mathematics

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Relative Gorenstein Dimensions over Triangular Matrix Rings

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Abstract: Let A and B be rings, U a (B, A)-bimodule, and $T = \begin{pmatrix} A & 0 \\ U & B \end{pmatrix}$ the triangular matrix ring. In this paper, several notions in relative Gorenstein algebra over a triangular matrix ring are investigated. We first study how to construct w-tilting (tilting, semidualizing) over T using the corresponding ones over A and B. We show that when U is relative (weakly) compatible, we are able to describe the structure of G_C-projective modules over T. As an application, we study when a morphism in T-Mod is a special $G_CP(T)$-precover and when the class $G_CP(T)$ is a special precovering class. In addition, we study the relative global dimension of T. In some cases, we show that it can be computed from the relative global dimensions of A and B. We end the paper with a counterexample to a result that characterizes when a T-module has a finite projective dimension.

Keywords: triangular matrix ring; weakly Wakamatsu tilting modules; relative Gorenstein dimensions

Citation: Bennis, D.; El Maaouy, R.; García Rozas, J.R.; Oyonarte, L. Relative Gorenstein Dimensions over Triangular Matrix Rings. *Mathematics* **2021**, *9*, 2676. https://doi.org/10.3390/math9212676

Academic Editor: Xiao-Wu Chen

Received: 18 September 2021
Accepted: 18 October 2021
Published: 22 October 2021

1. Introduction

Let A and B be rings and U be a (B, A)-bimodule. The ring $T = \begin{pmatrix} A & 0 \\ U & B \end{pmatrix}$ is known as the formal triangular matrix ring with usual matrix addition and multiplication. Such rings play an important role in the representation theory of algebras. The modules over such rings can be described in a very concrete fashion. Therefore, formal triangular matrix rings and modules over them have proven to be a rich source of examples and counterexamples. Some important Gorenstein notions over formal triangular matrix rings have been studied by many authors (see [1–3]). For example, Zhang [1] introduced compatible bimodules and explicitly described the Gorenstein projective modules over triangular matrix Artin algebra. Enochs, Izurdiaga, and Torrecillas [2] characterized when a left module over a triangular matrix ring is Gorenstein projective or Gorenstein injective under the "Gorenstein regular" condition. Under the same condition, Zhu, Liu, and Wang [3] investigated Gorenstein homological dimensions of modules over triangular matrix rings. Mao [4] studied Gorenstein flat modules over T (without the "Gorenstein regular" condition) and gave an estimate of the weak global Gorenstein dimension of T.

Semidualizing modules were independently studied (under different names) by Foxby [5], Golod [6], and Vasconcelos [7] over a commutative Noetherian ring. Golod used these modules to study the G_C-dimension for finitely generated modules. Motivated (in part) by Enochs and Jenda's extensions of the classical G-dimension given in [8], Holm and Jørgensen extended in [9] this notion to arbitrary modules. After that, several generalizations of semidualizing and the G_C-dimension have been made by several authors [10–12].

As the authors mentioned in [13], to study the Gorenstein projective modules and dimension relative to a semidualizing (R, S)-bimodule C, the condition $\mathrm{End}_S(C) \cong R$ seems to be too restrictive and in some cases unnecessary. Therefore, the authors introduced weakly Wakamatsu tilting as a weak notion of semidualizing, which made the theory of relative Gorenstein homological algebra wider and less restrictive, but still consistent. Weakly Wakamatsu tilting modules were the subject of many publications that showed how important these modules could become in developing the theory of relative (Gorenstein) homological algebra [13–15].

The main objective of the present paper is to study relative Gorenstein homological notions (w-tilting, relative Gorenstein projective modules, relative Gorenstein projective dimensions, and the relative global projective dimension) over triangular matrix rings.

This article is organized as follows:

In Section 2, we give some preliminary results.

In Section 3, we study how to construct w-tilting (tilting, semidualizing) over T using w-tilting (tilting, semidualizing) over A and B under the condition that U is relative (weakly) compatible. We introduce (weakly) C-compatible (B, A)-bimodules for a T-module C (Definition 4). Given two w-tilting modules ${}_AC_1$ and ${}_BC_2$, we prove in Proposition 2 that $C = \begin{pmatrix} C_1 \\ (U \otimes_A C_1) \oplus C_2 \end{pmatrix}$ is a w-tilting T-module when U is weakly C-compatible.

In Section 4, we first describe relative Gorenstein projective modules over T. Let $C = \begin{pmatrix} C_1 \\ (U \otimes_A C_1) \oplus C_2 \end{pmatrix}$ be a T-module. We prove in Theorem 1 that if U is C-compatible, then a T-module $M = \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}_{\varphi^M}$ is G_C-projective if and only if M_1 is a G_{C_1}-projective A-module, Cokerφ^M is a G_{C_2}-projective B-module, and $\varphi^M : U \otimes_A M_1 \to M_2$ is injective. As an application, we prove the converse of Proposition 2 and refine in the relative setting (Proposition 4), a result of when T is left (strongly) CM-free due to Enochs, Izurdiaga, and Torrecillas in [2]. Furthermore, when C is w-tilting, we characterize when a T-morphism is a special precover (see Proposition 5). Then, in Theorem 2, we prove that the class of G_C-projective T-modules is a special precovering if and only if so are the classes of G_{C_1}-projective A-modules and G_{C_2}-projective B-modules, respectively.

Finally, in Section 5, we give an estimate of the G_C-projective dimension of a left T-module and the left G_C-projective global dimension of T. First, it is proven that, given a T-module $M = \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}_{\varphi^M}$, if $C = \mathbf{p}(C_1, C_2) := \begin{pmatrix} C_1 \\ (U \otimes_A C_1) \oplus C_2 \end{pmatrix}$ is w-tilting, U is C-compatible, and:

$$SG_{C_2} - PD(B) := sup\{G_{C_2} - pd(U \otimes_A G) \mid G \in G_{C_1}P(A)\} < \infty,$$

then:

$$\begin{gathered} max\{G_{C_1} - pd(M_1), (G_{C_2} - pd(M_2)) - (SG_{C_2} - PD(B))\} \\ \leq G_C - pd(M) \leq \\ max\{(G_{C_1} - pd(M_1)) + (SG_{C_2} - PD(B)) + 1, G_{C_2} - pd(M_2)\}. \end{gathered}$$

As an application, we prove that, if $C = \mathbf{p}(C_1, C_2)$ is w-tilting and U is C-compatible, then:

$$\begin{gathered} max\{G_{C_1} - PD(A), G_{C_2} - PD(B)\} \\ \leq G_C - PD(T) \leq \\ max\{G_{C_1} - PD(A) + SG_{C_2} - PD(B) + 1, G_{C_2} - PD(B)\}. \end{gathered}$$

Some cases in which this estimation becomes an exact formula are also given.

The authors in [16] established a relationship between the projective dimension of modules over T and modules over A and B. Given an integer $n \geq 0$ and $M = \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}_{\varphi^M}$ a T-module, they proved that $pd_T(M) \leq n$ if and only if $pd_A(M_1) \leq n$, $pd_B(\overline{M}_2) \leq n$

and the map related to the n-th syzygy of M is injective. We end the paper by giving a counterexample to this result (Example 4).

2. Preliminaries

Throughout this paper, all rings are associative (not necessarily commutative) with identity, and all modules are, unless otherwise specified, unitary left modules. For a ring R, we use $\mathrm{Proj}(R)$ (resp., $\mathrm{Inj}(R)$) to denote the class of all projective (resp., injective) R-modules. The category of all left R-modules is denoted by R-Mod. For an R-module C, we use $\mathrm{Add}_R(C)$ to denote the class of all R-modules that are isomorphic to direct summands of direct sums of copies of C, and $\mathrm{Prod}_R(C)$ denotes the class of all R-modules that are isomorphic to direct summands of direct products of copies of C.

Given a class of modules $\mathcal{F}$ (which are always considered closed under isomorphisms), an $\mathcal{F}$-precover of $M \in R$-Mod is a morphism $\varphi : F \to M$ ($F \in \mathcal{F}$) such that $\mathrm{Hom}_R(F', \varphi)$ is surjective for every $F' \in \mathcal{F}$. If, in addition, any solution of the equation $\mathrm{Hom}_R(F, \varphi)(g) = \varphi$ is an automorphism of F, then φ is said to be an $\mathcal{F}$-cover. The $\mathcal{F}$-precover φ is said to be special if it is surjective and $\mathrm{Ext}^1(F, \ker\varphi) = 0$ for every $F \in \mathcal{F}$. The class $\mathcal{F}$ is said to be a special (pre)covering if every module has a special $\mathcal{F}$-(pre)cover.

Given the class $\mathcal{F}$, the class of all modules N such that $\mathrm{Ext}^i_R(F, N) = 0$ for every $F \in \mathcal{F}$ is denoted by $\mathcal{F}^{\perp_i}$ (similarly, ${}^{\perp_i}\mathcal{F} = \{N;\ \mathrm{Ext}^i_R(N, F) = 0\ \forall F \in \mathcal{F}\}$). The right and left orthogonal classes $\mathcal{F}^{\perp}$ and ${}^{\perp}\mathcal{F}$ are defined as follows:

$$\mathcal{F}^{\perp} = \cap_{i\geq 1}\mathcal{F}^{\perp_i} \text{ and } {}^{\perp}\mathcal{F} = \cap_{i\geq 1}\, {}^{\perp_i}\mathcal{F}$$

It is immediate to see that if C is any module, then $\mathrm{Add}_R(C)^{\perp} = \{C\}^{\perp}$ and ${}^{\perp}\mathrm{Prod}_R(C) = {}^{\perp}\{C\}$.

Given a class of R-modules $\mathcal{F}$, an exact sequence of R-modules:

$$\cdots \to X^1 \to X^0 \to X_0 \to X_1 \to \cdots$$

is called $\mathrm{Hom}_R(-, \mathcal{F})$-exact (resp., $\mathrm{Hom}_R(\mathcal{F}, -)$-exact) if the functor $\mathrm{Hom}_R(-, F)$ (resp., $\mathrm{Hom}_R(F, -)$) leaves the sequence exact whenever $F \in \mathcal{F}$. If $\mathcal{F} = \{F\}$, we simply say $\mathrm{Hom}_R(-, F)$-exact. Similarly, we can define $\mathcal{F}\otimes_R$-exact sequences when $\mathcal{F}$ is a class of right R-modules.

We now recall some concepts needed throughout the paper.

Definition 1.

1. ([17], Definition 2.1) *A semidualizing bimodule is an (R, S)-bimodule C satisfying the following properties:*
   (a) *${}_RC$ and C_S both admit a degreewise finite projective resolution in the corresponding module categories (R-Mod and Mod-S);*
   (b) $\mathrm{Ext}^{\geq 1}_R(C, C) = \mathrm{Ext}^{\geq 1}_S(C, C) = 0$;
   (c) *The natural homothety maps $R \overset{{}_R\gamma}{\to} \mathrm{Hom}_S(C, C)$ and $S \overset{\gamma_S}{\to} \mathrm{Hom}_R(C, C)$ both are ring isomorphisms.*
2. ([18], Section 3) *A Wakamatsu tilting module, simply tilting, is an R-module C satisfying the following properties:*
   (a) *${}_RC$ admits a degreewise finite projective resolution;*
   (b) $\mathrm{Ext}^{\geq 1}_R(C, C) = 0$;
   (c) *There exists a $\mathrm{Hom}_R(-, C)$-exact exact sequence of R-modules:*

   $$\mathbf{X} =\ 0 \to R \to C^0 \to C^1 \to \cdots$$

   where $C^i \in \mathrm{add}_R(C)$ for every $i \in \mathbb{N}$.

It was proven in ([18], Corollary 3.2), that an (R, S)-bimodule C is semidualizing if and only if $_RC$ is tilting with $S = \mathrm{End}_R(C)$. Therefore, the following notion, which is crucial in this paper, generalizes both concepts.

Definition 2 ([13], Definition 2.1)**.** *An R-module C is weakly Wakamatsu tilting (w-tilting for short) if it has the following two properties:*

1. $\mathrm{Ext}_R^{i\geq 1}(C, C^{(I)}) = 0$ *for every set* I;
2. *There exists a* $\mathrm{Hom}_R(-, \mathrm{Add}_R(C))$*-exact exact sequence of R-modules:*

$$\mathbf{X} = \ 0 \to R \to A^0 \to A^1 \to \cdots$$

where $A^i \in \mathrm{Add}_R(C)$ *for every* $i \in \mathbb{N}$;

If C satisfies 1 *but perhaps not* 2*, then C will be said to be* Σ*-self-orthogonal.*

Definition 3 ([13], Definition 2.2)**.** *Given any* $C \in R$*-Mod, an R-module M is said to be* G_C*-projective if there exists a* $\mathrm{Hom}_R(-, \mathrm{Add}_R(C))$*-exact exact sequence of R-modules:*

$$\mathbf{X} = \ \cdots \to P_1 \to P_0 \to A^0 \to A^1 \to \cdots$$

where the P_i'*s are all projective,* $A^i \in \mathrm{Add}_R(C)$ *for every* $i \in \mathbb{N}$, $M \cong \mathrm{Im}(P_0 \to A^0)$.

We use $G_CP(R)$ *to denote the class of all* G_C*-projective R-modules.*

It is immediate from the definitions that w-tilting modules can be characterized as follows.

Lemma 1. *An R-module C is w-tilting if and only if both C and R are* G_C*-projective modules.*

Now, we recall some facts about triangular matrix rings. Let A and B be rings and U a (B, A)-bimodule. We shall denote by $T = \begin{pmatrix} A & 0 \\ U & B \end{pmatrix}$ the generalized triangular matrix ring. By [19], Theorem 1.5, the category T-Mod of left T-modules is equivalent to the category $_T\Omega$ whose objects are triples $M = \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}_{\varphi^M}$, where $M_1 \in A$-Mod, $M_2 \in B$-Mod, and $\varphi^M : U \otimes_A M_1 \to M_2$ is a B-morphism and whose morphisms from $\begin{pmatrix} M_1 \\ M_2 \end{pmatrix}_{\varphi^M}$ to $\begin{pmatrix} N_1 \\ N_2 \end{pmatrix}_{\varphi^N}$ are pairs $\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$ such that $f_1 \in \mathrm{Hom}_A(M_1, N_1)$, $f_2 \in \mathrm{Hom}_B(M_2, N_2)$ satisfying that the following diagram is commutative.

$$\begin{array}{ccc} U \otimes_A M_1 & \xrightarrow{\varphi^M} & M_2 \\ \downarrow{\scriptstyle 1_U \otimes f_1} & & \downarrow{\scriptstyle f_2} \\ U \otimes_A N_1 & \xrightarrow{\varphi^N} & N_2 \end{array}$$

Since we have the natural isomorphism:

$$\mathrm{Hom}_B(U \otimes_A M_1, M_2) \cong \mathrm{Hom}_A(M_1, \mathrm{Hom}_B(U, M_2)),$$

there is an alternative way of defining T-modules and T-homomorphisms in terms of maps $\widetilde{\varphi^M} : M_1 \to \mathrm{Hom}_B(U, M_2)$ given by $\widetilde{\varphi^M}(x)(u) = \varphi^M(u \otimes x)$ for each $u \in U$ and $x \in M_1$.

