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Subdomains-like Notions in Relative Homological Algebra

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SUBDOMAINS-LIKE NOTIONS IN RELATIVE HOMOLOGICAL ALGEBRA

NOCIONES ASOCIADAS A SUBDOMINIOS EN ÁLGEBRA HOMOLÓGICA RELATIVA

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Resumen

Propiedades homológicas de los módulos tales como la inyectividad, proyectividad, planitud, etc. se han considerado clásicamente como atributos que los módulos pueden tener o no tener. Al igual que los interruptores clásicos, sólo tienen las posiciones de encendido y apagado. Pero esto ha cambiado últimamente y una nueva tendencia comenzó hace algunos años: la idea es no etiquetar un módulo como "tiene la propiedad" o "no la tiene", sino estudiar hasta qué punto el módulo tiene la propiedad. En esta tesis doctoral se desarrollarán estos conceptos en ámbitos nuevos y muy interesantes del álgebra homológica.

El primer objetivo de esta tesis es introducir una perspectiva nueva y fresca sobre la planitud de los módulos. Sin embargo, primero investigamos un contexto más general introduciendo dominios relativos a una clase precovering $\mathcal{X}$. Llamamos a estos dominios de completación de $\mathcal{X}$-precubiertas y los denotamos por $\mathcal{X}^{-1}(\mathcal{L})$ para una clase de módulos $\mathcal{L}$. En particular, cuando $\mathcal{X}$ es la clase de módulos planos, los llamamos dominios de completación de precubiertas planas. Este enfoque nos permite unificar algunos conceptos homológicos conocidos. Eso conduce a la generalización de algunos resultados importantes, así como a la caracterización de algunos anillos clásicos en términos de estos dominios.

El segundo objetivo de esta tesis es investigar cuando cada módulo de una clase $\mathcal{L}$ tiene una $\mathcal{X}^{-1}(\mathcal{L})$-preenvolvente. También investigamos las $\mathcal{X}^{-1}(\mathcal{L})$-preenvolventes épicas y mónicas. Este estudio juega un papel clave en el establecimiento de un marco general para varios resultados clásicos. Luego, para una clase de módulos $\mathcal{M}$ finitamente generados, introducimos la noción de módulos $\mathcal{M}$-R-Mittag-Leffler como una extensión natural de los módulos R-Mittag-Leffler. Esto nos permite encontrar pruebas más fáciles de algunos resultados conocidos y también establecer otros nuevos.

Palabras claves. Dominios de subproyectividad, dominios de subinyectividad, dominios de pura subproyectividad, dominios de completación de $\mathcal{X}$-precubiertas, dominios de completación de precubiertas planas, precubiertas, preenvolventes, R-Mittag-Leffler modulos

Résumé

Certaines propriétés homologiques des modules telles que l'injectivité, la projectivité, la platitude, etc. ont été classiquement considérées comme des attributs que les modules peuvent avoir ou ne pas avoir. Tout comme le interrupteurs classiques, ils n'ont que les positions marche et arrêt. Mais cela a changé récemment et une nouvelle tendance est apparue il y a quelques années: l'idée n'est pas de considérer un module comme "il a la propriété" ou "il ne l'a pas", mais plutôt d'étudier jusqu'à quel point le module a la propriété. Ces types de concepts sont développés dans cette thèse dans des contextes nouveaux et intéressants dans l'algèbre homologique.

Le premier objectif de cette thèse est d'introduire une nouvelle perspective sur la platitude des modules. Cependant, nous étudions d'abord un contexte plus général en introduisant des domaines relatifs à une classe précouvrante $\mathcal{X}$. Ces domaines sont appelés domaines de $\mathcal{X}$-précouvertures complétées et on les note par $\mathcal{X}^{-1}(\mathcal{L})$ pour une classe de modules $\mathcal{L}$. En particulier, lorsque $\mathcal{X}$ est la classe des modules plats, nous les appelons domaines de précouvertures plates complétées. Cette approche nous permet d'unifier certains concepts homologiques connus. Ceci induit la généralisation de certains résultats importants ainsi que la caractérisation de certains anneaux classiques en fonction de ces domaines.

