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Quotient probabilistic normed spaces and completeness results

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Abstract. We introduce the concept of quotient in PN spaces and give some examples. We prove some theorems with regard to the completeness of a quotient.

Keywords. Probabilistic normed space; probabilistic norm; triangle functions; quotient probabilistic normed space; σ-product.

1. Introduction

In the literature devoted to the theory of probabilistic normed spaces (PN spaces, briefly), topological and completeness questions, boundedness and compactness concepts [4, 5, 7], linear operators, probabilistic norms for linear operators [6], product spaces [3] and fixed point theorems have been studied by various authors. However quotient spaces of PN spaces have never been considered. This note is a first attempt to fill this gap.

The present paper is organized as follows. In §2 all necessary preliminaries are recalled and notation is established. In §3, the quotient space of a PN space with respect to one of its subspaces is introduced and its properties are studied. Finally, in §4, we investigate the completeness relationship among the PN spaces considered.

2. Definitions and preliminaries

In the sequel, the space of all probability distribution functions (briefly, d.f.'s) is $\Delta^+ = \{F\colon \mathbf{R} \cup \{-\infty, +\infty\} \longrightarrow [0, 1]\colon F$ is left-continuous and non-decreasing on $\mathbf{R}$, $F(0) = 0$ and $F(+\infty) = 1\}$ and the subset $D^+ \subseteq \Delta^+$ is the set $D^+ = \{F \in \Delta^+\colon l^- F(+\infty) = 1\}$. Here $l^- f(x)$ denotes the left limit of the function f at the point x, $l^- f(x) = \lim_{t\to x^-} f(t)$. The space Δ^+ is partially ordered by the usual point-wise ordering of functions, i.e., $F \leq G$ if and only if $F(x) \leq G(x)$ for all x in $\mathbf{R}$. The maximal element for Δ^+ in this order is the d.f. given by

$$\varepsilon_0 = \begin{cases} 0, & \text{if } x \leq 0, \\ 1, & \text{if } x > 0. \end{cases}$$

Also the minimal element for Δ^+ in this order is the d.f. given by

$$\varepsilon_\infty = \begin{cases} 0, & \text{if } x \leq \infty, \\ 1, & \text{if } x = \infty. \end{cases}$$

We assume that Δ is metrized by the Sibley metric d_S, which is the modified Lévy metric [8, 9]. If F and G are d.f.'s and h is in $(0, 1]$, let $(F, G; h)$ denote the condition

$$F(x-h) - h \leq G(x) \leq F(x+h) + h$$

for all x in $(-1/h, 1/h)$. Then the modified Lévy metric (Sibley metric) is defined by

$$d_S(F, G) := \inf\{h > 0\colon \text{both } (F, G; h) \text{ and } (G, F; h) \text{ hold}\}.$$

For any F in Δ^+,

$$d_S(F, \varepsilon_0) = \inf\{h > 0\colon (F, \varepsilon_0; h) \text{ holds}\}$$

$$= \inf\{h > 0\colon F(h^+) > 1 - h\},$$

and for any $t > 0$,

$$F(t) > 1 - t \Longleftrightarrow d_S(F, \varepsilon_0) < t.$$

It follows that, for every F, G in Δ^+,

$$F \leq G \Longrightarrow d_S(G, \varepsilon_0) \leq d_S(F, \varepsilon_0).$$

A sequence (F_n) of d.f.'s converges weakly to a d.f. F if and only if the sequence $(F_n(x))$ converges to $F(x)$ at each continuity point x of F. For the proof of the next theorem see Theorem 4.2.5 of [8].

Theorem 2.1. *Let (F_n) be a sequence of functions in Δ, and let F be in Δ. Then $F_n \to F$ weakly if and only if $d_S(F_n, F) \to 0$.*

DEFINITION 2.2

A *triangular norm* T (briefly, a *t-norm*) is an associative binary operation on $[0, 1]$ (henceforth, I) that is commutative, nondecreasing in each place, such that $T(a, 1) = a$ for all $a \in I$.

