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Probabilistic norms and convergence of random variables ☆

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Abstract

We prove that the probabilistic norms of suitable Probabilistic Normed spaces induce convergence in probability, L^p convergence and almost sure convergence.

Keywords: Convergence in probability; Almost sure convergence; Complete convergence; Probabilistic normed space; Probabilistic norm

1. History and introduction

The study of possible topologies for the various modes of convergence of sequences of random variables has a long history.

Let a probability space $(\Omega, \mathcal{A}, P)$ be given and let $L^0(\mathcal{A})$ be the linear space of (equivalence classes of) E-valued random variables, viz. E-valued measurable functions defined on it. Here $(E, |\cdot|)$ is a normed space.

For a sequence $\{f_n\colon n \in \mathbf{N}\}$ we shall consider the following modes of convergence: convergence in probability, convergence in L^p and almost sure convergence.

Ky Fan [7] showed that the topology of convergence in probability can be metrized and introduced the metric that now bears his name. Fréchet [6] exhibited several metrics whose

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topology coincides with that of convergence in probability. Later, it was proved by Dugué [3], Marczewski [14], Thomasian [20,21] that, in general, no norm exists that generates this topology. Moreover, in general, there exists no metric whose convergence coincides with almost sure convergence [4]. These results are summarized in [13].

Finally, Fernique [5], using and generalizing a famous theorem of Skorohod's [19] showed how one can build a probability space and a proper subset of $L^0(\mathcal{A})$ in which almost sure convergence can be defined through a metric topology.

As far as L^p convergence is concerned, we shall make reference to the paper [10], but see also the previous paper [9].

We now plan to examine the whole question from the point of view of probabilistic normed spaces; we shall then recognize that there are *probabilistic norms* that imply either type of convergence.

A *probabilistic normed space* (PN space in the following) (V, ν, τ, τ^*) is a quadruple in which V is a real linear space, the *probabilistic norm* ν is a mapping from V into Δ^+, the space of the distance distribution functions (i.e., those distribution functions F that vanish at the origin, $F(0) = 0$), and τ and τ^*, with $\tau \leqslant \tau^*$ are triangle functions; ν, τ and τ^* are subject to the following conditions:

(N1) $\nu_p = \varepsilon_0$ if, and only if, $p = \theta$ (the null vector of V);
(N2) $\nu_{-p} = \nu_p$ for every $p \in V$;
(N3) for all p and q in V, $\nu_{p+q} \geqslant \tau(\nu_p, \nu_q)$;
(N4) for every $p \in V$ and for every $\alpha \in [0, 1]$, $\nu_p \leqslant \tau^*(\nu_{\alpha p}, \nu_{(1-\alpha)p})$.

An important class of PN spaces is that of *Šerstnev spaces*: a PN space is called a Šerstnev space if (N1) and (N3) are satisfied along with the following condition:

$$\forall \alpha \in \mathbf{R} \setminus \{0\}\ \forall x > 0\ \nu_{\alpha p}(x) = \nu_p\left(\frac{x}{|\alpha|}\right), \tag{Š}$$

which implies (N2) and (see [1]) (N4) in the strengthened form

$$\forall \alpha \in [0, 1]\ \forall p \in V\ \nu_p = \tau_M(\nu_{\alpha p}, \nu_{(1-\alpha)p}). \tag{N4$'$}$$

A Šerstnev space will be denoted by (V, ν, τ), since the rôle of τ^* is played by a fixed triangle function, τ_M, which then satisfies (N4$'$).

For more information about PN spaces and for their properties see [1,9–12]. PN spaces were first introduced in a slightly less general form in [17]. The notation adopted in the present note is mainly from the book [16]. A PN space is endowed with the *strong topology* (see [16]) and this latter is metrizable. A sequence $\{p_n\}$ of elements of V converges to θ_V, the null element of V, in the strong topology if, and only if,

$$d_S(\nu_{p_n}, \varepsilon_0) \xrightarrow[n \to +\infty]{} 0,$$

where d_S denotes the Lévy metric as modified by Sibley [15,18].