Le deuxième objectif de cette thèse est d'étudier quand est ce que chaque module d'une classe $\mathcal{L}$ possède une $\mathcal{X}^{-1}(\mathcal{L})$-préenveloppe. Les $\mathcal{X}^{-1}(\mathcal{L})$-préenveloppes surjectives et injectives sont également étudiées. Cette étude joue un rôle clé dans la mise en place d'un cadre général pour plusieurs résultats classiques. Ensuite, pour une classe de modules $\mathcal{M}$ de type fini, nous introduisons la notion de modules $\mathcal{M}$-R-Mittag-Leffler comme extension naturelle des modules R-Mittag-Leffler. Cela nous permet de retrouver plus facilement certains résultats connus, ainsi que d'en établir de nouveaux.

Mots Clés. Domaine de sous-projectivité, domaine de sous-injectivité, domaine de pure-subprojectivité, domaine de $\mathcal{X}$-précouvertures complétées, domaine de précouvertures plates complétées, précouvertures, préenvelopes, R-Mittag-Leffler modules

Summary

Homological properties of modules such as injectivity, projectivity, flatness etc. have been classically considered as attributes that the modules can either have or not to have. Just like the classical switches, they only have the on and off positions. But this has changed lately and a new trend started some years ago: the idea is not to label a module as “it has the property” or “it doesn’t have it”, but to study up to what degree the module has the property instead. These type of concepts are developed in this doctoral thesis in new and interesting contexts for the homological algebra.

The first aim of this thesis is to introduce a new and fresh perspective on flatness of modules. However, we first investigate a more general context by introducing domains relative to a precovering class $\mathcal{X}$. We call these domains $\mathcal{X}$-precover completing domains and denote them $\mathcal{X}^{-1}(\mathcal{L})$ for a class of modules $\mathcal{L}$. In particular, when $\mathcal{X}$ is the class of flat modules, we call them flat-precover completing domains. This approach allows us to unify some known homological concepts. This leads to the generalization of some important results as well as the characterization of some classical rings in terms of these domains.

The second aim of this thesis is to investigate when every module of a class $\mathcal{L}$ has an $\mathcal{X}^{-1}(\mathcal{L})$-preenvelope. Epic and monic $\mathcal{X}^{-1}(\mathcal{L})$-preenvelopes are also investigated. This study plays a key role in setting a general framework for several classical results. Then, for a class of finitely generated modules $\mathcal{M}$, we introduce the notion of $\mathcal{M}$-R-Mittag-Leffler modules as a natural extension of R-Mittag-Leffler modules. This enables us to find easier proofs of some known results and also to establish new ones.

Key Words. Subprojectivity domains, subinjectivity domains, pure-projectivity domains, $\mathcal{X}$-precover completing domains, flat-precover completing domains, precovers, preenvelopes, R-Mittag-Leffler modules

Contents

Introduction

Tout au long de cette thèse, R désigne un anneau associatif avec identité et les modules sont des R-modules unitaire à gauche, sauf mention contraire. Comme d'habitude, on note la catégorie des R-modules à gauche par R-Mod *et la catégorie des R-modules à droite par* Mod-R.

L'étude d'un point de vue relatif de certaines propriétés telles que la projectivité, l'injectivité et la platitude a été le centre d'intérêt principal de nombreux auteurs au fil des années (voir par exemple [9], [19], [22] et [23]). L'idée n'est pas de seulement dire si un module possède une propriété ou non, mais plutôt d'étudier à quel point il est proche de l'avoir. Ainsi, chaque module est assigné un domaine qui mesure à quel point il se rapproche de posséder la propriété.

D'abord, les notions relatives de projectivité et d'injectivité ont été introduites comme un outil pour évaluer l'étendue de ces propriétés pour un module donné. En effet, la projectivité et l'injectivité relatives sont définies comme suit (voir [19]):
Soit U un module. Si M est un module, alors U est projectif par rapport à M (ou U est M-projectif) si pour chaque épimorphisme $g : M \to N$ et chaque homomorphisme $f : U \to N$, il existe un homomorphisme $h : U \to M$ tel que le diagramme

$$\begin{array}{ccccc} & & U & & \\ & \overset{h}{\swarrow} & \downarrow f & & \\ M & \xrightarrow[g]{} & N & \longrightarrow & 0 \end{array}$$

commute. Le domaine de projectivité d'un module U est la classe de tous les modules M tels que U est M-projectif.