DEFINITION 2.3

Let T be a binary operation on I. Denote by T^* the function defined by $T^*(a, b) := 1 - T(1-a, 1-b)$ for all $a, b \in I$. If T is a t-norm, then T^* will be called the *t-conorm* of T. A function S is a t-conorm if there is a t-norm T such that $S = T^*$.

Clearly, T^* is itself a binary operation on I, and $T^{**} = T$. Instances of such t-norms and t-conorms are M and M^*, respectively, defined by $M(x, y) = \min(x, y)$ and $M^*(x, y) = \max(x, y)$.

DEFINITION 2.4

A *triangle function* τ is an associative binary operation on Δ^+ that is commutative, nondecreasing in each place, and has ε_0 as identity.

Also we let $\tau^1 = \tau$ and

$$\tau^n(F_1, \ldots, F_{n+1}) = \tau(\tau^{n-1}(F_1, \ldots, F_n), F_{n+1}) \quad \text{for } n \geq 2.$$

Let T be a left-continuous t-norm and T^* a right-continuous t-conorm. Then instances of such triangle functions are τ_T and τ_{T^*} defined for all $F, G \in \Delta^+$ and every $x \in \mathbf{R}^+$, respectively, by

$$\tau_T(F, G)(x) = \sup\{T(F(u), G(v)) \mid u + v = x\}$$

and

$$\tau_{T^*}(F, G)(x) = \ell^- \inf\{T^*(F(u), G(v)) \mid u + v = x\}.$$

The triangular function τ is said to be *Archimedean* on Δ^+ if $\tau(F, G) < F$ for any F, G in Δ^+, such that $F \neq \varepsilon_\infty$ and $G \neq \varepsilon_0$.

DEFINITION 2.5

Let τ_1, τ_2 be two triangle functions. Then τ_1 dominates τ_2, and we write $\tau_1 \gg \tau_2$, if for all $F_1, F_2, G_1, G_2 \in \Delta^+$,

$$\tau_1(\tau_2(F_1, G_1), \tau_2(F_2, G_2)) \geq \tau_2(\tau_1(F_1, F_2), \tau_1(G_1, G_2)).$$

In 1993, Alsina, Schweizer and Sklar [1] gave a new definition of a probabilistic normed space as follows:

DEFINITION 2.6

A *probabilistic normed space*, briefly a PN space, is a quadruple (V, ν, τ, τ^*) in which V is a linear space, τ and τ^* are continuous triangle functions with $\tau \leq \tau^*$ and ν, the probabilistic norm, is a map $\nu\colon V \to \Delta^+$ such that

(N1) $\nu_p = \varepsilon_0$ if and only if $p = \theta$, θ being the null vector in V;
(N2) $\nu_{-p} = \nu_p$ for every $p \in V$;
(N3) $\nu_{p+q} \geq \tau(\nu_p, \nu_q)$ for all $p, q \in V$;
(N4) $\nu_p \leq \tau^*(\nu_{\alpha p}, \nu_{(1-\alpha)p})$ for every $\alpha \in [0, 1]$ and for every $p \in V$.

If, instead of (N1), we only have $\nu_\theta = \varepsilon_0$, then we shall speak of a *probabilistic pseudo normed space*, briefly a PPN space. If the inequality (N4) is replaced by the equality $\nu_p = \tau_M(\nu_{\alpha p}, \nu_{(1-\alpha)p})$, then the PN space is called a *Šerstnev space*; in this case, a condition stronger than (N2) holds, namely

$$\nu_{\lambda p} = \nu_p\left(\frac{j}{|\lambda|}\right), \qquad \forall \lambda \neq 0,\ \forall p \in V;$$

here j is the identity map on $\mathbf{R}$. A Šerstnev space is denoted by (V, ν, τ).

There is a natural topology in a PN space (V, ν, τ, τ^*), called the *strong topology*; it is defined, for $t > 0$, by the neighbourhoods

$$N_p(t) := \{q \in V\colon d_S(\nu_{q-p}, \epsilon_0) < t\} = \{q \in V\colon \nu_{q-p}(t) > 1 - t\}.$$

The strong neighbourhood system for V is the union $\bigcup_{p\in V}\mathcal{N}_p(\lambda)$ where $\mathcal{N}_p = \{N_p(\lambda)\colon \lambda > 0\}$. The strong neighborhood system for V determines a Hausdorff topology for V.

