A common fixed point for operators in probabilistic normed spaces

M.B. Ghaemi a, b, Bernardo Lafuerza-Guillen b, A. Razani c

a Faculty of Mathematics, Iran University of Science and Technology, Narmak, Tehran, Iran
b Department of Applied Mathematics, University of Almeria, Almeria, Spain
c Department of Mathematics, Faculty of Science, I. K. International University, P.O. Box 14194-288, Qazvin, Iran

Accepted 7 September 2007

Abstract

Probabilistic Metric spaces was introduced by Karl Menger. Alsina, Schweizer and Sklar gave a general definition of probabilistic normed space based on the definition of Menger [Alsina C, Schweizer B, Sklar A. On the definition of a probabilistic normed spaces. Archivaeae Math 1993;46:91–8]. Here, we consider the equicontinuity of a class of linear operators in probabilistic normed spaces and finally, a common fixed point theorem is proved. Application to quantum Mechanics is considered.

© 2007 Elsevier Ltd. All rights reserved.

1. Introduction

The theory of probabilistic metric spaces, introduced in 1942 by Menger [10], was developed by numerous authors, for instance [4], as well as those in [12,13]. The notion of a probabilistic metric space corresponds to the situations when we do not know exactly the distance between two points, we know only probabilities of possible values of this distance. Such a probabilistic generalization of metric spaces appears to be well adapted for the investigation of quantum physics as shown by El Naschie [2]. The notion of a probabilistic normed space was introduced by Senyshyn [14]. Alsina, Schweizer and Sklar gave a general definition of probabilistic normed space based on the definition of Menger [13] for probabilistic metric spaces in [3,2]. Linear operator in probabilistic normed spaces has been studied by Gähler, Lalena and Sempi in [5,8]. In this paper, we consider the equicontinuity of a class of linear operators on probabilistic normed space and applying these results, a common fixed point theorem is proved. In order to do this, we recall some definitions from [1–5,6].

Definition 1.1. A distribution function is a function \( P: \mathbb{R} \to [0, 1] \) that is nondecreasing and left continuous on \( \mathbb{R} \); moreover, \( P(-\infty) = 0 \) and \( P(\infty) = 1 \).

* Corresponding author.

E-mail addresses: mghaemi@just.ac.ir (M.B. Ghaemi), b.lafuerza@uniof.es (B. Lafuerza-Guillen), razani@khu.ac.ir (A. Razani).

069SS0757/8 - see front matter © 2007 Elsevier Ltd. All rights reserved.
The set of all distribution functions is denoted by $\mathcal{D}$ and the set of those distribution functions such that $F(0) = 0$ is denoted by $\mathcal{D}^\star$. The distance distribution functions are denoted by $D^*$ and $D^\star = \{ F \in \mathcal{D}^*: \lim_{x \to \pm \infty} F(x) = 1 \}$. A natural ordering in $\mathcal{D}$ and $\mathcal{D}^\star$ is defined by $F \preceq G$ whenever $F(x) \leq G(x)$ for every $x \in \mathbb{R}$. The maximal element in this order for $D^\star$ is $\varepsilon_0$, where for $a \in \mathbb{R}$, the distribution function $\varepsilon_a$ is defined by

$$
\varepsilon_a = \begin{cases} 
0, & \text{if } t \leq a, \\
1, & \text{if } t > a.
\end{cases}
$$

**Definition 1.2.** A triangle function is a binary operation on $\mathcal{D}^\star$ that is commutative, associative, nondecreasing in each place, and has $\varepsilon_0$ as identity.

Note that the continuity of a triangle function means continuity with respect to the topology of weak convergence in $\mathcal{D}^\star$.

**Example 1.3.** Let $T^*$ be a continuous $\tau$-norm, i.e., a continuous binary operation in $[0,1]$ that is associative, nondecreasing and has 1 as identity; $T^*$ is a continuous $\tau$-conorm, namely a continuous binary operation on $[0,1]$ that is related to a continuous $\tau$-norm through

$$
T^*(x,y) = 1 - T(1-x,1-y).
$$

Typical continuous triangle functions are convolution, the operations $\tau_T$ and $\tau_T^*$, which are given by

$$
\tau_T(F,G)(x) = \sup_{t \in \text{supp}(F)} T(F(x),G(t))
$$

and

$$
\tau_T^*(F,G)(x) = \inf_{t \in \text{supp}(F)} T^*(F(x),G(t))
$$

for all $F, G$ in $\mathcal{D}^\star$ and all $x \in \mathbb{R}$ [13, Sections 7.2 and 7.3], respectively.