De même, on dit que U est injectif par rapport à M (ou U est M-injectif) dans le cas où pour chaque monomorphisme $g : K \to M$ et chaque homomorphisme

$f : K \to U$, il existe un morphisme $h : M \to U$ tel que le diagramme

$$\begin{array}{ccccc} & & U & & \\ & & f\uparrow & \nwarrow h & \\ 0 & \longrightarrow & K & \xrightarrow[g]{} & M \end{array}$$

commute. Le domaine d'injectivité d'un module U est la classe de tous les modules M tels que U est M-injectif.

Ensuite, parallèlement à la notion bien connue de l'injectivité relative, Aydoğdu et López-Permouth ont introduit dans [5] la notion de sous-injectivité. Puis, Holston et al. ont introduit dans [23] l'analogue projectif de la sous-injectivité et l'ont appelé la sous-projectivité. Rappelons la définition.

Soient M et N des modules. Alors, M est dit N-sous-projectif si pour tout épimorphisme $g : B \to N$ et tout homomorphisme $f : M \to N$, il existe un homomorphisme $h : M \to B$ tel que $gh = f$ (voir [23, Définition 2.1]). Le domaine de sous-projectivité d'un module M, dénoté $\underline{\mathfrak{Pr}}^{-1}(M)$, est défini comme étant la classe

$$\underline{\mathfrak{Pr}}^{-1}(M) := \{N \in R\text{-Mod} \ : M \text{ est } N\text{-sous-projectif}\}.$$

Il est facile de voir qu'un module est projectif si et seulement si son domaine de sous-projectivité est constitué de tous les modules (voir [23, Proposition 2.4]). Ainsi, les domaines de sous-projectivité mesurent la projectivité des modules.

D'autre part, l'étude de la platitude a été abordée dans [9] et [16] de deux différentes perspectives mais légèrement similaires car les deux sont basées sur le produit tensoriel. En effet, la platitude relative étudiée dans [9] est définie comme suit : Soient N un module à droite. Un R-module à gauche M est dit plat par rapport à N, relativement plat par rapport à N, ou N-plat si le morphisme canonique $K \otimes_R M \to N \otimes_R M$ est un monomorphisme pour tout sous-module K de N (voir [9, Définition 2.1]). Le domaine de platitude d'un module M, $\mathcal{F}^{-1}(M)$, est défini comme suit:

$$\mathcal{F}^{-1}(M) := \{N \in \text{Mod-}R \ : M \text{ est } N\text{-plat}\}.$$

Il est clair d'après la définition qu'un module est plat si et seulement si son domaine de platitude est égal à toute la catégorie des modules.

Durğun modifie dans [16] de manière subtile la notion de domaines de platitude et définit les domaines absolument purs. Cette fois, la définition est basée sur la notion de pureté. En effet, étant donné un module à gauche M et un module à droite N, N est dit absolument M-pur si $N \otimes_R M \to B \otimes_R M$ est un monomorphisme pour chaque extension B de N (voir [16, Définition 2.2]). Pour un module M, le domaine absolument pur de M, $\mathcal{A}p(M)$, est défini comme suit:

$$\mathcal{A}p(M) := \{N \in \text{Mod-}R \ : N \text{ est absolument } M\text{-pur}\}.$$

Clairement, un module M est plat si et seulement si $\mathcal{A}p(M) = \text{Mod-}R$.

Dans cette thèse, nous introduisons une nouvelle perspective sur la platitude des modules sans utiliser le produit tensoriel. Cependant, nous commençons d'abord par définir plusieurs domaines dans un cadre général et introduisons des domaines relatifs à une classe précouvrante $\mathcal{X}$. Ces domaines sont appelés les domaines de $\mathcal{X}$-précouvertures complétées car un diagramme impliquant des précouvertures peut être complété.

La thèse est divisée en trois chapitres.

Le **Chapitre 1** est dédié aux préliminaires. Nous fixons la terminologie adoptée et introduisons quelques résultats utiles pour le reste des chapitres.