Analogously, the category Mod-T of right T-modules is equivalent to the category Ω_T whose objects are triples $M = (M_1, M_2)_{\varphi^M}$, where $M_1 \in$ Mod-A, $M_2 \in$ Mod-B, and $\varphi^M : M_2 \otimes_B U \to M_1$ is an A-morphism and whose morphisms from $(M_1, M_2)_{\phi^M}$ to

$(N_1, N_2)_{\phi^N}$ are pairs (f_1, f_2) such that $f_1 \in \mathrm{Hom}_A(M_1, N_1)$, $f_2 \in \mathrm{Hom}_B(M_2, N_2)$ satisfying that the following diagram:

$$\begin{array}{ccc} M_2 \otimes_B U & \xrightarrow{\phi^M} & M_1 \\ {\scriptstyle f_2 \otimes 1_U}\downarrow & & \downarrow{\scriptstyle f_1} \\ M_2 \otimes_B U & \xrightarrow{\phi^N} & N_1 \end{array}$$

is commutative.

In the rest of the paper, we shall identify T-Mod (resp. Mod-T) with ${}_T\Omega$ (resp. Ω_T). Consequently, through the paper, a left (resp. right) T-module will be a triple $M = \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}_{\varphi^M}$ (resp. $M = (M_1, M_2)_{\phi^M}$), and whenever there is no possible confusion, we shall omit the morphisms φ^M and ϕ^M. For example, ${}_TT$ is identified with $\begin{pmatrix} A \\ U \oplus B \end{pmatrix}$ and T_T is identified with $(A \oplus U, B)$.

A sequence of left T-modules $0 \to \begin{pmatrix} M_1 \\ M_2 \end{pmatrix} \to \begin{pmatrix} M_1' \\ M_2' \end{pmatrix} \to \begin{pmatrix} M_1'' \\ M_2'' \end{pmatrix} \to 0$ is exact if and only if both sequences $0 \to M_1 \to M_1' \to M_1'' \to 0$ and $0 \to M_2 \to M_2' \to M_2'' \to 0$ are exact.

Throughout this paper, $T = \begin{pmatrix} A & 0 \\ U & B \end{pmatrix}$ is a generalized triangular matrix ring. Given a T-module $M = \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}_{\varphi^M}$, the B-module Cokerφ^M is denoted as $\overline{M}_2$ and the A-module Ker$\widetilde{\varphi^M}$ as $\underline{M_1}$. A T-module $N = \begin{pmatrix} N_1 \\ N_2 \end{pmatrix}_{\varphi^N}$ is a submodule of M if N_1 is a submodule of M_1, N_2 is a submodule of M_2, and $\varphi^M|_{U \otimes_A N_1} = \varphi^N$.

As an interesting and special case of triangular matrix rings, we recall that the T_2-extension of a ring R is given by:

$$T(R) = \begin{pmatrix} R & 0 \\ R & R \end{pmatrix}$$

and the modules over $T(R)$ are triples $M = \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}_{\varphi^M}$ where M_1 and M_2 are R-modules and $\varphi^M : M_1 \to M_2$ is an R-homomorphism.

There are some pairs of adjoint functors $(\mathbf{p}, \mathbf{q})$, $(\mathbf{q}, \mathbf{h})$ and $(\mathbf{s}, \mathbf{r})$ between the category T-Mod and the product category A-Mod $\times$ B-Mod, which are defined as follows:

1. $\mathbf{p}$: A-Mod $\times$ B-Mod$\to$ T-Mod is defined as follows: for each object (M_1, M_2) of A-Mod$\times$$B$-Mod, let $\mathbf{p}(M_1, M_2) = \begin{pmatrix} M_1 \\ (U \otimes_A M_1) \oplus M_2 \end{pmatrix}$ with the obvious map, and for any morphism (f_1, f_2) in A-Mod$\times$$B$-Mod, let $\mathbf{p}(f_1, f_2) = \begin{pmatrix} f_1 \\ (1_U \otimes_A f_1) \oplus f_2 \end{pmatrix}$;
2. $\mathbf{q}$: T-Mod$\to$ A-Mod $\times$$B$-Mod is defined, for each left T-module $\begin{pmatrix} M_1 \\ M_2 \end{pmatrix}$ as $\mathbf{q}(\begin{pmatrix} M_1 \\ M_2 \end{pmatrix}) = (M_1, M_2)$ and for each morphism $\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$ in T-Mod as $\mathbf{q}(\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}) = (f_1, f_2)$;
3. $\mathbf{h}$: A-Mod $\times$ B-Mod$\to$ T-Mod is defined as follows: for each object (M_1, M_2) of A-Mod$\times$$B$-Mod, let $\mathbf{h}(M_1, M_2) = \begin{pmatrix} M_1 \oplus \mathrm{Hom}_B(U, M_2) \\ M_2 \end{pmatrix}$ with the obvious map, and for any morphism (f_1, f_2) in A-Mod$\times$$B$-Mod, let $\mathbf{h}(f_1, f_2) = \begin{pmatrix} f_1 \oplus \mathrm{Hom}_B(U, f_2) \\ f_2 \end{pmatrix}$;

4. $\mathbf{r}: A\text{-Mod} \times B\text{-Mod} \to T\text{-Mod}$ is defined as follows: for each object (M_1, M_2) of $A\text{-Mod}\times B\text{-Mod}$, let $\mathbf{r}(M_1, M_2) = \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}$ with the zero map, and for any morphism (f_1, f_2) in $A\text{-Mod}\times B\text{-Mod}$, let $\mathbf{r}(f_1, f_2) = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$;
5. $\mathbf{s}: T\text{-Mod} \to A\text{-Mod} \times B\text{-Mod}$ is defined, for each left T-module $\begin{pmatrix} M_1 \\ M_2 \end{pmatrix}$ as $\mathbf{s}(\begin{pmatrix} M_1 \\ M_2 \end{pmatrix}) = (M_1, \overline{M}_2)$ and for each morphism $\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$ in T-Mod as $\mathbf{s}(\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}) = (f_1, \overline{f}_2)$, where $\overline{f}_2$ is the induced map.

It is easy to see that $\mathbf{q}$ is exact. In particular, $\mathbf{p}$ preserves projective objects and $\mathbf{h}$ preserves injective objects. Note that the pairs of adjoint functors $(\mathbf{p}, \mathbf{q})$ and $(\mathbf{q}, \mathbf{h})$ were defined in [2]. In general, the three pairs of adjoint functors defined above can be found in [20].

For a future reference, we list these adjointness isomorphisms:

$$\mathrm{Hom}_T(\begin{pmatrix} M_1 \\ (U \otimes_A M_1) \oplus M_2 \end{pmatrix}, N) \cong \mathrm{Hom}_A(M_1, N_1) \oplus \mathrm{Hom}_B(M_2, N_2).$$

$$\mathrm{Hom}_T(N, \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}_0) \cong \mathrm{Hom}_A(N_1, M_1) \oplus \mathrm{Hom}_B(\overline{N}_2, M_2).$$

$$\mathrm{Hom}_T(M, \begin{pmatrix} N_1 \oplus \mathrm{Hom}_B(U, N_2) \\ N_2 \end{pmatrix}) \cong \mathrm{Hom}_A(M_1, N_1) \oplus \mathrm{Hom}_B(M_2, N_2).$$

Now, we recall the characterizations of projective, injective, and finitely generated T-modules.

Lemma 2. *Let* $M = \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}_{\varphi^M}$ *be a T-module.*

(1) ([21], Theorem 3.1) *M is projective if and only if M_1 is projective in A-Mod, $\overline{M}_2 = \mathrm{Coker}\varphi^M$ is projective in B-Mod, and φ^M is injective.*
(2) ([22], Proposition 5.1) *M is injective if and only if $\underline{M_1} = \mathrm{Ker}\widetilde{\varphi^M}$ is injective in A-Mod, M_2 is injective in B-Mod, and $\widetilde{\varphi^M}$ is surjective.*
(3) ([23]) *M is finitely generated if and only if M_1 and $\overline{M}_2$ are finitely generated.*

The following Lemma improves [24], Lemma 3.2.

Lemma 3. *Let* $M = \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}_{\varphi^M}$ *and* $N = \begin{pmatrix} N_1 \\ N_2 \end{pmatrix}_{\varphi^N}$ *be two T-modules and $n \geq 1$ be an integer number. Then, we have the following natural isomorphisms:*

1. *If* $\mathrm{Tor}^A_{1\leq i\leq n}(U, M_1) = 0$, *then* $\mathrm{Ext}^n_T(\begin{pmatrix} M_1 \\ U \otimes_A M_1 \end{pmatrix}, N) \cong \mathrm{Ext}^n_A(M_1, N_1)$;
2. $\mathrm{Ext}^n_T(\begin{pmatrix} 0 \\ M_2 \end{pmatrix}, N) \cong \mathrm{Ext}^n_B(M_2, N_2)$;
3. $\mathrm{Ext}^n_T(M, \begin{pmatrix} N_1 \\ 0 \end{pmatrix}) \cong \mathrm{Ext}^n_A(M_1, N_1)$;
4. *If* $\mathrm{Ext}^{1\leq i\leq n}_B(U, N_2) = 0$, *then* $\mathrm{Ext}^n_T(M, \begin{pmatrix} \mathrm{Hom}_B(U, N_2) \\ N_2 \end{pmatrix}) \cong \mathrm{Ext}^n_B(M_2, N_2)$.

Proof. We prove only 1, since 2 is similar and 3 and 4 are dual. Assume that $\mathrm{Tor}^A_{1\le i\le n}(U, M_1) = 0$, and consider an exact sequence of A-modules:

$$0 \to K_1 \to P_1 \to M_1 \to 0$$

where P_1 is projective. Therefore, there exists an exact sequence of T-modules:

$$0 \to \begin{pmatrix} K_1 \\ U \otimes_A K_1 \end{pmatrix} \to \begin{pmatrix} P_1 \\ U \otimes_A P_1 \end{pmatrix} \to \begin{pmatrix} M_1 \\ U \otimes_A M_1 \end{pmatrix} \to 0$$

where $\begin{pmatrix} P_1 \\ U \otimes_A P_1 \end{pmatrix}$ is projective by Lemma 2.

Let $n = 1$. By applying the functor $\mathrm{Hom}_T(-, N)$ to the above short exact sequence and since $\begin{pmatrix} P_1 \\ U \otimes_A P_1 \end{pmatrix}$ and P_1 are projectives, we obtain a commutative diagram with exact rows:

$$\begin{array}{ccccc}
\mathrm{Hom}_T(\begin{pmatrix} P_1 \\ U \otimes_A P_1 \end{pmatrix}, N) & \longrightarrow & \mathrm{Hom}_T(\begin{pmatrix} K_1 \\ U \otimes_A K_1 \end{pmatrix}, N) & \twoheadrightarrow & \mathrm{Ext}^1_T(\begin{pmatrix} M_1 \\ U \otimes_A M_1 \end{pmatrix}, N) \\
\downarrow \cong & & \downarrow \cong & & \downarrow \\
\mathrm{Hom}_A(P_1, N_1) & \longrightarrow & \mathrm{Hom}_A(K_1, N_1) & \twoheadrightarrow & \mathrm{Ext}^1_A(M_1, N_1)
\end{array}$$

where the first two columns are just the natural isomorphisms given by adjointness and the last two horizontal rows are epimorphisms. Thus, the induced map:

$$\mathrm{Ext}^1_T(\begin{pmatrix} M_1 \\ U \otimes_A M_1 \end{pmatrix}, N) \to \mathrm{Ext}^1_A(M_1, N_1)$$

is an isomorphism such that the above diagram is commutative.

Assume that $n > 1$. Using the long exact sequence, we obtain a commutative diagram with exact rows:

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathrm{Ext}^{n-1}_T(\begin{pmatrix} K_1 \\ U \otimes_A K_1 \end{pmatrix}, N) & \xrightarrow[\cong]{f} & \mathrm{Ext}^n_T(\begin{pmatrix} M_1 \\ U \otimes_A M_1 \end{pmatrix}, N) & \longrightarrow & 0 \\
 & & \sigma \downarrow \cong & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Ext}^{n-1}_A(K_1, N_1) & \xrightarrow[\cong]{g} & \mathrm{Ext}^n_A(M_1, N_1) & \longrightarrow & 0
\end{array}$$

where σ is a natural isomorphism by induction, since $\mathrm{Tor}^A_k(U, K_1) = 0$ for every $k \in \{1, \cdots, n-1\}$ because of the exactness of the following sequence:

$$0 = \mathrm{Tor}^A_{k+1}(U, M_1) \to \mathrm{Tor}^A_k(U, K_1) \to \mathrm{Tor}^A_k(U, P_1) = 0.$$

Thus, the composite map:

$$g\sigma f^{-1} : \mathrm{Ext}^n_T(\begin{pmatrix} M_1 \\ U \otimes_A M_1 \end{pmatrix}, N) \to \mathrm{Ext}^n_A(M_1, N_1)$$

is a natural isomorphism, as desired. □

Since T can be viewed as a trivial extension (see [20,25] for more details), the following lemma can be easily deduced from [25], Theorems 3.1 and 3.4. For the convenience of the reader, we give a proof.