2. Convergence in probability

In the notation we have fixed in the previous section, let $\nu : S \to \Delta^+$ be defined, for every $f \in L^0(\mathcal{A})$ and for every $x \in \overline{\mathbf{R}}_+$, by

$$\nu_f(x) := P\{\omega \in \Omega \colon |f(\omega)| < x\}. \tag{1}$$

The couple $(L^0(\mathcal{A}), \nu)$ is called an EN space. It was proved in [9] that the relationship, defined in the EN space $(L^0(\mathcal{A}), \nu)$ by $f \sim g$ if, and only if, $\nu_f = \nu_g$ is an equivalence relationship. Moreover, if $\overline{L^0(\mathcal{A})} := L^0(\mathcal{A})/\sim$ is the quotient space and if the *quotient mapping* $\bar{\nu} : \overline{L^0(\mathcal{A})} \to \Delta^+$ is defined via

$$\bar{\nu}_{\bar{f}} := \nu_f$$

for every f in the equivalence class $\bar{f}$, then $(\overline{L^0(\mathcal{A})}, \bar{\nu})$ is a PN space under the triangle functions τ_W and τ_M [16]; in fact, more is true: $(\overline{L^0(\mathcal{A})}, \bar{\nu}, \tau_W)$ is a Šerstnev space [8, Theorem 2.2.8].

As usual in probability theory, one writes f even when one means the equivalence class $\bar{f}$ of f. The equivalence between convergence in probability and convergence with respect to the probabilistic norm (1) is now almost obvious.

Theorem 1. *For a sequence of (equivalence classes of) E-valued random variables* $\{f_n\}$, *the following statements are equivalent*:

(a) $\{f_n\}$ *converges in probability to* θ_S, $f_n \xrightarrow[n\to+\infty]{P} \theta_S$;
(b) *the corresponding sequence* $\{\nu_{f_n}\}$ *of probabilistic norms converges weakly to* ε_0, *viz.* $d_S(\nu_{f_n}, \varepsilon_0) \xrightarrow[n\to+\infty]{} 0$;
(c) $\{f_n\}$ *converges to* θ_S *in the strong topology of the Šerstnev space* (L^0, ν, τ_W).

Proof. Since (b) and (c) are equivalent by definition, it suffices to establish the equivalence of (a) and (b).

The sequence $\{f_n\}$ converges to θ_S in probability if, and only if, for every $x > 0$,

$$\lim_{n\to+\infty} P(|f_n| < x) = 1,$$

or, equivalently, on account of (1), if, and only if, for every $x > 0$,

$$\lim_{n\to+\infty} \nu_{f_n}(x) = 1.$$

But this latter statement, in its turn, is equivalent (see [16]) to

$$\lim_{n\to+\infty} d_S(\nu_{f_n}, \varepsilon_0) = 0,$$

which proves the theorem. □

Of course, there is nothing special about θ_S as a limit; if one wishes to consider the convergence in probability of the sequence $\{f_n\}$ to the (E-valued) random variable f, then it suffices to consider the sequence $\{f_n - f\}$ and its convergence to θ_S.

3. L^p convergence

In order to consider convergence in L^p with $p \in [1, +\infty]$, the following result connecting the L^p norms with the probabilistc norm (1) will be needed (see [10]).

Theorem 2. *For $p \in [1, +\infty[$ the norm $\|\cdot\|_p$ in the space*

$$L^p(\mathcal{A}) := \left\{ f \in L^0(\mathcal{A}):\ \int |f|^p \, dP < +\infty \right\}$$

is given, for f in $L^p(\mathcal{A})$, by

$$\|f\|_p = \left(\int_{\mathbf{R}_+} t^p \, d\nu_f(t) \right)^{1/p};$$

for $p = +\infty$, the norm $\|\cdot\|_\infty$ in $L^\infty(\mathcal{A}) := \{f \in L^0(\mathcal{A}):\ \operatorname{ess\,sup} |f| < +\infty\}$ is given, for f in $L^\infty(\mathcal{A})$, by

$$\|f\|_\infty = \sup\big\{t > 0:\ \nu_f(t) < 1\big\}.$$

With the help of the previous result one can characterize L^p convergence. As in the previous section, there is no loss of generality in considering only convergence to θ_S, for, if one wishes to study the convergence of a sequence $\{f_n\}$ to $f \neq \theta_S$, it suffices to replace $\{f_n\}$ by $\{f_n - f\}$.