A linear map $T\colon (V, \nu, \tau, \tau^*) \to (V', \nu', \sigma, \sigma^*)$, is said to be *strongly bounded*, if there exists a constant $k > 0$ such that, for all $p \in V$ and $x > 0$,

$$\nu'_{Tp}(x) \geq \nu_p(x/k).$$

DEFINITION 2.7

A *Menger PN space* is a PN space (V, ν, τ, τ^*) in which $\tau = \tau_T$ and $\tau^* = \tau_{T^*}$ for some t-norm T and its t-conorm T^*. It will be denoted by (V, ν, T).

DEFINITION 2.8

Let (V, ν, τ, τ^*) be a PN space. A sequence $(p_n)_n$ in V is said to be *strongly convergent* to p in V if for each $\lambda > 0$, there exists a positive integer N such that $p_n \in N_p(\lambda)$, for $n \geq N$. Also the sequence $(p_n)_n$ in V is called a *strong Cauchy sequence* if, for every $\lambda > 0$, there is a positive integer N such that $\nu_{p_n - p_m}(\lambda) > 1 - \lambda$, whenever $m, n > N$. A PN space (V, ν, τ, τ^*) is said to be *strongly complete* in the strong topology if and only if every strong Cauchy sequence in V is strongly convergent to a point in V.

Lemma 2.9 [2]. *If* $|\alpha| \leq |\beta|$, *then* $\nu_{\beta p} \leq \nu_{\alpha p}$ *for every* p *in* V.

DEFINITION 2.10

Let $(V_1, \nu_1, \tau, \tau^*)$ and $(V_2, \nu_2, \tau, \tau^*)$ be two PN spaces under the same triangle functions τ and τ^*. Let σ be a triangle function. The σ-product of the two PN spaces is the quadruple

$$(V_1 \times V_2, \nu_1 \sigma \nu_2, \tau, \tau^*),$$

where

$$\nu_1 \sigma \nu_2\colon V_1 \times V_2 \longrightarrow \Delta^+$$

is a probabilistic semi-norm given by

$$(\nu_1 \sigma \nu_2)(p, q) := \sigma(\nu_1(p), \nu_2(q))$$

for all $(p, q) \in V_1 \times V_2$.

3. Quotient PN space

According to [8] (see Definition 12.9.3 in p. 215), one has the following:

DEFINITION 3.1

A triangle function τ is *sup-continuous* if, for every family $\{F_\lambda\colon \lambda \in \Lambda\}$ of d.f.'s in Δ^+ and every $G \in \Delta^+$,

$$\sup_{\lambda\in\Lambda} \tau(F_\lambda, G) = \tau\left(\sup_{\lambda\in\Lambda} F_\lambda, G\right).$$

In view of Lemma 4.3.5 of [8], this supremum is in Δ^+. An example of a *sup-continuous* triangle function is τ_T, where T is a left continuous t-norm.

DEFINITION 3.2

Let W be a linear subspace of V and denote by $\sim_W$ a relation on the set V defined via

$$p \sim_W q \Leftrightarrow p - q \in W,$$

for every $p, q \in V$.

Obviously this relationship is an equivalence relation and therefore the set V is partitioned into equivalence classes, $V/\sim_W$.

PROPOSITION 3.3

Let (V, ν, τ, τ^*) be a PN space. Suppose that τ and τ^* are sup-continuous. Let W be a subspace of V and $V/\sim_W$ its quotient defined by means of the equivalence relation $\sim_W$. Let ν' be the restriction of ν to W and define the mapping $\bar{\nu}$: $V/\sim_W \to \Delta^+$, for all $p \in V$, by

$$\bar{\nu}_{p+W}(x) := \sup_{q \in W} \{\nu_{p+q}(x)\}.$$

Then, (W, ν', τ, τ^*) is a PN space and $(V/\sim_W, \bar{\nu}, \tau, \tau^*)$ is a PPN space.