Lemma 4. *Let* $X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}_{\varphi^X}$ *be a T-module and* $(C_1, C_2) \in A\text{-}Mod \times B\text{-}Mod$*:*

1. $X \in \mathrm{Add}_T(\boldsymbol{p}(C_1, C_2))$ *if and only if:*
   - *(i)* $X \cong \boldsymbol{p}(X_1, \overline{X}_2)$*;*
   - *(ii)* $X_1 \in \mathrm{Add}_A(C_1)$ *and* $\overline{X}_2 \in \mathrm{Add}_B(C_2)$.

   In this case, φ^X *is injective;*
2. $X \in \mathrm{Prod}_T(\boldsymbol{h}(C_1, C_2))$ *if and only if:*
   - *(i)* $X \cong \boldsymbol{h}(\underline{X_1}, X_2)$*;*
   - *(ii)* $\underline{X_1} \in \mathrm{Prod}_A(C_1)$ *and* $X_2 \in \mathrm{Prod}_B(C_2)$.

   In this case, $\widetilde{\varphi^X}$ *is surjective.*

Proof. We only need to prove (1), since (2) is dual.

For the "if" part: if $X_1 \in \mathrm{Add}_A(C_1)$ and $\overline{X}_2 \in \mathrm{Add}_B(C_2)$, then $X_1 \oplus Y_1 = C^{(I_1)}$ and $\overline{X}_2 \oplus Y_2 = C_2^{(I_2)}$ for some $(Y_1, Y_2) \in A\text{-Mod} \times B\text{-Mod}$ and some sets I_1 and I_2. Without loss of generality, we may assume that $I = I_1 = I_2$. Then:

$$\begin{aligned}
X \oplus \mathbf{p}(Y_1, Y_2) &\cong \mathbf{p}(X_1, \overline{X}_2) \oplus \mathbf{p}(Y_1, Y_2) \\
&= \begin{pmatrix} X_1 \\ (U \otimes_A X_1) \oplus \overline{X}_2 \end{pmatrix} \oplus \begin{pmatrix} Y_1 \\ (U \otimes_A Y_1) \oplus Y_2 \end{pmatrix} \\
&\cong \begin{pmatrix} C_1^{(I)} \\ (U \otimes_A C_1^{(I)}) \oplus C_2^{(I)} \end{pmatrix} \\
&\cong \begin{pmatrix} C_1 \\ (U \otimes_A C_1) \oplus C_2 \end{pmatrix}^{(I)} \\
&= \mathbf{p}(C_1, C_2)^{(I)}.
\end{aligned}$$

Hence, $X \in \mathrm{Add}_T(\mathbf{p}(C_1, C_2))$.

Conversely, let $X \in \mathrm{Add}_T(\mathbf{p}(C_1, C_2))$ and $Y = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}_{\varphi^Y}$ be a T-module such that $X \oplus Y = \mathbf{p}(C_1, C_2)^{(I)}$ for some set I. Then, φ^X is injective, as X is a submodule of $C := \mathbf{p}(C_1, C_2)^{(I)}$ and φ^C is injective. Consider now the split exact sequence:

$$0 \to Y \overset{\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}}{\longrightarrow} C \overset{\begin{pmatrix} p_1 \\ p_2 \end{pmatrix}}{\longrightarrow} X \to 0$$

which induces the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & U \otimes_A Y_1 & \xrightarrow{1_U \otimes \lambda_1} & U \otimes_A C_1^{(I)} & \xrightarrow{1_U \otimes p_1} & U \otimes_A X_1 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \varphi^Y} & & \downarrow{\scriptstyle \varphi^C} & & \downarrow{\scriptstyle \varphi^X} & & \\
0 & \longrightarrow & Y_2 & \xrightarrow{\lambda_2} & U \otimes_A C_1^{(I)} \oplus C_2^{(I)} & \xrightarrow{p_2} & X_2 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \overline{\varphi^Y}} & & \downarrow{\scriptstyle \overline{\varphi^C}} & & \downarrow{\scriptstyle \overline{\varphi^X}} & & \\
0 & \longrightarrow & \overline{Y_2} & \xrightarrow{\overline{\lambda_2}} & C_2^{(I)} & \xrightarrow{\overline{p_2}} & \overline{X_2} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where $\overline{\varphi^X}$, $\overline{\varphi^C}$ and $\overline{\varphi^X}$ are the canonical projections. Clearly, $p_1 : C_1^{(I)} \to X_1$ and $\overline{p_2} : C_2^{(I)} \to \overline{X}_2$ are split epimorphisms. Then, $X_1 \in \mathrm{Add}_A(C_1)$ and $\overline{X}_2 \in \mathrm{Add}_B(C_2)$. It remains to prove that $X \cong \mathbf{p}(X_1, \overline{X}_2)$. For this, it suffices to prove that the short exact sequence:

$$0 \to U \otimes_A X_1 \overset{\varphi^X}{\to} X_2 \overset{\overline{\varphi^X}}{\to} \overline{X}_2 \to 0$$

splits. Let r_2 be the retraction of $\overline{p_2}$. If $i : C_2^{(I)} \to (U \otimes_A C_1^{(I)}) \oplus C_2^{(I)}$ denotes the canonical injection, then $\overline{\varphi^X} p_2 i r_2 = \overline{p_2}\,\overline{\varphi^C} i r_2 = \overline{p_2} r_2 = 1_{\overline{X}_2}$, and the proof is finished. □

Remark 1.

1. *Since the class of projective modules over T is nothing but the class $\mathrm{Add}_T(T)$, when we take $C_1 = A$ and $C_2 = B$ in Lemma 4, we recover the characterization of projective T-modules. On the other hand, note that the class of injective T-modules coincides with the class $\mathrm{Prod}_T(T^+)$. If we take $T_T = (A \oplus U, B)$, then the injective cogenerator T-module $T^+ = \mathrm{Hom}(T, \mathbb{Q}/\mathbb{Z})$ can be identified with $\begin{pmatrix} A^+ \oplus U^+ \\ B^+ \end{pmatrix} \cong \boldsymbol{q}(A^+, B^+)$. Therefore, by taking $C_1 = A^+$ and $C_2 = B^+$ in Lemma 4(A), we recover the characterization of injective T-modules;*
2. *Let (C_1, C_2) be a module over A-Mod $\times$ B-Mod. By Lemma 4(2), every module in $\mathrm{Add}_T(\boldsymbol{p}(C_1, C_2))$ has the form $\boldsymbol{p}(X_1, X_2)$ for some $X_1 \in \mathrm{Add}_A(C_1)$ and $X_2 \in \mathrm{Add}_B(C_2)$.*

3. w-Tilting Modules

In this section, we study when the functor $\mathbf{p}$ preserves w-tilting modules.

It is well known that the functor $\mathbf{p}$ preserves projective modules. However, the functor $\mathbf{p}$ does not preserve w-tilting modules in general, as the following example shows.

Example 1. *Let Q be the quiver:*

$$e_1 \overset{\alpha}{\longrightarrow} e_2,$$

and let $R = kQ$ be the path algebra over an algebraic closed field k. Put $P_1 = Re_1$, $P_2 = Re_2$, $I_1 = \mathrm{Hom}_k(e_1 R, k)$, and $I_2 = \mathrm{Hom}_k(e_2 R, k)$. Note that C_1 and C_2 are projective and injective R-modules, respectively. By [12], Example 2.3,

$$C_1 = P_1 \oplus P_2 (= R) \quad \textit{and} \quad C_2 = I_1 \oplus I_2$$

are semidualizing (R, R)-bimodules and, then, w-tilting R-modules. Now, consider the triangular matrix ring:

$$T(R) = \begin{pmatrix} R & 0 \\ R & R \end{pmatrix}.$$

We claim that $\boldsymbol{p}(C_1, C_2)$ is not a w-tilting $T(R)$-module. Note that I_1 is not projective. Since R is left hereditary by [26], Proposition 1.4, $\mathrm{pd}_R(I_1) = 1$. Hence, $\mathrm{Ext}^1_R(I_1, R) \neq 0$. Using Lemma 3, we obtain that $\mathrm{Ext}^1_{T(R)}(\boldsymbol{p}(C_1, C_2), \boldsymbol{p}(C_1, C_2)) \cong \mathrm{Ext}^1_R(C_1, C_1) \oplus \mathrm{Ext}^1_R(C_2, C_1) \oplus \mathrm{Ext}^1_R(C_2, C_2) \cong \mathrm{Ext}^1_R(I_1, R) \neq 0$. Therefore, $\boldsymbol{p}(C_1, C_2)$ is not a w-tilting $T(R)$-module.

Motivated by the definition of compatible bimodules in [1], Definition 1.1, we introduce the following definition, which will be crucial throughout the rest of this paper.

Definition 4. *Let $(C_1, C_2) \in A$-Mod $\times$ B-Mod and $C = \boldsymbol{p}(C_1, C_2)$. The bimodule ${}_BU_A$ is said to be C-compatible if the following two conditions hold:*

(a) *The complex $U \otimes_A \boldsymbol{X}_1$ is exact for every exact sequence in A-Mod:*

$$\boldsymbol{X}_1 : \cdots \to P_1^1 \to P_1^0 \to C_1^0 \to C_1^1 \to \cdots$$

where the P_1^i's are all projective and $C_1^i \in \mathrm{Add}_A(C_1)$ $\forall i$;

(b) *The complex* $\mathrm{Hom}_B(\mathbf{X}_2, U \otimes_A \mathrm{Add}_A(C_1))$ *is exact for every* $\mathrm{Hom}_B(-, \mathrm{Add}_B(C_2))$*-exact exact sequence in B-Mod:*

$$\mathbf{X}_2 : \cdots \to P_2^1 \to P_2^0 \to C_2^0 \to C_2^1 \to \cdots$$

where the P_2^i*'s are all projective and* $C_2^i \in \mathrm{Add}_B(C_2)$ $\forall i$.

Moreover, U is called weakly C-compatible if it satisfies (b) *and the following condition:*

(a') *The complex* $U \otimes_A \mathbf{X}_1$ *is exact for every* $\mathrm{Hom}_A(-, \mathrm{Add}_A(C_1))$*-exact exact sequence in A-Mod*

$$\mathbf{X}_1 : \cdots \to P_1^1 \to P_1^0 \to C_1^0 \to C_1^1 \to \cdots$$

where the P_1^i*'s are all projective and* $C_1^i \in \mathrm{Add}_A(C_1)$ $\forall i$.

When $C = {}_T T = \boldsymbol{p}(A, B)$, *the bimodule U will be called simply (weakly) compatible.*

Remark 2.

1. *It is clear by the definition that every C-compatible is weakly C-compatible;*
2. *The* (B, A)*-bimodule U is weakly compatible if and only if the functor* $U \otimes_A - : A\text{-Mod} \to B\text{-Mod}$ *is weakly compatible (see [27]);*
3. *If A and B are Artin algebras, and since* ${}_T T = \begin{pmatrix} A \\ U \oplus \end{pmatrix} = \boldsymbol{p}(A, B)$, *it is easy to see that* ${}_T T$*-compatible bimodules are nothing but compatible* (B, A)*-bimodules as defined in [1].*

The following can be applied to produce examples of (weakly) C-compatible bimodules later on.

Lemma 5. *Let* $C = \boldsymbol{p}(C_1, C_2)$ *be a T-module:*

1. *Assume that* $\mathrm{Tor}_1^A(U, C_1) = 0$. *If* $\mathrm{fd}_A(U) < \infty$, *then U satisfies* (a);
2. *Assume that* $\mathrm{Ext}_B^1(C_2, U \otimes_A C_1^{(I)}) = 0$ *for every set I. If* $\mathrm{id}_B(U \otimes_A C_1) < \infty$, *then U satisfies* (b);
3. *If* $U \otimes_A C_1 \in \mathrm{Add}_B(C_2)$, *then U satisfies* (b).

Proof. (3) is clear. We only prove (1), as (2) is similar. Consider an exact sequence of A-modules:

$$\mathbf{X}_1 : \cdots \to P_1^1 \to P_1^0 \to C_1^0 \to C_1^1 \to \cdots$$

where the P_1^i's are all projective and $C_1^i \in \mathrm{Add}_A(C_1)$ $\forall i$. We use induction on $\mathrm{fd}_A U$. If $\mathrm{fd}_A U = 0$, then the result is trivial. Now, suppose that $\mathrm{fd}_A U = n \geq 1$. Then, there exists an exact sequence of right A-modules:

$$0 \to L \to F \to U \to 0$$

where $\mathrm{fd}_A L = n - 1$ and F is flat. Applying the functor $- \otimes \mathbf{X}_1$ to the above short exact sequence, we obtain the commutative diagram with exact rows:

$$
\begin{array}{ccccccccc}
\vdots & & \vdots & & \vdots & & \vdots & & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & L \otimes P_1^0 & \longrightarrow & F \otimes_A P_1^0 & \longrightarrow & U \otimes_A P_1^0 & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\mathrm{Tor}_1^A(U, C_1^0) & \longrightarrow & L \otimes_A C_1^0 & \longrightarrow & F \otimes_A C_1^0 & \longrightarrow & U \otimes_A C_1^0 & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\mathrm{Tor}_1^A(U, C_1^1) & \longrightarrow & L \otimes_A C_1^1 & \longrightarrow & F \otimes_A C_1^1 & \longrightarrow & U \otimes_A C_1^1 & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\vdots & & \vdots & & \vdots & & \vdots & &
\end{array}
$$

Since $\mathrm{Tor}_1^A(U, C_1) = 0$, the above diagram induces an exact sequence of complexes:

$$0 \to L \otimes_A \mathbf{X}_1 \to F \otimes_A \mathbf{X}_1 \to U \otimes_A \mathbf{X}_1 \to 0.$$

By the induction hypothesis, the complexes $L \otimes_A \mathbf{X}_1$ and $F \otimes_A \mathbf{X}_1$ are exact. Thus, $U \otimes_A \mathbf{X}_1$ is exact, as well. □

Given a T-module $C = \mathbf{p}(C_1, C_2)$, we have simple characterizations of Conditions (a') and (b) if C_1 and C_2 are w-tilting.

Proposition 1. *Let $C = \boldsymbol{p}(C_1, C_2)$ be a T-module:*

1. *If C_1 is w-tilting, then the following assertions are equivalent:*
   - *(i)* *U satisfies (a');*
   - *(ii)* $\mathrm{Tor}_1^A(U, G_1) = 0$, $\forall G_1 \in G_{C_1}P(A)$;
   - *(iii)* $\mathrm{Tor}_{i\geq 1}^A(U, G_1) = 0$, $\forall G_1 \in G_{C_1}P(A)$.

