UNIVERSIDAD DE ALMERÍA

DISTRIBUTION FUNCTIONS AND PROBABILITY MEASURES ON TOPOLOGICAL STRUCTURES

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Almería, Diciembre 2021

Distribution functions and probability measures on topological structures

(Funciones de distribución y medidas de probabilidad en estructuras topológicas)

PhD thesis in Mathematics by

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December 2021

University of Almería

La presente memoria, con título “Distribution functions and probability measures on topological structures”, ha sido realizada por D. José Fulgencio Gálvez Rodríguez en el Área de Geometría y Topología del Departamento de Matemáticas de la Universidad de Almería, bajo la dirección del profesor D. Miguel Ángel Sánchez Granero, para la obtención del título de Doctor en Matemáticas de acuerdo con el Programa de Doctorado en Matemáticas por la Universidad de Almería; la Universidad de Cádiz; la Universidad de Granada; la Universidad de Jaén y la Universidad de Málaga.

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Agradecimientos

La etapa de mi vida que transcurre de manera paralela a la investigación realizada hasta la elaboración de esta memoria ha quedado marcada por muchas personas.

Primeramente, quisiera agradecer a mi director de tesis, el profesor Miguel Ángel Sánchez Granero, su confianza depositada en mí desde el primer momento. Su inmenso conocimiento, su paciencia y su entera disposición para enseñarme y aconsejarme han posibilitado, con creces, el desarrollo de este trabajo. Gracias, una vez más, por hacerme saber cómo disfrutar de las matemáticas.

Al impecable trabajo de mi director hay que sumar, sin duda, la motivación recibida por parte de mi familia y amigos, en forma de apoyo constante. En particular, tengo que destacar la excelente labor de mis padres, que desde pequeño me han transmitido el interés por aprender y trabajar en lo que más me gusta. También, he de mencionar a mi hermano, a mis abuelos y a mi novia, que siempre han estado ahí.

Finalmente, he de citar al equipo humano del Departamento de Matemáticas, con el que he tenido la oportunidad de compartir profesión en estos últimos años, por contribuir a mi desarrollo personal, académico y profesional. Asimismo, dicho departamento ha hecho posible, junto con el apoyo de los grupos de investigación SEJ-296 (*Finanzas empresariales*) y el ya extinto FQM-305 (*Grafos, Topología General y sus aplicaciones*), la difusión de los resultados de mi investigación en distintos eventos celebrados en el ámbito nacional e internacional.

A todos, muchas gracias.

Almería, 5 de diciembre de 2021

A mi familia.

Abstract

The present work has two fundamental goals: the construction of a probability measure from a fractal structure and, on the other hand, the development of a theory of probability distribution functions in linearly ordered topological spaces. That is why this dissertation is divided into two parts, each of them with the task of treating one of the two mentioned objectives. However, the first two chapters, which do not belong to either of the two parts as such, serve as a starting point, since they provide the theoretical framework and an introduction to the problems which are faced in the rest of the dissertation (see Chapter 1), as well as the mathematical concepts and results which are used along it (see Chapter 2). The second chapter will have to do with quasi-pseudometrics, fractal structures, measure theory, ordered sets, as well as with the Dedekind-MacNeille completion.

The content of each of the parts of the doctoral thesis is detailed below, in a fairly synthetic way, according to the chapters into which its content is divided, commenting on the sub-goals and main results treated in each of them.

The first part, whose aim is, as it was already mentioned, to introduce a method to construct probability measures from a space with a fractal structure, begins with a chapter (the third of the work) whose first goal is the construction of the completion of the space provided with that structure. We will see that this completion, which always exists, is unique up to fractal isomorphism. This completion will be the starting point until a probability measure is defined on the original space. In fact, in Chapter 4 we see how to define a probability measure on the completion of the space. We can do this in two ways: first, starting from a family of balls and a pre-measure defined on it and, moreover, we can build a new measure, also in the completion, but starting from the elements of a tiling fractal structure. Once we have done this, the next step in this theoretical development is to explore conditions so that the restriction of the

probability measure that we already have to the original space is, actually, a probability measure. Furthermore, we will prove that each probability measure defined on the space can be constructed from a certain pre-measure following the procedure we have introduced. That is, precisely, what the fifth chapter will be devoted to, together with several examples in order to illustrate the construction that has been developed, as well as the suitability of certain hypotheses imposed until a probability measure is achieved in the starting space. To finish with this first part, Chapter 6 shows some applications that arise from the theory that has been developed in the previous chapters. For example, the generation of probability measures from fractal structures allows us to develop a new parameter estimation method which, while it supports the predictions made by the well-known maximum likelihood method, offers better results when the given sample contains data which is completely unrelated to the distribution whose parameters we want to estimate. Furthermore, it is possible to generate samples of a certain probability distribution or design a goodness-of-fit test, both based on the construction exposed in Chapters 4 and 5.

The second part deals with the elaboration of a theory of distribution functions in a more general context than the one known on the real line, which we will refer to as the classical case. This context has to do with linearly ordered topological spaces. In Chapter 7 we start from a probability measure on a separable linearly ordered topological space to define a probability distribution function in that space. Furthermore, the pseudo-inverse of a distribution function is defined and the properties of both functions are studied, comparing them with those that they have in the classical case. In fact, one of the limitations of the pseudo-inverse of a distribution function is that it is not always defined for all values of the unit interval, since the existence of the infimum and the supremum of any subset of the starting space is not guaranteed. Therefore, we need an environment where the definition of the inverse makes complete sense, since it is a tool that will allow, among other things, to generate samples of a certain probability distribution by using a similar procedure to the inverse transform method, known in the case of real distribution functions of real variable, and used to generate random samples. Precisely, in Chapter 8 we study the Dedekind-MacNeille completion (or completion by cuts) of a separable linearly ordered topological space, and we see how to extend the distribution function, defined on the original space, to that completion. In fact, that completion turns out to be a compactification of the original space. This completion makes it possible to define the inverse of a distribution function in terms of cuts and so that it is well defined for each point at $[0, 1]$. Following this research line, it makes sense to ask ourselves under what conditions we can guarantee that there is a one-to-one relationship between probability measures and distribution functions in the context in which we are working. This study is carried out in Chapter 9. Furthermore, conditions will be determined to guarantee that a certain function is the inverse of the distribution function of a certain probability measure on the Borel σ-algebra of the space. In Chapter 10 we introduce two applications that have arisen from the theory developed in Chapters 7, 8 and 9. On the one hand, it can be shown that each distribution function in a separable linearly ordered topological space can be written as the convex sum of two distribution functions, each one with a certain peculiarity in terms of its continuity and, on the other hand, we describe a goodness-of-fit test, similar to the Kolmogorov-Smirnov test, known in the classical case. Finally, Chapter 11 serves as a meeting point between both parts of the work. Specifically, it shows how to build a fractal structure from a linearly ordered topological space and vice versa. This connection between both topological structures makes it possible to move from a probability measure to a distribution function, and vice versa, in both environments, so that the applications shown in each part of the work make perfect sense when they are developed in the context that the researcher prefers.

The work finishes with the pertinent conclusions and the corresponding bibliography, among whose references there are six research articles, in which the candidate for the title of doctor through this work appears as author: [29], [30], [31], [32], [33] and [34]. These works support most of the content of this thesis.

Resumen

El presente trabajo tiene dos objetivos fundamentales: la construcción de una medida de probabilidad a partir de una estructura fractal y, por otra parte, el desarrollo de una teoría de funciones de distribución de probabilidad en espacios topológicos linealmente ordenados. Es por ello que esta memoria se encuentra estructurada en dos partes, cada una de ellas con el cometido de abordar uno de los dos objetivos mencionados previamente. No obstante, los dos primeros capítulos, no pertenecientes a ninguna de las dos partes como tal, sirven como punto de partida, puesto que proporcionan el marco teórico y una introducción a los problemas a los que nos enfrentamos en el resto de la memoria (véase el Capítulo 1), así como los conceptos y resultados matemáticos utilizados a lo largo de ésta (véase el Capítulo 2). El segundo capítulo tendrá que ver con casi-seudométricas, estructuras fractales, teoría de la medida, conjuntos ordenados, así como con la completación de Dedekind-MacNeille.

A continuación se detalla, de una manera bastante sintética, el contenido de cada una de las partes de la tesis doctoral de acuerdo con los capítulos en que se divide su contenido, comentando los subobjetivos y resultados principales tratados en cada uno de ellos.

La primera parte, en la que se pretende, como ya se ha comentado, presentar un método para construir medidas de probabilidad a partir de un espacio con una estructura fractal, comienza con un capítulo (el tercero del trabajo) cuyo objetivo primero es la construcción de la completación del espacio dotado de dicha estructura. Veremos que dicha completación, que siempre existe, es única salvo isomorfismos fractales. Dicha completación será el punto de partida hasta conseguir definir una medida de probabilidad en el espacio original. De hecho, en el Capítulo 4 vemos cómo definir una medida de probabilidad en la completación del espacio. Esto lo podemos hacer de dos formas: primero partiendo de una familia de bolas y de una premedida definidas sobre éste y, además, podemos construir una nueva medida, también en la completación, pero a partir de los elementos de una estructura fractal teselación. Hecho esto, el siguiente paso de este desarrollo teórico es explorar condiciones para que la restricción de la medida de probabilidad de la que ya disponemos al espacio original sea, efectivamente, una medida de probabilidad. Es más, probaremos que cualquier medida de probabilidad definida en el espacio puede construirse a partir de cierta premedida siguiendo el procedimiento presentado. Eso es, precisamente, a lo que se dedicará el quinto capítulo, el cual se acompañará de varios ejemplos con el fin de ilustrar la construcción llevada a cabo, así como la idoneidad de ciertas hipótesis impuestas hasta la consecución de una medida de probabilidad en el espacio de partida. Para finalizar esta primera parte, en el Capítulo 6 se ponen de manifiesto algunas aplicaciones que surgen de la teoría desarrollada en los capítulos previos. Por ejemplo, la generación de medidas de probabilidad a partir de estructuras fractales nos permite desarrollar un nuevo método de estimación de parámetros que, si bien apoya las predicciones realizadas por el conocido método de máxima verosimilitud, ve mejorados los resultados cuando la muestra de la que se parte contiene datos completamente ajenos a la distribución cuyos parámetros queremos estimar. Además, es posible generar muestras de una determinada distribución de probabilidad o diseñar un test de bondad de ajuste, ambos basados en la construcción expuesta en los Capítulos 4 y 5.

La segunda parte trata sobre la elaboración de una teoría de funciones de distribución en un contexto más general que el conocido sobre la recta real, al que nos referiremos como caso clásico. Dicho contexto tiene que ver con los espacios topológicos linealmente ordenados. En el Capítulo 7 se parte de una medida de probabilidad en un espacio topológico linealmente ordenado y separable para definir una función de distribución de probabilidad en dicho espacio. Además, se define la inversa de una función de distribución y se estudian las propiedades de ambas funciones, comparándolas con las que éstas presentan en el caso clásico. De hecho, una de las limitaciones que presenta la inversa de una función de distribución es que no siempre está definida para todos los valores del intervalo unidad, dado que no tenemos garantizada la existencia del ínfimo y del supremo de cualquier subconjunto del espacio de partida. Por ello, surge la necesidad de disponer de un ambiente donde la definición de la inversa tenga completo sentido, puesto que es una herramienta que permitirá, entre otras cosas, generar muestras de una determinada distribución de probabilidad a partir de un procedimiento similar al de la transformada inversa, conocido en el caso de funciones de distribución reales de variable real, y utilizado para generar muestras aleatorias. Precisamente, en el Capítulo 8 estudiamos la completación de Dedekind-MacNeille (o completación por cortaduras) de un espacio topológico linealmente ordenado y separable, y vemos cómo extender la función de distribución, definida en el espacio original, a dicha completación. De hecho, dicha completación resulta ser una compactación del espacio de partida. Dicha completación posibilita la definición de la inversa de una función de distribución en términos de cortaduras y de forma que está bien definida para todo punto en $[0,1]$. Siguiendo esta línea de investigación, tiene sentido plantearse bajo qué condiciones podemos garantizar que existe una relación biunívoca entre medida de probabilidad y función de distribución en el contexto en el que nos enmarcamos. Este estudio se lleva a cabo en el Capítulo 9. Es más, se determinarán condiciones para garantizar que cierta función es la inversa de la función de distribución de cierta medida de probabilidad en la sigma álgebra de Borel del espacio. En el Capítulo 10 se presentan dos aplicaciones que han surgido de la teoría desarrollada en los Capítulos 7, 8 y 9. Por una parte, se puede demostrar que toda función de distribución en un espacio topológico linealmente ordenado y separable se puede expresar como la suma convexa de dos funciones de distribución, cada una de ellas con cierta peculiaridad en cuanto a su continuidad y, por otro lado, diseñamos un test de bondad de ajuste, similar al test de Kolmogorov-Smirnov, conocido en el caso clásico. Finalmente, el Capítulo 11 sirve como punto de encuentro entre ambas partes del trabajo realizado. Concretamente, se muestra cómo construir una estructura fractal a partir de un espacio topológico linealmente ordenado y viceversa. Esta conexión entre ambas estructuras topológicas hace posible dar el paso de medida de probabilidad a función de distribución, y al contrario, tanto en un ambiente como en otro, de forma que las aplicaciones mostradas en cada parte del trabajo cobran perfecto sentido a la hora de desarrollarse en el contexto preferido por el investigador.

El trabajo culmina con las conclusiones pertinentes y la correspondiente bibliografía, entre cuyas referencias se encuentran seis artículos de investigación, en los que figura como autor el que es el aspirante al título de doctor mediante este trabajo: [29], [30], [31], [32], [33] y [34]. Dichos trabajos respaldan la mayor parte del contenido de esta tesis.

Contents

Chapter 1

Introduction

The theory of probability measures and cumulative distribution functions is basic and well established in Statistics, Probability and Mathematics in general. Indeed, as it is well known, there is an equivalence between probability measures, cumulative distribution functions and random variables in the classical theory. For further reference about the classical theory on probability measures and cumulative distribution functions see, for example, [19] and [52].

However, while it is true that a probability distribution is characterized in terms of any of the previous three tools, working with a cumulative distribution is quite handy, since it is a usual function and not a set function (probability measure) or a function which is linked to structures like a σ-algebra (random variable). Moreover, defining a probability measure on a given space is not simple. Recall that it is a function which must satisfy some properties: it must be a non-negative and σ-additive set mapping defined on a σ-algebra such that the measure of the empty set is zero. That is the reason why, in the first part of this dissertation (see Chapters 3, 4, 5 and 6), we give a new way to construct probability measures as well as some applications of this theory. For that purpose, the first step is considering a fractal structure on the space where we want the probability measure to be defined. In what follows, we motivate the use of the fractal structure as a starting point in the new way to construct probability measures. Fractal structures were introduced by F. G. Arenas and M. A. Sánchez-Granero in [2] in order to study the structure of self-similar sets and fractals in general. Furthermore, this type of topological structure became of great interest, since it lets us study topological properties of the space and, in particular, it is related to structures which have to do with Asymetric Topology, such as non-archimedean quasi-pseudometrics. In fact, while fractal structures have been used when studying metrization (see [4] and [5]), completeness (see [6] and [7]), topological dimension (see [8]) and other topological properties (see [55]), one of the most interesting properties of them is their recursive character. This recursive nature is, indeed, what lets us start from a given mass (equal to 1) on the space, which will be distributed along it by dividing it into different sets induced by the fractal structure. Although this procedure seems direct, we have to be careful so that the mass is not lost while dividing it into the different sets in which the space is divided according to the fractal structure. For that purpose, we need some notions and results which involve completeness, so we first need to ensure that the mass gives us a probability measure on the completion of the space. Once we have done this, we look for conditions so that the mass is not lost in the following steps or, equivalently, when we work in the original space where we want the measure to be defined. In the literature, similar procedures can be found. For example, in [24] a method to construct a mass distribution on a subset of $\mathbb{R}^n$, which consists of repeating a subdivision of a mass between parts of a bounded Borel set, can be found. Another similar construction is the one made to define multiplicative cascades. [17], [18] and [46] are good references for further study of this topic.

On the other hand, a cumulative distribution function lets us calculate the measure of certain (measurable) sets in a quite simple way. Hence, it becomes preferable to work with a cumulative distribution function (once we know the properties that characterize it) instead of with the probability measure as such. Another reason to work with a cumulative distribution function, instead of its probability measure, has to do with its application to the generation of samples of a random variable. For that purpose, we use the pseudo-inverse of the corresponding cumulative distribution function. Generating samples of a given distribution is essential, for example, in Montecarlo simulations. What is more, in Statistics, several goodness-of-fit tests arise from considering a cumulative distribution function as the null-hypothesis of the test. One example of this is the Kolmogorov-Smirnov test. We can also describe some methods to estimate the unknown parameters of a distribution once we are given a random sample of it. Definitely, a cumulative distribution function plays a key role in several applications of Probability Theory. Moreover, note that working with a cumulative distribution function implies considering an order on the space and a topology so that we can talk about continuity of it. Hence, it seems natural to consider, as a more general context, the one which has to do with linearly ordered topological spaces in order to define a cumulative distribution function. In fact, it is worth noting that the study of measures on topological spaces (see [13]) lies in the intersection of Functional Analysis, Measure Theory, General Topology and Probability Theory and is a very wide research field, with multiple connections between fields. According to the previous comments, the main goal of the second part of this dissertation (see Chapters 7, 8, 9 and 10) is developing a theory on a cumulative distribution function on this type of topological space. Recall that, in the classical theory, a cumulative distribution function F is a real-valued function which is monotonically non-decreasing, right-continuous (which is equivalent to $F(a) = \lim_{x\to a^+} F(x)$ for each $a \in \mathbb{R}$) and such that $\lim_{x\to -\infty} F(x) = 0$ and $\lim_{x\to\infty} F(x) = 1$. However, these conditions are not enough to define a cumulative distribution function in the context of linearly ordered topological spaces, so it does make sense to explore sufficient and necessary conditions so that, as it happens in the classical case, there is a one-to-one relationship between probability measures and cumulative distribution functions. Given a cumulative distribution, we can define its pseudo-inverse and use it to generate samples of a certain distribution. The main problem of this function is that, when working with a linearly ordered topological space, it is not defined for each point in the unit interval, so we need additional properties of the original space such as compactness. In fact, we can solve the problem by extending the definition of the cumulative distribution function and the pseudo-inverse to the Dedekind-MacNeille completion of the space, and looking for conditions so that the equivalence between probability measures and cumulative distribution functions holds.

Finally, in Chapter 11 we establish the connection between both parts of the work. Indeed, we show that given a space with a fractal structure, we can define an order which is compatible with this structure and, conversely, given a linearly ordered topological space, we can define a fractal structure so that everything works fine, since the reasearcher can work in the preferred context.

Chapter 2

Preliminaries

2.1 Fractal structures and quasi-pseudometrics

Fractal structures were introduced in [2] for a topological space. The definition that we use in this work has been previously used in other works and is defined on a set instead of a topological space.

We will use the following notations:

- Let X be a nonempty set and Γ a covering of X. We define $St(x, \Gamma) = \bigcup\{A \in \Gamma : x \in A\}$ and $St(A, \Gamma) = \bigcup\{B \in \Gamma : B \cap A \neq \emptyset\}$ for each $x \in X$ and $A \subseteq X$.
- $U_\Gamma = \{(x, y) \in X \times X : y \notin \bigcup\{A \in \Gamma : x \notin A\}\}$, where Γ is a cover of a set X.
- A cover Γ_2 is a strong refinement of another cover Γ_1, written as $\Gamma_2 \prec\prec \Gamma_1$, if Γ_2 is a refinement of Γ_1 (that is, each element of Γ_2 is contained in some element of Γ_1), denoted by $\Gamma_2 \prec \Gamma_1$, and for each $B \in \Gamma_1$ it holds that $B = \bigcup\{A \in \Gamma_2 : A \subseteq B\}$.

Definition 2.1. *A fractal structure on a set X is a countable family of coverings $\mathbf{\Gamma} = \{\Gamma_n : n \in \mathbb{N}\}$ such that $\Gamma_{n+1} \prec\prec \Gamma_n$. Cover Γ_n is called the level n of the fractal structure.*

A fractal structure induces a transitive base of a quasi-uniformity given by $\{U_{\Gamma_n} : n \in \mathbb{N}\}$.

Moreover, we say that $\mathbf{\Gamma}$ is a finite fractal structure if Γ_n has a finite number of elements for each $n \in \mathbb{N}$.

In what follows, we introduce two simple examples of fractal structures. The first one is defined on $[0,1]$ and its levels are given by $\Gamma_n = \{[\frac{k}{2^n}, \frac{k+1}{2^n}] : k = 0, \ldots, 2^n - 1\}$ for each $n \in \mathbb{N}$. Note that the previous fractal structure is finite. However, if we consider the Euclidean space $\mathbb{R}$, it is defined as the countable family of coverings $\mathbf{\Gamma} = \{\Gamma_n : n \in \mathbb{N}\}$, where $\Gamma_n = \{[\frac{k}{2^{n-1}}, \frac{k+1}{2^{n-1}}] : k \in \mathbb{Z}\}$ for each $n \in \mathbb{N}$. In both cases, $\mathbf{\Gamma}$ is known as the natural fractal structure.

Let now $\mathbf{\Gamma} = \{\Gamma_n : n \in \mathbb{N}\}$ be a fractal structure on a set X. For each $n \in \mathbb{N}$, we define $U_{xn} = U_{\Gamma_n}(x) = X \setminus \bigcup_{x \notin A, A \in \Gamma_n} A$.

Let $d : X \times X \to \mathbb{R}_0^+$ be a function on X, where $\mathbb{R}_0^+$ denotes, as usual, the set of non-negative reals. Next, we include some properties of d for any $x, y, z \in X$ in order to characterize different types of distance functions.

1. $d(x,x) = 0$.

2. $d(x,z) \leq d(x,y) + d(y,z)$.

3. $d(x,y) = d(y,x)$.

4. $d(x,y) = 0$ implies that $x = y$.

5. $d(x,y) = 0 = d(y,x)$ implies that $x = y$.

6. $d(x,z) \leq \max\{d(x,y), d(y,z)\}$.

Definition 2.2. *d is a metric on X if $1, 2, 3$ and 4 are satisfied.*

d is a pseudometric on X if $1, 2$ and 3 are satisfied.

d is a quasi-metric on X if $1, 2$ and 4 are satisfied.

d is a quasi-pseudometric on X if 1 and 2 are satisfied.

d is a T_0-quasi-metric on X if $1, 2$ and 5 are satisfied.

Moreover, if one of the previous distance functions satisfies 6, it will be called non-archimedean. A non-archimedean metric is also called an ultrametric.

The non-archimedean quasi-pseudometric $d_{\mathbf{\Gamma}}$ induced by $\mathbf{\Gamma}$ is defined by (see [2])

$$d_{\mathbf{\Gamma}}(x,y)=\begin{cases}\frac{1}{2^n} & \text{if} \quad y\in U_{xn}\backslash U_{x,n+1}\\ \\ 1 & \text{if} \quad y\notin U_{x1}\\ \\ 0 & \text{otherwise}\end{cases}$$

We will denote it simply by d if there is no confusion on the fractal structure $\mathbf{\Gamma}$.

First, note that $B(x,\frac{1}{2^n})=U_{x,n+1}$ and, also, that d satisfies the inequality $d(x,z)\leq\max\{d(x,y),d(y,z)\}$ for each $x,y,z\in X$, which gives us that d is a non-archimedean quasi-pseudometric. In addition, we can consider the conjugate quasi-pseudometric, $d^{-1}(x,y)=d(y,x)$, and the supremum pseudometric, $d^*(x,y)=\max\{d(x,y),d(y,x)\}$, which is a non-archimedean pseudometric (or ultrapseudometric).

We will use the notation $U_{xn}^{-1}=\{y\in X:x\in U_{yn}\}$ and $U_{xn}^*=U_{xn}\cap U_{xn}^{-1}$. Note that $U_{xn}^{-1}=B_{d^{-1}}(x,\frac{1}{2^{n-1}})$ and $U_{xn}^*=B_{d^*}(x,\frac{1}{2^{n-1}})$.

When we work with a fractal structure, we can consider the topologies induced by d, d^{-1} or d^*. If we refer to a topological property in a space with a fractal structure, we will always refer to the topology induced by d (a neighborhood base of a point x is $\{U_{xn}:n\in\mathbb{N}\}$), unless a direct reference to another topology is used.

Next, we gather some properties of these sets.

Proposition 2.3. *Let $\mathbf{\Gamma}$ be a fractal structure on X, $x,y\in X$ and $n\in\mathbb{N}$. Then:*

1. *([2, Prop. 3.2]) $U_{xn}^{-1}=\bigcap_{x\in A,A\in\Gamma_n}A$.*
2. *([2, Prop. 2.1]) $\{U_{xn}^*:x\in X\}$ is a partition of X, that is, it covers X and, given $x,y\in X$, it follows that $U_{xn}^*=U_{yn}^*$ or $U_{xn}^*\cap U_{yn}^*=\emptyset$.*
3. *([2, Prop. 3.5]) Let $x,y\in X$ and $n\in\mathbb{N}$. Suppose that $y\in U_{xn}$. Then $U_{yn}\subseteq U_{xn}$.*
4. *([2, Lemma 3.7]) $U_{xm}\subseteq U_{xn}$ for each $m\geq n$. As a consequence, $U_{xm}^{-1}\subseteq U_{xn}^{-1}$ and $U_{xm}^*\subseteq U_{xn}^*$ for each $m\geq n$.*
5. *$U_{xn}^*=\bigcap_{x\in A,A\in\Gamma_n}A\backslash\bigcup_{x\notin A,A\in\Gamma_n}A$.*
6. *$U_{xn}^*\cap U_{yn}^*\neq\emptyset\Leftrightarrow y\in U_{xn}^*\Leftrightarrow U_{xn}^*=U_{yn}^*\Leftrightarrow x\in U_{yn}^*$.*

Note that the last two items are direct consequences of the previous ones.

There are different notions of completeness available for a fractal structure. For example, we can use any of the completeness notions for the induced quasi-uniformity (bicomplete, complete, convergent complete, left or right K-complete, etc.). We refer the reader to [27] as the basic reference for quasi-uniformities and quasi-pseudometrics, where some concepts of completeness are discussed, including the construction of the bicompletion. If we want to study a completeness property based only on the fractal structure, we can use the following one introduced in [7].

Definition 2.4. *Let $\mathbf{\Gamma} = \{\Gamma_n : n \in N\}$ be a fractal structure on X. $\mathbf{\Gamma}$ is said to be Cantor complete if for each decreasing sequence (A_n) (which means that $A_{n+1} \subseteq A_n$ for each $n \in \mathbb{N}$) of subsets of X with $A_n \in \Gamma_n$, it holds that $\bigcap_{n\in\mathbb{N}} A_n \neq \emptyset$.*

2.2 Measure theory

Let X be a set.

Definition 2.5. *Suppose that $\mathcal{R}$ is a nonempty collection of subsets of X. Then $\mathcal{R}$ is said to be a ring if $A \cup B \in \mathcal{R}$ and $A \setminus B \in \mathcal{R}$ for each $A, B \in \mathcal{R}$.*

Definition 2.6. *A σ-ring is a ring closed under the formation of countable unions.*

Definition 2.7. *Suppose that $\mathcal{Q}$ is a nonempty collection of subsets of X. Then $\mathcal{Q}$ is said to be an algebra if it is a ring such that $X \in \mathcal{Q}$.*

Definition 2.8. *Suppose that $\mathcal{A}$ is a nonempty collection of subsets of X. Then $\mathcal{A}$ is said to be a σ-algebra if it is closed under complement and countable union and $X \in \mathcal{A}$.*

Definition 2.9. *Let (X, τ) be a topological space. Then $\mathcal{B} = \sigma(\tau)$ is the Borel σ-algebra of the space, that is, it is the σ-algebra generated by the open sets of X.*

Definition 2.10. *Given a measurable space $(\Omega, \mathcal{A})$, a measure μ is a non-negative and σ-additive set mapping defined on a σ-algebra, $\mathcal{A}$, such that $\mu(\emptyset) = 0$. Moreover, if it holds that $\mu(\Omega) = 1$, then μ is said to be a probability measure on Ω.*

A set mapping is said to be σ-additive if $\mu(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n)$ for each countable collection $\{A_n\}_{n=1}^{\infty}$ of pairwise disjoint sets in $\mathcal{A}$.

A measure is monotonic (which means that if $A, B \in \mathcal{A}$ are such that $A \subseteq B$, then $\mu(A) \leq \mu(B)$). It is also continuous from below: if (A_n) is a monotonically non-decreasing sequence of sets (which means that $A_n \subseteq A_{n+1}$ for each $n \in \mathbb{N}$), then $\mu(A_n) \to \mu(\bigcup_{n\in\mathbb{N}} A_n)$. Moreover, it is continuous from above: if (A_n) is monotonically non-increasing (which means that $A_{n+1} \subseteq A_n$ for each $n \in \mathbb{N}$) and $\mu(A_1) < \infty$, then $\mu(A_n) \to \mu(\bigcap_{n\in\mathbb{N}} A_n)$.

Now, if $\mathcal{P}(X)$ denotes the power set of X (that is, the set of all subsets of X including $\emptyset$ and X itself), a given space, an outer measure on X is simply a map $\mu : \mathcal{P}(X) \to [0, \infty]$ such that it is monotone, sub-σ-additive (that is, whenever $A \in \mathcal{P}(X)$, and $(A_n)_{n=1}^{\infty}$ is a sequence in $\mathcal{P}(X)$ with $A \subset \bigcup_{n=1}^{\infty} A_n$, it follows that $\mu(A) \leq \sum_{n=1}^{\infty} \mu(A_n)$) and $\mu(\emptyset) = 0$.

Definition 2.11. *Let (X, d) be a metric space. An outer measure μ on X is said to be a metric outer measure if for each pair of sets $A, B \subset X$ such that $d(A, B) > 0$, it holds that $\mu(A \cup B) = \mu(A) + \mu(B)$, where $d(A, B)$ denotes the distance between A and B, that is, $d(A, B) = \inf\{d(x, y) : x \in A, y \in B\}$.*

Next, we recall from [22] two theorems on construction of outer measures.

Let X be a set and let $\mathcal{A}$ be a family of subsets of X that covers X. Let $c : \mathcal{A} \to [0, \infty]$ be any function. The theorem on construction of outer measures (Method I) is as follows:

Theorem 2.12. *([22, Th. 5.2.2]) There is a unique outer measure $\overline{\mathcal{M}}$ on X such that*

1. *$\overline{\mathcal{M}}(A) \leq c(A)$ for each $A \in \mathcal{A}$.*
2. *If $\overline{\mathcal{N}}$ is any outer measure on X with $\overline{\mathcal{N}}(A) \leq c(A)$ for each $A \in \mathcal{A}$, then $\overline{\mathcal{N}}(B) \leq \overline{\mathcal{M}}(B)$ for each $B \subseteq X$.*

Furthermore, for any subset B of X, the definition of the outer measure $\overline{\mathcal{M}}$ is given by $\overline{\mathcal{M}}(B) = \inf \sum_{A\in\mathcal{D}} c(A)$, where the infimum is over all countable covers $\mathcal{D}$ of B by sets of $\mathcal{A}$.

Proposition 2.13. *([22, Ex. 5.4.1]) Let $\mathcal{A} \subseteq \mathcal{B}$ two covers of X and let $c : \mathcal{B} \to [0, \infty]$ be a set function. If $\overline{\mathcal{M}}$ is the method I outer measure defined by c and $\mathcal{A}$, and if $\overline{\mathcal{N}}$ is the method I outer measure defined by c and $\mathcal{B}$, then $\overline{\mathcal{M}}(A) \geq \overline{\mathcal{N}}(A)$ for each $A \subseteq X$.*

Now, we recall Method II. Let $\mathcal{A}$ be a family of subsets of a metric space S and suppose that, for each $x \in S$ and $\varepsilon > 0$, there exists $A \in \mathcal{A}$ with $x \in A$ and $\operatorname{diam}(A) \leq \varepsilon$.

Suppose $c : \mathcal{A} \to [0, \infty]$ is a given function. An outer measure will be constructed based on this data. For each $\varepsilon > 0$, let $\mathcal{A}_\varepsilon = \{A \in \mathcal{A} : \operatorname{diam}(A) \leq \varepsilon\}$. Let $\overline{\mathcal{M}}_\varepsilon$ be the Method I outer measure determined by c using the family $\mathcal{A}_\varepsilon$. Then, by Proposition 2.13, for a given set E, when ε decreases, $\overline{\mathcal{M}}_\varepsilon(E)$ increases. Define $\overline{\mathcal{M}}(E) = \lim_{\varepsilon\to 0} \overline{\mathcal{M}}_\varepsilon(E) = \sup_{\varepsilon>0} \overline{\mathcal{M}}_\varepsilon(E)$.

Theorem 2.14. *([22, Th. 5.4.2]) The set function $\overline{\mathcal{M}}$ defined by the previous method is a metric outer measure.*

Let us take into account the next theorem which is related to the uniqueness of a measure.

Theorem 2.15. *([35, Th. A Section 13]) If μ is a σ-finite measure on an algebra R, then there is a unique measure $\overline{\mu}$ on the σ-algebra $\sigma(R)$ such that, for E in R, $\overline{\mu}(E) = \mu(E)$; the measure $\overline{\mu}$ is σ-finite.*

Moreover, each measure on an algebra gives us an outer measure, as the next result shows.

Theorem 2.16. *([35, Th. A Section 10]) If μ is a measure on an algebra $\mathcal{R}$ and if, for every $E \subseteq X$, $\mu^*(E) = \inf\{\sum_{n=1}^{\infty} \mu(E_n) : E_n \in \mathcal{R}, n = 1, 2, \ldots, E \subset \bigcup_{n=1}^{\infty} E_n\}$ then μ^* is an extension of μ to an outer measure on X. μ^* is called the outer measure induced by the measure μ.*

Definition 2.17. *([14, Def. 1.5.1]) Suppose that μ is a non-negative set function on domain $\mathcal{A} \subset \mathcal{P}(X)$. A set A is called μ-measurable (or Lebesgue measurable with respect to μ) if, for any $\varepsilon > 0$, there exists $A_\varepsilon \in \mathcal{A}$ such that $\mu^*(A \triangle A_\varepsilon) < \varepsilon$, where μ^* is the outer measure defined by $\mu^*(A) = \inf\left\{\sum_{n=1}^{\infty} \mu(A_n) : A_n \in \mathcal{A}, A \subset \bigcup_{n=1}^{\infty} A_n\right\}$ and $\triangle$ denotes the symmetric difference, that is, $A \triangle B = (A \setminus B) \cup (B \setminus A)$. The class of μ-measurable sets is denoted by $\mathcal{A}_\mu$.*

Proposition 2.18. *([14, Section 1.5]) Every set $A \in \mathcal{A}_\mu$ can be made a measure space by restricting μ to the class of μ-measurable subsets of A, which is a σ-algebra in A.*

Definition 2.19. *([14, Def. 2.1.3]) Let $(X_1, \mathcal{A}_1)$ and $(X_2, \mathcal{A}_2)$ be two spaces with σ-algebras. A mapping $f : X_1 \to X_2$ is called measurable (with respect to the pair $(\mathcal{A}_1, \mathcal{A}_2)$) if $f^{-1}(B) \in \mathcal{A}_1$ for all $B \in \mathcal{A}_2$.*

Definition 2.20. *([14, Section 3.6]) Let X and Y be two spaces with σ-algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ and let $f : X \to Y$ be a measurable function. Then, for any bounded (or bounded from below) measure μ on $\mathcal{A}_1$, the formula $\mu \circ f^{-1}$ given by $\mu(f^{-1}(B))$ for each $B \in \mathcal{A}_2$, defines a measure on $\mathcal{A}_2$ called the image of the measure μ under the mapping f.*

2.3 Ordered sets

First, we recall the definition of a linear order and a linearly ordered topological space.

Definition 2.21. *([53, Chapter 1]) A partially ordered set $(P, \leq)$ (that is, a set P with the binary relation $\leq$ that is reflexive, antisymmetric and transitive) is totally ordered if every $x, y \in P$ are comparable, that is, $x \leq y$ or $y \leq x$. In this case, the order is said to be total or linear.*

For further reference about partially ordered sets see, for example, [21]. Moreover, [15] is a useful reference about ordered sets.

Definition 2.22. *([44, Section 1]) A linearly ordered topological space (abbreviated LOTS) is a triple $(X, \tau, \leq)$, where $(X, \leq)$ is a linearly ordered set and where τ is the topology of the order $\leq$.*

The definition of the order topology is the following one.

Definition 2.23. *([64, Part II, 39]) Let X be a set which is linearly ordered by $<$. We define the order topology τ on X by taking the subbase $\{\{x \in X : x < a\} : a \in X\} \cup \{\{x \in X : x > a\} : a \in X\}$.*

Given a linear order $\leq$ on X, we define the next sets.

Definition 2.24. *Let $a, b \in X$ with $a \leq b$, we define the set $]a, b] = \{x \in X : a < x \leq b\}$. Analogously, we define $]a, b[$, $[a, b]$ and $[a, b[$. Moreover, $(\leq a)$ is given by $(\leq a) = \{x \in X : x \leq a\}$. $(< a), (\geq a)$ and $(> a)$ are defined similarly.*

Definition 2.25. *Let $a \in X$. We will also use $]a, \infty[$ and $[a, \infty[$ to denote $(> a)$ and $(\geq a)$, respectively. Similarly, $]-\infty, a[$ and $]-\infty, a]$ will also denote $(< a)$ and $(\leq a)$, respectively.*

Remark 2.26. *Note that an open base of X with respect to τ is given by $\{]a,b[: a < b, a, b \in (X \cup \{-\infty, \infty\})\}$.*

2.4 Cuts and the Dedekind-MacNeille completion

First, we recall the definition of a complete lattice.

Definition 2.27. *([60, Def. 8.1, Section 8.1]) Let L be a partially ordered set. Then L is called a lattice if and only if any two elements of L have a supremum and an infimum. L is called a complete lattice if and only if any subset of L has a supremum and an infimum.*

Definition 2.28. *([60, Defs. 2.16, 2.17]) Let P be an ordered set and let $A \subseteq P$. Then:*

1. *l is called a lower bound of A if and only if we have $l \leq a$ for each $a \in A$.*

2. *u is called an upper bound of A if and only if we have $u \geq a$ for each $a \in A$.*

Definition 2.29. *Given an ordered set X and $A \subseteq X$, we denote by A^l and A^u, respectively, the set of lower and upper bounds of A.*

Definition 2.30. *([60, Def. 3.18]) Let P be an ordered set and let $A \subseteq P$. Then:*

1. *The point u is called the lowest upper bound or supremum or join of A if and only if u is the minimum of the set A^u.*

2. *The point u is called the greatest lower bound or infimum or meet of A if and only if l is the maximum of the set A^l.*

Proposition 2.31. *([64, Part II, 39]) The order topology on X is compact if and only if the order is complete, that is, if and only if every nonempty subset of X has a greatest lower bound and a least upper bound.*

The Dedekind-MacNeille completion of X consists of all subsets $A \subseteq X$ for which $(A^u)^l = A$. Such subsets are called cuts. More formally, it can be defined as follows:

Definition 2.32. *([60, Def. 8.21 Section 8.3]) Let P be a partially ordered set. We define the Dedekind-MacNeille completion of P to be $DM(P) = \{A \subseteq P : A = (A^u)^l\}$ ordered by inclusion, that is, given $A, B \in DM(X)$, it holds that $A \leq B$ if and only if $A \subseteq B$. It is also referred to as the MacNeille completion or the completion by cuts.*

From now on, we will denote the order topology on $DM(X)$ by τ'.

Definition 2.33. *([60, Def. 1.30]) Let P, Q be ordered sets. Then $f : P \to Q$ is called an (order) embedding if and only if the following hold.*

1. *f is injective.*

2. *For all $p_1, p_2 \in P$, we have $p_1 \leq p_2$ if and only if $f(p_1) \leq f(p_2)$.*

Theorem 2.34. *([60, Th. 8.23]) Let P be an ordered set. Then $DM(P)$ is a complete lattice. Moreover, the map $\phi_{DM} : P \to DM(P)$, which is defined by $\phi_{DM}(p) = (\leq p)$ is an embedding that preserves all suprema and infima that exist in P. Throughout what follows, we write $\phi := \phi_{DM}$ for simplicity.*

Another way to describe the Dedekind-MacNeille completion is by using cuts in pairs. We say that (A, B) is a cut if $A^u = B$ and $B^l = A$. Cuts let us give an alternative definition of Dedekind-MacNeille completion. Indeed, if (A, B) is a cut, then $(A^u)^l = B^l = A$. We will work with the notation we have introduced previously. See [45] for more reference about cuts and [53] for more about the Dedekind-MacNeille completion.

For our study we need to introduce some terminology related to a linearly ordered set $(X, \leq)$.

Definition 2.35. *([44, Section 1]) A subset $C \subseteq X$ is said to be convex in X if, whenever $a, b \in C$ with $a \leq b$, then $\{x \in X : a \leq x \leq b\}$ is a subset of C.*

Proposition 2.36. *([64, Part II, 39]) Any subset $A \subseteq X$ can be uniquely expressed as a union of disjoint, nonempty, maximal convex sets in A, called convex components.*

Definition 2.37. *([27, Section 4.1]) We say that a set $A \subseteq X$ is decreasing (respectively increasing) if given $a \in A$ and $x \leq a$ (respectively $x \geq a$), then $x \in A$.*

The definition of interval is the following one.

Definition 2.38. *([44, Section 1]) An interval of X is a convex subset of X with two endpoints in X, which can belong to the interval or not.*

For convention, we will assume that ∞ and $-\infty$ can be the endpoints of intervals.

Part I

Generating a probability measure from a fractal structure

Fractal structures were introduced in [2] and allow us to characterize non-archimedean quasi-metrization, though the analysis and study of fractal structures have been extended to a wide range of applications in General Topology and different areas where fractals have been detected. Some of this applications include metrization, topological and fractal dimension, self-similar sets (fractals), compactification, completeness, transitive quasi-uniformities, or inverse limits (see, for example, [54] and its references). However, one of the most interesting aspects of a fractal structure is its recursive character, which allows to make iterative constructions. An example of this can be found in [54], where it is given a recursive method to construct continuous mappings, which can be useful to get filling space curves or other continuous maps.

Similar ideas exist in the literature. For example, in [24] a method to construct a mass distribution on a subset of $\mathbb{R}^n$ can be found. It consists of repeating a subdivision of a mass between parts of a bounded Borel set E. Let ε_0 consist of the single set E. For $k = 1, 2, \ldots$ we let ε_k be a collection of disjoint Borel subsets of E such that each set $U \in \varepsilon_k$ is contained in one of the sets of ε_{k-1} and contains a finite number of the sets in ε_{k+1}. We suppose that the maximum diameter of the sets in ε_k tends to 0 as $k \to \infty$. We define a mass distribution on E by repeated subdivision as we see in Figure 2.1.

We let $\mu(E)$ satisfy $0 < \mu(E) < \infty$, and we slipt this mass between the sets $U_1, \ldots, U_m$ in ε_1 by defining $\mu(U_i)$ in such a way that $\sum_{i=1}^m \mu(U_i) = \mu(E)$. Similarly, we assign masses to the sets of ε_2 so that if $U_1, \ldots, U_m$ are the sets of ε_2 contained in a set U of ε_1, then $\sum_{i=1}^m \mu(U_i) = \mu(U)$. In general, we assign masses so that $\sum_i \mu(U_i) = \mu(U)$ for each set U of ε_k, where the $\{U_i\}$ are the disjoint sets in ε_{k+1} contained in U.

For each k, we let E_k be the union of the sets in ε_k, and we let $\mu(\mathbb{R}^n \setminus E_k) = 0$. Let ε denote the collection of sets that belong to ε_k for some k together with the sets $\mathbb{R}^n \setminus E_k$. The above procedure defines the mass $\mu(A)$ of every set $A \in \varepsilon$, and it should seem reasonable that, by building up sets from the sets in ε, it specifies enough about the distribution of the mass μ across E to determine $\mu(A)$ for any Borel set A. [24, Prop. 1.7] shows that $\mu(A)$ can be determined as $\mu(A) = \inf\{\sum_i \mu(U_i) : A \subset \cup_i U_i \text{ and } U_i \in \varepsilon\}$.

A similar procedure is used to define multiplicative cascades (see, for example, [17], [18] and [46]).

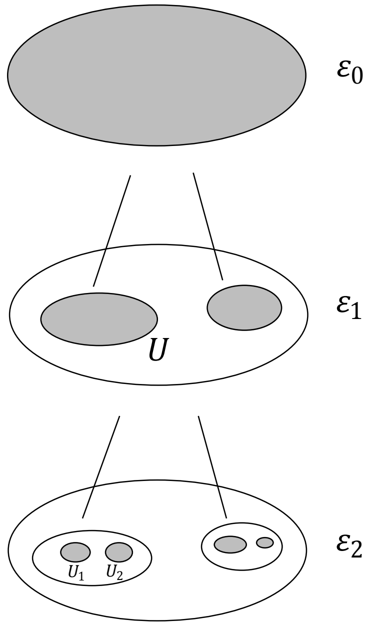


Figure 2.1: Mass distribution

In the first part of this work we use the recursive nature of a fractal structure in order to get a mass distribution on a space. One of the keys of this approach is the use of the completion of a space with a fractal structure (see Chapter 3). From the construction and study of the completion, and according to our purpose, we need to follow some steps to construct a probability measure on the original space, which will be described in Chapters 4 and 5.

Chapter 3

Completion of a fractal structure

The content of this chapter corresponds to [29].

A fractal structure induces a non-archimedean quasi-pseudometric (see the previous chapter), so we can always work with the bicompletion (or any other kind of completion) of the induced quasi-pseudometric and, then, we can have the fractal structure induced by the (non-archimedean) quasi-pseudometric on the bicompletion.

From the point of view of quasi-uniformities everything works fine with that approach, but if we need to preserve a stronger structure like the fractal structure itself, we need a new way to construct a completion of a fractal structure. In fact, we will face this issue in Chapter 4, when trying to define a probability measure on a space from a pre-measure defined on the elements of each level of a fractal structure, we need to define it first on the completion and all the structures induced by the fractal structure which are involved.

The goal of this chapter is to provide such a construction in a way that all structures induced by the fractal structure (quasi-pseudometric, metric, topology, the fractal structure itself, etc.) are extended nicely.

First of all, we gather other properties of the sets induced by d, d^{-1} and d^* that have not been explicitly stated before, though some of them may have been implicitly used.

Proposition 3.1.

1. *Let $n \in \mathbb{N}$, $A \in \Gamma_n$ and $x \in A$. Then $U^*_{xm} \subseteq U^{-1}_{xm} \subseteq A$ for each $m \geq n$.*
2. *Let $n \in \mathbb{N}$ and $A \in \Gamma_n$. Then $A = \bigcup_{x \in A} U^*_{xn}$ and $A = \bigcup_{x \in A} U^{-1}_{xn}$.*

3. *A is open in $\tau_{d^{-1}}$ and closed in τ_d for each $A \in \Gamma_n$ and each $n \in \mathbb{N}$. Consequently, A is open and closed in τ_{d^*} for each $A \in \Gamma_n$ and each $n \in \mathbb{N}$.*

4. *U^*_{xn} is open and closed in (X, d^*) for each $x \in X$ and $n \in \mathbb{N}$.*

5. *Given $n \in \mathbb{N}$, $A \in \Gamma_n$ and $x \in X$, it follows that $U^*_{xn} \subseteq A$ or $U^*_{xn} \cap A = \emptyset$. As a consequence, $U^*_{xn} \subseteq A$ if and only if $x \in A$.*

Proof. 1. Let $n \in \mathbb{N}$, $A \in \Gamma_n$ and $x \in A$. By definition of U^*_{xm}, it is clear that $U^*_{xm} \subseteq U^{-1}_{xm}$ and, by Proposition 2.3, it follows that $U^{-1}_{xm} \subseteq U^{-1}_{xn} = \bigcap_{x \in B, B \in \Gamma_n} B \subseteq A$.

2. Let $n \in \mathbb{N}$ and $A \in \Gamma_n$. It is obvious that $A \subseteq \bigcup_{x \in A} U^*_{xn} \subseteq \bigcup_{x \in A} U^{-1}_{xn}$, so let us show that $\bigcup_{x \in A} U^{-1}_{xn} \subseteq A$.

   Let $x \in A$. Then $U^{-1}_{xn} \subseteq A$ by the previous item and, hence, $\bigcup_{x \in A} U^{-1}_{xn} \subseteq A$.

3. Let $n \in \mathbb{N}$ and $A \in \Gamma_n$. First, we show that A is closed in τ_d. Let $x \in Cl_{\tau_d} A$ ($Cl_{\tau_d} A$ is the closure of A with respect to τ_d). Then $U_{xn} \cap A \neq \emptyset$. Since $U_{xn} = X \setminus \bigcup_{B \in \Gamma_n; x \notin B} B$, then $x \in A$ and, hence, $Cl_{\tau_d} A = A$, so A is closed in τ_d.

   Next, we show that A is open in $\tau_{d^{-1}}$. Let $x \in A$. Since $U^{-1}_{xn} \subseteq A$ by the first item, it follows that A is open in $\tau_{d^{-1}}$, since it is a neighborhood of each of its points in $\tau_{d^{-1}}$.

   Finally, since $\tau_d \subseteq \tau_{d^*}$ and $\tau_{d^{-1}} \subseteq \tau_{d^*}$, it follows that A is open and closed in τ_{d^*}.

4. Let $n \in \mathbb{N}$ and $x \in X$. Given $y \in U^*_{xn}$, then $U^*_{yn} = U^*_{xn}$ by Proposition 2.3. In particular, $U^*_{yn} \subseteq U^*_{xn}$ and U^*_{xn} is open in τ_{d^*}. On the other hand, if $y \in Cl_{\tau_{d^*}}(U^*_{xn})$, then $U^*_{yn} \cap U^*_{xn} \neq \emptyset$ and, by Proposition 2.3, it follows that $y \in U^*_{xn}$. It follows that $Cl_{\tau_{d^*}}(U^*_{xn}) = U^*_{xn}$ and, hence, U^*_{xn} is closed in τ_{d^*}.

5. Let $n \in \mathbb{N}$, $A \in \Gamma_n$ and $x \in X$ and suppose that $U^*_{xn} \cap A \neq \emptyset$. Then there exists $y \in U^*_{xn} \cap A$ and, hence, by item 1, $x \in U^*_{yn} \subseteq A$. Since $U^*_{xn} = U^*_{yn}$, by Proposition 2.3, we conclude that $U^*_{xn} \subseteq A$.

□

Let $\mathbf{\Gamma}$ be a fractal structure on X. Next, we introduce some characterizations for the properties T_0, T_1 and T_2 of X in terms of the fractal structure.

Proposition 3.2. *X is T_0 if and only if for each $x, y \in X$ with $x \neq y$, there exist $n \in \mathbb{N}$ and $A \in \Gamma_n$ such that A contains one of the points (x or y) but not the other one.*

Proof. $\Rightarrow$) Let X be a T_0 space and $x, y \in X$. Then there exists $n \in \mathbb{N}$ such that $y \notin U_{xn}$ or $x \notin U_{yn}$. Suppose that $y \notin U_{xn}$. Then $x \notin U_{yn}^{-1} = \bigcap_{A \in \Gamma_n, y \in A} A$, which implies that there exists $A \in \Gamma_n$ such that $y \in A$ and $x \notin A$. Analogously, suppose that $x \notin U_{yn}$. Then $y \notin U_{xn}^{-1}$, which implies that there exists $A \in \Gamma_n$ such that $x \in A$ and $y \notin A$.

$\Leftarrow$) Conversely, let $x, y \in X$ with $x \neq y$. By hypothesis, there exists $A \in \Gamma_n$ such that $x \in A$ and $y \notin A$ (or $y \in A$ and $x \notin A$), which implies that $y \notin \bigcap_{B \in \Gamma_n, x \in B} B = U_{xn}^{-1}$ and, then, $x \notin U_{yn}$ (analogously in the other case). It follows that X is T_0. $\square$

Proposition 3.3. *X is T_1 if and only if for each $x, y \in X$ with $x \neq y$, there exist $n \in \mathbb{N}$ and $A \in \Gamma_n$ such that $x \in A$ and $y \notin A$.*

Proof. $\Rightarrow$) Let X be a T_1 space and $x, y \in X$ with $x \neq y$. Then there exists $n \in \mathbb{N}$ such that $x \notin U_{yn}$. Since $x \notin U_{yn}$, $y \notin U_{xn}^{-1}$ and, then, there exists $A \in \Gamma_n$ such that $x \in A$ and $y \notin A$.

$\Leftarrow$) Let $x, y \in X$ with $x \neq y$. By hypothesis, there exist $n \in \mathbb{N}$ and $A \in \Gamma_n$ such that $x \in A$ and $y \notin A$, which implies that $y \notin U_{xn}^{-1}$ and, then, $x \notin U_{yn}$.

On the other hand, there exist $m \in \mathbb{N}$ and $B \in \Gamma_m$ such that $y \in B$ and $x \notin B$, which implies that $x \notin U_{ym}^{-1}$ and, then, $y \notin U_{xm}$.

Consequently, X is T_1.

$\square$

The characterization of the T_2 property is a bit more cumbersome.

Proposition 3.4. *X is T_2 if and only if for each $x, y \in X$ with $x \neq y$, there exists $n \in \mathbb{N}$ such that for each $z \in X$, there exists $A \in \Gamma_n$ with $z \in A$ which does not contain both x and y.*

Proof. As we know, $U_{xn} \cap U_{yn} = \emptyset$ if and only if there exists no $z \in U_{xn} \cap U_{yn}$ if and only if there exists no z such that $x, y \in U_{zn}^{-1}$ if and only if there exists no z such that, for each $A \in \Gamma_n$ with $z \in A$, it holds that A contains both x and y.

$\square$

A sufficient condition for the T_2 property is the following one.

Proposition 3.5. *X is T_2 if for each $x, y \in X$ with $x \neq y$, there exists $n \in \mathbb{N}$ such that there exists no element $A \in \Gamma_n$ which contains both x and y.*

Proof. Suppose that X is not T_2. Then there exist $x, y \in X$ with $x \neq y$ and such that $U_{xn} \cap U_{yn} \neq \emptyset$ for each $n \in \mathbb{N}$.

Given $n \in \mathbb{N}$, it follows that $U_{xn} \cap U_{yn} \neq \emptyset$, so there exists $z \in U_{xn} \cap U_{yn}$ and, hence, $x, y \in U_{zn}^{-1}$. Then any $A \in \Gamma_n$ with $z \in A$ must contain both x and y, but this contradices the hypothesis and, hence, X is T_2.

□

Note that if the fractal structure is irreducible (that is, each level is an irreducible cover, which means that it has no proper subcovers), then the converse of the previous proposition is also true. Thus, we can prove the next result.

Proposition 3.6. *Let X be a T_2 space and $\mathbf{\Gamma}$ be an irreducible fractal structure on X. Then, for each $x, y \in X$ with $x \neq y$, there exists $n \in \mathbb{N}$ such that there exists no element $A \in \Gamma_n$ which contains both x and y.*

Proof. Suppose that there exist $x, y \in X$ with $x \neq y$ and such that, for each $n \in \mathbb{N}$, there exists $A \in \Gamma_n$ with $x, y \in A$. Since $\mathbf{\Gamma}$ is irreducible, there exists $z \in A \setminus \bigcup_{B \in \Gamma_n, B \neq A} B$. By Proposition 3.4, we have that X is not T_2, which is a contradiction. □

Remark 3.7. *Let $\mathbf{\Gamma}$ be a fractal structure on X. Note that $d_{\mathbf{\Gamma}}$ is T_0 if and only if $d^*_{\mathbf{\Gamma}}$ is T_2.*

3.1 Fractal preserving maps

Given a fractal structure $\mathbf{\Gamma}$ on a space X and $Y \subseteq X$, the fractal structure induced by $\mathbf{\Gamma}$ on Y is defined by $\mathbf{\Gamma}|_Y = \{\{A \cap Y : A \in \Gamma_n\} : n \in \mathbb{N}\}$. Note that $\mathbf{\Gamma}|_Y$ is a fractal structure by [2, Prop. 3.3].

On the other hand, given $\mathbf{\Gamma}$ a fractal structure on X and $f : X \to Y$, the fractal structure induced by f and $\mathbf{\Gamma}$ on $f(X)$ is defined by $f(\mathbf{\Gamma}) = \{f(\Gamma_n) : n \in \mathbb{N}\}$, where $f(\Gamma_n) = \{f(A) : A \in \Gamma_n\}$ for each $n \in \mathbb{N}$.

By a lack of reference we prove the next result.

Proposition 3.8. *Let $\mathbf{\Gamma}$ be a fractal structure on X and $f : X \to Y$ be a map. Then $f(\mathbf{\Gamma})$ is a fractal structure on $f(X)$.*

Proof. Let $\mathbf{\Delta} = f(\mathbf{\Gamma})$. Next, we check that $\Delta_{n+1} \prec\prec \Delta_n$ for each n.

- $\Delta_{n+1} \prec \Delta_n$.

  Let $C \in \Delta_{n+1}$. Then $C = f(A)$ for some $A \in \Gamma_{n+1}$. Now, since $\mathbf{\Gamma}$ is a fractal structure on X, there exists $B \in \Gamma_n$ such that $A \subseteq B$. Thus, $f(A) \subseteq f(B) \in \Delta_n$.

- $C = \bigcup\{D \in \Delta_{n+1} : D \subseteq C\}$ for each $C \in \Delta_n$.

  $\subseteq$) Let $C \in \Delta_n$ and $x \in C$. By definition of $\mathbf{\Delta}$, we have that $C = f(A)$ for some $A \in \Gamma_n$. Then $x \in f(A)$, which implies that $x = f(a)$ for some $a \in A$. Moreover, there exists $B \in \Gamma_{n+1}$ such that $a \in B \subseteq A$. That means that $x \in f(B) \subseteq f(A)$. It follows that $x \in \bigcup\{D \in \Delta_{n+1} : D \subseteq C\}$, since $f(B) \in \Delta_{n+1}$.

  $\supseteq$) Let $x \in \bigcup\{D \in \Delta_{n+1} : D \subseteq C\}$. It is clear that $x \in C$.

$\square$

Now, we introduce new maps that preserve fractal structures.

Definition 3.9. *Let $\mathbf{\Delta}$ be a fractal structure on Y, $\mathbf{\Gamma}$ a fractal structure on X and $f : X \to Y$ a mapping. Then f is fractal-preserving if $\mathbf{\Delta}|_{f(X)} = f(\mathbf{\Gamma})$.*

Moreover, if f is injective, f is said to be a fractal embedding. If f is bijective, f is said to be a fractal isomorphism.

Remark 3.10. *Note that if f is a fractal isomorphism, then f^{-1} is also a fractal isomorphism.*

Proposition 3.11. *Let $\mathbf{\Delta}$ be a fractal structure on Y, $\mathbf{\Gamma}$ a fractal structure on X and $f : X \to Y$ be a fractal-preserving mapping such that $f(x) \in f(A)$ implies that $x \in A$ for each $A \in \Gamma_n$ and for each $n \in \mathbb{N}$. Then the following statements are true:*

1. *$y \in U_{xn}$ if and only if $f(y) \in U_{f(x)n}$.*

2. *$y \in U_{xn}^{-1}$ if and only if $f(y) \in U_{f(x)n}^{-1}$.*

3. *$y \in U_{xn}^{*}$ if and only if $f(y) \in U_{f(x)n}^{*}$.*

4. *f is an isometry with respect to d. It also follows that f is an isometry with respect to d^{-1} and d^*.*

5. *$f(U_{xn}) = U_{f(x)n} \cap f(X)$.*

6. *$f(U_{xn}^{-1}) = U_{f(x)n}^{-1} \cap f(X)$.*

7. *$f(U_{xn}^{*}) = U_{f(x)n}^{*} \cap f(X)$.*

Proof. We prove that $y \in U_{xn}$ if and only if $f(y) \in U_{f(x)n}$, since this item implies all the other ones.

$\Rightarrow$) Let $y \in U_{xn}$. Then $x \in U_{yn}^{-1}$. Now, let $B \in \Delta_n$ be such that $f(y) \in B$. Then $f(y) \in B \cap f(X) = f(A)$ for some $A \in \Gamma_n$. Since $f(y) \in f(A)$, by hypothesis it follows that $y \in A$, which implies that $x \in A$ (since $x \in U_{yn}^{-1}$). Consequently, $f(x) \in f(A) = B \cap f(X)$, so $f(x) \in B$. It follows that $f(x) \in \bigcap_{B \in \Delta_n, f(y) \in B} B = U_{f(y)n}^{-1}$, which implies that $f(y) \in U_{f(x)n}$.

$\Leftarrow$) Let $f(y) \in U_{f(x)n}$. Then $f(x) \in U_{f(y)n}^{-1}$. Let $A \in \Gamma_n$ be such that $y \in A$. Then $f(y) \in f(A) = B \cap f(X)$ for some $B \in \Delta_n$. Since $f(y) \in B$, it follows that $f(x) \in B$ (since $f(x) \in U_{f(y)n}^{-1}$), this means that $f(x) \in B \cap f(X) = f(A)$. Finally, since $f(x) \in f(A)$, $x \in A$ by hypothesis, so $x \in \bigcap_{A \in \Gamma_n, y \in A} A = U_{yn}^{-1}$. Therefore, $y \in U_{xn}$.

□

In particular, if f is injective, then we have the following result.

Remark 3.12. *Note that if $f : X \to Y$ is a fractal embedding, then the previous statements are true.*

3.2 The construction of a completion for a fractal structure

In this section we recall the construction of $\widetilde{X}$ from [2] and, then, we define a fractal structure on it. In what follows, we assume that $\mathbf{\Gamma} = \{\Gamma_n : n \in \mathbb{N}\}$ is a fractal structure on a set X.

For each $n \in \mathbb{N}$, let $G_n = \{U^*_{xn} : x \in X\}$. In G_n we define the partial order $U^*_{xn} \leq U^*_{yn}$ if and only if $y \in U_{xn}$. Then $(G_n, \leq)$ is a poset (that is, a partially ordered set).

Now, we define the maps $\rho_n : X \to G_n$ by $\rho_n(x) = U^*_{xn}$ and $\phi_n : G_{n+1} \to G_n$, given by $\phi_n(\rho_{n+1}(x)) = \rho_n(x)$. Finally, the inverse limit, $\varprojlim G_n$, is defined, as usual, by $\varprojlim G_n = \{(g_1, g_2, \ldots) \in \prod_{n=1}^{\infty} G_n : \phi_n(g_{n+1}) = g_n, \forall n \in \mathbb{N}\}$. We will use the notation $\widetilde{X} = \varprojlim G_n$ from now on. Finally, we define the mapping $\rho : X \to \widetilde{X}$ by $\rho(x) = (\rho_n(x))_{n\in\mathbb{N}}$.

Now, we see how to extend a fractal structure on X to a fractal structure on $\widetilde{X}$.

Let $\widetilde{\mathbf{\Gamma}} = \{\widetilde{\Gamma}_n : n \in \mathbb{N}\}$, where $\widetilde{\Gamma}_n = \{\widetilde{A} : A \in \Gamma_n\}$ and, for each $A \in \Gamma_n$, $\widetilde{A}$ is defined by $\widetilde{A} = \{(\rho_n(x_n))_{n\in\mathbb{N}} \in \widetilde{X} : x_n \in A\}$.

Next, we prove that $\widetilde{\mathbf{\Gamma}}$ is a fractal structure on $\widetilde{X}$.

Lemma 3.13. *Let $n \in \mathbb{N}$, $A \in \Gamma_n$ and $a = (\rho_n(x_n))_{n\in\mathbb{N}} \in \widetilde{X}$ be such that $x_k \in A$ for some $k \geq n$. Then $x_i \in A$ for each $i \geq n$.*

Proof. Let $i \geq n$. By definition of inverse limit, $U^*_{x_i n} = U^*_{x_n n}$ (1). Moreover, $U^*_{x_n n} = U^*_{x_k n}$ (2) due to the same fact, since $k \geq n$.

On the other hand, $U^*_{x_k n} \subseteq U^{-1}_{x_k n} \subseteq A$ (3). If we join expressions (1), (2) and (3), since $x_i \in U^*_{x_i n}$, $x_i \in A$. $\square$

Lemma 3.14. *Let $A \in \Gamma_{n+1}$ and $B \in \Gamma_n$ be such that $A \subseteq B$. Then $\widetilde{A} \subseteq \widetilde{B}$.*

Proof. Let $a \in \widetilde{A}$. Then $a = (\rho_i(x_i))_{i\in\mathbb{N}}$ with $x_{n+1} \in A$, so $x_{n+1} \in A \subseteq B$. By Lemma 3.13, $x_n \in B$. It follows that $a = (\rho_i(x_i))_{i\in\mathbb{N}} \in \widetilde{B}$.

$\square$

The above statements will help us proving the following result.

Proposition 3.15. $\widetilde{\mathbf{\Gamma}}$ *is a fractal structure on* $\widetilde{X}$.

Proof. It is clear that $\widetilde{\Gamma}_n$ is a covering of $\widetilde{X}$, since Γ_n is a covering of X. Now, let $n \in \mathbb{N}$ and let us prove that $\widetilde{\Gamma}_{n+1} \prec\prec \widetilde{\Gamma}_n$. This will be true if the next statements are satisfied:

1. $\widetilde{\Gamma}_{n+1} \prec \widetilde{\Gamma}_n$.

Let $\widetilde{A} \in \widetilde{\Gamma}_{n+1}$. Then $A \in \Gamma_{n+1}$ so that there exists $B \in \Gamma_n$ such that $A \subseteq B$. By Lemma 3.14, we conclude that $\widetilde{A} \subseteq \widetilde{B}$, and it is clear that $\widetilde{B} \in \widetilde{\Gamma}_n$.

2. $\widetilde{B} = \bigcup\{\widetilde{A} \in \widetilde{\Gamma}_{n+1} : \widetilde{A} \subseteq \widetilde{B}\}$ for each $B \in \Gamma_n$.

   Let $B \in \Gamma_n$.

   $\supseteq$) It is obvious.

   $\subseteq$) Let $b \in \widetilde{B}$. Then $b = (\rho_i(x_i))_{i\in\mathbb{N}}$, with $x_n \in B$. By Lemma 3.13, $x_{n+1} \in B$. Since $\Gamma_{n+1} \prec\prec \Gamma_n$, there exists $A \in \Gamma_{n+1}$ such that $x_{n+1} \in A \subseteq B$. It follows that $b \in \widetilde{A}$. Therefore, $b \in \widetilde{A} \subseteq \widetilde{B}$ by Lemma 3.14, and it is clear that $\widetilde{A} \in \widetilde{\Gamma}_{n+1}$.