Proof. The first statement is immediate. The remainder of the theorem is guaranteed by the fact that W is not necessarily closed in the strong topology. □

Notice that by Lemma 4.3.5 of [8], $\bar{\nu}_{p+W}$ is in Δ^+.

Hereafter we denote by p_W the subset $p + W$ of V, i.e. an element of quotient, and the strong neighbourhood of p_W by $N'_{p_W}(t)$.

Theorem 3.4. *Let W be a linear subspace of V. Then the following statements are equivalent:*

(a) *$(V/\sim_W, \bar{\nu}, \tau, \tau^*)$ is a PN space;*
(b) *W is closed in the strong topology of (V, ν, τ, τ^*).*

Proof. Let (V, ν, τ, τ^*) be a PN space. For every p in the closure of W and for each $n \in \mathbb{N}$ choose $q_n \in N_p(1/n) \cap W$. Then

$$\bar{\nu}_{p_W}(1/n) = \sup_{q \in W} \nu_{p+q}(1/n) \geq \nu_{p-q_n}(1/n) > 1 - 1/n,$$

and hence, $d_S(\bar{\nu}_{p_W}, \varepsilon_0) < 1/n$. Thus $p_W = W$ and hence, $p \in W$ and W is closed.

Conversely, if W is closed, let $p \in V$ be such that $\bar{\nu}_{p_W} = \varepsilon_0$. If $p \notin W$, then $N_p(t) \cap W = \varnothing$, for some $t > 0$. That is to say, for every $q \in W$, $\nu_{p-q}(t) \leq 1 - t$. Therefore $\bar{\nu}_{p_W}(t) = \sup_{q \in W} \nu_{p+q}(t) \leq 1 - t$, which is a contradiction. □

It is of interest to know whether a PN space can be obtained from a PPN space. An affirmative answer is provided by the following proposition.

PROPOSITION 3.5

Let (V, ν, τ, τ^*) be a PPN space and define

$$C = \{p \in V \colon \nu_p = \varepsilon_0\}.$$

Then C is the smallest closed subspace of (V, ν, τ, τ^*).

Proof. If $p, q \in C$, then $p + q \in C$ because $\nu_{p+q} \geq \tau(\nu_p, \nu_q) = \varepsilon_0$. Now suppose $p \in C$. For $\alpha \in [0, 1]$ one has $\nu_{\alpha p} \geq \nu_p$ by Lemma 2.9. For $\alpha > 1$, let $k = [\alpha] + 1$. Then, using the iterates of (N3) one has, $\nu_{kp} \geq \tau^{k-1}(\nu_p, \ldots, \nu_p) = \varepsilon_0$. By the above-mentioned lemma one has $\nu_{\alpha p} \geq \nu_{kp}$. As a consequence, αp belongs to C for all $\alpha \in \mathbf{R}$.

Furthermore it is easy to check that the set C is closed because of the continuity of the probabilistic norm, ν (see Theorem 1 in [2]).

Now, let W be a closed linear subspace of V and $p \in C$. Suppose that for some $t > 0$, $N_p(t) \cap W = \varnothing$, then $\nu_p(t) \leq 1 - t$, which is a contradiction; hence $C \subseteq W$. □

Remark 3.6. Moreover, with V and C as in Proposition 3.5, for all $p \in V$ and $r \in C$, one has

$$\bar{\nu}_{pw} \geq \nu_p = \nu_{p+r-r} \geq \tau(\nu_{p+r}, \nu_{-r}) = \nu_{p+r}.$$

Thus the probabilistic norm $\bar{\nu}$ in $(V/\!\sim_C, \bar{\nu}, \tau, \tau^*)$ coincides with that of (V, ν, τ, τ^*).

Example 3.7. Let (V, ν, T) be a Menger PN space. Suppose that W is a closed subspace of V, and $V/\!\sim_W$ its quotient. Then (W, ν', T) and $(V/\!\sim_W, \bar{\nu}, T)$ are Menger PN spaces.