   In this case, $\mathrm{Tor}_{i\geq 1}^A(U, C_1) = 0$;
2. *If C_2 is w-tilting, then the following assertions are equivalent:*
   - *(i)* *U satisfies (b);*
   - *(ii)* $\mathrm{Ext}_B^1(G_2, U \otimes_A X_1) = 0$, $\forall G_2 \in G_{C_2}P(B)$, $\forall X_1 \in \mathrm{Add}_A(C_1)$;
   - *(iii)* $\mathrm{Ext}_B^{i\geq 1}(G_2, U \otimes_A X_1) = 0$, $\forall G_2 \in G_{C_2}P(B)$, $\forall X_1 \in \mathrm{Add}_A(C_1)$;

   In this case, $\mathrm{Ext}_B^{i\geq 1}(C_2, U \otimes_A X_1) = 0$, $\forall X_1 \in \mathrm{Add}_A(C_1)$.

Proof. We only prove (1), since (2) is similar.

$(i) \Rightarrow (iii)$ Let $G_1 \in G_{C_1}P(R)$. There exists a $\mathrm{Hom}_A(-, \mathrm{Add}_A(C_1))$-exact exact sequence in A-Mod:

$$\mathbf{X}_1 : \cdots \to P_1^1 \to P_1^0 \to C_1^0 \to C_1^1 \to \cdots$$

where the P_1^i's are all projective, $G_1 \cong \mathrm{Im}(P_1^0 \to C_1^0)$ and $C_1^i \in \mathrm{Add}_A(C_1)$ $\forall i$. By Condition (a'), $U \otimes_A \mathbf{X}_1$ is exact, which means in particular that $\mathrm{Tor}_{i\geq 1}^A(U, G_1) = 0$.

$(iii) \Rightarrow (ii)$ Clear.

$(ii) \Rightarrow (i)$ Follows by [13], Corollary 2.13.

Finally, to prove that $\mathrm{Tor}_{i\geq 1}^A(U, C_1) = 0$, note that $C_1 \in G_{C_1}P(A)$ by [13], Theorem 2.12. □

In the following proposition, we study when $\mathbf{p}$ preserves w-tilting (tilting) modules.

Proposition 2. *Let $C = \boldsymbol{p}(C_1, C_2)$ be a T-module and assume that U is weakly C-compatible. If C_1 and C_2 are w-tilting (tilting), then $\boldsymbol{p}(C_1, C_2)$ is w-tilting (tilting).*

Proof. By Lemma 2, the functor $\mathbf{p}$ preserves finitely generated modules, so we only need to prove the statement for w-tilting. Assume that C_1 and C_2 are w-tilting, and let I be a set. Then, $\mathrm{Ext}_A^{i\geq 1}(C_1, C_1^{(I)}) = 0$ and $\mathrm{Ext}_B^{i\geq 1}(C_2, C_2^{(I)}) = 0$. By Proposition above, we have $\mathrm{Ext}_B^{i\geq 1}(C_2, U \otimes_A C_1^{(I)}) = 0$ and $\mathrm{Tor}_{i\geq 1}^A(U, C_1) = 0$. Using Lemma 3, for every $n \geq 1$, we obtain that:

$$\begin{aligned}\mathrm{Ext}_T^n(C, C^{(I)}) &= \mathrm{Ext}_T^n(\mathbf{p}(C_1, C_2), \mathbf{p}(C_1, C_2)^{(I)}) \\ &\cong \mathrm{Ext}_A^n(C_1, C_1^{(I)}) \oplus \mathrm{Ext}_B^n(C_2, U \otimes_A C_1^{(I)}) \oplus \mathrm{Ext}_B^n(C_2, C_2^{(I)}) \\ &= 0.\end{aligned}$$

Moreover, there exist exact sequences:

$$\mathbf{X}_1 : \ 0 \to A \to C_1^0 \to C_1^1 \to \cdots$$

and:

$$\mathbf{X}_2 : \ 0 \to B \to C_2^0 \to C_2^1 \to \cdots$$

which are $\mathrm{Hom}_A(-, \mathrm{Add}_A(C_1))$-exact and $\mathrm{Hom}_B(-, \mathrm{Add}_B(C_2))$-exact, respectively, and such that $C_1^i \in \mathrm{Add}_A(C_1)$ and $C_2^i \in \mathrm{Add}_B(C_2)$ for every $i \in \mathbb{N}$. Since U is weakly C-compatible, the complex $U \otimes_A \mathbf{X}_1$ is exact. Therefore, we construct in T-Mod the exact sequence:

$$\mathbf{p}(\mathbf{X}_1, \mathbf{X}_2) : \ 0 \to T \to \mathbf{p}(C_1^0, C_2^0) \to \mathbf{p}(C_1^1, C_2^1) \to \cdots$$

where $\mathbf{p}(C_1^i, C_2^i) = \begin{pmatrix} C_1^i \\ (U \otimes_A C_1^i) \oplus C_2^i \end{pmatrix} \in \mathrm{Add}_T(\mathbf{p}(C_1, C_2)), \forall i \in \mathbb{N}$, by Lemma 4(1).

Let $X \in \mathrm{Add}_T(\mathbf{p}(C_1, C_2))$. As a consequence of Lemma 4(1), $X = \mathbf{p}(X_1, X_2)$ where $X_1 \in \mathrm{Add}_A(C_1)$ and $X_2 \in \mathrm{Add}_B(C_2)$. Using the adjointness $(\mathbf{p}, \mathbf{q})$, we obtain an isomorphism of complexes:

$$\mathrm{Hom}_T(\mathbf{p}(\mathbf{X}_1, \mathbf{X}_2), X) \cong \mathrm{Hom}_A(\mathbf{X}_1, X_1) \oplus \mathrm{Hom}_B(\mathbf{X}_2, U \otimes X_1) \oplus \mathrm{Hom}_B(\mathbf{X}_2, X_2).$$

However, the complexes $\mathrm{Hom}_A(\mathbf{X}_1, X_1)$ and $\mathrm{Hom}_B(\mathbf{X}_2, X_2)$ are exact, and the complex $\mathrm{Hom}_B(\mathbf{X}_2, U \otimes X_1)$ is also exact since U is weakly C-compatible. Then, $\mathrm{Hom}_T(\mathbf{p}(\mathbf{X}_1, \mathbf{X}_2), X)$ is exact, as well, and the proof is finished. □

Now, we illustrate Proposition 2 with two applications.

Corollary 1. *Let $C = \mathbf{p}(C_1, C_2)$ be a T-module and A' and B' be two rings such that ${}_AC_{A'}$ and ${}_AC_{B'}$ are bimodules, and assume that U is weakly C-compatible. If ${}_AC_{A'}$ and ${}_AC_{B'}$ are semidualizing bimodules, then $\mathbf{p}(C_1, C_2)$ is a semidualizing $(T, \mathrm{End}_T(C))$-bimodule.*

Proof. This follows by Proposition 2 and [18], Corollary 3.2. □

Corollary 2. *Let R and S be rings, $\theta : R \to S$ be a homomorphism with S_R flat, and $T = T(\theta) =: \begin{pmatrix} R & 0 \\ S & S \end{pmatrix}$. Let C_1 be an R-module such that $S \otimes_R C_1 \in \mathrm{Add}_R(C_1)$ (for instance, if R is commutative or $R = S$). If ${}_RC_1$ is w-tilting, then:*

1. *$S \otimes_R C_1$ is a w-tilting S-module;*
2. *$C = \begin{pmatrix} C_1 \\ (S \otimes_R C_1) \oplus (S \otimes_R C_1) \end{pmatrix}$ is a w-tilting $T(\theta)$-module.*

Proof. 1. Let $C_2 = S \otimes_R C_1$, and note that $C = \mathbf{p}(C_1, C_2)$ and that ${}_SS_R$ is C-compatible. Therefore, by Proposition 2, we only need to prove that C_2 is a w-tilting S-module. Since ${}_RC_1$ is w-tilting, there exist $\mathrm{Hom}_R(-, \mathrm{Add}_R(C_1))$-exact exact sequences:

$$\mathbf{P}: \cdots \to P_1 \to P_0 \to C \to 0$$

and:

$$\mathbf{X}: 0 \to R \to C_0 \to C_1 \to \cdots$$

with each ${}_RP_i$ projective and ${}_RC_i \in \mathrm{Add}_R(C_1)$. Since S_R is flat, we obtain an exact sequence:

$$S \otimes_R \mathbf{P}: \cdots \to S \otimes_R P_1 \to S \otimes_R P_0 \to S \otimes_R C \to 0$$

and:

$$S \otimes_R \mathbf{X}: 0 \to S \to S \otimes_R C_0 \to S \otimes_R C_1 \to \cdots$$

with each $S \otimes_R P_i$ a projective S-module and $S \otimes_R C_i \in \mathrm{Add}_R(C_2)$.

We prove now that $S \otimes_R \mathbf{P}$ and $S \otimes_R \mathbf{X}$ are $\mathrm{Hom}_S - (,\mathrm{Add}_S(C_2))$-exact. Let I be a set. Then, $\mathrm{Hom}_S(S \otimes_R \mathbf{P}, S \otimes_R C_1^{(I)}) \cong \mathrm{Hom}_R(\mathbf{P}, \mathrm{Hom}_S(S, S \otimes_R C_1^{(I)})) \cong \mathrm{Hom}_R(\mathbf{P}, S \otimes_R C_1^{(I)})$ is exact since $S \otimes_R C_1^{(I)} \in \mathrm{Add}_R(C_1)$. Similarly, $S \otimes_R \mathbf{X}$ is $\mathrm{Hom}_S(-, \mathrm{Add}_S(C_2))$-exact;

2. This assertion follows from Proposition 2. We only need to note that S is weakly C-compatible since S_R is flat and $S \otimes_R C_1 \in \mathrm{Add}_R(C_2)$. □

We end this section with an example of a w-tilting module that is neither projective nor injective.

Example 2. *Take R and C_2 as in Example 1. Therefore, by Corollary 2, $C = \begin{pmatrix} C_2 \\ C_2 \oplus C_2 \end{pmatrix}$ is a w-tilting $T(R)$-module. By Lemma 2, C is not projective since C_2 is not, and it is not injective since the map $\widetilde{\varphi^C}: C_2 \to C_2 \oplus C_2$ is not surjective.*

Moreover, by [26], Proposition 2.6, $gl.dim(T(R)) = gl.dim(R) + 1 \le 2$. Therefore, if $0 \to T(R) \to E^0 \to E^1 \to E^2 \to 0$ is an injective resolution of $T(R)$, then $C^1 = E^0 \oplus E^1 \oplus E^2$ is a w-tilting $T(R)$-module. Note that $T(R)$ has at least three w-tilting modules, C^1, $C^2 = T(R)$ and $C^3 = C$.

4. Relative Gorenstein Projective Modules

In this section, we describe G_C-projective modules over T. Then, we use this description to study when the class of G_C-projective T-modules is a special precovering class.

Clearly, the functor $\mathbf{p}$ preserves the projective module. Therefore, we start by studying when the functor $\mathbf{p}$ also preserves relative Gorenstein projective modules. However, first, we need the following:

Lemma 6. *Let $C = \mathbf{p}(C_1, C_2)$ be a T-module and U be weakly C-compatible:*

1. *If $M_1 \in G_{C_1}P(A)$, then $\begin{pmatrix} M_1 \\ U \otimes_A M_1; \end{pmatrix} \in G_CP(T)$.*
2. *If $M_2 \in G_{C_2}P(B)$, then $\begin{pmatrix} 0 \\ M_2 \end{pmatrix} \in G_CP(T)$.*

Proof. 1. Suppose that $M_1 \in G_{C_1}P(A)$. There exists a $\mathrm{Hom}_A(-, \mathrm{Add}_A(C_1))$-exact exact sequence:

$$\mathbf{X}_1: \cdots \to P_1^1 \to P_1^0 \to C_1^0 \to C_1^1 \to \cdots$$

where the P_1^i's are all projective, $C_1^i \in \mathrm{Add}_A(C_1)$ $\forall i$ and $M_1 \cong \mathrm{Im}(P_1^0 \to C_1^0)$. Using the fact that U is weakly C-compatible, we obtain that the complex $U \otimes_A \mathbf{X}_1$ is exact in B-Mod, which implies that the complex $\mathbf{p}(\mathbf{X}_1, 0)$:

$$\cdots \to \begin{pmatrix} P_1^1 \\ U \otimes_A P_1^1 \end{pmatrix} \to \begin{pmatrix} P_1^0 \\ U \otimes_A P_1^0 \end{pmatrix} \to \begin{pmatrix} C_1^0 \\ U \otimes_A C_1^0 \end{pmatrix} \to \begin{pmatrix} C_1^1 \\ U \otimes_A C_1^1 \end{pmatrix} \to \cdots$$

is exact with
$$\begin{pmatrix} M_1 \\ U\otimes_A M_1 \end{pmatrix} \cong \mathrm{Im}(\begin{pmatrix} P_1^0 \\ U\otimes_A P_1^0 \end{pmatrix} \to \begin{pmatrix} C_1^0 \\ U\otimes_A C_1^0 \end{pmatrix}).$$

Clearly, $\mathbf{p}(P_1^i,0) = \begin{pmatrix} P_1^i \\ U\otimes_A P_1^i \end{pmatrix} \in \mathrm{Proj}(T)$ and $\mathbf{p}(C_1^i,0) = \begin{pmatrix} C_1^i \\ U\otimes_A C_1^i \end{pmatrix} \in \mathrm{Add}_T(C)\ \forall i \in \mathbb{N}$ by Lemmas 2(1) and 4(1). If $X \in \mathrm{Add}_T(C)$, then $X_1 \in \mathrm{Add}_A(C_1)$ by Lemma 4(1), and using the adjointness, we obtain that the complex $\mathrm{Hom}_T(\mathbf{p}(\mathbf{X}_1,0),X) \cong \mathrm{Hom}_A(\mathbf{X}_1,X_1)$ is exact. Hence, $\begin{pmatrix} M_1 \\ U\otimes_A M_1 \end{pmatrix}$ is G_C-projective;