Dans le **Chapitre 2**, nous définissons une nouvelle perspective sur la platitude des modules inspirée par des idées similaires étudiées dans plusieurs articles sur les domaines de sous-projectivité. Dans ce processus, les modules projectifs doivent en général être remplacés par les modules plats. Toutefois, les notions de projectivité et de platitude sont curieusement différentes, de sorte que les domaines peuvent être remarquablement uniques. Nous commençons d'abord par étudier un contexte général en introduisant des domaines relatifs à une classe précouvrante $\mathcal{X}$.

Ce chapitre est organisé comme suit :

Dans la *Section 2.1*, on définit le domaine de $\mathcal{X}$-précouvertures complétées $\mathcal{X}^{-1}(\mathcal{L})$ d'une classe de modules $\mathcal{L}$ (voir Définition 2.1.1). Puis on démontre quelques propriétés basiques de ces domaines. Dans cette thèse, nous mettons l'accent sur le domaine de $\mathcal{X}$-précouvertures complétées $\mathcal{X}^{-1}(\mathcal{L})$ d'une classe de modules $\mathcal{L}$ au lieu de $\mathcal{X}^{-1}(M)$ pour un module M. Cette approche, également adoptée précédemment dans [4], permet non seulement de retrouver plusieurs résultats connus mais conduit également à plus d'applications.

Lorsque $\mathcal{X}$ est la classe des modules projectifs, les domaines de $\mathcal{X}$-précouvertures complétées ne sont rien d'autre que les domaines de sous-projectivité [23]. De plus, il est facile de montrer qu'un module $M \in \mathcal{X}$ si et seulement si son domaine de $\mathcal{X}$-précouvertures complétées est constitué de la classe entière des modules R-Mod (Proposition 2.1.3). Et si $N \in \mathcal{X}$, alors N appartient aux domaines de $\mathcal{X}$-précouvertures complétées de tout module.

Nous étendons l'étude faite dans [4] et [23] au cas relatif et rassemblons différentes propriétés de stabilité que vérifient les domaines de $\mathcal{X}$-précouvertures complétées (voir Propositions 2.1.5, 2.1.6, 2.1.7,2.1.8, 2.1.9 et 2.1.10). Ensuite, nous étudions le domaine de $\mathcal{X}$-précouvertures complétées d'un module inclut dans un module de la classe $\mathcal{X}$. A savoir, nous démontrons que étant donné une suite exacte courte de la forme $0 \to M \to X \to M' \to 0$, nous avons $M'^{\perp} \subseteq \mathcal{X}^{-1}(M)$ (Proposition 2.1.12). Dans cette situation, $\mathcal{X}^{-1}(M)$ contient la classe des modules injectifs. Cela nous amène aux propositions 2.1.13 et 2.1.14 où nous établissons des conditions nécessaires et suffisantes pour que les domaines de $\mathcal{X}$-précouvertures complétées contiennent la classe particulière des modules injectifs $\mathcal{I}$ d'une part et celle des modules injectifs purs $\mathcal{PI}$ d'une autre part, respectivement. Enfin, dans la proposition 2.1.15, on compare les domaines relatifs à deux classes précouvrantes $\mathcal{X}$ et $\mathcal{Y}$ telles que $\mathcal{X} \subseteq \mathcal{Y}$.

Dans la *Section 2.2*, nous mettons en lumière les domaines de précouvertures plates complétées, obtenus en prenant $\mathcal{X}$ comme la classe des modules plats. Cela nous permet d'obtenir de nouvelles caractérisations de notions connues. Par exemple, dans la proposition 2.2.10, nous démontrons que pour tout anneau R, nous avons les assertions suivantes:

1. $\mathrm{wdim}(R) \leq 1$ si et seulement si le domaine de précouvertures plates complétées de n'importe quel module est stable par sous-modules.

2. L'anneau R est cohérent à droite si et seulement si le domaine de précouvertures plates complétées de tout module est stable par produits directs.

3. L'anneau R est semi-héréditaire à gauche si et seulement si le domaine de précouvertures plates complétées de tout module à droite est stable par produits directs et que le domaine de précouvertures plates complétées de n'importe quel module à gauche est stable par sous-modules.