It follows without difficulty from the above that

$$
\tau_T(\varepsilon_a, \varepsilon_b) = \varepsilon_{a+b} = \tau_T^*(\varepsilon_a, \varepsilon_b)
$$

for any continuous $\tau$-norm $T$, any continuous $\tau$-conorm $T^*$ and any $a,b \geq 0$.

**Definition 1.4.** A probabilistic normed space (briefly, PN-space) is a quadruple $(X, \Theta, \tau, \tau^*)$, where $X$ is a real vector space, $\tau$ and $\tau^*$ are continuous triangle functions, and $\Theta$ is a mapping from $X$ into $\mathcal{D}^\star$ such that, for $p,q \in X$, the following conditions hold:

1. (PN1) $\Theta_p = \varepsilon_0$ if $p = 0$ is the null vector in $X$;
2. (PN2) $\Theta_{-x} = \Theta_x$;
3. (PN3) $\Theta_{p+q} \succeq \tau(\Theta_p, \Theta_q)$;
4. (PN4) $\Theta_{\alpha} = \tau^*(\Theta_{\alpha p}, \Theta_{\alpha q})$, for all $\alpha \in \mathbb{R}$.

If $(X,\|\cdot\|)$ is a real normed space, $\tau$ a triangle function such that $\tau(\varepsilon_a, \varepsilon_b) \leq \varepsilon_{a+b}$ for all $a, b \geq 0$, and if $\Theta : X \to \mathcal{D}^\star$ is defined via $\Theta_p = \Theta_{\|p\|}$, then $(X,\|\cdot\|)$ is a PN-space.

Note that if $\tau^* = \tau_M$ (where $\tau_M$ is the $\tau$-norm defined as $\tau_M(x,v) = \text{Min}(x,v)$) and equality holds in (PN4), then $(X,\Theta,\tau,\tau_M)$ is a Šerstnev PN-space [1]. In this case, as shown in [3], the conditions

$$
\Theta_p = \tau_M(\Theta_{\alpha p}, \Theta_{\alpha q})
$$

for all $p$ in $X$ and $x \in I$ and (PN2), taken together, are equivalent to Šerstnev's condition

$$
\Theta_{\alpha}(x) = \Theta_{\frac{\alpha}{\|x\|}}(x)
$$

for all $x \in \mathbb{R} \setminus \{0\}$ and $\alpha$ in $\mathbb{R}$.

Every PN-space $(V, \Theta, \tau, \tau^*)$ can be endowed with the strong topology; this topology is generated by the strong neighborhood, which are defined as follows: for every $r > 0$, the neighborhood $N_r(\eta)$ at a point $\eta$ of $V$ is defined by

$$
N_r(\eta) := \{ q \in V : d_\tau(\Theta_{\eta q}, \varepsilon_r) < r \} = \{ q \in V : \Theta_{\eta q}(\tau_r) > 1 - r \}.
$$
Definition 1.5. A topological vector space (TVS) is a vector space $V$ together with a topology such that with respect to this topology:

(i) The map of $V \times V \to V$ defined by $(x, y) \to x + y$ is continuous.
(ii) The map of $F \times X \to X$ defined by $(x, x) \to xx$ is continuous. Where $F$ is $\mathbb{R}$ or $\mathbb{C}$.

Definition 1.6. A topological vector space $X$ is called locally convex if the neighborhood filter around 0 has a basis of convex sets.

Note that every PN-space $(V, \Theta, \tau, \tau')$, when it is endowed with the strong topology induced by the probabilistic norm $\Theta$ is a topological vector space if, and only if, for every $p \in V$ the map from $\mathbb{R}$ into $V$ defined by $x \to xp$ is continuous (see [2]) for more details. Henceforth a PN-space $(V, \Theta, \tau, \tau')$ which is a topological vector space is denoted by TV PN-space.

Remark 1.7. It was proved [3, Theorem 4] that if the triangle function $\tau'$ is Archimedean, i.e. if $\tau'$ admits no idempotents other than 0 and 1, then the mapping $x \to xp$ is continuous and as a consequence of this, the PN-space $(V, \tau, \tau', \tau)$ is a TV PN-space.

Definition 1.8. A PN-space $(V, \tau, \tau', \tau)$ is characteristic if $v(V) \subseteq \mathbb{R}$, or equivalently $v_p \in \mathbb{R}$ for every $p \in V$.

Theorem 1.9. A characteristic Šerstnev space $(V, \tau, \tau')$ with $\tau = \tau_M$ is locally convex.

Proof. (11). We prove it here because the notation in Prochaska's thesis is different from the one that has become usual after the publication of [12].