2. Suppose that M_2 is G_{C_2}-projective. There exists a $\mathrm{Hom}_B(-,\mathrm{Add}_B(C_2))$-exact exact sequence:
$$\mathbf{X}_2: \ \cdots \to P_2^1 \to P_2^0 \to C_2^0 \to C_2^1 \to \cdots$$
where the P_2^i's are all projective, $C_2^i \in \mathrm{Add}_B(C_2)\ \forall i$ and $M_2 \cong \mathrm{Im}(P_2^0 \to C_2^0)$. Clearly, the complex:
$$\mathbf{p}(0,\mathbf{X}_2): \cdots \to \begin{pmatrix} 0 \\ P_2^1 \end{pmatrix} \to \begin{pmatrix} 0 \\ P_2^0 \end{pmatrix} \to \begin{pmatrix} 0 \\ C_2^0 \end{pmatrix} \to \begin{pmatrix} 0 \\ C_2^1 \end{pmatrix} \to \cdots$$
is exact with $\begin{pmatrix} 0 \\ M_2 \end{pmatrix} \cong \mathrm{Im}(\begin{pmatrix} 0 \\ P_2^1 \end{pmatrix} \to \begin{pmatrix} 0 \\ C_2^0 \end{pmatrix})$, $\mathbf{p}(0,P_2^i) = \begin{pmatrix} 0 \\ P_2^i \end{pmatrix} \in \mathrm{Proj}(T)$ and $\mathbf{p}(0,C_2^i) = \begin{pmatrix} 0 \\ C_2^i \end{pmatrix} \in \mathrm{Add}_T(C)\ \forall i$, by Lemmas 2(1) and 4(1). Let $X \in \mathrm{Add}_T(C)$. Then, by Lemma 4(1), $X = \mathbf{p}(X_1,X_2)$ where $X_1 \in \mathrm{Add}_A(C_1)$ and $X_2 \in \mathrm{Add}_B(C_2)$. Using adjointness, we obtain that:
$$\mathrm{Hom}_T(\mathbf{p}(0,\mathbf{X}_2),X) \cong \mathrm{Hom}_B(\mathbf{X}_2, U\otimes_A X_1) \oplus \mathrm{Hom}_B(\mathbf{X}_2,X_2)$$
The complex $\mathrm{Hom}_B(\mathbf{X}_2,X_2)$ is exact, and since U is weakly C-compatible, the complex $\mathrm{Hom}_B(\mathbf{X}_2,U\otimes_A X_1)$ is also exact. This means that $\mathrm{Hom}_T(\mathbf{p}(0,\mathbf{X}_2),X)$ is exact as well and $\begin{pmatrix} 0 \\ M_2 \end{pmatrix}$ is G_C-projective. □

Proposition 3. *Let $C = \mathbf{p}(C_1,C_2)$ be a T-module. If ${}_BU_A$ is weakly C-compatible, then the functor $\mathbf{p}$ sends $G_{(C_1,C_2)}$-projectives to G_C-projectives. The converse holds provided that C_1 and C_2 are w-tilting.*

In particular, $\mathbf{p}$ preserves Gorenstein projective modules if and only if U is weakly compatible.

Proof. Note that:
$$\mathbf{p}(M_1,M_2) = \begin{pmatrix} M_1 \\ U\otimes_A M_1 \end{pmatrix} \oplus \begin{pmatrix} 0 \\ M_2 \end{pmatrix}.$$
Therefore, this direction follows from Lemma 6 and [13], Proposition 2.5.

Conversely, assume that C_1 and C_2 are w-tilting. By Proposition 1, it suffices to prove that $\mathrm{Tor}_1^A(U, G_{C_1}P(A)) = 0 = \mathrm{Ext}_B^1(G_{C_2}P(B), U\otimes_A \mathrm{Add}_A(C_1))$.

Let $G_1 \in G_{C_1}P(A)$. By [13], Corollary 2.13, there exits an exact and a $\mathrm{Hom}_A(-,\mathrm{Add}_A(C_1))$-exact sequence $0 \to L_1 \xrightarrow{\iota} P_1 \to G_1 \to 0$, where ${}_AP_1$ is projective and L_1 is G_{C_1}-projective. Note that $A, C_1 \in G_{C_1}P(A)$ and $B, C_2 \in G_{C_2}P(B)$ by Lemma 1. Then, ${}_TT = \mathbf{p}(A,B)$ and $C = \mathbf{p}(C_1,C_2)$ are G_C-projective, which imply by Lemma 1 that C is w-tilting. Moreover $\begin{pmatrix} L_1 \\ U\otimes_A L_1 \end{pmatrix} = \mathbf{p}(L_1,0)$ is also G_C-projective, and by [13], Corollary 2.13, there exists a short exact sequence:
$$0 \to \begin{pmatrix} L_1 \\ U\otimes_A L_1 \end{pmatrix} \to \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}_{\varphi^X} \to \begin{pmatrix} H_1 \\ H_2 \end{pmatrix}_{\varphi^H} \to 0$$
where $X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}_{\varphi^X} \in \mathrm{Add}_T(C)$ and $H = \begin{pmatrix} H_1 \\ H_2 \end{pmatrix}_{\varphi^H}$ is G_C-projective.

Since $X_1 \in \mathrm{Add}_A(C_1)$, we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L_1 & \xrightarrow{\iota} & P_1 & \longrightarrow & G_1 & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & L_1 & \longrightarrow & X_1 & \longrightarrow & H_1 & \longrightarrow & 0 \end{array}$$

Therefore, if we apply the functor $U \otimes_A -$ to the above diagram, we obtain the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} & & U \otimes_A L_1 & \xrightarrow{1_U \otimes \iota} & U \otimes_A P_1 & \longrightarrow & U \otimes_A G_1 & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ & & U \otimes_A L_1 & \longrightarrow & U \otimes_A X_1 & \longrightarrow & U \otimes_A H_1 & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & U \otimes_A L_1 & \longrightarrow & X_2 & \longrightarrow & H_2 & \longrightarrow & 0 \end{array}$$

The commutativity of this diagram implies that the map $1_U \otimes \iota$ is injective, and since P_1 is projective, $\mathrm{Tor}_1^A(U, G_1) = 0$.

Now, let $G_2 \in G_{C_2}P(B)$ and $Y_2 \in \mathrm{Add}_A(C_1)$. By hypothesis, $\begin{pmatrix} 0 \\ G_2 \end{pmatrix} = \mathbf{p}(0, G_2)$ is G_C-projective, and by Lemma 4, $\begin{pmatrix} Y_1 \\ U \otimes Y_1 \end{pmatrix} = \mathbf{p}(Y_1, 0) \in \mathrm{Add}_T(C)$. Hence, $\mathrm{Ext}^1_B(G_2, U \otimes_A Y_1) = \mathrm{Ext}^1_T(\begin{pmatrix} 0 \\ G_2 \end{pmatrix}, \begin{pmatrix} Y_1 \\ U \otimes Y_1 \end{pmatrix}) = 0$ by Lemma 3 and [13], Proposition 2.4. □

Theorem 1. *Let* $M = \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}_{\varphi^M}$ *and* $C = \boldsymbol{p}(C_1, C_2)$ *be two T-modules. If U is C-compatible, then the following assertions are equivalent:*

1. *M is G_C-projective;*
2. *(i) φ^M is injective;*
   (ii) M_1 is G_{C_1}-projective and $\overline{M}_2 := \mathrm{Coker}\ \varphi^M$ is G_{C_2}-projective.

In this case, if C_2 is Σ-self-orthogonal, then $U \otimes_A M_1$ is G_{C_2}-projective if and only if M_2 is G_{C_2}-projective.

Proof. 2. ⇒ 1. Since φ^M is injective, there exists an exact sequence in T-Mod:

$$0 \to \begin{pmatrix} M_1 \\ U \otimes_A M_1 \end{pmatrix} \to M \to \begin{pmatrix} 0 \\ \overline{M}_2 \end{pmatrix} \to 0$$

Note that $\begin{pmatrix} M_1 \\ U \otimes_A M_1 \end{pmatrix}$ *and* $\begin{pmatrix} 0 \\ \overline{M}_2 \end{pmatrix}$ are G_C-projective T-modules by Lemma 6. Therefore, M is G_C-projective by [13], Proposition 2.5.

1. ⇒ 2. There exists a $\mathrm{Hom}_T(-, \mathrm{Add}_T(C))$-exact sequence in T-Mod:

$$\mathbf{X} = \cdots \to \begin{pmatrix} P_1^1 \\ P_2^1 \end{pmatrix}_{\varphi^{P^1}} \to \begin{pmatrix} P_1^0 \\ P_2^0 \end{pmatrix}_{\varphi^{P^0}} \to \begin{pmatrix} C_1^0 \\ C_2^0 \end{pmatrix}_{\varphi^{C^0}} \to \begin{pmatrix} C_1^1 \\ C_2^1 \end{pmatrix}_{\varphi^{C^1}} \to \cdots$$

where $C^i = \begin{pmatrix} C_1^i \\ C_2^i \end{pmatrix}_{\varphi^{C^i}} \in \mathrm{Add}_T(C)$, $P^i = \begin{pmatrix} P_1^i \\ P_2^i \end{pmatrix}_{\varphi^{P^i}} \in \mathrm{Proj}(T)$ $\forall i \in \mathbb{N}$, and such that $M \cong \mathrm{Im}(P^0 \to C^0)$. Then, we obtain the exact sequence:

$$\mathbf{X}_1 = \cdots \to P_1^1 \to P_1^0 \to C_1^0 \to C_1^1 \to \cdots$$

where $C_1^i \in \mathrm{Add}_A(C_1)$, $P_1^i \in \mathrm{Proj}(A)$ $\forall i \in \mathbb{N}$ by Lemmas 2(1) and 4(1) and such that $M_1 \cong \mathrm{Im}(P_1^0 \to C_1^0)$. Since U is C-compatible, the complex $U \otimes_A \mathbf{X}_1$ is exact with $U \otimes_A M_1 \cong Im(U \otimes_A P_1^0 \to U \otimes_A C_1^0)$. If $\iota_1 : M_1 \to C_1^0$ and $\iota_2 : M_2 \to C_2^0$ are the inclusions, then $1_U \otimes \iota_1$ is injective, and the following diagram commutes:

$$\begin{array}{ccc} U \otimes_A M_1 & \xrightarrow{1_U \otimes \iota_1} & U \otimes_A C_1^0 \\ \downarrow{\scriptstyle \varphi^M} & & \downarrow{\scriptstyle \varphi^{C^0}} \\ M_2 & \xrightarrow{\iota_2} & C_2^0 \end{array}$$

By Lemma 4(1), φ^{C^0} is injective, then φ^M is also injective. Moreover, for every $i \in \mathbb{N}$, φ^{P^i} and φ^{C^i} are injective by Lemmas 2 and 4(1). Then, the following diagram with exact columns:

$$\begin{array}{ccccccccccc} & & 0 & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \to & U \otimes_A P_1^1 & \to & U \otimes_A P_1^0 & \to & U \otimes_A C_1^0 & \to & U \otimes_A C_1^1 & \to & \cdots \\ & & \downarrow{\scriptstyle \varphi^{P^1}} & & \downarrow{\scriptstyle \varphi^{P^0}} & & \downarrow{\scriptstyle \varphi^{C^0}} & & \downarrow{\scriptstyle \varphi^{C^1}} & & \\ \cdots & \to & P_2^1 & \to & P_2^0 & \to & C_2^0 & \to & C_2^1 & \to & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \to & \overline{P_2^1} & \to & \overline{P_2^0} & \to & \overline{C_2^0} & \to & \overline{C_2^1} & \to & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & 0 & & \end{array}$$

is commutative. Since the first row and the second row are exact, we obtain the exact sequence of B-modules:

$$\overline{\mathbf{X}}_2 : \cdots \to \overline{P_2^1} \to \overline{P_2^0} \to \overline{C_2^0} \to \overline{C_2^1} \to \cdots$$

where $\overline{P_2^i} \in \mathrm{Proj}(B)$, $\overline{C_2^i} \in \mathrm{Add}_B(C_2)$ by Lemmas 2 and 4(1) and such that $\overline{M_2} = \mathrm{Im}(\overline{P_2^0} \to \overline{C_2^0})$. It remains to see that $\mathbf{X}_1$ and $\overline{\mathbf{X}}_2$ are $\mathrm{Hom}_A(-, \mathrm{Add}(C_1))$-exact and $\mathrm{Hom}_B(-, \mathrm{Add}_B(C_2))$-exact, respectively. Let $X_1 \in \mathrm{Add}_A(C_1)$ and $X_2 \in \mathrm{Add}_B(C_2)$. Then, $\mathbf{p}(X_1, 0) = \begin{pmatrix} X_1 \\ U \otimes_A X_1 \end{pmatrix} \in \mathrm{Add}_T(C)$ and $\mathbf{p}(0, X_2) = \begin{pmatrix} 0 \\ X_2 \end{pmatrix} \in \mathrm{Add}_T(C)$ by Lemma 4(1). Therefore, by using adjointness, we obtain that $\mathrm{Hom}_B(\overline{\mathbf{X}}_2, X_2) \cong \mathrm{Hom}_T(\mathbf{X}, \begin{pmatrix} 0 \\ X_2 \end{pmatrix})$ is exact. Using adjointness again, we obtain that:

$$\mathrm{Hom}_T(\mathbf{X}, \begin{pmatrix} 0 \\ U \otimes_A X_1 \end{pmatrix}) \cong \mathrm{Hom}_B(\overline{\mathbf{X}}_2, U \otimes_A X_1)$$

and:

$$\mathrm{Hom}_T(\mathbf{X}, \begin{pmatrix} X_1 \\ 0 \end{pmatrix}) \cong \mathrm{Hom}_A(\mathbf{X}_1, X_1).$$

Note that $C^i \cong \mathbf{p}(C_1^i, \overline{C_2^i})$ by Lemma 4(1). Hence, $\mathrm{Ext}^1_T(C^i, \begin{pmatrix} 0 \\ U\otimes_A X_1 \end{pmatrix}) \cong \mathrm{Ext}^1_B(\overline{C_2^i}, U\otimes_A X_1) = 0$ by Lemma 3. Therefore, if we apply the functor $\mathrm{Hom}_T(\mathbf{X}, -)$ to the sequence:

$$0 \to \begin{pmatrix} 0 \\ U\otimes_A X_1 \end{pmatrix} \to \begin{pmatrix} X_1 \\ U\otimes_A X_1 \end{pmatrix} \to \begin{pmatrix} X_1 \\ 0 \end{pmatrix} \to 0,$$

we obtain the following exact sequence of complexes:

$$0 \to \mathrm{Hom}_B(\overline{\mathbf{X}}_2, U\otimes_A X_1) \to \mathrm{Hom}_T(\mathbf{X}, \begin{pmatrix} X_1 \\ U\otimes_A X_1 \end{pmatrix}) \to \mathrm{Hom}_A(\mathbf{X}_1, X_1) \to 0.$$

Since U is C-compatible, it follows that $\mathrm{Hom}_B(\overline{\mathbf{X}}_2, U\otimes_A X_1)$ is exact, and since C is w-tilting, $\mathrm{Hom}_T(\mathbf{X}, \begin{pmatrix} X_1 \\ U\otimes_A X_1 \end{pmatrix})$ is also exact. Thus, $\mathrm{Hom}_A(\mathbf{X}_1, X_1)$ is exact, and the proof is finished. □

The following consequence of the above theorem gives the converse of Proposition 2.