Theorem 3. *For a sequence of (equivalence classes of) E-valued random variables $\{f_n\}$ in L^p, the following statements are equivalent:*
if $p \in [1, +\infty[$:

(a) *$\{f_n\}$ converges to θ_S in L^p, $f_n \xrightarrow[n \to +\infty]{L^p} \theta_S$;*
(b) *the sequence of the p-th moments of the probabilistic norms $\{\nu_{f_n}\}$ tends to 0, viz.*

$$\int_{\mathbf{R}_+} t^p \, d\nu_{f_n}(t) \xrightarrow[n \to +\infty]{} 0;$$

if $p = +\infty$;

(c) *$\{f_n\}$ converges to θ_S in L^∞, $f_n \xrightarrow[n \to +\infty]{L^\infty} \theta_S$;*
(d) *for every $t > 0$, the sequence $\{\nu_{f_n}(t)\}$ is definitely equal to 1, viz., for all $t > 0$, there exists $n_0 = n_0(t) \in \mathbf{N}$ such that $\nu_{f_n}(t) = 1$ if $n \geqslant n_0$.*

Proof. Only the equivalence of (c) and (d) is not immediately obvious; thus we limit ourselves to proving it.

(c) ⇒ (d) Assume $\|f_n\|_\infty \xrightarrow[n\to+\infty]{} 0$ and let t be strictly positive; then, for every $\varepsilon \in]0, t[$ there exists $n_0 = n_0(\varepsilon)$ in $\mathbf{N}$ such that, for every $n \geqslant n_0$, one has

$$\sup\{t > 0\colon\ \nu_{f_n}(t) < 1\} = \|f_n\|_\infty < \varepsilon,$$

so that $\nu_{f_n}(\varepsilon) = 1$; but then, for all $n \geqslant n_0$, $\nu_{f_n}(t) \geqslant \nu_{f_n}(\varepsilon) = 1$.

(d) ⇒ (c) For $t > 0$, let $n_0 = n_0(t) \in \mathbf{N}$ be such that $\nu_{f_n}(t) = 1$ for all $n \geqslant n_0$; therefore, if $n \geqslant n_0$, $\|f_n\|_\infty < t$, which yields $\|f_n\|_\infty \xrightarrow[n\to+\infty]{} 0$. □

In comparing Theorem 3 with Theorem 1, it should be noticed that, if the sequence $\{\nu_{f_n}(t)\}$ is definitely equal to 1 for every $t > 0$, then, *a fortiori*, one has $\nu_{f_n}(t) \xrightarrow[n\to+\infty]{} 1$, or, equivalently $d_S(\nu_{f_n}, \varepsilon_0) \xrightarrow[n\to+\infty]{} 0$; this is the translation in the language of PN spaces of the fact that convergence in L^∞ implies convergence in probability. The converse is not true, as is well known: and, in fact, a sequence $\{\nu_{f_n}(t)\}$ may well converge to 1 without being definitely equal to 1.

4. Almost sure convergence

Consider the family $V := \{L^0(\mathcal{A})\}^{\mathbf{N}}$ of all the sequences of (equivalence classes of) E-valued random variables. The set V is a real vector space with respect to the componentwise operations; specifically, if $s = \{f_n\}$ and $s' = \{g_n\}$ are two sequences in V and if α is a real number, then the sum $s \oplus s'$ of s and s' and the scalar product $\alpha \odot s$ of α and s are defined via

$$s \oplus s' = \{f_n\} \oplus \{g_n\} := \{f_n + g_n\}, \qquad \alpha \odot s = \alpha \odot \{f_n\} := \{\alpha f_n\}.$$