$\square$

Let us denote by $\widetilde{d} = d_{\widetilde{\mathbf{\Gamma}}}$ the non-archimedean quasi-pseudometric induced by $\widetilde{\mathbf{\Gamma}}$ on $\widetilde{X}$, $\widetilde{U}_n$ to be the transitive base of the quasi-uniformity induced by $\widetilde{\mathbf{\Gamma}}$ on $\widetilde{X}$, so $\widetilde{U}_{xn} = X \setminus \bigcup_{x\notin\widetilde{A},\widetilde{A}\in\widetilde{\Gamma}_n} \widetilde{A}$ for each $x \in \widetilde{X}$ and $n \in \mathbb{N}$, $\widetilde{\tau}$ to be the topology induced by $\widetilde{d}$ and $\widetilde{\tau}^*$ to be the topology induced by $\widetilde{d}^*$.

The next proposition gather some relations between elements of X and its extensions to $\widetilde{X}$.

Proposition 3.16.

1. *Let $(\rho_n(x_n))_{n\in\mathbb{N}}, (\rho_n(y_n))_{n\in\mathbb{N}} \in \widetilde{X}$ be such that $(\rho_n(x_n))_{n\in\mathbb{N}} = (\rho_n(y_n))_{n\in\mathbb{N}}$, and let $m \in \mathbb{N}$ and $A \in \Gamma_m$. Then $x_m \in A$ if and only if $y_m \in A$ (roughly speaking, the definition of $\widetilde{A}$ does not depend on the sequence (x_n)).*

2. *Let $x = (\rho_n(x_n)) \in \widetilde{X}$ and $A \in \Gamma_k$. Then $x \in \widetilde{A}$ if and only if $x_k \in A$.*

3. *Let $x = (\rho_n(x_n))_{n\in\mathbb{N}} \in \widetilde{X}$. Then $\rho(x_n) \in \widetilde{U}^*_{xn}$ for each $n \in \mathbb{N}$.*

4. *Let $x = (\rho_n(x_n))_{n\in\mathbb{N}} \in \widetilde{X}$. Then $\rho(x_n) \xrightarrow{\widetilde{\tau}^*} x$.*

5. *Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X such that $x_{n+1} \in U^*_{xn}$ for each $n \in \mathbb{N}$. Then $(\rho_n(x_n))_{n\in\mathbb{N}} \in \widetilde{X}$.*

6. *Let $x \in X$, $n \in \mathbb{N}$ and $A \in \Gamma_n$. Then $\rho(x) \in \widetilde{A}$ if and only if $x \in A$. In particular, $\widetilde{A} \cap \rho(X) = \rho(A)$ for each $A \in \Gamma_n$ and each $n \in \mathbb{N}$ (roughly speaking, $\mathbf{\Gamma}$ is the restriction of $\widetilde{\mathbf{\Gamma}}$ to X). This means that ρ is a fractal-preserving mapping between X and $\widetilde{X}$.*

7. *Let $x, y \in X$ and $n \in \mathbb{N}$. Then $\rho(y) \in \widetilde{U}_{\rho(x)n}$ if and only if $y \in U_{xn}$. In particular, $\widetilde{U}_{\rho(x)n} \cap \rho(X) = \rho(U_{xn})$ for each $x \in X$ and $n \in \mathbb{N}$ (roughly speaking, U_{xn} is the restriction of $\widetilde{U}_{\rho(x)n}$ to X). It also follows that $\widetilde{U}^{-1}_{\rho(x)n} \cap \rho(X) = \rho(U^{-1}_{xn})$ and $\widetilde{U}^{*}_{\rho(x)n} \cap \rho(X) = \rho(U^{*}_{xn})$ for each $x \in X$ and $n \in \mathbb{N}$.*

8. *Let $x = (\rho_n(x_n))_{n\in\mathbb{N}} \in \widetilde{X}$. Then $\widetilde{U}^{*}_{xn} = \widetilde{U}^{*}_{\rho(x_n)n}$.*

9. *$\widetilde{A} = Cl_{\widetilde{\tau}^*}(\rho(A))$ for each $A \in \Gamma_n$.*

10. *$\widetilde{d}(\rho(x), \rho(y)) = d(x, y)$ for each $x, y \in X$ (that is, $\rho : (X, d) \to (\widetilde{X}, \widetilde{d})$ is an isometry). It also follows that $\rho : (X, d^{-1}) \to (\widetilde{X}, \widetilde{d}^{-1})$ and $\rho : (X, d^*) \to (\widetilde{X}, \widetilde{d}^*)$ are isometries.*

11. *$\rho(X)$ is dense in $(\widetilde{X}, \widetilde{d}^*)$.*

12. *If X is T_0, then ρ is injective and, hence, ρ is a fractal embedding.*

13. *$(\widetilde{X}, \widetilde{\mathbf{\Gamma}})$ is T_0.*

14. *If X is not T_0, then $\rho(X)$ is homeomorphic to the T_0-reflection of X.*

Proof. 1. By hypothesis, $U^*_{x_n n} = U^*_{y_n n}$ for each $n \in \mathbb{N}$. Now, let $m \in \mathbb{N}$ and $A \in \Gamma_m$:

$\Rightarrow$) Suppose that $x_m \in A$. Then $y_m \in U^*_{x_m m} \subseteq U^{-1}_{x_m m} = \bigcap_{B\in\Gamma_m, x_m\in B} B$, which implies that $y_m \in A$.

$\Leftarrow$) Conversely, suppose that $y_m \in A$. Then $x_m \in U^*_{y_m m} \subseteq U^{-1}_{y_m m} = \bigcap_{B\in\Gamma_m, y_m\in B} B$, which implies that $x_m \in A$.

2. $\Leftarrow$) If $x_k \in A$, since $x = (\rho_n(x_n))$, we have that $x \in \widetilde{A}$ by definition of elements of $\mathbf{\Gamma}$.

   $\Rightarrow$) If $x \in \widetilde{A}$, there exists $y_n \in X$ such that $x = (\rho_n(y_n))$ with $y_k \in A$. By the previous item, since $y_k \in A$, $x_k \in A$.

3. Let $x = (\rho_n(x_n)) \in \widetilde{X}$. We have that $\rho(x_n) \in \widetilde{U}^*_{xn}$ if and only if $\rho(x_n) \in \widetilde{U}_{xn}$ and $\rho(x_n) \in \widetilde{U}^{-1}_{xn}$.

   - On the one hand, $\rho(x_n) \in \widetilde{U}_{xn}$ if and only if $x \in \widetilde{U}^{-1}_{\rho(x_n)n} = \bigcap_{\widetilde{A}\in\widetilde{\Gamma}_n, \rho(x_n)\in\widetilde{A}} \widetilde{A}$. Let now $\widetilde{A} \in \widetilde{\Gamma}_n$ be such that $\rho(x_n) = (\rho_k(x_n))_{k\in\mathbb{N}} \in \widetilde{A}$. Then $x_n \in A$ by item 2 and, hence, $x \in \widetilde{A}$.

- On the other hand, let $\widetilde{A} \in \widetilde{\Gamma}_n$ be such that $x \in \widetilde{A}$, which implies that, by item 2, $x_n \in A$ and this means that $\rho(x_n) \in \widetilde{A}$. Therefore, $\rho(x_n) \in \bigcap_{\widetilde{A} \in \widetilde{\Gamma}_n, x \in \widetilde{A}} \widetilde{A} = \widetilde{U}_{xn}^{-1}$.

4. Let $x = (\rho_n(x_n)) \in \widetilde{X}$. By the third item, $\rho(x_n) \in \widetilde{U}_{xn}^*$, so that $\widetilde{d}^*(\rho(x_n), x) \leq \frac{1}{2^n}$ for each $n \in \mathbb{N}$, and, hence, $\rho(x_n) \xrightarrow{\tau_{d^*}} x$.

5. By definition of inverse limit it follows that $(\rho_n(x_n))_{n\in\mathbb{N}} \in \widetilde{X}$ if $\phi_n(\rho_{n+1}(x)) = \rho_n(x)$. Moreover, $\phi_n(\rho_{n+1}(x)) = \rho_n(x)$ if and only if $U_{x_{n+1}n}^* = U_{x_n n}^*$. If we prove the last equality, we will have that $(\rho_n(x_n))_{n\in\mathbb{N}} \in \widetilde{X}$.

   Since $x_{n+1} \in U_{x_n n}^*$ it follows that $U_{x_{n+1}n}^* = U_{x_n n}^*$ by Proposition 2.3.

6. Let $x \in X$, $n \in \mathbb{N}$ and $A \in \Gamma_n$.

   Suppose that $x \in A$. Then $\rho(x) = (\rho_i(x))_{i\in\mathbb{N}} \in \widetilde{X}$ and, hence, $\rho(x) \in \widetilde{A}$.

   Conversely, suppose that $\rho(x) \in \widetilde{A}$. Then $\rho(x) = (\rho_i(x))_{i\in\mathbb{N}}$ and, by item 2, $x \in A$.

7. It is clear by Proposition 3.11.

8. By item 3, we have that $\rho(x_n) \in \widetilde{U}_{xn}^*$ for each $n \in \mathbb{N}$. Moreover, by Proposition 2.3.3, it follows that $\widetilde{U}_{xn}^* = \widetilde{U}_{\rho(x_n)n}^*$.

9. Let $A \in \Gamma_n$.

   $\subseteq$) Let $x \in \widetilde{A}$. Then $x = (\rho_i(x_i))_{i\in\mathbb{N}}$ with $x_n \in A$. By item 4, $\rho(x_i) \xrightarrow{\widetilde{\tau}^*} x$. On the other hand, by Lemma 3.13, $x_m \in A$ for each $m \geq n$. We conclude that $x \in Cl_{\widetilde{\tau}^*}(\rho(A))$.

   $\supseteq$) $\rho(A) = \widetilde{A} \cap \rho(X)$ by item 6. Thus, $\rho(A) \subseteq \widetilde{A}$ and, hence, $Cl_{\tau_{d^*}}(\rho(A)) \subseteq Cl_{\tau_{d^*}}(\widetilde{A})$. Since $\widetilde{A}$ is a closed set with respect to τ_{d^*}, $Cl_{\tau_{d^*}}(\widetilde{A}) = \widetilde{A}$ and, consequently, $Cl_{\tau_{d^*}}(\rho(A)) \subseteq \widetilde{A}$.

10. It follows from Proposition 3.11.

11. Let $x = (\rho_n(x_n))_{n\in\mathbb{N}} \in \widetilde{X}$. By item 4, $\rho(x_n) \xrightarrow{\widetilde{\tau}_{d^*}} x$. It follows that $\rho(X)$ is dense in $(\widetilde{X}, \widetilde{d}^*)$.

12. Suppose that X is T_0 and let $\rho(x) = \rho(y)$. Then $\rho_n(x) = \rho_n(y)$ for each $n \in \mathbb{N}$, that is, $U_{xn}^* = U_{yn}^*$ for each $n \in \mathbb{N}$. Suppose that $x \neq y$. Since X is T_0, by Proposition 3.2, it follows that there exist n and $A \in \Gamma_n$ such that $x \in A$ and $y \notin A$ (or $x \notin A$

and $y \in A$). If $x \in A$, then $U^*_{xn} \subseteq A$, that is, $U^*_{yn} \subseteq A$ and this means that $y \in A$, which is a contradiction. Therefore, $x = y$ and ρ is injective.

13. Let $x = (\rho_n(x_n))$ and $y = (\rho_n(y_n))$ with $x \neq y$. Then $U^*_{x_n n} \neq U^*_{y_n n}$ for some $n \in \mathbb{N}$. It follows that $y_n \notin U^*_{x_n n} = U_{x_n n} \cap U^{-1}_{x_n n}$. This implies one of the following facts:

    - $y_n \notin U_{x_n n}$, which implies that $x_n \notin U^{-1}_{y_n n}$ and, hence, there exists $A \in \Gamma_n$ such that $y_n \in A$ and $x_n \notin A$. It follows, by item 2, that $y \in \widetilde{A}$ and $x \notin \widetilde{A}$. Thus, $x \notin \widetilde{U}^{-1}_{yn} = \bigcap_{y \in \widetilde{A}, \widetilde{A} \in \widetilde{\Gamma}_n} \widetilde{A}$ and, consequently, $y \notin \widetilde{U}_{xn}$.
    - $y_n \notin U^{-1}_{x_n n}$, which implies (analogously to the previous case) that $x \notin \widetilde{U}_{yn}$.

    Hence, $\widetilde{X}$ is T_0.

14. Let the equivalence relationship $x \sim y$ if and only if $\overline{\{x\}} = \overline{\{y\}}$. Then $X/\sim$ is the T_0-reflection of X. Next, we show that $f : \rho(X) \to X/\sim$ such that $f(\rho(x)) = [x]$ is an homeomorphism:

    First, we show that f is well defined. Let $x, y \in X$ be such that $\rho(x) = \rho(y)$, then, $U^*_{xn} = U^*_{yn}$ for each $n \in \mathbb{N}$. Thus, $x \in U^*_{yn}$ and $y \in U^*_{xn}$ for each $n \in \mathbb{N}$, which implies that $\overline{\{x\}} = \overline{\{y\}}$, so $[x] = [y]$ and, hence, $f(\rho(x)) = f(\rho(y))$.

    Now, we prove that f is continuous. Let G be an open set in $X/\sim$ and $y \in f^{-1}(G)$. Then there exists $x \in X$ such that $\rho(x) = y$. Moreover, $f(\rho(x)) = [x] \in G$. Let $\Pi : X \to X/\sim$ be the quotient map. Since Π is continuous, $\Pi^{-1}(G)$ is an open set. Therefore, there exists n such that $U_{xn} \subseteq \Pi^{-1}(G)$.

    Next, we see that $\rho(X) \cap \widetilde{U}_{\rho(x)n} \subseteq f^{-1}(G)$. We know that $\rho(X) \cap \widetilde{U}_{\rho(x)n} = \rho(U_{xn})$. Let $z \in \rho(U_{xn})$. Then we can write $z = \rho(z')$ with $z' \in U_{xn}$. Furthermore $z' \in U_{xn} \subseteq \Pi^{-1}(G)$. Consequently, $\Pi(z') \in G$ and $f(z) = f(\rho(z')) = \Pi(z') \in G$. Thus, $z \in f^{-1}(G)$. Therefore, $\rho(X) \cap \widetilde{U}_{\rho(x)n} = \rho(U_{xn}) \subseteq f^{-1}(G)$ and, hence, $f^{-1}(G)$ is a neighborhood of each of its points and, then, it is an open set. Therefore, f is continuous.

    Next, we prove that f is bijective.

    On the one hand, it is clear that f is onto. Let $[x] \in X/\sim$. If we consider $\rho(x)$, we have that $f(\rho(x)) = [x]$.

    On the other hand, let $[x] = [y]$, then:

- $y \in \overline{\{x\}}$.
- $x \in \overline{\{y\}}$.

From the first item it follows that $U_{yn} \cap \{x\} \neq \emptyset$ and, hence, $x \in U_{yn}$ for each $n \in \mathbb{N}$. From the second item, we conclude that $y \in U_{xn}$ for each $n \in \mathbb{N}$. Consequently, $U^*_{xn} = U^*_{yn}$ for each $n \in \mathbb{N}$, which implies that $\rho(x) = \rho(y)$. Therefore, f is injective.

To conclude we show that f^{-1} is continuous. We first recall item 10 in order to remember that ρ is an isometry and, hence, it is continuous. Moreover, $\Pi(x) = \Pi(y)$ implies $\rho(x) = \rho(y)$, as we proved previously. Thus, f^{-1} is continuous due to the universal property of the quotient topology. □

3.3 The bicompletion

First, we recall a well-known result which is useful in order to prove that a metric space is complete.

Lemma 3.17. *Let A be a dense subset of a metric space X. Then X is complete if and only if each Cauchy sequence $(a_n)_{n\in\mathbb{N}}$ with $a_n \in A$ is convergent.*

Now, we can prove that our completion is the bicompletion of the (non-archimedean) quasi-pseudometric induced by the fractal structure.

Proposition 3.18. *If (X, d) is T_0, then $(\widetilde{X}, \widetilde{d^*})$ is the completion of (X, d^*).*

Proof. By Proposition 3.16, we only have to prove that $(\widetilde{X}, \widetilde{d^*})$ is complete.

We will use Lemma 3.17, so let $(\rho(x_n))_{n\in\mathbb{N}}$ be a Cauchy sequence in $(\rho(X), \widetilde{d^*})$. Since ρ is an isometry by Proposition 3.16.10, $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in (X, d^*). Then:

There exists $n_0 \in \mathbb{N}$ such that $d^*(x_p, x_q) < \frac{1}{2}$ for each $p, q \geq n_0$. We define $\sigma(1) = n_0$.

There exists $n_1 \geq \sigma(1)$ such that $d^*(x_p, x_q) < \frac{1}{2^2}$ for each $p, q \geq n_1$. We define $\sigma(2) = n_1$.

By recursion we can define $\sigma : \mathbb{N} \to \mathbb{N}$ such that $d^*(x_{\sigma(p)}, x_{\sigma(q)}) < \frac{1}{2^{\min\{p,q\}}}$ for each p, q and $\sigma(n+1) \geq \sigma(n)$. In particular, σ is increasing and $d^*(x_{\sigma(n+1)}, x_{\sigma(n)}) < \frac{1}{2^n}$. This implies that $x_{\sigma(n+1)} \in U^*_{x_{\sigma(n)}n}$ and, by Proposition 3.16.5, $z = (\rho_n(x_{\sigma(n)})) \in \widetilde{X}$. By

Proposition 3.16.4, $\rho(x_{\sigma(n)}) \xrightarrow{\widetilde{\tau}^*} z$. Since $(\rho(x_{\sigma(n)}))_{n\in\mathbb{N}}$ is convergent, $(\rho(x_n))_{n\in\mathbb{N}}$ is also convergent (each Cauchy sequence (x_n) which has a convergent subsequence $x_{\sigma(n)} \to x$ is convergent and its limit is x). We conclude that $\widetilde{d}^*$ is complete. □

If we take into account that a quasi-pseudometric d is said to be bicomplete if the pseudometric d^* is complete, we get, as an immediate consequence of the previous result, the following one.

Corollary 3.19. *If (X, d) is T_0, then $(\widetilde{X}, \widetilde{d})$ is the bicompletion of (X, d).*

3.4 Other completeness properties of the completion

In this section we find conditions in order to get that the completion constructed in the previous sections is Cantor complete. First, we will need some definitions.

Recall that a cover Γ of a topological space is said to be point finite if each point belong to a finite number of elements of Γ. Γ is said to be locally finite if each point has a neighborhood which only meets a finite number of elements of Γ.

For our purposes, we need a concept that is related to the previous ones, which is the following one.

Definition 3.20. *Let Γ be a covering of X. Γ is said to be cover-finite if the set $\{B \in \Gamma : B \cap A \neq \emptyset\}$ is finite for each $A \in \Gamma$.*

Note that, by the previous definitions, if Γ is a finite cover, then Γ is cover-finite. Moreover, if Γ is cover-finite, it follows that Γ is point finite.

Definition 3.21. *A fractal structure $\mathbf{\Gamma} = \{\Gamma_n : n \in \mathbb{N}\}$ is said to be locally finite (respectively finite, cover-finite) if Γ_n is locally finite (respectively finite, cover-finite) for each $n \in \mathbb{N}$.*

Note that a fractal structure $\mathbf{\Gamma} = \{\Gamma_n : n \in \mathbb{N}\}$ such that each level Γ_n is point finite, is locally finite, since for each $A \in \Gamma_n$ it follows that a point $x \in X$ belongs to A if and only if $U_{xn} \cap A \neq \emptyset$ (and, of course, U_{xn} is a neighborhood of x). It follows that a finite fractal structure is cover-finite and a cover-finite fractal structure is locally finite.

From the definition of a cover-finite fractal structure we have the next result.

Lemma 3.22. *If $\mathbf{\Gamma}$ is a cover-finite fractal structure, then $\{U^*_{xn} : U^*_{xn} \subseteq A\}$ is finite for each $A \in \Gamma_n$ and each $n \in \mathbb{N}$.*

Proof. Let $n \in \mathbb{N}$, $A \in \Gamma_n$ and $x \in X$. By Proposition 2.3, $U^*_{xn} \subseteq A$ or $U^*_{xn} \cap A = \emptyset$. Now, we will show how to construct an injective map between $\{U^*_{xn} : U^*_{xn} \subseteq A\}$ and $\mathcal{P}(\{B \in \Gamma_n : B \cap A \neq \emptyset\})$, whence it will follow that $\{U^*_{xn} : U^*_{xn} \subseteq A\}$ is finite, where $\mathcal{P}(Y)$ is the family of all subsets of Y.

Note that $U^*_{xn} \subseteq A$ if and only if $x \in A$, so let $x \in A$. Then $U^*_{xn} = \bigcap_{x \in B, B \in \Gamma_n} B \setminus \bigcup_{x \notin B, B \in \Gamma_n} B$. Moreover, if $x \in B$, then $B \cap A \neq \emptyset$. Now, we define $\phi : \{U^*_{xn} : U^*_{xn} \subseteq A\} \to \mathcal{P}(\{B \in \Gamma_n : B \cap A \neq \emptyset\})$ given by $\phi(U^*_{xn}) = \{B : x \in B\}$. Now, we show that it is injective.

Let $U^*_{xn} \neq U^*_{yn}$. Then $y \notin U^{-1}_{xn} = \bigcap_{x \in B, B \in \Gamma_n} B$ or $x \notin U^{-1}_{yn}$. Hence, there exists $B \in \Gamma_n$ such that $x \in B$ and $y \notin B$, or $x \notin B$ and $y \in B$. This implies that $\phi(U^*_{xn}) \neq \phi(U^*_{yn})$ and ϕ is injective. □

Now, we can prove that for a cover-finite fractal structure the completion is Cantor complete.

Theorem 3.23. *Let $\mathbf{\Gamma}$ be a cover-finite fractal structure on X such that (X, d) is T_0. Then $(\widetilde{X}, \widetilde{\mathbf{\Gamma}})$ is Cantor complete.*

Proof. First, recall from Proposition 3.1.5 that for each $n \in \mathbb{N}$, $A \in \Gamma_n$ and $x \in X$, $U^*_{xn} \subseteq A$ if and only if $x \in A$.

Let $(\widetilde{A}_n)_{n \in \mathbb{N}}$ be a sequence with $\widetilde{A}_n \in \widetilde{\Gamma}_n$ satisfying $\widetilde{A}_{n+1} \subseteq \widetilde{A}_n$. We prove that $\bigcap_{n \in \mathbb{N}} \widetilde{A}_n \neq \emptyset$. Note that $A_{n+1} \subseteq A_n$, since $\rho(A_n) = \widetilde{A}_n \cap \rho(X)$ by Proposition 3.16.6 and ρ is injective by Proposition 3.16.12.

Let us construct a sequence of points in A_n so that we get an element in $\bigcap_{n \in \mathbb{N}} \widetilde{A}_n$. By Proposition 3.1, $A_1 = \bigcup \{U^*_{x1} : U^*_{x1} \subseteq A_1\}$. By Lemma 3.22, the set $\{U^*_{x1} : U^*_{x1} \subseteq A_1\}$ is finite, so there exists $x_1 \in A$ such that $U^*_{x_1 1} \cap A_n \neq \emptyset$ for each $n \in \mathbb{N}$. Now, $U^*_{x_1 1} \cap A_2 = \bigcup \{U^*_{y2} : y \in U^*_{x_1 1} \cap A_2\}$. Since the set $\{U^*_{y2} : y \in U^*_{x_1 1} \cap A_2\} \subseteq \{U^*_{y2} : U^*_{y2} \subseteq A_2\}$, then it is finite by Lemma 3.22 and nonempty by construction. So there exists $x_2 \in A_2 \cap U^*_{x_1 1}$ such that $U^*_{x_2 2} \cap A_n \neq \emptyset$ for each $n \in \mathbb{N}$. Recursively we define x_n such that $x_n \in A_n \cap U^*_{x_{n-1}, n-1}$ and $U^*_{x_n n} \cap A_k \neq \emptyset$ for each $k \in \mathbb{N}$.

Since $x_{n+1} \in U^*_{x_n n}$, it follows from Proposition 3.16 that $(\rho_i(x_i)) \in \widetilde{X}$ and we also have that $(\rho_i(x_i)) \in \widetilde{A}_n$ for each $n \in \mathbb{N}$, since $x_n \in A_n$ for each $n \in \mathbb{N}$. Therefore, $(\rho_i(x_i)) \in \bigcap_{n\in\mathbb{N}} \widetilde{A}_n$ and, hence, $\widetilde{\mathbf{\Gamma}}$ is Cantor complete. $\square$

The following example shows that being locally finite is not enough to ensure that $\widetilde{\mathbf{\Gamma}}$ is Cantor complete.

Example 3.24. *Let* $\mathbf{\Gamma} = \{\Gamma_n : n \in \mathbb{N}\}$, *where* $\Gamma_n = \{]-\infty, -k] : k \geq n\} \cup \{[k, \infty[: k \geq n\} \cup \{\left[\frac{i}{2^n}, \frac{i+1}{2^n}\right] : i = -n2^n, \ldots, n2^n - 1\}$ *for each* $n \in \mathbb{N}$. *Then* $\mathbf{\Gamma}$ *is a locally finite fractal structure but* $\widetilde{\mathbf{\Gamma}}$ *is not Cantor complete.*

First, we prove that $\mathbf{\Gamma}$ is locally finite. Let $n \in \mathbb{N}$ and $x \in X$. We have the following options:

- $x \in \left[\frac{i}{2^n}, \frac{i+1}{2^n}\right]$ for some i, and, hence, x belong to at most two elements of Γ_n.

- $x \in \cup\{[k, \infty[: k \geq n\}$, which implies that x can only belongs to $[n, \infty[, \ldots [\lfloor x \rfloor, \infty[$ or $\left[\frac{n2^n-1}{2^n}, n\right]$ if $x = n$, which is a finite number of elements of Γ_n, where $\lfloor a \rfloor$ is the largest integer not greater than x.

- $x \in \cup\{]-\infty, -k] : k \geq n\}$, which implies that x can only belongs to $]-\infty, -n], \ldots]-\infty, -\lfloor x \rfloor]$ or $\left[-n, \frac{-n2^n+1}{2^n}\right]$ if $x = -n$, which is a finite number of elements of Γ_n.

So Γ_n is locally finite by the previous discussion on the relation between locally finite and point finite for a fractal structure.

Now, we show that $\widetilde{\mathbf{\Gamma}}$ is not Cantor complete.

Let $A_n =]-\infty, -n] \in \Gamma_n$. Then $\widetilde{A}_n \in \widetilde{\mathbf{\Gamma}}_n$ and, by Lemma 3.14, since $A_{n+1} \subseteq A_n$, it follows that $\widetilde{A}_{n+1} \subseteq \widetilde{A}_n$. Let $x = (\rho_n(x_n)) \in \widetilde{X}$ and suppose that $x \in \bigcap_{n\in\mathbb{N}} \widetilde{A}_n$. Then $x \in \widetilde{A}_n$ for each $n \in \mathbb{N}$. Moreover, by Proposition 3.16 it follows that $x_n \in A_n$ for each $n \in \mathbb{N}$ and, hence, $U^*_{x_n n} = \{x_n\}$ if $x_n = -n$, $U^*_{x_n n} =]\lfloor x_n \rfloor - 1, \lfloor x_n \rfloor]$ if $x_n \in \mathbb{Z}$ or $U^*_{x_n n} =]\lfloor x_n \rfloor, \lfloor x_n \rfloor + 1]$ otherwise. Since $x = (\rho_n(x_n)) \in \widetilde{X}$, it follows that $x_k \in U^*_{x_n n}$ for each $k \geq n$. For $k \geq -\lfloor x_n \rfloor + 1$ we get that $x_k \in A_k =]-\infty, -k] \subseteq]-\infty, \lfloor x_n \rfloor - 1]$ which is a contradiction with $x_k \in U^*_{x_n n}$. Consequently, $\bigcap \widetilde{A}_n = \emptyset$ and $\widetilde{\mathbf{\Gamma}}$ is not Cantor complete.

3.5 Uniqueness of the bicompletion

Next, we give a theorem of uniqueness related to the bicompletion of a fractal structure. First, we give a definition of bicompletion.

Definition 3.25. *Let $\mathbf{\Gamma}$ be a fractal structure on X, $\mathbf{\Delta}$ a fractal structure on Y, $i : X \to Y$ a fractal embedding and suppose that $d_{\mathbf{\Delta}}$ is bicomplete, $i(X)$ is dense in $(Y, d^*_{\mathbf{\Delta}})$ and Y is T_0. Then we say that $(Y, \mathbf{\Delta}, i)$ is a bicompletion of $(X, \mathbf{\Gamma})$.*

Corollary 3.26. *If $(X, \mathbf{\Gamma})$ is T_0, then $(\widetilde{X}, \widetilde{\mathbf{\Gamma}}, \rho)$ is a bicompletion of $(X, \mathbf{\Gamma})$.*

Proof. It follows from Sections 3.2 and 3.3. □

The next result is about the uniqueness of the bicompletion of a fractal structure.

Theorem 3.27. *If $(Y, \mathbf{\Delta}, i)$ is a bicompletion of $(X, \mathbf{\Gamma})$, then there exists $I : \widetilde{X} \to Y$, a fractal isomorphism such that $I \circ \rho = i$. Roughly speaking, the bicompletion is unique up to fractal isomorphism.*

Proof. 1. Definition of I.

In order to define I, let $x = (\rho_n(x_n)) \in \widetilde{X}$. On the one hand, since i is fractal preserving and injetive, $y \in U_{xn}$ if and only if $i(y) \in U_{i(x)n}$ by Remark 3.12.

On the other hand, since $x \in \widetilde{X}$, $x_{n+1} \in U^*_{x_n n}$ for each $n \in \mathbb{N}$, which implies that $i(x_{n+1}) \in U^*_{i(x_n)n}$ by Remark 3.12. This means that $(i(x_n))$ is a Cauchy sequence in $(Y, d^*_{\mathbf{\Delta}})$. Thus, there exists $y \in Y$ such that $(i(x_n))$ converges to y in $(Y, d^*_{\mathbf{\Delta}})$.

Now, we can define $I(x) = y$.

2. I is well defined.

Let $(\rho_n(x_n)) = (\rho_n(y_n)) \in \widetilde{X}$. Then $U^*_{x_n n} = U^*_{y_n n}$ for each $n \in \mathbb{N}$ and, hence, $y_n \in U^*_{x_n n}$ for each $n \in \mathbb{N}$.

On the one hand, since $(\rho_n(x_n)) \in \widetilde{X}$, $x_{n+1} \in U^*_{x_n n}$ for each $n \in \mathbb{N}$. Consequently, $i(x_{n+1}) \in U^*_{i(x_n)n}$ by Remark 3.12, which implies that $(i(x_n))$ is a Cauchy sequence in $(Y, d^*_{\mathbf{\Delta}})$. Thus, there exists $z \in Y$ such that $i(x_n) \to z$ in $(Y, d^*_{\mathbf{\Delta}})$ (and this limit is unique, since $(Y, d^*_{\mathbf{\Delta}})$ is T_2). Then $I((\rho_n(x_n))) = z$.

On the other hand, since $y_n \in U^*_{x_n n}$, $d^*(i(y_n), i(x_n)) \leq \frac{1}{2^n}$ for each $n \in \mathbb{N}$ and, hence, $i(y_n) \to z$ in $(Y, d^*_{\mathbf{\Delta}})$. Thus, $I((\rho_n(x_n))) = z = I((\rho_n(y_n)))$.

3. $I \circ \rho = i$.

   Let $x \in X$. Then $\rho(x) = (\rho_n(x))$. Now, since $x \in U^*_{xn}$, $i(x) \in U^*_{i(x)n}$. This means that $(i(x))$ is a Cauchy sequence in $(Y, d^*_{\mathbf{\Delta}})$. Moreover, $i(x) \xrightarrow{d^*} i(x)$. Hence, $I \circ \rho = i$.

4. $I(\widetilde{\mathbf{\Gamma}}) = \mathbf{\Delta}$.

   Let $n \in \mathbb{N}$ and $A \in \Gamma_n$. Then we claim that $I(\widetilde{A}) \in \Delta_n$. In fact, $i(A) = B \cap i(X)$ for some $B \in \Delta_n$, so let us prove that $I(\widetilde{A}) = B$.

   $\subseteq$) Let $x = (\rho_k(x_k)) \in \widetilde{A}$. Then $x_n \in A$ and, by Lemma 3.13, $x_k \in A$ for each $k \geq n$. It follows that $i(x_k) \in i(A) \subseteq B$ for each $k \geq n$. Thus, $I(x) \in \overline{B}^* = B$.

   $\supseteq$) Now, let $y \in B$. By density of $i(X)$, for each $k \in \mathbb{N}$ there exists $y_k \in i(X)$ such that $y_k \in U^*_{yk}$. It follows that $y_{k+1} \in U^*_{y,k+1} \subseteq U^*_{yk} = U^*_{y_k k}$ and it is also clear that $y_k \xrightarrow{d^*_\Delta} y$. Then there exists $x_k \in X$ such that $y_k = i(x_k)$ for each $k \in \mathbb{N}$. It follows that $i(x_{k+1}) \in U^*_{i(x_k)k}$, so, by Remark 3.12, we have that $x_{k+1} \in U^*_{x_k k}$, and, by Proposition 3.16, $x = (\rho_k(x_k)) \in \widetilde{X}$. Note that, by definition of I, we have that $I(x) = y$.

   Since $U^*_{y_n n} = U^*_{yn} \subseteq B$, then $y_n \in B$ and $i(x_n) = y_n \in B \cap i(X) = i(A)$, which implies that $x_n \in A$ (since i is injective) and, hence, $x = (\rho_k(x_k)) \in \widetilde{A}$. Therefore, $y = I(x) \in I(\widetilde{A})$.

   Conversely, given $B \in \Delta_n$, since i is fractal preserving, there exists $A \in \Gamma_n$ such that $B \cap i(X) = i(A)$. It follows that $B = I(\widetilde{A})$, so $B \in I(\widetilde{\Gamma}_n)$. Note that, since B is open in (Y, d^*) and $i(X)$ is dense in (Y, d^*), $B \cap i(X) \neq \emptyset$.

5. I is injective.

   Let $x = (\rho_n(x_n))$ and $y = (\rho_n(y_n))$ with $I(x) = I(y)$. Then $i(x_n) \xrightarrow{d^*_\Delta} I(x)$ and $i(y_n) \xrightarrow{d^*_\Delta} I(y)$. Given $n \in \mathbb{N}$, there exists $k \geq n$ such that $i(x_k) \in U^*_{I(x)n}$ and $i(y_k) \in U^*_{I(y)n}$. It follows that $U^*_{i(x_k)n} = U^*_{I(x)n} = U^*_{I(y)n} = U^*_{i(y_k)n}$. Hence, $i(y_k) \in U^*_{i(x_k)n}$, and, by Remark 3.12, $y_k \in U^*_{x_k n}$, so $U^*_{y_n n} = U^*_{y_k n} = U^*_{x_k n} = U^*_{x_n n}$. We conclude that $U^*_{y_n n} = U^*_{x_n n}$ for each $n \in \mathbb{N}$, and, hence, $\rho_n(x_n) = \rho_n(y_n)$ for each $n \in \mathbb{N}$, so $x = y$. Therefore, I is injective.

6. I is onto.

Let $y \in Y$. By density of $i(X)$, for each $k \in \mathbb{N}$ there exists $y_k \in i(X)$ such that $y_k \in U^*_{yk}$. It follows that $y_{k+1} \in U^*_{y,k+1} \subseteq U^*_{yk} = U^*_{y_k k}$ and it is also clear that $y_k \xrightarrow{d^*_\Delta} y$. Then there exists $x_k \in X$ such that $y_k = i(x_k)$ for each $k \in \mathbb{N}$. It follows that $i(x_{k+1}) \in U^*_{i(x_k)k}$, so, by Remark 3.12, we have that $x_{k+1} \in U^*_{x_k k}$, and, by Proposition 3.16, $x = (\rho_k(x_k)) \in \widetilde{X}$. Note that, by definition of I, we have that $I(x) = y$. Therefore, I is onto.

□

Chapter 4

Generating a probability measure on the completion of a fractal structure

The content of this chapter corresponds to [32].

According to the main goal of this part of the work, we will consider a space with a fractal structure and a pre-measure defined on some families of subsets determined by the fractal structure. Moreover, we will suppose, in the rest of this part, that this space is T_0 with respect to the topology induced by the fractal structure (note that this is equivalent to d^* being a metric and, consequently, d^* is an ultrametric). Recall that the property T_0 can be characterized in terms of the fractal structure according to Proposition 3.2. Indeed, X is T_0 if and only if for each $x, y \in X$ with $x \neq y$, there exist $n \in \mathbb{N}$ and $A \in \Gamma_n$ such that A contains one of the points (x or y) but not the other one.

Under the assumptions above, we will show that this pre-measure can be extended to a measure on the Borel σ-algebra of the completion of the space and, also, that this measure is unique. That construction is given by two key elements: the iterative character of the fractal structure and the use of the completion of the space. What is more, that construction can be made from the collection of balls with respect to the ultrametric induced by the fractal structure (see Section 4.1) or from the elements of the fractal structure (see Section 4.2).

4.1 Defining a measure on the completion

In addition to the assumption of X being T_0 (which is equivalent to d^* being an ultrametric), we will assume in the rest of this chapter that $G_n = \{U^*_{xn} : x \in X\}$ is countable for each $n \in \mathbb{N}$. Note that this is equivalent to d^* being separable.

Examples of fractal structures with G_n countable for each $n \in \mathbb{N}$ include finite fractal structures and locally finite countable fractal structures (that is, fractal structures in which each level is a countable covering and each point belongs only to a finite number of elements of level n of the fractal structure for each $n \in \mathbb{N}$).

In this section we show how to construct a probability measure on $\widetilde{X}$ from a pre-measure defined on the sets $\rho_n(x) = U^*_{xn}$.

Before doing this construction, it is also worth noting that there exists a correspondence between infinite trees and Polish ultrametric spaces (that is, ultrametric spaces which are separable and complete). As a consequence, the completion of a space with a fractal structure can be seen as a tree. For further reference about ultrametric trees see, for example, [36] and [37]. Moreover, this procedure is similar to the Carathéodory's construction for which [25], [47] and [51] are good references.

Due to the properties of the isometry ρ that we showed in the previous chapter (see Proposition 3.16), in order to simplify the notation, we will use the identifications $x \equiv \rho(x)$, $X \equiv \rho(X)$, $U_{xn} \equiv \rho(U_{xn})$ and so on.

Let $\mathbf{\Gamma} = \{\Gamma_n : n \in \mathbb{N}\}$ be a fractal structure on X and let us denote by $\mathcal{G} = \bigcup_{n\in\mathbb{N}} G_n = \{U^*_{xn} : x \in X, n \in \mathbb{N}\}$. Let ω be a function $\omega : \mathcal{G} \to [0,1]$ (a pre-measure). We say that ω satisfies the mass distribution conditions if:

1. $\sum_{\rho_1(x)\in G_1} \omega(\rho_1(x)) = 1$.
2. $\omega(\rho_n(x)) = \sum_{\rho_{n+1}(y)\in G_{n+1}, \rho_n(y)=\rho_n(x)} \omega(\rho_{n+1}(y))$ for each $n \in \mathbb{N}$ and each $\rho_n(x) \in G_n$.

Mass distribution conditions can be written, alternatively, as

1. $\sum_{x\in X} \omega(U^*_{x1}) = 1$.
2. $\omega(U^*_{xn}) = \sum_{y\in U^*_{xn}} \omega(U^*_{y,n+1})$ for each $n \in \mathbb{N}$ and each $x \in X$.

Take into account that, in that case, we are using the agreement introduced in Remark 4.1 in order to simplify the notation.

Remark 4.1. *In the sum* $\sum_{x\in A}\omega(U^*_{xn})$ *the index set does not refer to the points of* A *but it does to* $\{U^*_{xn}\in G_n : x\in A\}$*, since many points of* A *have the same* U^*_{xn}*.*

Remark 4.2. *Note that the fact that* $\sum_{\rho_n(x)\in G_n}\omega(\rho_n(x))=1$ *for each* $n\in\mathbb{N}$ *follows from the mass distribution conditions.*

Roughly speaking, the mass distribution conditions let us distribute the pre-measure along the subsets of G_n so that everything works fine from the point of view of the σ-additivy. Indeed, since G_1 is a disjoint collection of sets, the first condition ensures that the sum of the pre-measure of the balls of radius 1 coincides with the whole mass we want to distribute ($=1$). Now, since G_1 can be decomposed into a countable collection of balls in G_2 and, in general, $G_{n+1}\prec\prec G_n$ for each $n\in\mathbb{N}$, it does make sense to impose that the mass in some U_{xn} can be expressed as the sum of mass of the balls of G_{n+1} which are contained in U_{xn}. For example, in Figure 4.1 a distribution of the mass can be seen for the balls of the ultrametric induced by the natural fractal structure on $[0,1]$.

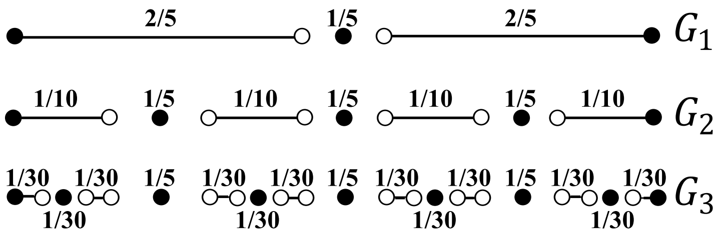


Figure 4.1: Mass distribution conditions on G_n for each $n=1,2,3$ and $([0,1],\mathbf{\Gamma})$, where $\mathbf{\Gamma}$ is the natural fractal structure

Now, let $\widetilde{\mathcal{G}}=\{\widetilde{\rho}_n(x):x\in\widetilde{X},n\in\mathbb{N}\}=\{\widetilde{U}^*_{xn}:x\in\widetilde{X},n\in\mathbb{N}\}$, where $\widetilde{\rho}_n(x)=\widetilde{U}^*_{xn}$. Note that, by Proposition 3.16.8, $\widetilde{\mathcal{G}}=\{\widetilde{U}^*_{xn}:x\in X,n\in\mathbb{N}\}$, and consider the function $\widetilde{\omega}:\widetilde{\mathcal{G}}\to[0,1]$ defined by $\widetilde{\omega}(\widetilde{U}^*_{xn})=\omega(U^*_{xn})$. It is straightforward to check that $\widetilde{\omega}$ satisfies the mass distribution conditions for $\widetilde{X}$.

Let μ be the outer measure on X given by Method I (Theorem 2.12) from $\mathcal{G}$ and ω. Then μ is defined for a subset A of X as follows: $\mu(A)=\inf\{\sum_{i=1}^{\infty}\omega(U^*_{x_in_i}):A\subseteq\bigcup U^*_{x_in_i}\}$, where the infimum is taken over all countable coverings of A by elements of $\mathcal{G}$. Analogously, we define $\widetilde{\mu}$ as the outer measure on $\widetilde{X}$ given by Method I from $\widetilde{\mathcal{G}}$ and $\widetilde{\omega}$.

Note that μ is the restriction of $\widetilde{\mu}$ to X, as the next result shows.

Remark 4.3. *$\widetilde{\mu}(A) = \mu(A)$ for each $A \subseteq X$.*

Proof. Let $A \subseteq X$.

$\leq$) Suppose that $A \subseteq \bigcup U^*_{x_i n_i}$ with $x_i \in X$. Since $U^*_{x_i n_i} \subseteq \widetilde{U}^*_{x_i n_i}$, it follows that $A \subseteq \bigcup \widetilde{U}^*_{x_i n_i}$. Moreover, the definition of $\widetilde{\omega}$ gives us that $\sum_{i=1}^{\infty} \omega(U^*_{x_i n_i}) = \sum_{i=1}^{\infty} \widetilde{\omega}(\widetilde{U}^*_{x_i n_i})$ so we conclude that $\widetilde{\mu}(A) \leq \mu(A)$.

$\geq$) Suppose that $A \subseteq \bigcup \widetilde{U}^*_{x_i n_i}$, with $x_i \in X$. Since $U^*_{x_i n_i} = \widetilde{U}^*_{x_i n_i} \cap X$, it follows that $A \subseteq \bigcup U^*_{x_i n_i}$. Moreover, the definition of $\widetilde{\omega}$ gives us that $\sum_{i=1}^{\infty} \omega(U^*_{x_i n_i}) = \sum_{i=1}^{\infty} \widetilde{\omega}(\widetilde{U}^*_{x_i n_i})$ so we conclude that $\mu(A) \leq \widetilde{\mu}(A)$. □

In fact, by the mass distribution conditions on ω (respectively on $\widetilde{\omega}$), we will show in the next proposition that μ (respectively $\widetilde{\mu}$) coincides with the outer measure provided by Method II.

Proposition 4.4. *μ is a metric outer measure on (X, d^*).*

Proof. Given $\varepsilon > 0$, let μ_ε be the outer measure provided by Method I, determined by ω using the family $\mathcal{G}_\varepsilon = \{A \in \mathcal{G} : \operatorname{diam}(A) \leq \varepsilon\}$, where the diameter is considered with respect to the metric d^*. Define $\overline{\mathcal{M}}(E) = \lim_{\varepsilon \to 0} \mu_\varepsilon(E)$. Then $\overline{\mathcal{M}}$ is the outer measure provided by Method II which, by Theorem 2.14, is a metric outer measure. So we only have to prove that $\overline{\mathcal{M}} = \mu$.

Let $0 < \varepsilon \leq \delta$. Then, $\mathcal{G}_\varepsilon \subseteq \mathcal{G}_\delta$ and, by Proposition 2.13, we have that $\mu_\varepsilon(E) \geq \mu_\delta(E)$ for each $E \subseteq \widetilde{X}$.

On the other hand, by using equalities

$$\omega(\rho_n(x)) = \sum_{\phi_n(\rho_{n+1}(y)) = \rho_n(x)} \omega(\rho_{n+1}(y))$$

and

$$\rho_n(x) = \bigcup_{\phi_n(\rho_{n+1}(y)) = \rho_n(x)} \rho_{n+1}(y)$$

recursively, it is clear that, given $m \in \mathbb{N}$ such that $\frac{1}{2^m} < \varepsilon$, we can replace the sets U^*_{xn} with $\varepsilon < \operatorname{diam}(U^*_{xn}) \leq \delta$ by the family $\{U^*_{ym} : y \in U^*_{xn}\}$, since $\operatorname{diam}(U^*_{ym}) \leq \frac{1}{2^m} < \varepsilon$. We can conclude that $\mu_\varepsilon = \mu_\delta$ and, hence, $\overline{\mathcal{M}} = \mu$ (since $\mu = \mu_1 = \overline{\mathcal{M}}$). □

From the previous proposition follows that $\widetilde{\mu}$ is a metric outer measure on $(\widetilde{X}, \widetilde{d}^*)$.

Note that $\widetilde{\tau} \subseteq \widetilde{\tau}^*$, so the Borel σ-algebra of $\widetilde{\tau}$ is contained in the Borel σ-algebra of $\widetilde{\tau}^*$.

In fact, $\sigma(\widetilde{\tau}) = \sigma(\widetilde{\tau}^*)$, which is a consequence of the next result.

Remark 4.5. $\sigma(\tau_{d^*}) = \sigma(\tau_d)$.

Proof. $\supseteq$) It is clear that $\tau_d \subseteq \tau_{d^*}$, which implies that $\sigma(\tau_d) \subseteq \sigma(\tau_{d^*})$.

$\subseteq$) Since Γ_n is a closure-preserving closed cover of X (recall that a family of subsets of X is said to be closure preserving if the closure of the union of any subfamily is equal to the union of the closure of each member of the subfamily) for each $n \in \mathbb{N}$ (see [5, Prop. 2.4]), it is clear, by definition of U_{xn}, that it is open in τ_d and it is clear, by Proposition 2.3, that U_{xn}^{-1} is closed in τ_d. Hence, $U_{xn}, U_{xn}^{-1} \in \sigma(\tau_d)$ and, consequently, $U_{xn} \cap U_{xn}^{-1} = U_{xn}^* \in \sigma(\tau_d)$. Now, let G be an open set in τ_{d^*}. Then G can be written as a countable union (here we are using our global assumption that $\mathcal{G}$ is countable) as follows: $G = \bigcup_{n\in\mathbb{N}} \{U_{xn}^* : x \in G, U_{xn}^* \subseteq G\}$. Hence, $\sigma(\tau_{d^*}) \subseteq \sigma(\tau_d)$. □

Corollary 4.6. *μ is a measure on the Borel σ-algebras of (X, τ^*) and (X, τ).*

Proof. It is clear, by Proposition 4.4, since a metric outer measure is a measure on the Borel σ-algebra (see [22, Th. 5.2.6]). □

Next, we prove some properties of $\widetilde{\mu}$.

Proposition 4.7. *$\widetilde{\mu}$ is an extension of $\widetilde{\omega}$. In fact, $\widetilde{\mu}(\widetilde{U}_{xn}^*) = \omega(U_{xn}^*) = \widetilde{\omega}(\widetilde{U}_{xn}^*)$ for each $x \in X$, $n \in \mathbb{N}$ and $\widetilde{\mu}(\widetilde{U}_{xn}^*) = \widetilde{\omega}(\widetilde{U}_{xn}^*)$ for each $x \in \widetilde{X}$, $n \in \mathbb{N}$.*

Proof. 1. Let $x \in X$ and $n \in \mathbb{N}$. First, note that $\widetilde{\omega}(\widetilde{U}_{xn}^*) = \omega(U_{xn}^*)$ by definition of $\widetilde{\omega}$. So, let us prove the equality $\widetilde{\mu}(\widetilde{U}_{xn}^*) = \widetilde{\omega}(\widetilde{U}_{xn}^*)$.

On the one hand, it is clear that $\widetilde{\mu}(\widetilde{U}_{xn}^*) \leq \widetilde{\omega}(\widetilde{U}_{xn}^*)$ by using the first point of Theorem 2.12 (remember that $\widetilde{\mu}$ is the outer measure given by Method I).

On the other hand, suppose that $\widetilde{\mu}(\widetilde{U}_{xn}^*) < \widetilde{\omega}(\widetilde{U}_{xn}^*)$. Then, by definition of $\widetilde{\mu}$, there exists a covering $\mathcal{A}' \subseteq \widetilde{\mathcal{G}}$ of $\widetilde{U}_{xn}^*$ such that $\sum_{A\in\mathcal{A}'} \widetilde{\omega}(A) < \widetilde{\omega}(\widetilde{U}_{xn}^*)$. If there exists $\widetilde{U}_{yk}^* \in \mathcal{A}'$ such that $y \notin \widetilde{U}_{xn}^*$, then $\widetilde{U}_{yk}^* \cap \widetilde{U}_{xn}^* = \emptyset$, so by defining $\mathcal{A} = \{\widetilde{U}_{yk}^* \in \mathcal{A}' : y \in \widetilde{U}_{xn}^*\}$, we have that $\mathcal{A} \subseteq \mathcal{A}' \subseteq \widetilde{\mathcal{G}}$, $\mathcal{A}$ is a covering of $\widetilde{U}_{xn}^*$ and

$\sum_{A\in\mathcal{A}}\widetilde{\omega}(A) \leq \sum_{A\in\mathcal{A}'}\widetilde{\omega}(A) < \widetilde{\omega}(\widetilde{U}^*_{xn})$, so we can assume that $y \in \widetilde{U}^*_{xn}$ for each $\widetilde{U}_{yk} \in \mathcal{A}$.

Since $\widetilde{\omega}(\widetilde{\rho}_n(x)) = \sum_{\widetilde{\phi}_n(\widetilde{\rho}_{n+1}(y))=\widetilde{\rho}_n(x)}\widetilde{\omega}(\widetilde{\rho}_{n+1}(y))$ and $\widetilde{\rho}_n(x) = \bigcup_{\widetilde{\phi}_n(\widetilde{\rho}_{n+1}(y))=\widetilde{\rho}_n(x)}\widetilde{\rho}_{n+1}(y)$, there exists $y \in U^*_{xn}$ such that $\sum\{\widetilde{\omega}(U) : U \in \mathcal{A}, U \subseteq \widetilde{U}^*_{y,n+1}\} < \widetilde{\omega}(\widetilde{U}^*_{y,n+1})$. Define $x_{n+1} = y$. By defining $x_k = x$ for each $k \leq n$ and x_k for $k > n+1$ by recursion analogously to the definition of x_{n+1}, we can define a sequence (x_k) such that $x_{k+1} \in U^*_{x_k k}$ for each $k \in \mathbb{N}$ and $\sum\{\widetilde{\omega}(U) : U \in \mathcal{A}, U \subseteq \widetilde{U}^*_{x_k k}\} < \widetilde{\omega}(\widetilde{U}^*_{x_k k})$ for each $k \geq n$.

By Proposition 3.16.5, it follows that $z = (\rho_n(x_n))_{n\in\mathbb{N}} \in \widetilde{X}$. On the other hand, since $x_k \in \widetilde{U}^*_{xn}$ for each $k \in \mathbb{N}$, $\widetilde{U}^*_{xn}$ is closed in $(\widetilde{X}, \widetilde{d}^*)$ (Proposition 2.3) and (x_k) converges to z in $(\widetilde{X}, \widetilde{d}^*)$ (Proposition 3.16.4), it follows that $z \in \widetilde{U}^*_{xn}$. Since $\mathcal{A}$ is a covering of $\widetilde{U}^*_{xn}$, there exists $\widetilde{U}^*_{yk} \in \mathcal{A}$ (with $y \in \widetilde{U}^*_{xn}$) such that $z \in \widetilde{U}^*_{yk}$. Since $y \in \widetilde{U}^*_{xn}$, it follows that $k \geq n+1$. By Proposition 2.3, $\widetilde{U}^*_{zk} = \widetilde{U}^*_{yk}$, and, by Proposition 3.16.8, $\widetilde{U}^*_{zk} = \widetilde{U}^*_{x_k k}$ and, hence, $\widetilde{U}^*_{x_k k} = \widetilde{U}^*_{yk} \in \mathcal{A}$, but then $\sum\{\widetilde{\omega}(U) : U \in \mathcal{A}, U \subseteq \widetilde{U}^*_{x_k k}\} \geq \widetilde{\omega}(\widetilde{U}^*_{x_k k})$, a contradiction with the definition of the sequence (x_k).

We conclude that $\widetilde{\mu}(\widetilde{U}^*_{xn}) = \widetilde{\omega}(\widetilde{U}^*_{xn})$.

2. Now, let $x = (\rho_k(x_k))_{k\in\mathbb{N}} \in \widetilde{X}$ and $n \in \mathbb{N}$. Note that, by Proposition 3.16.8, $\widetilde{U}^*_{xn} = \widetilde{U}^*_{x_n n}$. By the previous item, $\widetilde{\omega}(\widetilde{U}^*_{xn}) = \widetilde{\omega}(\widetilde{U}^*_{x_n n}) = \widetilde{\mu}(\widetilde{U}^*_{x_n n}) = \widetilde{\mu}(\widetilde{U}^*_{xn})$.

□

Proposition 4.8. $\widetilde{\mu}(\widetilde{X}) = 1$ *and, hence,* $\widetilde{\mu}$ *is a probability measure on* $\widetilde{X}$.

Proof. By the mass distribution conditions, it is clear that $\widetilde{\mu}(\widetilde{X}) = \widetilde{\mu}(\bigcup_{x\in\widetilde{X}}\widetilde{U}^*_{x1}) = \sum\widetilde{\mu}(\widetilde{U}^*_{x1}) = \sum\widetilde{\omega}(\widetilde{U}^*_{x1}) = \sum\omega(U^*_{x1}) = 1$. □

Proposition 4.9. *Let* $A \in \Gamma_n$. *Then* $\widetilde{\mu}(\widetilde{A}) = \sum\omega(U^*_{xn}) = \sum\widetilde{\omega}(\widetilde{U}^*_{xn})$, *where the first sum is on the family* $\{U^*_{xn} : x \in A\}$ *and the second one is on the family* $\{\widetilde{U}^*_{xn} : x \in A\} = \{\widetilde{U}^*_{xn} : x \in \widetilde{A}\}$.

Proof. Indeed, by Proposition 2.3 and Proposition 3.16.2, for each $m \geq n$, the family $\mathcal{A} = \{\widetilde{U}^*_{xm} : x \in A\}$ is a partition of $\widetilde{A}$, so by Proposition 4.7 and Corollary 4.6 (note that $\widetilde{A}$ is closed in $\widetilde{d}^*$, since it is closed in $\widetilde{d}$), $\widetilde{\mu}(\widetilde{A}) = \sum_{\mathcal{A}}\widetilde{\mu}(\widetilde{U}^*_{xm}) = \sum_{\mathcal{A}}\widetilde{\omega}(\widetilde{U}^*_{xm})$ and, by definition of $\widetilde{\omega}$, $\omega(U^*_{xm}) = \widetilde{\omega}(\widetilde{U}^*_{xm})$. □

Now, we look for a simpler way to calculate the measure of an open or closed set in $(\widetilde{X}, \widetilde{d}^*)$ by using the pre-measure $\widetilde{\omega}$.

Lemma 4.10. *Let F be a closed set in (X, d^*). Then $F = \bigcap_{n\in\mathbb{N}} U_n^*(F)$, where $U_n^*(F) = \bigcup_{x\in F} U_{xn}^*$.*

Proof. $\subseteq$) It is clear that $F \subseteq \bigcap_{n\in\mathbb{N}} U_n^*(F)$.

$\supseteq$) Now, let $x \in \bigcap_{n\in\mathbb{N}} U_n^*(F)$. Then, for each $n \in \mathbb{N}$, there exists $x_n \in F$ such that $x \in U_{x_n n}^*$ and, hence, $d^*(x, x_n) \leq \frac{1}{2^n}$. Then $(x_n)_{n\in\mathbb{N}}$ is a sequence in F which converges to x in (X, d^*) and, since F is closed in (X, d^*), then $x \in F$. □

The next result allows us to calculate the measure of a closed set in $(\widetilde{X}, \widetilde{d}^*)$ by levels.

Proposition 4.11. *Let F be a closed set in $(\widetilde{X}, \widetilde{d}^*)$. Then $\widetilde{\mu}(F) = \lim \widetilde{\mu}_n(F)$, where $\widetilde{\mu}_n(F) = \sum \widetilde{\omega}(\widetilde{U}_{xn}^*)$ and the sum is on the family $\{\widetilde{U}_{xn}^* : x \in F\}$.*

Proof. By Lemma 4.10, $F = \bigcap_{n\in\mathbb{N}} \widetilde{U}_n^*(F)$, and it is clear that $\widetilde{U}_{n+1}^*(F) \subseteq \widetilde{U}_n^*(F)$. Moreover, $\widetilde{\mu}_n(F) = \sum_{\mathcal{A}} \widetilde{\omega}(\widetilde{U}_{xn}^*) = \widetilde{\mu}(\bigcup_{\mathcal{A}} \widetilde{U}_{xn}^*) = \widetilde{\mu}(\widetilde{U}_n^*(F))$, where $\mathcal{A} = \{\widetilde{U}_{xn}^* : x \in F\}$. Since $\widetilde{\mu}$ is a measure (and, hence, continuous from above), then $\widetilde{\mu}_n(F) = \widetilde{\mu}(\widetilde{U}_n^*(F)) \to \widetilde{\mu}(F)$. □

Next, we introduce a proposition in order to calculate the measure of an open set in $(\widetilde{X}, \widetilde{d}^*)$ by levels.

Proposition 4.12. *Let O be an open set in $(\widetilde{X}, \widetilde{d}^*)$. Then $\widetilde{\mu}(O) = \lim \widetilde{\mu}_n(O)$, where $\widetilde{\mu}_n(O) = \sum_{\widetilde{U}_{xn}^* \in \widetilde{\mathcal{G}}_n; x \in O_n} \widetilde{\omega}(\widetilde{U}_{xn}^*)$ and $O_n = \{x \in O : \widetilde{U}_{xn}^* \subseteq O\}$.*

Proof. Let O be an open set in $(\widetilde{X}, \widetilde{d}^*)$. Then we can write $O = \bigcup_{n\in\mathbb{N}} \widetilde{U}_n^*(O_n)$, where $O_n = \{x \in O : \widetilde{U}_{xn}^* \subseteq O\}$. It is clear that $\widetilde{U}_n^*(O_n) \subseteq \widetilde{U}_{n+1}^*(O_{n+1})$. Moreover, $\widetilde{\mu}_n(O) = \sum_{O_n} \widetilde{\omega}(\widetilde{U}_{xn}^*) = \widetilde{\mu}(\bigcup_{O_n} \widetilde{U}_{xn}^*) = \widetilde{\mu}(\widetilde{U}_n^*(O_n))$. Since $\widetilde{\mu}$ is a measure (and, hence, continuous below), it follows that $\widetilde{\mu}_n(O) = \widetilde{\mu}(\widetilde{U}_n^*(O_n)) \to \widetilde{\mu}(O)$. □

Now, we prove the uniqueness of the measure.

Proposition 4.13. *Let δ be a measure satisfying $\delta(\widetilde{U}_{xn}^*) = \widetilde{\omega}(\widetilde{U}_{xn}^*)$ for each $x \in \widetilde{X}$ and $n \in \mathbb{N}$. Then $\delta = \widetilde{\mu}$ on the Borel σ-algebra of $(\widetilde{X}, \widetilde{d}^*)$.*

Proof. Let $\mathfrak{S} = \sigma(\mathfrak{A})$, where $\mathfrak{A} = \{\widetilde{U}_n^*(F) : F \subseteq \widetilde{X}, n \in \mathbb{N}\}$. Then $\mathfrak{A}$ is an algebra. Indeed, for each $F, F' \subseteq \widetilde{X}$ and $n \leq m$:

1. $\widetilde{U}^*_n(F) \cup \widetilde{U}^*_m(F') \in \mathfrak{A}$. Indeed, this is true due to the fact that $\widetilde{U}^*_n(F) \cup \widetilde{U}^*_m(F') = \widetilde{U}^*_m(\widetilde{U}^*_n(F) \cup F') \in \mathfrak{A}$, since $\widetilde{U}^*_m(\widetilde{U}^*_n(F)) = \widetilde{U}^*_n(F)$.

2. $\widetilde{U}^*_n(F) \cap \widetilde{U}^*_m(F') \in \mathfrak{A}$. Indeed, this is true due to the fact that $\widetilde{U}^*_n(F) \cap \widetilde{U}^*_m(F') = \widetilde{U}^*_m(\widetilde{U}^*_n(F) \cap \widetilde{U}^*_m(F')) \in \mathfrak{A}$.

3. $\widetilde{X} \backslash \widetilde{U}^*_n(F) \in \mathfrak{A}$. Indeed, this is true because $\widetilde{X} \backslash \widetilde{U}^*_n(F) = \widetilde{U}^*_n(\widetilde{X} \backslash \widetilde{U}^*_n(F)) \in \mathfrak{A}$.

Note that each element in $\mathfrak{A}$ is open in $(\widetilde{X}, \widetilde{d}^*)$, so $\mathfrak{S}$ is contained in the Borel σ-algebra of $(\widetilde{X}, \widetilde{d}^*)$ and, hence, $\widetilde{\mu}$ and δ are measures on $\mathfrak{S}$. Furthermore, given $n \in \mathbb{N}$ and $F \subseteq \widetilde{X}$, $\widetilde{\mu}(\widetilde{U}^*_n(F)) = \widetilde{\mu}(\bigcup_{x \in F} \widetilde{U}^*_{xn}) = \sum_{x \in F} \widetilde{\mu}(\widetilde{U}^*_{xn}) = \sum_{x \in F} \widetilde{\omega}(\widetilde{U}^*_{xn}) = \sum_{x \in F} \delta(\widetilde{U}^*_{xn}) = \delta(\bigcup_{x \in F} \widetilde{U}^*_{xn}) = \delta(\widetilde{U}^*_n(F))$, what shows that $\widetilde{\mu}(A) = \delta(A)$ for each $A \in \mathfrak{A}$. By Theorem 2.15, we conclude that $\widetilde{\mu} = \delta$ on $\mathfrak{S}$.

Finally, if O is an open set in $(\widetilde{X}, \widetilde{d}^*)$, then $O = \bigcup_{n \in \mathbb{N}} \widetilde{U}^*_n(O_n)$, where $O_n = \{x \in O : \widetilde{U}^*_{xn} \subseteq O\}$ and, hence, $O \in \mathfrak{S}$. We conclude that $\mathfrak{S}$ is the Borel σ-algebra of $(\widetilde{X}, \widetilde{d}^*)$. $\square$

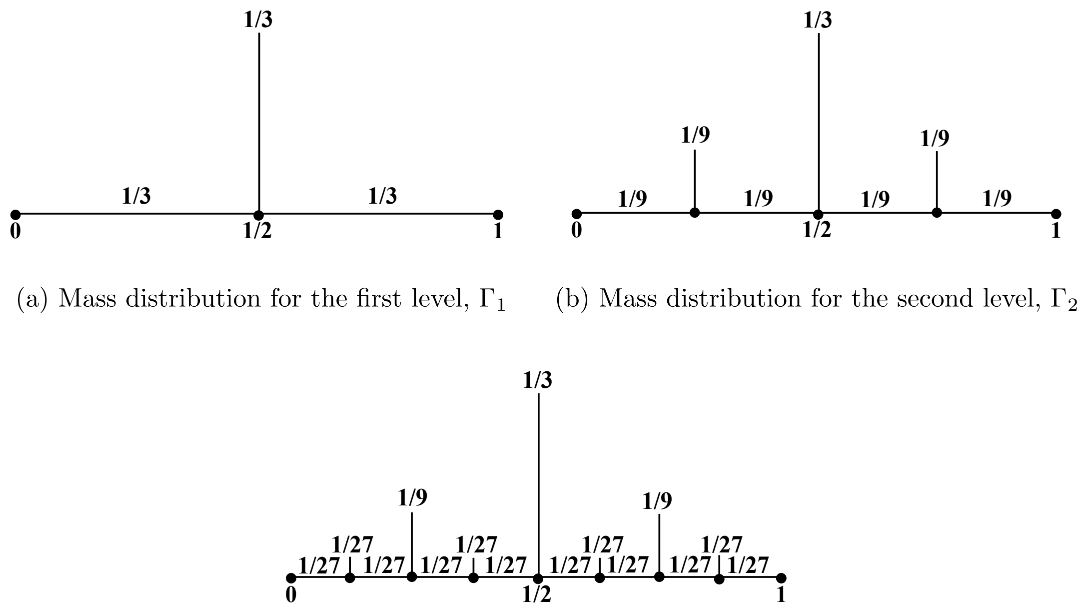
To end this section, we give an example where a definition of the pre-measure on G_n gives us a probability measure on $\widetilde{X}$.