Corollary 3. *Let $C = \boldsymbol{p}(C_1, C_2)$ and assume that U is C-compatible. Then, C is w-tilting if and only if C_1 and C_2 are w-tilting.*

Proof. An easy application of Proposition 1 and Theorem 1 on the T-modules $C = \begin{pmatrix} C_1 \\ (U\otimes_A C_1)\oplus C_2 \end{pmatrix}$ and ${}_TT = \begin{pmatrix} A \\ U\oplus B \end{pmatrix}$. □

One would like to know if every w-tilting T-module has the form $\mathbf{p}(C_1, C_2)$ where C_1 and C_2 are w-tilting. The following example gives a negative answer to this question.

Example 3. *Let R be a quasi-Frobenius ring and $T(R) = \begin{pmatrix} R & 0 \\ R & R \end{pmatrix}$. Consider the exact sequence of T-modules:*

$$0 \to T \to \begin{pmatrix} R\oplus R \\ R\oplus R \end{pmatrix} \to \begin{pmatrix} R \\ 0 \end{pmatrix} \to 0.$$

By Lemma 2, $I^0 = \begin{pmatrix} R\oplus R \\ R\oplus R \end{pmatrix}$ and $I^1 = \begin{pmatrix} R \\ 0 \end{pmatrix}$ are both injective $T(R)$-modules. Note that $T(R)$ is Noetherian ([23], Proposition 1.7), and then, we can see that $C := I^0 \oplus I^1$ is a w-tilting $T(R)$-module, but does not have the form $\boldsymbol{p}(C_1, C_2)$ where C_1 and C_2 are w-tilting by Lemma 4 since $I^1 \in \mathrm{Add}_{T(R)}(C)$ and φ^{I^1} is not injective.

As an immediate consequence of Theorem 1, we have the following.

Corollary 4. *Let R be a ring and $T(R) = \begin{pmatrix} R & 0 \\ R & R \end{pmatrix}$. If $M = \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}_{\varphi^M}$ and $C = \boldsymbol{p}(C_1, C_1)$ are two $T(R)$-modules with C_1 Σ-self-orthogonal, then the following assertions are equivalent:*

1. *M is G_C-projective $T(R)$-module;*
2. *M_1 and $\overline{M}_2$ are G_{C_1}-projective R-modules, and φ^M is injective;*
3. *M_1 and M_2 are G_{C_1}-projective R-modules, and φ^M is injective.*

An Artin algebra Λ is called Cohen–Macaulay-free (CM-free) if any finitely generated Gorenstein projective module is projective. The authors in [2] extended this definition to arbitrary rings and defined strongly CM-free as rings over which every Gorenstein projective module is projective. Now, we introduce a relative notion of these rings and give a characterization of when T is such rings.

Definition 5. *Let R be a ring. Given an R-module C, R is called CM-free (relative to C) if $G_CP(R) \cap R$-mod $= \mathrm{add}_R(C)$, and it is called strongly CM-free (relative to C) if $G_CP(R) = \mathrm{Add}_R(C)$.*

Remark 3. *Let R be a ring and C a Σ-self-orthogonal R-module. Then, $\mathrm{Add}_R(C) \subseteq G_CP(R)$ and $\mathrm{add}_R(C) \subseteq G_CP(R) \cap R$-mod by [13], Propositions 2.5 and 2.6 and Corollary 2.10, then R is CM-free (relative to C) if and only if every finitely generated G_C-projective is in $\mathrm{add}_R(C)$, and it is strongly CM-free (relative to C) if every G_C-projective is in $\mathrm{Add}_R(C)$.*

Using the above results, we refine and extend [2], Theorem 4.1, to our setting. Note that the condition B is left Gorenstein regular is not needed.

Proposition 4. *Let ${}_AC_1$ and ${}_BC_2$ be Σ-self-orthogonal, and $C = \mathbf{p}(C_1, C_2)$. Assume that U is weakly C-compatible, and consider the following assertions:*

1. *T is (strongly) CM-free relative to C;*
2. *A and B are (strongly) CM-free relative to C_1 and C_2, respectively.*

Then, 1. $\Rightarrow$ 2. If U is C-compatible, then 1. $\Leftrightarrow$ 2.

Proof. We only prove the result for relative strongly CM-free, since the case of relative CM-free is similar.

1. $\Rightarrow$ 2. By the remark above, we only need to prove that $G_{C_1}P(A) \subseteq \mathrm{Add}_A(C_1)$ and $G_{C_2}P(B) \subseteq \mathrm{Add}_B(C_2)$. Let M_1 be a G_{C_1}-projective A-module and ${}_BM_2$ a G_{C_2}-projective B-module. By the assumption and Proposition 3, $\mathbf{p}(M_1, M_2) \in G_CP(T) = \mathrm{Add}_T(C)$. Hence, $M_1 \in \mathrm{Add}_A(C_1)$ and $M_2 \in \mathrm{Add}_B(C_2)$ by Lemma 4.

2. $\Rightarrow$ 1. Assume U is C-compatible. Clearly, C is Σ-self-orthogonal, then by Remark above, we only need to prove that $G_CP(T) \subseteq \mathrm{Add}_T(C)$. Let $M = \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}_{\varphi^M}$ be a G_C-projective T-module. By the assumption and Theorem 1, $M_1 \in G_{C_1}P(A) = \mathrm{Add}_A(C_1)$ and $\overline{M}_2 \in G_{C_2}P(B) = \mathrm{Add}_B(C_2)$, and the map φ^M is injective. By the assumption, we can easily see that $\mathrm{Ext}_B^{i\geq 1}(U \otimes_A M_1, \overline{M}_2) = 0$. Therefore, the map $0 \to U \otimes_A M_1 \overset{\varphi^M}{\to} M_2 \to \overline{M}_2 \to 0$ splits. Hence, $M \cong \mathbf{p}(M_1, \overline{M}_2) \in \mathrm{Add}_T(C)$ by Lemma 4. □

Our aim now is to study special $G_CP(T)$-precovers in T-Mod. We start with the following result.

Proposition 5. *Let $C = \mathbf{p}(C_1, C_2)$ be w-tilting, U be C-compatible, and $M = \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}_{\varphi^M}$ and $G = \begin{pmatrix} G_1 \\ G_2 \end{pmatrix}_{\varphi^G}$ two T-modules with G G_C-projective. Then:*

$$f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} : G \longrightarrow M$$

is a special $G_CP(T)$-precover if and only if:

(i) *$G_1 \overset{f_1}{\to} M_1$ is a special $G_{C_1}P(A)$-precover;*

(ii) *$G_2 \overset{f_2}{\to} M_2$ is surjective with its kernel lying in $G_{C_2}P(B)^{\perp_1}$.*

In this case, if $G_2 \in G_{C_2}P(B)$, then $G_2 \overset{f_2}{\to} M_2$ is a special $G_{C_2}P(B)$-precover.

Proof. First of all, let $K = \mathrm{Ker} f = \begin{pmatrix} K_1 \\ K_2 \end{pmatrix}_{\varphi^K}$, and note that, since C_1 is w-tilting, $\mathrm{Tor}_1^A(U, H_1) = 0$ for every $H_1 \in G_{C_1}P(A)$ by Proposition 1(1).

$\Rightarrow$ Since the map f is surjective, so are f_1 and f_2. Let $H_1 \in G_{C_1}P(A)$ and $H_2 \in G_{C_2}P(B)$. Then, $\begin{pmatrix} H_1 \\ U \otimes_A H_1 \end{pmatrix}, \begin{pmatrix} 0 \\ H_2 \end{pmatrix} \in G_C P(T)$ by Theorem 1. Using Lemma 3 and the fact that K lies in $G_C P(R)^{\perp_1}$, we obtain that:

$$\mathrm{Ext}^1_A(H_1, K_1) \cong \mathrm{Ext}^1_T(\begin{pmatrix} H_1 \\ U \otimes_A H_1 \end{pmatrix}, K) = 0$$

and:

$$\mathrm{Ext}^1_B(H_2, K_2) \cong \mathrm{Ext}^1_T(\begin{pmatrix} 0 \\ H_2 \end{pmatrix}, K) = 0.$$

It remains to see that $G_1 \in G_{C_1}P(A)$, which is true by Theorem 1, since G is G_C-projective.

$\Leftarrow$ The morphism f is surjective since f_1 and f_2 are. Therefore, we only need to prove that K lies in $G_C P(R)^{\perp_1}$. Let $H \in G_C P(R)$. By Theorem 1, we have the short exact sequence of T-modules:

$$0 \to \begin{pmatrix} H_1 \\ U \otimes_A H_1 \end{pmatrix} \to H \to \begin{pmatrix} 0 \\ \overline{H}_2 \end{pmatrix} \to 0$$

where H_1 is G_{C_1}-projective and $\overline{H}_2$ is G_{C_2}-projective. Therefore, by hypothesis and Lemma 3, we obtain that $\mathrm{Ext}^1_T(\begin{pmatrix} H_1 \\ U \otimes_A H_1 \end{pmatrix}, K) \cong \mathrm{Ext}^1_A(H_1, K_1) = 0$ and $\mathrm{Ext}^1_T(\begin{pmatrix} 0 \\ \overline{H}_2 \end{pmatrix}, K) \cong \mathrm{Ext}^1_B(\overline{H}_2, K_2)$ = 0. Then, the exactness of this sequence:

$$\mathrm{Ext}^1_T(\begin{pmatrix} H_1 \\ U \otimes_A H_1 \end{pmatrix}, K) \to \mathrm{Ext}^1_T(H, K) \to \mathrm{Ext}^1_T(\begin{pmatrix} 0 \\ \overline{H}_2 \end{pmatrix}, K)$$

implies that $\mathrm{Ext}^1_T(H, K) = 0$. □

Theorem 2. *Let $C = \mathbf{p}(C_1, C_2)$ be w-tilting and U C-compatible. Then, the class $G_C P(T)$ is special precovering in T-Mod if and only if the classes $G_{C_1}P(A)$ and $G_{C_2}P(B)$ are special precovering in A-Mod and B-Mod, respectively.*

Proof. $\Rightarrow$ Let M_1 be an A-module and $\begin{pmatrix} G_1 \\ G_2 \end{pmatrix}_{\varphi^G} \to \begin{pmatrix} M_1 \\ 0 \end{pmatrix}$ be a special $G_C P(T)$-precover in T-Mod. Then, by Proposition 5, $G_1 \to M_1$ is a special $G_{C_1}P(A)$-precover in A-Mod.

Let M_2 be a B-module and $\begin{pmatrix} 0 \\ f_2 \end{pmatrix} : \begin{pmatrix} G_1 \\ G_2 \end{pmatrix}_{\varphi^G} \to \begin{pmatrix} 0 \\ M_2 \end{pmatrix}$ be a special $G_C P(T)$-precover in T-Mod. By Proposition 5, $G_1 \to 0$ is a special $G_{C_1}P(A)$-precover. Then, $\mathrm{Ext}^1_A(G_{C_1}P(A), G_1) = 0$. On the other hand, by [13], Proposition 2.8, there exists an exact sequence of A-modules:

$$0 \to G_1 \to X_1 \to H_1 \to 0$$

where $X_1 \in \mathrm{Add}_A(C_1)$ and H_1 is G_{C_1}-projective. However, this sequence splits, since $\mathrm{Ext}^1_A(H_1, G_1) = 0$, which implies that $G_1 \in \mathrm{Add}_A(C_1)$. Let $K = \begin{pmatrix} K_1 \\ K_2 \end{pmatrix}_{\varphi^K}$ be the kernel of $\begin{pmatrix} 0 \\ f_2 \end{pmatrix}$. Note that $K_1 = G_1$. Therefore, there exists a commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & U\otimes_A G_1 & = & U\otimes_A G_1 & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \varphi^K} & & \downarrow{\scriptstyle \varphi^G} & & \downarrow & & \\
0 & \longrightarrow & K_2 & \longrightarrow & G_2 & \longrightarrow & M_2 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
 & & \overline{K}_2 & \longrightarrow & \overline{G}_2 & \longrightarrow & M_2 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Using the snake lemma, there exists an exact sequence of B-modules:

$$0 \to \overline{K}_2 \to \overline{G}_2 \to M_2 \to 0$$

where $\overline{G}_2$ is G_{C_2}-projective by Theorem 1. It remains to see that $\overline{K}_2$ lies in $G_{C_2}P(B)^{\perp_1}$. Let $H_2 \in G_{C_2}P(B)$. Then, $\mathrm{Ext}_B^1(H_2, K_2) = 0$ by Proposition 5 and $\mathrm{Ext}_B^{i\geq 1}(H_2, U\otimes_A G_1) = 0$ by Proposition 1(2). From the above diagram, φ^K is injective. Therefore, if we apply the functor $\mathrm{Hom}_B(H_2, -)$ to the short exact sequence:

$$0 \to U\otimes_A G_1 \to K_2 \to \overline{K}_2 \to 0,$$

we obtain an exact sequence:

$$\mathrm{Ext}_B^1(H_2, K_2) \to \mathrm{Ext}_B^1(H_2, \overline{K}_2) \to \mathrm{Ext}_B^2(H_2, U\otimes_A G_1)$$

which implies that $\mathrm{Ext}_B^1(H_2, \overline{K}_2) = 0$.