COROLLARY 3.8

Let (V, ν, τ, τ^*) be a Šerstnev PN space. Suppose that τ is sup-continuous. Let W be a closed subspace of V and $V/\!\sim_W$ its quotient. Then, (W, ν', τ, τ^*) and $(V/\!\sim_W, \bar{\nu}, \tau, \tau^*)$ are Šerstnev PN spaces.

Theorem 3.9. *Let (V, ν, τ, τ^*) be a PN space. Suppose that τ and τ^* are sup-continuous. Let W be a closed subspace of V with respect to the strong topology of (V, ν, τ, τ^*). Let*

$$\pi \colon V \to V/\!\sim_W$$

be the canonical projection. Then π is strongly bounded, open, and continuous with respect to the strong topologies of (V, ν, τ, τ^*) and $(V/\!\sim_W, \bar{\nu}, \tau, \tau^*)$. In addition, the strong topology and the quotient topology on $V/\!\sim_W$, induced by π, coincide.

Proof. One has that $\bar{\nu}_{pw} \geq \nu_p$ which implies π is strongly bounded, and hence continuous (see Theorem 3.3 in [5]).

The map π is open because of the equality $\pi(N_p(t)) = N'_{pw}(t)$. □

Example 3.10. Let $(V, \|\cdot\|)$ be a normed space and define $\nu \colon V \to \Delta^+$ via $\nu_p := \varepsilon_{\|p\|}$ for every $p \in V$. Let τ, τ^* be continuous triangle functions such that $\tau \leq \tau^*$ and $\tau(\varepsilon_a, \varepsilon_b) = \varepsilon_{a+b}$, for all $a, b > 0$. For instance, it suffices to take $\tau = \tau_T$ and $\tau^* = \tau_{T^*}$, where T is a continuous t-norm and T^* is its t-conorm. Then (V, ν, τ, τ^*) is a PN space (see Example 1.1 of [5]).

Th 8.1.3

Assume that τ is sup-continuous. Let W be a closed linear subspace of V with respect to the strong topology of (V, ν, τ, τ^*). By Theorem 3.4, $(V/\sim_W, \mu, \tau, \tau^*)$ is a PN space in which, $\mu_{pW} = \sup_{w\in W} \varepsilon_{\|p+w\|}$. On the other hand, if one considers the normed space $(V/\sim_W, \|\cdot\|')$, where $\| pW \|' = \inf_{w\in W} \| p + w \|$, then one can easily prove that the PN structure given to the normed space $(V/\sim_W, \|\cdot\|')$ by means of $\eta_{pW} := \varepsilon_{\|pW\|'}$ coincides with $(V/\sim_W, \eta, \tau, \tau^*)$.

4. Completeness results

Here we study the completeness of a quotient PN space. When a PN space (V, ν, τ, τ^*) is strongly complete, then we say that it is a probabilistic normed Banach (henceforth PNB) space.

Lemma 4.1. *Given the PN space $(V/\sim_W, \bar{\nu}, \tau, \tau^*)$ in which τ and τ^* are sup-continuous, let W be a closed subspace of V.*

(i) *If $p \in V$, then for every $\epsilon > 0$ there is a p' in V such that $p' + W = p + W$ and*
$$d_S(\nu_{p'}, \varepsilon_0) < d_S(\bar{\nu}_{p+W}, \varepsilon_0) + \epsilon.$$

(ii) *If p is in V and $\bar{\nu}_{p+W} \geq G$ for some d.f. $G \neq \varepsilon_0$, then there exists $p' \in V$ such that $p + W = p' + W$ and $\nu_{p'} \geq \tau(\bar{\nu}_{p+W}, G)$.*

Proof.