A mapping $\phi : V \to \Delta^+$ will be defined on V via

$$\phi_s(x) := P\Big(\sup_{k\in\mathbf{N}} |f_k| < x\Big) = P\Big(\bigcap_{k\in\mathbf{N}} \{|f_k| < x\}\Big),$$

where $x > 0$ and $s = \{f_k\}$. Then one can prove

Theorem 4. *The triple (V, ϕ, τ_W) is a Šerstnev space.*

Proof. Let O denote the null sequence, $O := \{o_n\}$, where, for every $n \in \mathbf{N}$, $o_n \equiv \theta_E$; here, θ_E is the null vector of E. Obviously, O is the null element of the vector space V, $O = \theta_V$ and $\phi_O = \varepsilon_0$. In the other direction, assume $\phi_s = \varepsilon_0$; then one has, for every $x > 0$,

$$\phi_s(x) := P\Big(\sup_{k\in\mathbf{N}} |f_k| < x\Big) = 1,$$

so that, for every $x > 0$ and for every $k \in \mathbf{N}$,

$$P\big(|f_k| < x\big) = 1;$$

in other words, $\nu_{f_k} = \varepsilon_0$ for every $k \in \mathbf{N}$. As already noticed in the previous section, (L^0, ν, τ_W) is a Šerstnev space; therefore, $f_k = \theta_E$ for all $k \in \mathbf{N}$, or, equivalently, $s = O$, which concludes the proof of (N1).

Property (N2) is obvious.

Notice that, for $\alpha \in \mathbf{R}$, $\alpha \neq 0$, for every $s \in V$ and for all $x > 0$, one has

$$\phi_{\alpha \odot s}(x) = P\Big(\sup_{k\in\mathbf{N}} |\alpha f_k| < x\Big) = P\Big(\sup_{k\in\mathbf{N}} |f_k| < \frac{x}{|\alpha|}\Big) = \phi_s\Big(\frac{x}{|\alpha|}\Big).$$

This proves (Š).

For all sequences $s = \{f_k\}$ and $s' = \{g_k\}$, for all $x > 0$ and $t \in\,]0, x[$, one has

$$\begin{aligned}
P\Big(\bigcup_{k\in\mathbf{N}}\{|f_k + g_k| \geqslant x\}\Big) &\leqslant P\Big(\bigcup_{k\in\mathbf{N}}\{|f_k| + |g_k| \geqslant x\}\Big)\\
&\leqslant P\Big[\bigcup_{k\in\mathbf{N}}\Big(\{|f_k| \geqslant t\} \cup \{|g_k| \geqslant x - t\}\Big)\Big]\\
&= P\Big(\bigcup_{k\in\mathbf{N}}\{|f_k| \geqslant t\}\Big) + P\Big(\bigcup_{k\in\mathbf{N}}\{|g_k| \geqslant x - t\}\Big)\\
&\quad - P\Big[\Big(\bigcup_{k\in\mathbf{N}}\{|f_k| \geqslant t\}\Big) \cap \Big(\bigcup_{k\in\mathbf{N}}\{|g_k| \geqslant x - t\}\Big)\Big]
\end{aligned}$$

so that

$$\begin{aligned}
P\Big(\bigcap_{k\in\mathbf{N}}\{|f_k + g_k| < x\}\Big) &\geqslant P\Big(\bigcap_{k\in\mathbf{N}}\{|f_k| + |g_k| < x\}\Big)\\
&= 1 - P\Big(\bigcup_{k\in\mathbf{N}}\{|f_k| + |g_k| \geqslant x\}\Big)\\
&\geqslant 1 - P\Big(\bigcup_{k\in\mathbf{N}}\{|f_k| \geqslant t\}\Big) - P\Big(\bigcup_{k\in\mathbf{N}}\{|g_k| \geqslant x - t\}\Big)\\
&\quad + P\Big[\Big(\bigcup_{k\in\mathbf{N}}\{|f_k| \geqslant t\}\Big) \cap \Big(\bigcup_{k\in\mathbf{N}}\{|g_k| \geqslant x - t\}\Big)\Big]\\
&= P\Big(\bigcap_{k\in\mathbf{N}}\{|f_k| < t\}\Big) + P\Big(\bigcap_{k\in\mathbf{N}}\{|g_k| < x - t\}\Big)\\
&\quad - 1 + P\Big[\Big(\bigcup_{k\in\mathbf{N}}\{|f_k| \geqslant t\}\Big) \cap \Big(\bigcup_{k\in\mathbf{N}}\{|g_k| \geqslant x - t\}\Big)\Big]\\
&\geqslant P\Big(\bigcap_{k\in\mathbf{N}}\{|f_k| < t\}\Big) + P\Big(\bigcap_{k\in\mathbf{N}}\{|g_k| < x - t\}\Big) - 1.
\end{aligned}$$

