff and V. Totik, *Logarithmic Potentials with External Fields*, vol. 316 of Grundlehren der Mathematischen Wissenschaften, Springer-Verlag, Berlin, 1997.

[20] J. F. Sánchez Lara, *On the asymptotic expansion of the entropy of Gegenbauer polynomials*, J. Comput. Appl. Math., 142 (2002), pp. 401–409.

[21] G. Szegő, *Orthogonal Polynomials*, vol. 23 of Amer. Math. Soc. Colloq. Publ., Amer. Math. Soc., Providence, RI, fourth ed., 1975.

[22] R. J. Yáñez, W. Van Assche, and J. S. Dehesa, *Position and momentum information entropies of the D-dimensional harmonic oscillator and hydrogen atom*, Physical Rev. A, 50 (1994), pp. 3065–3079.

[23] R. J. Yañez, W. Van Assche, R. González-Férez, J. S. Dehesa, *Entropic integrals of hyperspherical harmonics and spatial entropy of D-dimensional central potentials*, J. Math. Phys. **40** (1999), pp. 5675–5686.