De plus, nous avons pu déterminer les domaines de précouvertures plates complétées de certains types de modules intéressants. En effet, dans l'exemple 2.2.2, nous démontrons que:

1. Si M est de Ding projectif, alors il existe un module Ding projectif M' tel que le domaine de précouvertures plates complétées de M est $M'^{\perp}$.

2. Si M est un module de présentation finie et plat de Gorenstein fort alors le domaine de précouvertures plates complétées de M est $M^{\perp}$.

Ensuite, de manière similaire au contexte homologique classique où la relation entre les modules plats et projectifs est minutieusement étudiée, nous étudions dans cette thèse la relation entre les domaines de précouvertures plates complétées et les domaines de sous-projectivité (Proposition 2.2.4). Cela nous dévoile un nouvel aspect de notions bien connues. Rappelons qu'un anneau R est dit parfait si tout module plat est projectif. Dans cette thèse, nous donnons une nouvelle caractérisation d'anneaux parfaits en fonction de ces domaines (Corollaire 2.2.5).

Dans la Proposition 2.2.9, nous donnons une nouvelle perspective sur le résultat connu suivant : Un module est plat si et seulement si son module caractère est injectif (voir [26, Theorem]). Ou de manière équivalente, un module est plat si et seulement si son module caractère est absolument pur. Dans ce nouveau contexte, on étudie la relation entre les domaines absolument purs et de domaines de précouvertures plates complétées (Proposition 2.2.9).

Enfin, dans [11], les anneaux cohérents sont caractérisés par l'équivalence entre la pureté absolue d'un module et la platitude de son module caractère. Ici, nous démontrons le résultat homologue dans notre contexte. À savoir, nous caractérisons les anneaux cohérents à droite au moyen de domaines de précouvertures plates complétées et de domaines absolument purs (voir la proposition 2.2.11).

Dans la *Section 2.3*, nous nous intéressons à la relation entre les domaines de précouvertures plates complétées et les domaines de sous-injectivité. Nous obtenons une nouvelle perspective au résultat [26, Theorem] énoncé comme suit: un module est plat si et seulement si son module caractère est injectif. Dans notre nouveau contexte, nous étudions la relation entre les domaines de précouvertures plates complétées et les domaines de sous-injectivité (Proposition 2.3.1).

L'une des caractérisations classiques des anneaux quasi-Frobenius est celle que tout module plat est injectif. Ici, nous caractérisons les anneaux quasi-Frobenius en termes de domaines de précouvertures plates complétées et de domaines de sous-injectivité (Proposition 2.3.3). Il est également bien connu que lorsque la classe des modules plats et la classe des modules injectifs coïncident, alors l'anneau n'est rien d'autre que l'anneau quasi-Frobenius. Nous étudions ici l'homologue de ce résultat dans notre contexte. Pour cela, nous considérons la question suivante :

Q1. Quelle est la structure d'un anneau sur lequel coïncident les domaines de précouvertures plates complétées et les domaines de sous-injectivité ?

Nous prouvons qu'un anneau satisfait cette condition si et seulement si tout anneau quotient de R est quasi-Frobenius si et seulement si R est isomorphe à un produit direct fini d'anneaux des matrices $n \times n$ sur des anneaux artiniens à idéaux principaux locaux; pour des entiers positifs n (voir Théorème 2.3.7). Ensuite, nous donnons des caractérisations équivalentes pour que les domaines de précouvertures plates complétées contiennent la classe des modules injectifs (Proposition 2.3.9). En conséquence, ce résultat nous permet de donner une preuve directe des caractérisations des anneaux IF précédemment établies par Colby dans [12, Theorem 1] (Corollaire 2.3.10).

Dans [5], un module avec le plus petit domaine de sous-injectivité possible est dit indigent. Nous introduisons ici le concept analogue; c'est-à-dire celui des modules pour lesquels les domaines de précouvertures plates complétées sont aussi petits que possible. Nous appelons ces nouveaux modules des modules f-robustes. Nous démontrons qu'il existe des modules f-robustes pour tout anneau arbitraire et enfin, nous établissons une connexion entre les modules f-robustes et les modules indigents (Proposition 2.3.11).