It suffices to consider the family of neighborhoods of the origin 0, $N_0(t)$, with $t > 0$. Let $t > 0$, $p, q \in N_0(t)$ and $x \in [0, 1]$. Then

$$v_{t+1}(t) = \sup_{\beta \in [0, 1]} \{v_{t+1}(\beta) : v_{t+1}(\beta) > t\} = \sup_{\beta \in [0, 1]} \{v_{t+1}(\beta) : v_{t+1}(\beta) > t\} = \sup_{\beta \in [0, 1]} \{v_{t+1}(\beta) : v_{t+1}(\beta) > t\}$$

Thus $v_{t+1}(t) = \sup_{\beta \in [0, 1]} \{v_{t+1}(\beta) : v_{t+1}(\beta) > t\}$ for every $x \in [0, 1]$. □

Theorem 1.10. Let $(V, \tau, \tau', \tau)$ be a TV PN-space. If $A : X \to X$ is linear and continuous at 0, then $A$ is continuous.

A linear operator $T : V_1 \to V_2$ where $(V_1, \tau_1, \tau'_1)$ and $(V_2, \tau_2, \tau'_2)$ are TV PN-spaces, is bounded if it transform bounded subset of $V_1$ into bounded subset of $V_2$. Note that continuous linear operators are bounded.

Remark 1.11. A linear operator between two locally convex TV PN-spaces is continuous if and only if it is bounded, see [5, p. 477].

2. Common fixed point

In this section, we prove some theorems and as a result of these theorems, one can prove the existence of a common fixed point theorem. Due to this, the next result is a uniform boundedness theorem for TV PN spaces.

Theorem 2.1. If $\Gamma$ is a collection of continuous linear maps between two TV PN-spaces $(V_1, \tau_1, \tau'_1)$ and $(V_2, \tau_2, \tau'_2)$ and if the set

$$\Gamma(t) = \{A : A \in \Gamma\}$$

is a bounded subset of $V_2$, for every $x \in V_1$, then $\Gamma$ is equicontinuous.

Proof. Let $N_0(t) = \{p \in V_1 : t_p(t) > 1 - t\}$ be a neighborhood of 0, then

$$N_0\left(\frac{1}{3}\right) = \left\{p : t_p(t) > 1 - \frac{1}{3}\right\} = \left\{p \in V_1 : d_p(t, t_0) \leq \frac{1}{3}\right\}$$
and we have
\[ \mathcal{A}\left( \frac{1}{3} \right) + N_x\left( \frac{1}{3} \right) \subseteq \{ y \in \mathbb{R}^d : d(y, y_0) \leq 1 \} . \]

Put
\[ E = \bigcap_{x \in \mathbb{R}^d} A^{-1}\left( N_x\left( \frac{1}{3} \right) \right) . \]

\( V_1 \) is a complete metric space and \( E \subseteq \bigcup_{x \in \mathbb{R}^d} A^{-1}\left( N_x\left( \frac{1}{3} \right) \right) . \) Therefore \( E \) is a closed subset of \( V_1 \) also the interior of \( E \) is not empty. Hence \( x - E \) contains a neighborhood \( N_x(\varepsilon) \) of 0 such that
\[ A( N_x(\varepsilon) ) \subseteq A(x) - A(E) \subseteq N_x\left( \frac{1}{3} \right) + N_x\left( \frac{1}{3} \right) \]
for every \( A \in E \). This proves \( E \) is equicontinuous. \( \square \)

**Corollary 2.2.** If \( \Gamma \) is a collection of continuous linear maps from \( \mathbb{R}^n \)-space \( (V_1, v_1, \tau_1) \) onto \( \mathbb{R}^n \)-space \( (V_2, v_2, \tau_2) \), where \( \tau_1 \) and \( \tau_2 \) are Archimedean and \( \Gamma(x) = \{ Ax : A \in \Gamma \} \) is a bounded subset of \( V_2 \) for every \( x \in V_1 \), then \( \Gamma \) is equicontinuous.

**Proof.** By Remark 1.7 the \( \mathbb{R}^n \)-spaces \( (V_1, v_1, \tau_1) \) and \( (V_2, v_2, \tau_2) \) are TV \( \mathbb{R}^n \)-spaces. Now the result follows from Theorem 2.1. \( \square \)

**Corollary 2.3.** If \( \Gamma = \{ A : V_1 \rightarrow V_2 \} \) is a collection of continuous linear maps, where \( (V_1, v_1, \tau_1) \) and \( (V_2, v_2, \tau_2) \) are characteristic \( \mathcal{S} \)-spaces with \( \tau_1 = \tau_2 = \tau_M \) and if \( \Gamma(x) = \{ Ax : A \in \Gamma \} \) is a bounded subset of \( V_2 \) for every \( x \in V_1 \), then \( \Gamma \) is equicontinuous.