Suppose that G_n is finite for each $n \in \mathbb{N}$. We can define a measure on $\widetilde{X}$ by defining a pre-measure, ω, such that $\omega(U^*_{xn})$ has a uniform value for each $x \in X$ and $n \in \mathbb{N}$. For G_1, we can write $\omega(U^*_{x1}) = \frac{1}{\# G_1}$, where $\# G_1$ denotes the cardinality of G_1 for each $x \in \mathbb{R}$. Now, for the second level $\omega(U^*_{x2}) = \frac{\omega(U^*_{x1})}{\#\{U^*_{y2} \in G_2 : U^*_{y2} \subseteq U^*_{x1}\}}$. Analogously, let $U^*_{x,n+1} \in G_{n+1}$ and $U^*_{xn} \in G_n$ then $\omega(U^*_{x,n+1}) = \frac{\omega(U^*_{xn})}{\#\{U^*_{y,n+1} \in G_{n+1} : U^*_{y,n+1} \subseteq U^*_{xn}\}}$.

Example 4.14. *Let* $([0,1], \mathbf{\Gamma})$*, where* $\mathbf{\Gamma}$ *is the natural fractal structure, which means that its levels are defined by* $\Gamma_n = \{[\frac{k}{2^n}, \frac{k+1}{2^n}] : k = 0, \ldots, 2^n - 1\}$ *for each* $n \in \mathbb{N}$*. If we want to define a measure uniformly with respect to the sets* U^*_{xn}*, we can proceed as follows:*

Since $G_1 = \{U^*_{\frac{1}{2}1}, U^*_{01}, U^*_{11}\}$*,* $\omega(U^*_{x1}) = \frac{1}{3}$ *for each* $x \in [0,1]$*. Analogously,* $\omega(U^*_{x2}) = \frac{1}{9}$ *for each* $x \in [0,1] \setminus \{\frac{1}{2}\}$ *and* $\omega(U^*_{\frac{1}{2}2}) = \frac{1}{3}$*. Moreover,* $\omega(U^*_{x3}) = \frac{1}{27}$ *for each* $x \in [0,1] \setminus \{\frac{1}{2}, \frac{1}{4}, \frac{3}{4}\}$*,* $\omega(U^*_{\frac{1}{2}2}) = \frac{1}{3}$ *and* $\omega(U^*_{\frac{1}{4}3}) = \omega(U^*_{\frac{3}{4}3}) = \frac{1}{9}$*.*

The distribution of the mass can be seen in Figure 4.2.



(a) Mass distribution for the first level, Γ_1 (b) Mass distribution for the second level, Γ_2

(c) Mass distribution for the third level, Γ_3

Figure 4.2: Mass distribution by levels

Note that the pre-measure is distributed among the points of the form $\frac{k}{2^n}$ for each $k = 1, \ldots, 2^n - 1$. In fact, $\widetilde{\mu}(\{\frac{k}{2^n} : k \in \{1, \ldots, 2^n - 1\}) = \sum_{i=1}^{2^n-1} \omega(\{\frac{i}{2^n}\})$. Hence, we can write $\widetilde{\mu}(X) = \widetilde{\mu}(\bigcup_n \{\frac{k}{2^n} : k = 1, \ldots, 2^n - 1\}) = \sum_{i=1}^{\infty} \frac{2^{i-1}}{3^i} = 1$.

4.2 Defining a measure on the completion from the fractal structure

Next step is showing an alternative way to define a probability measure on $\widetilde{X}$. It consist of constructing it from the elements of the fractal structure. For that purpose, we need a condition on the fractal structure, which is related to the concept we recall next.

Definition 4.15. *A cover Γ of X is said to be a tiling if the elements of Γ are regularly closed ($A = \overline{A^\circ}$ for each $A \in \Gamma$) and its interiors are pairwise disjoint ($A^\circ \cap B^\circ = \emptyset$ for each $A, B \in \Gamma$ with $A \neq B$).*

A fractal structure $\mathbf{\Gamma} = \{\Gamma_n : n \in \mathbb{N}\}$ is said to be tiling if Γ_n is a tiling for each

$n \in \mathbb{N}$.

In the rest of this section we will assume that $\mathbf{\Gamma}$ is a tiling fractal structure on X such that Γ_n is countable for each $n \in \mathbb{N}$ together with the assumptions made in the previous section that τ_d is T_0 and G_n is countable for each $n \in \mathbb{N}$. Moreover, we will use the notations and definitions of the previous section.

Lemma 4.16. *Let Γ be a tiling. Then $A^\circ \cap B = \emptyset$ for each $A, B \in \Gamma$ with $A \neq B$.*

Proof. Suppose that there exist $A, B \in \Gamma$ with $A \neq B$ and such that $A^\circ \cap B \neq \emptyset$, and let $x \in A^\circ \cap B = A^\circ \cap \overline{B^\circ}$ (since $B = \overline{B^\circ}$). Thus, $x \in A^\circ$ and $x \in \overline{B^\circ}$. The fact that $x \in \overline{B^\circ}$, means that each neighborhood of x meets B°. Since A° is an open set, it is a neighborhood of all its points, in particular of x. Hence, $A^\circ \cap B^\circ \neq \emptyset$, which is a contradiction with the fact that Γ is a tiling. It follows that $A^\circ \cap B = \emptyset$ for each A, $B \in \Gamma$ with $A \neq B$. □

For each $A \in \Gamma_n$, let us define $i_n(A) = A \setminus \bigcup_{B \in \Gamma_n; B \neq A} B$. On the one hand, note that, since $\mathbf{\Gamma}$ is tiling, by the previous lemma, we have that $\emptyset \neq A^\circ \subseteq i_n(A)$ for each $A \in \Gamma_n$ and $n \in \mathbb{N}$. On the other hand, since Γ_n is a closure-preserving closed cover of X for each $n \in \mathbb{N}$ (see [5, Prop. 2.4]), then $\bigcup_{B \in \Gamma_n; B \neq A} B$ is a closed set and, hence, $i_n(A)$ is open. It is clear that $i_n(A) \subseteq A$ and, hence, $i_n(A)^\circ \subseteq A^\circ$. Since $i_n(A)$ is an open set, it follows that $i_n(A) \subseteq A^\circ$. Consequently, $i_n(A) = A^\circ$ for each $A \in \Gamma_n$ and $n \in \mathbb{N}$. Furthermore, if $x \in i_n(A)$, then $U^*_{xn} = A^\circ$.

Let $\omega : \bigcup \Gamma_n \to [0, 1]$ be a function. We say that ω satisfies the mass distribution conditions if:

1. $\sum_{A \in \Gamma_1} \omega(A) = 1$.

2. $\omega(A) = \sum_{B \in \Gamma_{n+1}, B \subseteq A} \omega(B)$ for each $A \in \Gamma_n$.

From ω, we can define a function (which we will call ω too) on $\mathcal{G}$ as follows:

$$\omega(\rho_n(x)) = \omega(U^*_{xn}) = \begin{cases} \omega(A) & \text{if} \quad x \in A^\circ \\ \\ 0 & \text{if} \quad x \in A \setminus A^\circ \end{cases}$$

Note that given $x \in X$ and $n \in \mathbb{N}$, if there exists only one element A of Γ_n which contains x, then $\omega(U^*_{xn}) = \omega(A)$, while if there is more than one element of Γ_n which contains x, then $\omega(U^*_{xn}) = 0$.

Proposition 4.17. $\omega : \mathcal{G} \to [0, 1]$ *satisfies the mass distribution conditions.*

Proof. First, note that for each $n \in \mathbb{N}$ and $A \in \Gamma_n$ it follows that $\omega(A) = \sum_{x \in A} \omega(U^*_{xn}) = \omega(U^*_{zn})$ (by Proposition 2.3) for any $z \in A^\circ$, where $\omega(A) = \sum_{x \in A} \omega(U^*_{xn})$ stands for the sum $\sum\{\omega(U^*_{xn}) : U^*_{xn} \in G_n \text{ with } x \in A\}$.

1. $\sum \omega(U^*_{x1}) = 1$.

   $\sum_{x \in X} \omega(U^*_{x1}) = \sum_{A \in \Gamma_1; x \in A^\circ} \omega(U^*_{x1}) = \sum_{A \in \Gamma_1} \omega(A) = 1$.

2. $\omega(U^*_{xn}) = \sum_{y \in U^*_{xn}} \omega(U^*_{y,n+1})$ for each $x \in X$ and $n \in \mathbb{N}$.

   Let $x \in X$ and $n \in \mathbb{N}$.

   On the one hand, suppose that there exist $A, B \in \Gamma_n$ with $A \neq B$ and such that $x \in A \cap B$. Then $\omega(U^*_{xn}) = 0$.

   Let $y \in U^*_{xn}$ and suppose that there exists $C \in \Gamma_{n+1}$ such that $y \in C^\circ$. Since $y \in U^*_{xn}$, then $y \in A \cap B$. Since $\Gamma_{n+1} \prec\prec \Gamma_n$ and $y \in C^\circ$, it follows that $y \in C \subseteq A$ and $y \in C \subseteq B$, and, hence, $C^\circ \subseteq A^\circ$ and $C^\circ \subseteq B^\circ$, so $\emptyset \neq C^\circ \subseteq A^\circ \cap B^\circ$, a contradiction. We conclude that there exists no $C \in \Gamma_{n+1}$ such that $y \in C^\circ$. Therefore, $\omega(U^*_{y,n+1}) = 0$ for each $y \in U^*_{xn}$ and the equality holds.

   On the other hand, suppose that there exists $A \in \Gamma_n$ such that $x \in A^\circ$. Then $\omega(U^*_{xn}) = \omega(A) = \sum_{B \in \Gamma_{n+1}, B \subseteq A} \omega(B)$. By the observation at the beginning of the proof, given $B \in \Gamma_{n+1}$, $\omega(B) = \sum_{y \in B} \omega(U^*_{y,n+1})$ and, hence, $\sum_{B \in \Gamma_{n+1}, B \subseteq A} \omega(B) = \sum_{y \in A} \omega(U^*_{y,n+1})$. Since $\omega(U^*_{y,n+1}) = 0$ if $y \notin A^\circ$ and $A^\circ = U^*_{xn}$, then $\sum_{y \in A} \omega(U^*_{y,n+1}) = \sum_{y \in U^*_{xn}} \omega(U^*_{y,n+1})$. Therefore, $\omega(U^*_{xn}) = \sum_{y \in U^*_{xn}} \omega(U^*_{y,n+1})$.

□

From the previous proposition we can apply the constructions on the previous section so we can define $\widetilde{\omega}$, $\widetilde{\mu}$, etc.

Proposition 4.18. $\widetilde{\mu}(\widetilde{A}) = \omega(A)$ *for each* $A \in \Gamma_n$ *and* $n \in \mathbb{N}$.

Proof. Let $n \in \mathbb{N}$, $A \in \Gamma_n$ and $z \in A^\circ$. Then $\widetilde{\mu}(\widetilde{A}) = \widetilde{\mu}(\bigcup_{x \in A} \widetilde{U}^*_{xn}) = \sum_{x \in A} \widetilde{\omega}(\widetilde{U}^*_{xn}) = \sum_{x \in A} \omega(U^*_{xn}) = \omega(U^*_{zn}) = \omega(A)$.

□

Definition 4.19. *We define $\widetilde{\omega}(\widetilde{A}) = \omega(A)$ for each $\widetilde{A} \in \widetilde{\Gamma}_n$ and each $n \in \mathbb{N}$.*

Now, we show that the previous pre-measure satisfies the mass distribution conditions.

Proposition 4.20. *$\widetilde{\omega} : \bigcup \widetilde{\Gamma}_n \to [0, 1]$ satisfies the mass distribution conditions.*

Proof.

1. $\sum_{\widetilde{A} \in \widetilde{\Gamma}_1} \widetilde{\omega}(\widetilde{A}) = \sum_{A \in \Gamma_1} \omega(A) = 1$ by hyphothesis on ω.

2. Let $\widetilde{A} \in \widetilde{\Gamma}_n$. Then $\widetilde{\omega}(\widetilde{A}) = \omega(A)$. Now, since ω satisfies the mass distribution conditions on $\bigcup \Gamma_n$, it follows that $\widetilde{\omega}(\widetilde{A}) = \sum_{B \in \Gamma_{n+1}; B \subseteq A} \omega(B)$. The definition of $\widetilde{\omega}$ implies that
$$\sum_{B \in \Gamma_{n+1}; B \subseteq A} \omega(B) = \sum_{\widetilde{B} \in \widetilde{\Gamma}_{n+1}; \widetilde{B} \subseteq \widetilde{A}} \omega(\widetilde{B})$$
which means that $\widetilde{\omega}(\widetilde{A}) = \sum_{\widetilde{B} \in \widetilde{\Gamma}_{n+1}} \omega(\widetilde{B})$. Note that we have used that for $B \in \Gamma_{n+1}$ and $A \in \Gamma_n$ it holds that $B \subseteq A$ if and only if $\widetilde{B} \subseteq \widetilde{A}$.

□

Now, we define a subset of X that will be used in the following results.

Definition 4.21. *For each $n \in \mathbb{N}$ we define $\mathcal{C}_n = \bigcup \{A \cap B : A, B \in \Gamma_n; A \neq B\}$.*

Proposition 4.22. *Let $\mathbf{\Gamma}$ be a fractal structure on X and let δ be a measure defined on $\mathbf{\Gamma}$ such that $\delta(A) = \omega(A)$ for each $A \in \Gamma_n$ and $n \in \mathbb{N}$, where ω is a pre-measure satisfying the mass distribution conditions. Then $\delta(\mathcal{C}_n) = 0$ for each $n \in \mathbb{N}$.*

Proof. Let $n \in \mathbb{N}$ and $A \in \Gamma_n$. Then $A = \bigcup \{B \in \Gamma_{n+1} : B \subseteq A\}$. Let $B \in \Gamma_{n+1}$ be such that $B \subseteq A$. Then $\delta(B \cup (\bigcup_{C \in \Gamma_{n+1}; C \subseteq A; C \neq B} C)) = \delta(B) + \delta(\bigcup_{C \in \Gamma_{n+1}; C \subseteq A; C \neq B} C) - \delta(B \cap (\bigcup_{C \in \Gamma_{n+1}; C \subseteq A; C \neq B} C)) \leq \delta(B) + \sum_{C \in \Gamma_{n+1}; C \subseteq A; C \neq B} \delta(C) - \delta(B \cap (\bigcup_{C \in \Gamma_{n+1}; C \subseteq A; C \neq B} C))$. Note that in the last inequality we use the sub-σ-additivity of δ. Moreover, the hypothesis $\delta(A) = \omega(A)$ for each $A \in \Gamma_n$ implies that $\delta(A) = \omega(A) = \sum_{B \in \Gamma_{n+1}; B \subseteq A} \omega(B) = \sum_{B \in \Gamma_{n+1}; B \subseteq A} \delta(B)$ and, hence,
$$\delta\left(B \cup \left(\bigcup_{C \in \Gamma_{n+1}; C \subseteq A; C \neq B} C\right)\right) = \delta(A) = \delta(B) + \sum_{C \in \Gamma_{n+1}; C \subseteq A; C \neq B} \delta(C)$$

so it follows that

$$\delta\left(B\cap\left(\bigcup_{C\in\Gamma_{n+1};C\subseteq A;C\neq B}C\right)\right)=0$$

Hence, if we define $C_n(A)=\bigcup_{B\in\Gamma_{n+1},B\subseteq A}(B\cap\bigcup_{C\in\Gamma_{n+1};C\subseteq A;C\neq B}C)$, then $\delta(C_n(A))=0$ for each $A\in\Gamma_n$ and each $n\in\mathbb{N}$. On the other hand, it holds the next equality.

Claim 4.23. *$\mathcal{C}_{n+1}=\mathcal{C}_n\cup\left(\bigcup_{A\in\Gamma_n}C_n(A)\right)$ for each $n\in\mathbb{N}$.*

Proof. $\subseteq$) Let $n\in\mathbb{N}$ and $x\in\mathcal{C}_{n+1}$. Then it can happen that $x\in\mathcal{C}_n$ or $x\notin\mathcal{C}_n$. The fact that $x\in\mathcal{C}_{n+1}$ implies that there exist $A,B\in\Gamma_{n+1}$ with $A\neq B$ such that $x\in A\cap B$. Moreover, if $x\notin\mathcal{C}_n$, there exists a unique $C\in\Gamma_n$ such that $x\in C$. Consequently, $A\cap B\subseteq C$ and $x\in C_n(C)$.

$\supseteq$) Let $n\in\mathbb{N}$ and $x\in\mathcal{C}_n$. Then there exist $A,B\in\Gamma_n$ with $A\neq B$ such that $x\in A\cap B$. Since $x\in A$, by definition of fractal structure, there exists $A_{n+1}\in\Gamma_{n+1}$ such that $x\in A_{n+1}\subseteq A$. Analogously, there exists $B_{n+1}\in\Gamma_{n+1}$ such that $x\in B_{n+1}\subseteq B$. Note that, since $\mathbf{\Gamma}$ is a tiling fractal structure and $A\neq B$, it follows that $A_{n+1}\neq B_{n+1}$ (indeed, if $A_{n+1}=B_{n+1}$, then $A_{n+1}\subseteq A\cap B$, and, since A_{n+1} is regularly closed, its interior is nonempty and it is contained in $A^\circ\cap B^\circ$, but $A^\circ\cap B^\circ$ is empty, since Γ_n is a tiling, a contradiction). Since $x\in A_{n+1}\cap B_{n+1}$ and $A_{n+1}\neq B_{n+1}$, we conclude that $x\in\mathcal{C}_{n+1}$. Now, consider $A\in\Gamma_n$ and $x\in C_n(A)$. It immediately follows that $x\in\mathcal{C}_{n+1}$. □

Now, note that $\delta(\mathcal{C}_1)=0$. Indeed, since ω is a pre-measure satisfying the mass distribution conditions, $\sum_{A\in\Gamma_1}\omega(A)=1$. Additionally, given $A\in\Gamma_1$, we can write $X=A\cup\left(\bigcup_{B\in\Gamma_1;B\neq A}B\right)$, so $\delta(X)=\delta\left(A\cup\left(\bigcup_{B\in\Gamma_1;B\neq A}B\right)\right)\leq\delta(A)+\delta\left(\bigcup_{B\in\Gamma_1;B\neq A}B\right)\leq\sum_{B\in\Gamma_1}\delta(B)=\sum_{B\in\Gamma_1}\omega(B)=1=\delta(X)$ and, hence, $\delta\left(A\cup\left(\bigcup_{B\in\Gamma_1;B\neq A}B\right)\right)=\delta(A)+\delta\left(\bigcup_{B\in\Gamma_1;B\neq A}B\right)$, which lets us conclude that $\delta\left(A\cap\left(\bigcup_{B\in\Gamma_1;B\neq A}B\right)\right)=0$ for each $A\in\Gamma_1$. Therefore, $\delta(\mathcal{C}_1)=0$.

Hence, it holds, by the sub-σ-additivity of δ, that $\delta(\mathcal{C}_2)=\delta(\mathcal{C}_1\cup\bigcup_{A\in\Gamma_1}C_1(A))\leq\delta(\mathcal{C}_1)+\delta(\bigcup_{A\in\Gamma_1}C_1(A))\leq\sum_{A\in\Gamma_1}\delta(C_1(A))=0$. Recursively it can be proven that $\delta(\mathcal{C}_n)=0$ for each $n\in\mathbb{N}$.

□

Lemma 4.24. *Let $x=(\rho_k(x_k))\in\widetilde{X}$ and $n\in\mathbb{N}$. Then $x\in\widetilde{\mathcal{C}}_n$ if and only if $x_n\in\mathcal{C}_n$.*

Proof. Let $x \in \widetilde{X}$ and $n \in \mathbb{N}$.

Suppose that $x \in \widetilde{\mathcal{C}}_n$. Then there exist $\widetilde{A}, \widetilde{B} \in \widetilde{\Gamma}_n$ with $\widetilde{A} \neq \widetilde{B}$ such that $x \in \widetilde{A} \cap \widetilde{B}$, which is equivalent (by Proposition 3.16.2) to $x_n \in A \cap B$. Note that $A \neq B$ (since $\widetilde{A} \neq \widetilde{B}$), so it follows that $x_n \in \mathcal{C}_n$.

Conversely, suppose that $x_n \in \mathcal{C}_n$. Then there exist $A, B \in \Gamma_n$ with $A \neq B$ such that $x_n \in A \cap B$, which is equivalent (by Proposition 3.16.2) to $x \in \widetilde{A} \cap \widetilde{B}$. Therefore, $x \in \widetilde{\mathcal{C}}_n$.

□

Now, we prove the uniqueness of the measure.

Proposition 4.25. *Let δ be a measure on the Borel σ-algebra of $(\widetilde{X}, \widetilde{d}^*)$ satisfying $\delta(\widetilde{A}) = \omega(A)$ for each $\widetilde{A} \in \widetilde{\Gamma}_n$ and $n \in \mathbb{N}$. Then $\delta = \widetilde{\mu}$ on the Borel σ-algebra of $(\widetilde{X}, \widetilde{d}^*)$.*

Proof. We show that $\delta(\widetilde{U}^*_{xn}) = \widetilde{\omega}(\widetilde{U}^*_{xn})$ for each $x \in \widetilde{X}$ and $n \in \mathbb{N}$. By Proposition 4.18, it holds that $\widetilde{\mu}(\widetilde{A}) = \omega(A)$ for each $A \in \Gamma_n$ and each $n \in \mathbb{N}$, which means that $\widetilde{\mu}(\widetilde{A}) = \delta(\widetilde{A})$ for each $A \in \Gamma_n$ and each $n \in \mathbb{N}$. Moreover, we can write $\widetilde{\mu}(\widetilde{A}) = \sum_{x \in \widetilde{A}} \widetilde{\omega}(\widetilde{U}^*_{xn})$ and $\delta(\widetilde{A}) = \sum_{x \in \widetilde{A}} \delta(\widetilde{U}^*_{xn})$. Now, let $x \in \widetilde{X}$ and $n \in \mathbb{N}$. We distinguish two cases depending on whether $x \in \widetilde{\mathcal{C}}_n$ or not in order to show that $\delta(\widetilde{U}^*_{xn}) = \widetilde{\omega}(\widetilde{U}^*_{xn})$:

1. Let $x \in \widetilde{\mathcal{C}}_n$. Then $\widetilde{U}^*_{xn} \subseteq \widetilde{\mathcal{C}}_n$, which implies that $\delta(\widetilde{U}^*_{xn}) = 0$ due to the fact that $\delta(\widetilde{\mathcal{C}}_n) = 0$ by Proposition 4.22 (note that $\widetilde{\omega}(\widetilde{A}) = \omega(A)$, which means that $\delta(\widetilde{A}) = \widetilde{\omega}(\widetilde{A})$ and, morever, $\widetilde{\omega}$ satisfies the mass distribution conditions as it is shown in Proposition 4.20). On the other hand, by the previous lemma, $\widetilde{\omega}(\widetilde{U}^*_{xn}) = \widetilde{\omega}(\widetilde{U}^*_{x_n n})$, with $x_n \in \mathcal{C}_n$. Moreover, $\widetilde{\omega}(\widetilde{U}^*_{x_n n}) = \omega(U^*_{x_n n})$ and $\omega(U^*_{x_n n}) = 0$, since $x_n \in \mathcal{C}_n$. Hence, $\delta(\widetilde{U}^*_{xn}) = \widetilde{\omega}(\widetilde{U}^*_{xn}) = 0$.

2. Let $x = (\rho_n(x_n)) \notin \widetilde{\mathcal{C}}_n$. Then, by the previous lemma, $x_n \notin \mathcal{C}_n$, which means that $\widetilde{\omega}(\widetilde{U}^*_{xn}) = \omega(U^*_{x_n n}) = \omega(A)$, where A is the only element on Γ_n such that $x_n \in A$. By hypothesis, $\delta(\widetilde{A}) = \omega(A)$. Moreover, $\delta(\widetilde{A}) = \sum_{y \in \widetilde{A}} \delta(\widetilde{U}^*_{yn}) = \delta(\widetilde{U}^*_{xn})$. Note that the last equality follows from the fact that $\delta(\widetilde{\mathcal{C}}_n) = 0$ (by Proposition 4.22) and for each $y \in \widetilde{A}$ it follows that $y \notin \widetilde{\mathcal{C}}_n$ and, hence, $\widetilde{U}^*_{yn} = \widetilde{U}^*_{xn}$ or $y \in \widetilde{\mathcal{C}}_n$, which gives us that $\widetilde{U}^*_{yn} \subseteq \widetilde{\mathcal{C}}_n$, so $\delta(\widetilde{U}^*_{yn}) = 0$. If we join the previous expressions, we conclude that $\delta(\widetilde{U}^*_{xn}) = \widetilde{\omega}(\widetilde{U}^*_{xn})$.

By Proposition 4.13, we conclude that $\widetilde{\mu} = \delta$ on the Borel σ-algebra of $(\widetilde{X}, \widetilde{d^*})$.

□

Once we have developed the theory, the next step is to show some examples where it is possible to get a probability measure on a space from a pre-measure defined on the elements of the fractal structure of the space.

4.2.1 Getting a measure from a mass distribution in the fractal structure

The idea of this subsection is to define a pre-measure ω from a finite fractal structure such that $\widetilde{\mu}$ is a measure on the completion of X.

Example 4.26. *Let* $([0,1], \mathbf{\Gamma})$*, where* $\mathbf{\Gamma}$ *is the natural fractal structure.*

Let p_0 *and* p_1 *be two positive numbers such that* $p_0 + p_1 = 1$*. In the first level,* Γ_1*, the pre-measure* ω *spreads mass equal to* p_0 *on the subinterval* $[0, \frac{1}{2}]$ *and mass equal to* p_1 *on* $[\frac{1}{2}, 1]$*. In* Γ_2*, the set* $[0, \frac{1}{2}]$ *is split into two intervals,* $[0, \frac{1}{4}]$ *and* $[\frac{1}{4}, \frac{1}{2}]$*, which are given a fraction* p_0 *and* p_1 *of the whole mass* $\omega([0, \frac{1}{2}])$*. If we apply the same procedure to the set* $[\frac{1}{2}, 1]$*, we obtain:*

$$\omega\left(\left[0, \frac{1}{4}\right]\right) = p_0 p_0, \quad \omega\left(\left[\frac{1}{4}, \frac{1}{2}\right]\right) = p_0 p_1$$
$$\omega\left(\left[\frac{1}{2}, \frac{3}{4}\right]\right) = p_1 p_0, \quad \omega\left(\left[\frac{3}{4}, 1\right]\right) = p_1 p_1$$

If we iterate this procedure, we can define $\omega(A)$ *for each* $A \in \Gamma_n$ *and* $n \in \mathbb{N}$*. Then we have that* $\widetilde{\mu}$ *is a probability measure on the completion of* $[0,1]$*.*

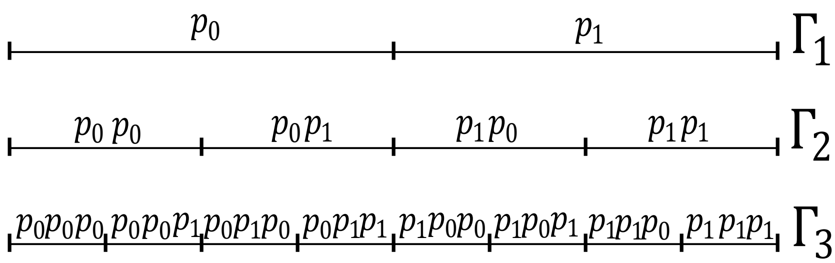


Figure 4.3: Mass distribution by levels

In fact, ω satisfies the mass distribution conditions:

1. $\sum_{A\in\Gamma_1}\omega(A)=\omega([0,\frac{1}{2}])+\omega([\frac{1}{2},1])=p_0+p_1=1$ *by hypothesis.*
2. *Let* $A\in\Gamma_n$. *Then* $A=[\frac{k}{2^n},\frac{k+1}{2^n}]$ *for some* $k\in\{0,\dots,2^n-1\}$. *Hence,* $\omega(A)=\omega([\frac{k}{2^n},\frac{k}{2^n}+\frac{1}{2^{n+1}}])+\omega([\frac{k}{2^n}+\frac{1}{2^{n+1}},\frac{k+1}{2^n}])=\sum_{B\in\Gamma_{n+1},B\subseteq A}\omega(B)$.

Figure 4.3 shows the mass distribution among the elements of the fractal structure by levels.

Note that the previous procedure is similar to the one introduced in [46, Section 3.2] to define multiplicative cascades.

4.2.2 Defining a measure on an attractor of an iterated function system

Now, we recall the definition of attractor of an iterated function system from [38].

Definition 4.27. *Let X be a complete metric space and $\{f_i : i\in I\}$ be a finite family of contractions from X into itself. Then there exists a unique nonempty compact subset K of X such that $K=\bigcup_{i\in I} f_i(K)$. $\{f_i : i\in I\}$ is said to be an iterated function system (IFS), and K is its attractor.*

The definition of a fractal structure on an attractor for an IFS is given in [3]. Let $\{f_i : i\in I\}$ be the family of mappings from a topological space X to itself such that $X=\bigcup_{i\in I} f_i(X)$. We define $\mathbf{\Gamma}(\{f_i : i\in I\})$ to be the structure $\mathbf{\Gamma}=\{\Gamma_n : n\in\mathbb{N}\}$. The first level of this structure is $\Gamma_1=\{f_i(X) : i\in I\}$. The second level is $\Gamma_2=\{f_{ij}(X) : i,j\in I\}$. Recursively we define $\Gamma_n=\{f_w^n(X) : w\in I^n\}$, where $f_w^n=f_{w_1}\circ\ldots\circ f_{w_n}$, with $w=w_1\ldots w_n$. Hence, the fractal structure associated with an attractor of an IFS $(K,\{f_i : i\in I\})$ is $\mathbf{\Gamma}(K)=\mathbf{\Gamma}(\{f_i : i\in I\})$.

On the other hand, we introduce the dimension of these sets according to [24, Section 9.2]. Let $f_1,\dots,f_k:\mathbb{R}^n\to\mathbb{R}^n$ be an iterated function system of n similarities. The contraction factor of f_i is a number $0<c_i<1$ such that $|f_i(x)-f_i(y)|=c_i|x-y|$ for each $x,y\in\mathbb{R}^n$. It is shown that, under certain conditions, the attractor F has Hausdorff and box dimensions equal to the value of s satisfying $\sum_{i=1}^m c_i^s=1$ and, further, that F

has positive and finite $\mathcal{H}^s$-measure. If $F = \bigcup_{i=1}^m f_i(F)$ with the union "nearly disjoint", we have that $\mathcal{H}^s(F) = \sum_{i=1}^m \mathcal{H}^s(f_i(F)) = \sum_{i=1}^m c_i^s \mathcal{H}^s(F)$. Moreover, we need a condition such that ensures that the components $f_i(F)$ do not overlap "too much". We say that f_i satisfies the open set condition if there exists a nonempty bounded open set V such that $\bigcup_{i=1}^m f_i(V) \subset V$ with the union disjoint.

Theorem 4.28 (Moran Theorem). *([24, Th. 9.3]) Suppose that the open set condition holds for the similarities f_i on $\mathbb{R}^n$ with ratios c_i $(1 \leq i \leq m)$. If F is the invariant set satisfying $F = \bigcup_{i=1}^m f_i(F)$, then $\dim_H F = \dim_B F = s$, where s is given by*

$$\sum_{i=1}^m c_i^s = 1$$

Moreover, for this value of s, $0 < \mathcal{H}^s(F) < \infty$.

Now, let X be the attractor for the iterated function system $\{f_i : i \in I\}$, that is, the set given by $X = \bigcup_{i \in I} f_i(X)$, where $\{f_i : i \in I\}$ is a family of contractions of X to itself, and let $\mathbf{\Gamma}$ be the respective fractal structure. Then the pre-measure of the elements of Γ_1 is distributed as $\omega(f_i(X)) = c_i^s$ for each $i \in I$, where s is the solution of $\sum_{i \in I} c_i^s = 1$. For the second level, we have that $\omega(f_w^2(X)) = \omega(f_{w_1 w_2}^2(X)) = \omega(f_{w_1} \circ f_{w_2}(X)) = c_{w_1}^s c_{w_2}^s$ for each $w = w_1 w_2 \in I^2$. Analogously, we can define the pre-measure of elements of level n as $\omega(f_w^n(X)) = \omega(f_{w_1 \ldots w_n}^n(X)) = \omega(f_{w_1} \circ \ldots \circ f_{w_n})(X) = c_{w_1}^s \ldots c_{w_n}^s$ for each $w = w_1 \ldots w_n \in I^n$.

Next example shows the distribution of the mass among the subsets of the fractal structure related to the Sierpinski triangle.

Example 4.29. *Let X be the Sierpinski triangle and $f_1, f_2, f_3 : \mathbb{R}^2 \to \mathbb{R}^2$ be the similarities that define this fractal: $f_1(x,y) = \frac{1}{2}(x,y)$, $f_2(x,y) = \frac{1}{2}(x+1,y)$ and $f_2(x,y) = \frac{1}{2}(x+\frac{1}{2}, y+1)$.*

Since we have a "non-overlaping" iterated function system made up of 3 similarities with contraction factors $c_1 = \frac{1}{2}, c_2 = \frac{1}{2}$ and $c_3 = \frac{1}{2}$, then the fractal dimension s of the attractor of the iterated function system satisfies the equation $c_1^s + c_2^s + c_3^s = 1$, that is, $s = \frac{\ln 3}{\ln 2}$.

Hence, we can define $\omega(f_{i1} \circ \ldots \circ f_{in}(X)) = c_{i1}^s \ldots c_{in}^s$.

Thus, we have a uniform distribution of ω according to the previous construction:

$$\omega(f_1(X)) = \omega(f_2(X)) = \omega(f_3(X)) = \left(\frac{1}{2}\right)^s = \frac{1}{3}$$

$$\omega(f_{11}(X)) = \omega(f_{12}(X)) = \ldots = \omega(f_{32}(X)) = \omega(f_{33}(X)) = \left(\frac{1}{2}\right)^s \left(\frac{1}{2}\right)^s = \frac{1}{9}$$

And so on.

Note that, in this example, the mass is uniformly distributed on each level of the fractal structure.

4.2.3 Getting a measure from a uniform definition of the pre-measure

The developed idea in this subsection is defining a pre-measure ω from a finite fractal structure, $\mathbf{\Gamma}$, such that $\omega(A)$ has a uniform value for each $A \in \Gamma_n$ and each $n \in \mathbb{N}$.

Note that this idea is similar to the one developed at the end of Section 4.1 but with the elements of the fractal structure. For the first level we write $\omega(A) = \frac{1}{\#\Gamma_1}$ for each $A \in \Gamma_1$, where $\#A$ denotes the cardinality of A. Now, let $A \in \Gamma_2$ with $A \subseteq B \in \Gamma_1$. We can define $\omega(A) = \frac{\omega(B)}{\#\{C \in \Gamma_2 : C \subseteq B\}}$ in order to make sure that each of the elements of B have the same pre-measure. Analogously, let $A \in \Gamma_{n+1}$ and $B \in \Gamma_n$ be such that $A \subseteq B$. Then $\omega(A) = \frac{\omega(B)}{\#\{C \in \Gamma_{n+1} : C \subseteq B\}}$.

Roughly speaking, the mass of A is uniformly distributed among its subsets of Γ_{n+1}.

Note that the previous example of the Sierpinski triangle follows this pattern.

Chapter 5

Generating a probability measure on X

The content of this chapter corresponds to [31].

In the previous chapter we proved that the recursive character of fractal structures allowed us to construct a probability measure, $\widetilde{\mu}$, on the bicompletion of a space, $\widetilde{X}$, with a fractal structure as a first step. In fact, note that $\widetilde{\mu}$ is a probability measure on $\widetilde{X}$ and μ is a measure on X, but not necessarily a probability measure as the next example shows.

Example 5.1. *Let $\mathbf{\Gamma}$ be the natural fractal structure on $[0, 1]$ and suppose that the mass ω is distributed as Figure 5.1 shows.*

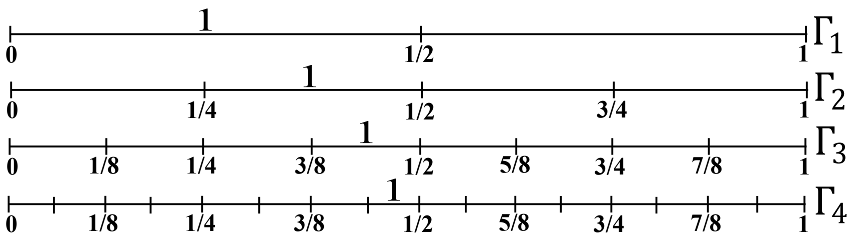


Figure 5.1: Mass distribution by levels

Note that, in this case, the only point of $\widetilde{X}$ that has a positive mass is $(]\frac{1}{2} - \frac{1}{2^n}, \frac{1}{2}[)_{n \in \mathbb{N}}$. In fact, the whole mass is concentrated in that point. Moreover, it is clear that this point does not belong to X. Hence, $\widetilde{\mu}(\widetilde{X} \setminus X) = 1$ and $\widetilde{\mu}(X) = \mu(X) = 0$ (since μ is the restriction of $\widetilde{\mu}$ to X).

Hence, we have to be careful so that the mass is not lost in the remainder of the space in the completion, as it happens in the previous example. So in this chapter we investigate which conditions are needed in order to keep all the mass in the original space so we can get a probability measure on it. More specifically, we will explore conditions on ω such that the restriction of $\widetilde{\mu}$ to X is a probability measure.

5.1 Defining a probability measure on X

Recall from [22] that, given an outer measure, $\overline{\mathcal{M}}$, defined on a set, X, a set $A \subseteq X$ is $\overline{\mathcal{M}}$-measurable (in the sense of Carathéodory) if and only if $\overline{\mathcal{M}}(E) = \overline{\mathcal{M}}(E \cap A) + \overline{\mathcal{M}}(E \setminus A)$ for each set $E \subseteq X$.

Moreover, $\overline{\mathcal{M}}$-measurable sets form a σ-algebra such that the restriction of $\overline{\mathcal{M}}$ to that σ-algebra is a measure.

Theorem 5.2. *([35, Th. C Section 11]) If μ^* is an outer measure and if $\overline{S}$ is the class of all μ^*-measurable sets, then every set of outer measure zero belongs to $\overline{S}$ and the set function $\overline{\mu}$, defined for E in $\overline{S}$ by $\overline{\mu}(E) = \mu^*(E)$, is a complete measure on $\overline{S}$.*

First of all, we show that if $\widetilde{\mu}(\widetilde{X} \setminus X) = 0$, then μ is a probability measure on X and an extension not just of $\widetilde{\omega}$, but also of ω.

Theorem 5.3. *Suppose that $\widetilde{\mu}(\widetilde{X} \setminus X) = 0$. Then X is $\widetilde{\mu}$-measurable and $\mu(X) = 1$. Furthermore $\mu(U^*_{xn}) = \omega(U^*_{xn})$ for each $x \in X$ and $n \in \mathbb{N}$ and if ω is defined from $\mathbf{\Gamma}$, then $\mu(A) = \omega(A)$ for each $A \in \Gamma_n$ and each $n \in \mathbb{N}$. Therefore, μ is an extension of ω.*

Proof. Firstly, let us justify that X is $\widetilde{\mu}$-measurable. By Theorem 5.2, the fact that $\widetilde{\mu}(\widetilde{X} \setminus X) = 0$ means that $\widetilde{X} \setminus X$ is $\widetilde{\mu}$-measurable and, since X is the complement of $\widetilde{X} \setminus X$ in $\widetilde{X}$, it follows that X is $\widetilde{\mu}$-measurable.

Moreover, μ is a probability measure on X. Indeed, $\widetilde{\mu}$ is a probability measure on $\widetilde{X}$ and, hence, $1 = \widetilde{\mu}(\widetilde{X}) = \widetilde{\mu}(X) + \widetilde{\mu}(\widetilde{X} \setminus X)$, it follows that $\widetilde{\mu}(X) = 1$, since, by hypothesis, $\mu(\widetilde{X} \setminus X) = 0$. The fact that μ is the restriction of $\widetilde{\mu}$ to X gives us that $\widetilde{\mu}(X) = \mu(X) = 1$.

Now, we show that μ is an extension of ω. Let $x \in X$ and $n \in \mathbb{N}$. Note that $\widetilde{\mu}(\widetilde{U}^*_{xn}) = \widetilde{\mu}((\widetilde{U}^*_{xn} \cap X) \cup (\widetilde{U}^*_{xn} \cap (\widetilde{X} \setminus X)) = \widetilde{\mu}(\widetilde{U}^*_{xn} \cap X) = \mu(U^*_{xn})$ due to the facts that

$\widetilde{U}^*_{xn} \cap X = U^*_{xn}$ and that $\widetilde{\mu}$ is an extension of μ, and $\widetilde{U}^*_{xn} \cap (\widetilde{X} \setminus X) \subseteq \widetilde{X} \setminus X$ implies that $\widetilde{\mu}(\widetilde{U}^*_{xn} \cap (\widetilde{X} \setminus X)) = 0$ because $\widetilde{\mu}(\widetilde{X} \setminus X) = 0$ by hypothesis. Moreover, by Proposition 4.7, it follows that $\mu(U^*_{xn}) = \widetilde{\omega}(\widetilde{U}^*_{xn}) = \omega(U^*_{xn})$.

Finally, suppose that ω is defined from $\mathbf{\Gamma}$ and let $n \in \mathbb{N}$ and $A \in \Gamma_n$. Then $\widetilde{\mu}(\widetilde{A}) = \widetilde{\mu}((\widetilde{A} \cap X) \cup (\widetilde{A} \cap (\widetilde{X} \setminus X)) = \widetilde{\mu}(\widetilde{A} \cap X) = \mu(A)$, since $\widetilde{\mathbf{\Gamma}}$ and $\widetilde{\mu}$ are, respectively, extensions of $\mathbf{\Gamma}$ and μ, and $\widetilde{A} \cap (\widetilde{X} \setminus X) \subseteq \widetilde{X} \setminus X$ implies that $\widetilde{\mu}(\widetilde{A} \cap (\widetilde{X} \setminus X)) = 0$ because $\widetilde{\mu}(\widetilde{X} \setminus X) = 0$ by hypothesis. Recall that $\widetilde{\mathbf{\Gamma}}$ is an extension of $\mathbf{\Gamma}$ if $\widetilde{A} \cap X = A$ for each $\widetilde{A} \in \widetilde{\Gamma}_n$ and each $n \in \mathbb{N}$.

Moreover, by Proposition 4.18, $\widetilde{\mu}(\widetilde{A}) = \omega(A)$, and, since $\widetilde{\mu}(\widetilde{A}) = \mu(A)$, it follows that $\mu(A) = \omega(A)$ for each $A \in \Gamma_n$ and each $n \in \mathbb{N}$. Consequently, μ is an extension of ω.

□

Recall that a quasi-pseudometric d is said to be bicomplete if the pseudometric d^* is complete. In our context X is bicomplete if and only if $X = \widetilde{X}$. From the previous theorem we can also get the following results.

Corollary 5.4. *If X is bicomplete, then μ is a probability measure on X.*

Proof. Suppose that X is bicomplete. Then $X = \widetilde{X}$, which implies that $\widetilde{\mu} = \mu$. Hence, μ is a probability measure on X. □

Example 5.5. *Let $\sum$ be a finite set (alphabet) and denote by $X = \Sigma^{\mathbb{N}} \cup \bigcup_{n \in \mathbb{N}} \sum^n$ the collection of infinite and finite sequences (words) over $\sum$. Recall from [62] that a (non-archimedean) quasi-metric d can be defined on X by $d(x, y) = 0$ if $x \sqsubseteq y$ and $d(x, y) = 2^{-l(x \sqcap y)}$ otherwise, where $l(x)$ denotes the length of x for each $x \in X$ and $x \sqcap y$ is the common prefix of x and y. Moreover, $x \sqsubseteq y$ if and only if x is a prefix of y.*

Since any non-archimedean quasi-pseudometric defined on a topological space X induces a fractal structure, which can be defined from the collection of balls with respect to d^{-1} of radius $\frac{1}{2^n}$ (see [2]), we can consider the fractal structure $\mathbf{\Gamma}$ defined by d. A wider description of $\mathbf{\Gamma}$ is provided in [26]. For example, if $\Sigma = \{a, b\}$ the first level, Γ_1, consists of two elements: the first one includes all the words beginning with a and the second one those which begin with b. The second level Γ_2 consists of six elements which are given by the words beginning with aa, ab, ba and bb and the elements $\{a\}$ and $\{b\}$. Note that the fact that we consider infinite words in our space, lets us conclude that X is *bicomplete. Hence, if we define a pre-measure satisfying the mass distribution conditions on $\mathcal{G}$, we can claim that μ is a probability measure on X.*

The next result lets us claim that each probability measure constructed from a pre-meaure is, indeed, the extension of that pre-measure.

Theorem 5.6. *If $\mathbf{\Gamma}$ is a fractal structure on X, then μ is a probability measure on X if and only if μ is an extension of ω.*

Proof. $\Leftarrow$) We have to distinguish two cases depending on the structures we are considering in order to get that μ is an extension of ω:

1. Suppose that $\mu(U_{xn}^*) = \omega(U_{xn}^*)$. Obviously, $X = \bigcup_{x\in X} U_{x1}^*$. Therefore, $\mu(X) = \mu(\bigcup_{x\in X} U_{x1}^*) = \sum \mu(U_{x1}^*) = \sum \omega(U_{x1}^*) = 1$.
2. Suppose that $\mu(A) = \omega(A)$ for each $n \in \mathbb{N}$ and each $A \in \Gamma_n$. Given $A, B \in \Gamma_1$, we can write $A \cup B = i_1(A) \cup i_1(B) \cup U_{x_1 1}^* \cup \ldots \cup U_{x_n 1}^* \cup \ldots$ (recall that we are assuming that $\{U_{xn}^* : x \in X, n \in \mathbb{N}\}$ is countable), where $x_i \in \bigcup\{C \cap D : C, D \in \Gamma_1, C \neq D\}$, which implies that $\mu(A \cup B) = \mu(i_1(A)) + \mu(i_1(B))$. Now, since $\mu(A) = \omega(A)$ and $\mu(A) = \mu(i_1(A))$, and analogously for B, it follows that $\mu(A \cup B) = \mu(i_1(A)) + \mu(i_1(B)) = \omega(A) + \omega(B)$. Consider an enumeration of the elements of $\Gamma_1 = \{A_1, A_2, \ldots\}$. Then, in a similar way, it can be proven that $\mu(A_1 \cup \ldots \cup A_n) = \sum_{i=1}^n \mu(i_1(A_i)) = \sum_{i=1}^n \omega(A_i)$. If Γ_1 is not finite (if it is finite, the reasoning is similar, but easier), we can take limit when n tends to infinity and, then, $\sum_{i=1}^n \omega(A_i) \to \sum_{i=1}^\infty \omega(A_i) = 1$. Moreover, $\mu(A_1 \cup \ldots \cup A_n) \to \mu(\bigcup_{i=1}^\infty A_i) = \mu(X)$. If we join both expressions $\mu(A_1 \cup \ldots \cup A_n) \to \mu(X)$ and $\mu(A_1 \cup \ldots \cup A_n) = \sum_{i=1}^n \omega(A_i) \to 1$, we conclude that $\mu(X) = 1$.

In both cases we get $\mu(X) = 1$.

$\Rightarrow$) On the one hand, by construction (see Method I on construction of outer measures) we have that $\mu(U_{xn}^*) \leq \omega(U_{xn}^*)$ for each $x \in X$ and each $n \in \mathbb{N}$. Suppose now that $\mu(U_{xn}^*) < \omega(U_{xn}^*)$ for some $x \in X$ and $n \in \mathbb{N}$. Then $\mu(X) = \sum_{y\in X} \mu(U_{yn}^*) < \sum_{y\in X} \omega(U_{yn}^*)$. Since ω satisfied the mass distribution conditions, $\sum_{y\in X} \omega(U_{yn}^*) = 1$, and it follows that $\mu(X) < 1$, a contradiction with the fact that μ is a probability measure on X.

Analogously, it can be proven that $\mu(A) = \omega(A)$ for each $n \in \mathbb{N}$ and each $A \in \Gamma_n$. $\square$

Lemma 5.7. *Let $U^*_{x_n n}$ be a decreasing sequence. Then $\widetilde{x} = (\rho_n(x_n)) \in \widetilde{X}$ and $\{\widetilde{x}\} = \bigcap_{n\in\mathbb{N}} \widetilde{U}^*_{x_n n}$. Furthermore, $\widetilde{x} \in X$ if and only if $\bigcap_{n\in\mathbb{N}} U^*_{x_n n} \neq \emptyset$.*

Proof. Let $U^*_{x_n n}$ be a decreasing sequence and choose $\widetilde{x} = (\rho_n(x_n))$. Then $\widetilde{x} \in \widetilde{X}$ and $\widetilde{x} \in \bigcap \widetilde{U}^*_{x_n n}$ (since, by Proposition 3.16.8, $\widetilde{U}^*_{\widetilde{x} n} = \widetilde{U}^*_{x_n n}$ for each $n \in \mathbb{N}$). Indeed, $\bigcap \widetilde{U}^*_{x_n n}$ is a singleton due to the fact that $\widetilde{X}$ is T_0 (see Proposition 3.16.13) and, hence, $\widetilde{d}^*$ is a metric. Hence, $\{\widetilde{x}\} = \bigcap_{n\in\mathbb{N}} \widetilde{U}_{x_n n}$.

Now, we prove the equivalence:

$\Rightarrow$) Suppose that $\widetilde{x} \in X$. Note that $\bigcap_{n\in\mathbb{N}} U^*_{x_n n} = \bigcap_{n\in\mathbb{N}} (\widetilde{U}^*_{x_n n} \cap X) = \left(\bigcap_{n\in\mathbb{N}} \widetilde{U}^*_{x_n n}\right) \cap X = \{\widetilde{x}\} \cap X = \{\widetilde{x}\} \neq \emptyset$.

$\Leftarrow$) Let $z \in \bigcap_{n\in\mathbb{N}} U^*_{x_n n}$. Then $z \in \bigcap_{n\in\mathbb{N}} \widetilde{U}^*_{x_n n}$. Since $\{\widetilde{x}\} = \bigcap_{n\in\mathbb{N}} \widetilde{U}^*_{x_n n}$, it follows that $z = \widetilde{x}$, which gives us that $\widetilde{x} \in X$.

□

Next proposition shows a necessary condition to ensure that μ is a probability measure on X.

Proposition 5.8. *Suppose that μ is a probability measure on X and X is $\widetilde{\mu}$-measurable. Then $\omega(U^*_{x_n n}) \to 0$ for each decreasing sequence $U^*_{x_n n}$ with $\bigcap U^*_{x_n n} = \emptyset$.*

Proof. Let $U^*_{x_n n}$ be a decreasing sequence such that $\bigcap U^*_{x_n n} = \emptyset$. By Lemma 5.7, it holds that $\widetilde{x} = (\rho_n(x_n)) = \bigcap_n \widetilde{U}_{x_n n}$. Now, since $\widetilde{\mu}$ is continuous from above, we have that $\lim \widetilde{\mu}(\widetilde{U}^*_{x_n n}) = \widetilde{\mu}(\bigcap \widetilde{U}^*_{x_n n}) = \widetilde{\mu}(\{\widetilde{x}\}) = 0$ due to the fact that μ is a probability measure on X and X is $\widetilde{\mu}$-measurable (and, hence, $\widetilde{\mu}(\widetilde{X} \setminus X) = 0$). □

Next example shows that the converse of the previous proposition is not true.

Example 5.9. *Let $\mathbf{\Gamma}$ be the natural fractal structure on $[0,1] \times [0,1]$, that is, $\Gamma_n = \{[\frac{k_1}{2^n}, \frac{k_1+1}{2^n}] \times [\frac{k_2}{2^n}, \frac{k_2+1}{2^n}] : k_1, k_2 \in \{0, \ldots, 2^n - 1\}\}$ for each $n \in \mathbb{N}$, and suppose that the mass ω is distributed as Figure 5.2 shows.*

Note that it is satisfied that $\omega(U^*_{x_n n}) \to 0$ for each decreasing sequence $U^*_{x_n n}$ with $\bigcap U^*_{x_n n} = \emptyset$. Moreover, since the mass is concentrated next to the vertical line which divides the unit square into two equal parts, we have that $\widetilde{\mu}(\widetilde{X} \setminus X) = 1$, which means that $\mu(X) = 0$ and, consequently, μ is not a probability measure on X.

0	1/2
0	1/2

(a) Distribution of ω in Γ_1

0	0	1/4	0
0	0	1/4	0
0	0	1/4	0
0	0	1/4	0

(b) Distribution of ω in Γ_2

0	0	0	0	1/8	0	0	0
0	0	0	0	1/8	0	0	0
0	0	0	0	1/8	0	0	0
0	0	0	0	1/8	0	0	0
0	0	0	0	1/8	0	0	0
0	0	0	0	1/8	0	0	0
0	0	0	0	1/8	0	0	0
0	0	0	0	1/8	0	0	0

(c) Distribution of ω in Γ_3

Figure 5.2: Mass distribution by levels

Corollary 5.10. *Suppose that $\widetilde{X} \setminus X$ is countable and $\omega(U^*_{x_n n}) \to 0$ for each sequence (x_n) satisfying $x_{n+1} \in U^*_{x_n n}$ with $\bigcap U^*_{x_n n} = \emptyset$. Then μ is a measure on the Borel σ-algebra of (X, d) and $\mu(X) = 1$.*

Proof. First of all, by Lemma 5.7, it holds that, given $\widetilde{x} = (\rho_n(x_n)) \in \widetilde{X} \setminus X$, the corresponding sequence of sets $U^*_{x_n n}$ satisfies that $\bigcap_n U^*_{x_n n} = \emptyset$.

If $\widetilde{X} \setminus X$ is countable, then we can write $\widetilde{X} \setminus X = \bigcup_{\widetilde{x} \in N} \{\widetilde{x}\}$, where N is a countable set. Since μ is a measure on $\widetilde{X}$, by its σ-additivity, we can write $\widetilde{\mu}(\widetilde{X} \setminus X) = \widetilde{\mu}(\bigcup_{\widetilde{x} \in N} \{\widetilde{x}\}) = \sum_{\widetilde{x} \in N} \widetilde{\mu}(\{\widetilde{x}\}) = 0$. We have to show that $\widetilde{\mu}(\{\widetilde{x}\}) = 0$, with $\widetilde{x} = (x_n) \in \widetilde{X}$. Observe that $\{\widetilde{x}\} = \bigcap_n \widetilde{U}^*_{x_n n}$ and, since $U^*_{x_n n}$ is a decreasing sequence of sets and $\widetilde{\mu}$ is continuous from above, it follows that $\widetilde{\mu}(\widetilde{U}^*_{x_n n}) \to \widetilde{\mu}(\bigcap_n \widetilde{U}^*_{x_n n}) = \mu(\{\widetilde{x}\})$. Moreover, since $\widetilde{\mu}(\widetilde{U}^*_{x_n n}) =$

$\widetilde{\omega}(\widetilde{U}^*_{x_n n}) = \omega(U^*_{x_n n})$ and, by hypothesis, $\omega(U^*_{x_n n}) \to 0$, we conclude that $\widetilde{\mu}(\{\widetilde{x}\}) = 0$. Hence, by Theorem 5.3, we have that μ is a measure on the Borel σ-algebra of (X, d) and $\mu(X) = 1$. $\square$

One way to define a probability measure distributed uniformly by using the sets U^*_{xn} is as it is shown in the next example.

Example 5.11. *Let $\mathbf{\Gamma}$ be a finite fractal structure defined on X, a space such that $\widetilde{X} \setminus X$ is countable. We can write $\omega(U^*_{x1}) = \frac{1}{\# G_1}$ for each $x \in X$, where $\# A$ denotes the cardinality of A. Now, for the second level, $\omega(U^*_{x2}) = \frac{\omega(U^*_{x1})}{\#\{U^*_{y2} \in G_2 : U^*_{y2} \subseteq U^*_{x1}\}}$. Analogously, let $U^*_{x,n+1} \in G_{n+1}$ for some $x \in X$. Then $U^*_{xn} \in G_n$ and we can define $\omega(U^*_{x,n+1}) = \frac{\omega(U^*_{xn})}{\#\{U^*_{y,n+1} \in G_{n+1} : U^*_{y,n+1} \subseteq U^*_{xn}\}}$. Since $\widetilde{X} \setminus X$ is countable, Corollary 5.10 lets us claim that μ is a probability measure on X. Indeed, let (x_n) be a sequence satisfying that $x_{n+1} \in U^*_{x_n n}$ and $\bigcap U^*_{x_n n} = \emptyset$. Set $n \in \mathbb{N}$. Then there exists $k > n$ such that $\#\{U^*_{xk} \in G_k : U^*_{xk} \subseteq U^*_{x_n n}\} > 1$. What is more, since the mass is distributed uniformly, $\omega(U^*_{x_k k}) \leq \frac{\omega(U^*_{x_n n})}{2}$. Recursively it can be proven that, given $m \in \mathbb{N}$, there exists $k > n$ such that $\omega(U^*_{x_k k}) \leq \frac{\omega(U^*_{x_n n})}{2^m}$. Therefore, we have proven that for each $m \in \mathbb{N}$ there exists $k \in \mathbb{N}$ such that $\omega(U^*_{x_k k}) < \frac{1}{2^m}$, which means that $\omega(U^*_{x_n n}) \to 0$.*

Next example shows the previous construction for a certain fractal structure on $[0, 1]$.

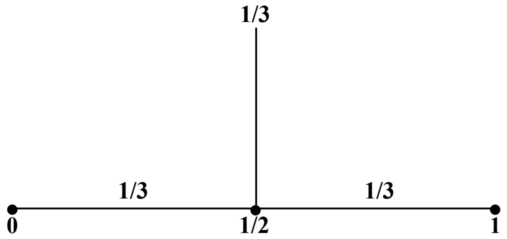
Example 5.12. *Let $([0,1], \mathbf{\Gamma})$, where $\mathbf{\Gamma}$ is the natural fractal structure. Now, define ω uniformly in each G_n as follows:*

Since $G_1 = \{U^*_{\frac{1}{2}1}, U^*_{01}, U^*_{11}\}$, $\omega(U^*_{x1}) = \frac{1}{3}$ for each $x \in [0,1]$. Analogously, $\omega(U^*_{x2}) = \frac{1}{9}$ for each $x \in [0,1] \setminus \{\frac{1}{2}\}$ and $\omega(U^*_{\frac{1}{2}2}) = \frac{1}{3}$. Moreover, $\omega(U^*_{x3}) = \frac{1}{27}$ for each $x \in [0,1] \setminus \{\frac{1}{2}, \frac{1}{4}, \frac{3}{4}\}$, $\omega(U^*_{\frac{1}{2}2}) = \frac{1}{3}$ and $\omega(U^*_{\frac{1}{4}3}) = \omega(U^*_{\frac{3}{4}3}) = \frac{1}{9}$.

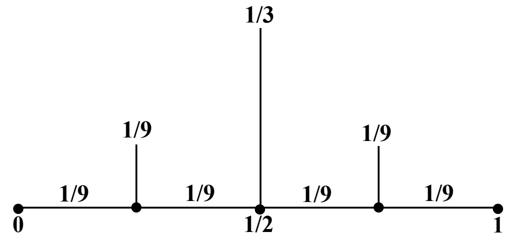
The distribution of the mass can be seen in Figure 5.3.

Note that, in this case, $\widetilde{X} \setminus X$ is countable and that the pre-measure is distributed between the points of the form $\frac{k}{2^n}$ for each $k = 1, \ldots, 2^n - 1$. Indeed, note that, for each sequence (x_n) in X satisfying $x_{n+1} \in U^*_{x_n n}$ and $\bigcap_n U^*_{x_n n} = \emptyset$, we have that $\omega(U^*_{x_n n}) = \frac{1}{3^n} \to 0$. Hence, Corollary 5.10 gives us that μ is a probability measure on X.

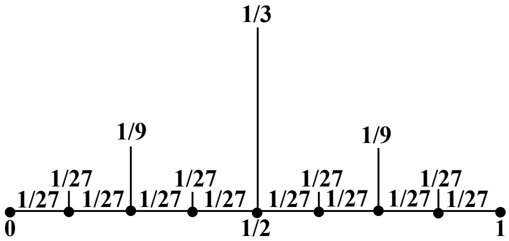
Definition 5.13. *([7, Def. 3.1]) Let $\mathbf{\Gamma} = \{\Gamma_n : n \in \mathbb{N}\}$ be a fractal structure on X. $\mathbf{\Gamma}$ is said to be half-complete if d is, which means that each Cauchy sequence in (X, d^*) is convergent in (X, d).*



(a) Mass distribution for the first level, Γ_1



(b) Mass distribution for the second level, Γ_2



(c) Mass distribution for the third level, Γ_3

Figure 5.3: Mass distribution by levels

The following result relates the points of $\widetilde{X}$ to half-completeness.

Remark 5.14. *Let $\mathbf{\Gamma}$ be a half-complete fractal structure on X. Then, for each $\widetilde{x} = (\rho_n(x_n))_{n\in\mathbb{N}} \in \widetilde{X}$, there exists $x \in X$ such that $x_n \to x$.*

Proof. Indeed, given $\widetilde{x} = (\rho_n(x_n))_{n\in\mathbb{N}} \in \widetilde{X}$, it holds that $U^*_{x_{n+1},n+1} \subseteq U^*_{x_n n}$ for each $n \in \mathbb{N}$. In particular, for each $k \in \mathbb{N}$, we have that $x_{n+k} \in U^*_{x_{n+k},n+k} \subseteq U^*_{x_n n}$, which means that (x_n) is a Cauchy sequence in X with respect to d^*. Hence, by definition of half-complete, there exists $x \in X$ such that $x_n \to x$. $\square$

In fact, the previous result is a characterization. Essentially, the converse of Remark 5.14 is [6, Prop. 3.2.(1)].

Remark 5.15. *Let $\widetilde{x} = (\rho_n(x_n))_{n\in\mathbb{N}} \in \widetilde{X}$, and $x \in X$ be such that $x_n \to x$. Then $x_n \in U_{xn}$ for each $n \in \mathbb{N}$.*

Proof. Let $n \in \mathbb{N}$. Since $x_m \to x$, there exists $m_0 \in \mathbb{N}$ such that $x_m \in U_{xn}$ for each $m \geq m_0$. Now, the fact that $x_m \in U_{xn}$ implies that $U_{x_m n} \subseteq U_{xn}$ (see Proposition 2.3.3) for each $m \geq m_0$. Moreover, the fact that $\widetilde{x} \in \widetilde{X}$ gives us that $x_k \in U^*_{x_n n}$ for each $k \geq n$.

Hence, by Proposition 2.3.6, $U^*_{x_k n} = U^*_{x_n n}$ for each $k \geq n$ and, thus, $U_{x_k n} = U_{x_n n}$ for each $k \geq n$. Consequently, for $m \geq \max\{n, m_0\}$, it follows that $x_n \in U_{x_n n} = U_{x_m n} \subseteq U_{xn}$. $\square$

Next, we recall, from Definition 4.21, that $\mathcal{C}_n = \bigcup\{A \cap B : A, B \in \Gamma_n; A \neq B\}$. This subset of X will be crucial to give new sufficient conditions to ensure that μ is a probability measure on X.

Note that, by the properties of a fractal structure, it follows that $\mathcal{C}_n \subseteq \mathcal{C}_{n+1}$ for each $n \in \mathbb{N}$.

Lemma 5.16. *Let* $\mathbf{\Gamma}$ *be a half-complete fractal structure on* X*. Then it holds that* $\widetilde{X} \setminus X \subseteq \bigcup_{n \in \mathbb{N}} \bigcap_{m \in \mathbb{N}} St(\mathcal{C}_n, \widetilde{\Gamma}_m)$.

Proof. Let $x = (\rho_n(x_n))_{n \in \mathbb{N}} \in \widetilde{X} \setminus X$ and suppose that $x \notin \bigcup_{n \in \mathbb{N}} \bigcap_{m \in \mathbb{N}} St(\mathcal{C}_n, \widetilde{\Gamma}_m)$. Then, for each $n \in \mathbb{N}$, there exists $m \in \mathbb{N}$ (we can suppose that $m \geq n$) such that $x \notin St(\mathcal{C}_n, \widetilde{\Gamma}_m)$.

It follows that $x_m \notin \mathcal{C}_n$. Indeed, in case $x_m \in \mathcal{C}_n$, if we consider $A \in \Gamma_m$ such that $x_m \in A$, by Proposition 3.16.2, $x \in \widetilde{A}$, and, since $x_m \in A \subseteq \widetilde{A}$, it follows that $x \in St(\mathcal{C}_n, \widetilde{\Gamma}_m)$, a contradiction.

Now, we prove that for each $n \in \mathbb{N}$, $x_n \notin \mathcal{C}_n$, that is, there exists only one element $A_n \in \Gamma_n$ such that $x_n \in A_n$. Indeed, let $n \in \mathbb{N}$ and let us suppose that there exist $A, B \in \Gamma_n$ such that $A \neq B$ and $x_n \in A \cap B$. Let $m \geq n$ be such that $x \notin St(\mathcal{C}_n, \widetilde{\Gamma}_m)$. Then $x_m \notin \mathcal{C}_n$, but, since $U^*_{x_n n} = U^*_{x_m n}$, we have that $x_m \in A \cap B$, a contradiction.

Since $\mathbf{\Gamma}$ is half-complete, by Remark 5.14, there exists $z \in X$ such that $x_n \to z$. By Remark 5.15, it follows that $x_n \in U_{zn}$ for each $n \in \mathbb{N}$. Consequently, $z \in U^{-1}_{x_n n} \subseteq A_n$ for each $n \in \mathbb{N}$, which implies that $z \in A_n$ for each $n \in \mathbb{N}$. Since $x \notin X$, there exists $n \in \mathbb{N}$ such that $U^*_{x_n n} \neq U^*_{zn}$ and, hence, there exists $B_n \in \Gamma_n$ such that $z \in B_n$ but $x_n \notin B_n$ (see items 2 and 5 in Proposition 2.3). Also, note that $z \in U^{-1}_{x_n n}$ and, hence, $z \in C$ for each $C \in \Gamma_n$ with $x_n \in C$). It follows that $B_n \neq A_n$ and $z \in A_n \cap B_n$, so $z \in \mathcal{C}_n$. On the one hand, since $x, z \in \bigcap_{k \in \mathbb{N}} \widetilde{A}_k$, we have that $x \in St(z, \widetilde{\Gamma}_k) \subseteq St(\mathcal{C}_n, \widetilde{\Gamma}_k)$ for each $k \in \mathbb{N}$. On the other hand, by hypothesis, there exists $m \geq n$ such that $x \notin St(\mathcal{C}_n, \widetilde{\Gamma}_m)$, a contradiction.