(i) We know
$$\bar{\nu}_{p+W} = \sup\{\nu_{p-q} \colon q \in W\}.$$
Now, let q be an element of W such that
$$\bar{\nu}_{p+W} < \nu_{p-q} + \frac{\epsilon}{2}.$$
We put $p - q = p'$. Now,
$$\begin{aligned} d_S(\bar{\nu}_{p+W}, \varepsilon_0) &= \inf\{h > 0 \colon \bar{\nu}_{p+W}(h^+) > 1 - h\} \\ &\geq \inf\left\{h > 0 \colon \nu_{p'}(h^+) + \frac{\epsilon}{2} > 1 - h\right\} \\ &= \inf\left\{h > 0 \colon \nu_{p'}(h^+) > 1 - \left(h + \frac{\epsilon}{2}\right)\right\} \\ &\geq \inf\left\{h > 0 \colon \nu_{p'}\left(\left(h + \frac{\epsilon}{2}\right)^+\right) > 1 - \left(h + \frac{\epsilon}{2}\right)\right\} \\ &> d_S(\nu_{p'}, \varepsilon_0) - \epsilon. \end{aligned}$$

(ii) Because of the definition of supremum and sup-continuity of τ, there exists a $q_n \in W$ such that $q_n \to q$ if $n \to +\infty$ and
$$\nu_{p+q_n} > \tau(\bar{\nu}_{pW}, \varepsilon_0) - \frac{1}{n} \geq \tau(\bar{\nu}_{pW}, G) - \frac{1}{n}.$$
Now it is enough to put $p' = p + q$ and see that, when $n \to +\infty$, one has $\nu_{p+q} \geq \tau(\bar{\nu}_{pW}, G)$. □

Let p, q be elements of V such that $d_S(\nu_{(p-q)+W}, \varepsilon_0) < \delta$ for some positive δ. By Lemma 4.1, there is a $q' \in V$ such that $(p-q') + W = (p-q) + W$ and

$$d_S(\nu_{p-q'}, \varepsilon_0) < \delta.$$

Theorem 4.2. *Let W be a closed subspace of V and suppose that (V, ν, τ, τ^*) is a PNB space with τ and τ^* sup-continuous. Then, $(V/\sim_W, \bar{\nu}, \tau, \tau^*)$ is also a PNB space.*

Proof. Let (a_n) be a strong Cauchy sequence in $(V/\sim_W, \bar{\nu}, \tau, \tau^*)$, i.e. for every $\delta > 0$, there exists $n_0 = n_0(\delta) \in \mathbf{N}$ such that, for all $m, n > n_0$,

$$d_S(\bar{\nu}_{a_n - a_m}, \varepsilon_0) < \delta.$$

Now, define a strictly decreasing sequence (δ_n) with $\delta_n > 0$ in the following way: let $\delta_1 > 0$ be such that $\tau(B_{d_S}(\varepsilon_0; \delta_1) \times B_{d_S}(\varepsilon_0; \delta_1)) \subseteq B_{d_S}(\varepsilon_0; 1)$ where $B_{d_S}(\varepsilon_0; \lambda) = \{F \in \Delta^+; d_S(F, \varepsilon_0) < \lambda\}$. For $n > 1$, define δ_n by induction in such a manner that

$$\tau(B_{d_S}(\varepsilon_0; \delta_n) \times B_{d_S}(\varepsilon_0; \delta_n)) \subseteq B_{d_S}\left(\varepsilon_0; \min\left(\frac{1}{n}, \delta_{n-1}\right)\right). \tag{1}$$

There is a subsequence (a_{n_i}) of (a_n) with

$$d_S(\bar{\nu}_{a_{n_i} - a_{n_{i+1}}}, \varepsilon_0) < \delta_{i+1}. \tag{2}$$

Because of the definition of the canonical projection π one can say that $\pi^{-1}(N'_{pW}(t)) = N_p(t)$ and consequently $\pi^{-1}(a_{n_i}) = x_i$ exists. Inductively, from Lemma 4.1 we can find $x_i \in V$ such that $\pi(x_i) = a_{n_i}$ and then

$$d_S(\nu_{x_i - x_{i+1}}, \varepsilon_0) < \delta_{i+1} \tag{3}$$

holds. We claim that (x_i) is a strong Cauchy sequence in (V, ν, τ, τ^*). By applying the relations (1), (2) and (3) to $i = m-1$ and $i = m-2$, and using Lemma 4.3.4 of [8], one obtains the inequalities

$$\begin{aligned} d_S(\nu_{x_m - x_{m-2}}, \varepsilon_0) &\leq d_S(\tau(\nu_{x_{m-1} - x_m}, \nu_{x_{m-2} - x_{m-1}}), \varepsilon_0) \\ &< \min\left(\frac{1}{m-1}, \delta_{m-2}\right). \end{aligned}$$