Dans le **Chapitre 3**, nous poursuivons l'étude des domaines de $\mathcal{X}$-précouvertures complétées. L'un des problèmes classiques dans le contexte des précouvertures et des préenveloppes est celui de caractériser quand chaque module possède une préenveloppe. Ce problème est davantage intéressent lorsque cette préenveloppe est d'un type spécifique tel qu'une préenveloppe surjective ou injective. Dans ce chapitre, nous répondons à la question principale suivante :

Q2. Quand est-ce que chaque module d'une classe de modules $\mathcal{L}$ possède une $\mathcal{X}^{-1}(\mathcal{L})$-préenveloppe ?

Cela donne une nouvelle perspective de certaines notions n'impliquant pas de domaines de $\mathcal{X}$-précouvertures complétées. Par exemple, en prenant $\mathcal{L}$ comme étant la classe des modules de type fini et $\mathcal{X}$ celle des modules pure-projectifs, nous déterminons quand est-ce que chaque module de type fini a-t-il une R-Mittag-Leffler préenveloppe. Ceci sera amplement développé dans la dernière section.

De plus, les études faites dans [28] et [31] se présentent comme de bons exemples de ce travail. En effet, Parra et Rada dans [31] ont défini $\mathcal{S}$-proj, pour une classe de modules de type fini $\mathcal{S}$, comme étant la classe de modules N telle que tout le morphisme $f : S \to N$, où $S \in \mathcal{S}$, se factorise par un module libre. Ensuite, ils ont étudié quand est-ce que chaque module de $\mathcal{S}$ possède une $\mathcal{S}$-proj-préenveloppe. Il s'est avéré que la classe $\mathcal{S}$-proj est précisément le domaine de sous-projectivité $\underline{\mathfrak{Pr}}^{-1}(\mathcal{S})$ (voir [4, Proposition 2.7]). Et dans [28], on peut voir que Mao a étudié quand est-ce que chaque module simple possède une $\underline{\mathfrak{Pr}}^{-1}(\mathfrak{S})$-préenveloppe, où $\mathfrak{S}$ désigne la classe des modules simples.

Par conséquent, le chapitre 3 fournit un contexte unifié répondant aux questions relatives à l'existence de préenveloppes pour les modules d'une classe. Et une telle étude couvre plusieurs applications possibles.

On rappelle qu'un module M est dit f-projectif si, pour tout sous-module C de type fini de M, l'injection canonique se factorise à travers un module libre de type fini. Maintenant, on sait par [4, Proposition 2.22] que la classe $\underline{\mathfrak{Pr}}^{-1}(\mathcal{S})$ pour une classe $\mathcal{S}$ de modules de type fini généralise le concept des modules f-projectifs. Cela nous inspire à définir une extension naturelle du concept de modules R-Mittag-Leffler. Ici, nous nous appuyons sur les domaines $\mathcal{PP}$-précouvertures complétées, où $\mathcal{PP}$

désigne la classe des modules purs projectifs. Nous utilisons une caractérisation appropriée des modules R-Mittag-Leffler dus à Goodearl dans [21]. Nous appelons ces modules $\mathcal{M}$-R-modules Mittag-Leffler, relatifs à diverses classes de modules de type fini $\mathcal{M}$. Avec cette approche, nous retrouvons des résultats connus et donnons aussi une nouvelle perspective sur les modules R-Mittag-Leffler.

Le **Chapitre 3** est divisé en deux sections.

Dans la *Section 3.1*, nous étudions les domaines de $\mathcal{X}$-précouvertures complétées en tant que classes pré-enveloppantes. Un outil important pour caractériser quand est-ce que chaque module d'une classe $\mathcal{L}$ a une $\mathcal{X}^{-1}(\mathcal{L})$-préenveloppe est le concept relatif des classes localement initialement petites que nous introduisons comme une extension de la notion classique de classes localement initialement petites (voir [32, Définition 2.1]). Nous démontrons l'un des résultats principaux de cette section qui répond à la Question **Q2** (Théorème 3.1.3). Ensuite, nous étudions quand est-ce que chaque module de $\mathcal{L}$ a une $\mathcal{X}^{-1}(\mathcal{L})$-préenveloppe surjective d'une part, et une $\mathcal{X}^{-1}(\mathcal{L})$-préenveloppe injective d'une autre part (voir les Théorèmes 3.1.7 et 3.1.9, respectivement).