We conclude that $x \in \bigcup_{n \in \mathbb{N}} \bigcap_{m \in \mathbb{N}} St(\mathcal{C}_n, \widetilde{\Gamma}_m)$. $\square$

Corollary 5.17. *Let $\mathbf{\Gamma}$ be a half-complete fractal structure and suppose that the set $\bigcap_{m\in\mathbb{N}} St(\mathcal{C}_n, \widetilde{\Gamma}_m)$ is countable for each $n \in \mathbb{N}$ and $\omega(U^*_{x_n n}) \to 0$ for each sequence (x_n) satisfying $x_{n+1} \in U^*_{x_n n}$ with $\bigcap U^*_{x_n n} = \emptyset$. Then μ is a measure on the Borel σ-algebra of (X, d) and $\mu(X) = 1$.*

Proof. Since, by hypothesis, $\bigcap_{m\in\mathbb{N}} St(\mathcal{C}_n, \widetilde{\Gamma}_m)$ is countable for each $n \in \mathbb{N}$, we have that $\bigcup_{n\in\mathbb{N}} \bigcap_{m\in\mathbb{N}} St(\mathcal{C}_n, \widetilde{\Gamma}_m)$ is countable. By Lemma 5.16, $\widetilde{X} \setminus X \subseteq \bigcup_{n\in\mathbb{N}} \bigcap_{m\in\mathbb{N}} St(\mathcal{C}_n, \widetilde{\Gamma}_m)$, which means that $\widetilde{X} \setminus X$ is countable. By Corollary 5.10, we have that μ is a measure on the Borel σ-algebra of (X, d) and $\mu(X) = 1$. $\square$

Example 5.18. *Let $\mathbf{\Gamma} = \{\Gamma_n : n \in \mathbb{N}\}$ be the natural fractal structure on $X = \mathbb{R}$ or $X = [0, 1]$, and let ω be a pre-measure such that $\omega(U^*_{x_n n}) \to 0$ for each sequence (x_n) satisfying $x_{n+1} \in U^*_{x_n n}$ with $\bigcap U^*_{x_n n} = \emptyset$. Note that $\bigcap_m St(\mathcal{C}_n, \widetilde{\Gamma}_m)$ is countable. Hence, by the previous corollary, we have that μ is a measure on the Borel σ-algebra of (X, d) and $\mu(X) = 1$.*

Now, we apply the previous example in a certain case where the pre-measure is known and given by a random variable.

Example 5.19. *Consider $(\mathbb{R}, \mathbf{\Gamma})$, where $\mathbf{\Gamma}$ is the natural fractal structure (see Figure 5.4). Note that the topology induced by $\mathbf{\Gamma}$ is the usual topology and $\mathbf{\Gamma}$ is a tiling finite fractal structure.*

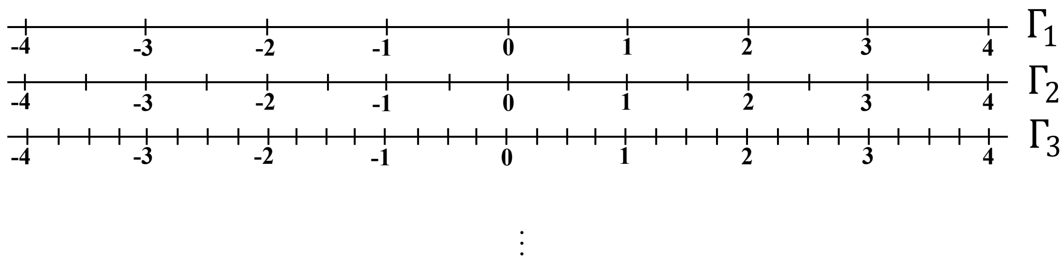


Figure 5.4: First levels of the natural fractal structure on $\mathbb{R}$

We define, for this fractal structure, $\omega(A) = \omega([a, b]) = F(b) - F(a)$, where F denotes the cumulative distribution function of a continuous random variable for each $A \in \Gamma_n$ and $n \in \mathbb{N}$. Next, we check that, in fact, the measure μ that can be got from the pre-measure ω according to the construction made in the previous section is a probability measure on X.

First, we describe $\bigcap_m St(\mathcal{C}_n, \widetilde{\Gamma}_m)$ *for each* $n \in \mathbb{N}$*. Note that* $U^*_{xn} = \{x\}$ *for each* $x \in \mathcal{C}_n$ *and each* $n \in \mathbb{N}$*, and* $U^*_{xn} =]\frac{k'}{2^{n-1}}, \frac{k'+1}{2^{n-1}}[$ *for each* $x \in \mathbb{R} \setminus \mathcal{C}_n$*, each* $n \in \mathbb{N}$ *and for some* $k' \in \mathbb{Z}$*. Let* $k, n \in \mathbb{N}$ *and let* $y_m = \frac{k}{2^{n-1}} + \frac{1}{2^m}$ *and* $z_m = \frac{k}{2^{n-1}} - \frac{1}{2^m}$ *for each* $m \geq n$ *and* $y_m = y_n$ *and* $z_m = z_n$ *for* $m < n$*. We define* $u_{k,n} = (\rho_m(y_m)) \in \widetilde{X}$ *and* $l_{k,n} = (\rho_m(z_m)) \in \widetilde{X}$*. Note that* $\mathcal{C}_n = \{\frac{k}{2^{n-1}} : k \in \mathbb{Z}\}$ *and* $\bigcap_m St(\mathcal{C}_n, \widetilde{\Gamma}_m) = \mathcal{C}_n \cup \{u_{k,n} : k \in \mathbb{Z}\} \cup \{l_{k,n} : k \in \mathbb{Z}\}$*. It follows that* $\bigcap_m St(\mathcal{C}_n, \widetilde{\Gamma}_m)$ *is countable for each* $n \in \mathbb{N}$*.*

On the other hand, $\omega(U^*_{x_n n}) \to 0$ *for each sequence* (x_n) *such that* $x_{n+1} \in U^*_{x_n n}$ *with* $\bigcap U^*_{x_n n} = \emptyset$*. Indeed, let* (x_n) *be a sequence such that* $x_{n+1} \in U^*_{x_n n}$*. Then* $U^*_{x_n n} \in]\frac{k_n}{2^{n-1}}, \frac{k_n+1}{2^{n-1}}[$ *for each* $n \in \mathbb{N}$ *and some* $k_n \in \mathbb{Z}$*, and it follows that* $\omega(U^*_{x_n n}) = \omega(]\frac{k_n}{2^{n-1}}, \frac{k_n+1}{2^{n-1}}[) = F(\frac{k_n+1}{2^{n-1}}) - F(\frac{k_n}{2^{n-1}}) \to 0$ *due to the fact that* F *is the cumulative distribution function of a continuous random variable.*

Hence, by Corollary 5.17, $\mu(X) = 1$*, that is,* μ *is a probability measure on* X*.*

It can also be applied to other fractal structures as the next example shows.

Example 5.20. *Consider* $(\mathbb{R}, \mathbf{\Gamma})$*, where* $\mathbf{\Gamma}$ *is a finite fractal structure whose levels are defined by* $\Gamma_n = \{]-\infty, -n]\} \cup \{[\frac{k}{2^{n-1}}, \frac{k+1}{2^{n-1}}] : k = -n2^{n-1}, \ldots, n2^{n-1} - 1\} \cup \{[n, +\infty[\}$ *(see Figure 5.5).*

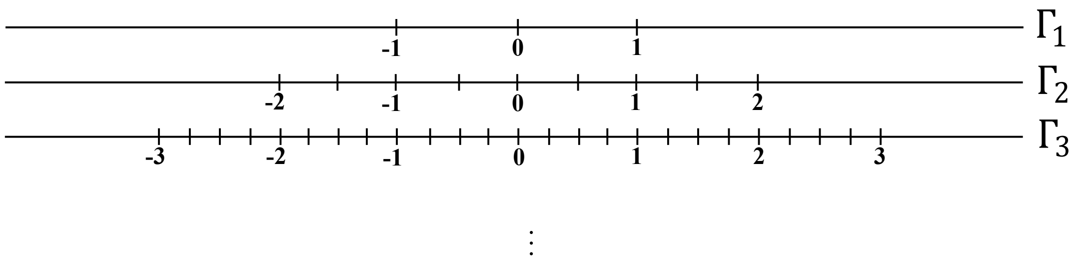


Figure 5.5: First levels of the finite fractal structure $\mathbf{\Gamma}$ on $\mathbb{R}$

We define $\omega(A) = \omega([a, b]) = F(b) - F(a)$*, where* F *denotes the cumulative distribution function of a continuous random variable for each* $A \in \Gamma_n$ *and each* $n \in \mathbb{N}$*.*

Note that, once again, $\bigcap_m St(\mathcal{C}_n, \widetilde{\Gamma}_m)$ *is countable for each* $n \in \mathbb{N}$ *by an argument similar to the previous example. In fact, it is finite for each* $n \in \mathbb{N}$*. Moreover,* $\omega(U^*_{x_n n}) \to 0$ *for each sequence* (x_n) *satisfying* $x_{n+1} \in U^*_{x_n n}$ *with* $\bigcap U^*_{x_n n} = \emptyset$*. Indeed, let* (x_n) *be a sequence satisfying* $x_{n+1} \in U^*_{x_n n}$*. Then two things may happen:*

1. *$x_1 < -1$ or $x_1 > 1$. Suppose that $x_1 < -1$. Then two things may happen and we proceed analogously in case $x_1 > 1$:*

   (a) *There exists $m \in \mathbb{N}$ such that $x_m \notin]-\infty, -m[$. Then $U^*_{x_n n} =]\frac{k_n}{2^{n-1}}, \frac{k_n+1}{2^{n-1}}[$ for some $k_n \in \{-n2^{n-1}, \dots, n2^{n-1}-1\}$ and each $n \geq m$, which means that $\omega(U^*_{x_n n}) = F(\frac{k_n+1}{2^{n-1}}) - F(\frac{k_n}{2^{n-1}})$ for each $n \geq m$ and, consequently, $\omega(U^*_{x_n n}) \to 0$.*

   (b) *$x_n \in]-\infty, -n[$ for each $n \in \mathbb{N}$. Then $\omega(U^*_{x_n n}) = F(-n) - F(-\infty) = F(-n) \to 0$.*

2. *Suppose that $x_1 \notin]-\infty, -1[\cup]1, \infty[$. Then $U^*_{x_n n} =]\frac{k_n}{2^{n-1}}, \frac{k_n+1}{2^{n-1}}[$ for some $k_n \in \{-n2^{n-1}, \dots, n2^{n-1}-1\}$ and each $n \in \mathbb{N}$, which means that $\omega(U^*_{x_n n}) = F(\frac{k_n+1}{2^{n-1}}) - F(\frac{k_n}{2^{n-1}}) \to 0$.*

Hence, by Corollary 5.17, $\mu(X) = 1$, that is, μ is a probability measure on X.

Lemma 5.21. *Let $\mathbf{\Gamma}$ be a fractal structure on X. Then $\widetilde{U}^*_{xm} \subseteq St(A, \widetilde{\Gamma}_m)$ if and only if $U^*_{xm} \subseteq St(A, \Gamma_m)$ for each $x \in X, m \in \mathbb{N}$ and each $A \subseteq X$.*

Proof. Let $x \in X$, $A \subseteq X$ and $m \in \mathbb{N}$.

$\Rightarrow$) Let $y \in U^*_{xm}$. Then $y \in \widetilde{U}^*_{xm}$ due to the fact that $U^*_{xm} \subseteq \widetilde{U}^*_{xm}$. Now, the fact that $\widetilde{U}^*_{xm} \subseteq St(A, \widetilde{\Gamma}_m)$ lets us claim that there exists $\widetilde{B} \in \widetilde{\Gamma}_m$ such that $y \in \widetilde{B}$ and $\widetilde{B} \cap A \neq \emptyset$. Now, by taking into account that $y \in X$, it follows that $y \in \widetilde{B} \cap X = B$. It also holds that $B \cap A = \widetilde{B} \cap A \neq \emptyset$ so we conclude that $y \in St(A, \Gamma_m)$.

$\Leftarrow$) Let $y \in \widetilde{U}^*_{xm}$, and let $y_k \in X$ be such that $y = (\rho_k(y_k))$. By Propositions 2.3.6, 3.16.7 and 3.16.8, it follows that $y_m \in U^*_{xm}$. Now, the fact that $U^*_{xm} \subseteq St(A, \Gamma_m)$ gives us that $y_m \in St(A, \Gamma_m)$, which means that there exists $B \in \Gamma_m$ such that $y_m \in B$ and $A \cap B \neq \emptyset$. Hence, by Proposition 3.16.2, it follows that $y \in \widetilde{B}$. Since $B \subseteq \widetilde{B}$, we have that $\widetilde{B} \cap A \neq \emptyset$. Hence, $y \in St(A, \widetilde{\Gamma}_m)$. □

In what follows, if $A \subseteq X$ is such that $A = \bigcup_{x \in A} U^*_{xn}$, then $\omega(A)$ means $\sum_{U^*_{xn} \subseteq A} \omega(U^*_{xn})$, where the last sum stands for $\sum\{\omega(U^*_{xn}) : U^*_{xn} \in G_n; U^*_{xn} \subseteq A\}$. Note that if we define ω from $\mathbf{\Gamma}$, it follows that $\sum_{U^*_{xm} \subseteq St(\mathcal{C}_n, \Gamma_m)} \omega(U^*_{xm}) = \sum_{A \in \Gamma_m; A \cap \mathcal{C}_n \neq \emptyset} \omega(A)$.

Theorem 5.22. *Let $\mathbf{\Gamma}$ be a half-complete fractal structure on X and suppose that for each $n \in \mathbb{N}$ the sequence $\omega(St(\mathcal{C}_n, \Gamma_m)) \to 0$. Then μ is a measure on the Borel σ-algebras of (X, d) and (X, d^*) and $\mu(X) = 1$.*

Proof. Given $n \in \mathbb{N}$, since $St(\mathcal{C}_n, \widetilde{\Gamma}_{m+1}) \subseteq St(\mathcal{C}_n, \widetilde{\Gamma}_m)$, by the continuity from above of $\widetilde{\mu}$, we have that $\widetilde{\mu}(St(\mathcal{C}_n, \widetilde{\Gamma}_m)) \to \widetilde{\mu}(\bigcap_{m\in\mathbb{N}} St(\mathcal{C}_n, \widetilde{\Gamma}_m))$.

On the other hand,

$$\widetilde{\mu}(St(\mathcal{C}_n, \widetilde{\Gamma}_m)) = \sum_{\widetilde{U}^*_{xm} \subseteq St(\mathcal{C}_n, \widetilde{\Gamma}_m)} \widetilde{\omega}(\widetilde{U}^*_{xm}) = \sum_{U^*_{xm} \subseteq St(\mathcal{C}_n, \Gamma_m)} \omega(U^*_{xm}) = \omega(St(\mathcal{C}_n, \Gamma_m))$$

Note that the second equality follows from the fact that $\widetilde{U}^*_{xm} \subseteq St(\mathcal{C}_n, \widetilde{\Gamma}_m)$ is equivalent to $U^*_{xm} \subseteq St(\mathcal{C}_n, \Gamma_m)$ for each $x \in X$ (see Lemma 5.21). Also note that the fact that $U^*_{xn} \neq U^*_{yn}$, implies that $\widetilde{U}^*_{xn} \neq \widetilde{U}^*_{yn}$ for each $x, y \in X$ and each $n \in \mathbb{N}$. Indeed, given $x, y \in X$ and $n \in \mathbb{N}$, if $U^*_{xn} \neq U^*_{yn}$, then the fact that $\widetilde{U}^*_{xn} \cap X = U^*_{xn}$ (see Proposition 3.16.7) lets us conclude that $\widetilde{U}^*_{xn} \neq \widetilde{U}^*_{yn}$.

In connection with the first and the third equalities, note that $St(\mathcal{C}_n, \widetilde{\Gamma}_m)$ and $St(\mathcal{C}_n, \Gamma_m)$ can be decomposed, respectively, into the disjoint union of $\widetilde{U}^*_{xm}$ and U^*_{xm} if we recall that $St(A, \Gamma) = \bigcup\{B \in \Gamma : A \cap B \neq \emptyset\}$ for each $A \subseteq X$ and each covering Γ, and that, by Proposition 3.1.2, $A = \bigcup_{x\in A} U^*_{xn}$ for each $A \in \Gamma_n$ and each $n \in \mathbb{N}$.

Now, by hypothesis, $\omega(St(\mathcal{C}_n, \Gamma_m)) \to 0$, which means that $\widetilde{\mu}(St(\mathcal{C}_n, \widetilde{\Gamma}_m)) \to 0$ and, hence, $\widetilde{\mu}(\bigcap_{m\in\mathbb{N}} St(\mathcal{C}_n, \widetilde{\Gamma}_m)) = 0$ for each $n \in \mathbb{N}$.

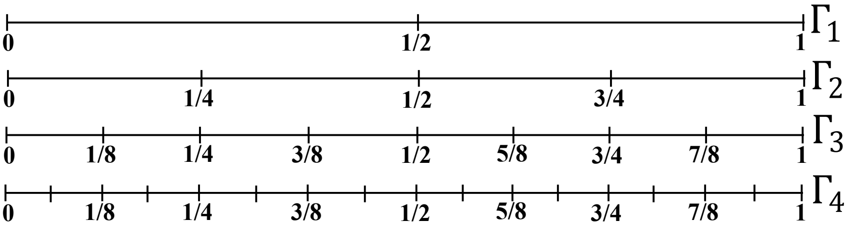
Finally, by Lemma 5.16, $\widetilde{X} \setminus X \subseteq \bigcup_{n\in\mathbb{N}} \bigcap_{m\in\mathbb{N}} St(\mathcal{C}_n, \widetilde{\Gamma}_m)$, so we have that $\widetilde{\mu}(\widetilde{X} \setminus X) \leq \widetilde{\mu}(\bigcup_{n\in\mathbb{N}} \bigcap_{m\in\mathbb{N}} St(\mathcal{C}_n, \widetilde{\Gamma}_m)) \leq \sum_{n=1}^{\infty} \widetilde{\mu}(\bigcap_{m\in\mathbb{N}} St(\mathcal{C}_n, \widetilde{\Gamma}_m))) = 0$. Therefore, $\widetilde{\mu}(\widetilde{X} \setminus X) = 0$ and, by Theorem 5.3, X is $\widetilde{\mu}$-measurable and $\mu(X) = 1$, that is, μ is a probability measure on the Borel σ-algebras of (X, d) and (X, d^*).

□

Definition 5.23. *([54, Def. 2.3]) Let* $\mathbf{\Gamma}$ *be a fractal structure on* X*. We will say that* $\mathbf{\Gamma}$ *is starbase if* $\{St(x, \Gamma_n) : n \in \mathbb{N}\}$ *is a neighborhood base of* x *for each* $x \in X$*.*

The next example shows another situation where, starting from a fractal structure on $[0, 1]$, we can define a pre-measure such that its extension is a probability measure generated by a known continuous random variable and given by its cumulative distribution function.

Example 5.24. *Consider* $([0, 1], \mathbf{\Gamma})$*, where* $\mathbf{\Gamma}$ *is the natural fractal structure (see Figure 5.6). The topology induced by* $\mathbf{\Gamma}$ *is the usual one and* $\mathbf{\Gamma}$ *is a tiling finite fractal structure.*



$\vdots$

Figure 5.6: First levels of the natural fractal structure on $[0, 1]$

We define, for this fractal structure, $\omega(A) = \omega([a, b]) = F(b) - F(a)$ for each $A \in \Gamma_n$ and each $n \in \mathbb{N}$, where F denotes a cumulative distribution function of a continuous random variable such that $F(x) = 0$ for each $x < 0$ and $F(x) = 1$ for each $x \geq 1$. It gives us a measure on X according to the construction made previously.

Note that, under the above conditions, $\mathbf{\Gamma}$ is a half-complete tiling starbase fractal structure. Moreover, $\mathcal{C}_n$ is finite for each $n \in \mathbb{N}$, which means that the convergence of $\omega(St(\mathcal{C}_n, \Gamma_m))$ to 0 can be proven easily for each $n \in \mathbb{N}$. In fact, observe that the number of points in $\mathcal{C}_n$ is $2^n - 1$ for each $n \in \mathbb{N}$. Thus, we can write $\mathcal{C}_n = \{a_k : k = 1, \ldots, 2^n - 1\}$ for each $n \in \mathbb{N}$. Hence, $\omega(St(\mathcal{C}_n, \Gamma_m)) = \sum_{A \in \Gamma_m, A \cap \mathcal{C}_n \neq \emptyset} \omega(A) = \sum_{k=1}^{2^n-1} (F(a_k + \frac{1}{2^m}) - F(a_k - \frac{1}{2^m})) \to \sum_{k=1}^{2^n-1} (F(a_k) - F(a_k)) = 0$. Hence, by Theorem 5.22 we have that μ is a probability measure on X.

Note that if $A \cap B = \emptyset$ for each $A, B \in \Gamma_n$ and each n, then $\mathcal{C}_n = \emptyset$ for each n, which implies that $\omega(St(\mathcal{C}_n, \Gamma_m)) \to 0$, and, consequently, by Theorem 5.22, we have proven the next result.

Corollary 5.25. *If $\mathbf{\Gamma}$ is a half-complete fractal structure on X such that $A \cap B = \emptyset$ for each $A, B \in \Gamma_n$ with $A \neq B$ and for each n, then μ is a probability measure on X.*

From $\mathcal{C}_n$ we define a set that will let us give some necessary and sufficient conditions in order to get a probability measure on X. It can bee seen next.

Definition 5.26. *For each $n \in \mathbb{N}$ we define $\widehat{\mathcal{C}_n} = \bigcup_{x \in \mathcal{C}_n} \bigcap_{m \in \mathbb{N}} St(x, \widetilde{\Gamma}_m)$.*

Lemma 5.27. *Let $\mathbf{\Gamma}$ be a half-complete starbase fractal structure on X. Then $\widetilde{X} \setminus X = \bigcup_{n \in \mathbb{N}} \widehat{\mathcal{C}_n} \setminus \mathcal{C}_n$.*

Proof. $\subseteq$) Let $\widetilde{x} \in \widetilde{X} \setminus X$. Then there exists $x_n \in X$ such that $\widetilde{x} = (\rho_n(x_n))$. Since $\mathbf{\Gamma}$ is half-complete, by Remark 5.14, there exists $y \in X$ such that $x_n \to y$. Suppose now that $y \notin \bigcup_{m \in \mathbb{N}} \mathcal{C}_m$. By Remark 5.15, $x_n \in U_{yn}$ for each $n \in \mathbb{N}$ and, hence, $y \in U_{x_n n}^{-1}$ for each $n \in \mathbb{N}$. Thus, $x_n \notin \mathcal{C}_n$ for each $n \in \mathbb{N}$ (indeed, if $x_n \in A_n \cap B_n$, then $y \in A_n \cap B_n$, a contradiction with the initial assumption). Now, the fact that $x_n \notin \mathcal{C}_n$ for each $n \in \mathbb{N}$ lets us claim that for each $n \in \mathbb{N}$ there exists a unique $A \in \Gamma_n$ such that $x_n \in A$. Thus, $U_{x_n n}^{-1} = \bigcap_{x \in B, B \in \Gamma_n} B = A$ and, since $y \in U_{x_n n}^{-1}$ and $y \notin \mathcal{C}_n$, A is the only element in Γ_n that contains y. Hence, x_n and y belong to the same elements of the fractal structure, which implies that $U_{yn}^* = U_{x_n n}^*$ for each $n \in \mathbb{N}$. It follows that $\widetilde{x} = y \in X$, a contradiction. Consequently, there exists $m \in \mathbb{N}$ such that $y \in \mathcal{C}_m$. Now, given $n \in \mathbb{N}$, $y \in U_{x_n n}^{-1}$, which implies that $x_n \in St(y, \Gamma_n)$ for each $n \in \mathbb{N}$, that is, $\widetilde{x} \in St(y, \widetilde{\Gamma}_n)$ for each $n \in \mathbb{N}$ and, hence, $\widetilde{x} \in \bigcap_{n \in \mathbb{N}} St(y, \widetilde{\Gamma}_n)$. If we join this fact with $y \in \mathcal{C}_m$, it follows that $\widetilde{x} \in \widehat{\mathcal{C}}_m$.

$\supseteq$) Let $x \in \widehat{\mathcal{C}}_n \setminus \mathcal{C}_n$ for some $n \in \mathbb{N}$. Then $x \in St(y, \widetilde{\Gamma}_m) \setminus \mathcal{C}_n$ for some $y \in \mathcal{C}_n$ and each $m \in \mathbb{N}$. Now, suppose that $x \in X$. Then $x \in St(y, \widetilde{\Gamma}_m) \cap X = St(y, \Gamma_m)$ for each $m \in \mathbb{N}$. Since $\mathbf{\Gamma}$ is starbase, it follows that $y = x$, which means that $x \in \mathcal{C}_n$, which contradicts the initial assumption.

□

Theorem 5.28. *Let $\mathbf{\Gamma}$ be a half-complete starbase fractal structure on X such that, for each $n \in \mathbb{N}$, $\mathcal{C}_n$ is countable and, for each $x \in \mathcal{C}_n$, suppose that $\omega(St(x, \Gamma_m)) \to 0$. Then μ is a measure on the Borel σ-algebras of (X, d) and (X, d^*) and $\mu(X) = 1$.*

Proof. Given $n \in \mathbb{N}$ and $x \in \mathcal{C}_n$, since $St(x, \widetilde{\Gamma}_{m+1}) \subseteq St(x, \widetilde{\Gamma}_m)$, by the continuity from above of $\widetilde{\mu}$ and the monotonicity of the sequence $St(x, \widetilde{\Gamma}_m)$, we have that $\widetilde{\mu}(St(x, \widetilde{\Gamma}_m)) \to \widetilde{\mu}(\bigcap_{m \in \mathbb{N}} St(x, \widetilde{\Gamma}_m))$.

On the other hand,

$$\widetilde{\mu}(St(x, \widetilde{\Gamma}_m)) = \sum_{\widetilde{U}_{ym}^* \subseteq St(x, \widetilde{\Gamma}_m)} \widetilde{\omega}(\widetilde{U}_{ym}^*) = \sum_{U_{ym}^* \subseteq St(x, \Gamma_m)} \omega(U_{ym}^*) = \omega(St(x, \Gamma_m))$$

Note that the second equality follows from the fact that $\widetilde{U}_{ym}^* \subseteq St(x, \widetilde{\Gamma}_m)$ is equivalent to $U_{ym}^* \subseteq St(x, \Gamma_m)$ for each $y \in X$ (see Lemma 5.21).

By hypothesis, $\omega(St(x, \Gamma_m)) \to 0$, so $\widetilde{\mu}(St(x, \widetilde{\Gamma}_m)) \to 0$ and, hence, $\widetilde{\mu}(\bigcap_{m \in \mathbb{N}} St(x, \widetilde{\Gamma}_m)) = 0$ for each $n \in \mathbb{N}$.

Finally, by the previous lemma, we have that $\widetilde{X} \setminus X = \bigcup_{n\in\mathbb{N}} \widehat{\mathcal{C}_n} \setminus \mathcal{C}_n$, so $\widetilde{\mu}(\widetilde{X} \setminus X) = \widetilde{\mu}(\widehat{\mathcal{C}_n} \setminus \mathcal{C}_n) \leq \widetilde{\mu}(\widehat{\mathcal{C}_n}) = \widetilde{\mu}(\bigcup_{x\in\mathcal{C}_n} \bigcap_{m\in\mathbb{N}} St(x, \widetilde{\Gamma}_m)) \leq \sum_{x\in\mathcal{C}_n} \widetilde{\mu}(\bigcap_{m\in\mathbb{N}} St(x, \widetilde{\Gamma}_m))) = 0$. Note that we have taken into account the fact that $\mathcal{C}_n$ is countable for each $n \in \mathbb{N}$ when writing the last inequality. Therefore, $\widetilde{\mu}(\widetilde{X} \setminus X) = 0$ and, by Theorem 5.3, X is $\widetilde{\mu}$-measurable and $\mu(X) = 1$, that is, μ is a probability measure on the Borel σ-algebras of (X, d) and (X, d^*). □

For the next example, we can define a fractal structure, $\mathbf{\Delta}$, on X from a finite fractal structure, $\mathbf{\Gamma}$, on $[0, 1]$ and a known random variable such that $\omega(A)$ is uniform for each $A \in \Delta_n$ and each $n \in \mathbb{N}$.

Example 5.29. *Let X be the extended real line $X = \mathbb{R} \cup \{-\infty, \infty\}$ and $F : X \to [0, 1]$ be the extension of an injective probability distribution function of a continuous random variable and let $\mathbf{\Gamma}$ be the natural fractal structure. The levels of the new fractal structure, $\mathbf{\Delta}$, are determined from $\mathbf{\Gamma}$ as follows: $\Delta_n = \{F^{-1}(A) : A \in \Gamma_n\}$ for each $n \in \mathbb{N}$.*

Let $B \in \Delta_n$. It is clear that we can write $B = \left[F^{-1}\left(\frac{k}{2^n}\right), F^{-1}\left(\frac{k+1}{2^n}\right)\right] = F^{-1}\left(\left[\frac{k}{2^n}, \frac{k+1}{2^n}\right]\right)$ and, thus, $\omega(B) = \frac{1}{2^n}$ for each $B \in \Delta_n$ and each level n of the fractal structure $\mathbf{\Delta}$.

We prove that $\mathbf{\Delta}$ is starbase. At first, observe that $U_{xn} \subseteq St(x, \Delta_n)$ for each $x \in X$ and $n \in \mathbb{N}$. Moreover, given $n \in \mathbb{N}$, $U_{xn} =]a, b[$, where $a = F^{-1}(\frac{k}{2^n})$ and $b = F^{-1}(\frac{k'}{2^n})$ for some $k \in \{0, \ldots 2^n - 1\}$ and $k' = k + 1$ or $k' = k + 2$. It is clear that $a < x < b$, while $St(x, \Delta_n) = [a, b]$. Hence, it is sufficient to consider $m \in \mathbb{N}$ such that $F(a) < F(x) - \frac{1}{2^m}$ and $F(x) + \frac{1}{2^m} < F(b)$, since, in this case, it holds that $F(St(x, \Delta_m)) \subseteq [F(x) - \frac{1}{2^m}, F(x) + \frac{1}{2^m}] \subseteq]F(a), F(b)[= F(U_{xn})$. Since $F : X \to [0, 1]$ is bijective, it follows that $St(x, \Delta_m) \subseteq U_{xn}$. We conclude that $\{St(x, \Delta_n) : n \in \mathbb{N}\}$ is a neighborhood base of x for each $x \in X$.

Note that $\mathbf{\Gamma}$ is half-complete on X. Indeed, if $x_n \in X$ is such that $x_{n+1} \in U^*_{x_n n}$, then $F(x_{n+1}) \in U^*_{F(x_n)n}$ and, since $\mathbf{\Gamma}$ is half-complete on $[0, 1]$, there exists $y \in [0, 1]$ such that $(F(x_n))$ converges to y. Since X is compact and F is bijective, F^{-1} is continuous, and, hence, (x_n) converges to $F^{-1}(y)$. Therefore, $\mathbf{\Gamma}$ is half-complete on X.

On the other hand, $\mathcal{C}_n = \{F^{-1}(\frac{k}{2^n}) : 0 < k < 2^n\}$ and, hence, it is countable. It also holds that if $x \in \mathcal{C}_n$, $St(x, \Gamma_m)$ is the union of exactly two elements of Γ_m and, hence, $\omega(St(x, \Gamma_m)) = \frac{1}{2^{m-1}}$. Therefore, we can apply the previous theorem to get that μ is a probability measure on X.

A particular instance of the previous example is the following one.

Example 5.30. *Consider an exponential random variable $R \sim \varepsilon(1)$. Then its probability distribution function $F \to [0,1]$ is defined by*

$$F_R(x) = \begin{cases} 0 & \text{if} \quad x < 0 \\ \\ 1 - e^{-x} & \text{if} \quad x \geq 0 \end{cases}$$

If we define the natural fractal structure on $[0,1]$, we can define on $X = [0,\infty]$ the fractal structure $\mathbf{\Delta}$, where $\Delta_n = \{F^{-1}(A) : A \in \Gamma_n\}$ is described as:

$\Delta_1 = \{F^{-1}\left(\left[0, \frac{1}{2}\right]\right), \left(\left[\frac{1}{2}, 1\right]\right)\} = \{[0, \ln 2], [\ln 2, +\infty]\}$

$\Delta_2 = \{[0, \ln \frac{4}{3}], [\ln \frac{4}{3}, \ln 2], [\ln 2, \ln 4], [\ln 4, +\infty]\}$

$$\vdots$$

Hence, $\omega(A) = \frac{1}{2^n}$ for each $A \in \Delta_n$ and each $n \in \mathbb{N}$, and μ is a probability measure on X.

Lemma 5.31. *Let $\mathbf{\Gamma}$ be a starbase fractal structure on X. Then $\bigcap_{m\in\mathbb{N}} St(A, \Gamma_m) = A$ for each closed subset A of X.*

Proof. $\supseteq$) It is clear that $A \subseteq St(A, \Gamma_m)$ for each $m \in \mathbb{N}$.

$\subseteq$) Let $x \in \bigcap_{m\in\mathbb{N}} St(A, \Gamma_m)$. The case in which $x \in A$ is clear. Suppose that $x \notin A$. The fact that A is closed implies that there exists $m \in \mathbb{N}$ such that $St(x, \Gamma_m) \cap A = \emptyset$, which is a contradiction with the fact that $x \in St(A, \Gamma_m)$. Hence, $x \in A$. □

Lemma 5.32. *Let $\mathbf{\Gamma}$ be a starbase and half-complete fractal structure on X. Then $\bigcup_{A,B\in\Gamma_n; A\neq B} \bigcap_{m\in\mathbb{N}} St(A\cap B, \widetilde{\Gamma}_m) = \widehat{\mathcal{C}}_n$ for each $n \in \mathbb{N}$.*

Proof. $\subseteq$) Let $n \in \mathbb{N}$ and $A, B \in \Gamma_n$ with $A \neq B$ and consider $\widetilde{x} \in \bigcap_{m\in\mathbb{N}} St(A\cap B, \widetilde{\Gamma}_m)$. Then there exists a sequence (x_m) such that $\widetilde{x} = (\rho_m(x_m))$. Since $\mathbf{\Gamma}$ is half-complete, by Remark 5.14, there exists $x \in X$ such that $x_m \to x$. By Remark 5.15, $x_m \in U_{xm}$. Consequently, $x \in U_{x_m m}^{-1}$ for each $m \in \mathbb{N}$. Since $\widetilde{x} \in St(A\cap B, \widetilde{\Gamma}_m)$ for each $m \in \mathbb{N}$, given $m \in \mathbb{N}$, $x_m \in St(A \cap B, \Gamma_m)$. Let A_m be such that $x_m \in A_m$ with $A_m \cap (A \cap B) \neq \emptyset$.

Then $x \in A_m$ and $x \in St(A \cap B, \Gamma_m)$. It follows that $x \in \bigcap_{m\in\mathbb{N}} St(A \cap B, \Gamma_m) = A \cap B$ by Lemma 5.31.

Since $x \in U^{-1}_{x_m m} \subseteq St(x_m, \Gamma_m) \subseteq St(\widetilde{x}, \widetilde{\Gamma}_m)$ for each $m \in \mathbb{N}$, it follows that $\widetilde{x} \in \bigcap_{m\in\mathbb{N}} St(x, \widetilde{\Gamma}_m)$. Since $x \in A \cap B \subseteq \mathcal{C}_n$, it follows that $\widetilde{x} \in \widehat{\mathcal{C}}_n$.

$\supseteq$) Let $\widetilde{x} \in \widehat{\mathcal{C}}_n$. Then, by definition of $\widehat{\mathcal{C}}_n$, there exists $x \in \mathcal{C}_n$ such that $\widetilde{x} \in St(x, \widetilde{\Gamma}_m)$ for each $m \in \mathbb{N}$. Since $x \in \mathcal{C}_n$, $x \in A \cap B$ for some $A, B \in \Gamma_n$ with $A \neq B$. The fact that there exist $A, B \in \Gamma_n$ such that $x \in A \cap B$ means that $\widetilde{x} \in St(x, \widetilde{\Gamma}_m) \subseteq St(A \cap B, \widetilde{\Gamma}_m)$ for each $m \in \mathbb{N}$. Hence, $\widetilde{x} \in \bigcup_{A,B\in\Gamma_n; A\neq B} \bigcap_{m\in\mathbb{N}} St(A \cap B, \widetilde{\Gamma}_m)$.

□

Lemma 5.33. *Let $\mathbf{\Gamma}$ be a fractal structure on X and consider $A, B \in \Gamma_n$ for some $n \in \mathbb{N}$. Then $A \cap B = \bigcup_{x\in A\cap B} U^*_{xn}$.*

Proof. Let $n \in \mathbb{N}$ and consider $A, B \in \Gamma_n$.

$\subseteq$) Given $y \in A \cap B$, it is clear that $y \in \bigcup_{x\in A\cap B} U^*_{xn}$.

$\supseteq$) Let $y \in \bigcup_{x\in A\cap B} U^*_{xn}$. Then there exists $x \in A \cap B$ such that $y \in U^*_{xn}$. By Proposition 2.3.5, we can write $U^*_{xn} = \bigcap_{x\in D, D\in\Gamma_n} D \setminus \bigcup_{x\notin D, D\in\Gamma_n} D$. Since $\bigcap_{x\in D, D\in\Gamma_n} D \subseteq A \cap B$, we conclude that $U^*_{xn} \subseteq A \cap B$ and, hence, $y \in A \cap B$.

□

Recall that, for a set A such that $A = \bigcup_{x\in A} U^*_{xn}$, we are using the notation $\omega(A) = \sum_{U^*_{xn}\subseteq A} \omega(U^*_{xn})$. Note that the sets U^*_{xn} and U^*_{yn} are mutually disjoint or they are the same set and that in the sum, $\omega(U^*_{xn})$ appears only once for each set U^*_{xn} with $x \in A$.

It follows that if μ is an extension of ω and A is a set such that $A = \bigcup_{x\in A} U^*_{xn}$, then $\mu(A) = \omega(A)$. Indeed, since μ is a measure, the union is disjoint and μ is an extension of ω, it follows that $\mu(A) = \sum_{U^*_{xn}\subseteq A} \mu(U^*_{xn}) = \sum_{U^*_{xn}\subseteq A} \omega(U^*_{xn}) = \omega(A)$.

Examples of sets A such that $A = \bigcup_{x\in A} U^*_{xn}$ are the following ones: A for $A \in \Gamma_n$ with $n \in \mathbb{N}$; $A \cap B$ for $A, B \in \Gamma_n$ with $n \in \mathbb{N}$ (by the previous lemma); $St(A, \Gamma_n)$ for $n \in \mathbb{N}$ and $A \subseteq X$.

Another necessary condition of μ being a probability measure is the next one:

Proposition 5.34. *Let $\mathbf{\Gamma}$ be a starbase fractal structure on X, a $\widetilde{\mu}$-measurable space. If $\mu(X) = 1$, then $\omega(St(A \cap B, \Gamma_m)) \to \omega(A \cap B)$ for each $A, B \in \Gamma_n$ with $A \neq B$ and* *each $n \in \mathbb{N}$.*

Proof. Note that $\mu(X) = 1$ implies that $\widetilde{\mu}(\widetilde{X} \setminus X) = 0$, since X is supposed to be $\widetilde{\mu}$-measurable. Since $\widetilde{\mu}(\widetilde{X} \setminus X) = 0$, by Theorem 5.3, $\omega(U^*_{xn}) = \mu(U^*_{xn})$ for each $x \in X$ and $n \in \mathbb{N}$, that is, μ is an extension of ω. By the previous discussion, $\mu(St(A \cap B, \Gamma_m)) = \omega(St(A \cap B, \Gamma_m))$. Note that $A \cap B$ is closed, since A and B are both closed (see Proposition 3.1.3). Hence, by Lemma 5.31, $\bigcap_{m\in\mathbb{N}} St(A \cap B, \Gamma_m) = A \cap B$. By the continuity from above of the measure μ, it follows that $\mu(St(A \cap B, \Gamma_m)) \to \mu(\bigcap_{m\in\mathbb{N}} St(A \cap B, \Gamma_m)) = \mu(A \cap B)$. Finally, Lemma 5.33 and the previous discussion let us claim that $\mu(A \cap B) = \omega(A \cap B)$, what concludes the proof.

□

As it has already been clarified, the main goal of this chapter is looking for conditions to ensure that μ is a probability measure on X. So far, we have given, mostly, some necessary conditions to ensure that μ is a probability measure. Moreover, so far, all the results involved conditions on $\widetilde{X}$ or structures related to the completion of the space. However, it is more convenient to have conditions on X or functions defined on X in order to characterize the fact that μ is a probability measure on X. By taking advantage of the previous proposition we can give some results in this line.

Corollary 5.35. *Let $\mathbf{\Gamma}$ be a half-complete and starbase fractal structure on X such that $\omega(\mathcal{C}_n) = 0$ and Γ_n is countable for each $n \in \mathbb{N}$. Then $\mu(X) = 1$ and X is $\widetilde{\mu}$-measurable if and only if $\omega(St(A \cap B, \Gamma_m)) \to 0$ for each $A, B \in \Gamma_n$ with $A \neq B$ and each $n \in \mathbb{N}$.*

Proof. $\Rightarrow$) Note that, given $n \in \mathbb{N}$, $\omega(A \cap B) = 0$ for each $A, B \in \Gamma_n$ with $A \neq B$, since $\omega(\mathcal{C}_n) = 0$, so the implication immediately follows from Proposition 5.34.

$\Leftarrow$) If we prove that $\widetilde{\mu}(\widetilde{X} \setminus X) = 0$, we will have, by Theorem 5.3, that X is $\widetilde{\mu}$-measurable and that μ is a probability measure on X. On the one hand, note that $\widetilde{\mu}(\widetilde{X} \setminus X) \leq \sum_{n\in\mathbb{N}} \widetilde{\mu}(\widehat{\mathcal{C}}_n \setminus \mathcal{C}_n) \leq \sum_{n\in\mathbb{N}} \widetilde{\mu}(\widehat{\mathcal{C}}_n)$ as a consequence of the fact that $\widetilde{X} \setminus X = \bigcup_{n\in\mathbb{N}} \widehat{\mathcal{C}}_n \setminus \mathcal{C}_n$ (see Lemma 5.27). Now, by Lemma 5.32, it holds that $\widetilde{\mu}(\widehat{\mathcal{C}}_n) \leq \sum_{A,B\in\Gamma_n; A\neq B} \widetilde{\mu}(\bigcap_{m\in\mathbb{N}} St(A \cap B, \widetilde{\Gamma}_m))$. Indeed, recall that $\{(A, B) \in \Gamma_n \times \Gamma_n : A \neq B\}$ is countable due to the initial assumption that Γ_n is countable for each $n \in \mathbb{N}$.

On the other hand, since $St(A \cap B, \widetilde{\Gamma}_m)$ is a monotonically non-decreasing sequence of sets, the continuity from above of $\widetilde{\mu}$ gives us that $\widetilde{\mu}(St(A \cap B, \widetilde{\Gamma}_m)) \to \widetilde{\mu}(\bigcap_{m\in\mathbb{N}} St(A \cap$

$B, \widetilde{\Gamma}_m)$). It also holds that

$$\widetilde{\mu}(St(A\cap B, \widetilde{\Gamma}_m)) = \sum_{\widetilde{U}^*_{ym} \subseteq St(A\cap B, \widetilde{\Gamma}_m)} \widetilde{\omega}(\widetilde{U}^*_{ym}) = \sum_{U^*_{ym} \subseteq St(A\cap B, \Gamma_m)} \omega(U^*_{ym}) = \omega(St(A\cap B, \Gamma_m))$$

for each $A, B \in \Gamma_n$ and each $n \in \mathbb{N}$.

Note that the second equality follows from the fact that $\widetilde{U}^*_{xm} \subseteq St(A \cap B, \widetilde{\Gamma}_m)$ is equivalent to $U^*_{xm} \subseteq St(A \cap B, \Gamma_m)$ for each $x \in X$ (see Lemma 5.21). Since, by hypothesis, $\omega(St(A \cap B, \Gamma_m)) \to 0$ for each $A, B \in \Gamma_n$ and each $n \in \mathbb{N}$, it follows that $\widetilde{\mu}(\bigcap_{m\in\mathbb{N}} St(A \cap B, \widetilde{\Gamma}_m)) = 0$.

Finally, $\widetilde{\mu}(\widehat{\mathcal{C}}_n) \leq \sum_{A,B\in\Gamma_n; A\neq B} \widetilde{\mu}(\bigcap_{m\in\mathbb{N}} St(A \cap B, \widetilde{\Gamma}_m)) = 0$. Hence, $\widetilde{\mu}(\widetilde{X} \setminus X) = 0$ and we conclude the proof. □

Example 5.36. *Consider the space* $([0,1]^2, \mathbf{\Gamma})$*, where* $\mathbf{\Gamma}$ *is the fractal structure whose levels are defined by* $\Gamma_n = \{[\frac{k_1}{2^n}, \frac{k_1+1}{2^n}] \times [\frac{k_2}{2^n}, \frac{k_2+1}{2^n}] : k_1, k_2 \in \{0, \ldots, 2^n - 1\}\}$*. Now, consider the pre-measure of each element in* Γ_n *given by* $\omega([a,b] \times [c,d]) = (b-a)(d-c)$*. It is easy to check that* $\omega(\mathcal{C}_n) = 0$ *for each* $n \in \mathbb{N}$ *and that* $\omega(St(A \cap B, \Gamma_m)) \to 0$ *for each* $A, B \in \Gamma_n$ *with* $A \neq B$ *and each* $n \in \mathbb{N}$*. Hence, by the previous corollary, we can claim that* μ *is a probability measure on* $[0,1]^2$*.*

We can also give a characterization of the fact that μ is a probability measure on X for the case in which μ is constructed from a pre-measure defined from a fractal structure.

Corollary 5.37. *Let* $\mathbf{\Gamma}$ *be a half-complete, starbase (and tiling) fractal structure on* X *and suppose that* ω *is defined from* $\mathbf{\Gamma}$*. Then* $\mu(X) = 1$ *and* X *is* $\widetilde{\mu}$*-measurable if and only if* $\omega(St(A \cap B, \Gamma_m)) \to 0$ *for each* $A, B \in \Gamma_n$ *with* $A \neq B$ *and each* $n \in \mathbb{N}$*.*

Proof. Note that the fact that ω is defined from $\mathbf{\Gamma}$ implies that $\omega(A \cap B) = 0$ for each $A, B \in \Gamma_n$ with $A \neq B$ and each $n \in \mathbb{N}$. Hence, $\omega(\mathcal{C}_n) = 0$ for each $n \in \mathbb{N}$ and we can take into account the previous corollary to justify the equivalence to proof. □

Example 5.38. *Let* X *be the Sierpinski triangle and* $f_1, f_2, f_3 : X \to X$ *the similitudes that define this fractal. Recall, from Example 4.29, the way to distribute the mass uniformly on this set. According to that construction,* $\omega(A) = \frac{1}{3^n}$ *for each* $A \in \Gamma_n$ *and* $n \in \mathbb{N}$*, where the fractal structure* $\mathbf{\Gamma}$ *is defined from the iterated system and, hence, given* *by the levels $\Gamma_n = \{f_w^n(X) : w \in I^n\}$ for each $n \in \mathbb{N}$, where $f_w^n = f_{w_1} \circ \ldots \circ f_{w_n}$ with $w = w_1 \ldots w_n$. Hence, this is an example where ω is defined from $\mathbf{\Gamma}$.*

Note that the topology induced by $\mathbf{\Gamma}$ on X is the usual topology. Since $St(x, \Gamma_n)$ has diameter (with respect to the Euclidean metric) at most $\frac{1}{2^{n-1}}$, it follows that $\mathbf{\Gamma}$ is starbase. On the other hand, since X is a Hausdorff compact space and $\mathbf{\Gamma}$ is finite, it follows that d is half-complete by [7, Cor. 5.6].

On the other hand, note that $St(A \cap B, \Gamma_m)$ consists of two triangles for each $A, B \in \Gamma_n$ and each $n \in \mathbb{N}$ with $A \cap B \neq \emptyset$. Hence, $\omega(St(A \cap B, \Gamma_m)) = \frac{2}{3^m}$. It holds that $\omega(St(A \cap B, \Gamma_m)) \to 0$, so Corollary 5.37 lets us claim that μ is a probability measure on X.

5.2 Uniqueness of the measure

Proposition 5.39. *Let δ be a probability measure on the Borel σ-algebra of X and let $\omega(U_{xn}^*) = \delta(U_{xn}^*)$ for each $x \in X$ and $n \in \mathbb{N}$. The next statements are satisfied:*

1. *ω satisfies the mass distribution conditions.*
2. *$\mu = \delta$ on the Borel σ-algebra of X.*

Proof. 1. (a) $\sum_{\rho_1(x) \in G_1} \omega(\rho_1(x)) = 1$. Indeed, it holds that $\sum_{\rho_1(x) \in G_1} \omega(\rho_1(x)) = \sum_{\rho_1(x) \in G_1} \delta(\rho_1(x)) = \delta(X) = 1$, since δ is a probability measure on X.

(b) $\omega(\rho_n(x)) = \sum_{\rho_{n+1}(y) \in G_{n+1}, \rho_n(y) = \rho_n(x)} \omega(\rho_{n+1}(y))$ for each $n \in \mathbb{N}$ and each $\rho_n(x) \in G_n$. Let $x \in X$ and $n \in \mathbb{N}$. Then $\omega(\rho_n(x)) = \delta(\rho_n(x))$. Now, by the σ-additivity of δ as a measure, $\omega(\rho_n(x)) = \sum_{\rho_{n+1}(y) \in G_{n+1}, \rho_n(y) = \rho_n(x)} \delta(\rho_{n+1}(y)) = \sum_{\rho_{n+1}(y) \in G_{n+1}, \rho_n(y) = \rho_n(x)} \omega(\rho_{n+1}(x))$, what concludes the proof of this item.

2. First of all, we show that $\mu(U_{xn}^*) = \delta(U_{xn}^*)$ for each $x \in X$ and $n \in \mathbb{N}$.

$\geq$) Let δ^* be the outer measure induced by δ. If we consider the pre-measure ω and the family U_{xn}^*, by the first point of Method I, we have that $\delta^*(U_{xn}^*) \leq \mu(U_{xn}^*)$. Now, the fact that δ^* is an extension of δ (see Theorem 2.16) gives us that $\delta(U_{xn}^*) \leq \mu(U_{xn}^*)$.

$\leq$) Now, note that $U_{xn}^* \subseteq \widetilde{U}_{xn}^*$ so it follows that $\mu(U_{xn}^*) \leq \widetilde{\omega}(\widetilde{U}_{xn}^*)$. Moreover, $\widetilde{\omega}(\widetilde{U}_{xn}^*) = \omega(U_{xn}^*) = \delta(U_{xn}^*)$.

Now, recall that $U_n^*(F) = \bigcup_{y \in F} U_{yn}^*$, and this is a countable union of mutually disjoint sets. Since μ and δ are measures, and taking into account that $\mu(U_{xn}^*) = \delta(U_{xn}^*)$, it follows that $\delta(U_n^*(F)) = \mu(U_n^*(F))$.

Finally, let $\mathfrak{A} = \{U_n^*(F) : F \subseteq X, n \in \mathbb{N}\}$. It can be proven that $\mathfrak{A}$ is an algebra (where μ and δ coincide, as proved before) generating the Borel σ-algebra of (X, d^*). By Theorem 2.15, it follows that $\mu = \delta$ on the Borel σ-algebra of (X, d^*) (a more detailed proof can be found in the proof of Proposition 4.13). Finally, note that, according to Remark 4.5, the Borel σ-algebras of (X, d^*) and (X, d) are the same, so it follows that $\mu = \delta$ on X with respect to $\sigma(\tau_d)$ and $\sigma(\tau_d^*)$.

$\square$

Corollary 5.40. *If δ is a probability measure on the Borel σ-algebra of X and ω is defined as $\omega(U_{xn}^*) = \delta(U_{xn}^*)$, then $\mu(U_{xn}^*) = \delta(U_{xn}^*)$ and $\mu(U_n^*(F)) = \delta(U_n^*(F))$ for each $x \in X$, $n \in \mathbb{N}$ and $F \subseteq X$.*

Proposition 5.39 lets us claim that each probability measure defined on a space with a fractal structure can be construted by using the procedure based on a pre-measure that we develop in this chapter together with the results given in the previous one. It also lets us claim that, in case that μ is a probability measure defined from ω, it is the unique one defined from that pre-measure.

Chapter 6

Applications

To end this first part of the work, we show some applications that have arisen from the theory that was developed in the previous chapters. All of these applications are based on the recursive nature of the fractal structure together with the construction of probability measures shown before. First of all, once we have defined a pre-measure on the elements of the fractal structure, it does make sense to create an iterative method to generate samples of a distribution on a space equipped with that topological structure. What is more, since a fractal structure can be defined on a n-dimensional space, we can use that procedure, which we explain in Section 6.1, to generate samples of random vectors. Secondly, in Section 6.2 we introduce an estimation method to get the parameters of a certain distribution once we are given a random sample of that. That method is based on a similar idea to the maximum likelihood estimation method, although we will explore some situations for which the new method becomes better than the classical one. Finally, Section 6.3 shows a way to test if a random sample comes from a certain distribution. The idea is similar to the one used by the χ^2 test, also well known in the classical case.

6.1 Generating samples of a distribution

The first application we introduce in this chapter consists of generating samples of a certain distribution. That distribution is associated with a probability measure that can be defined from a pre-measure defined on a space with a fractal structure by following the procedure which has been developed in the previous two chapters.

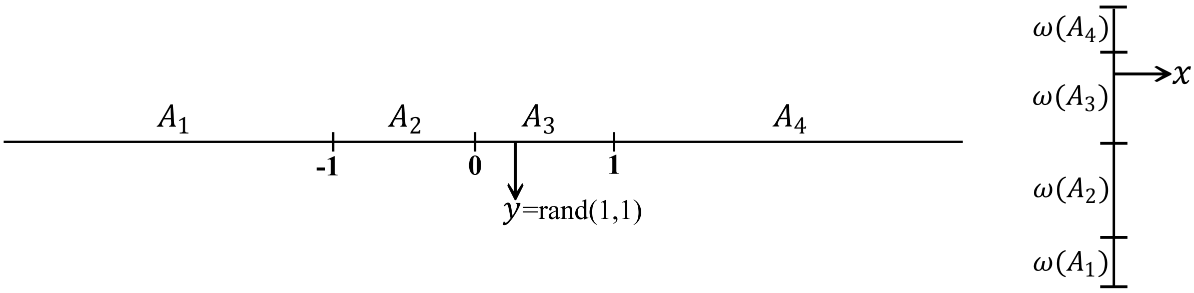
We start from a finite fractal structure, $\mathbf{\Gamma}$, on a space, X. Now, given $n \in \mathbb{N}$, we can define the pre-measure of $A \in \Gamma_n$, which we will denote by $\omega(A)$, according to the cumulative distribution function of the random variable Y for what we want a sample. Indeed, in Example 5.20 we showed that when F is the cumulative distribution function of a continuous random variable and the elements of the fractal structure can be written as $A = [a, b]$, we define ω by $\omega(A) = F(b) - F(a)$ and it gives us a probability measure on X according to the construction made in the previous chapters. What is more, if the elements of each level on the fractal structure can be written as $A =]a, b]$, then $\omega(A)$ defined as before gives us a probability measure from a random variable regardless of whether it is continuous or not. Note that the proof of the previos fact is similar to the one made in Example 5.20. Also, observe that we can enumerate the elements in Γ_n for each $n \in \mathbb{N}$. What is more, if l_n denotes the number of elements in Γ_n, we can write $\Gamma_n = \{A_1, \ldots, A_{l_n}\}$.

Next, we generate a sample of m random numbers in $[0, 1]$. For a chosen level of the fractal structure, n, we get the pre-measure of each element on it. After that, we get the cumulative sum of $\omega(A_i)$, where $i = 1, \ldots, l_n$. Each of the random numbers we have generated will be assigned a random number in the element of the level n according to the cumulative sum of the pre-measures. More precisely, given a number in the sample we have generated, x, it can happen:

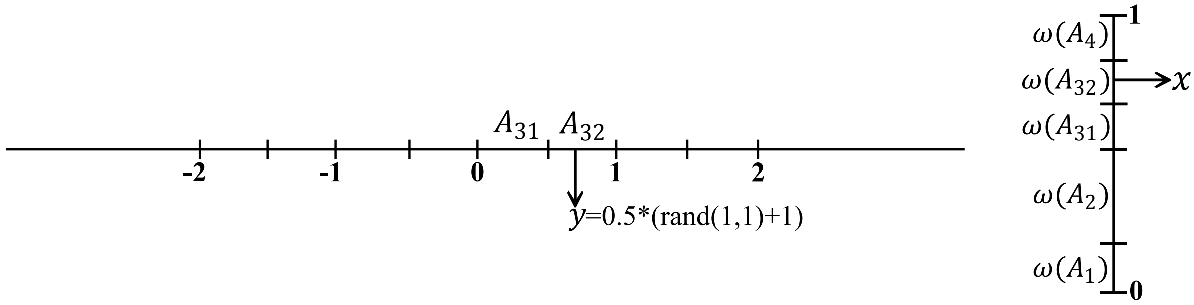
1. If $x \leq \omega(A_1)$, then $F(y) \approx x$, where y is a random point in A_1.
2. If $\sum_{i=1}^{j} \omega(A_i) < x \leq \sum_{i=1}^{j+1} \omega(A_i)$ (where $j = 1, \ldots, l_n - 1$), then we can approximate $F(y) \approx x$, where y is a random point in A_{j+1}.

As it is natural, when we repeat that procedure for all the numbers in the $[0, 1]$-sample, we get a random sample in X.

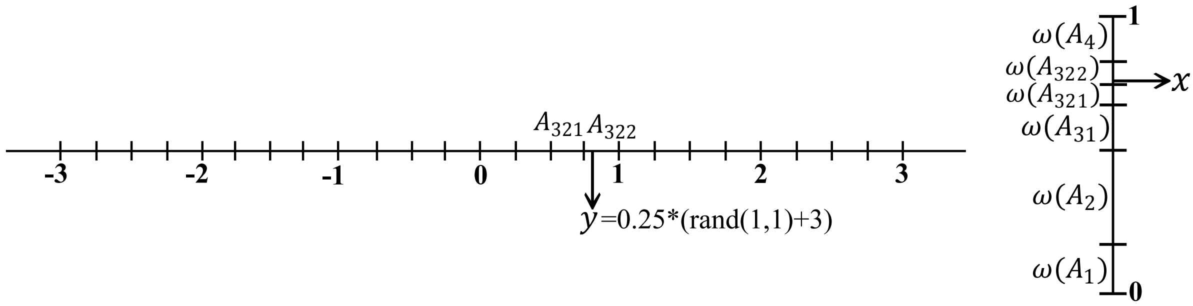
For example, let X be the real line and consider the finite fractal structure $\mathbf{\Gamma}$, which is defined by the levels $\Gamma_n = \{] - \infty, -n]\} \cup \{]\frac{k}{2^{n-1}}, \frac{k+1}{2^{n-1}}] : k = -n2^{n-1}, \ldots, n2^{n-1} - 1\} \cup \{]n, +\infty[\}$. Now, we define the pre-measure of each element in a certain level by taking into account the cumulative distribution function of a normal distribution variable $\mathcal{N}(0, 1)$. In Figure 6.1 one can see how the method works when choosing one of the first three levels of $\mathbf{\Gamma}$. Take into account that $rand(1, 1)$ means a random number in $[0, 1]$.



(a) Approximation for the first level, Γ_1



(b) Approximation for the second level, Γ_2

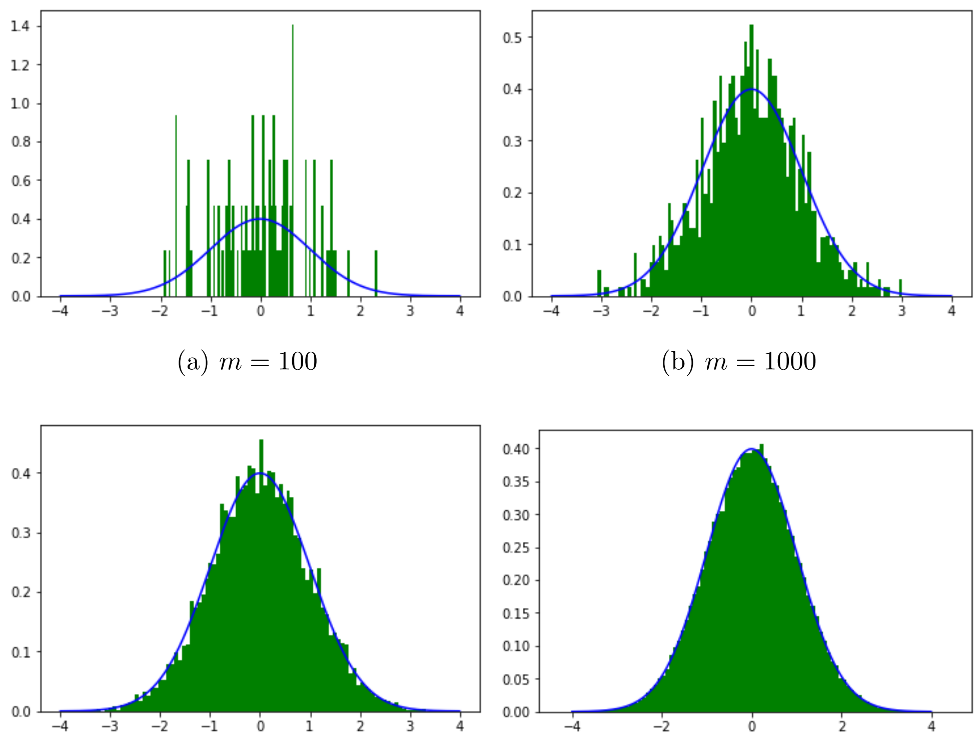


(c) Approximation for the third level, Γ_3

Figure 6.1: $F(y) \approx x$ according to the method to generate samples

See Figure 6.2 to have a look at the normalized histograms that we can get when we generate a sample of $100, 1000, 10000$ and 100000 data respectively, according to the procedure explained before when we consider the level 10 of $\mathbf{\Gamma}$.

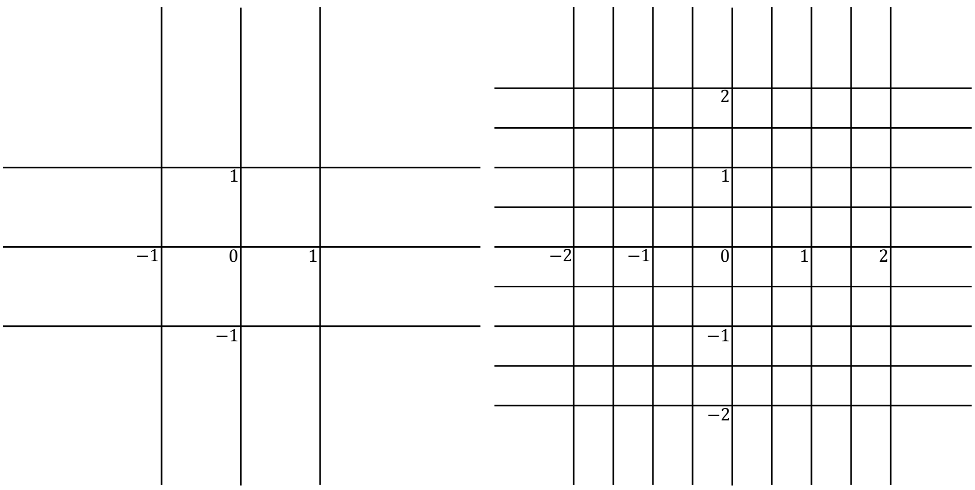
However, the method we have just introduced does not only let us generate samples of a univariate probability distribution but also from a multivariate one (or random vector). That is the reason why we introduce an example where, by using a finite fractal structure on $\mathbb{R}^2$, it is possible to generate a sample from a bivariate normal distribution. Let us consider, for example, the finite fractal structure $\mathbf{\Delta}$ whose levels are defined by $\Delta_n = \{A \times B : A, B \in \Gamma_n\}$. Its first two levels can be seen in Figure 6.3.



(a) $m = 100$ (b) $m = 1000$

(c) $m = 10000$ (d) $m = 100000$

Figure 6.2: Normalized histograms for level 10 of the fractal structure according to the sample size, m, of $\mathcal{N}(0,1)$



(a) Level 1 (b) Level 2

Figure 6.3: Levels of the finite fractal structure $\mathbf{\Delta}$ on $\mathbb{R}^2$

Now, if F is the cumulative distribution function of a bivariate normal vector, (X, Y), with mean vector and covariance matrix

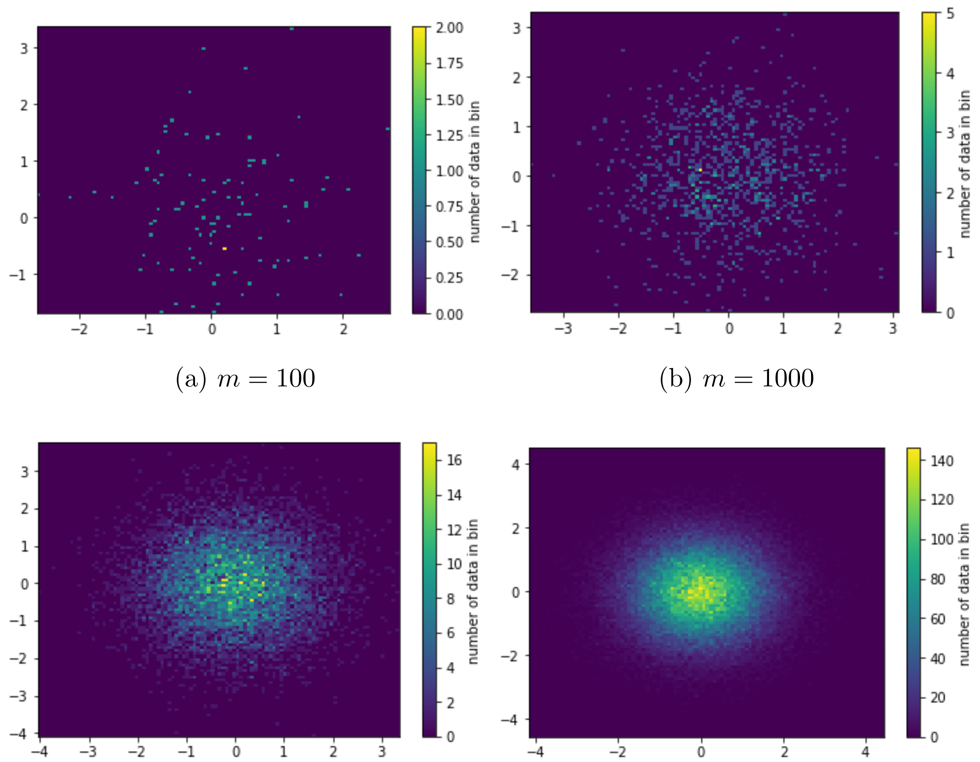
$$\boldsymbol{\mu} = (\mu_X, \mu_Y) = (0,0), \qquad \boldsymbol{\Sigma} = \begin{pmatrix} \sigma_X^2 & \sigma_{X,Y} \\ \sigma_{X,Y} & \sigma_Y^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

where $\sigma_{X,Y}$ denotes the covariance between X and Y, then the pre-measure of each element of the fractal structure can be defined by

$$\omega(A_i) = \omega([a_i, b_i] \times [c_i, d_i]) = F(b_i, d_i) + F(a_i, c_i) - F(a_i, d_i) - F(b_i, c_i)$$

for each $A_i \in \Delta_n$ and each $i = 1, \ldots, (n2^n + 2)^2$. The proof that ω gives us a probability measure on $\mathbb{R}^2$ according to the construction method introduced in this work is analogous to the one made in Example 5.20.