Following this reasoning, we obtain that $d_S(\nu_{x_m - x_n}, \varepsilon_0) < 1/n$ and therefore, (x_i) is a strong Cauchy sequence. Since it was assumed that (V, ν, τ, τ^*) is strongly complete, (x_i) is strongly convergent and hence, by the continuity of π, (a_{n_i}) is also strongly convergent. From this and taking into account the continuity of τ and Lemma 4.3.4 of [8], one sees that the whole sequence (a_n) strongly converges. □

The converse of the above theorem also holds.

Theorem 4.3. *Let (V, ν, τ, τ^*) be a PN space in which τ and τ^* sup-continuous, and let $(V/\sim_W, \bar{\nu}, \tau, \tau^*)$ be its quotient space with respect to the closed subspace W. If any two of the three spaces V, W and $V/\sim_W$ are strongly complete, so is the third.*

Proof. If V is a strongly complete PN space, so are $V/\sim_W$ and W. Therefore all one needs to check is that V is strongly complete whenever both W and $V/\sim_W$ are strongly complete. Suppose W and $V/\sim_W$ are strongly complete PN spaces and (p_n) be a strong Cauchy sequence in V. Since

$$\bar{\nu}_{(p_m-p_n)+W} \geq \nu_{p_m-p_n}$$

whenever $m, n \in \mathbf{N}$, the sequence $(p_n + W)$ is strong Cauchy in $V/\sim_W$ and, therefore, it strongly converges to $q + W$ for some $q \in V$. Thus there exists a sequence of d.f.'s (H_n) such that $H_n \longrightarrow \varepsilon_0$ and $\bar{\nu}_{(p_n-q)+W} > H_n$. Now by Lemma 4.1 there exists (q_n) in V such that $q_n + W = (p_n - q) + W$ and

$$\nu_{q_n} > \tau(\bar{\nu}_{(p_n-q)+W}, H_n).$$

Thus $\nu_{q_n} \longrightarrow \varepsilon_0$ and consequently $q_n \longrightarrow \theta$. Therefore $(p_n - q_n - q)$ is a strong Cauchy sequence in W and is strongly convergent to a point $r \in W$ and implies that (p_n) strongly converges to $r + q$ in V. Hence V is strongly complete. □

Theorem 4.4. *Let $(V_1, \nu^1, \tau, \tau^*), \ldots, (V_n, \nu^n, \tau, \tau^*)$ be PNB spaces in which τ and τ^* are sup-continuous. Suppose that there is a triangle function σ such that $\tau^* \gg \sigma$ and $\sigma \gg \tau$. Then their σ-product is a PNB space.*

Proof. One proves for $n = 2$ (see Theorem 2 in [3]), and then we apply induction for an arbitrary n. Since the quotient norm of

$$\frac{V_1 \times V_2}{V_1 \times \theta_2} (\simeq V_2)$$

is the same as ν^2 and the restriction of the product norm of $V_1 \times V_2$ to $V_1 \times \theta_2 (\simeq V_1)$ is the same as ν^1 (see [3]), and in view of Theorem 4.3, the proof is complete. □

By Theorem 3.9 the following corollaries can be proved easily.

COROLLARY 4.5

Under the assumptions of Proposition 3.3 *and if W is a closed subset of V, the probabilistic norm $\bar{\nu}$: $V/\sim_W \to \Delta^+$ in $(V/\sim_W, \bar{\nu}, \tau, \tau^*)$ is uniformly continuous.*

Proof. Let η be a positive real number, $\eta > 0$. By Theorem 3.9 there exists a pair (p', q') in $(V \times V)$ such that $d_S(\bar{\nu}_{\pi(p-p')}, \varepsilon_0) < \eta$ and $d_S(\bar{\nu}_{\pi(q-q')}, \varepsilon_0) < \eta$, whenever $d_S(\nu_{p-p'}, \varepsilon_0) < \eta$ and $d_S(\nu_{q-q'}, \varepsilon_0) < \eta$.