$\Leftarrow$ Note that the functor $U\otimes_A - : A$-Mod$\to B$-Mod is $G_{C_1}P(A)$-exact since $\mathrm{Tor}_1^A(U, G_{C_1}P(A)) = 0$ by Proposition 1. Therefore, this direction follows by [27], Theorem 1.1, since $G_CP(T) = \{M = \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}_{\varphi^M} \in T\text{-Mod} \mid M_1 \in G_{C_1}P(A),\ \overline{M_2} \in G_{C_2}P(B)$ and φ^M is injective$\}$ by Theorem 1. □

Corollary 5. *Let R be a ring, $T(R) = \begin{pmatrix} R & 0 \\ R & R \end{pmatrix}$, and $C = \mathbf{p}(C_1, C_1)$ a w-tilting $T(R)$-module. Then, $G_CP(T(R))$ is a special precovering class if and only if $G_{C_1}P(R)$ is a special precovering class.*

5. Relative Global Gorenstein Dimension

In this section, we investigate the G_C-projective dimension of T-modules and the left G_C-projective global dimension of T.

Let R be a ring. Recall [13] that a module M is said to have a G_C-projective dimension less than or equal to n, $G_C{-}pd(M) \leq n$, if there is an exact sequence:

$$0 \to G_n \to \cdots \to G_0 \to M \to 0$$

with $G_i \in G_CP(R)$ for every $i \in \{0, \cdots, n\}$. If n is the least nonnegative integer for which such a sequence exists, then $G_C{-}pd(M) = n$, and if there is no such n, then $G_C{-}pd(M) = \infty$.

The left G_C-projective global dimension of R is defined as:

$$G_C - PD(R) = sup\{G_C{-}pd(M) \mid M \text{ is an } R\text{-module}\}$$

Lemma 7. *Let $C = \mathbf{p}(C_1, C_2)$ be w-tilting and U C-compatible.*

1. $G_{C_2}-pd(\mathrm{M}_2) = G_C-pd(\begin{pmatrix} 0 \\ M_2 \end{pmatrix})$.
2. $G_{C_1}-pd(\mathrm{M}_1) \leq G_C-pd(\begin{pmatrix} M_1 \\ U\otimes_A M_1 \end{pmatrix})$, *and the equality holds if* $\mathrm{Tor}^A_{i\geq 1}(U, M_1) = 0$.

Proof. 1. Let $n \in \mathbb{N}$, and consider an exact sequence of B-modules:

$$0 \to K_2^n \to G_2^{n-1} \to \cdots \to G_2^0 \to M_2 \to 0$$

where each G_2^i is G_{C_2}-projective. Thus, there exists an exact sequence of T-modules:

$$0 \to \begin{pmatrix} 0 \\ K_2^n \end{pmatrix} \to \begin{pmatrix} 0 \\ G_2^{n-1} \end{pmatrix} \to \cdots \to \begin{pmatrix} 0 \\ G_2^0 \end{pmatrix} \to \begin{pmatrix} 0 \\ M_2 \end{pmatrix} \to 0$$

where each $\begin{pmatrix} 0 \\ G_2^i \end{pmatrix}$ is G_C-projective by Theorem 1. Again, by Theorem 1, $\begin{pmatrix} 0 \\ K_2^n \end{pmatrix}$ is G_C-projective if and only if K_2^n is G_{C_1}-projective, which means that $G_C-pd(\begin{pmatrix} 0 \\ M_2 \end{pmatrix}) \leq n$ if and only if $G_{C_2}-pd(M_2) \leq n$ by [13], Theorem 3.8. Hence $G_C-pd(\begin{pmatrix} 0 \\ M_2 \end{pmatrix}) = G_{C_2}-pd(M_2)$;

2. We may assume that $n = G_C-pd(\begin{pmatrix} M_1 \\ U\otimes_A M_1 \end{pmatrix}) < \infty$. By Definition, there exists an exact sequence of T-modules:

$$0 \to G^n \to G^{n-1} \to \cdots \to G^0 \to \begin{pmatrix} M_1 \\ U\otimes_A M_1 \end{pmatrix} \to 0$$

where each $G^i = \begin{pmatrix} G_1^i \\ G_2^i \end{pmatrix}_{\varphi^{G^i}}$ is G_C-projective. Thus, there exists an exact sequence of A-modules:

$$0 \to G_1^n \to G_1^{n-1} \to \cdots \to G_1^0 \to M_1 \to 0$$

where each G_1^i is G_{C_1}-projective by Theorem 1. Therefore, $G_{C_1}-pd(M_1) \leq n$. Conversely, we prove that $G_C-pd(\begin{pmatrix} M_1 \\ U\otimes_A M_1 \end{pmatrix}) \leq G_{C_1}-pd(M_1)$. We may assume that $m := G_{C_1}-pd(M_1) < \infty$. The hypothesis means that if:

$$\mathbf{X}_1 \ : \ 0 \to K_1^m \to P_1^{m-1} \to \cdots \to P_1^0 \to M_1 \to 0$$

is an exact sequence of A-modules where each P_1^i is projective, then the complex $U \otimes_A \mathbf{X}_1$ is exact. Since C_1 is w-tilting, each P_i is G_{C_1}-projective by [13], Proposition 2.11, and then, K^m is G_{C_1}-projective by [13], Theorem 3.8. Thus, there exists an exact sequence of T-modules

$$0 \to \begin{pmatrix} K_1^m \\ U\otimes_A K_1^m \end{pmatrix} \to \begin{pmatrix} P_1^{m-1} \\ U\otimes_A P_1^{m-1} \end{pmatrix} \to \cdots \to \begin{pmatrix} P_1^0 \\ U\otimes_A P_1^0 \end{pmatrix} \to \begin{pmatrix} M_1 \\ U\otimes_A M_1 \end{pmatrix} \to 0,$$

where $\begin{pmatrix} K_1^m \\ U\otimes_A K_1^m \end{pmatrix}$ and all $\begin{pmatrix} P_1^i \\ U\otimes_A P_1^i \end{pmatrix}$ are G_C-projectives by Theorem 1. Therefore, $G_C-pd(\begin{pmatrix} M_1 \\ U\otimes_A M_1 \end{pmatrix}) \leq m = G_{C_1}-pd(M_1)$. □

Given a T-module $C = \mathbf{p}(C_1, C_2)$, we introduce a strong notion of the G_{C_2}-projective global dimension of B, which will be crucial when we estimate the G_C-projective dimension of a T-module and the left global G_C-projective dimension of T. Set:

$$SG_{C_2} - PD(B) = sup\{G_{C_2}-pd_B(U\otimes_A G) \mid G \in G_{C_1}P(A)\}.$$

Remark 4.

1. *Clearly,* $SG_{C_2} - PD(B) \leq G_{C_2} - PD(B)$*;*
2. *Note that* $pd_B(U) = sup\{pd_B(U \otimes_A P) \mid {}_AP$ *is projective* $\}$*. Therefore, in the classical case, the strong left global dimension of* B *is nothing but the projective dimension of* ${}_BU$*.*

Theorem 3. *Let* $C = \boldsymbol{p}(C_1, C_2)$ *be w-tilting,* U *C-compatible,* $M = \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}_{\varphi^M}$ *a T-module, and* $SG_{C_2} - PD(B) < \infty$*. Then:*

$$max\{G_{C_1}-pd_A(M_1), (G_{C_2}-pd_B(M_2)) - (SG_{C_2} - PD(B))\}$$
$$\leq G_C-pd(M) \leq$$
$$max\{(G_{C_1}-pd_A(M_1)) + (SG_{C_2} - PD(B)) + 1, G_{C_2}-pd_B(M_2)\}$$

Proof. First of all, note that C_1 and C_2 are w-tilting by Proposition 3, and let $k := SG_{C_2} - PD(B)$.

Let us first prove that:

$$max\{G_{C_1}-pd(M_1), G_{C_2}-pd(M_2) - k\} \leq G_C-pd(M).$$

We may assume that $n := G_C-pd(M) < \infty$. Then, there exists an exact sequence of T-modules:

$$0 \to G^n \to G^{n-1} \to \cdots \to G^0 \to M \to 0$$

where each $G^i = \begin{pmatrix} G_1^i \\ G_2^i \end{pmatrix}_{\varphi^{G^i}}$ is G_C-projective. Thus, there exists an exact sequence of A-modules:

$$0 \to G_1^n \to G_1^{n-1} \to \cdots \to G_1^0 \to M_1 \to 0$$

where each G_1^i is G_{C_1}-projective by Theorem 1. Therefore, $G_{C_1}-pd(M_1) \leq n$. By Theorem 1, for each i, there exists an exact sequence of B-modules:

$$0 \to U \otimes_A G_1^i \to G_2^i \to \overline{G_2^i} \to 0$$

where $\overline{G_2^i}$ is G_{C_2}-projective. Then, $G_{C_2}-pd(G_2^i) = G_{C_2}-pd(U \otimes_A G_1^i) \leq k$ by [13], Proposition 3.11. Therefore, using the exact sequence of B-modules:

$$0 \to G_2^n \to G_2^{n-1} \to \cdots \to G_2^0 \to M_2 \to 0$$

and [13], Proposition 3.11(4), we obtain that $G_{C_2}-pd(M_2) \leq n + k$.

Next we prove that:

$$G_C-pd(M) \leq max\{G_{C_1}-pd(M_1) + k + 1, G_{C_2}-pd(M_2)\}.$$

We may assume that:

$$m := max\{G_{C_1}-pd(M_1) + k + 1, G_{C_2}-pd(M_2)\} < \infty.$$

Then, $n_1 := G_{C_1}-pd(M_1) < \infty$ and $n_2 := G_{C_2}-pd(M_2) < \infty$. Since $G_{C_1}-pd(M_1) = n_1 \leq m - k - 1$, there exists an exact sequence of A-modules:

$$0 \to G_1^{m-k-1} \to \cdots \to G_1^{n_2-k} \to \cdots \xrightarrow{f_1^1} G_1^0 \xrightarrow{f_1^0} M_1 \to 0$$

where each G_1^i is G_{C_1}-projective. Since C_2 is w-tilting, there exists an exact sequence of B-modules $G_2^0 \xrightarrow{g_2^0} M_2 \to 0$ where G_2^0 is G_{C_2}-projective by [13], Corollary 2.14. Let $K_1^i = \mathrm{Ker} f_1^i$, and define the map $f_2^0 : U \otimes_A G_1^0 \oplus G_2^0 \to M_2$ to be $(\varphi^M(1_U \otimes f_1^0)) \oplus g_2^0$. Then, we obtain an exact sequence of T-modules:

$$0 \to \begin{pmatrix} K_1^1 \\ K_2^1 \end{pmatrix}_{\varphi^{K^1}} \to \begin{pmatrix} G_1^0 \\ (U \otimes_A G_1^0) \oplus G_2^0 \end{pmatrix} \overset{\begin{pmatrix} f_1^0 \\ f_2^0 \end{pmatrix}}{\to} M \to 0.$$

Similarly, there exists an exact sequence of B-modules $G_2^1 \overset{g_2^1}{\to} K_2^1 \to 0$ where G_2^1 is G_{C_2}-projective, and then, we obtain an exact sequence of T-modules:

$$0 \to \begin{pmatrix} K_1^2 \\ K_2^2 \end{pmatrix}_{\varphi^{K^2}} \to \begin{pmatrix} G_1^1 \\ (U \otimes_A G_1^1) \oplus G_2^1 \end{pmatrix} \to \begin{pmatrix} K_1^1 \\ K_2^1 \end{pmatrix}_{\varphi^{K^1}} \to 0.$$

Repeating this process, we obtain the exact sequence of T-modules:

$$0 \to \begin{pmatrix} 0 \\ K_2^{m-k} \end{pmatrix} \to \begin{pmatrix} G_1^{m-k-1} \\ (U \otimes_A G_1^{m-k-1}) \oplus G_2^{m-k-1} \end{pmatrix} \overset{\begin{pmatrix} f_1^{m-k-1} \\ f_2^{m-k-1} \end{pmatrix}}{\longrightarrow}$$

$$\cdots \to \begin{pmatrix} G_1^1 \\ (U \otimes_A G_1^1) \oplus G_2^1 \end{pmatrix} \overset{\begin{pmatrix} f_1^1 \\ f_2^1 \end{pmatrix}}{\longrightarrow} \begin{pmatrix} G_1^0 \\ (U \otimes_A G_1^0) \oplus G_2^0 \end{pmatrix} \overset{\begin{pmatrix} f_1^0 \\ f_2^0 \end{pmatrix}}{\longrightarrow} M \to 0$$

Note that $G_{C_2}-pd((U \otimes_A G_1^i) \oplus G_2^i) = G_{C_2}-pd(U \otimes_A G_1^i) \leq k$, for every $i \in \{0, \cdots, m-k-1\}$. Therefore, by [13], Proposition 3.11(2), and the exact sequence $0 \to K_2^{m-k} \to (U \otimes_A G_1^{m-k-1}) \oplus G_2^{m-k-1} \overset{f_2^{m-k-1}}{\longrightarrow} \cdots \to (U \otimes_A G_1^0) \oplus G_2^0 \overset{f_2^0}{\to} M_2 \to 0$, we obtain that $G_{C_2}-pd(K_2^{m-k}) \leq k$. This means that there exists an exact sequence of B-modules:

$$0 \to G_2^m \to \cdots \to G_2^{m-k+1} \to G_2^{m-k} \to K_2^{m-k} \to 0.$$

Thus, there exists an exact sequence of T-modules:

$$0 \to \begin{pmatrix} 0 \\ G_2^m \end{pmatrix} \to \cdots \to \begin{pmatrix} 0 \\ G_2^{m-k+1} \end{pmatrix} \to$$

$$\begin{pmatrix} 0 \\ G_2^{m-k} \end{pmatrix} \to \begin{pmatrix} G_1^{m-k-1} \\ (U \otimes_A G_1^{m-k-1}) \oplus G_2^{m-k-1} \end{pmatrix} \overset{\begin{pmatrix} f_1^{m-k-1} \\ f_2^{m-k-1} \end{pmatrix}}{\longrightarrow}$$

$$\cdots \to \begin{pmatrix} G_1^1 \\ (U \otimes_A G_1^1) \oplus G_2^1 \end{pmatrix} \overset{\begin{pmatrix} f_1^1 \\ f_2^1 \end{pmatrix}}{\longrightarrow} \begin{pmatrix} G_1^0 \\ (U \otimes_A G_1^0) \oplus G_2^0 \end{pmatrix} \overset{\begin{pmatrix} f_1^0 \\ f_2^0 \end{pmatrix}}{\longrightarrow} M \to 0.$$

By Theorem 1, all $\begin{pmatrix} G_1^i \\ (U \otimes_A G_1^i) \oplus G_2^i \end{pmatrix}$ and all $\begin{pmatrix} 0 \\ G_2^j \end{pmatrix}$ are G_C-projectives. Thus, $G_C-pd(M) \leq m$. □

The following consequence of Theorem 3 extends [2], Proposition 2.8(1), and [3], Theorem 2.7(1), to the relative setting.