Thus, one has, for every $t \in\,]0, x[$,

$$\phi_{s\oplus s'}(x) \geqslant \phi_s(t) + \phi_{s'}(x - t) - 1;$$

therefore

$$\phi_{s\oplus s'} \geqslant \tau_W(\phi_s, \phi_{s'}),$$

which proves (N3) and concludes the proof. □

Given an element s of V, viz. given a sequence $s = \{f_k\colon k \in \mathbf{N}\}$ of E-valued random variables, $f_k \in L^0(\mathcal{A})$ for every $k \in \mathbf{Z}_+ := \{0, 1, \ldots\}$, consider the *n-shift* s_n of s, $s_n := \{f_{k+n}\colon k \in \mathbf{N}\}$, which again belongs to V. We are now in a position to state our main result.

Theorem 5. *A sequence $s = \{f_k\colon k \in \mathbf{N}\}$ of E-valued random variables converges almost surely to θ_E, the null vector of E, if, and only if, the sequence $\{\phi_{s_n}\colon n \in \mathbf{Z}_+\}$ of the probabilistic norms of the n-shifts of s converges weakly to ε_0, or, equivalently, if, and only if, the sequence $\{s_n\}$ of the n-shifts of s converges to $O := \{\theta_E, \theta_E, \ldots\}$ in the strong topology of (V, ϕ, τ_W).*

Proof. All the statements are equivalent to the assertion, which holds for every $x > 0$,

$$\begin{aligned}
1 = \lim_{n\to+\infty} \phi_{s_n}(x) &= \lim_{n\to+\infty} P\Big(\bigcap_{k\in\mathbf{N}} \big\{|f_{k+n}| < x\big\}\Big) \\
&= \lim_{n\to+\infty} P\Big(\bigcap_{k\geqslant n} \big\{|f_k| < x\big\}\Big) = P\Big(\bigcup_{n\in\mathbf{N}} \bigcap_{k\geqslant n} \big\{|f_k| < x\big\}\Big) \\
&= P\Big(\liminf_{n\to+\infty} \big\{|f_k| < x\big\}\Big).
\end{aligned}$$

This proves the result. □

Given an element s in V, define $n : V \to \Delta^+$ via

$$n_s(x) := \sup_{n\in\mathbf{N}} \phi_{s_n}(x) = \lim_{n\to+\infty} P\Big(\bigcap_{k\geqslant n} \big\{|f_k| < x\big\}\Big);$$

then Theorem 5 says that s converges to θ_E if, and only if, $n_s = \varepsilon_0$. An equivalence relationship can be defined on V by stipulating that $s = \{f_n\} \sim s' = \{g_n\}$ if, and only if, $\{f_n - g_n\}$ converges almost surely to the null element $O = \{\theta_E, \theta_E, \ldots\}$ of V, or, equivalently, if, and only if, $n_s = \varepsilon_0$. In the quotient space $V_0 := V/\sim$, the constant sequence $\{f, f, \ldots\}$, where f is in $L^0(\mathcal{A})$, is the family of the sequences of E-valued random variables that converge almost surely to f.

As a final remark, notice that we have shown that the strong topology of the Šerstnev space (V, ϕ, τ_W) induces a convergence equivalent to almost sure convergence in $L^0(\mathcal{A})$. This does not contradict the well-known fact that, in general, almost sure convergence does not derive from a topology on $L^0(\mathcal{A})$.

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