On the other hand, we have

$$\bar{\nu}_{\pi(p'-q')} \geq \tau(\tau(\bar{\nu}_{\pi(p-p')}, \bar{\nu}_{\pi(q-q')}), \bar{\nu}_{\pi(p-q)})$$

and

$$\bar{\nu}_{\pi(p-q)} \geq \tau(\tau(\bar{\nu}_{\pi(p-p')}, \bar{\nu}_{\pi(q-q')}), \bar{\nu}_{\pi(p'-q')}).$$

Thus, from the relationship (12.1.5) and Lemma 12.2.1 in [8] it follows that for any $h > 0$ there is an appropriate $t > 0$ such that

$$d_S(\bar{\nu}_{\pi(p-q)}, \bar{\nu}_{\pi(p'-q')}) < h,$$

whenever $p' \in N_p(\eta)$ and $q' \in N_q(\eta)$. This implies that $\bar{\nu}$ is a uniformly continuous mapping from $V/\sim_W$ into Δ^+. □

Also the inequality $d_S(\bar{\nu}_{\pi((p+q)-(p'+q'))}, \varepsilon_0) \leq d_S(\nu_{(p+q)-(p'+q')}, \varepsilon_0)$ implies that $(V/\sim_W, +)$ is a topological group.

COROLLARY 4.6

Let (V, ν, τ, τ^*) be a PN space such that τ^* is Archimedean, τ and τ^* are sup-continuous, and $\nu_p \neq \varepsilon_\infty$ for all $p \in V$. If we define quotient probabilistic norm via Proposition 3.3, *then $(V/\sim_W, \bar{\nu}, \tau, \tau^*)$ is a PPN space where the scalar multiplication is a continuous mapping from $R \times V/\sim_W$ into $V/\sim_W$.*

Proof. For any $p \in V$ and $\alpha, \beta \in R$ we know $d_S(\bar{\nu}_{\pi(\alpha p)}, \nu_{\pi(\beta p)})$ is small whenever $d_S(\bar{\nu}_{\pi((\alpha-\beta)p)}, \varepsilon_0)$ is small. But

$$d_S(\bar{\nu}_{\pi((\alpha-\beta)p)}, \varepsilon_0) \leq d_S(\nu_{(\alpha-\beta)p}, \varepsilon_0)$$

and by Lemma 3 of [2], $d_S(\nu_{(\alpha-\beta)p}, \varepsilon_0)$ is small whenever $|\alpha - \beta|$ is small. □

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References

[1] Alsina C, Schweizer B and Sklar A, On the definition of a probabilistic normed space, *Aequationes Math.* **46** (1993) 91–98

[2] Alsina C, Schweizer B and Sklar A, Continuity properties of probabilistic norms, *J. Math. Anal. Appl.* **208** (1997) 446–452

[3] Lafuerza-Guillén B, Finite products of probabilistic normed spaces, *Rad. Mat.* **13** (2004) 111–117

[4] Lafuerza-Guillén B, Rodríguez-Lallena J A and Sempi C, Completion of probabilistic normed spaces, *Int. J. Math. Math. Sci.* **18** (1995) 649–652

[5] Lafuerza-Guillén B, Rodríguez-Lallena J A and Sempi C, A study of boundedness in probabilistic normed spaces, *J. Math. Anal. Appl.* **232** (1999) 183–196

[6] Lafuerza-Guillén B, Rodríguez-Lallena J A and Sempi C, Probabilistic norms for linear operators, *J. Math. Anal. Appl.* **220** (1998) 462–476

[7] Saadati R and Amini M, *D*-boundedness and *D*-compactness in finite dimensional probabilistic normed spaces, *Proc. Indian Acad. Sci. (Math. Sci.)* **115** (2005) 483–492

[8] Schweizer B and Sklar A, Probabilistic metric spaces (New York: North–Holland) (1983)

[9] Sibley D A, A metric for weak convergence of distribution functions, *Rocky Mountain J. Math.* **1** (1971) 427–430