Corollary 6. *Let $C = \boldsymbol{p}(C_1, C_2)$ be w-tilting, U C-compatible and $M = \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}_{\varphi^M}$ a T-module. If $SG_{C_2} - PD(B) < \infty$, then $G_C-pd(M) < \infty$ if and only if $G_{C_1}-pd(\mathrm{M}_1) < \infty$ and $G_{C_2}-pd(\mathrm{M}_2) < \infty$.*

The following theorem gives an estimate of the left G_C-projective global dimension of T.

Theorem 4. *Let $C = \boldsymbol{p}(C_1, C_2)$ be w-tilting and U C-compatible. Then:*

$$max\{G_{C_1} - PD(A), G_{C_2} - PD(B)\}$$
$$\leq G_C - PD(T) \leq$$
$$max\{G_{C_1} - PD(A) + SG_{C_2} - PD(B) + 1, G_{C_2} - PD(B)\}.$$

Proof. We prove first that $max\{G_{C_1} - PD(A), G_{C_2} - PD(B)\} \leq G_C - PD(T)$. We may assume that $n := G_C - PD(T) < \infty$. Let M_1 be an A-module and M_2 be a B-module. Since $G_C - pd(\begin{pmatrix} M_1 \\ U \otimes_A M_2 \end{pmatrix}) \leq n$ and $G_C - pd(\begin{pmatrix} 0 \\ M_2 \end{pmatrix}) \leq n$, $G_{C_1} - pd(M_1) \leq n$ and $G_{C_2} - pd(M_2) \leq n$ by Lemma 7. Thus, $G_{C_1} - PD(A) \leq n$ and $G_{C_2} - PD(B) \leq n$.

Next, we prove that:

$$G_C - PD(T) \leq max\{G_{C_1} - PD(A) + 1 + SG_{C_2} - PD(B), G_{C_2} - PD(B)\}.$$

We may assume that:

$$m := max\{G_{C_1} - PD(A) + 1 + SG_{C_2} - PD(B), G_{C_2} - PD(B)\} < \infty.$$

Then, $n_1 := G_{C_1} - PD(A) < \infty$ and $k := SG_{C_2} - PD(B) \leq n_2 := G_{C_2} - PD(B) < \infty$

Let $M = \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}_{\varphi^M}$ be a T-module. By Theorem 3,

$$G_C - pd(M) \leq max\{n_1 + k + 1, n_2\} \leq m.$$

□

Corollary 7. *Let $C = \boldsymbol{p}(C_1, C_2)$ be w-tilting and U C-compatible. Then, $G_C - PD(T) < \infty$ if and only if $G_{C_1} - PD(A) < \infty$ and $G_{C_2} - PD(B) < \infty$*

Recall that a ring R is called left Gorenstein regular if the category R-Mod is Gorenstein ([2], Definition 2.1, and [28], Definition 2.18).

We know by [29], Theorem 1.1, that the following equality holds:

$$sup\{Gpd_R(M) \mid M \in R\text{-Mod}\} = sup\{\mathrm{Gid}_R(M) \mid M \in R\text{-Mod}\}.$$

and this common value is call the left global Gorenstein dimension of R, denoted by $l.\mathrm{Ggldim}(R)$. As a consequence of [28], Theorem 2.28, a ring R is left Gorenstein regular if and only if the global Gorenstein dimension of R is finite.

We shall say that a ring R is left n-Gorenstein regular if $n = l.\mathrm{Ggldim}(R) < \infty$.

Enochs, Izurdiaga, and Torrecillas characterized in [2], Theorem 3.1, when T is left Gorenstein regular under the conditions that ${}_BU$ has finite projective dimension and U_A has finite flat dimension. As a direct consequence of Corollary 7, we refine this result.

Corollary 8. *Assume that U is compatible. Then, T is left Gorenstein regular if and only if so are A and B.*

There are some cases when the estimate in Theorem 4 becomes an exact formula, which computes left the G_C-projective global dimension of T.

Recall that an injective cogenerator E in R-Mod is said to be strong if any R-module embeds in a direct sum of copies of E.

Corollary 9. *Let $C = \boldsymbol{p}(C_1, C_2)$ be w-tilting and U C-compatible.*

1. *If $U = 0$ then:*

$$G_C - PD(T) = max\{G_{C_1} - PD(A), G_{C_2} - PD(B)\};$$

2. *If A is left Noetherian and ${}_AC_1$ is a strong injective cogenerator, then:*

$$G_C - PD(T) = \begin{cases} G_{C_2} - PD(B) & \text{if } U = 0 \\ max\{SG_{C_2} - PD(B) + 1, G_{C_2} - PD(B)\} & \text{if } U \neq 0. \end{cases}$$

Proof. 1. Using a similar way as we do in the proof of Theorems 3 and 4, we can prove this statement. We only need to notice that if $U = 0$, then a T-module $M = \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}_{\varphi^M}$ is G_C-projective if and only if M_1 is G_{C_1}-projective and M_2 is G_{C_2}-projective (since φ^M is always injective and $M_2 = \overline{M}_2$) by Theorem 1;

2. Note first that $G_{C_1} - PD(A) = 0$ by [14], Corollary 2.3. Then, the case $U = 0$ follows by 1. Assume that $U \neq 0$. Note that by Theorem 1, $\begin{pmatrix} A \\ 0 \end{pmatrix}$ is not G_C-projective since $U \neq 0$. Hence, $G_{C_2} - PD(B) \geq G_C - pd_T(\begin{pmatrix} A \\ 0 \end{pmatrix}) \geq 1$.

By Theorem 4, we have the inequality:

$$G_{C_2} - PD(B) \leq G_C - PD(T) \leq max\{SG_{C_2} - PD(B) + 1, G_{C_2} - PD(B)\}.$$

Therefore, the case $SG_{C_2} - PD(B) + 1 \leq G_{C_2} - PD(B)$ is clear, and we only need to prove the result when $SG_{C_2} - PD(B) + 1 > n := G_{C_2} - PD(B)$. Since $G_{C_2} - pd(U \otimes_A G) \leq G_{C_2} - PD(B) = n$ for every $G \in G_{C_1}P(A)$, $SG_{C_2} - PD(B) = n$. Let G_1 be a G_{C_1}-projective A-module with $G_{C_2} - pd(U \otimes_A G_1) = n$, and consider the following short exact sequence:

$$0 \to \begin{pmatrix} 0 \\ U \otimes_A G_1 \end{pmatrix} \to \begin{pmatrix} G_1 \\ U \otimes_A G_1 \end{pmatrix} \to \begin{pmatrix} G_1 \\ 0 \end{pmatrix} \to 0.$$

By Theorem 1, $\begin{pmatrix} G_1 \\ U \otimes_A G_1 \end{pmatrix}$ is G_C-projective and by Lemma 7:

$$G_C - pd(\begin{pmatrix} 0 \\ U \otimes_A G_1 \end{pmatrix}) = G_{C_2} - pd(U \otimes_A G) = n.$$

Thus, by [13], Proposition 3.11(4):

$$G_C - pd(\begin{pmatrix} G_1 \\ 0 \end{pmatrix}) = G_C - pd(\begin{pmatrix} 0 \\ U \otimes_A G_1 \end{pmatrix}) + 1 = n + 1 = SG_{C_2} - PD(B) + 1.$$

This shows that $G_C - PD(T) = SG_{C_2} - PD(B) + 1$, and the proof is finished. □

Corollary 10. *Let R be a ring, $T(R) = \begin{pmatrix} R & 0 \\ R & R \end{pmatrix}$ and $C = \mathbf{p}(C_1, C_1)$ where C_1 is w-tilting. Then:*

$$G_C - PD(T(R)) = G_{C_1} - PD(R) + 1.$$

Proof. Note first that C is a w-tilting $T(R)$-module, R is C-compatible, and $SG_{C_1} - PD(R) = 0$. Therefore, by Theorem 4,

$$G_{C_1} - PD(R) \leq G_C - PD(T(R)) \leq G_{C_1} - PD(R) + 1.$$

The case $G_{C_1} - PD(R) = \infty$ is clear. Assume that $n := G_{C_1} - PD(R) < \infty$.

There exists an R-module M with $G_{C_1}-pd(M) = n$ and $\operatorname{Ext}^n_R(M,X) \neq 0$ for some $X \in \operatorname{Add}_R(C_1)$ by [13], Theorem 3.8. If we apply the functor $\operatorname{Hom}_{T(R)}(-, \begin{pmatrix} 0 \\ X \end{pmatrix})$ to the exact sequence of $T(R)$-modules:

$$0 \to \begin{pmatrix} 0 \\ M \end{pmatrix} \to \begin{pmatrix} M \\ M \end{pmatrix}_{1_M} \to \begin{pmatrix} M \\ 0 \end{pmatrix} \to 0$$

we obtain an exact sequence

$$\cdots \to \operatorname{Ext}^n_{T(R)}(\begin{pmatrix} M \\ M \end{pmatrix}, \begin{pmatrix} 0 \\ X \end{pmatrix}) \to \operatorname{Ext}^n_{T(R)}(\begin{pmatrix} 0 \\ M \end{pmatrix}, \begin{pmatrix} 0 \\ X \end{pmatrix}) \to$$

$$\operatorname{Ext}^{n+1}_{T(R)}(\begin{pmatrix} M \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ X \end{pmatrix}) \to \operatorname{Ext}^{n+1}_{T(R)}(\begin{pmatrix} M \\ M \end{pmatrix}, \begin{pmatrix} 0 \\ X \end{pmatrix}) \to \cdots$$

By Lemma 3, $\operatorname{Ext}^{i\geq 1}_{T(R)}(\begin{pmatrix} M \\ M \end{pmatrix}, \begin{pmatrix} 0 \\ X \end{pmatrix}) \cong \operatorname{Ext}^{i\geq 1}_R(M,0) = 0$. Again, by Lemma 3 and the above exact sequence,

$$\operatorname{Ext}^{n+1}_{T(R)}(\begin{pmatrix} M \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ X \end{pmatrix}) \cong \operatorname{Ext}^n_{T(R)}(\begin{pmatrix} 0 \\ M \end{pmatrix}, \begin{pmatrix} 0 \\ X \end{pmatrix}) \cong \operatorname{Ext}^n_R(M,X) \neq 0.$$

since $\begin{pmatrix} 0 \\ X \end{pmatrix} \in \operatorname{Add}_{T(R)}(C)$ by Lemma 4(1), it follows that $n < G_C-pd(\begin{pmatrix} M \\ 0 \end{pmatrix})$ by [13], Theorem 3.8. However, $G_C-pd(\begin{pmatrix} M \\ 0 \end{pmatrix}) \leq G_C - PD(T(R)) \leq n+1$. Thus, $G_C-pd(\begin{pmatrix} M \\ 0 \end{pmatrix}) = n+1$, which means that $G_C - PD(T(R)) = n+1$. □

Corollary 11. *Let R be a ring, $T(R) = \begin{pmatrix} R & 0 \\ R & R \end{pmatrix}$, and $n \geq 0$ an integer. Then, $T(R)$ is left $(n+1)$-Gorenstein regular if and only if R is left n-Gorenstein regular .*

The authors in [16] established a relationship between the projective dimension of modules over T and modules over A and B. Given an integer $n \geq 0$ and $M = \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}_{\varphi^M}$ a T-module, they proved that $pd_T(M) \leq n$ if and only if $pd_A(M_1) \leq n$, $pd_B(\overline{M}_2) \leq n$, and the map related to the n-th syzygy of M is injective. The following example shows that this is not true in general.

Example 4. *Let R be a left hereditary ring that is not semisimple, and let $T(R) = \begin{pmatrix} R & 0 \\ R & R \end{pmatrix}$. Then, $lD(T(R)) = lD(R) + 1 = 2$ by [24], Corollary 3.4(3). This means that there exists a $T(R)$-module $M = \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}_{\varphi^M}$ with $pd_{T(R)}(M) = 2$. If $K^1 = \begin{pmatrix} K^1_1 \\ K^1_2 \end{pmatrix}_{\varphi^{K^1}}$ is the first syzygy of M, then there exists an exact sequence of $T(R)$-modules:*

$$0 \to K^1 \to P \to M \to 0$$

where $P = \begin{pmatrix} P_1 \\ P_2 \end{pmatrix}_{\varphi^P}$ is projective. Then, we obtain the following commutative diagram:

$$
\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \downarrow & & & & \\
0 & \longrightarrow & K_1^1 & \longrightarrow & P_1 & \longrightarrow & M_1 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \varphi^{K^1_1}} & & \downarrow{\scriptstyle \varphi^{P}} & & \downarrow{\scriptstyle \varphi^{M}} & & \\
0 & \longrightarrow & K_2^1 & \longrightarrow & P_2 & \longrightarrow & M_2 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \overline{K_2^1} & \longrightarrow & \overline{P}_2 & \longrightarrow & \overline{M}_2 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

By the snake lemma, φ^{K^1} is injective. On the other hand, since $lD(R) = 1$, $pd_R(M_1) \leq 1$ and $pd_R(\overline{M}_2) \leq 1$. However, $pd_{T(R)}(M) = 2 > 1$.

Author Contributions: Investigation, D.B., R.E.M., J.R.G.R. and L.O.; Writing—original draft, R.E.M.; Writing—review & editing, J.R.G.R. and L.O. All authors have read and agreed to the published version of the manuscript.

Funding: This research was partially funded by MINISTERIO DE ECONOMÍA Y COMPTETITIVIDAD grant number PID2020-113552GB-I00 and Junta de Andalucía, grant number P20-00770.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: The authors would like to thank Javad Asadollahi for the discussion on Example 4.

Conflicts of Interest: The authors declare no conflict of interest.

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