**Proof.** Note that \( V_1 \) and \( V_2 \) are locally convex spaces by [8, Theorem 7]. Now the result is immediate consequence of Theorem 2.1. \( \square \)

**Theorem 2.4.** If \( \{ A_n \} \) is a sequence of continuous linear mapping from a TV \( \mathbb{R}^n \)-space \( (V_1, v_1, \tau_1) \) into a TV \( \mathbb{R}^n \)-space \( (V_2, v_2, \tau_2) \) and if \( Ax = \lim_{n \to \infty} A_n x \) exist for every \( x \in V_1 \), then \( A \) is continuous.

**Proof.** Theorem 2.1 implies that \( \{ A_n \} \) is equicontinuous. Suppose \( U_1 \) is a neighborhood of 0 in \( V_1 \), then \( A_n(U_1) - U_1 \) for all \( n \in \mathbb{N} \) and some neighborhood \( U_1 \) of 0 in \( V_1 \). It follows that \( A(U_1) \subseteq U_1 \), hence \( A \) is continuous. \( \square \)

**Definition 2.5.** A linear operator \( T \) of \( V_1 \) into \( V_2 \) is called bounded if it transform bounded subset of \( V_1 \) into bounded subset of \( V_2 \) [5, p. 63].

**Corollary 2.6.** If \( \{ A_n \} \) is a sequence of bounded linear mapping from a characteristic \( \mathcal{S} \)-space \( (V_1, v_1, \tau_1) \) into a characteristic \( \mathcal{S} \)-space \( (V_2, v_2, \tau_2) \) with \( \tau_1 = \tau_2 = \tau_M \) and if \( Ax = \lim_{n \to \infty} A_n x \) exist for every \( x \in V_1 \), then \( A \) is bounded.

**Proof.** By Remark 1.1.1, \( A_n \) is continuous for all \( n \). Now the result is a immediate consequence of Theorem 2.4. \( \square \)

**Lemma 2.7.** In a characteristic \( \mathcal{S} \)-space \( (V, v, \tau) \) the following statement are equivalent for a subset \( A \) of \( V 

(a) \( A \) is \( \mathcal{S} \)-bounded
(b) \( A \) is topologically bounded

**Proof.** (a) \( \Rightarrow \) (b) Let \( A \) any \( \mathcal{S} \)-bounded subset of \( V \) and let \( p_n \) be any sequence of elements of \( A \) and \( x_n \) any sequence of real numbers that converges to 0; there is no loss of generality in assuming \( x_n = 0 \) for every \( n \in \mathbb{N} \).

\[ e_n(x) = e\left( \frac{x}{|x_n|} \right) \Rightarrow R_n\left( \frac{x}{|x_n|} \right) \rightarrow 1 \]
as \( e \rightarrow +\infty. \)
Thus \( x_0 \rightarrow 0 \) in the strong topology and \( A \) is topologically bounded. (b) \( \Rightarrow \) (a) Let \( A \) be a subset of \( V \) which is not \( \mathcal{S} \)-bounded. Then
\[
R_A(x) - \gamma < 1
\]
as \( x \rightarrow +\infty \).

By definition of \( R_A \) for every \( n \in \mathbb{N} \) there is \( p_n \in A \) such that
\[
v_{p_n}(r^2) < \frac{1 + \gamma}{2} < 1.
\]

If \( z_n = \frac{1}{n} \), then
\[
v_{p_n}(r^2) = v_{p_n}(r) - v_{p_n}(r^2) < \frac{1 + \gamma}{2} < 1,
\]
which shows that \( v_{p_n} \) does not tend to \( z_n \), even if it has a weak limit. viz. \( x, p_n \) does not tend to \( \ell \) in the strong topology, in other words, \( A \) is not topologically bounded.

The following corollary is immediate from Corollary 2.6 and Lemma 2.5. 

**Corollary 2.8.** If \( \Gamma = \{ A : A \in \mathcal{N} \} \) is a collection of continuous linear maps, where \( (V_1, t_1, v_1) \) and \( (V_2, \tau_2, v_2) \) are characteristic \( \mathcal{S} \)-spaces with \( t_1 = \tau_2 = \tau_M \) and if \( \Gamma(x) = \{ Ax : \lambda \in \Gamma \} \) is a \( \mathcal{S} \)-bounded subset of \( V_2 \) for every \( x \in V_1 \), then \( \Gamma \) is equicontinuous.