See Figure 6.4 to have a look at the normalized histograms that we can get when we generate a sample of $100, 1000, 10000$ and 100000 data respectively, according to the procedure explained before when we consider the level 5 of $\boldsymbol{\Delta}$.



(a) $m = 100$

(b) $m = 1000$

(c) $m = 10000$

(d) $m = 100000$

Figure 6.4: Histograms for level 5 of the fractal structure $\boldsymbol{\Delta}$ according to the sample size, m, of a standard bivariate normal distribution

6.2 A estimation method based on fractal structures

Let $\mathbf{\Gamma}$ be a finite fractal structure on a space and consider a sample of length m from a certain probability distribution. Our goal is to estimate the value of a certain parameter of the distribution (or vector of parameters, that will be denoted by $\boldsymbol{\theta}$). For that purpose, the estimation method will try to maximize the probability that, in each element of the fractal structure (for a given level), there are as much data of the sample as there are actually. Since $\mathbf{\Gamma}$ is finite, we can denote the number of elements in the level n by l_n for each $n \in \mathbb{N}$. Note that if we call $\omega_{\boldsymbol{\theta}}$ the pre-measure induced by the distribution whose parameter(s) we want to estimate, the random variable which gives us the probability that r of these data belong to a certain element in Γ_n, A_i, is a binomial, $\mathcal{B}(m, \omega_{\boldsymbol{\theta}}(A_i))$ for each $i \in \{1, \ldots, l_n\}$ and each $n \in \mathbb{N}$. Now, for each level $n \in \{1, \ldots, n_M\}$ (where n_M is the maximum level considered in the estimation), we define

$$h_n(\boldsymbol{\theta}) = \prod_{i=1}^{l_n} P[X = r_i | X \sim \mathcal{B}(m, \omega_{\boldsymbol{\theta}}(A_i))]$$

where r_i is the number of elements in the sample which belong to A_i. The estimated parameter is the one that maximizes $h(\boldsymbol{\theta}) = \prod_{n=1}^{n_M} h_n(\boldsymbol{\theta})$, and will be denoted by $\widehat{\boldsymbol{\theta}}$.

6.2.1 Estimation based on a known random variable

Consider a random variable, Y, on a space with a finite fractal structure, $\mathbf{\Gamma}$. Moreover, assume that the sample is given over the real line and that the fractal structure is the one defined by the levels $\Gamma_n = \{]-\infty, -n]\} \cup \{]\frac{k}{2^{n-1}}, \frac{k+1}{2^{n-1}}] : k = -n2^{n-1}, \ldots, n2^{n-1} - 1\} \cup \{]n, +\infty[\}$. In this case, we can define the pre-measure $\omega(A_i) = F_Y(b_i) - F_Y(a_i)$, where F_Y is the cumulative distribution function of the random variable Y. Indeed, as it was stated in the previous subsection, ω gives us a probability measure on X according to the construction made in the previous chapters.

In what follows, we introduce an example of estimation for a sample of a normal distribution whose mean and standard deviation are, respectively, 1 and 2. We show the results for several maximum levels of the fractal structure (from the first to the fifth). If we repeat the procedure 100 times, we get the mean and the standard deviation of the results and collect them in the following tables. In Tables 6.1 and 6.2 we consider a sample of 100 data and give, respectively, the estimated value for the mean (denoted by $\widehat{\mu}$) and the standard deviation (denoted by $\widehat{\sigma}$). Apart from that, in Tables 6.3 and 6.4 we get results of the estimations for a sample of size 1000. Moreover, we compare those results with those given by the maximum likelihood estimator which, as it is well known, is one of the best known estimators. In order to get an estimator for the mean and standard deviation of a normal distribution $\mathcal{N}(\mu, \sigma)$, the estimators given by this method, for the sample $x_1, \ldots, x_n$, are

$$\widehat{\mu} = \overline{x} = \frac{1}{n}\sum_{i=1}^{n} x_i, \qquad \widehat{\sigma} = \sqrt{\sum_{i=1}^{n} \frac{(x_i - \overline{x})^2}{n}}$$

However, while the estimator for the mean keeps on being the mean of the sample when the standard deviation in known, the one for the standard deviation when the mean is known is given by

$$\widehat{\sigma} = \sqrt{\sum_{i=1}^{n} \frac{(x_i - \mu)^2}{n}}$$

In the results we introduce next we will work with the case that we want to estimate one parameter when the other one is known, although it is possible to estimate both jointly.

For further reference about the maximum likelihood estimator see, for example, [52, Section 8.7].

$\widehat{\mu}$	Maximum level	Mean	Standard deviation
Maximum likelihood		1.03564	0.19986
New	1	1.04987	0.23353
	2	1.04684	0.21391
	3	1.04261	0.20798
	4	1.04083	0.20556
	5	1.03959	0.20367

Table 6.1: Estimations of the mean for a sample of size 100

$\widehat{\sigma}$	Maximum level	Mean	Standard deviation
Maximum likelihood		1.99196	0.14360
New	1	2.01570	0.28990
	2	2.01384	0.21313
	3	2.00256	0.17654
	4	1.99936	0.15796
	5	1.99693	0.14952

Table 6.2: Estimations of the standard deviation for a sample of size 100

$\widehat{\mu}$	Maximum level	Mean	Standard deviation
Maximum likelihood		1.00041	0.06219
New	1	1.00101	0.06479
	2	0.99988	0.06322
	3	1.00005	0.06115
	4	1.00024	0.06088
	5	1.00018	0.06078

Table 6.3: Estimations of the mean for a sample of size 1000

$\widehat{\sigma}$	Maximum level	Mean	Standard deviation
Maximum likelihood		2.00309	0.04645
New	1	1.99776	0.08607
	2	2.00048	0.06678
	3	2.00134	0.05777
	4	2.00238	0.05331
	5	2.00247	0.05060

Table 6.4: Estimations of the standard deviation for a sample of size 1000

Once we see the results in the previous tables, we can conclude some aspects:

- The new estimation method gives us results that are very similar to those we can get by using the maximum likelihood method.

- The higher the maximum level of the fractal structure is, the better the estimate.

- The larger the sample length of the fractal structure is, the better the estimate.

6.2.2 Estimation from a uniform distribution of the pre-measure

Moreover, we can define a finite fractal structure, $\mathbf{\Gamma}$, on a space by taking into account some random variable such that the pre-measure is the same for each $A \in \Gamma_n$ and each $n \in \mathbb{N}$. Indeed, that construction is made in Example 5.29 in detail. In that example, we also proved that the pre-measure defined uniformly gives us a probability measure on the space according to the procedure of construction of measures that was developed in the previous chapters. In this subsection we use this finite fractal structure together with the pre-measure defined by $\omega(A_n) = \frac{1}{2^n}$ for each $A_n \in \Gamma_n$ and each $n \in \mathbb{N}$, to estimate the parameters of a normal random variable when we are given a sample of it.

However, there is a significative difference between the estimation made in this subsection with respect to the one introduced in the previous one: the parameter does not only give us the definition of the pre-measure, but also we consider a finite fractal structure whose levels are given by the cumulative distribution function which depends on the parameter(s) $\boldsymbol{\theta}$.

$\widehat{\mu}$	Maximum level	Mean	Standard deviation
Maximum likelihood		1.00374	0.19144
New	1	0.95989	0.22088
	2	0.97545	0.22562
	3	0.99099	0.23048
	4	1.00766	0.22096
	5	0.99925	0.24274

Table 6.5: Estimations of the mean for a sample of size 100

$\widehat{\sigma}$	Maximum level	Mean	Standard deviation
Maximum likelihood		1.98390	0.12941
New	1	x	x
	2	1.95974	0.22666
	3	1.98571	0.20760
	4	1.98639	0.18443
	5	1.99893	0.18065

Table 6.6: Estimations of the standard deviation for a sample of size 100

$\widehat{\mu}$	Maximum level	Mean	Standard deviation
Maximum likelihood		1.00182	0.06706
New	1	0.99452	0.08291
	2	1.00339	0.07713
	3	1.00314	0.07741
	4	0.99852	0.07600
	5	1.00280	0.07926

Table 6.7: Estimations of the mean for a sample of size 1000

$\widehat{\sigma}$	Maximum level	Mean	Standard deviation
Maximum likelihood		2.00443	0.04395
New	1	x	x
	2	1.99487	0.06735
	3	1.99594	0.05770
	4	1.99593	0.05555
	5	1.99826	0.05704

Table 6.8: Estimations of the standard deviation for a sample of size 1000

Once we have shown the results of the estimations, note that, although the fractal structure is considered in a different way with respect to the previous subsection, the estimations get better as we consider a higher level of the fractal structure. Moreover, the size of the sample implies differences in the results, since the estimation is better when the size is greater. However, the estimation made for the standard deviation has not been collected when considering the first level of the fractal structure (see x in Tables 6.6 and 6.8). The reason why we do not collect that data in the table is due to the fact that $h(\sigma)$ is constant in this case. Indeed, by the construction made of the fractal structure, it holds that $\Gamma_1 = \{]-\infty, 1], [1, \infty[\}$ regardless of σ.

6.2.3 The estimation method against outliers

As it was stated in the previous subsections as conclusions, the new estimation method lets us get similar results to the given by the maximum likelihood method. Hence, it does make sense to ask ourselves if there exist some situations in which the method based on fractal structures improves the estimation of the parameters. For example, we can add some completely unrelated data to the distribution whose sample has been considered. This data is known as outliers.

Since in the previous two subsections we have shown the results for the estimation of a mean and the standard deviation of a normal sample, we will keep the same kind of sample, with 1000 data, and add it 10 outliers according to:

- A normal distribution with mean 1 and standard deviation 5. See Tables 6.9 and 6.10.

- A normal distribution with mean 3 and standard deviation 5. See Tables 6.11 and 6.12.

- An exponential distribution with mean 10. See Tables 6.13 and 6.14.

$\widehat{\mu}$	Maximum level	Mean	Standard deviation
Maximum likelihood		1.00146	0.05925
New	1	1.00021	0.06915
	2	1.00223	0.06508
	3	1.00185	0.06289
	4	1.00139	0.06131
	5	1.00106	0.06054

Table 6.9: Estimations of the mean for a 1000 data sample with 10 $\mathcal{N}(1,5)$ outliers

$\widehat{\sigma}$	Maximum level	Mean	Standard deviation
Maximum likelihood		2.04455	0.05215
New	1	2.01766	0.09947
	2	2.01814	0.07073
	3	2.01629	0.05984
	4	2.01671	0.05468
	5	2.01791	0.05151

Table 6.10: Estimations of the standard deviation for a 1000 data sample with 10 $\mathcal{N}(1,5)$ outliers

$\widehat{\mu}$	Maximum level	Mean	Standard deviation
Maximum likelihood		1.02992	0.06127
New	1	1.01461	0.07898
	2	1.01605	0.07122
	3	1.01679	0.06812
	4	1.01791	0.06608
	5	1.01871	0.06501

Table 6.11: Estimations of the mean for a 1000 data sample with 10 $\mathcal{N}(3,5)$ outliers

$\widehat{\sigma}$	Maximum level	Mean	Standard deviation
Maximum likelihood		2.05728	0.05375
New	1	2.00155	0.08630
	2	2.00935	0.06600
	3	2.01267	0.05588
	4	2.01625	0.05140
	5	2.01901	0.04932

Table 6.12: Estimations of the standard deviation for a 1000 data sample with 10 $\mathcal{N}(3,5)$ outliers

$\widehat{\mu}$	Maximum level	Mean	Standard deviation
Maximum likelihood		1.08863	0.07425
New	1	1.01802	0.08195
	2	1.02167	0.07442
	3	1.02362	0.07146
	4	1.02538	0.07024
	5	1.02737	0.06949

Table 6.13: Estimations of the mean for a 1000 data sample with 10 exponential outliers

$\widehat{\sigma}$	Maximum level	Mean	Standard deviation
Maximum likelihood		2.35858	0.22481
New	1	1.99370	0.09067
	2	2.00123	0.07055
	3	2.00425	0.06112
	4	2.00764	0.05565
	5	2.01220	0.05239

Table 6.14: Estimations of the standard deviation for a 1000 data sample with 10 exponential outliers

Once we have had a look at the results collected in the previous tables, it is clear that, in the presence of outliers, our method offers better estimations that the maximum likelihood method.

6.3 A goodness-of-fit test based on fractal structures

The last application that arises from the theory that has been developed in this first part of the work consists of designing a test to check if some given sample comes from a certain distribution or not. The idea of this goodness-of-fit test is based on the χ^2 distribution test, for which [52, Section 10.3] is a good reference. The difference between the new goodness-of-fit test and the classical one is that our test takes advantage of the recursive nature of the fractal structure to be more exact than the known one.

Let Y be a space with a finite fractal structure, $\mathbf{\Gamma}$, such that each element in Γ_{n+1} is contained in only one element of level n. The idea of the test is comparing the number of observations that belong to each of the elements in the fractal structure (for a certain level) with the number of observations that are expected to be got in case the data really come from a certain distribution.

Let us consider a random sample with m data of a known distribution that is given by a random variable, X. Since the fractal structure is finite, we can enumerate the elements in each level. For example, $\Gamma_1 = \{A_1, \ldots, A_k\}$ and each element in Γ_1 is divided into a finite number of elements in Γ_2. Hence, we can write $\Gamma_2 = \{A_{11}, A_{12}, \ldots, A_{k1}, A_{k2}, \ldots\}$ and recursively, we call the elements in Γ_n for each $n \in \mathbb{N}$.

Now, for the m data we have generated, N_i will denote the random variable which describes the number of sample values that belong to A_i and by $p_i = P[X \in A_i]$ for each $i = 1, \ldots, k$, where X is the random variable for what we want to test the sample. Hence, $N_i \sim \mathcal{B}(m, p_i)$. Recursively N_{ij} will denote the random variable which describes the number of sample values that belong to A_{ij} when the number of data in A_i is known, and $p_{ij} = P[X \in A_{ij}]$ for each $j = 1, \ldots, l$ and each $i = 1, \ldots, k$, and so on. This implies that $N_{ij} \sim \mathcal{B}(n_i, \frac{p_{ij}}{p_i})$, where n_i is the number of data in A_i. Recursively it can be known each probability on the form $p_{w_1 \ldots w_n}$ for each $n \in \mathbb{N}$ and, consequently, the distribution of $N_{w_1 \ldots w_n}$.

Our first goal is defining an statistic that measures the deviation of the distribution of the sample from the hypothetical one. For that purpose, by following a similar idea to the one by the χ^2 test, we define one statistic for each level of the fractal structure as follows:

$$E_1 = \sum_{i=1}^{k} \frac{(N_i - mp_i)^2}{mp_i}$$

$$E_2 = \sum_{i=1}^{k} \sum_{j=1}^{l} \frac{(N_{ij} - N_i \frac{p_{ij}}{p_i})^2}{N_i \frac{p_{ij}}{p_i}}$$

$$\vdots$$

$$E_n = \sum_{i_1=1}^{j_1} \cdots \sum_{i_n=1}^{j_n} \frac{(N_{i_1 \ldots i_n} - N_{i_1 \ldots i_{n-1}} \frac{p_{i_1 \ldots i_n}}{p_{i_1 \ldots i_{n-1}}})^2}{N_{i_1 \ldots i_{n-1}} \frac{p_{i_1 \ldots i_n}}{p_{i_1 \ldots i_{n-1}}}}, \forall n > 0$$

Note that, as it is well known in the classical case, it holds that

$$E_1 \sim \chi^2_{k-1}, E_2 \sim \chi^2_{lk-1}, \ldots, E_n \sim \chi^2_{j_1 \ldots j_n - 1}$$

However, it is not clear the independence between N_i, N_{ij} and the rest of binomial random variables that we consider in different sums. Nevertheless, the random variables we are considering to get the statistic are not only the binomial, but the discrepancy between expected frequencies and the observed ones. That error seems to be independent when considering E_i and E_j with $i \neq j$. The proof of this last fact opens a new research line to continue in the future.

Anyway, under the assumption that $E_1, \ldots, E_n$ are independent, the test consists of considering the null hypothesis which claims that the sample values follow the same distribution as the random variable X. That is why, for a significance level α, we will reject the null hypothesis if $\sum_{i=1}^{n} E_i > \chi^2_{s,1-\alpha}$, where s is the sum of the degrees of freedom of E_i from $i = 1$ to the chosen level.

Part II

Distribution functions and probability measures on linearly ordered topological spaces

The theory of a cumulative distribution function (in short, cdf) of a random variable is a basic and well established theory in Probability and Mathematics. This theory is interesting by several reasons: the pseudo-inverse of the cdf can be used to generate random samples of the random variable, which is essential in Montecarlo simulations, for example; it provides an equivalence between probability measures on the reals and distribution functions, which allows to forget about the measure (a set function) to focus on the cdf (a usual function). This allows to describe random variables in a simple way by providing its cdf.

Recall that, in the classical case, the cdf of a real-valued random variable X is the function defined by $F_X(x) = P[X \leq x]$ and it satisfies the following properties:

1. F is non-decreasing, which means that for each $x, y \in \mathbb{R}$ with $x < y$, we have $F(x) \leq F(y)$.
2. F is right-continuous, which means that $F(a) = \lim_{x \to a^+} F(x)$ for each $a \in \mathbb{R}$. Furthermore, $\lim_{x \to -\infty} F(x) = 0$ and $\lim_{x \to +\infty} F(x) = 1$.

Moreover, when a cdf, F, is strictly increasing and continuous, it holds that $F^{-1}(p)$ is the unique real number x such that $F(x) = p$ for each $p \in [0, 1]$. In that case, this defines the inverse of the distribution function.

The pseudo-inverse is not unique for some distributions (for example, in the case that the density function is $f_X(x) = 0$ for each $a < x < b$, causing F_X to be constant). This problem can be solved by defining, for each $p \in [0, 1]$, the pseudo-inverse of the distribution function by $F^{-1}(p) = \inf\{x \in \mathbb{R} : F(x) \geq p\}$.

The inverse of a cdf lets us generate samples of a distribution. Indeed, let X be a random variable whose distribution can be described by the cdf F. Our goal is to generate values of X according to this distribution. The inverse transform sampling method is as follows: generate a random number, u, from the standard uniform distribution in the interval $[0, 1]$ and, then, consider $x = F^{-1}(u)$.

Roughly speaking, given a continuous uniform random variable, U, in $[0, 1]$ and a cdf, F, the random variable $X = F^{-1}(U)$ has distribution F (or, X is distributed F). For further reference about the pseudo-inverse of F see, for example, [20, Chapter 1].

As we can see before, since the main properties of a cdf are related to order and continuity, a linearly ordered topological space (in short, LOTS) seems to be the natural place where such a theory could be developed. The study of measures on LOTS is also of interest (see [9], [12], [39], [56], [57] and [58]). In fact, another generalization of the concept of distribution function can be found in [65].

In [43] (see also [16], [42] and [49]) it is proved the equivalence between probability measures and fuzzy intervals in the real line. On the other hand, probabilistic metric and normed spaces, which were introduced in [48] and [61] respectively, provide a different (more studied) relationship between topology and probability measures. For further reference about this topic see, for example, [1], [41] and [50].

In the second part of this work we describe a theory of a cdf on a separable LOTS from a probability measure defined on this space. This function can be extended to the Dedekind-MacNeille completion of the space where it does make sense to define the pseudo-inverse. Moreover, we study the properties of both functions (the cdf and the pseudo-inverse) and get results that are similar to those which are well known in the classical case. For example, the pseudo-inverse of a cdf allows us to generate samples of a distribution and give us the chance to calculate integrals with respect to the related probability measure. Finally, we give some conditions such that there is an equivalence between probability measures and distribution functions defined on a separable LOTS, like it happens in the classical case. What is more, we prove that there is a one-to-one relationship between the pseudo-inverse of a cdf and its probability measure. From this theory, some applications have arisen, such as a goodness-of-fit test.

In the rest of this part, unless otherwise stated, X will be a separable LOTS and μ will be a measure on X with respect to the Borel σ-algebra of X.

Chapter 7

The cdf of a probability measure on a LOTS

The content of this chapter corresponds to [33].

The main goal of this chapter is to provide a first study of the definition and properties of a cdf on a separable LOTS.

Our study begins with the proof of several properties of the order topology on a separable LOTS by taking advantage of certain types of sequences in some cases (see Section 7.1). In Section 7.2 we define the cumulative distribution function (cdf) of a probability measure on a separable LOTS and prove that its properties are quite similar to those which are well known in the classical case, described in the introduction to this part of the work. Furthermore, from a cdf F, we will define F_- which plays a similar role to that played by $\lim_{x\to a^-} F(x)$ in the classical theory of distribution functions. We also use F and F_- to get the measure of an interval. On the other hand, Section 7.3 is dedicated to proving some aspects related to the discontinuities of a cdf. Section 7.4 introduces the concept of pseudo-inverse for a cdf defined on a separable LOTS, which is a measurable function. Finally, Section 7.5 shows that, by using the pseudo-inverse of a cdf, it is possible to calculate integrals with respect to the initial probability measure and generate samples of a distribution in case that the separable LOTS is compact.

7.1 The order on X

In this section we study some properties (mainly topological) of a separable LOTS.

The definition of the order topology, τ, (see Definition 2.23) suggests the next

Definition 7.1. *Given $x \in X$, it is said to be a left-isolated (respectively right-isolated) point if $(< x) = \emptyset$ (respectively $(> x) = \emptyset$) or there exists $z \in X$ such that $]z, x[= \emptyset$ (respectively there exists $z \in X$ such that $]x, z[= \emptyset$). Moreover, we will say that $x \in X$ is isolated if it is both right and left-isolated.*

Lemma 7.2. *Let $A, B \subseteq X$ be such that $A^l = B^l$ (respectively $A^u = B^u$). If there exists* $\inf A$ *(respectively* $\sup A$*), then there exists* $\inf B$ *(respectively* $\sup B$*) and* $\inf A = \inf B$ *(respectively* $\sup A = \sup B$*).*

Proof. Let $A, B \subseteq X$ be such that $A^l = B^l$ and suppose that there exists $\inf A$. It holds that $x \leq \inf A$ for each $x \in A^l$. Now, since $A^l = B^l$, we have that $\inf A \in A^l = B^l$ and $x \leq \inf A$ for each $x \in B^l$, that is, $\inf A = \inf B$.

The case in which $A^u = B^u$ and there exists $\sup A$ can be proven analogously. $\square$

Proposition 7.3. *Let $A \subseteq X$ be a nonempty subset such that it does not have a minimum (respectively a maximum). Then there exists a sequence (a_n) with $a_n \in A$ such that $a_{n+1} < a_n$ for each $n \in \mathbb{N}$ and $A^l = \{a_n : n \in \mathbb{N}\}^l$ (respectively $a_{n+1} > a_n$ for each $n \in \mathbb{N}$ and $A^u = \{a_n : n \in \mathbb{N}\}^u$).*

Proof. Let D be a dense and countable subset of X and consider $D_A = \{d \in D : d \notin A^l\}$. Note that the fact that $d \notin A^l$ is equivalent to the existence of $a \in A$ such that $a < d$. Moreover, $D_A \subseteq D$, so D_A is countable, so we can enumerate it as $D_A = \{d_n : n \in \mathbb{N}\}$. Given $d_1 \in D_A$, there exists $a_1 \in A$ such that $a_1 < d_1$. Suppose that $a_n \in A$ is a sequence defined by $a_n < d_n$ and $a_n < a_{n-1}$ for each $n \in \mathbb{N}$. We define a_{n+1} as follows. Since there does not exist the minimum of A, we can choose $a \in A$ such that $a < a_n$. Apart from that, there exists $a' \in A$ such that $a' < d_{n+1}$. Hence, if we consider $a_{n+1} = \min\{a, a'\}$, then $a_{n+1} < a_n$ and $a_{n+1} < d_{n+1}$. Recursively we have defined a sequence $a_n \in A$ such that $a_{n+1} < a_n$ and $a_n < d_n$ for each $n \in \mathbb{N}$.

Now, we prove that $A^l = \{a_n : n \in \mathbb{N}\}^l$.

$\subseteq$) This is obvious.

$\supseteq$) Let $x \in X$ be such that $x \leq a_n$ for each $n \in \mathbb{N}$. Now, we prove that $x \leq a$ for each $a \in A$. For that purpose, let $a \in A$. Since there does not exist the minimum of A, there exist $a' \in A$ such that $a' < a$ and $a'' \in A$ such that $a'' < a'$. Consequently, $]a'', a[$ is a nonempty open set in X with respect to τ, so we can choose $d \in D \cap]a'', a[$. Hence, $d > a''$, which implies that $d \in D_A$. It follows that there exists $n_0 \in \mathbb{N}$ such that $d = d_{n_0}$. Therefore, $x \leq a_{n_0} < d_{n_0} < a$, which lets us conclude that $x \leq a$. □

Convex subsets can be described as countable union of intervals.

Corollary 7.4. *Let $A \subseteq X$ be a convex subset. Then it holds that:*

1. *If there exist both minimum and maximum of A, then $A = [\min A, \max A]$.*
2. *If there does not exist the minimum of A but it does its maximum, then there exists a decreasing sequence $a_n \in A$ such that $A = \bigcup_{n\in\mathbb{N}}]a_n, \max A]$.*
3. *If there does not exist the maximum of A but it does its minimum, then there exists an increasing sequence $b_n \in A$ such that $A = \bigcup_{n\in\mathbb{N}} [\min A, b_n[$.*
4. *If there does not exist the minimum of A nor its maximum, then there exist a decreasing sequence $a_n \in A$ and an increasing one $b_n \in A$ such that $A = \bigcup_{n\in\mathbb{N}}]a_n, b_n[$.*

Proof. 1. It is clear.

2. Since A is nonempty and there does not exist the minimum of A, by Proposition 7.3, we can choose a sequence $a_n \in A$ such that $a_{n+1} < a_n$ for each $n \in \mathbb{N}$ and $A^l = \{a_n : n \in \mathbb{N}\}^l$. Now, we prove that $A = \bigcup_{n\in\mathbb{N}}]a_n, \max A]$.

   $\subseteq$) Let $x \in A$. Since A does not have a minimum, then $x \notin A^l$, which implies that $x \notin \{a_n : n \in \mathbb{N}\}^l$. Hence, there exists $n \in \mathbb{N}$ such that $a_n < x$. Consequently, $x \in \bigcup_{n\in\mathbb{N}}]a_n, \max A]$.

   $\supseteq$) Let $x \in \bigcup_{n\in\mathbb{N}}]a_n, \max A]$. Then there exists $n \in \mathbb{N}$ such that $a_n < x \leq \max A$. Hence, the fact that A is convex together with the fact that $a_n \in A$ give us that $x \in A$.

3. It can be proven similarly to the previous item.

4. Since A is nonempty and there does not exist the minimum of A nor its maximum, by Proposition 7.3, we can choose two sequences $a_n, b_n \in A$ such that $a_{n+1} < a_n$ and $b_{n+1} > b_n$ for each $n \in \mathbb{N}$ and $A^l = \{a_n : n \in \mathbb{N}\}^l$, $B^u = \{b_n : n \in \mathbb{N}\}^u$. Now, we prove that $A = \bigcup_{n\in\mathbb{N}}]a_n, b_n[$.

   $\subseteq$) Let $x \in A$. Since A does not have a minimum nor a maximum then $x \notin A^l$ and $x \notin A^u$, which implies that $x \notin \{a_n : n \in \mathbb{N}\}^l$ and $x \notin \{b_n : n \in \mathbb{N}\}^u$, then there exist $n_1 \in \mathbb{N}$ and $n_2 \in \mathbb{N}$ such that $a_{n_1} < x < b_{n_2}$. If we define $n = \max\{n_1, n_2\}$, then it holds that $a_n < x < b_n$ and we conclude that $x \in \bigcup_{n\in\mathbb{N}}]a_n, b_n[$.

   $\supseteq$) Let $x \in \bigcup_{n\in\mathbb{N}}]a_n, b_n[$. Then there exists $n \in \mathbb{N}$ such that $a_n < x < b_n$. Hence, the fact that A is convex together with the fact that $a_n, b_n \in A$ give us that $x \in A$.

$\square$

Similarly, convex open subsets can be described as countable union of open intervals.

Corollary 7.5. *Let A be an open and convex subset of X. Then A is the countable union of open intervals.*

Proof. We distinguish some cases depending on whether there exist the maximum or the minimum of A:

1. Suppose that there does not exist the maximum of A nor its minimum. Then, by Corollary 7.4, it holds that A can be written as the countable union of open intervals.

2. Suppose that there does not exist the minimum of A but it does its maximum. By the previous corollary, it holds that $A = \bigcup_{n\in\mathbb{N}}]a_n, \max A]$. Now, note that the fact that A is open means that $\max A$ is right-isolated so we can write $A = \bigcup_{n\in\mathbb{N}}]a_n, b[$, where b is the following point to $\max A$. Hence, A is the countable union of open intervals.

3. If there exists the minimum of A but not its maximum, we can proceed analogously to claim that $A = \bigcup_{n\in\mathbb{N}}]a, b_n[$, where a is the previous point to $\min A$ and (b_n) is an increasing sequence in A.

4. If there exist both minimum and maximum of A, then $A =]a, b[$, where a is the previous point to $\min A$ and b is the following one to $\max A$.

□

Next, we prove that a separable LOTS is first countable.

Proposition 7.6. *τ is first countable.*

Proof. Since X is separable with respect to the topology τ, there exists a countable dense subset D of X. Now, we prove that given $x \in X$, each of the following countable families is a countable neighborhood base of x with respect to the topology τ.

- $\mathcal{B}_x = \{\{x\}\}$ if x is isolated.
- $\mathcal{B}_x = \{]a, b[: a < x < b; a, b \in D\}$ if x is not left-isolated nor right-isolated.
- $\mathcal{B}_x = \{[x, b[: x < b; b \in D\}$ if x is left-isolated but it is not right-isolated.
- $\mathcal{B}_x = \{]a, x] : a < x; a \in D\}$ if x is right-isolated but it is not left-isolated.

For that purpose, we prove the next two items:

- Each element of $\mathcal{B}_x$ is a neighborhood of x for each $x \in X$. This is clear if we take into account that each element in $\mathcal{B}_x$ is an open set with respect to the topology τ (see Remark 2.26). Indeed, if x is left-isolated, then, given $B \in \mathcal{B}_x$, we can write $B = [x, b[$ for some $b \in D$ with $b > x$. Equivalently, $B =]a, b[$, where a is the previous point to x according to the order. The other cases are similar.
- For each neighborhood of x, U, there exists $B \in \mathcal{B}_x$ such that $B \subseteq U$. Indeed, let U be a neighborhood of x. Then there exists an open set G such that $x \in G \subseteq U$. Since G is open and $\mathcal{B} = \{]a, b[: a < b\}$ is an open base, we can consider a, b such that $]a, b[\subseteq G$ and $a < x < b$. Now, we distinguish some cases depending on whether x is isolated or not:

  1. Suppose that x is isolated. Then there exist $y, z \in X$ such that $y < x < z$ and $]y, z[= \{x\}$. In this case, $\{x\}$ is an element of $\mathcal{B}_x$ which is contained in U.
  2. Suppose that x is not left-isolated nor right-isolated. Since $]a, x[$ and $]x, b[$ are both open in τ and D is dense in τ, we can choose $c \in]a, x[\cap D$ and $d \in]x, b[\cap D$. Furthermore, it holds that $x \in]c, d[\subseteq]a, b[\subseteq G \subseteq U$, which finishes the proof.

3. Suppose that x is left-isolated but it is not right-isolated. Then there exists $y \in X$ such that $]y, x[= \emptyset$ and $]x, z[\neq \emptyset$ for each $z > x$. Since $]x, b[$ is open in τ and D is dense in τ, we can choose $d \in]x, b[\cap D$. Furthermore, it holds that $x \in [x, d[\subseteq]a, b[\subseteq G \subseteq U$.
4. Suppose that x is not left-isolated but it is right-isolated. Then there exists $z \in X$ such that $]y, x[\neq \emptyset$ and $]x, z[= \emptyset$ for each $y < x$. Since $]a, x[$ is a neighborhood in τ and D is dense in τ, we can choose $c \in]a, x[\cap D$. Furthermore, it holds that $x \in]c, x] \subseteq]a, b[\subseteq G \subseteq U$.

□

We can choose a countable neighborhood base of each point such that its elements are ordered, as the next remark shows.

Remark 7.7. *Let $x \in X$. Then there exists a countable neighborhood base of x, $\mathcal{B}'_x = \{]a'_n, b'_n[: a'_n < x < b'_n; n \in \mathbb{N}\}$ such that (a_n) is a non-decreasing sequence and (b_n) a non-increasing one.*

Proof. Indeed, since τ is first countable, there exists a countable base of each point. According to the previous proposition, in case that x is not left-isolated nor right-isolated, we have that $\mathcal{B}_x = \{]a_1, b_1[: a_1 < x < b_1; a_1, b_1 \in D\}$ is a countable base of x. Since D is a dense subset in τ and $]x, b_1[$ and $]a_1, x[$ are nonempty open sets in τ, there exist $d_{a_1} \in D\cap]a_1, x[$ and $d_{b_1} \in]x, b_1[\cap D$. Now, define $a_2 = d_{a_1}$ and $b_2 = d_{b_1}$. Moreover, there exist $d_{a_2} \in D\cap]a_2, x[$ and $d_{b_2} \in D\cap]x, b_2[$. Now, we define $a_3 = d_{a_2}$ and $b_3 = d_{b_2}$. Recursively we have that $\mathcal{B}'_x = \{]a_n, b_n[: a_n < x < b_n; n \in \mathbb{N}, a_n, b_n \in D\}$, where $a_n = d_{a_{n-1}}$ and $b_n = d_{b_{n-1}}$. It is clear that $\mathcal{B}_x$ is a neighborhood base of x. Moreover, given $n \in \mathbb{N}$ it holds that $]a_{n+1}, b_{n+1}[\subseteq]a_n, b_n[$ by definition of a'_n and b'_n. We can proceed analogously to get a base for the right-isolated or left-isolated points. Moreover, note that if x is isolated, the base given in the previous proposition satisfies the condition given in this remark. □

There exists an equivalence between the property of second countable for τ and the countability of the set of isolated points.

Proposition 7.8. *Let X be a LOTS. X is second countable with respect to the topology τ if and only if X is separable and the set of points which are right-isolated or left-isolated is countable.*

Proof. Let us define C_1 and C_2 to be, respectively, the set of left-isolated points and the set of right-isolated points.

$\Leftarrow$) Let D be a countable dense subset of X. Moreover, suppose that C_1 and C_2 are countable subsets. Consider the family $\mathcal{B} = \{\{x\} : x \in C_1 \cap C_2\} \cup \{]a, x] : a < x, x \in C_2, a \in D\} \cup \{[x, b[: x < b, x \in C_1, b \in D\} \cup \{]a, b[: a < b, a, b \in D\}$ and note that it is an open base of X with respect to τ. Furthermore, the countability of the set of right-isolated and left-isolated points gives us that $\mathcal{B}$ is countable. Hence, τ is second countable.

$\Rightarrow$) Suppose that X is second countable with respect to τ. Then there exists a countable open base, $\mathcal{B} = \{U_n : n \in \mathbb{N}\}$. Since second countable spaces are separable, we only must prove that C_1 and C_2 are countable subsets, which gives us that $C_1 \cup C_2$ is also countable.

- C_1 is countable: let $x \in C_1$ and $b_1 > x$ with $b_1 \in D$. Since $\mathcal{B}$ is an open base and $[x, b_1[$ is an open set containing x, there exists $n_x \in \mathbb{N}$ such that $x \in U_{n_x} \subseteq [x, b_1[$. Now, let $y \in C_1$ with $y \neq x$ and $b_2 \in D$ with $y < b_2$. Then there exists $n_y \in \mathbb{N}$ such that $y \in U_{n_y} \subseteq [y, b_2[$ for $b_2 > y$. Consequently, $f : C_1 \to \mathbb{N}$ given by $f(x) = n_x$ is an injective function, which proves the countability of C_1.
- The countability of C_2 can be proven similarly to the countability of C_1.

$\square$

Now, we define the concept of right convergent and left convergent sequence.

Definition 7.9. *Let $x \in X$ and ν be a topology defined on X. We say that a sequence $x_n \in X$ is right ν-convergent (respectively left ν-convergent) to x if $x_n \xrightarrow{\nu} x$ and $x_n \geq x$ (respectively $x_n \leq x$) for each $n \in \mathbb{N}$.*

Now, we define the concept of monotonically right convergent and monotonically left convergent sequence.

Definition 7.10. *Let $x \in X$ and ν be a topology defined on X. We say that a sequence $x_n \in X$ is monotonically right ν-convergent (respectively monotonically left ν-convergent) to x if $x_n \xrightarrow{\nu} x$ and $x < x_{n+1} < x_n$ (respectively $x_n < x_{n+1} < x$) for each $n \in \mathbb{N}$.*

Proposition 7.11. *Let $x \in X$. Then x is not left-isolated (respectively right-isolated) if and only if there exists a monotone sequence which left τ-converges (respectively right τ-converges) to x.*

Proof. $\Rightarrow$) Let x be a non-left-isolated point. Then $x \neq \min X$. Since τ is first countable (by Proposition 7.6), we can consider a countable neighborhood base of x, $\mathcal{B}_x = \{]a_n, b_n[: n \in \mathbb{N}\}$. Now, let $a, b \in X$ be such that $a < x < b$. Then there exists $n_1 \in \mathbb{N}$ such that $a \leq a_{n_1} < x$ due to the fact that $\mathcal{B}_x$ is a neighborhood base of x. Since x is not left-isolated, we can choose $z_1 \in]a_{n_1}, x[$. Now, we can consider $n_2 \in \mathbb{N}$ such that $z_1 \leq a_{n_2} < x$ due to the fact that $\mathcal{B}_x$ is a neighborhood base of x. Recursively we can construct a subsequence of (a_n), $(a_{\sigma(n)})$, such that $a_{\sigma(n)} < a_{\sigma(n+1)} < x$ and $a_{\sigma(n)} \to x$, that is, $(a_{\sigma(n)})$ is a monotone sequence which left τ-converges to x.

The proof is analogous in case that x is not right-isolated.

$\Leftarrow$) Let $x \in X$ and suppose that it is a left-isolated point. If $x = \min X$, the proof is easy. Suppose that $x \neq \min X$. Then there exists $z \in X$ such that $]z, x[= \emptyset$. Suppose that there exists a monotone sequence which left τ-converges to x, (x_n). Then it holds that there exists $n_0 \in \mathbb{N}$ such that $x_n > z$ for each $n \geq n_0$. Moreover, since $x_n < x$, we have that $x_n \in]z, x[= \emptyset$, which is a contradiction. Hence, x is not left-isolated.

The case in which there exists a monotone sequence which right τ-converges to x can be proven analogously. $\square$

Lemma 7.12.

1. *If (a_n) is a monotone sequence which left τ-converges to a, then $\cup(< a_n) = (< a) = \cup(\leq a_n)$.*

2. *If (a_n) is a monotone sequence which right τ-converges to a, then $\cap(< a_n) = (\leq a) = \cap(\leq a_n)$.*

Proof. 1. Next, we prove both equalities:

- $\cup(< a_n) = (< a)$. On the one hand, since $a_n < a$, we have that $(< a_n) \subseteq (< a)$. Therefore, $\cup(< a_n) \subseteq (< a)$.

  On the other hand, let $x < a$. Since $a_n \xrightarrow{\tau} a$ and $a_n < a$, there exists $n \in \mathbb{N}$ such that $x < a_n < a$ and, hence, $x \in \cup(< a_n)$.

- $\cup(< a_n) = \cup(\leq a_n)$. On the one hand, let $x \in \cup(< a_n)$. Then there exists $n \in \mathbb{N}$ such that $x \in (< a_n)$. It is clear that $x \in (\leq a_n)$ and, hence, $x \in \cup(\leq a_n)$.

  On the other hand, let $x \in \cup(\leq a_n)$. Then there exists $n \in \mathbb{N}$ such that $x \in (\leq a_n)$. Since $a_n < a$ and $a_n \xrightarrow{\tau} a$, it holds that there exists $m > n$ such that $a_n < a_m < a$. The fact that $x \in (\leq a_n)$ gives us that $x \in (< a_m)$. We conclude that $x \in \cup(< a_n)$.

2. Next, we prove both equalities:

   On the one hand, let $x \leq a_n$ for each $n \in \mathbb{N}$ and suppose that $x > a$. Then there exists $m \in \mathbb{N}$ such that $a < a_m < x$, which is a contradiction with the fact that $x \leq a_n$ for each $n \in \mathbb{N}$. Hence, $x \leq a$ and $\cap(\leq a_n) \subseteq (\leq a)$.

   Moreover, since $a < a_n$ for each $n \in \mathbb{N}$, we have that $(\leq a) \subseteq (< a_n)$. Therefore, $(\leq a) \subseteq \cap(< a_n)$.

   Furthermore, it is clear that $(< a_n) \subseteq (\leq a_n)$, so we conclude that $\cap(< a_n) \subseteq \cap(\leq a_n)$ and we finish the proof.

□

Proposition 7.13. *Each connected set in τ is convex.*

Proof. Let $A \subseteq X$ be a connected set. Suppose that A is not convex, which means that there exist $a, b \in A$ with $a < b$ such that there exists $x \in X \setminus A$ with $a < x < b$. Note that $(< x)$ and $(> x)$ are both open sets in τ, which implies that $U = (< x) \cap A$ and $V = (> x) \cap A$ are both open in A with the topology induced by τ in A. Note that $U, V \neq \emptyset$, since $a \in U, b \in V$ and $U \cup V = A$, which implies that A is not connected. Hence, A is convex. □

7.2 Defining the cumulative distribution function

The definition of the cumulative distribution function of a probability measure defined on the Borel σ-algebra of X is the next one.

Definition 7.14. *The cumulative distribution function (in short, cdf) of a probability measure μ on X is a function $F : X \to [0, 1]$ defined by $F(x) = \mu(\leq x)$ for each $x \in X$.*

Lemma 7.15. *Let τ' be a first countable topology on X such that $\tau \subseteq \tau'$. Let $f : X \to [0,1]$ be a monotonically non-decreasing function and $x \in X$ and suppose that $f(x_n) \to f(x)$ for each monotone sequence which right τ'-converges (respectively left τ'-converges) to x. Then f is right τ'-continuous (respectively f is left τ'-continuous).*

Proof. Let $x \in X$ and $x_n \xrightarrow{\tau'} x$ be a right τ'-convergent sequence. If (x_n) is eventually constant (there exists $k \in \mathbb{N}$ such that $x_n = x$ for each $n \geq k$), the proof is easy. Otherwise, using that $\tau \subseteq \tau'$, we can recursively define a decreasing subsequence $(x_{\sigma(n)})$ of (x_n) such that $x < x_{\sigma(n+1)} < x_{\sigma(n)}$ for each $n \in \mathbb{N}$.

It follows that $(x_{\sigma(n)})$ is a monotone sequence which right τ'-converges to x and, hence, by hypothesis, $f(x_{\sigma(n)}) \to f(x)$.

Given $k \in \mathbb{N}$, we have that $x < x_{\sigma(k)}$. Since $\tau \subseteq \tau'$, it follows that $x_n \xrightarrow{\tau} x$, which gives us that there exists $n_0 \in \mathbb{N}$ such that $x \leq x_n < x_{\sigma(k)}$ for each $n \geq n_0$.

Now, the monotonicity of f gives us that $f(x) \leq f(x_n) \leq f(x_{\sigma(k)})$ for each $n \geq n_0$. We conclude that $f(x_n) \to f(x)$ and, hence, f is right τ'-continuous.

We can proceed analogously to show that f is left τ'-continuous when (x_n) is left τ'-convergent to x. □

Corollary 7.16. *Let τ' be a first countable topology on X and $f : X \to [0,1]$ a function. If f is right and left τ'-continuous, then f is τ'-continuous.*

Proof. Let $x \in X$ and $x_n \xrightarrow{\tau'} x$. Let $\sigma_1, \sigma_2 : \mathbb{N} \to \mathbb{N}$ be two increasing functions such that $x_{\sigma_1(n)} \geq x$ and $x_{\sigma_2(n)} \leq x$ with $\sigma_1(\mathbb{N}) \cup \sigma_2(\mathbb{N}) = \mathbb{N}$. If either $\sigma_1(\mathbb{N})$ or $\sigma_2(\mathbb{N})$ is finite, then the proof is easy. Otherwise, $(x_{\sigma_1(n)})$ is a right subsequence of (x_n) and $(x_{\sigma_2(n)})$ is a left subsequence of (x_n). By hypothesis, it holds that $f(x_{\sigma_1(n)}) \to f(x)$ and $f(x_{\sigma_2(n)}) \to f(x)$. It easily follows that $f(x_n) \to f(x)$, which means that f is continuous with respect to the topology τ'. □

Remark 7.17. *Note that Lemma 7.15 and Corollary 7.16 can be both applied to the topology τ.*

Corollary 7.18. *Let τ' be a first countable topology on X with $\tau \subseteq \tau'$ and let $f : X \to [0,1]$ be a monotonically non-decreasing function. Suppose that $f(x_n) \to f(x)$ for each monotone sequence which right τ'-converges to x and each monotone sequence which left τ'-converges to x, (x_n). Then f is continuous (with respect to the topology τ').*

Proof. It follows from Lemma 7.15 and Corollary 7.16. □

Proposition 7.19. *Let F be a cdf. Then:*

1. *F is monotonically non-decreasing.*

2. *F is right τ-continuous.*

3. *If there does not exist $\min X$, then $\inf F(X) = 0$.*

4. *$\sup F(X) = 1$.*

Proof. 1. This is obvious if we take into account the monotonicity of μ that follows from the fact that μ is a measure.

2. For the purpose of proving that F is right τ-continuous, let (x_n) be a monotone sequence which right τ-converges to x. Let us see that $F(x_n) \to F(x)$.

   First, note that the fact that (x_n) is a monotone sequence which right τ-converges to x implies, by Lemma 7.12, that $\bigcap_n (\leq x_n) = (\leq x)$. Moreover, $(\leq x_n)$ is a monotonically non-increasing sequence, so $(\leq x_n) \to \bigcap_n (\leq x_n) = (\leq x)$. Thus, from the continuity from above of the measure μ, it follows that $\mu(\leq x_n) \to \mu(\leq x)$, that is, $F(x_n) \to F(x)$. Therefore, by Lemma 7.15 and Remark 7.17, we have that F is right τ-continuous.

3. If there exists $\min X$, then $F_-(\min X) = \mu(< \min X) = \mu(\emptyset) = 0$. Suppose that there does not exist $\min X$. Suppose that there does not exist $\min X$. Then, by Proposition 7.3, we can consider a sequence (x_n) in X such that $x_{n+1} < x_n$ for each $n \in \mathbb{N}$ and $\{x_n : n \in \mathbb{N}\}^l = X^l = \emptyset$. Then we have that $\bigcap (\leq x_n) = \emptyset$. Now, note that $(\leq x_n)$ is a monotonically non-increasing sequence, which implies that $(\leq x_n) \to \bigcap (\leq x_n) = \emptyset$. By the continuity from above of the measure μ, it holds that $\mu(\leq x_n) = F(x_n) \to \mu(\emptyset) = 0$. Hence, $\inf\{F(x_n) : n \in \mathbb{N}\} = 0$. Finally, if we join the previous equality with the fact that $0 \leq \inf F(X) \leq \inf\{F(x_n) : n \in \mathbb{N}\}$, we conclude that $\inf F(X) = 0$.

4. We distinguish two cases depending on whether there exists the maximum of X or not:

(a) Suppose that there exists $\max X$. In this case, $\sup F(X) = F(\max X) = \mu(X) = 1$.

(b) Suppose that there does not exist $\max X$. By Proposition 7.3, we can consider a sequence (x_n) in X such that $x_{n+1} > x_n$ for each $n \in \mathbb{N}$ and $\{x_n : n \in \mathbb{N}\}^u = X^u = \emptyset$. Then we have that $\bigcup(\leq x_n) = X$. Now, note that $(\leq x_n)$ is a monotonically non-decreasing sequence, which implies that $(\leq x_n) \to \bigcup(\leq x_n) = X$. By the continuity from below of the measure μ, it holds that $\mu(\leq x_n) = F(x_n) \to \mu(X) = 1$. Hence, $\sup\{F(x_n) : n \in \mathbb{N}\} = 1$. Finally, if we join the previous equality with the fact that $\sup\{F(x_n) : n \in \mathbb{N}\} \leq \sup F(X) \leq 1$, we conclude that $\sup F(X) = 1$.

□

The previous proposition makes us wonder the next question.

Question 7.20. *Let $F : X \to [0,1]$ be a function satisfying the properties collected in Proposition 7.19. Does there exist a probability measure μ on X such that its cdf, F_μ, is F?*

According to the previous results we can conclude the following.

Corollary 7.21. *Let F be a cdf and $x \in X$. Then F is τ-continuous at x if and only if F is left τ-continuous at x.*

Proposition 7.22. *Let $x \in X$ and f be a monotonically non-decreasing function. If x is left-isolated (respectively right-isolated), then f is left τ'-continuous (respectively right τ'-continuous) at x, where τ' is a first countable topology such that $\tau \subseteq \tau'$.*

Proof. Let $x \in X$ and suppose that it is left-isolated. The case in which $x = \min X$ is obvious. Suppose that $x \neq \min X$. Then there exists $z \in X$ such that $]z, x[= \emptyset$. Hence, $(> z)$ is open in τ and, consequently, a neighborhood of x. Let (x_n) be a sequence which left τ'-converges to x. Then it is also left τ-convergent to x. Hence, there exists $n_0 \in \mathbb{N}$ such that $x_n \in (> z)$ for each $n \geq n_0$. Since $x_n \leq x$, we have that $x_n = x$ for each $n \geq n_0$. Consequently, $f(x_n) \to f(x)$ and f is left τ'-continuous at x.

The case in which x is right-isolated can be proven analogously. □

Corollary 7.23. *Let μ be a probability measure on X and F its cdf. Let $x \in X$. If x is left-isolated, then F is τ-continuous at x.*

Proof. It immediately follows from Proposition 7.19, Corollary 7.21 and Proposition 7.22. □

Definition 7.24. *Let μ be a probability measure on X and F its cdf. We define $F_- : X \to [0,1]$, by $F_-(x) = \mu(< x)$ for each $x \in X$.*

Note that F_- is monotonically non-decreasing by the monotonicity of the measure.

Next, we introduce two results which relate F_- to F.

Proposition 7.25. *Let μ be a probability measure on X and F its cdf. Then $\sup F(< x) = F_-(x)$ for each $x \in X$ with $x \neq \min X$.*

Proof. $\geq$) Let $x \in X$ with $x \neq \min X$. We distinguish two cases depending on whether x is left-isolated or not:

1. Suppose that x is not left-isolated. Then, by Proposition 7.11, there exists a monotone sequence, (a_n), which left τ-converges to x. This implies that $(\leq a_n) \to \cup(\leq a_n)$. Moreover, Lemma 7.12 gives us that $\cup(\leq a_n) = (< x) = \cup(< a_n)$. Hence, $(\leq a_n) \to (< x)$ and, consequently, $F(a_n) \to \mu(< x)$. Now, since $a_n < x$, $F(a_n) \leq \sup F(< x)$. If we take limits, we have that $\mu(< x) = F_-(x) \leq \sup F(< x)$.

2. Suppose that x is left-isolated. Then there exists $z \in X$ such that $z < x$ and $]z, x[= \emptyset$, which implies that $F(z) \leq \sup F(< x)$. Moreover, note that $(< x) = (\leq z)$, which means that $\mu(< x) = F(z)$. We conclude that $\mu(< x) = F_-(x) \leq \sup F(< x)$.

$\leq$) Let $y \in X$ with $y < x$. Then $F(y) \leq \mu(< x)$ and, hence, $\sup F(< x) \leq \mu(< x) = F_-(x)$. □

We can recover the cdf F from F_-.

Proposition 7.26. *Let F be a cdf. Then $F(x) = \inf F_-(> x)$ for each $x \in X$ with $x \neq \max X$.*

Proof. $\leq$) Let $x \in X$ with $x \neq \max X$ and $y \in X$ be such that $y > x$. Then $\mu(< y) \geq \mu(\leq x)$, that is, $F(x) \leq F_-(y)$, which gives us that $F(x) \leq \inf F_-(> x)$.

$\geq$) Let $x \in X$ with $x \neq \max X$. We distinguish two cases depending on whether x is right-isolated or not:

1. Suppose that x is right-isolated. Then there exists $z \in X$ such that $z > x$ and $]x, z[= \emptyset$, which implies that $\inf F_-(> x) \leq F_-(z)$. Moreover, note that $(> x) = (\geq z)$, which means that $\mu(> x) = \mu(\geq z)$ or, equivalently, $\mu(\leq x) = \mu(< z)$. Hence, $F(x) = F_-(z)$. We conclude that $\inf F_-(> x) \leq F(x)$.

2. Suppose that x is not right-isolated. Then, by Proposition 7.11, there exists a monotone sequence, (a_n), which right τ-converges to x. Since F is right τ-continuous, we have that $F(a_n) \to F(x)$. Now, the fact that $a_n > x$ gives us that $\inf F_-(> x) \leq F_-(a_n) \leq F(a_n)$. Finally, if we take limits, we have that $\inf F_-(> x) \leq F(x)$.

□

Lemma 7.27. *Let μ be a probability measure on X and F its cdf. Given $x \in X$, it holds that $F(x) = F_-(x) + \mu(\{x\})$.*

Proof. Indeed, given $x \in X$, by definition of cdf, we have that $F(x) = \mu(\leq x)$. Now, since μ is σ-additive, $F(x) = \mu(< x) + \mu(\{x\})$. We conclude that $F(x) = F_-(x) + \mu(\{x\})$. □

A cdf lets us calculate the measure of $]a, b]$ for each $a \leq b$ according to the next proposition and Lemma 7.27.

Proposition 7.28. *Let μ be a probability measure on X and F its cdf. Then $\mu(]a, b]) = F(b) - F(a)$ for each $a, b \in X$ with $a < b$.*

Proof. Let $a, b \in X$ be such that $a < b$. Note that we can write $(\leq b) = (\leq a) \cup]a, b]$. Now, since μ is a measure (and, hence, σ-additive) it holds that $\mu(\leq b) = \mu(\leq a) + \mu(]a, b])$, that is, $\mu(]a, b]) = F(b) - F(a)$. □

Corollary 7.29. *Let μ be a probability measure on X and F its cdf. Then:*

1. $\mu([a,b]) = F(b) - F_-(a)$.

2. $\mu(]a,b[) = F_-(b) - F(a)$.

3. $\mu([a,b[) = F_-(b) - F_-(a)$.

Proof. The proof is immediate if we take into account the previous proposition and Lemma 7.27. □

Proposition 7.30. *Let μ be a probability measure on X and F its cdf. Let $x \in X$ and (x_n) be a monotone sequence which left τ-converges to x. Then $F(x_n) \to F_-(x)$.*

Proof. Let $x \in X$ and (x_n) be a monotone sequence which left τ-converges to x. Lemma 7.12 gives us that $\bigcup_n (\leq x_n) = (< x)$. Note that $(\leq x_n)$ is a monotonically non-decreasing sequence, which means that $(\leq x_n) \to \bigcup_n (\leq x_n) = (< x)$. Finally, by the continuity from below of μ, it follows that $\mu(\leq x_n) \to \mu(< x) = F_-(x)$, that is, $F(x_n) \to F_-(x)$. □

Next, we collect the properties of F_-.

Proposition 7.31. *Let μ be a probability measure on X and F its cdf. Then:*

1. *F_- is monotonically non-decreasing.*

2. *F_- is left τ-continuous.*

3. $\inf F_-(X) = 0$.

4. *If there does not exist the maximum of X, then $\sup F_-(X) = 1$. Otherwise, $F_-(\max X) = 1 - \mu(\{\max X\})$.*

Proof. 1. This is obvious if we take into account the monotonicity of μ that follows from the fact that it is a measure.

2. Let (x_n) be a monotone sequence which left τ-converges to x. Then, by Proposition 7.30, it holds that $F(x_n) \to F_-(x)$. Since (x_n) is monotonically left τ-convergent, it holds that $x_n < x_{n+1} < x$, so the fact that F_- is monotonically non-decreasing implies that $F(x_n) \leq F_-(x_{n+1}) \leq F_-(x)$. By taking limits, we conclude that $F_-(x_n) \to F_-(x)$ and, by Lemma 7.15 and Remark 7.17, F_- is left τ-continuous.

3. By Proposition 7.3, we can consider a sequence (x_n) in X such that $x_{n+1} < x_n$ for each $n \in \mathbb{N}$ and $\{x_n : n \in \mathbb{N}\}^l = X^l = \emptyset$. Then we have that $\bigcap(< x_n) = \emptyset$. Now, note that $(< x_n)$ is a monotonically non-increasing sequence, which implies that $(< x_n) \to \bigcap(< x_n) = \emptyset$. By the continuity from above of the measure μ, it holds that $\mu(< x_n) = F_-(x_n) \to \mu(\emptyset) = 0$. Hence, $\inf\{F_-(x_n) : n \in \mathbb{N}\} = 0$. Finally, if we join the previous equality with the fact that $0 \leq \inf F_-(X) \leq \inf\{F_-(x_n) : n \in \mathbb{N}\}$, we conclude that $\inf F_-(X) = 0$.

4. We distinguish two cases depending on whether there exists the maximum of X or not:

   (a) Suppose that there does not exist $\max X$. By Proposition 7.3, there exists a sequence (x_n) in X such that $x_{n+1} > x_n$ for each $n \in \mathbb{N}$ and $\{x_n : n \in \mathbb{N}\}^u = X^u = \emptyset$. Then we have $\bigcup(< x_n) = X$. Now, note that $(< x_n)$ is a monotonically non-decreasing sequence, which implies that $(< x_n) \to \bigcup(< x_n) = X$. By the continuity from below of the measure μ it holds that $\mu(< x_n) = F_-(x_n) \to \mu(X) = 1$. Hence, $\sup\{F_-(x_n) : n \in \mathbb{N}\} = 1$. Finally, if we join the previous equality with the fact that $\sup\{F_-(x_n) : n \in \mathbb{N}\} \leq \sup F_-(X) \leq 1$, we conclude that $\sup F_-(X) = 1$.

   (b) Now, suppose that there exists $\max X$. Then Lemma 7.27 lets us claim that $F_-(\max X) = F(\max X) - \mu(\{\max X\}) = 1 - \mu(\{\max X\})$.

   Thus, the item is proved.

□

7.3 Discontinuities of a cdf

In this section we prove some results which are analogous to those proven in [19, Chapter 1] and which are related to the discontinuities of a cdf.

First, we give a sufficient condition to ensure that a cdf is continuous at a point.

Proposition 7.32. *Let $x \in X$, μ be a probability measure on X and F its cdf. If $\mu(\{x\}) = 0$, then F is τ-continuous at x.*

Proof. Let (x_n) be a monotone sequence which left τ-converges to x. Then, by Proposition 7.30, it holds that $F(x_n) \to F_-(x)$. By Lemma 7.27, it holds that $F(x) = F_-(x)$, so $F(x_n) \to F(x)$ and, by Lemma 7.15 and Remark 7.17, F is left τ-continuous. Finally, by Corollary 7.21, F is τ-continuous. □

Next, we introduce a lemma that will be crucial to show that the set of discontinuity points of a cdf is at most countable.

Lemma 7.33. *Let μ be a probability measure on X and F its cdf. Then $\{x \in X : \mu(\{x\}) > 0\}$ is countable.*

Proof. For every integer N, the number of points satisfying $\mu(\{x\}) > \frac{1}{N}$ is at most N. Hence, there are no more than a countable number of points with positive measure. □

Next, we collect two properties of a cdf F_μ.

Proposition 7.34. *Let μ be a probability measure on X. Then:*

1. *F_μ is determined by a dense set, D, in X (with respect to the topology τ) in its points with null measure, that is, if for each $x \in D$ it holds that $F_\mu(x) = F_\delta(x)$, then $F_\mu(x) = F_\delta(x)$ for each $x \in X$ with $\mu(\{x\}) = 0$ and $\delta(\{x\}) = 0$, where F_δ is the cdf of a probability measure, δ, on X.*

2. *The set of discontinuity points of F_μ with respect to the topology τ is countable.*

Proof. 1. Let $x \in X$ with $\mu(\{x\}) = 0$ and $\delta(\{x\}) = 0$. We distinguish two cases:

(a) Suppose that x is left-isolated and right-isolated. Then there exist $y, z \in X$ such that $]y, z[= \{x\}$, which implies that $x \in D$ due to the fact that D is dense. Consequently, $F_\mu(x) = F_\delta(x)$.

(b) x is not left-isolated or it is not right-isolated. If x is not left-isolated, by Proposition 7.11, there exists a sequence $x_n \xrightarrow{\tau} x$ such that $x_n < x_{n+1} < x$. Now, since D is dense, it follows that there exists $d_n \in D$ such that $x_n < d_n < x_{n+1}$ and, hence, $d_n < d_{n+1}$ for each $n \in \mathbb{N}$. Hence, $d_n \to x$ in τ. By hypothesis, we have that $F_\mu(d_n) = F_\delta(d_n)$. By Proposition 7.30, $F_\mu(d_n) \to F_{\mu-}(x)$. However, $F_{\mu-}(x) = F_\mu(x)$, since $\mu(\{x\}) = 0$ (see Lemma 7.27). Analogously, $F_\delta(d_n) \to F_\delta(x)$. Consequently, $F_\mu(x) = F_\delta(x)$.
The case in which x is not right-isolated can be proven analogously.

2. Let $x \in X$. By Proposition 7.32, we know that the fact that F_μ is not continuous at x means that $\mu(\{x\}) > 0$. Since, by previous lemma, we have that $\{x \in X : \mu(\{x\}) > 0\}$ is countable, we conclude that the set of discontinuity points is at most countable.

□

7.4 The pseudo-inverse of a cdf

In this section we see how to define the pseudo-inverse of a cdf F defined on X and we gather some properties which relate this function to both F and F_-. Its properties are similar to those which characterize the pseudo-inverse in the classical case (see, for example, [20, Th. 1.2.5]). Moreover, we see that it is measurable.

Now, we recall the definition of this function in the classical case (see the introduction to this part of the work) to give a similar one in the context of a linearly ordered topological space. However, there exists a problem when we mention the infimum of a set, since there is no guarantee that every set has an infimum. Indeed, it is possible to extend the cdf to the Dedekind–MacNeille completion so that the pseudo-inverse is naturally defined from $[0,1]$ to the Dedekind–MacNeille completion as it can be seen in the next chapter. Hence, in this part of the work, we restrict that definition to those points which let us talk about the infimum of a set as the next definition shows.

Definition 7.35. *Let F be a cdf. We define the pseudo-inverse of F as $G : [0,1] \to X$ given by $G(x) = \inf\{y \in X : F(y) \geq x\}$ for each $x \in [0,1]$ such that there exists the infimum of $\{y \in X : F(y) \geq x\}$.*

Hereinafter, when we apply G to a point, we assume that G is defined at that point.

According to the previous definition, it is clear the next result.

Proposition 7.36. *G is monotonically non-decreasing.*

Proof. Let $x, y \in [0,1]$ with $x < y$. Note that $\{z \in X : F(z) \geq y\} \subseteq \{z \in X : F(z) \geq x\}$ and it follows that $\inf\{z \in X : F(z) \geq x\} \leq \inf\{z \in X : F(z) \geq y\}$, that is, $G(x) \leq G(y)$, which means that G is monotonically non-decreasing. □

Lemma 7.37. *Let $a = \inf\{a_n : n \in \mathbb{N}\}$ (respectively $a = \sup\{a_n : n \in \mathbb{N}\}$), where (a_n) is a sequence such that $a_{n+1} < a_n$ (respectively $a_{n+1} > a_n$) for each $n \in \mathbb{N}$. Then $a_n \xrightarrow{\tau} a$.*

Proof. Let (a_n) be a sequence in X such that $a_{n+1} < a_n$ for each $n \in \mathbb{N}$ and suppose that there exists $a = \inf\{a_n : n \in \mathbb{N}\}$. Let $b, c \in X \cup \{-\infty, \infty\}$ be such that $b < a < c$. Suppose that $a_n \geq c$ for each $n \in \mathbb{N}$. Then $\inf\{a_n : n \in \mathbb{N}\} \geq c > a$, a contradiction with the fact that $a = \inf\{a_n : n \in \mathbb{N}\}$. Hence, there exists $n_0 \in \mathbb{N}$ such that $a_{n_0} < c$. Furthermore, $a_n < c$ for each $n \geq n_0$, since $a_{n+1} < a_n$ for each $n \in \mathbb{N}$. Consequently, $a_n \xrightarrow{\tau} a$.

The case in which $a = \sup\{a_n : n \in \mathbb{N}\}$ and $a_{n+1} > a_n$ can be proven analogously. □

Proposition 7.38. *Let F be a cdf. Then:*

1. *$G(F(x)) \leq x$ for each $x \in X$.*

2. *$F(G(r)) \geq r$ for each $r \in [0, 1]$.*

Proof.

1. Indeed, $x \in \{z \in X : F(z) \geq F(x)\}$ and, hence, $\inf\{z \in X : F(z) \geq F(x)\} \leq x$, which is equivalent to $G(F(x)) \leq x$. This proves the first item.

2. Now, let $y = G(r) = \inf\{z \in X : F(z) \geq r\}$. If $y = \min\{z \in X : F(z) \geq r\}$, it is clear that $F(y) \geq r$. Suppose that $y \neq \min\{z \in X : F(z) \geq r\}$. Then, by Proposition 7.3, there exists a sequence $y_n \in \{z \in X : F(z) \geq r\}$ such that $y_{n+1} < y_n$ and $\{y_n : n \in \mathbb{N}\}^l = \{z \in X : F(z) \geq r\}^l$. Furthermore, by Lemma 7.2, it holds that $y = \inf\{y_n : n \in \mathbb{N}\}$. Hence, Lemma 7.37 lets us claim that $y_n \xrightarrow{\tau} y$. Consequently, the right τ-continuity of F gives us that $F(y_n) \to F(y)$. Moreover, $F(y_n) \geq r$, since $y_n \in \{z \in X : F(z) \geq r\}$. If we join this fact with the fact that $F(y_n) \to F(y)$, we conclude that $F(y) \geq r$. This proves the second item.

□

We get, as an immediate corollary, the following.

Corollary 7.39. *$G(r) \leq x$ if and only if $r \leq F(x)$ for each $x \in X$ and each $r \in [0, 1]$.*

Next result collects some properties of G which arise from some relationships between F and F_- and some conditions on them.

Proposition 7.40. *Let F be a cdf and let $x \in X$ and $r \in [0,1]$. Then:*

1. *$F(x) < r$ if and only if $G(r) > x$.*

2. *If $F_-(x) < r$, then $x \leq G(r)$.*

3. *If $F_-(x) < r \leq F(x)$, then G is defined at r and $G(r) = x$.*

4. *If $r < F_-(x)$, then $G(r) < x$.*

5. *If $r = F_-(x)$, then $G(r) \leq x$.*

Proof. 1. Note that it is an immediate consequence of Corollary 7.39.

2. Suppose that $G(r) < x$. Then $\mu(< x) \geq \mu(\leq G(r))$ or, equivalently, $F_-(x) \geq F(G(r)) \geq r$, that is, $F_-(x) \geq r$.

3. Let $x \in X$ and $r \in [0,1]$ be such that $F_-(x) < r \leq F(x)$. First, note that if $y < x$, then $F(y) \leq \sup F(< x) = F_-(x) < r$ and, hence, $x = \inf\{y \in X : F(y) \geq r\}$. It follows that G is defined at r and $x = G(r)$.

4. Let $x \in X$ and $r \in [0,1]$. Suppose that $r < F_-(x)$. Since $F_-(x) = \sup F(< x)$, there exists $y < x$ such that $r < F(y) \leq F_-(x)$. Since $F(y) > r$, then $y \geq \inf\{z \in X : F(z) \geq r\} = G(r)$. We conclude that $G(r) < x$.

5. Suppose that $r = F_-(x)$. The fact that $F_-(x) \leq F(x)$ for each $x \in X$ gives us that $F(x) \geq r$, which is equivalent, by Corollary 7.39, to $G(r) \leq x$.

□

We prove another property of G.

Proposition 7.41. *G is left τ-continuous.*

Proof. Let (r_n) be a sequence in $[0,1[$ which is left convergent to $r \in [0,1[$ with $r_n \neq r$. Since $r_n \leq r$, by the monotonicity of G (see Proposition 7.36), we have that $G(r_n) \leq G(r)$. Now, we prove that $G(r) = \sup\{G(r_n) : n \in \mathbb{N}\}$. For this purpose, let $x \in \{G(r_n) : n \in \mathbb{N}\}^u$ and suppose that $x < G(r)$. By Proposition 7.40.1, it holds that $F(x) < r$, so there exists $n \in \mathbb{N}$ such that $F(x) < r_n$. On the other hand, since $x \in \{G(r_n) : n \in \mathbb{N}\}^u$, then $G(r_n) \leq x$ for each $n \in \mathbb{N}$. By the monotonicity of

F, we have that $F(G(r_n)) \leq F(x)$ and, hence, by Proposition 7.38, $r_n \leq F(x)$, since $F(G(r_n)) \geq r_n$. If we join this fact with the fact that $F(x) < r_n$ for some $n \in \mathbb{N}$, we conclude that $r_n < r_n$, a contradiction.

It follows, by Lemma 7.37, that $(G(r_n))$ τ-converges to $G(r)$. □

Next proposition collects some properties of F and F_- which arise from considering some conditions on G.

Proposition 7.42. *Let F be a cdf and let $x \in X$ and $r \in [0,1]$. Then:*

1. *$G(r) > x$ if and only if $F(x) < r$.*

2. *If $G(r) = x$, then $F_-(x) \leq r \leq F(x)$.*

3. *If $G(r) < x$, then $r \leq F_-(x)$.*

Proof. 1. Note that this item is the same as Proposition 7.40.1.

2. Suppose that $G(r) = x$ and that $r > F(x)$. By item 1, it follows that $G(r) > x$, which is a contradiction with the fact that $G(r) = x$.

   Now, suppose that $r < F_-(x)$. Then Proposition 7.40.4 gives us that $G(r) < x$, which is a contradiction with the fact that $G(r) = x$.

   We conclude that $F_-(x) \leq r \leq F(x)$.

3. It is equivalent to Proposition 7.40.2.

□

Some consequences that arise from the previous propositions are collected next.

Corollary 7.43. *Let F be a cdf and $r \in [0,1]$. Then:*

1. *$F_-(G(r)) \leq r \leq F(G(r))$.*

2. *If $F(G(r)) > r$, then $\mu(\{G(r)\}) > 0$.*

Proof. 1. Let $r \in [0,1]$. On the one hand, suppose that $F_-(G(r)) > r$. Then, by Proposition 7.40.4, it holds that $G(r) < G(r)$, which is a contradiction. Hence, $F_-(G(r)) \leq r$.

On the other hand, the inequality $r \leq F(G(r))$ is clear if we take into account Proposition 7.38.

2. By Lemma 7.27, $F(x) = F_-(x) + \mu(\{x\})$ for each $x \in X$, so we have that $F(G(r)) = F_-(G(r)) + \mu(\{G(r)\})$. If $F(G(r)) > r$, it holds that $F_-(G(r)) + \mu(\{G(r)\}) > r$. Moreover, if we join this fact with the previous item, we conclude that $\mu(\{G(r)\}) > 0$.

□

Corollary 7.44. *Let $r \in [0,1]$. If $\mu(\{G(r)\}) = 0$, then $F(G(r)) = r$.*

Now, we introduce some results to characterize the injectivity of G and F.

Proposition 7.45. *$\mu(\{x\}) = 0$ for each $x \in X$ if and only if G is injective.*

Proof. $\Rightarrow$) It immediately follows from Proposition 7.42.2. Indeed, this proposition gives us that, if $G(r) = x$, then $F_-(x) \leq r \leq F(x)$. Suppose that there exist $r, s \in X$ such that $r \neq s$ with $G(r) = G(s) = x$. Then $F_-(G(r)) \leq r \leq F(G(r))$ and $F_-(G(r)) \leq s \leq F(G(r))$. Since $\mu(\{G(r)\}) = 0$, it holds that $F_-(G(r)) = F(G(r)) = r = s$ and, hence, G is injective.

$\Leftarrow$) Suppose that there exists $x \in X$ such that $\mu(\{x\}) > 0$. Then $F_-(x) < F(x)$. Now, let $r \in [0,1]$ be such that $F_-(x) < r < F(x)$. By Proposition 7.40.3, we have that G is defined at r and $G(r) = G(F(x)) = x$ for each $r \in]F_-(x), F(x)[$, which is a contradiction with the fact that G is injective. □

Proposition 7.46. *Let F be a cdf. Then F is injective if and only if $\mu(]a,b]) > 0$ for each $a < b$.*

Proof. Let $a, b \in X$ be such that $a < b$. Note that, by Proposition 7.28, $\mu(]a,b]) = 0$ is equivalent to $F(b) - F(a) = 0$, that is, $F(b) = F(a)$ if and only if F is not injective. □

And we get, as immediate corollary, the next one.

Corollary 7.47. *Let F be a cdf of a probability measure μ, and let $A \subseteq [0,1]$ be the subset of points where G is defined. The following statements are equivalent:*

1. *$F \circ G(r) = r$ for each $r \in A$, $F(X) \subseteq A$ and $G \circ F(x) = x$ for each $x \in X$.*

2. *F is injective and $F(X) = A$.*

3. *$G : A \to X$ is bijective.*

4. *$\mu(]a, b]) > 0$ for each $a < b$ and $\mu(\{a\}) = 0$ for each $a \in X$.*

Proof. First, we prove the following result.

Claim 7.48. *If F is injective, then $F(X) \subseteq A$ and $G(F(x)) = x$ for each $x \in X$.*

Proof. Suppose that there exists $x \in X$ such that G is not defined at $F(x)$, that is, there does not exist the infimum of $\{y \in X : F(y) \geq F(x)\}$. It follows that x is not the infimum of the latter set, so there exists $y < x$ with $F(y) \geq F(x)$. By the monotonicity of F, it follows that $F(y) = F(x)$ and, since F is injective, $y = x$, a contradiction. We conclude that $F(X) \subseteq A$.

Finally, let $x \in X$. Then $F(x) \in A$ and $G(F(x)) = \inf\{y \in X : F(y) \geq F(x)\}$. On the other hand, if $y < x$, then $F(y) \leq F(x)$ and, since F is injective, $F(y) < F(x)$. Therefore, $G(F(x)) = x$. □

(1) $\implies$ (2). Since $F(X) \subseteq A$ and $G(F(x)) = x$ for each $x \in X$, it follows that F is injective. Now, we prove that $A \subseteq F(X)$. Indeed, let $r \in A$. Then $F(G(r)) = r$, so $r \in F(X)$.

(1) $\implies$ (3). Since $G(F(x)) = x$ for each $x \in X$, it follows that G is surjective. Since $F(G(r)) = r$ for each $r \in A$, it follows that G is injective.

(1) $\implies$ (4). Since (1) implies (2) and (3), we have that F and G are both injective, so (4) follows from Propositions 7.45 and 7.46.

(2) $\implies$ (1). Let $r \in A$. Since $F(X) = A$ and F is injective, there is only one $x \in X$ such that $F(x) = r$. It follows, by definition of G, that $G(r) = x$ and, hence, $F(G(r)) = F(x) = r$. By the previous claim, we have the rest of item (1).

(3) $\implies$ (1). Let $r \in A$. Then $F(G(r)) \geq r$ by Proposition 7.38. Suppose that $F(G(r)) > r$. It easily follows that $]r, F(G(r))[\subseteq A$ and $G(]r, F(G(r))[) = G(r)$, but this is a contradiction, since G is injective. We conclude that $F(G(r)) = r$.

Now, let $x \in X$. Since G is bijective, there exists $r \in A$ such that $x = G(r)$. It follows that $F(x) = F(G(r)) = r$ and, hence, $F(x) \in A$. Therefore, $F(X) \subseteq A$.

Finally, let $x \in X$. Then $F(x) \in A$ and $G(F(x)) = \inf\{y \in X : F(y) \geq F(x)\}$. Suppose that there exists $y < x$ such that $F(y) \geq F(x)$. By the monotonicity of F, it follows that $F(y) = F(x)$. Since G is bijective, there exist $r, s \in [0, 1]$ such that $G(r) = y$ and $G(s) = x$. Note that $r < s$ by the monotonicity of G. It follows that $r = F(G(r)) = F(y) = F(x) = F(G(s)) = s$, a contradiction. We conclude that $x = \inf\{y \in X : F(y) \geq F(x)\} = G(F(x))$.

(4) $\implies$ (1). By Corollary 7.44, it follows that $F(G(r)) = r$ for each $r \in A$. By Proposition 7.46, F is injective and, by the previous claim, it follows that $F(X) \subseteq A$ and $G(F(x)) = x$ for each $x \in X$. □

Proposition 7.49. *Let $a, b \in X$ be such that $a < b$. Then $G^{-1}(]a, b[) =]F(a), F_{-}(b)| \cap A$, where $|$ means $]$ or $[$ and A is the subset of $[0, 1]$ where G is defined.*

Proof. First, we show that $G^{-1}(]a, b[) \subseteq]F(a), F_{-}(b)] \cap A$. For that purpose, let $r \in G^{-1}(]a, b[)$ and suppose that $r \notin]F(a), F_{-}(b)]$. Then it can happen:

- $r \leq F(a)$, which implies, by Corollary 7.39, that $G(r) \leq a$, which is a contradiction with the fact that $r \in G^{-1}(]a, b[)$.
- $r > F_{-}(b)$, which gives us, by Proposition 7.40.2, that $b \leq G(r)$, which implies that $r \notin G^{-1}(]a, b[)$, since $b \notin]a, b[$, a contradiction.

Now, we prove that $]F(a), F_{-}(b)[\cap A \subseteq G^{-1}(]a, b[)$. For that purpose, let $r \in]F(a), F_{-}(b)[$ where G is defined, and suppose that $r \notin G^{-1}(]a, b[)$. Then it can happen:

- $G(r) \leq a$, which implies, by Corollary 7.39, that $r \leq F(a)$, a contradiction with the fact that $r > F(a)$.
- $G(r) \geq b$ which gives us, by Proposition 7.40.4, that $r \geq F_{-}(b)$, a contradiction with the fact that $r < F_{-}(b)$.

We conclude that $r \in G^{-1}(]a, b[)$. □

According to Proposition 7.49, it is clear the next result.

Corollary 7.50. *Suppose that G is defined on $[0, 1]$. Let $a, b \in X$ be such that $a < b$. Then $G^{-1}(]a, b[) \in \sigma([0, 1])$, where $\sigma([0, 1])$ denotes de Borel σ-algebra with respect to the Euclidean topology.*

Proof. Since $G^{-1}(]a,b[) =]F(a), F_-(b)|$, it is an open set or the intersection of an open and a closed set. Consequently, $G^{-1}(]a,b[) \in \sigma([0,1])$. □

Proposition 7.51. *Each open set in τ is the countable union of open intervals.*

Proof. Let $G \subseteq X$ be an open set in τ. If $G = \emptyset$, the result is clear, since it can be written as $G =]a,a[$. Now, suppose that G is nonempty. Then $G = \bigcup_{i \in I} G_i$, where G_i is a convex component of G for each $i \in I$ (see Proposition 2.36). Now, we prove that G_i is open for each $i \in I$. Let $i \in I$ and $x \in G_i$. Since G is an open set and $\{]a,b[: a,b \in X, a < b\}$ is an open base of X with respect to τ, there exist $a,b \in X$ such that $x \in]a,b[\subseteq G$. Note that $G_i \cup]a,b[$ is a convex set contained in G, which implies that $G_i \cup]a,b[= G_i$, since G_i is a convex component of G. Consequently, $]a,b[\subseteq G_i$, which means that G_i is an open set. Now, let D be a countable dense subset of X. Then we can choose $d_i \in D \cap G_i$ for each $i \in I$, which gives us the countability of I, since the family $\{G_i : i \in I\}$ is pairwise disjoint.

Since G_i is convex and open, by Corollary 7.5, G_i can be expressed as a countable union of open intervals. Thus, G is the countable union of open intervals. □

Next result will be essential to show that G is measurable with respect to the Borel σ-algebra.

Theorem 7.52. *([10, Th. 1.7.2]) Let $(\Omega, \mathfrak{A})$ and $(\Omega', \mathfrak{A}')$ be measurable spaces; further let $\mathfrak{B}'$ be a generator of $\mathfrak{A}'$. A mapping $T : \Omega \to \Omega'$ is measurable if and only if $T^{-1}(A') \in \mathfrak{A}$ for each $A' \in \mathfrak{B}'$.*

Since $G^{-1}(]a,b[) \in \sigma([0,1])$ for each $a,b \in X$ with $a < b$ and, by taking into account Proposition 7.51, we conclude the next result.

Corollary 7.53. *Suppose that G is defined on $[0,1]$. Then G is measurable with respect to the Borel σ-algebras.*

Proof. To show that G is measurable we just have to use Corollary 7.50, Theorem 7.52 and the fact that each open set in τ can be written as the countable union of open intervals (see Proposition 7.51). □

7.5 Generating samples

Once we have studied the properties of the pseudo-inverse, the next step is to take advantage of this function in order to generate samples of a certain distribution.

Lemma 7.54. *The family* $\mathfrak{A} = \{\bigcup_{i=1}^{n} |a_i, b_i| : a_1 \leq b_1 < a_2 \leq b_2 < \ldots < a_n \leq b_n, a_1 \in X \cup \{-\infty\}, b_n \in X \cup \{\infty\}\}$ *is an algebra and the* σ*-algebra generated by it is the Borel* σ*-algebra.*

Proof. Now, we prove that $\mathfrak{A}$ is an algebra.

1. $A \cup B \in \mathfrak{A}$ for each $A, B \in \mathfrak{A}$. Indeed, this is true because the union of two intervals consists of two disjoint intervals in case $A \cap B = \emptyset$ or it is a new interval otherwise.

2. $A \cap B \in \mathfrak{A}$ for each $A, B \in \mathfrak{A}$. Indeed, this is true because the intersection of two intervals is $\emptyset$ or a new interval. Hence, $A \cap B$ is the finite union of disjoint intervals, which means that $A \cap B \in \mathfrak{A}$.

3. $X \backslash A \in \mathfrak{A}$ for each $A \in \mathfrak{A}$. Indeed, this is true due to the fact that $X \backslash A =]-\infty, a_1| \cup |b_1, a_2| \cup \ldots \cup |b_{n-1}, a_n| \cup |b_n, \infty[\in \mathfrak{A}$.

Note that each element in $\mathfrak{A}$ belongs to (X, τ). Indeed, this is true due to the fact that, given $A \in \mathfrak{A}$, it consists of the finite union of open intervals, semi-open intervals (which are the intersection of an open and a closed set) or closed intervals (which are closed). Hence, $\mathfrak{S}$ is contained in the Borel σ-algebra of (X, τ), where $\mathfrak{S} = \sigma(\mathfrak{A})$. Finally, if G is an open set in (X, τ), by Proposition 7.51, it can be written as the countable union of open intervals. Thus, G can be written as the countable union of elements in $\mathfrak{A}$, which means that $G \in \mathfrak{S}$. In conclusion, $\mathfrak{S}$ is the Borel σ-algebra of (X, τ). □

Now, we want to prove the uniqueness of the measure with respect to its cdf.

First, recall Theorem 2.15, which is about the uniqueness of a measure. As a consequence of it, we have that two measures that coincide in an algebra also coincide in its generated σ-algebra.

Proposition 7.55. *Let* F_μ *and* F_δ *be the cdfs of the measures* μ *and* δ *satisfying* $F_\mu = F_\delta$. *Then* $\mu = \delta$ *on the Borel* σ*-algebra of* (X, τ).

Proof. Let $a, b \in X$ be such that $a \leq b$. Then a cdf lets us determine the measure of the set $|a, b|$. Indeed, we distinguish four cases depending on whether a and b belong to $|a, b|$ or not:

1. $\mu(]a, b]) = F_\mu(b) - F_\mu(a) = F_\delta(b) - F_\delta(a) = \delta(]a, b])$.

2. $\mu([a, b]) = F_\mu(b) - F_{\mu-}(a) = F_\mu(b) - \sup F_\mu(< a) = F_\delta(b) - \sup F_\delta(< a) = F_\delta(b) - F_{\delta-}(a) = \delta([a, b])$, where we have taken into account that $F_-(x) = \sup F(< x)$ for each $x \in X$ (see Proposition 7.25).

3. $\mu(]a, b[) = F_{\mu-}(b) - F_\mu(a) = \sup F_\mu(< b) - F_\mu(a) = \sup F_\delta(< b) - F_\delta(a) = F_{\delta-}(b) - F_\delta(a) = \delta(]a, b[)$.

4. $\mu([a, b[) = F_{\mu-}(b) - F_{\mu-}(a) = \sup F_\mu(< b) - \sup F_\mu(< a) = \sup F_\delta(< b) - \sup F_\delta(< a) = F_{\delta-}(b) - F_{\delta-}(a) = \delta([a, b[)$.

Since $\mu(|a, b|) = \delta(|a, b|)$ for each $a, b \in X$ with $a \leq b$, it follows that $\mu(A) = \delta(A)$ for each $A \in \mathfrak{A}$ due to the σ-additivity of μ and δ as measures. Since $\mu = \delta$ on $\mathfrak{A}$, we conclude that $\mu = \delta$ on $\sigma(\mathfrak{A})$, that is, they coincide in the Borel σ-algebra of (X, τ) by the previous results. □

Theorem 7.56. *([59, Th. A. 81]) A measurable function f from one measure space $(S_1, \mathcal{A}_1, \mu_1)$ to a measurable space $(S_2, \mathcal{A}_2)$, $f : S_1 \to S_2$, induces a measure on the range S_2. For each, $A \in \mathcal{A}_2$, define $\mu_2(A) = \mu_1(f^{-1}(A))$. Integrals with respect to μ_2 can be written as integrals with respect to μ_1 in the following way: If $g : S_2 \to \mathbb{R}$ is integrable, then,*

$$\int g(y) d\mu_2(y) = \int g(f(x)) d\mu_1(x)$$

Proposition 7.57. *Let μ be a probability measure and suppose that G is defined on $[0, 1]$. Then $\mu(A) = l(G^{-1}(A))$ for each $A \in \sigma(X)$, where l is the Lebesgue measure and $\sigma(X)$ is the Borel σ-algebra of X.*

Proof. By Proposition 7.49, we have that $G^{-1}(]a, b[) =]F(a), F_-(b)|$ for each $a, b \in X$ with $a < b$.

Moreover, by Corollary 4.6, it holds that $\mu(]a, b[) = F_-(b) - F(a)$. It follows that $l(G^{-1}(]a, b[)) = \mu(]a, b[)$ for each $a, b \in X$ with $a < b$.

Now, let μ_2 be the measure defined by $\mu_2(A) = l(G^{-1}(A))$ for each $A \in \sigma(X)$. Indeed, μ_2 is a measure by Theorem 7.56 and Corollary 7.53. Note that the fact that $\mu(]a,b[) = \mu_2(]a,b[)$ for each $a, b \in X$ with $a < b$, implies that $\mu = \mu_2$ on the algebra $\mathfrak{A}$. Therefore, μ and μ_2 coincides in an algebra which generates $\sigma(X)$, so they are equal in $\sigma(X)$ (see, for example, Theorem 2.15).

Consequently, we can write $\mu(A) = l(G^{-1}(A))$ for each $A \in \sigma(X)$. □

Finally, by taking into account the previous results, we can generate samples with respect to the probability measure μ by following the classical procedure (see the introduction to Part II). In our case we will have to use G to do it.

Remark 7.58. *Suppose that G is defined on $[0,1]$. We can also calculate integrals with respect to μ by using Theorem 7.56, so for $g : X \to \mathbb{R}$,*

$$\int g(x)d\mu(x) = \int g(G(t))dt$$

Remark 7.59. *Suppose that X is compact. Then every subset of X has both infimum and supremum (see Proposition 2.31) and, hence, G is defined at each point of $[0,1]$. Therefore, in this case, we can generate samples with respect to a distribution based on a measure μ.*

Remark 7.60. *Note that the classical theory for the distribution function is a particular case of the one we have developed for a separable LOTS.*

Chapter 8

The cdf of a probability measure on the Dedekind-MacNeille completion of a LOTS

The content of this chapter corresponds to [34].

In the previous chapter we provided a theory of a cdf F (defined from a probability measure μ) on a separable LOTS X. When X is compact, the pseudo-inverse of the cdf is properly defined as $G(r) = \inf\{x \in X : F(x) \geq r\}$ for each $r \in [0,1]$, and can be used to generate samples in X, as in the classical case.

When X is not compact, there is no guarantee that there exists the infimum, so G may not be defined for each point of $[0,1]$. In this chapter we show (see Section 8.3) that G is naturally defined from $[0,1]$ to $DM(X)$, where $DM(X)$ is the Dedekind-MacNeille completion (indeed, compactification) of X. In fact, in Sections 8.2 and 8.3, we show that the probability measure μ on X can be extended to a probability measure $\widetilde{\mu}$ on $DM(X)$ such that its cdf $\widetilde{F}$ is an extension of F to $DM(X)$ and G is the pseudo-inverse of $\widetilde{F}$. These results allow us (see Section 8.4) to use G to generate samples of the probability measure on X, even if X is not compact.

It follows that $DM(X)$ is the right place to work when X is a separable LOTS. In fact, in Section 8.5 we study the equivalence between probability measures and distribution functions defined on $DM(X)$ by proving that each monotonically non-decreasing and right continuous function $F : DM(X) \to [0,1]$ with $\sup F(X) = 1$ is the cdf of a (unique) probability measure on $DM(X)$. As a corollary, we obtain that each monotonically non-decreasing and right continuous function $F : X \to [0,1]$ with $\sup F(X) = 1$ can be extended to a cdf on $DM(X)$.

Of course, for compact separable LOTSs, we also get the equivalence between probability measures and distribution functions.

8.1 The completion of the order

First of all, we will study and prove some properties of the Dedekind-MacNeille completion of a separable LOTS, which will be crucial in the rest of the chapter.

Given $x \in X$, it is clear that $(\leq x)$ is a cut, since $(\leq x)^u = (\geq x)$ and $(\geq x)^l = (\leq x)$, that is, $((\leq x)^u)^l = (\leq x)$.

Fist, we prove that we do not add (left or right) isolated points when making the completion, except for $\min DM(X)$ and $\max DM(X)$.

Lemma 8.1. *Let $A \in DM(X)$ be a left-isolated (respectively right-isolated) cut such that $A \neq \min DM(X)$ (respectively $A \neq \max DM(X)$). Then there exists $x \in X$ such that $A = (\leq x)$.*

Proof. Let A be a left-isolated cut such that $A \neq \min DM(X)$. Then there exists $B \in DM(X)$ such that $]B, A[= \emptyset$. Let $x \in A \setminus B$ and suppose that $A \neq (\leq x)$. Then $(\leq x) \in]B, A[$, which is a contradiction with the fact that A is left-isolated. Thus, $A = (\leq x)$.

Suppose now that A is right-isolated such that $A \neq \max DM(X)$. Then there exists $B \in DM(X)$ such that $]A, B[= \emptyset$. Since $A \subset B$, it holds that $B^u \subset A^u$. Now, let $x \in A^u \setminus B^u$ and suppose that $A^u \neq (\geq x)$. Then $B^u \subset (\geq x) \subset A^u$, which implies that $A \subset (\leq x) \subset B$, a contradiction with the fact that A is right-isolated. Thus, $A = (\leq x)$. □

Now, we prove that isolated points in X are related to isolated points in $DM(X)$.

Proposition 8.2. *Let LI and RI be, respectively, the set of left-isolated and right-isolated points of X. It holds that:*

1. $LI(DM(X)) = \phi(LI(X)) \cup \{\min DM(X)\}$.
2. $RI(DM(X)) = \phi(RI(X)) \cup \{\max DM(X)\}$.

Proof. 1. $\supseteq$) It is clear that $\min DM(X)$ is a left-isolated cut of $DM(X)$, since $(< \min DM(X)) = \emptyset$. Now, let $x \in X$ be a left-isolated point. Then there exists $a < x$ such that $]a, x[= \emptyset$. Hence, $](\leq a), (\leq x)[= \emptyset$, which means that $(\leq x)$ is left-isolated in $DM(X)$.

$\subseteq$) Let $A \in DM(X)$ be a left-isolated cut such that $A \neq \min DM(X)$. Then there exists $B \in DM(X)$ such that $]B, A[= \emptyset$. Note that B is right-isolated. Therefore, by Lemma 8.1, there exist $x, y \in X$ such that $(\leq x) = A$ and $B = (\leq y)$. Moreover, note that x is left-isolated due to the fact that $]y, x[= \emptyset$.

2. It can be proven analogously to the previous item.

□

Proposition 8.3. *Let D be a dense subset of X with respect to the order topology, τ. Then $\phi(D)$ is dense in $DM(X)$ with respect to τ' (the order topology on $DM(X)$).*

Proof. Let $C \in DM(X)$ be such that $C \neq \min DM(X)$ and $C \neq \max DM(X)$, and consider U, a neighborhood of C with respect to τ'. Then there exists $G \in \tau'$ such that $C \in G \subseteq U$. Since G is an open set containing C and the family $\{]A, B[: A < B; A, B \in DM(X)\} \cup \{(< A) : A \in DM(X)\} \cup \{(> A) : A \in DM(X)\}$ is an open base of $DM(X)$ (see Remark 2.26), we can consider $A, B \in DM(X)$ with $A < B$ such that $C \in]A, B[\subseteq G$. We distinguish some cases depending on whether C is isolated or not:

1. Suppose that C is not left-isolated nor right-isolated. Then we can consider $a \in C \setminus A$ and $b \in B \setminus C$ so that $]a, b[$ is an open set with respect to τ. Hence, the fact that D is dense gives us that there exists $d \in]a, b[\cap D$. What is more, $(\leq a) < (\leq d) < (\leq b)$, so we conclude that $\phi(d) \in]A, B[$, which implies that $U \cap \phi(D) \neq \emptyset$.

2. Suppose that C is left-isolated but it is not right-isolated. By Lemma 8.1, there exists $x \in X$ such that $C = (\leq x)$. Now, let $b \in B \setminus C$. Then it holds that $]x, b[$ is an open set in τ. Hence, the fact that D is dense gives us that there exists $d \in]x, b[\cap D$. We conclude that $\phi(d) \in]A, B[$, which implies that $U \cap \phi(D) \neq \emptyset$.

3. Suppose that C is right-isolated but it is not left-isolated. Then, by Lemma 8.1, there exists $x \in X$ such that $C = (\leq x)$. Now, let $a \in C \setminus A$. Then $]a, x[$ is an open set in τ. Hence, the fact that D is dense gives us that there exists $d \in]a, x[\cap D$. What is more, $(\leq a) < (\leq d) < (\leq x)$, so we conclude that $\phi(d) \in]A, B[$, which implies that $U \cap \phi(D) \neq \emptyset$.

4. Suppose that C is isolated. Hence, we can choose A and B such that $]A, C[= \emptyset$ and $]C, B[= \emptyset$. Since A is right-isolated, there exists $a \in X$ such that $A = (\leq a)$ (see Lemma 8.1). Moreover, the fact that B is left-isolated implies that there exists $b \in B$ such that $B = (\leq b)$. Finally, there exists $x \in X$ such that $C = (\leq x)$. Since $]a, b[$ is open with respect to τ and D is dense in X, it holds that $x \in D$. Consequently, $\phi(x) \in]A, B[\cap \phi(D)$.

The cases in which $C = \max DM(X)$ and $C = \min DM(X)$ can be justified analogously. Hence, $\phi(D)$ is dense in $DM(X)$. □

Corollary 8.4. *If X is a separable LOTS, then $DM(X)$ is also a separable LOTS.*

Proof. Let D be a countable dense subset of X. The separability of $DM(X)$ follows from the fact that $\phi(D)$ is dense in $DM(X)$ (see the previous proposition) and the fact that $\phi(D)$ is countable. □

Corollary 8.5. *If X is a separable LOTS, then $DM(X)$ is a first countable LOTS.*

Remark 8.6. *Note that $DM(X)$ is a compactification of X. In fact, it is the smallest order-compactification of X ([11], [40]).*

Corollary 8.7. *If X is a second countable LOTS, then $DM(X)$ is also second countable.*

Proof. Since X is second countable, by Proposition 7.8, it holds that X is separable and the set of points which are right-isolated or left-isolated is countable. Since X is separable, $DM(X)$ is also separable (see Corollary 8.4). Moreover, by Proposition 8.2, we have that the set of points which are right-isolated or left-isolated in $DM(X)$ is countable. If we join the previous fact with the fact that $DM(X)$ is separable, we conclude that $DM(X)$ is second countable by Proposition 7.8. □

Lemma 8.8. *Let X be a separable LOTS and $A \subseteq X$. If A is decreasing (respectively increasing) and it does not have a maximum (respectively a minimum), then there exists* *an increasing (respectively decreasing) sequence* (a_n) *in* A *such that* $\bigcup_{n\in\mathbb{N}}(\leq a_n) = A$ *(respectively* $\bigcup_{n\in\mathbb{N}}(\geq a_n) = A$*).*

Proof. Let A be a decreasing set such that it does not have a maximum and let D be a countable dense subset of X with respect to the topology τ. Since it is countable, we can enumerate it, $D = \{d_n : n \in \mathbb{N}\}$. Now, let k be the least natural number such that $d_k \in A$. Define $a_1 = d_k$. Now, suppose that we have defined a_n and define $a_{n+1} = d_m$, where m is the least natural such that $d_m \in A$ and $d_m > a_n$. By construction, it is clear that the sequence (a_n) is increasing. Now, we prove that $\bigcup_{n\in\mathbb{N}}(\leq a_n) = A$. Indeed,

$\subseteq$) Let $x \in \bigcup(\leq a_n)$. Then there exists $n \in \mathbb{N}$ such that $x \in (\leq a_n)$. Since $a_n \in A$ for each $n \in \mathbb{N}$ and A is decreasing, we conclude that $x \in A$.

$\supseteq$) Let $x \in A$. It holds that $(> x) \cap A$ is a nonempty open set in A. Indeed, it is nonempty due to the fact that A does not have a maximum. Let $a' \in (> x) \cap A$, and consider $a \in (> a') \cap A$. Then $]x, a[$ is a nonempty open set in τ, so there exists $k \in \mathbb{N}$ such that $d_k \in]x, a[$. Hence, by definition of the sequence (a_n), $d_n \leq a_n$ for each $n \in \mathbb{N}$ such that $d_n \in A$. This implies that $x \in (\leq a_k)$ and, consequently, $x \in \bigcup(\leq a_n)$.

We can proceed analogously to define a decreasing sequence in case that A is increasing and it does not have a minimum. $\square$

Remark 8.9. *Note that* A^l *is decreasing and* A^u *is increasing for each* $A \subseteq X$.

8.2 The extension of a cdf to $DM(X)$

In this section we extend the definition of a cdf to the Dedekind-MacNeille completion of a separable LOTS. For that purpose, in the rest of this chapter, X is a separable LOTS.

Lemma 8.10. *Let* F *be a non-decreasing function and let* (a_n) *be an increasing (respectively decreasing) sequence in a decreasing (respectively increasing) set* $A \subseteq X$ *such that* $A = \bigcup(\leq a_n)$ *(respectively* $A = \bigcup(\geq a_n)$*). Then* $F(a_n) \to \sup F(A)$ *(respectively* $F(a_n) \to \inf F(A)$*).*

Proof. Let A be a decreasing set of X and (a_n) an increasing sequence such that $A = \bigcup(\leq a_n)$. Let $r < \sup F(A)$. Then there exists $a \in A$ such that $r < F(a)$. Since $A = \bigcup(\leq a_n)$, there exists $n_0 \in \mathbb{N}$ such that $a_{n_0} \geq a$. Consequently, $a_n \geq a$ for each

$n \geq n_0$. The monotonicity of F gives us that $r < F(a) \leq F(a_n) < \sup F(A)$ for each $n \geq n_0$, which implies that $F(a_n) \to \sup F(A)$.

A similar proof lets us conclude that, in case A is increasing and a_n is a decreasing sequence satisfying that $A = \bigcup(\geq a_n)$, we have that $F(a_n) \to \inf F(A)$. □

Proposition 8.11. *Let μ be a probability measure on X and consider $A \subseteq X$. If A is decreasing, then $\mu(A) = \sup F(A)$.*

Proof. Let $A \subseteq X$ be a decreasing set. First, note that if a is the maximum of A, then it is clear that $\mu(A) = \mu(\leq a) = F(a) = \sup F(A)$.

If A does not have a maximum, by Lemma 8.8, there exists an increasing sequence (a_n) such that $\bigcup_{n\in\mathbb{N}}(\leq a_n) = A$. Since $(\leq a_n)$ is a monotonically non-decreasing sequence of sets, it holds that $(\leq a_n) \to \bigcup_{n\in\mathbb{N}}(\leq a_n) = A$. By the continuity from below of the measure μ, we have that $\mu(\leq a_n) \to \mu(A)$. Moreover, $\mu(\leq a_n) = F(a_n) \to \sup F(A)$ by Lemma 8.10, so we conclude that $\mu(A) = \sup F(A)$.

□

Proposition 8.12. *Let $F : X \to [0,1]$ a cdf defined from a probability measure μ. Then* $\sup F(A) = \inf F(A^u)$ *for each $A \in DM(X)$.*

Proof. We distinguish two cases depending on whether the intersection of A and A^u is empty or not:

1. Suppose that $A \cap A^u = \{x\}$ for some $x \in X$. Then $A = (\leq x)$ and $A^u = (\geq x)$, and the result is clear.

2. Suppose that $A \cap A^u = \emptyset$. Since A is decreasing and it does not have a maximum, by Lemma 8.8, we can consider an increasing sequence, a_n, in A such that $A = \bigcup(\leq a_n)$. Moreover, since A^u is increasing and it does not have a minimum, we can consider a decreasing sequence in A^u, (b_n), such that $A^u = \bigcup(\geq b_n)$. Therefore, $]a_n, b_n]$ is a monotonically non-increasing sequence, which implies that $]a_n, b_n] \to \bigcap_{n\in\mathbb{N}}]a_n, b_n]$. What is more, $\bigcap_{n\in\mathbb{N}}]a_n, b_n] = \emptyset$. Indeed, suppose that there exists $y \in]a_n, b_n]$ for each $n \in \mathbb{N}$. The fact that $a_n < y$ for each $n \in \mathbb{N}$ means that $y \notin A$. Hence, $y \in A^u$. If we join $y \in A^u$ with the fact that $y \leq b_n$ for each $n \in \mathbb{N}$, it follows that $y = \min A^u$, which is a contradiction. Then the continuity

from above of the measure μ gives us that $\mu(]a_n, b_n]) \to \mu(\emptyset) = 0$. Moreover, $\mu(]a_n, b_n]) = F(b_n) - F(a_n)$ and Lemma 8.10 gives us that $F(b_n) \to \inf F(A^u)$ and $F(a_n) \to \sup F(A)$, which let us conclude that $\inf F(A^u) = \sup F(A)$.

□

Proposition 8.13. *Let $A \in DM(X)$. Then $\phi^{-1}(\leq A) = A$.*

Proof. $\subseteq$) Let $x \in X$ be such that $\phi(x) \in (\leq A)$. Then $\phi(x) \leq A$, that is, $(\leq x) \subseteq A$ and, consequently, $x \in A$.

$\supseteq$) Let $x \in A$. Then $(\leq x) \subseteq A$ due to the fact that A is decreasing, so $\phi(x) \leq A$, which implies that $\phi(x) \in (\leq A)$.

□

Definition 8.14. *Given the cdf, F, of a probability measure μ defined on X, we define $\widetilde{F} : DM(X) \to [0,1]$ by $\widetilde{F}(A) = \inf F(A^u)$ for each $A \in DM(X)$.*

We can prove that F is the restriction of $\widetilde{F}$ to X.

Lemma 8.15. *$\widetilde{F} \circ \phi = F$.*

Proof. Given $x \in X$, it holds that $\widetilde{F} \circ \phi(x) = \widetilde{F}(\leq x) = \inf F(\geq x)$, since $(\leq x)^u = (\geq x)$. What is more, $\inf F(\geq x) = F(x)$, so we conclude that $\widetilde{F}(\leq x) = F(x)$ for each $x \in X$.

□

We can prove that, indeed, $\widetilde{F}$ is a cdf.

Definition 8.16. *Given a probability measure μ in X, we define its extension to $DM(X)$, $\widetilde{\mu}$, as follows, $\widetilde{\mu}(A) = \mu(\phi^{-1}(A))$ for each $A \in \sigma(DM(X))$.*

Next, we prove that, indeed, $\widetilde{\mu}$ is a probability measure.

Proposition 8.17. *$\widetilde{\mu}$ is a probability measure with respect to the Borel σ-algebra of $DM(X)$.*

Proof. Let us define the family $\mathcal{A} = \{A \in DM(X) : \phi^{-1}(A) \in \sigma(X)\}$. It can be easily proven that $\mathcal{A}$ is a σ-algebra in X. Define the set-function $\overline{\mu} : \mathcal{A} \to [0,1]$ by the formula

$\overline{\mu}(A) := \mu(\phi^{-1}(A))$ for each $A \in \mathcal{A}$. It can be easily seen that $\overline{\mu}$ is a probability measure on $\mathcal{A}$ (see [14, Section 3.6]), implying that its restriction $\widetilde{\mu}$ to $\sigma(DM(X))$ is a probability measure on the Borel σ-algebra $\sigma(DM(X))$.

□

Proposition 8.18. *$\widetilde{F}$ is the cdf defined from $\widetilde{\mu}$.*

Proof. Note that $\widetilde{F}(A) = \inf F(A^u) = \sup F(A)$ by definition of $\widetilde{F}$ and Proposition 8.12. Now, by Proposition 8.11, $\mu(A) = \sup F(A)$ for each $A \in DM(X)$, so we can write $\widetilde{F}(A) = \mu(A) = \mu(\phi^{-1}(\leq A)) = \widetilde{\mu}(\leq A)$ for each $A \in DM(X)$, where we have used the definition of $\widetilde{\mu}$ and the fact that $\phi^{-1}(\leq A) = A$ for each $A \in DM(X)$ (see Proposition 8.13). □

Remark 8.19. *$\widetilde{F}$ is monotonically non-decreasing, right continuous, $\sup \widetilde{F}(DM(X)) = 1$ and $\inf \widetilde{F}(DM(X)) = 0$ if there does not exist the minimum of X.*

Proof. It immediately follows from Propositions 7.19 and 8.18. □

Corollary 8.20. *$\widetilde{F}_- \circ \phi = F_-$.*

Proof. Let $x \in X$. By Proposition 8.18, $\widetilde{F}$ is the cdf of $\widetilde{\mu}$ so, by definition of $\widetilde{F}_-$, we have that $\widetilde{F}_-(\phi(x)) + \widetilde{\mu}(\phi(x)) = \widetilde{F}(\phi(x))$. Now, if we take into account Lemma 8.15 and the definition of $\widetilde{\mu}$, the previous equality is $\widetilde{F}_-(\phi(x)) + \mu(\phi^{-1}(\phi(x))) = F(x)$ if and only if $\widetilde{F}_-(\phi(x)) = F(x) - \mu(\{x\}) = F_-(x)$. □

8.3 The pseudo-inverse of a cdf

In this section we see that the pseudo-inverse of a cdf can be naturally defined from $[0,1]$ to the Dedekind-MacNeille completion of a separable LOTS. Indeed, the pseudo-inverse of the extension of a cdf to the Dedekind-MacNeille completion matches with that new definition of the pseudo-inverse.

The definition of the pseudo-inverse of a cdf defined on $DM(X)$ is the following one.

Definition 8.21. *Let F be a cdf. We define the mapping $G : [0,1] \to DM(X)$ by $G(r) = A$ for each $r \in [0,1]$, where $B = \{x \in X : F(x) \geq r\}$ and $A = B^l$.*

Proposition 8.22. *Given $r \in [0,1]$, it holds that $G(r) \in \phi(X)$ or $G(r) = \{x \in X : F(x) < r\}$.*

Proof. Let $B = \{x \in X : F(x) \geq r\}$ and $A = B^l$. We distinguish two cases depending on whether there exists the minimum of B or not:

1. Suppose that there exists the minimum of B an denote it by x. Since $x = \min B$, we have that $A = B^l = (\leq x)$, which implies that $G(r) \in \phi(X)$.

2. Suppose that there does not exist the minimum of B. We prove that $G(r) = B^l = \{x \in X : F(x) < r\}$.

   $\subseteq$) Let $x \in X$ be such that $x \in B^l$. Then $x \notin B$, since there does not exist the minimum of B. Consequently, $F(x) < r$.

   $\supseteq$) Let $x \in X$ be such that $F(x) < r$ and let $b \in B$. Then $F(b) \geq r$, which implies that $F(x) < F(b)$. Hence, by the monotonicity of F, we have that $x \leq b$. By the arbitrariness of b, we conclude that $x \in B^l$.

□

Next, we prove that G is well defined.

Lemma 8.23. *Let $B \subseteq X$ be an increasing set. Then $B^l \in DM(X)$.*

Proof. We distinguish two cases depending on whether there exists the infimum of B or not:

1. Suppose that there exists the infimum of B and denote it by a. Then $B^l = (\leq a) \in DM(X)$.

2. Suppose that there does not exist the infimum of B. We prove that $(B^l)^u = B$ in which case we have that $B^l \in DM(X)$.

   $\subseteq$) Let $b \in (B^l)^u$ and suppose that $b \notin B$. By definition of the set of upper bounds, we have that the fact that $b \in (B^l)^u$ implies that $b \geq x$ for each $x \in B^l$. Now, since there does not exist the infimum of B, $b \notin B^l$. Since $b \notin B^l$, then there exists $c \in B$ such that $c < b$ and, since B is increasing, $b \in B$, a contradiction.

   $\supseteq$) Let $b \in B$ and $x \in B^l$. Since $x \in B^l$, it holds that $x \leq b$, which means that $b \in (B^l)^u$.

□

Proposition 8.24. *$G(r) \in DM(X)$ for each $r \in [0, 1]$.*

Proof. Let $r \in [0, 1]$ and $B = \{x \in X : F(x) \geq r\}$. In order to prove that $G(r) \in DM(X)$, we must prove that $B^l \in DM(X)$. For that purpose, we show that B is increasing due to the fact that Lemma 8.23 lets us ensure that $B^l \in DM(X)$.

Let $b \in B$. Then it holds that $F(b) \geq r$. Now, let $x \in X$ be such that $x > b$. Then the monotonicity of F as a cdf gives us that $F(b) \leq F(x)$, so we conclude that $F(x) \geq r$, that is, $x \in B$, and, consequently, B is increasing. □

Proposition 8.25. *$G(r)^u = \{x \in X : F(x) \geq r\}$ for each $r \in [0, 1]$.*

Proof. Let $G(r) = A = B^l$, where $B = \{x \in X : F(x) \geq r\}$. In the proof of the previous proposition we have proven that B is increasing. If there does not exist the infimum of B, we have, by the proof of Lemma 8.23, that $(B^l)^u = B$ and, hence, $G(r)^u = B$ as wanted.

Suppose that there exists the infimum of B, and let $x = \inf B$. If $x = \min B$, then $B^l = (\leq x)$ and $B = (\geq x)$, so $G(r)^u = (\leq x)^u = (\geq x) = B$.

Finally, if $x \neq \min B$, by Lemma 7.2, [28, 4A2R(f)] and by Proposition 7.3, there exists a monotone sequence which right τ-converges to x, (x_n), with $x_n \in B$. By the right continuity of F, it follows that $(F(x_n))$ converges to $F(x)$. Since $x_n \in B$ for each $n \in \mathbb{N}$, then $F(x_n) \geq r$ for each $n \in \mathbb{N}$ and, hence, $F(x) \geq r$, so $x \in B$ and $x = \min B$, a contradiction.

We can conclude that $G(r)^u = B$. □

Next, we prove that G is the pseudo-inverse of the extension of F.

Proposition 8.26. *$G : [0, 1] \to DM(X)$ is the pseudo-inverse of $\widetilde{F}$.*

Proof. For that purpose, we prove that $G(r) = \inf\{C \in DM(X) : \widetilde{F}(C) \geq r\}$ for each $r \in [0, 1]$.

If $G(r) = A$, by Proposition 8.25, we have that $A^u = \{x \in X : F(x) \geq r\}$. Moreover, note that $G(r) = \inf \phi(A^u)$, which means that $G(r) \leq \phi(x)$ for each $x \in X$ such that $F(x) \geq r$.

$\leq$) Let $C \in DM(X)$ be such that $\widetilde{F}(C) \geq r$. Equivalently, $\inf F(C^u) \geq r$. Suppose that $C < A$. Then there exists $a \in A \setminus C$, which implies that $C < \phi(a)$ and, hence, the monotonicity of $\widetilde{F}$ as cdf gives us that $\widetilde{F}(C) \leq \widetilde{F}(\phi(a)) = F(a)$ by Lemma 8.15. We distinguish two cases depending on whether a is the minimum of A^u or not:

1. Suppose that $a \neq \min A^u$. Then $a \in A$ but $a \notin A^u$, which means that $F(a) < r$. Consequently, $\widetilde{F}(C) \leq F(a) < r$, a contradiction with the fact that $\widetilde{F}(C) \geq r$.

2. Suppose that $a = \min A^u$. Since $a \notin C$ and $C \in DM(X)$, we have that $a \notin (C^u)^l$, so we can choose $d \in C^u$ such that $d < a$, which implies that $d \notin A^u$ and $d \neq \max A$, so $F(d) < r$. By the monotonicity of $\widetilde{F}$ (note that $d \in C^u$ implies that $C < \phi(d)$) and by taking into account Lemma 8.15, it holds that $\widetilde{F}(C) \leq \widetilde{F}(\phi(d)) = F(d) < r$, which is a contradiction with the fact that $\widetilde{F}(C) \geq r$.

Consequently, $A \leq C$. By the arbitrariness of C, we conclude that $G(r) = A \leq \inf\{C \in DM(X) : \widetilde{F}(C) \geq r\}$.

$\geq$) Suppose that $G(r) < \inf\{C \in DM(X) : \widetilde{F}(C) \geq r\}$. Then there exists $c \in \inf\{C \in DM(X) : \widetilde{F}(C) \geq r\} \setminus A$. We distinguish two cases depending on whether $\phi(c) = \inf\{C \in DM(X) : \widetilde{F}(C) \geq r\}$ or not:

1. Suppose that $\phi(c) < \inf\{C \in DM(X) : \widetilde{F}(C) \geq r\}$. Since $c \notin A$, $c \in A^u$, that is, $F(c) \geq r$. By Lemma 8.15, $\widetilde{F}(\phi(c)) = F(c)$, which means that $\widetilde{F}(\phi(c)) \geq r$. The fact that $\widetilde{F}(\phi(c)) \geq r$ is a contradiction with the fact that $\phi(c) < \inf\{C \in DM(X) : \widetilde{F}(C) \geq r\}$.

2. Suppose that $\phi(c) = \inf\{C \in DM(X) : \widetilde{F}(C) \geq r\}$. Since $A < \phi(c)$, we distinguish two cases depending on whether $d \in A$ for each $d < c$ or not:

   (a) Suppose that there exists $d \in A^u$ such that $d < c$, which implies that $\phi(d) < \inf\{C \in DM(X) : \widetilde{F}(C) \geq r\}$. Moreover, $F(d) \geq r$ or, equivalently, $\widetilde{F}(\phi(d)) \geq r$, which is a contradiction with the fact that $\phi(d) < \inf\{C \in DM(X) : \widetilde{F}(C) \geq r\}$.

   (b) Suppose that $d \in A$ for each $d < c$. Then it holds that $A = (< c)$. Note that A is a right-isolated element of $DM(X)$, so Lemma 8.1 gives us that there exists $a \in A$ such that $A = (\leq a)$. Moreover, $]a, c[= \emptyset$. Thus, c is left-isolated.

What is more, $A^u = (\geq a)$, which implies that $a \in A^u$ and $F(a) \geq r$. Since $\widetilde{F}(\phi(a)) = F(a)$ by Lemma 8.15, we conclude, since c is the infimum, that $\phi(a) \geq \phi(c)$, which gives us that $a \geq c$, a contradiction.

□

8.4 Generating samples

Recall that, in Section 7.5, we proved that we can generate samples of a certain distribution when the pseudo-inverse of the cdf is defined for each $r \in [0,1]$. Now that we have proven that G can be naturally defined from $[0,1]$ to the Dedekind-MacNeille completion, we can generate samples of a distribution in each case, even if, for example, X is not compact. We just need to prove the next statement.

Proposition 8.27. *If $X \in \sigma(DM(X))$, then $\widetilde{\mu}(DM(X) \setminus X) = 0$, which means that $G(r) \in X$ almost surely for each $r \in [0,1]$.*

Proof. By Proposition 8.13, $\widetilde{\mu}(DM(X)\setminus X) = \mu(\phi^{-1}(DM(X)\setminus X))$. Hence, $\widetilde{\mu}(DM(X)\setminus X) = \mu(\emptyset) = 0$. □

Remark 8.28. *G lets us generate samples of a distribution with probability 1.*

8.5 Defining a probability measure from a cdf on a compact LOTS

Previously, we have studied several aspects related to a cdf which result from considering a probability measure on a separable LOTS. However, we have not answered Question 7.20 yet. In this section we prove that when X is a compact separable LOTS and $F : X \to [0,1]$ is a function satisfying the properties collected in Proposition 7.19, it is possible to define a probability measure, μ, on X such that its cdf, F_μ, is F.

Definition 8.29. *Let X be a compact LOTS and $F : X \to [0,1]$ a monotonically non-decreasing and right τ-continuous function satisfying $\sup F(X) = 1$. Let us define $G : [0,1] \to X$ as the function given by $G(r) = \inf\{x \in X : F(x) \geq r\}$ for each $r \in [0,1]$.*

Lemma 8.30. *Let X be a compact LOTS, $F : X \to [0,1]$ a monotonically non-decreasing and right τ-continuous function satisfying $\sup F(X) = 1$ and G as defined in Definition 8.29. Given $r \in [0,1]$, it holds that $G(r) = \min\{x \in X : F(x) \geq r\}$.*

Proof. Let $r \in [0,1]$ and consider $y = G(r)$. Suppose that y is not a minimum. Then, by Proposition 7.3, there exists a sequence $y_n \in \{z \in X : F(z) \geq r\}$ such that $y_{n+1} < y_n$ for each $n \in \mathbb{N}$ and $\{y_n : n \in \mathbb{N}\}^l = \{z \in X : F(z) \geq r\}^l$. What is more, by Lemma 7.2, it holds that $y = \inf\{y_n : n \in \mathbb{N}\}$. Hence, [28, 4A2R(f)] lets us claim that $y_n \xrightarrow{\tau} y$. Consequently, the right τ-continuity of F gives us that $F(y_n) \to F(y)$. Moreover, $F(y_n) \geq r$, since $y_n \in \{z \in X : F(z) \geq r\}$. If we join this fact with the fact that $F(y_n) \to F(y)$, we conclude that $F(y) \geq r$, which gives us that $y \in \{x \in X : F(x) \geq r\}$, a contradiction. □

Next, we collect some properties that relate F to G and which are similar to the properties obtained when F is the cdf of some probability measure and G is its pseudo-inverse (see the previous chapter).

Proposition 8.31. *Let X be a compact LOTS, $F : X \to [0,1]$ a monotonically non-decreasing and right τ-continuous function satisfying $\sup F(X) = 1$ and G as defined in Definition 8.29. Then it holds that:*

1. *G is monotonically non-decreasing.*
2. *$G(F(x)) \leq x$.*
3. *$F(G(r)) \geq r$.*
4. *$G(r) \leq x$ if and only if $r \leq F(x)$.*
5. *$F(x) < r$ if and only if $G(r) > x$.*
6. *If $\sup F(< x) < r$, then $x \leq G(r)$.*
7. *If $r < \sup F(< x)$, then $G(r) < x$.*

Proof. 1. Let $x, y \in [0,1]$ with $x < y$. Note that $\{z \in X : F(z) \geq y\} \subseteq \{z \in X : F(z) \geq x\}$ and it follows that $\inf\{z \in X : F(z) \geq x\} \leq \inf\{z \in X : F(z) \geq y\}$, that is, $G(x) \leq G(y)$, which means that G is monotonically non-decreasing.

2. Indeed, $x \in \{z \in X : F(z) \geq F(x)\}$ and, hence, $\inf\{z \in X : F(z) \geq F(x)\} \leq x$, which is equivalent to $G(F(x)) \leq x$.

3. Let $y = G(r) = \inf\{z \in X : F(z) \geq r\}$. By Lemma 8.30, we have that $y = \min\{x \in X : F(x) \geq r\}$ so it is immediate that $F(y) \geq r$.

4. It follows immediately from the previous three items and the monotonicity of F.

5. It is an immediate consequence of the previous item.

6. Suppose that $G(r) < x$. Then $G(r) \in (< x)$, which means that $\sup F(< x) \geq F(G(r))$. By the third item, $F(G(r)) \geq r$, so we conclude that $\sup F(< x) \geq r$.

7. Let $x \in X$ and $r \in [0, 1]$. Suppose that $r < \sup F(< x)$. Then there exists $y < x$ such that $r < F(y) \leq \sup F(< x)$. Since $F(y) > r$, then $y \geq \inf\{z \in X : F(z) \geq r\} = G(r)$. We conclude that $G(r) < x$.

□

Proposition 8.32. *Let X be a compact LOTS, $F : X \to [0, 1]$ a monotonically non-decreasing and right τ-continuous function satisfying $\sup F(X) = 1$ and G as defined in Definition 8.29. Then, given $a, b \in X$ such that $a < b$, it holds that $G^{-1}(]a, b[) =]F(a), \sup F(< b)|$, where $|$ means $]$ or $[$.*

Proof. First of all, we show that $G^{-1}(]a, b[) \subseteq]F(a), \sup F(< b)]$. For that purpose, let $r \in G^{-1}(]a, b[)$ and suppose that $r \notin]F(a), \sup F(< b)]$. Then it can happen:

- $r \leq F(a)$, which implies, by Proposition 8.31.4, that $G(r) \leq a$, which is a contradiction with the fact that $r \in G^{-1}(]a, b[)$.

- $r > \sup F(< b)$, which gives us, by Proposition 8.31.6, that $b \leq G(r)$, which implies that $r \notin G^{-1}(]a, b[)$, a contradiction.

Now, we prove that $]F(a), \sup F(< b)[\subseteq G^{-1}(]a, b[)$. For that purpose, let $r \in]F(a), \sup F(< b)[$, and suppose that $r \notin G^{-1}(]a, b[)$. Then it can happen:

- $G(r) \leq a$, which implies, by Proposition 8.31.4, that $r \leq F(a)$, a contradiction with the fact that $r > F(a)$.

- $G(r) \geq b$, which gives us, by Proposition 8.31.7, that $\sup F(< b) \leq r$ a contradiction with the fact that $r < \sup F(< b)$.

□

Corollary 8.33. *Let X be a compact LOTS, $F : X \to [0,1]$ a monotonically non-decreasing and right τ-continuous function satisfying $\sup F(X) = 1$ and G as defined in Definition 8.29. Then G is measurable with respect to the Borel σ-algebras.*

Proof. According to the previous proposition, we can claim that $G^{-1}(]a,b[)$ is an open set or the intersection of an open and a closed one. Hence, $G^{-1}(]a,b[) \in \sigma([0,1])$.

Since every open set in τ is a countable union of open intervals by Proposition 7.51, applying Theorem 7.52 we conclude the result. □

Theorem 8.34. *Let X be a compact LOTS and $F : X \to [0,1]$ a monotonically non-decreasing and right τ-continuous function satisfying $\sup F(X) = 1$. Then there exists a unique probability measure μ on $\sigma(X)$ such that $F = F_\mu$.*

Proof. Since F is a monotonically non-decreasing and right τ-continuous function satisfying $\sup F(X) = 1$, the function $G : [0,1] \to X$ defined by $G(r) = \inf\{x \in X : F(x) \geq r\}$ is measurable with respect to the Borel σ-algebras (see the previous corollary). Since G is measurable, by taking into account Proposition 7.57, if we define $\mu(A) = l(G^{-1}(A))$ for each $A \in \sigma(X)$, then μ is a probability measure on X, where l is the Lebesgue measure.

Claim 8.35. $G^{-1}(\leq x) = [0, F(x)]$ *for each* $x \in X$.

Proof. $\subseteq$) Let $r \in G^{-1}(\leq x)$. Then $G(r) \leq x$. By the monotonicity of F, it holds that $F(G(r)) \leq F(x)$. By Proposition 8.31.3, it follows that $r \leq F(x)$ due to the fact that $F(G(r)) \geq r$. Consequently, $r \in [0, F(x)]$.

$\supseteq$) Let $y \in [0, F(x)]$. Then $y \leq F(x)$ and, by the monotonicity of G (see Proposition 8.31.1), $G(y) \leq G(F(x))$. What is more, $G(y) \leq x$ due to the fact that $G(F(x)) \leq x$ (see Proposition 8.31.2). Consequently, $y \in G^{-1}(\leq x)$. □

Once we have proven the claim, it holds that $\mu(\leq x) = l(G^{-1}(\leq x)) = l([0, F(x)]) = F(x)$ which lets us conclude that F is the cdf of the measure μ. Moreover, the fact that $\sup F(X) = 1$ means that $\mu(X) = 1$, that is, μ is a probability measure on X.

Finally, the uniqueness of the measure follows immediately from Proposition 7.55. $\square$

Theorem 8.36. *Let $F : X \to [0,1]$ a monotonically non-decreasing and right τ-continuous function satisfying $\sup F(X) = 1$. Then the function $\widetilde{F} : DM(X) \to [0,1]$ given by $\widetilde{F}(A) = \inf F(A^u)$ is the cdf of a unique probability measure $\widetilde{\mu}$ on $DM(X)$. Moreover, $\widetilde{F}$ is an extension of F to $DM(X)$.*

Proof. First, we show that F is the restriction of $\widetilde{F}$ to X, that is, $\widetilde{F}(\phi(x)) = F(x)$ for each $x \in X$.

Indeed, let $x \in X$. Then $\widetilde{F}(\phi(x)) = \inf F((\leq x)^u) = \inf F(\geq x) = F(x)$.

Next, we prove the three properties of $\widetilde{F}$ which let us conclude that it is a cdf:

1. $\widetilde{F}$ is monotonically non-decreasing. Let $A, B \in DM(X)$ be such that $A \subset B$. Then $B^u \subseteq A^u$. Hence, $F(B^u) \subseteq F(A^u)$ and, consequently, $\inf F(A^u) \leq \inf F(B^u)$, that is, $\widetilde{F}(A) \leq \widetilde{F}(B)$.

2. $\widetilde{F}$ is right τ'-continuous. Let $A_n \in DM(X)$ be a monotone sequence which right τ'-converges to $A \in DM(X)$. Now, we prove that $\widetilde{F}(A_n) \to \widetilde{F}(A)$, which gives us the right τ'-continuity of $\widetilde{F}$. On the one hand, we can choose $a_n \in A_n \setminus A_{n+1}$ for each $n \in \mathbb{N}$. The next claim is crucial in this proof.

   Claim 8.37. $A = \{a_n : n \in \mathbb{N}\}^l$.

   Proof. $\subseteq$) Let $a \in A$. Then $a < a_n$ for each $n \in \mathbb{N}$ due to the fact that $A_n > A$ and $a_n \notin A_{n+1}$ for each $n \in \mathbb{N}$. Hence, $a \in \{a_n : n \in \mathbb{N}\}^l$.

   $\supseteq$) Let $a \in \{a_n : n \in \mathbb{N}\}^l$ and suppose that $a \notin A$. Then $A < \phi(a)$, which implies that there exists $n \in \mathbb{N}$ such that $A < A_n < \phi(a)$, which means that $A < \phi(a_n) \leq A_n < \phi(a)$, so we conclude that $a_n < a$, a contradiction with the fact that $a \in \{a_n : n \in \mathbb{N}\}^l$. $\square$

   Once we have proven the claim, we distinguish two cases depending on whether $A \cap A^u = \emptyset$ or not in order to show that $F(a_n) \to \inf F(A^u)$:

   (a) Suppose that $A \cap A^u = \{x\}$ for some $x \in X$. Then $A = (\leq x)$ and $A^u = (\geq x)$. In this case, the sequence (a_n) satisfies that $x < a_{n+1} < a_n$ for each $n \in \mathbb{N}$. What is more, $a_n \to x$ and the right τ-continuity of F gives us that $F(a_n) \to F(x) = \inf F(A^u)$.

(b) Suppose that $A \cap A^u = \emptyset$. On the one hand, by the previous claim, it holds that $A = \{a_n : n \in \mathbb{N}\}^l$, which means that $A^u = (\{a_n : n \in \mathbb{N}\}^l)^u$.

On the other hand, suppose that $a_n \notin A^u$ for some $n \in \mathbb{N}$. It holds that $a_n \in A$. By definition of a_n, $a_n \in A_n$. Hence, it is not possible that $a_n \in A$ due to the fact that $A < A_n$ and $a_n \notin A_{n+1}$ for each $n \in \mathbb{N}$. Therefore, $a_n \in A^u$ for each $n \in \mathbb{N}$ and, hence, $\lim F(a_n) \geq \inf F(A^u)$. Suppose now that $\lim F(a_n) > \inf F(A^u)$. Then there exists $a \in A^u$ such that $F(a) < \lim F(a_n) \leq F(a_n)$ for each $n \in \mathbb{N}$. Since $a \in A^u$ and $A^u = (\{a_n : n \in \mathbb{N}\}^l)^u$, there exists $m \in \mathbb{N}$ such that $A_m < \phi(a)$. Indeed, if $\phi(a) \leq A_n$ for each $n \in \mathbb{N}$, then $a \in \{a_n : n \in \mathbb{N}\}^l$, which gives us that $a \in (\{a_n : n \in \mathbb{N}\}^l)^u \cap \{a_n : n \in \mathbb{N}\}^l$. Hence, $\{a_n : n \in \mathbb{N}\}^l = (\leq a)$, which implies that $a_n \to a$ and, thus, $\lim F(a_n) = F(a)$, a contradiction.

From $A_m < \phi(a)$, it follows that $a_m < a$ and, by the monotonocity of F, $F(a_m) \leq F(a)$, which is a contradiction with the fact that $F(a) < F(a_n)$ for each $n \in \mathbb{N}$, as we have proven before. Consequently, $F(a_n) \to \inf F(A^u)$.

Now, since $\widetilde{F}$ is an extension of F, $\widetilde{F}(\phi(a_n)) = F(a_n)$. What is more, $\widetilde{F}(\phi(a_n)) \to \inf F(A^u) = \widetilde{F}(A)$. Moreover, the monotonicity of $\widetilde{F}$ lets us write $\widetilde{F}(A) \leq \widetilde{F}(A_{n+1}) \leq \widetilde{F}(\phi(a_n))$ so, by taking limits, we have that $\widetilde{F}(A_n) \to \widetilde{F}(A)$. By Lemma 7.15, we conclude that $\widetilde{F}$ is right τ'-continuous.

3. Note that $1 = \sup F(X) = \sup \widetilde{F}(\phi(X)) \leq \sup \widetilde{F}(DM(X)) \leq 1$, which lets us conclude that $\sup \widetilde{F}(DM(X)) = 1$.

Consequently, Theorem 8.34 lets us claim that $\widetilde{F}$ is the cdf of a unique probability measure $\widetilde{\mu}$ on $DM(X)$. $\square$

Chapter 9

Equivalence between cdfs and probability measures on a LOTS

The content of this chapter corresponds to [30].