In order to give the final result of this article, the following definition is given:

**Definition 2.9.** A set \( A \subseteq V \) is balanced, if \( x \in A \) whenever \( x \in V \) and \( |x| \leq 1. \) A set \( A \) is absorbing, if for each \( x \in V \) there is an \( e > 0 \), such that \( tx \in A \) for \( 0 < t < e.\)

**Theorem 2.10.** Suppose \( K \) is a nonempty convex compact subset of locally convex TV PN-space \( (V, v, \tau, \tau^*) \). Let \( G \) be a group of linear mapping such that
\[
\Gamma(x) = \{ Ax : \lambda \in G \}
\]
is bounded in \( V \) for every \( x \in V \). Also \( \mathcal{A}(K) \subseteq K \) for every \( \lambda \in G \). Then \( G \) has a common fixed point in \( K \); that is there exist \( p \in K \) such that \( Ap = p \) for every \( \lambda \in G \).

**Proof.** The group \( G \) is equicontinuous by Theorem 2.1 and \( V \) has a local base consisting of balanced convex set \( U \) which satisfies \( \mathcal{A}(U) \subseteq U \) for every \( \lambda \in G \). Let \( \overline{\Omega} \) be the collection of all nonempty convex sets \( H \subseteq K \) such that \( \mathcal{A}(H) = H \) for every \( \lambda \in G \). Now, we partially ordered the set \( \overline{\Omega} \) by set inclusion. Then Hahn-Banach's maximality theorem shows that \( \overline{\Omega} \) contains a maximal totally ordered subcollection \( \overline{\Omega}_0 \). The intersection \( H_0 \) of all members of \( \overline{\Omega}_0 \) is a minimal member of \( \overline{\Omega} \). Now, it is enough to show that \( H_0 \) has exactly one point. Otherwise, if \( H \in \overline{\Omega} \) contains more than one point. Then \( H - H = \{ 0 \} \) and there is a convex balanced member of the above local base which does not cover \( H - H \). Since \( H - H \) is compact, there exists some \( s > 0 \) such that \( H - H \subseteq s \). Let \( t \) be the greatest lower bound of these member \( s. \) Set \( W = tU \), then \( W \) is a convex balanced open set such that
\[
(1 - r)W \text{ does not cover } H - H \text{ if } 0 < r < 1.
\]

**Proof.** Let \( x_1, x_2, \ldots, x_n \in H \) such that \( H \subseteq \bigcup_{i=1}^{n} (x_i + 1/2W) \). Let \( r = 1/4n \) and define \( H_1 = \bigcap_{i=1}^{n} (x_i + (1 - r)W) \). \( H_1 \) is compact, convex and \( \mathcal{A}H_1 \subseteq H_1 \) for every \( \lambda \in G \). By (1) there are points \( x \in H \) and \( y \notin H \) such that \( x - y \) does not lie in \((1 - r)W \). Any such \( x \) is not in \( H_1 \). Thus \( H_1 \neq H \). The point \( x_0 = 1/n \sum_{i=1}^{n} x_i \) is in \( H_1 \) and therefore \( H_1 \neq \emptyset \). This shows that \( H_0 \) contains only one point.

The following corollaries are immediate consequences from Theorems 2.10 and 1.9.

**Corollary 2.11.** Suppose \( K \) is a nonempty convex subset of characteristic \( \mathcal{S} \)-space \( (V, v, \tau, \tau^*) \) with \( \tau = \tau_M \). Let \( G \) be a group of linear mapping such that \( \Gamma(x) = \{ Ax : \lambda \in G \} \) is bounded in \( V \) for every \( x \in V \). Moreover, \( \mathcal{A}(K) \subseteq K \) for every \( \lambda \in G \). Then \( G \) has a common fixed point in \( K \).

**Corollary 2.12.** Suppose \( K \) is a nonempty convex subset of characteristic \( \mathcal{S} \)-space \( (V, v, \tau, \tau^*) \) with \( \tau = \tau_M \) and \( \lambda \) is an invertible operator on \( V \) such that the set \( \{ Ax : \lambda^{-1}x \} \) is bounded for every \( x \in V \), then \( \lambda \) and \( \lambda^{-1} \) has a common fixed point.
3. Application in physics

Menger sponge is a random space which could be used for instance to predict the Background micro wave radiation (see El Naschie and also Ie’s Book [7]).

4. Conclusions

In this work we have analyzed in some detail the problem of equicontinuity of a class of linear operators on probabilistic normed spaces. We have shown for the class of characteristic Selivanov spaces if \( \{ A_\alpha \} \) is bounded then \( I \) is equicontinuous. A detailed study of how we can have a common fixed point for a group of linear operators is given.

References