In Chapter 7 we described a theory of a cumulative distribution function on a separable linearly ordered topological space. Moreover, we showed that this function plays a similar role to that played in the classical case and studied its pseudo-inverse, which allowed us to generate samples of the probability measure that we used to define the distribution function.

In Chapter 8 we extended a cdf defined on a separable linearly ordered topological space, X, to its Dedekind-MacNeille completion, $DM(X)$. That completion is, indeed, a compactification. Moreover, we proved that each function satisfying the properties of a cdf on $DM(X)$ is the cdf of a probability measure defined on $DM(X)$. Indeed, if X is compact, a similar result can be obtained in this context. Finally, the compactification $DM(X)$ lets us generate samples of a distribution in X.

By following this research line, the next step is to explore some conditions on X such that, given a function F with the properties of a cdf, we can ensure that there exists a unique probability measure on X such that its cdf is F. Furthermore, we will show that there is a one-to-one relationship between the pseudo-inverse of a cdf and its probability measure. This is the main goal of this chapter. Specifically, in Section 9.1, we claim and prove the main results in this context. The last section consists of some examples where it is interesting to study the relationship between probability measures and cdfs according to the theory that has been developed.

9.1 Defining a probability measure from a cdf

In Chapter 8 it was shown that if F is the cdf of the probability measure μ on X, then F can be extended to a cdf on $DM(X)$, $\widetilde{F}$, that is defined from the probability measure $\widetilde{\mu}$ which is defined by $\widetilde{\mu}(A) = \mu(\phi^{-1}(A))$ for each $A \in \sigma(DM(X))$. What is more, it holds that $\widetilde{F} \circ \phi = F$ and $\widetilde{F}_{-} \circ \phi = F_{-}$ (see Lemma 8.15 and Corollary 8.20). According to the properties that we proved in Section 7.4 about the pseudo-inverse of a cdf and, by taking into account that G is the pseudo-inverse of $\widetilde{F}$, if we extend F to $DM(X)$, we can relate G to F and $\widetilde{F}$ as the next proposition shows.

Proposition 9.1. *Let F be a cdf, $x \in X$ and $r \in [0,1]$. Then:*

1. $G(F(x)) \leq \phi(x)$.
2. $\widetilde{F}(G(r)) \geq r$.
3. $G(r) \leq \phi(x)$ *if and only if* $r \leq F(x)$.
4. $F(x) < r$ *if and only if* $G(r) > \phi(x)$.
5. $G(r) = \inf\{A \in DM(X) : \widetilde{F}(A) \geq r\}$.

In Chapter 7 it was shown the uniqueness of a measure with respect to its cdf (see Proposition 7.55). What is more, a cdf F can be defined from F_{-} as Proposition 7.26 states.

Additionally, we can prove that the value of $F(x)$ can be obtained from the pseudo-inverse, as shown next.

Proposition 9.2. *Let X be a separable LOTS. If F is the cdf of a probability measure on X, then $F(x) = \sup G^{-1}(\leq \phi(x))$ for each $x \in X$.*

Proof. Let $x \in X$. By Proposition 9.1.4, it holds that $G(r) \leq \phi(x)$ if and only if $F(x) \geq r$ for each $r \in [0,1]$. Hence, $\sup G^{-1}(\leq \phi(x)) = \sup\{r \in [0,1] : G(r) \leq \phi(x)\} = \sup\{r \in [0,1] : F(x) \geq r\} = F(x)$. $\square$

Now, we prove the uniqueness of the measure with respect to F_- and the pseudo-inverse.

Corollary 9.3. *Let F_μ and F_δ be respectively the cdfs of the measures μ and δ. If $F_{\mu-} = F_{\delta-}$, then $\mu = \delta$ on the Borel σ-algebra of (X, τ).*

Proof. By Proposition 7.26, $F(x) = \inf F_-(x)$ for each $x \in X$ and, consequently, $F_\mu = F_\delta$. Hence, by Proposition 7.55, $\mu = \delta$ on the Borel σ-algebra of (X, τ). $\square$

Corollary 9.4. *Let F_μ and F_δ be respectively the cdfs of the measures μ and δ. If $G_\mu = G_\delta$, then $\mu = \delta$ on the Borel σ-algebra of (X, τ).*

Proof. By Proposition 9.2, $F(x) = \sup G^{-1}(\leq \phi(x))$ for each $x \in X$ and, consequently, $F_\mu = F_\delta$. Hence, by Proposition 7.55, $\mu = \delta$ on the Borel σ-algebra of (X, τ). $\square$

Moreover, we prove a lemma which will be useful when proving some results.

Lemma 9.5. *Let X be a separable LOTS and let (x_n) be a sequence which τ-converges to x. Suppose that there exists $z \in X$ such that $x_n \leq z$ for each $n \in \mathbb{N}$. Then $x \leq z$.*

Proof. Suppose that $x > z$. The convergence of (x_n) gives us that there exists $n_0 \in \mathbb{N}$ such that $x_n > z$ for each $n \geq n_0$, a contradiction with the fact that $x_n \leq z$ for each $n \in \mathbb{N}$. $\square$

In this section we explore some conditions such that, given a function, F, with the properties of a cdf on a separable LOTS, then there exists a probability measure, μ, on X such that $F_\mu = F$. Indeed, the converse relationship between a probability measure and its cdf is well known. According to Chapter 7, the cdf of a probability measure on a separable LOTS, X, is right τ-continuous and monotonically non-decreasing. Moreover, it satisfies that $\sup F(X) = 1$ and, if there does not exist $\min X$, then $\inf F(X) = 0$ (see Proposition 7.19).

In what follows, when we write a statement like $\sup F_-(A) = \inf F_-(A^u)$ for each $A \in DM(X)$, we mean for each $A \in DM(X)$ such that the expression makes sense. In this case, A must be nonempty (so that $\sup F_-(A)$ makes sense) and A^u must be nonempty (so that $\inf F_-(A^u)$ makes sense). Note that A can be empty if X does not have a minimum and $A = \min DM(X)$ and A^u can be empty if X does not have a maximum and $A = \max DM(X)$.

In fact, Proposition 8.12 gives us that $\sup F(A) = \inf F(A^u)$ for each $A \in DM(X)$, where F is the cdf of a probability measure defined on a separable LOTS X. We now prove some results in this line, but which involve F_- and the pseudo-inverse of the cdf.

Proposition 9.6. *Let X be a separable LOTS and $F : X \to [0,1]$ a cdf defined from a probability measure μ. Then:*

1. $\sup F_-(A) = \inf F_-(A^u)$ *for each* $A \in DM(X)$.
2. $\sup F(A) = \sup F_-(A)$ *for each* $A \in DM(X) \setminus \phi(X)$.
3. $\inf F(A^u) = \inf F_-(A^u)$ *for each* $A \in DM(X) \setminus \phi(X)$.
4. $\sup G^{-1}(< A) = \sup F(A)$ *for each* $A \in DM(X) \setminus \phi(X)$.
5. $\inf F(A^u) = \inf G^{-1}(> A)$ *for each* $A \in DM(X) \setminus \phi(X)$.
6. $\sup G^{-1}(< A) = \inf G^{-1}(> A)$ *for each* $A \in DM(X) \setminus \phi(X)$.

Proof. 1. Let $A \in DM(X)$. In case that $A \in \phi(X)$, the equality is clear. Now, let $A \in DM(X) \setminus \phi(X)$. Then $A \cap A^u = \emptyset$. Since A is decreasing and it does not have a maximum, by Lemma 8.8, there exists an increasing sequence (a_n) in A such that $A = \bigcup(\leq a_n)$. Analogously, since A^u is increasing and it does not have a minimum, we can consider a decreasing sequence in A^u, (b_n), such that $A^u = \bigcup(\geq b_n)$. Moreover, Lemma 8.10 lets us claim that $F_-(a_n) \to \sup F_-(A)$ and $F_-(b_n) \to \inf F_-(A^u)$. Now, note that $[a_n, b_n[$ is a monotonically non-increasing sequence, which implies that $[a_n, b_n[\to \bigcap_{n\in\mathbb{N}}[a_n, b_n[$. Indeed, $\bigcap_{n\in\mathbb{N}}[a_n, b_n[= \emptyset$, which gives us that $\mu([a_n, b_n[) \to \mu(\emptyset) = 0$, that is, $F_-(b_n) - F_-(a_n) \to 0$. Both convergences $F_-(a_n) \to \sup F_-(A)$ and $F_-(b_n) \to \inf F_-(A^u)$ let us conclude that $\inf F_-(A^u) = \sup F_-(A)$.

2. Let $A \in DM(X) \setminus \phi(X)$ with $A \neq \min DM(X)$. By Lemma 8.1, A is not isolated.

   $\geq$) This inequality is clear if we take into account that $F_-(x) \leq F(x)$ for each $x \in X$.

   $\leq$) Since A is not left-isolated and $A \neq \min DM(X)$, we can consider a monotonically non-decreasing sequence (A_n) in $DM(X)$ such that $A_n \xrightarrow{\tau'} A$. Now, let $a_n \in A_{n+1} \setminus A_n$ for each $n \in \mathbb{N}$. Note that $a_n \in A$, since $A_n \subset A$ for each $n \in \mathbb{N}$.

Given $a \in A$, then there exists $n \in \mathbb{N}$ such that $a_n > a$ and, hence, it follows that $F(a) \leq F_-(a_n) \leq \sup F_-(A)$, which lets us conclude that $\sup F(A) \leq \sup F_-(A)$.

3. Let $A \in DM(X) \setminus \phi(X)$ with $A \neq \max DM(X)$. By Lemma 8.1, A is not isolated.

   $\geq$) This inequality is clear if we take into account that $F_-(x) \leq F(x)$ for each $x \in X$.

   $\leq$) Since A is not right-isolated and $A \neq \max DM(X)$, we can consider a monotonically non-increasing sequence (A_n) such that $A_n \xrightarrow{\tau'} A$. Now, let $a_n \in A^u_{n+1} \setminus A^u_n$ for each $n \in \mathbb{N}$. Note that $a_n \in A^u$, since $A \subset A_n$ for each $n \in \mathbb{N}$. Given $a \in A^u$, then there exists $n \in \mathbb{N}$ such that $a_n < a$ and, hence, it follows that $\inf F(A^u) \leq F(a_n) \leq F_-(a)$, which lets us conclude that $\inf F(A^u) \leq \inf F_-(A^u)$.

4. Let $A \in DM(X) \setminus \phi(X)$.

   $\geq$) Let $a \in A$. By Proposition 9.2, $F(a) = \sup G^{-1}(\leq \phi(a)) \leq \sup G^{-1}(< A)$, so we have that $\sup F(A) \leq \sup G^{-1}(< A)$.

   $\leq$) Let $r \in G^{-1}(< A)$. Then $G(r) < A$, so we can consider $a \in A \setminus G(r)$, and hence, $G(r) \leq \phi(a)$. By Proposition 9.2, $F(a) = \sup\{r' \in [0,1] : G(r') \leq \phi(a)\}$. Note that $r \leq F(a)$ and, consequently, $r \leq \sup F(A)$, which lets us conclude that $\sup G^{-1}(< A) \leq \sup F(A)$.

5. Let $A \in DM(X) \setminus \phi(X)$.

   $\leq$) Suppose that $\inf F(A^u) > \inf G^{-1}(> A)$. Then there exists $r \in [0,1]$ such that $r < \inf F(A^u)$ and $G(r) > A$. Since $r < \inf F(A^u)$, $r < F(a)$ for each $a \in A^u$. By Proposition 9.1.4, $G(r) \leq \phi(a)$ for each $a \in A^u$, which means that $G(r) \leq A$, a contradiction with the fact that $G(r) > A$.

   $\geq$) Suppose that $\inf F(A^u) < \inf G^{-1}(> A)$. Now, let $r = \inf G^{-1}(> A)$ and consider a sequence (r_n) such that $\inf F(A^u) < r_n < r$ and $r_n \to r$. By the left-continuity of G, $G(r_n) \to G(r)$. Moreover, the fact that $r_n < r$ implies that $G(r_n) \leq A$. Consequently, $G(r) \leq A$ by Lemma 9.5. What is more, the fact that $\inf F(A^u) < r$ means that there exists $a \in A^u$ such that $F(a) < r$ and, by Proposition 9.1.5, $G(r) > \phi(a) > A$. $G(r) > A$ is a contradiction with the fact that $G(r) \leq A$. Thus, $\inf F(A^u) \geq \inf G^{-1}(> A)$.

6. It immediately follows from Proposition 8.12 and items 4 and 5.

□

Moreover, a cdf always satisfies the next result.

Proposition 9.7. *Let X be a separable LOTS and $F : X \to [0,1]$ be a cdf. It follows that $G(0) = \min DM(X)$. Moreover, if X does not have a maximum, then $G^{-1}(\max DM(X)) \subseteq \{1\}$, and, if X does not have a minimum, then $G^{-1}(\min DM(X)) = \{0\}$.*

Proof. First, we prove that $G(0) = \min DM(X)$.

By Proposition 9.1.5, we have that $G(0) = \inf\{C \in DM(X) : \widetilde{F}(C) \geq 0\}$. Since $\widetilde{F}$ is a cdf, it holds that $\widetilde{F}(C) \geq 0$ for each $C \in DM(X)$. Moreover, the fact that $DM(X)$ is compact means that $\inf\{C \in DM(X) : \widetilde{F}(C) \geq 0\}$ is, indeed, a minimum, which lets us conclude that $G(0) = \min DM(X)$.

Now, suppose that X does not have a minimum. This implies, by Proposition 7.19, that $\inf F(X) = 0$. We prove that $G^{-1}(\min DM(X)) \subseteq \{0\}$.

Suppose that there exists $r \in]0,1]$ such that $G(r) = \min DM(X)$. By Proposition 9.1.5, $G(r) = \inf\{C \in DM(X) : \widetilde{F}(C) \geq r\}$. Thus, given $x \in X$, it holds that $\phi(x) > \min DM(X)$ due to the fact that there does not exist the minimum of X. Hence, $F(x) = \widetilde{F}(\phi(x)) \geq \widetilde{F}(G(r)) \geq r$ (see Proposition 9.1.2). Consequently, $F(x) \geq r$ for each $x \in X$, a contradiction with the fact that $\inf F(X) = 0$.

Finally, consider the case in which X does not have a maximum. We prove that $G^{-1}(\max DM(X)) \subseteq \{1\}$. Suppose that there exists $r \in [0,1[$ such that $G(r) = \max DM(X)$. Note that, given $x \in X$, it holds that $\phi(x) < \max DM(X) = G(r)$ due to the fact that there does not exist the maximum of X. Now, by Proposition 9.1.4, $\phi(x) < G(r)$ implies that $F(x) < r$, a contradiction with the fact that $\sup F(X) = 1$. We conclude that $G^{-1}(\max DM(X)) \subseteq \{1\}$.

□

Example 9.8. *Let $X = \mathbb{Q}_0^+$, that is, the set of non-negative rationals, and $F : X \to [0,1]$ the function given by $F(x) = 1 - e^{-x}$ for each $x \in X$. Consider $\leq$ as the usual order on X and suppose that there exists a probability measure, μ, on X such that $F_\mu = F$. Hence, $1 = \mu(X) = \mu(\bigcup_{x \in X}\{x\}) \leq \sum_{x \in X} \mu(\{x\}) = 0$, a contradiction, which means* *that there does not exist any probability measure such that its cdf is F. Note that, in this case, $DM(X) \setminus \phi(X)$ is not countable.*

The last example suggests considering the countability of $DM(X) \setminus \phi(X)$ in order to be able to get a probability measure on X such that its cdf is a function satisfying the properties in Proposition 7.19.

The main result of this chapter is the following one.

Theorem 9.9. *Let X be a separable LOTS such that $DM(X) \setminus \phi(X)$ is countable and $F : X \to [0, 1]$ a monotonically non-decreasing and right τ-continuous function satisfying $\sup F(X) = 1$ and $\sup F(A) = \inf F(A^u)$ for each $A \in DM(X)$. Moreover, $\inf F(X) = 0$ if there does not exist the minimum of X. Then there exists a unique probability measure on X, μ, such that $F = F_\mu$.*

Proof. By Theorem 8.36, the function $\widetilde{F} : DM(X) \to [0, 1]$ given by $\widetilde{F}(A) = \inf F(A^u)$ for each $A \in DM(X)$ is an extension of F, which means that $\widetilde{F}(\leq x) = F(x)$ for each $x \in X$ and $\widetilde{F}$ is the cdf of a probability measure, $\widetilde{\mu}$, on $DM(X)$. Now, define the measure μ by $\mu(A) = \widetilde{\mu}(\phi(A))$ for each $A \subseteq X$. We show that $\phi(X)$ is measurable with respect to the Borel σ-algebra of $DM(X)$. Indeed, note that given $A \in DM(X)$, $\{A\}$ is closed with respect to the order topology of $DM(X)$, which means that $\{A\} \in \sigma(DM(X))$. Hence, the fact that $DM(X) \setminus \phi(X)$ is countable lets us claim that $DM(X) \setminus \phi(X)$ is the countable union of elements in $\sigma(DM(X))$, which implies that $DM(X) \setminus \phi(X) \in \sigma(DM(X))$. Hence, its complement belongs to the Borel σ-algebra of $DM(X)$, that is $\phi(X) \in \sigma(DM(X))$, which lets us conclude that $\phi(X)$ is measurable. Hence, Proposition 2.18 lets us claim that $\widetilde{\mu}$ is a measure on $\sigma(\phi(X))$. Now, considering the map $\phi^{-1} : \phi(X) \to X$, Definition 2.20 gives us that μ is a measure with respect to $\sigma(X)$.

Now, we prove a claim which is crucial to show that μ is a probability measure on X.

Claim 9.10. *Let $A \in DM(X) \setminus \phi(X)$ be such that $A \neq \min DM(X)$. Then $\widetilde{F}_-(A) = \sup F(A)$.*

Proof. Let $A \in DM(X) \setminus \phi(X)$. Then Lemma 8.1 lets us claim that A is not left-isolated. Now, by Proposition 7.11, there exists a sequence (A_n) in $DM(X)$ such that $A_n \xrightarrow{\tau'} A$ and $A_n < A_{n+1} < A$ for each $n \in \mathbb{N}$. Now, let $a_n \in A_n \setminus A_{n-1}$ for each $n \geq 2$.

$\leq$) Let $B \in DM(X)$ be such that $B < A$. Then, by definition of a_n, there exists $n \in \mathbb{N}$ such that $B < \phi(a_n) < A$. What is more, $\sup F(A) \geq F(a_n) = \widetilde{F}(\phi(a_n)) \geq \widetilde{F}(B)$, where we have used the monotonicity of $\widetilde{F}$ as a cdf. Hence, $\sup F(A) \geq \sup_{B<A} \widetilde{F}(B) = \widetilde{F}_-(A)$ for each $A \in DM(X) \setminus \phi(X)$.

$\geq$) Let $a \in A$. Then $\phi(a) < A$. Moreover, $F(a) = \widetilde{F}(\phi(a)) \leq \sup_{B<A} \widetilde{F}(B) = \widetilde{F}_-(A)$. Therefore, $\widetilde{F}_-(A) \geq F(a)$ for each $a \in A$, which means that $\widetilde{F}_-(A) \geq \sup F(A)$. $\square$

Finally we prove that $\mu(X) = 1$. Note that we can write $DM(X) = \phi(X) \cup (DM(X) \setminus \phi(X))$, which implies that $\widetilde{\mu}(DM(X)) = \widetilde{\mu}(\phi(X) \cup (DM(X) \setminus \phi(X)))$. Now, the σ-additivity of $\widetilde{\mu}$ gives us that $\widetilde{\mu}(DM(X)) = \widetilde{\mu}(\phi(X) \cup (DM(X) \setminus \phi(X))) = \widetilde{\mu}(\phi(X)) + \widetilde{\mu}(DM(X) \setminus \phi(X))$. Since $\widetilde{\mu}$ is a probability measure on $DM(X)$, we have that $\widetilde{\mu}(DM(X)) = 1$. The fact that $DM(X) \setminus \phi(X)$ is countable implies that $\widetilde{\mu}(DM(X) \setminus \phi(X)) = 0$. Indeed, to prove that, we first show the next claim.

Claim 9.11. *$\widetilde{\mu}(\{A\}) = 0$ for each $A \in DM(X) \setminus \phi(X)$.*

Proof. Let $A \in DM(X) \setminus \phi(X)$. Then $\widetilde{\mu}(\{A\}) = \widetilde{F}(A) - \widetilde{F}_-(A)$. Now, we distinguish two cases depending on whether $A = \min DM(X)$ or not:

1. Suppose that $A = \min DM(X)$, in which case $A^u = X$. Note that there does not exist the minimum of X. Indeed, if there exists $\min X$, then $\phi(\min X) = A$, which contradicts the fact that $A \in DM(X) \setminus \phi(X)$. Hence, the definition of $\widetilde{F}$ and the initial assumption that $\inf F(X) = 0$ let us claim that $\widetilde{F}(A) = \inf F(A^u) = \inf F(X) = 0$. On the other hand, since $\widetilde{F}$ is a cdf on $DM(X)$, it holds, by Proposition 7.31, that $\inf \widetilde{F}_-(DM(X)) = 0$ and, consequently, $\widetilde{F}_-(A) = 0$. Therefore, $\widetilde{\mu}(\{A\}) = 0$.

2. If $A \neq \min DM(X)$, by taking into account the definition of $\widetilde{F}$ and Claim 9.10, it follows that $\widetilde{\mu}(\{A\}) = \inf F(A^u) - \sup F(A)$. Finally, $\inf F(A^u) - \sup F(A) = 0$ by the initial assumption in the theorem, which lets us conclude that $\widetilde{\mu}(\{A\}) = 0$.

$\square$

Hence, by the previous claim and the σ-additivity of $\widetilde{\mu}$ as a measure, we can write $\widetilde{\mu}(DM(X) \setminus \phi(X)) = \widetilde{\mu}(\bigcup_{A \in DM(X)\setminus\phi(X)} \{A\}) = \sum_{A \in DM(X)\setminus\phi(X)} \widetilde{\mu}(\{A\}) = 0$. Consequently, $1 = \widetilde{\mu}(DM(X)) = \widetilde{\mu}(\phi(X)) + \widetilde{\mu}(DM(X) \setminus \phi(X)) = \widetilde{\mu}(\phi(X)) = \mu(X)$.

The uniqueness of the measure immediately follows from Proposition 7.55. □

To end this section, we introduce some results whose main goal is to define a probability measure from a function F_- satisfying the properties that we collect in Proposition 7.31 and from a function G satisfying the properties of the pseudo-inverse of a cdf.

Corollary 9.12. *Let X be a separable LOTS such that $DM(X) \setminus \phi(X)$ is countable and let $F_- : X \to [0,1]$ be a monotonically non-decreasing, left τ-continuous function such that $\inf F_-(X) = 0$ and $\sup F_-(A) = \inf F_-(A^u)$ for each $A \in DM(X)$. Moreover, $\sup F_-(X) = 1$ if there does not exist the maximum of X. Then there exists a unique probability measure on X, μ, such that $F_{\mu-} = F_-$.*

Proof. Let us define $F : X \to [0,1]$ by $F(x) = \inf F_-(> x)$ if $(> x) \neq \emptyset$ and $F(x) = 1$ if $x = \max X$.

First of all, we prove the next claims which are crucial in the rest of the proof:

Claim 9.13. *$F_-(x) \leq F(x)$ for each $x \in X$.*

Proof. It immediately follows from the definition of F and the monotonicity of F_-. □

Claim 9.14. *Let $a, b \in X$ be such that $a < b$. Then $F(a) \leq F_-(b)$.*

Proof. Let $a, b \in X$ be such that $a < b$. Then $\inf F_-(> a) \leq F_-(b)$, that is, $F(a) \leq F_-(b)$. □

Claim 9.15. *Let (x_n) be a monotone sequence which right τ-converges to x. Then $F_-(x_n) \to F(x)$.*

Proof. Let (x_n) be a monotone sequence which right τ-converges to x. By Claim 9.14, it holds that $F(x) \leq F_-(x_n)$, since $x < x_n$. Note that $(F_-(x_n))$ is a monotonically non-increasing sequence with a lower bound, $F(x)$. Hence, $F_-(x_n) \to r'$ for some $r' \geq F(x)$. Note that $F_-(x_n) \geq r'$ for each $n \in \mathbb{N}$. Now, suppose that $F(x) < r'$. Then, by definition of F, there exists $y > x$ such that $F_-(y) < r'$. Since $x_n \to x$, there exists $m \in \mathbb{N}$ such that $x < x_m < y$ and, hence, $F_-(x_m) \leq F_-(y) < r'$, which contradicts the fact that $F_-(x_n) \geq r'$ for each $n \in \mathbb{N}$. Consequently, $r' = F(x)$. □

Secondly, we show that F is a cdf. Indeed,

1. The fact that F_- is monotonically non-decreasing gives us that F satisfies that property too.

2. F is right τ-continuous. Let (x_n) be a monotone sequence which right τ-converges to x. Then, by Claim 9.14, we have that $F(x) \leq F_-(x_{n+1})$ and $F(x_{n+1}) \leq F_-(x_n)$. Moreover, Claim 9.13 gives us that $F_-(x_{n+1}) \leq F(x_{n+1})$. Hence, if we join all the previous inequalities, it follows that $F(x) \leq F_-(x_{n+1}) \leq F(x_{n+1}) \leq F_-(x_n)$. Finally, by taking limits and using the fact that $F_-(x_n) \to F(x)$ (see Claim 9.15), we conclude that $F(x_n) \to F(x)$, that is, F is right τ-continuous.

3. $\sup F(X) = 1$. We distinguish two cases depending on whether there exists the maximum of X or not:

   (a) Suppose that there does not exist $\max X$. Then, by Claim 9.13, it holds that $F_-(x) \leq F(x)$ for each $x \in X$, which gives us that $\sup F_-(X) \leq \sup F(X)$. By taking into account that $\sup F_-(X) = 1$, we conclude that $\sup F(X) = 1$.

   (b) If there exists $\max X$, then, by definition of F, we have that $F(\max X) = 1$ and, consequently, $\sup F(X) = 1$.

4. $\inf F(X) = 0$ if there does not exist the minimum of X. Since X is increasing and it does not have a minimum, by Lemma 8.8, we can consider a decreasing sequence (a_n) in X such that $X = \bigcup(\geq a_n)$. Moreover, the fact that F is monotonically non-decreasing lets us claim, by Lemma 8.10, that $F(a_n) \to \inf F(X)$. What is more, the monotonicity of F_- implies that $F_-(a_n) \to \inf F_-(X) = 0$. By Claim 9.13, we have that $\inf F_-(X) \leq \inf F(X)$ and, by Claim 9.14, it holds that $F(a_{n+1}) \leq F_-(a_n)$. Therefore, the next inequality follows: $0 \leq \inf F(X) \leq F(a_{n+1}) \leq F_-(a_n)$. By taking limits, we conclude that $\inf F(X) = 0$.

Note that it is obvious that $\sup F(A) = \inf F(A^u)$ for each $A \in \phi(X)$. Now, we prove a claim that will be crucial to get the equality $\sup F(A) = \inf F(A^u)$ for each $A \in DM(X) \setminus \phi(X)$.

Claim 9.16. *Let $A \in DM(X) \setminus \phi(X)$. Then $\sup F(A) = \sup F_-(A)$ and $\inf F(A^u) = \inf F_-(A^u)$.*

Proof. Let $A \in DM(X) \setminus \phi(X)$. Then A is not isolated by Lemma 8.1. First, we prove that $\sup F(A) = \sup F_-(A)$ for each $n \in \mathbb{N}$

$\geq$) It is clear if we take into account Claim 9.13, which gives us that $F_{-}(x) \leq F(x)$ for each $x \in X$.

$\leq$) Since A is not left-isolated, we can consider a monotonically non-decreasing sequence (A_n) such that $A_n \xrightarrow{\tau'} A$. Now, let $a_n \in A_{n+1} \setminus A_n$ for each $n \in \mathbb{N}$. Note that $a_n \in A$, since $A_n \subseteq A$ for each $n \in \mathbb{N}$. Given $a \in A$, then there exists $n \in \mathbb{N}$ such that $a_n > a$ and, hence, by Claim 9.14, it follows that $F(a) \leq F_{-}(a_n) \leq \sup F_{-}(A)$, which lets us conclude that $\sup F(A) \leq \sup F_{-}(A)$.

Now, we prove the equality $\inf F(A^u) = \inf F_{-}(A^u)$.

$\geq$) It immediately follows from Claim 9.13 due to the fact that $F_{-}(x) \leq F(x)$ for each $x \in X$.

$\leq$) Since A is not right-isolated, we can consider a monotonically non-increasing sequence (A_n) such that $A_n \xrightarrow{\tau'} A$. Now, let $a_n \in A^u_{n+1} \setminus A^u_n$ for each $n \in \mathbb{N}$. Note that $a_n \in A^u$, since $A \subset A_n$ for each $n \in \mathbb{N}$. Given $a \in A^u$, then there exists $n \in \mathbb{N}$ such that $a_n < a$ and, hence by Claim 9.14, it follows that $\inf F(A^u) \leq F(a_n) \leq F_{-}(a)$, which lets us conclude that $\inf F(A^u) \leq \inf F_{-}(A^u)$. $\square$

The previous claim gives us that $\sup F(A) = \sup F_{-}(A)$ and $\inf F(A^u) = \inf F_{-}(A^u)$ for each $A \in DM(X)\backslash\phi(X)$, which means that the condition $\sup F_{-}(A) = \inf F_{-}(A^u)$ for each $A \in DM(X)\backslash\phi(X)$ implies that $\inf F(A^u) = \sup F(A)$ for each $A \in DM(X)\backslash\phi(X)$. Hence, Theorem 9.9 lets us conclude that F is the cdf of a probability measure, μ, defined on X.

Finally we show that $F_{\mu-} = F_{-}$. For that purpose, given $x \in X$, we distinguish two cases depending on whether x is left-isolated or not:

1. Suppose that x is not left-isolated. Then, by Proposition 7.11, there exists a monotone sequence which left τ-converges to x. Let (x_n) be that sequence. On the one hand, since $F = F_\mu$ is a cdf, we have that $F_\mu(x_n) \to F_{\mu-}(x)$ (see Proposition 7.30). Moreover, Claim 9.13 gives us that $F_{-}(x_n) \leq F(x_n)$ and, by Claim 9.14, $F(x_n) \leq F_{-}(x_{n+1})$. Hence, if we join the previous inequalities, we have that $F_\mu(x_n) = F(x_n) \leq F_{-}(x_{n+1}) \leq F(x_{n+1}) = F_\mu(x_{n+1})$. Now, by taking limits in the previous expression, since $F_\mu(x_n) \to F_{\mu-}(x)$, we have that $F_{-}(x_{n+1}) \to F_{\mu-}(x)$.

   On the other hand, the left τ-continuity of F_{-} means that $F_{-}(x_n) \to F_{-}(x)$.

The facts that $F_-(x_n) \to F_{\mu-}(x)$ and $F_-(x_n) \to F_-(x)$ let us conclude that $F_-(x) = F_{\mu-}(x)$.

2. Suppose that x is left-isolated. Then it can happen:

   (a) There exists $z \in X$ such that $]z, x[= \emptyset$. Note that the fact that F_μ is the cdf defined from μ gives us that $F_{\mu-}(x) = \mu(< x) = \mu(\leq z) = F_\mu(z)$. Now, Theorem 9.9 lets us claim that $F_\mu(z) = F(z)$. By definition of F, it holds that $F(z) = \inf F_-(> z) = \inf F_-(\geq x) = F_-(x)$, which finishes the proof.

   (b) If $(< x) = \emptyset$, then $x = \min X$ and, consequently, $F_{\mu-}(x) = 0 = F_-(x)$ by hypothesis.

Hence, $F_{\mu-} = F_-$. The uniqueness of the measure immediately follows from Corollary 9.3.

□

Corollary 9.17. *Let X be a separable LOTS such that $DM(X) \setminus \phi(X)$ is countable and let $G : [0,1] \to DM(X)$ be a monotonically non-decreasing and left τ-continuous function such that $\sup G^{-1}(< A) = \inf G^{-1}(> A)$ for each $A \in DM(X) \setminus \phi(X)$, $G(0) = \min DM(X)$, $G^{-1}(\max DM(X)) \subseteq \{1\}$ if there does not exist the maximum of X and $G^{-1}(\min DM(X)) = \{0\}$ if there does not exist the minimum of X. Then there exists a unique probability measure on X, μ, such that G is the pseudo-inverse of F_μ.*

Proof. First of all, we use the fact that $DM(X)$ is separable as a consequence of the separability of X (see Corollary 8.4).

Let us define $F : X \to [0,1]$ by $F(x) = \sup\{r \in [0,1] : G(r) \leq \phi(x)\} = \sup G^{-1}(\leq \phi(x))$.

Note that $0 \in G^{-1}(\leq \phi(x))$ for each $x \in X$, since $G(0) = \min DM(X)$ and, hence, F is well defined.

Now, we prove a claim which will be crucial to show the right continuity of F.

Claim 9.18. *Let $x \in X$ and $r \in [0,1]$. Then $F(x) < r$ if and only if $G(r) > \phi(x)$.*

Proof. First, note that if $r = 0$, the statement is trivial, so we can suppose that $r > 0$.

$\Rightarrow$) Suppose that $F(x) < r$. Then $\sup\{r' \in [0,1] : G(r') \leq \phi(x)\} < r$, which means that $r \notin \{r' \in [0,1] : G(r') \leq \phi(x)\}$, which implies that $G(r) > \phi(x)$.

$\Leftarrow$) Suppose now that $G(r) > \phi(x)$. We distinguish two cases:

1. Suppose that $F(x) > r$. Then $\sup\{r' \in [0,1] : G(r') \leq \phi(x)\} > r$, which means that there exists $r' \in [0,1]$ with $r' > r$ such that $G(r') \leq \phi(x)$. Hence, the monotonicity of G gives us that $G(r) \leq G(r') \leq \phi(x)$. Thus, $G(r) \leq \phi(x)$, a contradiction with the initial assumption.

2. Suppose now that $F(x) = r$ and let (r_n) be a left convergent sequence to r with $r_n \in [0, r[$ for each $n \in \mathbb{N}$. Then $\sup\{r' \in [0,1] : G(r') \leq \phi(x)\} > r_n$ for each $n \in \mathbb{N}$. Hence, given $n \in \mathbb{N}$, there exists $r' \in [0,1]$ with $r' > r_n$ and such that $G(r') \leq \phi(x)$. Hence, the monotonicity of G gives us that $G(r_n) \leq G(r') \leq \phi(x)$. Thus, $G(r_n) \leq \phi(x)$ for each $n \in \mathbb{N}$. Since G is left τ-continuous by hypothesis, by taking limits and using Lemma 9.5, we conclude that $G(r) \leq \phi(x)$, which is a contradiction with the initial assumption.

□

Secondly, we show that F is a cdf. For this purpose, we start proving its properties as cdf:

1. F is monotonically non-decreasing. Indeed, it immediately follows from the monotonicity of G and the definition of F.

2. F is right τ-continuous. Let (x_n) be a monotone sequence which right τ-converges to x. Note that $F(x) \leq F(x_n)$ for each $n \in \mathbb{N}$ and $F(x_{n+1}) \leq F(x_n)$, that is, $(F(x_n))$ is a monotonically non-increasing sequence with a lower bound, which means that $F(x_n) \to r'$ for some $r' \geq F(x)$. Suppose that $r' > F(x)$. Then there exists $r \in [0,1]$ such that $F(x) < r < r'$. The previous claim gives us that $\phi(x) < G(r)$ and $G(r) \leq G(r')$, since G is monotonically non-decreasing. Since (x_n) is a monotone sequence which right τ-converges to x, there exists $n \in \mathbb{N}$ such that $\phi(x_n) < G(r)$. By the previous claim, this fact implies that $F(x_n) < r$, which contradicts the fact that $F(x_n) \geq r$ for each $n \in \mathbb{N}$. Hence, $F(x) = r'$.

3. $\sup F(X) = 1$. Note that if there exists the maximum of X, then $F(\max X) = 1$ by definition of F. Suppose that there does not exist the maximum of X and that $\sup F(X) \neq 1$. Then we can consider $r \in [0,1[$ such that $r > \sup F(X)$. Now, we claim that $G(r) = \max DM(X)$. Indeed, suppose that $G(r) \neq \max DM(X)$. Then we can choose $x \in X$ such that $\phi(x) > G(r)$ and, hence, by Claim 9.18, we have that $F(x) \geq r$, which contradicts the fact that $r > \sup F(X)$. Consequently, $G(r) = \max DM(X)$, which is a contradiction with the initial assumption $G^{-1}(\max DM(X)) \subseteq \{1\}$.

4. $\inf F(X) = 0$ if there does not exist the minimum of X. Suppose that $\inf F(X) \neq 0$. Then we can consider $r \in]0,1]$ such that $r < \inf F(X)$. Now, we claim that $G(r) = \min DM(X)$. Indeed, suppose that $G(r) \neq \min DM(X)$. Then we can choose $x \in X$ such that $\phi(x) < G(r)$ and, hence, by Claim 9.18, we have that $F(x) < r$, a contradiction with the fact that $r < \inf F(X)$. Consequently, $G(r) = \min DM(X)$, which is a contradiction with the initial assumption $G^{-1}(\min DM(X)) = \{0\}$.

Now, we prove a claim that will be crucial to get the equality $\sup F(A) = \inf F(A^u)$ for each $A \in DM(X) \setminus \phi(X)$.

Claim 9.19. *Let $A \in DM(X) \setminus \phi(X)$. Then $\sup F(A) = \sup G^{-1}(< A)$ and $\inf F(A^u) = \inf G^{-1}(> A)$.*

Proof. Let $A \in DM(X) \setminus \phi(X)$. First, we prove that $\sup F(A) = \sup G^{-1}(< A)$.

$\leq$) Let $a \in A$. Then $F(a) = \sup G^{-1}(\leq \phi(a)) \leq \sup G^{-1}(< A)$, so we have that $\sup F(A) \leq \sup G^{-1}(< A)$.

$\geq$) Let $r \in G^{-1}(< A)$. Then $G(r) < A$, so we can consider $a \in A \setminus G(r)$, that is, $G(r) \leq \phi(a)$. Now, according to the definition of F from G, $F(a) = \sup\{r' \in [0,1] : G(r') \leq \phi(a)\}$. Note that $r \leq F(a)$. What is more, $r \leq F(a) \leq \sup F(A)$, which implies that $\sup G^{-1}(< A) \leq \sup F(A)$.

Now, we prove the equality $\inf F(A^u) = \inf G^{-1}(> A)$. Let $A \in DM(X) \setminus \phi(X)$.

$\leq$) Suppose that $\inf F(A^u) > \inf G^{-1}(> A)$. Then there exists $r \in [0,1]$ such that $r < \inf F(A^u)$ and $G(r) > A$. Since $r < \inf F(A^u)$, $r < F(a)$ for each $a \in A^u$. By Claim

9.18, $G(r) \leq \phi(a)$ for each $a \in A^u$, which means that $G(r) \leq A$, a contradiction with the fact that $G(r) > A$.

$\geq$) Suppose that $\inf F(A^u) < \inf G^{-1}(> A)$. Then there exists $b \in A^u$ such that $F(b) < \inf G^{-1}(> A)$. Since $\inf G^{-1}(> A) = \sup G^{-1}(< A)$ by hypothesis, we have that $F(b) < \sup G^{-1}(< A)$. Hence, there exists $r \in G^{-1}(< A)$ such that $F(b) < r$. Equivalently, there exists $r \in [0,1]$ with $G(r) < A$ such that $F(b) < r$. Now, Claim 9.18 gives us that $G(r) > \phi(b)$. The fact that $(< A)$ is decreasing together with the facts that $G(r) < A$ and $G(r) > \phi(b)$ let us conclude that $b \in A$, a contradiction.

□

By the previous claim, the condition $\sup G^{-1}(< A) = \inf G^{-1}(> A)$ for each $A \in DM(X) \setminus \phi(X)$ implies that $\sup F(A) = \inf F(A^u)$ for each $A \in DM(X) \setminus \phi(X)$, so Theorem 9.9 lets us conclude that F is the cdf of a probability measure, μ, defined on X.

Now, we prove another claim that will help us in showing the equality $G_\mu = G$. For that purpose, and by taking into account that F is a cdf, we will use its extension to $DM(X)$, $\widetilde{F}$.

Claim 9.20. *$\widetilde{F}(G(r)) \geq r$ for each $r \in [0,1]$.*

Proof. Let $r \in [0,1]$ and suppose that $\widetilde{F}(G(r)) < r$. Then $\inf_{x \in G(r)^u} F(x) < r$. Hence, there exists $x \in G(r)^u$ such that $F(x) < r$. Now, by Claim 9.18, it follows that $G(r) > \phi(x)$, which means that $x \notin G(r)^u$, a contradiction. Consequently, $\widetilde{F}(G(r)) \geq r$. □

Finally, we show that $G_\mu = G$.

$\geq$) Let $r \in [0,1]$ and $A \in DM(X)$ be such that $\widetilde{F}(A) \geq r$. Now, let $x \in A^u$. Then $\widetilde{F}(\phi(x)) \geq \widetilde{F}(A) \geq r$. The fact that $\widetilde{F}$ is an extension of F gives us that $\widetilde{F}(\phi(x)) = F(x)$. Since $F(x) \geq r$, by Claim 9.18, we have that $G(r) \leq \phi(x)$. By the arbitrariness of x, we conclude that $G(r) \leq A$ and, consequently, $\inf\{A \in DM(X) : \widetilde{F}(A) \geq r\} \geq G(r)$, that is, $G_\mu(r) \geq G(r)$.

$\leq$) By Claim 9.20, $r \leq \widetilde{F}(G(r))$ for each $r \in [0,1]$. Now, by taking into account that G_μ is the pseudo-inverse of $\widetilde{F}$ as a cdf, its monotonicity gives us that $G_\mu(r) \leq G_\mu(\widetilde{F}(G(r)))$. Finally, by taking into account Proposition 9.1.1, it follows that $G_\mu(\widetilde{F}(G(r))) \leq G(r)$, so we can conclude that $G_\mu(r) \leq G(r)$.

The uniqueness of the measure immediately follows from Corollary 9.4. □

Once we have proven that a measure can be determined from F_- and G when given some conditions on them and $DM(X)$, we get two immediate results.

Corollary 9.21. *Let X be a compact separable LOTS and $F_- : X \to [0,1]$ a monotonically non-decreasing, left τ-continuous function such that $\inf F_-(X) = 0$. Then there exists a unique probability measure μ on the Borel σ-algebra of X such that $F_{\mu -} = F_-$.*

Proof. Note that the fact that X is compact means that $DM(X) = \phi(X)$, which implies that $DM(X) \setminus \phi(X) = \emptyset$. Since given $A \in DM(X)$, there exists $a \in X$ such that $A = \phi(a)$, it is clear that $\sup F_-(A) = \inf F_-(A^u)$. Hence, by taking into account the hypothesis on F_- and Corollary 9.12, we conclude that there exists a probability measure μ on the Borel σ-algebra of X such that $F_{\mu -} = F_-$. Moreover, Corollary 9.3 ensures that μ is unique. □

Corollary 9.22. *Let X be a compact separable LOTS and $G : X \to [0,1]$ a monotonically non-decreasing and left τ-continuous function satisfying $G(0) = \min X$ and $G(1) = \max X$. Then there exists a unique probability measure μ on the Borel σ-algebra of X such that G is the pseudo-inverse of F_μ.*

Proof. Since X is compact, $DM(X) = \phi(X)$, which implies that $DM(X) \setminus \phi(X) = \emptyset$. Now, by taking into account the hypothesis on G and Corollary 9.17, we conclude that there exists a probability measure μ on the Borel σ-algebra of X such that G is the pseudo-inverse of F_μ. Moreover, Corollary 9.4 ensures that μ is unique. □

9.2 Examples

Next, we show some examples in which it is possible to define a probability measure on X from a function satisfying the properties of a cdf by taking into account the theory that has been developed in Part II previously.

Example 9.23. *Let $X = (\{0\} \cup \mathbb{N}) \times [0,1]$ and $\leq$ be the lexicographic order on X. Consider the function $F : X \to [0,1]$ given by $F(x,y) = 1 - \frac{1}{2}e^{-(x+y)} - \frac{1}{2}e^{-x}$ for each $(x,y) \in X$.*

Note that, in this case, $DM(X)\setminus\phi(X)=\{X\}$. Roughly speaking, $DM(X)$ coincides with the one-point compactification of X, since the only cut that we add when we consider $DM(X)$ is X.

We have already seen that $DM(X)\setminus\phi(X)$ is countable. Moreover, by definition of F, it holds that F is a monotonically non-decreasing and right τ-continuous function (indeed, F is continuous) satisfying $\sup F(X)=1$ and $\sup F(A)=\inf F(A^u)$ for each $A\in DM(X)$. Finally, Theorem 9.9 lets us conclude that there exists a unique probability measure μ on X such that F is its cdf.

Example 9.24. *Let $X=(\{0\}\cup\mathbb{N})\times[0,1]$ and $\leq$ be the lexicographic order on X. Consider the function $F_-:X\to[0,1]$ given by $F_-(x,y)=1-\frac{1}{2}e^{-(x+y)}-\frac{1}{2}e^{-x}$ for each $(x,y)\in X\setminus\{(x,0):x\in\mathbb{N}\}$ and $F_-(x,0)=1-\frac{e+1}{2}e^{-x}$ for each $x\in\mathbb{N}$.*

We have already seen that $DM(X)\setminus\phi(X)$ is countable. Note that F_- is continuous in $X\setminus\{(x,0):x\in\mathbb{N}\}$ so it is left τ-continuous. Moreover, given $(x,0)$ for some $x\in\mathbb{N}$, it holds that F_- is left τ-continuous at $(x,0)$, since this point is left-isolated.

On the other hand, by definition of F_-, it holds that F_- is monotonically non-decreasing and it satisfies $\sup F_-(X)=1$ and $\sup F_-(A)=\inf F_-(A^u)$ for each $A\in DM(X)$. Finally, Corollary 9.12 lets us conclude that there exists a unique probability measure, μ, on X such that $F_{\mu-}=F_-$.

Example 9.25. *Let $X=(\{0\}\cup\mathbb{N})\times[0,1]$ and $\leq$ be the lexicographic order on X. Consider the function $G:DM(X)\to[0,1]$ given by $G(r)=(\leq\min\{(x,y)\in X: x+y\geq\ln(1-r)^{-1}\})$ for each $r\in[0,1[$ and $G(1)=X$. Note that G satisfies the conditions of Corollary 9.17, which means that there exists a probability measure μ on X such that G is the pseudo-inverse of F_μ.*

Indeed, by taking into account Proposition 9.2, we can define F_μ by $F_\mu(x,y)=1-e^{-(x+y)}$ for each $(x,y)\in X$.

The next example shows a function that is not a cdf.

Example 9.26. *Let $X=(\{0\}\cup\mathbb{N})\times]0,1[$ and $\leq$ be the lexicographic order on X. Consider the function $F:X\to[0,1]$ given by $F(x,y)=1-\frac{1}{2^x}$ for each $(x,y)\geq(1,0)$ and $F(x,y)=0$ otherwise.*

Note that $DM(X)\setminus\phi(X)$ is countable, F is right τ-continuous, monotonically non-decreasing, $\inf F(X)=0$ and $\sup F(X)=1$.

However, if we consider the cut $A = (< (1,1))$*, the condition* $\sup F(A) = \inf F(A^u)$ *does not hold. In this case,* $A^u = (> (2,0))$*. Note that* $\sup F(A) = \frac{1}{2}$ *and* $\inf F(A^u) = \frac{3}{4}$*. Hence,* $\sup F(A) \neq \inf F(A^u)$ *and, by Proposition 8.12,* F *is not the cdf of a probability measure on* X*.*

To end this section, we introduce a simple real example where our theory is essential to get the probability distribution.

Example 9.27. *Consider three cdfs that are the lifetime of three different light bulbs. The distributions are exponential with means* $800, 1000$ *and* 1200 *hours. Consider a system with three light bulbs one of each type. Find the probability that, at least, one of the light bulbs of this type has a lifetime of more than* 900 *hours.*

In the classical case we can define the random variables $X_1 \sim \varepsilon(\frac{1}{800}), X_2 \sim \varepsilon(\frac{1}{1000})$ *and* $X_3 \sim \varepsilon(\frac{1}{1200})$*. Note that the corresponding cdfs,* F_1, F_2 *and* F_3 *are a particular case of a cdf according to the developed theory. Furthermore, the idea of modelling the case in which three light bulbs work together is considering the set* $X = [0, \infty[\times \{0,1,2\}$ *and* $\leq$ *as the lexicographic order on* X*. It holds that* X *is a separable LOTS and that* $DM(X) \setminus \phi(X) = \{X\}$*. The function* $F : X \to [0,1]$ *defined by* $F(x,y) = \frac{1}{3}(F_1(x) + F_2(x) + F_3(x))$ *is monotonically non-decreasing, right* τ*-continuous and* $\sup F(A) = \inf F(A^u)$ *for each* $A \in DM(X)$*. Since* $DM(X) \setminus \phi(X)$ *is countable, Theorem 9.9 lets us ensure that there exists a probability measure on* X *such that its cdf is* F*. Hence, it is possible for us to know the probability we want by calculating* $1 - F(900, y)$ *for any* $y \in \{0,1,2\}$*.*

Chapter 10

Applications

This chapter is split into two sections, which consist of some applications that have arisen from the theory that was developed in Chapters 7, 8 and 9. Indeed, in Section 10.1 we prove that each cdf on a separable LOTS can be decomposed as a convex sum of cdfs and, in Section 10.2, we give a goodness-of-fit test so that we can check if a given random sample comes from a certain distribution.

10.1 Decomposition of a cdf

Note that the convex sum of cdfs is a cdf, as a consequence of Theorem 9.9. This allows us to give a decomposition theorem for a cdf defined on a separable LOTS, X, where $DM(X) \setminus \phi(X)$ is countable. In the decomposition theorem we will use the condition $F = F_-$ instead of the continuity of F in order to get the uniqueness of the decomposition.

Definition 10.1. *Let μ be a probability measure on a separable LOTS X and F its cdf. We say that F is a step cdf if $\{x \in X : \mu(\{x\}) > 0\}$ is a nonempty set and $\sum_{x\in X:\mu(\{x\})>0} \mu(\{x\}) = 1$.*

Theorem 10.2. *Every cdf F_μ defined on a separable LOTS X such that $DM(X) \setminus \phi(X)$ is countable, can be decomposed into $F_\mu = \alpha F_d + (1-\alpha)F_c$ with $0 \le \alpha \le 1$, where F_d is a step cdf, and F_c is a cdf satisfying that $F_{c-} = F_c$. Moreover, the decomposition is unique.*

Proof. Let $X - C(F_\mu) = \{x \in X : \mu(\{x\}) > 0\}$. By Lemma 7.33, it is known that $X - C(F_\mu)$ is countable, which means that we can write $X - C(F_\mu) = \{x_n : n \geq 1\}$ and, if we define $H_d(x) = \sum_{x_n \leq x} \mu(\{x_n\})$ for each $x \in X$, it holds that H_d is a step function such that:

- H_d is monotonically non-decreasing. Let $x, y \in X$ with $x < y$. Then $H_d(y) = \sum_{x_n \leq y} \mu(\{x_n\}) = \sum_{x_n \leq x} \mu(\{x_n\}) + \sum_{x < x_n \leq y} \mu(\{x_n\}) = H_d(x) + \sum_{x < x_n \leq y} \mu(\{x_n\})$ so it clearly follows that $H_d(x) \leq H_d(y)$.

- H_d is right τ-continuous. Let $y \in X$ and $y_n \xrightarrow{\tau} y$ with $y_n > y_{n+1} > y$. Then we can write $H_d(y_n) = \sum_{x_k \leq y_n} H_d(x_k) = \sum_{x_k \leq y} H_d(x_k) + \sum_{y < x_k \leq y_n} \mu(\{x_k\}) = H_d(y) + \sum_{y < x_k \leq y_n} \mu(\{x_k\})$. By taking limits, we have that $H_d(y_n) \to H_d(y)$. Indeed, note that $\sum_{y < x_k \leq y_n} \mu(\{x_k\}) = \mu(\bigcup_{y < x_k \leq y_n} \{x_k\}) \leq \mu(]y, y_n])$. Now, the fact that (y_n) is monotonically non-increasing implies that $]y, y_n] \to \bigcap_{n \in \mathbb{N}}]y, y_n] = \emptyset$, so it follows, by the continuity from above of μ, that $\mu(]y, y_n]) \to 0$ and, consequently, $\sum_{y < x_n \leq y_n} \mu(\{x_n\}) \to 0$. Hence, by Lemma 7.15, H_d is right τ-continuous.

- $\sup H_d(A) = \inf H_d(A^u)$ for each $A \in DM(X)$. Note that the case in which $A \in \phi(X)$ is clear, since $A = (\leq x)$ and $A^u = (\geq x)$ for some $x \in X$ and, consequently, by definition of H_d, we have that $\sup H_d(A) = \inf H_d(A^u)$.

  Now, let $A \in DM(X) \setminus \phi(X)$. Since $A \notin \phi(X)$, there does not exist $\max A$. Hence, the fact that A is decreasing lets us consider an increasing sequence (a_n) in A such that $\bigcup_{n \in \mathbb{N}} (\leq a_n) = A$ (see Lemma 8.8). What is more, we can consider a decreasing sequence (b_n) in A^u such that $\bigcup_{n \in \mathbb{N}} (\geq b_n) = A^u$, since A^u is increasing and it does not have a minimum. Now, define A_1 to be the set of points of $X - C(F)$ that belong to A and $A_2 = (X - C(F)) \setminus A_1$.

  Note that $\inf H_d(A^u) - \sup H_d(A) = \inf H_d(b_n) - \sup H_d(a_n) \leq H_d(b_n) - H_d(a_n) = \sum_{x \in X - C(F) : a_n < x < b_n} \mu(\{x\}) = \mu(\bigcup_{x \in X - C(F) : a_n < x < b_n} \{x\}) \leq \mu(]a_n, b_n]) = F(b_n) - F(a_n)$. Now, by taking limits, it holds that $F(b_n) - F(a_n) \to \inf F(A^u) - \sup F(A)$. Since F is a cdf, it satisfies that $\sup F(A) = \inf F(A^u)$ for each $A \in DM(X)$, which lets us conclude that $\inf H_d(A^u) = \sup H_d(A)$.

- $\inf H_d(X) = 0$ if there does not exist $\min X$. Since $H_d(x) \leq F(x)$ for each $x \in X$, it holds that $0 \leq \inf H_d(X) \leq \inf F(X)$. Since F is a cdf, $\inf F(X) = 0$, which lets us conclude that $\inf H_d(X) = 0$.

On the other hand, it is clear that $H_d(x) \leq F(x)$ for each $x \in X$. Therefore, we can define $H(x) = F(x) - H_d(x)$ for each $x \in X$ and it holds that:

- H is monotonically non-decreasing. Indeed, let $x < y$. Then $H(y)-H(x) = F(y)-F(x) - [H_d(y) - H_d(x)] = F(y) - F(x) - \sum_{x<x_n\leq y} \mu(\{x_n\})$ and $H(y) - H(x) \geq 0$ if and only if $\sum_{x<x_n\leq y} \mu(\{x_n\}) \leq F(y) - F(x)$. Note that $\sum_{x<x_n\leq y} \mu(\{x_n\}) = \mu(\bigcup_{x<x_n\leq y}\{x_n\}) \leq \mu(]x, y]) = F(y) - F(x)$ by Proposition 7.28 and, consequently, H is monotonically non-decreasing.

- H is right τ-continuous. It is clear that H is right τ-continuous due to the fact that it is the subtraction of two right τ-continuous functions.

- $\sup H(A) = \inf H(A^u)$ for each $A \in DM(X)$. The case in which $A \in \phi(X)$ is clear, so we show that the equality is true in case that $A \in DM(X) \setminus \phi(X)$. Since A is decreasing and it does not have a maximum and A^u is increasing and it does not have a minimum, we can consider (a_n) and (b_n) with $a_n \in A$ and $b_n \in A^u$ to be, respectively, an increasing sequence and a decreasing one such that $\bigcup_{n\in\mathbb{N}}(\leq a_n) = A$ and $\bigcup_{n\in\mathbb{N}}(\geq b_n) = A^u$. What is more, the fact that F, H_d and H are monotonically non-decreasing lets us claim that $F(a_n) \to \sup F(A)$, $H_d(a_n) \to \sup G(A)$, $F(a_n) \to \inf F(A^u)$, $H_d(a_n) \to \inf H_d(A^u)$, $H(a_n) \to \sup H(A)$ and $H(a_n) \to \inf H(A^u)$. Since, by definition of H, we can write $H(a_n) = F(a_n) - H_d(a_n)$, by taking limits, it holds that $\sup H(A) = \sup F(A) - \sup H_d(A)$. Moreover, $H(b_n) = F(b_n) - H_d(b_n)$ gives us that $\inf H(A^u) = \inf F(A^u) - \inf H_d(A^u)$. Since F is a cdf, we have that $\sup F(A) = \inf F(A^u)$. H_d also satisfies that equality as it has been proven before. Hence, $\sup H(A) = \inf H(A^u)$.

- If there does not exist $\min X$, then $\inf H(X) = 0$. Indeed, the fact that $H(x) = F(x) - H_d(x)$ for each $x \in X$ implies that $H(x) \leq F(x)$ for each $x \in X$, so $\inf H(X) \leq \inf F(X) = 0$, which lets us conclude that $\inf H(X) = 0$.

Now, note that $\sum_{x_n\in X\setminus C(F)} \mu(\{x_n\})$ is an absolutely convergent series, that is, there exists a number α such that $\sup H_d(X) = \alpha$. Moreover, the facts that $H(x) = F(x) - H_d(x)$ and $\sup F(X) = 1$ mean that $\sup H(X) = 1 - \alpha$. Hence, if we define $F_d(x) = \frac{1}{\alpha}H_d(x)$ and $F_c = \frac{1}{1-\alpha}H(x)$ for each $x \in X$, it follows that $\sup F_d(X) = \sup F_c(X) = 1$. Moreover, in case that there does not exist $\min X$, $\inf F_d(X) = \inf F_c(X) = 0$ due to the fact that $\inf H_d(X) = \inf H(X) = 0$.

We conclude that F_d and F_c are both cdfs by Theorem 9.9 (since they are monotonically non-decreasing and right τ-continuous, $\sup F_d(A) = \inf F_d(A^u)$ and $\sup F_c(A) = \inf F_c(A^u)$ for each $A \in DM(X)$, because H_d and H satisfy these properties, $DM(X) \setminus \phi(X)$ is countable and $\inf F_d(X) = \inf F_c(X) = 0$ if there does not exist $\min X$).

Moreover, $F_{c-} = F_c$. Indeed, given $x \in X$, we distinguish two cases depending on whether the measure of x is null or not:

1. Let $x \in X$ be such that $\mu(\{x\}) = 0$. We distinguish two cases depending on whether x is left-isolated or not:

   (a) Suppose that x is left-isolated. Then there exists $y \in X$ with $y < x$ and such that $]y, x[= \emptyset$. Hence, $F_{c-}(x) = F_c(y)$. By definition of F_c, we have that $F_c(y) = \frac{1}{1-\alpha}\left(F(y) - \sum_{x_i \leq y} \mu(\{x_i\})\right)$. Moreover, the fact that $\mu(\{x\}) = 0$ gives us that $F(y) = F(x)$ and $\sum_{x_i \leq y} \mu(\{x_i\}) = \sum_{x_i \leq x} \mu(\{x_i\})$. Thus, $F_c(y) = \frac{1}{1-\alpha}\left(F(x) - \sum_{x_i \leq x} \mu(\{x_i\})\right) = F_c(x)$. We conclude that $F_{c-}(x) = F_c(x)$.

   (b) Suppose that x is not left-isolated. Then, by Proposition 7.11, there exists a monotone sequence which left τ-converges to x. Let (y_n) be the previous sequence. By Proposition 7.30, it holds that $F_c(y_n) \to F_{c-}(x)$ due to the fact that F_c is a cdf. Now, we prove that $F_c(y_n) \to F_c(x)$. We can write $F_c(y_n) = \frac{1}{1-\alpha}\left(F(y_n) - \sum_{x_i \leq y_n} \mu(\{x_i\})\right)$. By taking limits, $\sum_{x_i \leq y_n} \mu(\{x_i\}) \to \sum_{x_i < x} \mu(\{x_i\}) = \sum_{x_i \leq x} \mu(\{x_i\})$, since $\mu(\{x\}) = 0$. Moreover, $F(y_n) \to F_-(x)$, since F is a cdf (see Proposition 7.30). What is more, $F(y_n) \to F(x)$ because $\mu(\{x\}) = 0$. Consequently, $F_c(y_n) \to \frac{1}{1-\alpha}\left(F(x) - \sum_{x_i \leq x} \mu(\{x_i\})\right) = F_c(x)$. Since $F_c(y_n) \to F_c(x)$ and $F_c(y_n) \to F_{c-}(x)$, we conclude that $F_c(x) = F_{c-}(x)$.

2. Let $x \in X$ be such that $\mu(\{x\}) > 0$. We distinguish two cases depending on whether x is left-isolated or not:

   (a) Suppose that x is left-isolated. Then there exists $y \in X$ with $y < x$ such that $]y, x[= \emptyset$. Thus, $F_{c-}(x) = F_c(y)$. Now, by definition of F_c, we can write $F_c(y) = \frac{1}{1-\alpha}\left(F(y) - \sum_{x_i \leq y} \mu(\{x_i\})\right)$. Since $\mu(\{x\}) > 0$, it holds that $F(y) = F(x) - \mu(\{x\})$, so we have that $\frac{1}{1-\alpha}\left(F(y) - \sum_{x_i \leq y} \mu(\{x_i\})\right) =$

$\frac{1}{1-\alpha}\left(F(x)-\mu(\{x\})-\sum_{x_i\leq y}\mu(\{x_i\})\right)=\frac{1}{1-\alpha}\left(F(x)-\sum_{x_i\leq x}\mu(\{x_i\})\right)=F_c(x)$. We conclude that $F_{c-}(x)=F_c(x)$.

(b) Suppose that x is not left-isolated. Then there exists a monotone sequence which left τ-converges to x. Let (y_n) be the previous sequence. By Proposition 7.30, it holds that $F_c(y_n)\to F_{c-}(x)$ due to the fact that F_c is a cdf. Now, we prove that $F_c(y_n)\to F_c(x)$. We can write, by definition of F_c, $F_c(y_n)=\frac{1}{1-\alpha}\left(F(y_n)-\sum_{x_i\leq y_n}\mu(\{x_i\})\right)$. Now, by taking limits, we have that $F_c(y_n)\to\frac{1}{1-\alpha}\left(F_-(x)-\sum_{x_i<x}\mu(\{x_i\})\right)=\frac{1}{1-\alpha}\left(F_-(x)-(\sum_{x_i\leq x}\mu(\{x_i\})-\mu(\{x\}))\right)=\frac{1}{1-\alpha}\left(F(x)-\sum_{x_i\leq x}\mu(\{x_i\})\right)=F_c(x)$.

Finally, we prove the uniqueness of the previous decomposition. Suppose that we can write $F(x)=\alpha F_{d_1}(x)+(1-\alpha)F_{c_1}(x)=\beta F_{d_2}(x)+(1-\beta)F_{c_2}(x)$ for each $x\in X$, where F_{d_1} and F_{d_2} are both step cdfs and F_{c_1} and F_{c_2} are cdfs satisfying the hypothesis given for F_c. Moreover, $\alpha,\beta\in[0,1]$. Let $x\in X$. We distinguish two cases depending on whether x is left-isolated or not:

1. Suppose that x is not left-isolated. Then there exists a monotone sequence which left τ-converges to x. Let (x_n) be that sequence. Then it holds that $F(x_n)=\alpha F_{d_1}(x_n)+(1-\alpha)F_{c_1}(x_n)=\beta F_{d_2}(x_n)+(1-\beta)F_{c_2}(x_n)$ or, equivalently, $(1-\alpha)F_{c_1}(x_n)-(1-\beta)F_{c_2}(x_n)+\alpha F_{d_1}(x_n)-\beta F_{d_2}(x_n)=0$. Moreover, for x it holds that $(1-\alpha)F_{c_1}(x)-(1-\beta)F_{c_2}(x)+\alpha F_{d_1}(x)-\beta F_{d_2}(x)=0$. Now, if we substract both previous equalities, we have that $0=\alpha F_{d_1}(x_n)-\alpha F_{d_1}(x)-\beta F_{d_2}(x_n)+\beta F_{d_2}(x)+(1-\alpha)F_{c_1}(x_n)-(1-\alpha)F_{c_1}(x)-(1-\beta)F_{c_2}(x_n)+(1-\beta)F_{c_2}(x)$. Note that $(1-\alpha)F_{c_1}(x_n)-(1-\alpha)F_{c_1}(x)-(1-\beta)F_{c_2}(x_n)+(1-\beta)F_{c_2}(x)\to 0$, since $F_{c_1}(x_n)\to F_{c_1-}(x)$ and $F_{c_2}(x_n)\to F_{c_2-}(x)$, since they are both cdfs (see Proposition 7.30). Moreover, we take into account that $F_{c_1}=F_{c_1-}$ and $F_{c_2}=F_{c_2-}$. Hence, by taking limits in the expression $0=\alpha F_{d_1}(x_n)-\alpha F_{d_1}(x)-\beta F_{d_2}(x_n)+\beta F_{d_2}(x)+(1-\alpha)F_{c_1}(x_n)-(1-\alpha)F_{c_1}(x)-(1-\beta)F_{c_2}(x_n)+(1-\beta)F_{c_2}(x)$, it follows that $\alpha F_{d_1}(x_n)-\beta F_{d_2}(x_n)\to\alpha F_{d_1}(x)-\beta F_{d_2}(x)$. Since $F_{d_1}(x_n)\to F_{d_1-}(x)$ and $F_{d_2}(x_n)\to F_{d_2-}(x)$ due to the fact that F_{d_1} and F_{d_2} are both cdfs (see Proposition 7.30), we also have that $\alpha F_{d_1}(x_n)-\beta F_{d_2}(x_n)\to\alpha F_{d_1-}(x)-\beta F_{d_2-}(x)$. Hence, $\alpha F_{d_1-}(x)-\beta F_{d_2-}(x)=\alpha F_{d_1}(x)-\beta F_{d_2}(x)$, which implies that $\alpha\mu_{F_{d_1}}(\{x\})=\beta\mu_{F_{d_2}}(\{x\})$.

2. Suppose that x is left-isolated. Then there exists $z\in X$ with $z<x$ and such that

$]z, x[= \emptyset$. By using a similar reasoning to the one made in the previous item, it holds that $0 = \alpha F_{d_1}(z) - \alpha F_{d_1}(x) - \beta F_{d_2}(z) + \beta F_{d_2}(x) + (1-\alpha)F_{c_1}(z) - (1-\alpha)F_{c_1}(x) - (1-\beta)F_{c_2}(z) + (1-\beta)F_{c_2}(x)$. What is more, the previous expression becomes $0 = \alpha F_{d_1}(z) - \alpha F_{d_1}(x) - \beta F_{d_2}(z) + \beta F_{d_2}(x)$ if we take into account that $F_{c_1} = F_{c_1 -}$ and $F_{c_2} = F_{c_2 -}$. Hence, we have that $\alpha F_{d_1}(x) - \alpha F_{d_1}(z) = \beta F_{d_2}(x) - \beta F_{d_2}(z)$ if and only if $\alpha \mu_{F_{d_1}}(\{x\}) = \beta \mu_{F_{d_2}}(\{x\})$.

Therefore, $\alpha \mu_{F_{d_1}}(\{x\}) = \beta \mu_{F_{d_2}}(\{x\})$ for each $x \in X$. Now, observe that $\alpha = \alpha \sum_{x \in X} \mu_{F_{d_1}}(\{x\}) = \sum_{x \in X} \alpha \mu_{F_{d_1}}(\{x\}) = \sum_{x \in X} \beta \mu_{F_{d_2}}(\{x\}) = \beta \sum_{x \in X} \mu_{F_{d_2}}(\{x\}) = \beta$, where we have used that $\sum_{x \in X} \mu_{F_{d_1}}(\{x\}) = \sum_{x \in X} \mu_{F_{d_2}}(\{x\}) = 1$, since $\mu_{F_{d_1}}$ and $\mu_{F_{d_2}}$ are both probability measures on X.

Hence, $\mu_{F_{d_1}}(\{x\}) = \mu_{F_{d_2}}(\{x\})$ for each $x \in X$, so $F_{\mu_{F_{d_1}}}(x) = F_{\mu_{F_{d_2}}}(x)$ for each $x \in X$. By Theorem 9.9, we have that $F_{\mu_{F_{d_1}}} = F_{d_1}$ and $F_{\mu_{F_{d_2}}} = F_{d_2}$ and, consequently, $F_{c_1} = F_{c_2}$, which gives us the uniqueness of the decomposition.

□

Remark 10.3. *Continuity is not enough to ensure that the decomposition is unique. We also need the measure of each point to be null. That is the reason why the decomposition made in the previous theorem is not unique except if we suppose that the measure of each point is null according to the cdf* F_c.

The decomposition of a cdf in the classical case is unique (see [19, Th. 1.2.3]) due to the fact that, if a step function is continuous, then it is null in each point. However, that statement is not true when we work with a cdf defined on a separable LOTS.

Example 10.4. *Let* $X = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$ *with the usual order. Let* μ *be the measure defined on* X *by* $\mu(\{\frac{2}{3}\}) = 1$*, where* $\mu(X \setminus \{\frac{2}{3}\}) = 0$*, and let* F *be its cdf, that is,* $F : X \to [0, 1]$ *is given by*

$$F(x) = \begin{cases} 0 & \text{if} \quad x < \frac{2}{3} \\ \\ 1 & \text{if} \quad x \geq \frac{2}{3} \end{cases}$$

It is clear that F *is a step cdf. Note that* $\frac{2}{3}$ *is left-isolated due to the fact that* $]\frac{1}{3}, \frac{2}{3}[= \emptyset$. *Hence, by Proposition 7.23, it holds that* F *is continuous at* $\frac{2}{3}$. *Since* $\mu(\{x\}) = 0$ *for each* $x \in X \setminus \{\frac{2}{3}\}$*, by Proposition 7.32, it holds that* F *is also continuous in* $X \setminus \{\frac{2}{3}\}$. *We conclude that* F *is continuous.*

10.1.1 Example

Next cdf is a mixture of cdfs. Indeed, it can be decomposed as the convex sum of two different cdfs.

Example 10.5. *Let $X = (\{0\} \cup \mathbb{N}) \times [0,1]$ and $\leq$ be the lexicographic order on X. Consider the function $F : X \to [0,1]$ given by $F(x,y) = 1 - \frac{1}{2}e^{-(x+y)} - \frac{1}{2}e^{-x}$ for each $(x,y) \in X$.*

According to Example 9.23, there exists a probability measure μ such that $F = F_\mu$. Note that $X - C(F) = \{(x,y) \in X : \mu(\{(x,y)\}) > 0\} = \{(n,0) : n \in \mathbb{N}\}$. What is more, $\mu(\{(n,0)\}) = F(n,0) - F_-(n,0) = F(n,0) - \sup F(< (n,0)) = F(n,0) - F(n-1,1) = 1 - \frac{1}{2}e^{-n} - \frac{1}{2}e^{-n} - \left(1 - \frac{1}{2}e^{-n} - \frac{1}{2}e^{-(n-1)}\right) = \frac{e-1}{2}e^{-n}$.

To get the decomposition, firstly we define a step function that accumulates the mass in the points in X with non-zero probability. We define $H_d : X \to [0, \frac{1}{2}]$ by

$$H_d(x,y) = \sum_{(x_n,y_n) \leq (x,y)} \mu(\{(x_n,y_n)\}) = \sum_{k=1}^{x} \mu(\{(k,0)\}) = \sum_{k=1}^{x} e^{-k}\frac{e-1}{2} = \frac{1}{2}(1 - e^{-x})$$

Note that $\sup H_d(X) = \frac{1}{2}$.

On the other hand, we define $H : X \to [0, \frac{1}{2}]$ as the one given by $H(x,y) = F(x,y) - H_d(x,y)$, that is, $H(x,y) = \frac{1}{2}(1 - e^{-(x+y)})$ for each $(x,y) \in X$.

Note that $\sup H(X) = \frac{1}{2}$.

Hence, the decomposition theorem lets us ensure that F can be uniquely decomposed as the convex sum $F = \frac{1}{2}F_d + \frac{1}{2}F_c$, where $F_d, F_c : X \to [0,1]$ are both cdfs that are, respectively, a step one and one satisfying $F_{c-} = F_c$. Moreover, they are defined respectively by

$$F_d(x,y) = 1 - e^{-x}$$

and

$$F_c(x,y) = 1 - e^{-(x+y)}$$

for each $(x,y) \in X$.

Finally, observe that F_d and F_c are, indeed, both cdfs of a probability measure on X, since they satisfy the hypothesis of Theorem 9.9.

10.2 A goodness-of-fit test

In this section we introduce a statistic in order to establish a new goodness-of-fit test in a LOTS. For that purpose, we first need to introduce some concepts.

Definition 10.6. *Let X be a separable LOTS and consider $x_1, \ldots, x_n \in X$. The previous collection of points is said to be a random sample of the distribution given by a probability measure, μ, such that $\mu(\{x\}) = 0$ for each $x \in X$, if $F(x_1), \ldots, F(x_n)$ is a random sample of a uniform distribution on $[0,1]$.*

Moreover, given a random sample, we can define a cdf from it, as stated next.

Definition 10.7. *Let X be a separable LOTS. If $x_1, \ldots, x_n$ is an ordered random sample of the distribution given by a cdf F, then the function $F_n : X \to [0,1]$ defined by*

$$F_n(x) = \begin{cases} 0 & \text{if} \quad x < x_1 \\ \\ \frac{k}{n} & \text{if} \quad x_k \leq x < x_{k+1}, \forall k = 1, \ldots, n-1 \\ \\ 1 & \text{if} \quad x \geq x_n \end{cases}$$

is said to be the empirical cdf of the sample.

Remark 10.8. *Note that F_n is, indeed, a cdf, since it is defined from the probability measure given by $\mu_n(\{x_k\}) = \frac{1}{n}$ for each $k = 1, \ldots, n$.*

Definition 10.9. *Let F be a cdf on a separable LOTS, X, and G its pseudo-inverse. If $x_1, \ldots, x_n \in X$ is an ordered random sample whose empirical cdf is F_n, we define $H_n : [0,1] \to [0,1]$ by $H_n = \widetilde{F_n} \circ G$.*

Note that we can write

$$H_n(r) = \begin{cases} 0 & \text{if} \quad G(r) < \phi(x_1) \\ \\ \frac{k}{n} & \text{if} \quad \phi(x_k) \leq G(r) < \phi(x_{k+1}) \text{ for each } k = 1, \ldots, n-1 \\ \\ 1 & \text{if} \quad G(r) \geq \phi(x_n) \end{cases}$$

or, equivalently,

$$H_n(r) = \begin{cases} 0 & \text{if} \quad r < F(x_1) \\ \\ \frac{k}{n} & \text{if} \quad F(x_k) \le r < F(x_{k+1}) \text{ for each } k = 1, \ldots, n-1 \\ \\ 1 & \text{if} \quad r \ge F(x_n) \end{cases}$$

Remark 10.10. *From the previous expression, H_n is the empirical cdf of the sample $F(x_1), \ldots, F(x_n)$ and, by definition, that sample comes from a uniform-$[0,1]$ sample.*

Suppose that we are given a random sample on a separable LOTS according to a certain cumulative distribution function. Our purpose is testing if that sample comes from a given distribution, F.

First of all, there exists a relationship between a cdf and its extension to $DM(X)$ that involves the pseudo-inverse:

Lemma 10.11. *Let F be a cdf on a separable LOTS and G its pseudo-inverse. Then*

1. $\widetilde{F}(G(F(x))) = F(x)$ *for each* $x \in X$.

2. $\widetilde{\mu}(]G(F(x)), \phi(x)]) = 0$ *for each* $x \in X$.

Proof. Let $x \in X$.

1. On the one hand, the fact that $G(F(x)) \le \phi(x)$ for each $x \in X$ and the monotonicity of $\widetilde{F}$ as cdf give us $\widetilde{F}(G(F(x))) \le \widetilde{F}(\phi(x)) = F(x)$. On the other hand, since $\widetilde{F}$ is a cdf, it holds that $\widetilde{F}(G(r)) \ge r$ for each $r \in [0,1]$. In particular, we have that $\widetilde{F}(G(F(x))) \ge F(x)$, so we conclude the equality.

2. Since $\widetilde{F}(G(F(x))) = F(x)$ and $\widetilde{F}$ is an extension of F (that is, $\widetilde{F}(\phi(x)) = F(x)$), it follows that $\widetilde{\mu}(]G(F(x)), \phi(x)]) = 0$ for each $x \in X$.

□

We define the statistic

$$D_n = \sup_{x \in X} |F_n(x) - F(x)|$$

Theorem 10.12. *Let X be a separable LOTS. Also, let $x_1, \ldots, x_n$ be a random sample of μ, a probability measure on X such that $\mu(\{x\}) = 0$ for each $x \in X$, and whose empirical cdf is F_n. Then we can write*

$$D_n = \max_{0 \leq r \leq 1} |H_n(r) - r|$$

Proof. Note that

$$D_n = \sup_{x \in X} |F_n(x) - F(x)| = \sup_{x \in X} |\widetilde{F}_n(\phi(x)) - \widetilde{F}(\phi(x))|$$

since $\widetilde{F}_n$ and $\widetilde{F}$ are, respectively, extensions of F_n and F to $DM(X)$.

Moreover, since $\mu(\{x\}) = 0$ for each $x \in X$, we have that $\widetilde{\mu}(\{G(r)\}) = 0$ for each $r \in [0, 1]$. That implies that $\widetilde{F}(G(r)) = r$.

What is more,

$$\sup_{x \in X} |\widetilde{F}_n(\phi(x)) - \widetilde{F}(\phi(x))| = \max_{0 \leq r \leq 1} |\widetilde{F}_n(G(r)) - \widetilde{F}(G(r))|$$

Indeed, we can prove the previous equality as follows:

$\leq$) Let $x \in X$ and consider $r = F(x)$. Then $\widetilde{F}(G(r)) = \widetilde{F}(G(F(x))) = F(x)$ by Lemma 10.11.1. Since $\widetilde{F}$ is an extension of F, we can write the previous equality as $\widetilde{F}(G(r)) = \widetilde{F}(\phi(x))$. Moreover, note that $\widetilde{F}_n(G(r)) = \widetilde{F}_n(\phi(x))$ with probability 1 due to the fact that $\widetilde{\mu}(]G(r), \phi(x)]) = 0$ (see Lemma 10.11.2). Hence, $|\widetilde{F}_n(G(r)) - \widetilde{F}(G(r))| = |\widetilde{F}_n(\phi(x)) - \widetilde{F}(\phi(x))|$ with probability 1.

$\geq$) Let $r \in [0, 1]$ and $\varepsilon > 0$. We distinguish two cases:

1. If $G(r) \in \phi(X)$, then it is clear that $x = \phi^{-1}(G(r))$ is a point in X satisfying that $|\widetilde{F}_n(G(r)) - \widetilde{F}(G(r))| - \varepsilon \leq |\widetilde{F}_n(\phi(x)) - \widetilde{F}(\phi(x))|$.

2. Suppose that $G(r) \notin \phi(X)$. Then we can write $G(r) = \{x \in X : F(x) < r\}$ (see Proposition 8.22). Now, we can suppose the random sample to be ordered and, in this case, consider $n \in \mathbb{N}$ such that $\phi(x_n) < G(r) < \phi(x_{n+1})$. By Proposition 8.12, $\inf F(A^u) = \sup F(A)$ for each $A \in DM(X)$. In particular, it holds that $\inf F(G(r)^u) = \sup F(G(r))$, which is equivalent, by definition of $\widetilde{F}$, to $\widetilde{F}(G(r)) = \sup F(G(r))$. Moreover, since $\widetilde{F}(G(r)) = r$, $\widetilde{F}(G(r)) = \sup F(G(r))$ implies that $r = \sup F(G(r))$. Since $\sup F(G(r)) = r$, there exists $x \in X$ such that $\phi(x_n) <$

$\phi(x) < G(r)$ and $F(x) > r - \varepsilon$. That implies, firstly, that $\widetilde{F}_n(\phi(x)) = \widetilde{F}_n(G(r))$ because $\phi(x_n) < \phi(x) < G(r) < \phi(x_{n+1})$. Secondly, it holds that $\widetilde{F}(\phi(x)) = F(x)$, $\widetilde{F}(G(r)) = r$ and $|F(x) - r| < \varepsilon$. Finally, we conclude that $|F_n(\phi(x)) - \widetilde{F}(\phi(x))| \geq |\widetilde{F}_n(G(r)) - \widetilde{F}(G(r))| - \varepsilon$.

Once we have proven the equality, we conclude that

$$D_n = \max_{0 \leq r \leq 1} |\widetilde{F}_n(G(r)) - \widetilde{F}(G(r))| = \max_{0 \leq r \leq 1} |\widetilde{F}_n(G(r)) - r| = \max_{0 \leq r \leq 1} |H_n(r) - r|$$

□

We get, as an immediate consequence, the next one.

Corollary 10.13. *Given a separable LOTS X, the distribution of D_n is the same for each cdf, F_μ, satisfying that $\mu(\{x\}) = 0$ for each $x \in X$.*

Proof. It immediately follows from the fact that H_n is the empirical cdf of a uniform-$[0,1]$ distribution and the previous theorem. □

Recall that, in the classical case, the Kolmogorov-Smirnov test works when we are testing if the sample comes from a continuous distribution. However, in the context of a LOTS, continuity is not enough to ensure that everything works fine to prove that the distribution of D_n is the same for each F (null hypothesis).

Chapter 11

Fractal structures and separable LOTS

To end this work, we dedicate a chapter to establish the relationship between separable LOTS and spaces with a fractal structure. Hence, the whole theory that has been developed along the rest of the work can be indistinctly used in both contexts and it makes sense to talk about some applications that arise from it.

11.1 Defining a fractal structure from a LOTS

The main goal of this section is showing the construction of a fractal structure on a second countable topological space with a linear order.

It is known that the set of isolated points in a separable linearly ordered topological space is countable if and only if the space is second countable with respect to the order topology (see Proposition 7.8). We will suppose that X is second countable with respect to τ in order to be able to define a fractal structure from the set of points which are right-isolated or left-isolated.

Definition 11.1. *Let X be a LOTS. We define C_1 and C_2 to be, respectively, the set of left-isolated and the set of right-isolated points.*

Definition 11.2. *Let $x \in C_1$ (respectively $x \in C_2$). Then we define x^l (respectively x^r) to be the previous (respectively following) point in X such that $]x^l, x[= \emptyset$ (respectively*

$]x, x^r[= \emptyset)$.

Definition 11.3. *Let X be a second countable LOTS with respect to τ. Since X is second countable, it is separable, that is, there exists a countable dense subset of X, D'. Now, consider $D = D' \cup C_1 \cup C_2$. Since D is countable, we can enumerate it, $D = \{d_n : n \in \mathbb{N}\}$.*

Now, let $d_1 \in D$. In order to define the first level of the fractal structure, we distinguish some cases depending on whether d_1 is left-isolated, right-isolated, isolated or it is not right-isolated nor left-isolated:

1. *If $d_1 \in C_1 \setminus C_2$, we define $\Gamma_1 = \{(\leq d_1^l), (\geq d_1)\}$.*
2. *If $d_1 \in C_2 \setminus C_1$, we define $\Gamma_1 = \{(\leq d_1), (\geq d_1^r)\}$.*
3. *If $d_1 \in C_1 \cap C_2$, we define $\Gamma_1 = \{(\leq d_1^l), \{d_1\}, (\geq d_1^r)\}$.*
4. *If $d_1 \notin C_1 \cup C_2$, we define $\Gamma_1 = \{(\leq d_1), (\geq d_1)\}$.*

Recursively, once that Γ_n has been defined, we proceed as follows to define Γ_{n+1}:

Given $A \in \Gamma_n$, we distinguish some cases depending on the form of A:

1. *Suppose that $A = [a, b]$ for some $a, b \in D$ with $a < b$. Now, let i be the first natural number such that $d_i \in]a, b[$.*
   (a) *If $d_i \in C_1 \setminus C_2$, we define $S_A = \{[a, d_i^l], [d_i, b]\}$.*
   (b) *If $d_i \in C_2 \setminus C_1$, we define $S_A = \{[a, d_i], [d_i^r, b]\}$.*
   (c) *If $d_i \in C_1 \cap C_2$, we define $S_A = \{[a, d_i^l], \{d_i\}, [d_i^r, b]\}$.*
   (d) *If $d_i \notin C_1 \cup C_2$, we define $S_A = \{[a, d_i], [d_i, b]\}$.*
2. *Suppose that $A = (\leq a)$ for some $a \in D$. Let i be the first natural number such that $d_i \in (< a)$.*
   (a) *If $d_i \in C_1 \setminus C_2$, we define $S_A = \{(\leq d_i^l), [d_i, a]\}$.*
   (b) *If $d_i \in C_2 \setminus C_1$, we define $S_A = \{(\leq d_i), [d_i^r, a]\}$.*
   (c) *If $d_i \in C_1 \cap C_2$, we define $S_A = \{(\leq d_i^l), \{d_i\}, [d_i^r, a]\}$.*

(d) *If $d_i \notin C_1 \cup C_2$, we define $S_A = \{(\leq d_i), [d_i, a]\}$.*

3. *Suppose that $A = (\geq b)$ for some $b \in D$. Let i be the first natural number such that $d_i \in (> b)$. We proceed analogously to the previous case to define S_A.*

The fractal structure from a second countable LOTS is $\mathbf{\Gamma} = \{\Gamma_n : n \in \mathbb{N}\}$, where $\Gamma_{n+1} = \bigcup\{S_A : A \in \Gamma_n\}$ for each $n \in \mathbb{N}$.

Now, we prove that, indeed, $\mathbf{\Gamma}$ is a fractal structure on X.

Proposition 11.4. *$\mathbf{\Gamma}$ is a fractal structure on X.*

Proof. We check that $\Gamma_{n+1} \prec\prec \Gamma_n$ for each $n \in \mathbb{N}$. For that purpose, we have to prove that

- $\Gamma_{n+1} \prec \Gamma_n$ for each $n \in \mathbb{N}$. Let $n \in \mathbb{N}$ and $A \in \Gamma_{n+1}$. Then there exists $B \in \Gamma_n$ such that $A \in S_B$. It is clear, by construction of $\mathbf{\Gamma}$, that $A \subseteq B$.
- $A = \bigcup\{B \in \Gamma_{n+1} : B \subseteq A\}$ for each $A \in \Gamma_n$. Let $n \in \mathbb{N}$. By definition of $\mathbf{\Gamma}$, given $A \in \Gamma_n$, it holds that $B \subseteq A$ for each $B \in S_A$. What is more, $A = \bigcup_{B \in S_A} B$ and $B \in \Gamma_{n+1}$ for each $B \in S_A$.

□

To end this section, we show that the fractal structure we have just defined is compatible with the order, that is, the order topology on X, τ, coincides with the topology of the non-archimedean quasi-pseudometric induced by the fractal structure on X, d.

First of all, by construction of $\mathbf{\Gamma}$, the next statement holds.

Remark 11.5. *Let $i \in \mathbb{N}$. Then there exists a natural $n \leq i$ such that d_i is the greatest or the lower point of A for some $A \in \Gamma_n$.*

Proposition 11.6. *$\tau = \tau_d$, where τ_d is the topology of the non-archimedean quasi-pseudometric induced by $\mathbf{\Gamma}$.*

Proof. $\subseteq$) We start proving that $\tau \subseteq \tau_d$. Indeed, let $a, b, x \in X$ with $a < x < b$. Then there exist $i, j \in \mathbb{N}$ such that $a < d_i < x < d_j < b$. Now, if we consider $n = \max\{i, j\}$, it follows that $A \subseteq [d_i, d_j]$ for each $A \in \Gamma_n$ with $x \in A$. Note that $U_{xn} \subseteq [d_i, d_j]$, which means that $U_{xn} \subseteq]a, b[$, which lets us conclude that $]a, b[$ is an open set in τ_d.

$\supseteq$) Let $x \in X$ and $n \in \mathbb{N}$. Then $U_{xn} = X \setminus \bigcup_{A \in \Gamma_n : x \notin A} A$. Given $A \in \Gamma_n$, let us define A^- and A^+ to be, respectively, the set of elements in Γ_n whose points are lower than the points in A and greater that the points in A respectively. Next, we prove that U_{xn} is an open set with respect to τ. For that purpose, we distinguish two cases:

1. Suppose that there exist $A, B \in \Gamma_n$ such that $x \in A \cap B$ and $x = \max A = \min B$. Then the following cases may occur:
   (a) If $A^- \neq \emptyset$ and $B^+ \neq \emptyset$, then $U_{xn} =]\max A^-, \min B^+[$.
   (b) If $A^- = \emptyset$, then $U_{xn} = (< \min B^+)$.
   (c) If $B^+ = \emptyset$, then $U_{xn} = (> \max A^-)$.
2. There exists a unique $A \in \Gamma_n$ such that $x \in A$. In this case, it can happen:
   (a) If $A = [a, b]$ for some $a, b \in D$ such that $a < b$, then $U_{xn} =]\max A^-, \min A^+[$.
   (b) If $A = (\leq a)$ for some $a \in D$, then $U_{xn} = (< \min A^+)$.
   (c) If $A = (\geq a)$ for some $a \in D$, then $U_{xn} = (> \max A^-)$.

□

The idea of defining a fractal structure from a LOTS makes it possible to apply the theory of the first part of this dissertation when we are working with a LOTS. Recall, from Chapter 4, that the space must be T_0 with respect to the non-archimedean quasi-pseudometric induced by the fractal structure so that the theory on construction of probability measures makes sense. Indeed, it holds because a LOTS is always T_2 with respect to the order topology (which, by Proposition 11.6, coincides with the one given by non-archimedean quasi-pseudometric induced by the fractal structure).

Remark 11.7. *A LOTS is T_2 with respect to the order topology.*

Proof. Let X be a LOTS and consider $a, b \in X$ such that $a < b$. We distinguish two cases in order to find disjoint neighborhoods of a and b:

1. Suppose that b is the following point to a. Then $(< b)$ and $(> a)$ are, respectively, neighborhoods of a and b which are, in fact, disjoint.

2. Suppose that b is not the following point to a. Then there exists $c \in]a, b[$ and, hence, $(< c)$ and $(> c)$ are, respectively, neighborhoods of a and b which are, in fact, disjoint.

$\square$

11.2 Defining a LOTS from a fractal structure

In this part of the work we will show that, given a space, X, with a fractal structure, we can define an order so that X becomes a separable LOTS, where it does make sense the theory that has been described in previous chapters. For that purpose, we will assume that $\mathbf{\Gamma}$ is a fractal structure on X, which is T_0 with respect to the induced quasi-pseudometric, d. The fact that X is T_0 with respect to d implies that d^* is a metric (also called an ultrametric). In Subsection 11.2.1 we see how to define an order on an ultrametric space from the collection of balls. Once we have defined the conditions on the order and prove the properties of it, we show two examples of ultrametric spaces that we can adapt to the context of spaces with a fractal structure in a natural way (see Subsections 11.2.2 and 11.2.3). To end this section, we show an example where, starting from a space with a fractal structure, we define a linear order and see how to deal with probability measures and cdfs in this case.

11.2.1 Defining an order from an ultrametric

In this subsection we will assume that (X, d) is a separable ultrametric space. Given $x \in X$ and $n \in \mathbb{N}$, we will denote by $U_{xn} = \{y \in X : d(x, y) \leq \frac{1}{2^n}\}$ the closed ball, with respect to the ultrametric d, centered at x with radius $\frac{1}{2^n}$. The collection of these balls will be denoted by $\mathcal{G} = \bigcup_{n \in \mathbb{N}} G_n$, where $G_n = \{U_{xn} : x \in X\}$ for each $n \in \mathbb{N}$. Moreover, τ will be the topology of d.

Next, we collect some properties of an ultrametric space according to the notation we have just introduced and [22, Ex. 2.1.15]:

Proposition 11.8. *Let (X, d) be an ultrametric space. Then:*

1. *A ball U_{xn} has diameter at most $\frac{1}{2^n}$.*

2. *Every point of a ball is a center: that is, if $y \in U_{xn}$, then $U_{xn} = U_{yn}$ for each $x \in X$ and each $n \in \mathbb{N}$. Consequently, G_n is a partition of X, that is, it covers X and, given $x, y \in X$, it follows that $U_{xn} = U_{yn}$ or $U_{xn} \cap U_{yn} = \emptyset$.*

3. *U_{xn} is open and closed in τ for each $x \in X$ and $n \in \mathbb{N}$.*

Note that, according to the previous properties, G_{n+1} is a refinement of G_n for each $n \in \mathbb{N}$.

We first give a condition that the order must satisfy.

Definition 11.9. *Let (X, d) be a separable ultrametric space. An order is said to be ball-compatible or B-compatible if, given $x \leq z$ and $n \in \mathbb{N}$, it holds that $U_{xn} = U_{zn}$ or $y \leq t$ for each $y \in U_{xn}$ and each $t \in U_{zn}$.*

From now on, we will assume that (X, d) is a separable ultrametric space and that $\leq$ is a B-compatible order.

Definition 11.10. *Let $A, B \subseteq X$. We say that $A < B$ if and only if $a < b$ for each $a \in A$ and each $b \in B$.*

Next, we introduce a definition of order on G_n.

Definition 11.11. *Let $x, y \in X$ and $n \in \mathbb{N}$. We say that $x \leq_n y$ if and only if $U_{xn} \leq U_{yn}$. Analogously, we say that $x <_n y$ if and only if $U_{xn} < U_{yn}$.*

From the previous definitions it follows the next result.

Proposition 11.12. *Let $x, y \in X$ then $x \leq z$ if and only if $x \leq_n z$ for each $n \in \mathbb{N}$.*

Proof. $\Rightarrow$) It follows from Definition 11.11.

$\Leftarrow$) Let $x, z \in X$ be such that $x \leq_n z$ for each $n \in \mathbb{N}$. Suppose that $x > z$. Then $z \leq_n x$ for each $n \in \mathbb{N}$, which means that $U_{xn} = U_{zn}$ for each $n \in \mathbb{N}$. The last equality implies that $x = z$, which is a contradiction with the fact that $x > z$. Hence, $x \leq z$. $\square$

Corollary 11.13. *Let $x, y \in X$. Then $x < z$ if and only if there exists $n \in \mathbb{N}$ such that $x <_n z$.*

Proof. $\Leftarrow$) It follows from Proposition 11.12.

$\Rightarrow$) Suppose that $x \geq_n z$ for each $n \in \mathbb{N}$. Then, by Proposition 11.12, we have that $x \geq z$, which is a contradiction with the fact that $x < z$. Hence, there exists $n \in \mathbb{N}$ such that $x <_n z$. □

Remark 11.14. *Let $x, y \in X$.*

1. *If $x \leq_n y$ for some $n \in \mathbb{N}$, then $x \leq_k y$ for each $k \leq n$.*

2. *If $x <_n y$ for some $n \in \mathbb{N}$, then $x <_k y$ for each $k \geq n$.*

Indeed, the balls with respect to the ultrametric are convex according to the order, as the next result shows.

Proposition 11.15. *U_{xn} is convex for each $x \in X$ and each $n \in \mathbb{N}$.*

Proof. Let $x \in X$, $n \in \mathbb{N}$ and $a, b \in U_{xn}$ be such that $a \leq b$ and let $y \in X$ be such that $a \leq y \leq b$. Then $a \leq_n y \leq_n b$, which means that $U_{an} \leq U_{yn} \leq U_{bn}$. Since $U_{an} = U_{xn} = U_{bn}$ due to the fact that $a, b \in U_{xn}$ (see Proposition 11.8.2), we conclude that $y \in U_{xn}$ and, consequently, U_{xn} is convex. □

Now, we introduce some notation.

Definition 11.16. *τ_o is the order topology on X given by $\leq$.*

Recall, from Definition 2.23, that the order topology is given by the subbase $\{(< a) : a \in X\} \cup \{(> a) : a \in X\}$. Moreover, an open base of X with respect to τ_o is given by $\{]a, b[: a < b, a, b \in (X \cup \{-\infty, \infty\})\}$. We can prove that the elements in the open base and the subbase are, indeed, open sets with respect to the topology of the ultrametric.

Remark 11.17. *Let $a, b \in X$ with $a < b$, then $]a, b[, (< b)$ and $(> a)$ are open in τ.*

Proof. Let $a, b \in X$ with $a < b$ and let $x \in]a, b[$. Then there exists $n \in \mathbb{N}$ such that $a, b \notin U_{xn}$ and $a <_n x <_n b$, which means that $U_{xn} \subseteq]a, b[$. Since U_{xn} is an open set in τ (see Proposition 11.8.3), it follows that $]a, b[$ is a neighborhood of x with respect to τ. The proofs for $(< b)$ and $(> a)$ are similar. □

Moreover, the topology previously defined is related to the topology τ in the next sense.

Proposition 11.18. $\tau_o \subseteq \tau$.

Proof. Remark 11.17 gives us that $]a, b[, (< b)$ and $(> a)$ are open sets in τ. That means that all the elements of the subbase that defined the order topology are contained in τ. Consequently, $\tau_o \subseteq \tau$. $\square$

We get, as an immediate consequence, the following one.

Corollary 11.19. $\sigma(\tau_o) = \sigma(\tau)$.

Proof. $\subseteq$) Indeed, this is true due to the fact that $\tau_o \subseteq \tau$ (see the previous proposition) means that $\sigma(\tau_o) \subseteq \sigma(\tau)$.

$\supseteq$) Let G be an open set in τ. Since X is separable with respect to d, we can write $G = \bigcup_{n\in\mathbb{N}}\{U_{xn} : x \in G, U_{xn} \subseteq G\}$, a countable union. Moreover, since U_{xn} is convex for each $x \in X$ and $n \in \mathbb{N}$ by Proposition 11.15, U_{xn} can be written as the countable union of sets on the form $[a, b], [a, b[,]a, b[$ or $]a, b]$ (recall, from Corollary 7.4 that each convex set can be expressed as the countable union of intervals). It is clear that $[a, b],]a, b[\in \sigma(\tau_o)$, since they are, respectively, closed and open with respect to the order topology. Now, note that $]a, b]$ and $[a, b[$ can be written as the intersection of an open and a closed subset of X, so they both belong to $\sigma(\tau_o)$. Hence, given $x \in X$ and $n \in \mathbb{N}$, $U_{xn} \in \sigma(\tau_o)$ and, consequently, $G \in \sigma(\tau)$, which finishes the proof. $\square$

Remark 11.20. *A function $F : X \to [0, 1]$ is a cdf with respect to τ if and only if it is a cdf with respect to τ_o.*

Proof. Indeed, if F is a cdf with respect to τ, then there exists a probability measure μ on the Borel σ-algebra of X (with respect to τ) such that $F = F_\mu$. What is more, since $\sigma(\tau) = \sigma(\tau_o)$ (by the previous corollary), F is a cdf with respect to τ_o. $\square$

11.2.2 Defining a LOTS from a Polish ultrametric space

In this subsection we define a linear order from a Polish ultrametric space, that is, an ultrametric space which is complete and separable. For that purpose, we first need to define an order on G_n. Note that G_n is countable because (X, d) is separable.

Definition 11.21. *We can enumerate $G_1 = \{g_1, g_2, \ldots\}$. Since each element of G_1 can be decomposed into a countable number of elements of G_2, we can write $g_i = g_{i1} \cup g_{i2} \cup \ldots$ for each $g_i \in G_1$, and define the lexicographic order on G_2. Hence, we can enumerate G_2 by considering, first, the elements which are contained in g_1, then those which are contained in g_2, ... Recursively we define an order on G_n for each $n \in \mathbb{N}$.*

Given $n \in \mathbb{N}$, this order induces an order on X given by $x \leq_n y$ if and only if $U_{xn} \leq U_{yn}$. From that orders, we define an order on X given by $x \leq y$ if and only if $x \leq_n y$ for each $n \in \mathbb{N}$.

Remark 11.22. *$\leq$ is B-compatible.*

Proof. Let $x, z \in X$ be such that $x \leq z$ and consider $n \in \mathbb{N}$. By definition, it holds that $x \leq_n z$. Suppose that $U_{xn} \neq U_{zn}$ and let $y \in U_{xn}$ and $t \in U_{zn}$. Let us prove that $y \leq t$. It follows that $U_{yn} = U_{xn}$ and $U_{tn} = U_{zn}$ and, hence, $y \leq_n t$ (since $x \leq_n z$). If there exists $m > n$ with $t <_m y$, then it is clear that $t <_k y$ for each $k \geq m$ and $t \leq_k y$ for each $k < m$, because of the relationship between the order $\leq_{k+1}$ and $\leq_k$ given by the lexicographic order. It follows that $t < y$, but then $t \leq_n y$ and, hence, $t =_n y$, so $U_{xn} = U_{yn} = U_{tn} = U_{zn}$, a contradiction. Therefore, $y \leq_m t$ for each m and, hence, $y \leq t$.

$\square$

Example 11.23. *Let X be the Cantor set. As a topological space, this set is homeomorphic to the product of countably many copies of the space $\{0, 1\}$, where we consider the discrete topology on each copy. Hence, this is the space of all sequences in two digits $\{(x_n) : x_n \in \{0, 1\}, \text{ for } n \in \mathbb{N}\}$.*

Now, define the ultrametric

$$d(x, y) = \begin{cases} \frac{1}{2^n} & \text{if} \quad n \text{ is such that } x_k = y_k \text{ and } x_{n+1} \neq y_{n+1} \text{ for each } k \leq n \\ \\ 1 & \text{if} \quad x_1 \neq y_1 \end{cases}$$

Note that (X, d) is complete and separable so it is a Polish ultrametric space. Now, according to the previous definition, we can order the elements of G_n as follows:

$G_1 = \{g_0, g_1\}$, where $g_0 = \{0\} \times \{0, 1\} \times \{0, 1\} \times \ldots$ and $g_1 = \{1\} \times \{0, 1\} \times \{0, 1\} \times \ldots$.

Now, we can write $G_2 = \{g_{00}, g_{01}, g_{10}, g_{11}\}$*, where* $g_{00} = \{0\} \times \{0\} \times \{0,1\} \times \ldots$, $g_{01} = \{0\} \times \{1\} \times \{0,1\} \times \ldots$, $g_{10} = \{1\} \times \{0\} \times \{0,1\} \times \ldots$, $g_{11} = \{1\} \times \{1\} \times \{0,1\} \times \ldots$.

Proposition 11.24. $(G_n, \leq_n)$ *is a well ordered set (that is, it is a linear ordered set and each subset has a minimum).*

Proof. Note that $\leq_n$ is a linear order on G_n for each $n \in \mathbb{N}$, which follows from the fact that the elements in G_n are enumerated according to the lexicographic order.

Let us prove that each nonempty subset of G_n has a minimum for each n.

It is clear, by construction, that any subset of G_1 has a minimum, since we have started by enumerating G_1.

Reasoning by induction, we now suppose that there exists the minimum of each subset of G_n. Next, we show that, given $A \subseteq G_{n+1}$ with $A \neq \emptyset$, there exists the minimum of A in G_{n+1}. Indeed, let $B = \{U_{xn} : U_{x,n+1} \in A\}$. By the induction hypothesis, we have the existence of the minimum of B in G_n. Let $x \in X$ be such that U_{xn} is the minimum of B in G_n (note that, in particular, $U_{x,n+1} \in A$). Let $\{x_i : i \in I\} \subseteq X$, where $I \subseteq \mathbb{N}$ is such that $U_{xn} = \bigcup_{i \in I} U_{x_i,n+1}$. By definition of the order on G_{n+1}, the set $C = \{U_{x_i,n+1} : i \in I\}$ is well ordered in G_{n+1}. Moreover, $C \cap A \neq \emptyset$ (since $U_{x,n+1} \in A \cap C$) and the minimum of C is a lower bound of A (since, otherwise, U_{xn} is not the minimum of B). It follows that the minimum of $A \cap C$ is the minimum of A. □

Next, we recall a theorem which is useful to prove the next results.

Theorem 11.25. *([23, Th. 4.3.9]) A metric space* X *is complete if and only if for every decreasing sequence of nonempty closed subsets of* X, (F_n), *with* $F_{n+1} \subseteq F_n$ *for each* $n \in \mathbb{N}$, *and* $\mathrm{diam}(F_n) \to 0$, *there is a point* $x \in X$ *such that* $x \in \bigcap_{n \in \mathbb{N}} F_n$.

Proposition 11.26. *Let* (x_n) *be a sequence of points of* X *such that* $x_{n+1} \in U_{x_n n}$. *Then there exists* $x \in X$ *such that* $\bigcap_{n \in \mathbb{N}} U_{x_n n} = \{x\}$ *and* $U_{x_n n} = U_{xn}$.

Proof. Let (x_n) be a sequence of points of X such that $x_{n+1} \in U_{x_n n}$ for each $n \in \mathbb{N}$. Then $U_{x_{n+1},n+1} \subseteq U_{x_n n}$ for each $n \in \mathbb{N}$. Since, by Proposition 11.8.1, $\mathrm{diam}(U_{x_n n}) \leq \frac{1}{2^n} \to 0$, then, by Theorem 11.25, there exists $x \in \bigcap_{n \in \mathbb{N}} U_{x_n n}$. Hence, $U_{xn} = U_{x_n n}$. Suppose that there exists $y \in X$ such that $y \in \bigcap_{n \in \mathbb{N}} U_{x_n n}$. Then $d(x,y) \leq \frac{1}{2^n} \to 0$, which means that $y = x$. Consequently, $\{x\} = \bigcap_{n \in \mathbb{N}} U_{x_n n}$. □

Corollary 11.27. *Let $x \in X$. Then $\{x\} = \bigcap_{n\in\mathbb{N}} U_{xn}$.*

Proof. It immediately follows from the previous proposition. □

Lemma 11.28. *Let $A \subseteq X$. Then:*

1. *A has an infimum.*
2. *A has a supremum or $\sup A = \infty$. We say that $\sup A = \infty$ if for each $x \in X$, there exists $y \in A$ such that $y > x$ (that is, A does not have an upper bound).*

Proof.

1. By Proposition 11.24, there exists the minimum of each subset of G_n with the order $\leq_n$, so let $M_n = \min\{U_{an} : a \in A\}$, where the minimum is considered in $(G_n, \leq_n)$. Note that $M_{n+1} \subseteq M_n$ for each $n \in \mathbb{N}$, so it follows, by Proposition 11.26, that there exists $m \in X$ such that $\{m\} = \bigcap_{n\in\mathbb{N}} M_n$ and $U_{mn} = M_n$ for each $n \in \mathbb{N}$. Since $M_n = \min\{U_{an} : a \in A\}$ in G_n, it holds that $M_n \leq_n U_{an}$ for each $a \in A$, which gives us that $m \leq_n a$ for each $a \in A$ and each $n \in \mathbb{N}$ or, equivalently, $m \leq a$ for each $a \in A$, that is, m is a lower bound of A. Suppose that there exists $b \in X$ such that $m < b \leq a$ for each $a \in A$, then there exists $n \in \mathbb{N}$ such that $m <_n b \leq_n a$ for each $a \in A$, but this is a contradiction with the definition of M_n. Consequently, m is the infimum of A.

2. Let $A \subseteq X$ with $A \neq \emptyset$. Consider the set $Y = \{y \in X : y \geq x, \forall x \in A\}$. By the previous item, we have that there exists the infimum of Y or $Y = \emptyset$. Hence, we distinguish two cases:

   (a) Suppose that $Y = \emptyset$, then $\sup A = \infty$.

   (b) Now, suppose that $Y \neq \emptyset$, and let $m = \inf Y$. Then a standard argument can be used to prove that m is the supremum of A.

□

From the previous lemma, it immediately follows the next result.

Remark 11.29. *Let X be a linearly ordered topological space with respect to the order given in Definition 11.21. Then the Dedekind-MacNeille completion of X satisfies:*

1. *$DM(X) = \phi(X) \cup \{X\}$ if $\sup X = \infty$. Note that, indeed, $DM(X)$ is the one-point compactification of $\phi(X)$.*

2. *$DM(X) = \phi(X)$ (or, equivalently, (X, τ_o) is compact) if $\sup X \neq \infty$.*

Proposition 11.30. *$(X, \leq)$ is a totally ordered set with a bottom. If d is totally bounded, then it also has a top.*

Proof. Note that X is totally ordered under $\leq$, which follows from Remark 11.14 and the fact that $\leq_n$ is a total order on G_n for each $n \in \mathbb{N}$.

Given $n \in \mathbb{N}$, let M_n be the minimum of G_n. By Proposition 11.26, there exists $a \in X$ such that $a = \bigcap_{n\in\mathbb{N}} M_n$. It easily follows that a is the bottom of X.

Finally, note that d is totally bounded if and only if G_n is finite for each $n \in \mathbb{N}$. In this case, we can define M_n as the maximum of G_n for each $n \in \mathbb{N}$. By Proposition 11.26, there exists $b \in X$ such that $b = \bigcap_{n\in\mathbb{N}} M_n$. It easily follows that b is the top of X.

□

Proposition 11.31. *Let $x \in X$. Then $U_{xn} = [a, b|$, where $a = \min U_{xn}$, $b = \sup U_{xn}$ and $|$ means $[$ or $]$.*

Proof. Note that there always exists the minimum of U_{xn} for each $x \in X$ and $n \in \mathbb{N}$ by Proposition 11.24. Indeed, that proposition lets us claim that there exists the minimum of U_{xn} in G_m for each $m \in \mathbb{N}$. Let M_m be the minimum of U_{xn} in G_m. Then, by Proposition 11.26, there exists $m \in X$ such that $m = \bigcap_{m\geq n} M_m$. Note that m is the minimum of U_{xn} with respect to the order $\leq$. Moreover, Lemma 11.28 gives us the existence of the supremum of U_{xn} for each $x \in X$ and $n \in \mathbb{N}$. We define $a = \min U_{xn}$ and $b = \sup U_{xn}$ (note that b can be infinite) and now we show that $[a, b[\subseteq U_{xn} \subseteq [a, b]$:

- On the one hand, let $y \in [a, b[$ be such that $y \notin U_{xn}$. Then $y \neq_n x$, so it can happen:

  1. Suppose that $y <_n x$. In this case, y is a lower bound of U_{xn}, which implies that $y \leq \inf U_{xn} = a$. Since $y \neq a$, it holds that $y < a$, which is a contradiction with the fact that $y \in [a, b[$.

2. Suppose that $y >_n x$. In this case, y is an upper bound of U_{xn}, which implies that $y \geq \sup U_{xn} = b$, which is a contradiction with the fact that $y \in [a, b[$.

Therefore, we have that $[a, b[\subseteq U_{xn}$.

- On the other hand, it is clear that $U_{xn} \subseteq [a, b]$.

We conclude that $U_{xn} = [a, b|$. □

Lemma 11.32. *Let $x \in X$.*

1. *If x is a non-left-isolated point such that $x = \min U_{xm}$ for some $m \in \mathbb{N}$, then there exists $n \geq m$ such that U_{xn} does not have an immediately before element in G_n.*

2. *If $x = \max U_{xn}$ for some $n \in \mathbb{N}$, then x is right-isolated.*

Proof. Let $x \in X$.

1. Suppose that, for each $n \geq m$, there exists the element immediately before U_{xn}. Let $U_{x_n n}$ be the set immediately before U_{xn} for each $n \geq m$ and consider $x_i = x_m$ for $i \leq m$. Then, by Proposition 11.26, there exists $z \in X$ such that $\{z\} = \bigcap_{n\in\mathbb{N}} U_{x_n n} \in X$ and $U_{zn} = U_{x_n n}$. Note that $z < x$. What is more, $]z, x[= \emptyset$. Indeed, if there exists $y \in X$ such that $z < y < x$, then there exists $n \geq m$ such that $U_{zn} <_n U_{yn} <_n U_{xn}$, which is a contradiction with the fact that $U_{zn} = U_{x_n n}$ is the element immediately before U_{xn} in G_n. Consequently, x is left-isolated.

2. Let $x = \max U_{xn}$ for some $n \in \mathbb{N}$, and suppose that x is not right-isolated. Then $]x, z[\neq \emptyset$ for each $z \in X$ with $z > x$. Let y be the minimum of the the element immediately after U_{xn} in G_n. It holds that $]x, y[\neq \emptyset$ but it is not possible, since $x = \max U_{xn}$ and y is the minimum of the element immediately after U_{xn}.

□

Proposition 11.33. *If (x_n) is right τ_o-convergent to x, then $x_n \xrightarrow{\tau} x$.*

Proof. Let $x \in X$ and (x_n) be a sequence of points of X such that $x_n \xrightarrow{\tau_o} x$ with $x \leq x_n$. We distinguish two cases depending on whether x is the supremum of U_{xn} or not:

1. Suppose that there exists $n \in \mathbb{N}$ such that $x = \sup U_{xn}$. It follows that $x = \max U_{xn}$. By Lemma 11.32.2, we have that x is right-isolated. Now, let b be the minimum of the element immediately after U_{xn} in G_n. It holds that $]x, b[= \emptyset$. Therefore, there exists $n_0 \in \mathbb{N}$ such that $x_m = x$ for each $m \geq n_0$. Consequently, $x_n \xrightarrow{\tau} x$.

2. Suppose that $x \neq \sup U_{xn}$ for each $n \in \mathbb{N}$ and let $b_n = \sup U_{xn}$. Then $x < b_n$ for each $n \in \mathbb{N}$. Now, let $n \in \mathbb{N}$. Since $x_n \xrightarrow{\tau_o} x$, there exists $n_0 \in \mathbb{N}$ such that $x < x_m < b_n$ for each $m \geq n_0$, which means that $x_m \in U_{xn}$ for each $m \geq n_0$ and, consequently, $x_n \xrightarrow{\tau} x$.

□

Corollary 11.34. *(x_n) is a sequence which right τ_o-converges to x if and only if (x_n) is right τ-convergent to x.*

Proof. It immediately follows from the previous proposition and the fact that $\tau_o \subseteq \tau$ (see Proposition 11.18). □

Lemma 11.35. *Let $A \subseteq X$. The following properties are satisfied:*

1. *Let $a = \inf A$. Then $a = \min A$ or there exists a sequence of points of A which is monotonically right τ_o-convergent to a.*

2. *Let $a = \sup A$. Then $a = \max A$ or there exists a sequence of points of A which is monotonically left τ_o-convergent to a.*

3. *Let $a = \inf A$. Then there exists a sequence of points of A which is right τ_o-convergent to a.*

4. *Let $a = \sup A$. Then there exists a sequence of points of A which is left τ_o-convergent to a.*

Proof. 1. Let $A \subseteq X$ and a be the infimum of A. Suppose that a is not the minimum of A. Then a is not right-isolated, which means, by Proposition 7.11, that there exists a sequence (x_n) of points of X which is monotonically right τ_o-convergent to a. Now, we recursively construct a sequence (a_n) of points of A which is monotonically right τ_o-convergent to a.

Since $a < x_1$, there exists $a_1 \in A$ such that $a_1 < x_1$. Suppose that we have defined $a_n \in A$ with $a_n < \min\{x_n, a_{n-1}\}$ and let us define a_{n+1}. Since $a < a_n$, $a < x_{n+1}$ and $a = \inf A$, there exists $a_{n+1} \in A$ with $a_{n+1} < \min\{a_n, x_{n+1}\}$. Note that $a < a_{n+1} < a_n < x_n$ for each $n \in \mathbb{N}$. It follows that (a_n) is a sequence of points of A which is monotonically right τ_o-convergent to a.

2. The proof is similar to the previous item.

3. Let $a = \inf A$. Then it can happen:

   - Suppose that $a = \min A$. Then $x_n = a$ is a sequence of points of A which is right τ_o-convergent to a.
   - Suppose that $a \neq \min A$. Then, by item 1, there exists a sequence of points of A which is monotonically right τ_o-convergent to a. It is, in particular, a sequence which right τ_o-converges to a.

4. Let $a = \sup A$. Then it can happen:

   - Suppose that $a = \max A$. Then $x_n = a$ is a sequence of points of A which is left τ_o-convergent to a.
   - Suppose that $a \neq \max A$. Then, by item 2, there exists a sequence of points of A which is monotonically left τ_o-convergent to a. It is, in particular, a sequence which left τ_o-converges to a.

□

Proposition 11.36. *Let $f : X \to [0, 1]$ be a monotonically non-decreasing function. Then f is right τ-continuous if and only if f is right τ_o-continuous.*

Proof. It immediately follows from Corollary 11.34. □

11.2.3 Herrlich's construction

In this subsection we see how to define another order from an ultrametric. [63] is a good reference for this topic. Before defining the order, we give a concept that will be essential in the construction made next.

Definition 11.37. *A total order on X is discrete if all points of X are isolated.*

Let (X, d) a separable ultrametric space. Since d is separable, G_n is countable for each $n \in \mathbb{N}$. G_1 can be discretely ordered. Indeed, if G_1 is finite, then we are finished. If G_1 is not finite, let $\prec$ be the usual order over $Z = \{-\frac{1}{n} : n \in \mathbb{N}\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$. The fact that G_1 is countable let us define a bijection $f : G_1 \to Z$. Moreover, $U_{x_{n_1}1} \leq_1 U_{x_{n_2}1}$ if and only if $f(U_{x_{n_1}1}) \preceq f(U_{x_{n_2}1})$. Thus, we have shown that G_1 is discretely ordered. Since G_1 can be decomposed into a countable number of elements in G_2, we can write $g_i = g_{i1} \cup g_{i2} \cup \ldots$ for each $g_i \in G_1$. What is more, we can give a discrete order for the elements of G_2 which are contained in g_i by taking advantage of the order on Z, Indeed, we can define the lexicographic order on G_2. Roughly speaking, according to that order, an element g_{ij} is less than g_{ik} if, following the enumeration, $g_{ij} \leq_2 g_{ik}$. Recursively we define a discrete order on G_n for each $n \in \mathbb{N}$.

Next step is defining a linear order on X such that $\tau_o = \tau$. For this purpose, given $x \in X$, we first consider a point $a \in U_{xn}$ that, once we have constructed the order, it is the minimum of U_{xn}. Since U_{xn} can be decomposed into a countable union of elements in G_{n+1}, we order those elements such that a belongs to the first element of them. For the rest of elements in the subdivision we choose a point that, after constructing the order, will be the minimum of the element where we have considered it. Analogously, we proceed to define the maximum of U_{xn}. We proceed recursively to define the order $\leq$ in X.

Remark 11.38. *$\leq$ is B-compatible.*

Proof. The proof is similar to the one described in Remark 11.22. □

Proposition 11.39. *$(X, \leq)$ is a totally ordered set with a bottom and a top.*

Proof. Indeed, it is clear that $(X, \leq)$ is totally ordered if we take into account the previous construction. Moreover, the mimimum of the first element in G_1 is the minimum of X with the order. The maximum of the last element in G_1 is the maximum of X. □

Proposition 11.40. *Let $x \in X$. Then $U_{xn} = [a, b]$, where $a = \min U_{xn}$ and $b = \max U_{xn}$.*

Proof. It immediately follows from the way we have defined the order on X. □

Corollary 11.41. *Let $x \in X$ and $n \in \mathbb{N}$. If $a, b \in X$ are such that $[a, b] = U_{xn}$, then a is left-isolated and b is right-isolated.*

Proof. Let $x \in X$ and $n \in \mathbb{N}$ and consider U_{yn} and U_{zn} as the previous and the following elements to U_{xn}. By Proposition 11.40, we can write $U_{yn} = [a_1, b_1]$ and $U_{zn} = [a_2, b_2]$. Consequently, $]b_1, a[= \emptyset$ and $]b, a_2[= \emptyset$, which imply that a is left-isolated and b is right-isolated. □

Proposition 11.42. $\tau_o = \tau$.

Proof. According to Proposition 11.18, we have that $\tau_o \subseteq \tau$. Now, given $x \in X$ and $n \in \mathbb{N}$, suppose that U_{yn} and U_{zn} are, respectively, the previous and the following elements to U_{xn}. By Proposition 11.40, we can write $U_{yn} = [a_1, b_1]$ and $U_{xn} = [a, b]$ and $U_{zn} = [a_2, b_2]$. Consequently, $U_{xn} =]b_1, a_2[$, which gives us that $\tau \subseteq \tau_o$. □

11.2.4 Example

In the previous two subsections we saw how to define two possible B-compatible orders from (X, d), where d is an ultrametric. Moreover, recall that, from Chapter 4 onwards, we supposed the topology induced by the non-archimedean quasi-pseudometric given by a fractal structure to be T_0 (see Proposition 3.2 to recall how to characterize this property in terms of the fractal structure). This implies that the supremum pseudometric, d^*, defined from the induced non-archimedean quasi-pseudometric and its conjugate is, indeed, a non-archimedean metric (also called an ultrametric). Hence, from d^*, we can define a linear order on X. Consequently, there is an equivalence between fractal structures and LOTSs according to the theory that has been developed in this chapter previously.

Next, we show an example of order defined by taking into account the Herrlich's construction and the natural fractal structure on $\mathbb{R}$. Thus, we define $\mathbf{\Gamma} = \{\Gamma_n : n \in \mathbb{N}\}$, where $\Gamma_n = \left\{\left[\frac{k}{2^{n-1}}, \frac{k+1}{2^{n-1}}\right] : k \in \mathbb{Z}\right\}$ for each $n \in \mathbb{N}$.

Note that $U^*_{x1} = \{x\}$ for each $x \in \mathbb{Z}$ and $U^*_{x1} =]\lfloor x \rfloor, \lfloor x \rfloor + 1[$ for each $x \in \mathbb{R} \setminus \mathbb{Z}$, where $\lfloor x \rfloor$ is the floor function, that is, the largest integer not greater than x.

Now, we define the bijection $f : G_1 \to Z = \{-\frac{1}{n} : n \in \mathbb{N}\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$ such that

$$f(U^*_{x1}) = \begin{cases} -\frac{1}{2x+1} & \text{if} \quad x \in \mathbb{N} \cup \{0\} \\ -\frac{1}{2(\lfloor x \rfloor+1)} & \text{if} \quad x \in [0, \infty[\setminus \mathbb{N} \\ -\frac{1}{2x} & \text{if} \quad x \in \mathbb{Z}^- \\ -\frac{1}{2\lfloor x \rfloor+1} & \text{if} \quad x \in]-\infty, 0[\setminus \mathbb{Z} \end{cases}$$

The previous bijection assigns the elements in Z to each U^*_{x1}, as Figure 11.1 shows.

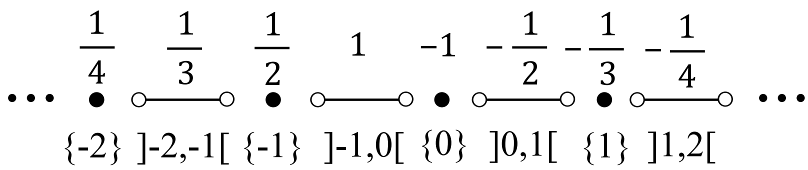


Figure 11.1: Bijection between G_1 and Z

Now, if we consider the usual order on Z, it induces an order on G_1. Moreover, observe that each $g_i \in G_1$ is decomposed into a finite number of elements in G_2. For example, note that $U^*_{x1} = U^*_{x2}$ for each $x \in \mathbb{Z}$, while U^*_{x1} gives us the collection $\left\{\left]\frac{\lfloor x \rfloor}{2}, \lfloor x \rfloor + \frac{1}{2}\right[, \{\lfloor x \rfloor + \frac{1}{2}\}, \left]\lfloor x \rfloor + \frac{1}{2}, \frac{\lfloor x \rfloor+1}{2}\right[\right\}$ in G_2 otherwise. Since that collection is finite, it is discretely ordered with the usual order and, hence, G_2 is ordered with the lexicographic order as explained previously. Therefore, if we list the elements of each G_n according to the order, we have that:

$$G_1 = \{\{0\},]0,1[, \{1\}, \ldots, \{-1\},]-1,0[\}$$

$$G_2 = \left\{\{0\}, \left]0, \frac{1}{2}\right[, \left\{\frac{1}{2}\right\}, \left]\frac{1}{2}, 1\right[, \{1\}, \ldots, \{-1\}, \left]-1, -\frac{1}{2}\right[, \left\{-\frac{1}{2}\right\}, \left]-\frac{1}{2}, 0\right[\right\}$$

$$\vdots$$

From that order, we can define a linear order on the completion of the space, $\widetilde{\mathbb{R}}$, whose topology we denote by τ_o. Note that $0 \in U^*_{0n}$ and $\left(\left]-\frac{1}{2^{n-1}}, 0\right[\right)_{n\in\mathbb{N}}$ are, respectively, the minimum and the maximum of it.

According to Proposition 11.42, it follows that $\tau_o = \tau_{\widetilde{d^*}}$ in $\widetilde{\mathbb{R}}$. What is more, we can restrict the topology given by the ultrametric in the completion to the original space and it holds that that restriction gives us the topology of the ultrametric in $\mathbb{R}$. Indeed, it is true due to Proposition 3.16.10. Figure 11.2 shows the linear order induced on $\mathbb{R}$ by the order we have defined on G_n for each $n \in \mathbb{N}$.

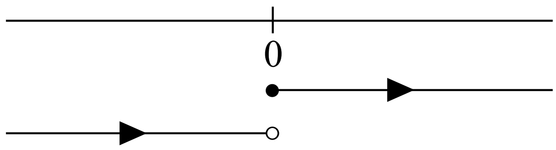


Figure 11.2: Linear order induced by the fractal structure $\mathbf{\Gamma}$ in $\mathbb{R}$

Note that 0 is the minimum of X with respect to the order and that points which are located on the left of this point in $\mathbb{R}$ are greater than those which are on the right (if we consider the usual order).

Once we have defined the order according to Herrlich's construction and the natural fractal structure on $\mathbb{R}$, we consider the cdf of a probability measure defined on $\mathbb{R}$ with respect to the usual order. Let us denote that cdf by F. Then the cdf given by the new order that we have defined on $\mathbb{R}$ (from the fractal structure), that we can denote by F_o, is defined by

$$F_o(x) = \begin{cases} F(x) - F_{-}(0) & \text{if} \quad 0 \le x < \infty \\ F(x) + 1 - F_{-}(0) & \text{if} \quad -\infty < x \le 0 \end{cases}$$

On the other hand, in order to define the pseudo-inverse according to the order, note that $DM(\mathbb{R}) = \phi(\mathbb{R}) \cup \{[0,\infty[\}$. Now, if G denotes the pseudo-inverse of F, then the pseudo-inverse of the cdf F_o, given by the new order, is defined by

$$G_o(r) = \begin{cases} G(F(0) + r) & \text{if} \quad r < 1 - F(0) \\ \{[0,\infty[\} & \text{if} \quad r = 1 - F(0) \\ G(r - (1 - F(0))) & \text{if} \quad r > 1 - F(0) \end{cases}$$

Conclusions

This chapter is dedicated to compiling the main results of this work, together with some comments on possible future research lines that arise from the theory and applications that have been developed throughout it.

First of all, we will collect the conclusions which are related to the first part of the thesis, whose main goal is the definition of a probability measure with the help of a fractal structure:

- In Chapter 3 we study the completion of a space with a fractal structure, as it is the starting point to be able to construct probability measures on the original space.

- Before studying the completion of a space with a fractal structure and its induced structures, we characterize the properties T_0, T_1 y T_2 of the space in terms of a fractal structure defined on it.

- Moreover, we describe the completion of a space with a fractal structure by using an inverse limit. In fact, that completion, which always exists, is the bicompletion of the non-archimedean quasi-pseudometric induced by the fractal structure. What is more, the completion of a space with a fractal structure is unique up to fractal isomorphism.

- There are two different ways to define a probability measure on the completion of the space, both exposed in Chapter 4: the first one is based on a pre-measure defined on the collection of balls with respect to the ultrametric induced by the fractal structure and, a second one, which starts from a pre-measure defined on the elements of the fractal structure according to its levels. In the last case, we assume that the fractal structure is tiling. In both cases, it can be proven that the generated probability measure is unique.

- Once we have defined a probability measure on the completion of the space, it can be proven that its restriction to the original space is a measure but there is no guarantee that the measure of the space is 1. Hence, in Chapter 5 we explore conditions to ensure that that restriction is, indeed, a probability measure on the original space. By following this research line, we give some results that let us ensure that we get a probability measure on the original space, some of them are quite handy, since they deal with conditions on the original space and the fractal structure on it, and not with the completion or the structures induced by it.

- The theory that has been developed along the first part of the thesis results in some applications: a new way to generate samples of a distribution, a new estimation method and a goodness-of-fit test, both based in the recursive nature of the fractal structure. All of them are developed in Chapter 6 together with some examples.

- We can generate samples of a given distribution by chosing a certain level of the fractal structure. We can generate, not only samples of a distribution in the real line, but also from some multivariate distributions.

- The new estimation method is based on the idea of maximizing the probability that in each element of a certain level of the fractal structure there are as many elements as there are actually. The estimations made for samples of the standard normal distribution shows that this method offers better results when we consider a higher maximum level of the fractal structure and, also, when the size of the data sample is bigger. Moreover, in the presence of outliers in the sample, the new estimation method is more robust than the maximum likelihood one.

- The last application is a goodness-of-fit test, based on the well-known χ^2 Pearson's test, to check if a random sample comes from a certain distribution or not. However, when trying to define the statistic of the test, although it seems to works fine, we need to guarantee the independence of the random variables we define in the sums that we use to create the statistic. That is, precisely, one of the open problems that arise from this part of the work and that will be the main goal in future research works.

The conclusions of the second part of the work, whose main aim is the elaboration of a theory of a cumulative distribution function (cdf) in a linearly ordered topological space (LOTS), are the following:

- Given a probability measure on a separable linearly ordered topological space, X, it is possible to define a function, the cumulative distribution function (cdf) of the measure, which is monotonically non-decreasing, right continuous with respect to the order topology and which satisfies that $\sup F(X) = 1$ and, if there does not exist the minimum of X, then $\inf F(X) = 0$. These properties are quite similar to those known for a cdf in the classical case. Moreover, the uniqueness of a probability measure with respect to its cdf holds.

- From a cdf F, we define a new function, that we denote by F_-, involving the probability measure defined on the space, which plays a similar role to that played by $\lim_{x\to a^-} F(x)$ in the classical case. Indeed, F and F_- let us calculate the measure of each interval in a separable LOTS.

- F and F_- let us calculate the measure of each interval in a LOTS. What is more, if $F(x) = F_-(x)$ for each $x \in X$, then F is continuous with respect to the order topology, but the converse is not true, contrary to what happens in the classical case where the continuity of a cdf is characterized in terms of the null measure of all points of the space.

- It does make sense to define the pseudo-inverse of a cdf, but it has an important limitation: that function is not defined for each number in $[0, 1]$, since the existence of the infimum and supremum is not guaranteed for each subset of a LOTS. For example, in case that X is compact, the pseudo-inverse is defined on the unit interval. The pseudo-inverse let us generate samples of a given distribution.

- Indeed, after presenting the results of Chapter 7, we can conclude that the theory of distribution functions in the real line is a particular case of the one that has been developed in this work.

- Since the pseudo-inverse is not always defined for each $r \in [0, 1]$, we look for a context which let us define it on $[0, 1]$ and such that the theory that has been previously developed makes sense with the new definition. The ideal environment to make this is the Dedekind-MacNeille completion, also known as the completion by cuts, of the original space. For a separable LOTS that completion is, indeed, a compactification. The first step to treat with this new context in our theory is to extend the cdf to the completion, study the properties of that extension and relate them to those which are known for the case in which the cdf is defined on the original space. That is the main aim of Chapter 8, where we show that the pseudo-inverse is naturally defined from $[0,1]$ to the Dedekind-Macneille completion. Hence, we can generate samples of a distribution defined on each separable LOTS.

- Once we have studied F, F_{-} and the pseudo-inverse of a given probability measure, we can ask ourselves if, given a function satisfying the properties of a cdf, there exists a probability measure whose cdf is that function. That is why in Chapter 9 we look for conditions to ensure that there exists a one-to-one relationship between probability measures and cdfs on a LOTS. What is more, we prove that there is an equivalence between F_{-} and a probability measure and it also happens with respect to the pseudo-inverse.

- The theory that has been developed in Chapters 7, 8 and 9 has some applications, which are described in detail in Chapter 10: first, under some assumptions, each cdf can de decomposed as the convex sum of a step cdf and one for what its measure is zero in each point and, secondly, we can give a goodness-of-fit test to check if a random sample comes from a certain distribution.

- Finally, in Chapter 11 we relate a space with a fractal structure to a linearly ordered topological space. More precisely, given a second countable LOTS, we can define a fractal structure from the set of left and right-isolated points such that the topology of the non-archimedean quasi-pseudometric coincides with the order topology. Analogously, from a space with a fractal structure, it is possible to define an order which is compatible with the balls of the ultrametric induced by the fractal structure. We show two examples of linear orders which are defined from an ultrametric space (and, hence, can be adapted to the case of a space with a fractal structure). The equivalence between fractal structures and LOTS lets us use the theory and applications that have been developed in each part of this thesis regardless of the context where we are working.

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