La *Section 3.2* est dédiée aux applications. Plusieurs résultats pour des classes importantes de modules peuvent être obtenus comme applications de nos résultats principaux. Dans cette section, pour une classe des modules de type fini $\mathcal{M}$, on définit les modules $\mathcal{M}$-R-Mittag-Leffler (Définition 3.2.1). Nous donnons quelques caractérisations équivalentes des modules $\mathcal{M}$-R-Mittag-Leffler (Proposition 3.2.3) puis rassemblons et démontrons leurs propriétés de stabilité (Propositions 3.2.5, 3.2.7 et 3.2.9). Nous démontrons que les modules R-mod-R-Mittag-Leffler ne sont que les modules R-Mittag-Leffler, et que les modules $\mathcal{C}$-R-Mittag-Leffler sont exactement des modules pure-projectifs séparé, où R-mod désigne la classe des modules de type fini et $\mathcal{C}$ celle des modules cycliques (voir Remarque 3.2.2 et Exemple 3.2.4).

En algèbre homologique classique, la relation entre les modules R-Mittag-Leffler, les modules f-projectifs et les modules plats a été étudiée et caractérisée ([25, Proposition 1.2]). Dans notre nouveau contexte, nous étudions le résultat homologue reliant la notion de modules $\mathcal{M}$-R-Mittag-Leffler à celle de $\mathcal{M}$-proj (voir la proposition 3.2.12). Enfin, basés sur des résultats prouvés dans la section précédente, nous donnons des caractérisations pour qu'un module de $\mathcal{M}$ aie une $\mathcal{M}$-R-Mittag-Leffler-préenveloppe (Proposition 3.2.13).
De même, l'existence de $\mathcal{M}$-R-Mittag-Leffler-préenveloppes surjectives et injectives pour les modules de la classe $\mathcal{M}$ est caractérisée (Propositions 3.2.15 et 3.2.16). Enfin, nous donnons une caractérisation des $\mathcal{M}$-R-Mittag-Leffler-enveloppes pour les modules dans $\mathcal{M}$ (Proposition 3.2.17).

Introduction

Throughout this thesis R will denote an associative ring with identity and modules will be unital left R-modules, unless otherwise explicitly stated. As usual, we denote by R-Mod *the category of left R-modules and by* Mod-R *the category of right R-modules.*

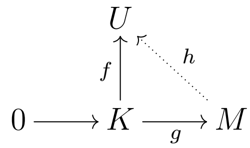
The study from a relative perspective of certain properties such as projectivity, injectivity and flatness was the main focus of many authors throughout the years (see for instance [9], [19], [22] and [23]). The idea is not to determine whether or not a module has a property, but rather to study how close it is to have it. That way, each module is assigned a relative domain that measures to what extent it has a certain property.

First, relative notions of projectivity and injectivity were introduced as a tool to evaluate the extent of these properties for any given module. Indeed, relative projectivity is defined as follows (see [19]). Let U be a module. If M is a module, then U is projecive relative to M (or U is M-projective) in case for each epimorphism $g : M \to N$ and each homomorphism $f : U \to N$, there is a homomorphism $h : U \to M$ such that the diagram

$$\begin{array}{ccccc} & \overset{h}{\swarrow} & U & & \\ & & \downarrow f & & \\ M & \xrightarrow[g]{} & N & \longrightarrow & 0 \end{array}$$

commutes. The projectivity domain of a module U is the class of all modules M such that U is M-projective.

On the other hand, U is said to be injective relative to M (or U is M-injective) in case for each monomorphism $g : K \to M$ and each homomorphism $f : K \to U$, there is a morphism $h : M \to U$ such that the diagram



commutes. The injectivity domain of a module U is the class of all modules M such that U is M-injective.

Then, in contrast to the well-known notion of relative injectivity, Aydoğdu and López-Permouth introduced in [5] the notion of subinjectivity. Then, Holston et al. introduced in [23] the projective analogue of subinjectivity and called it subprojectivity. We recall the definition of subprojectivity domains.

Given two modules M and N, M is said to be N-subprojective if for every epimorphism $g : B \to N$ and homomorphism $f : M \to N$, there exists a homomorphism $h : M \to B$ such that $gh = f$ (see [23, Definition 2.1]). The subprojectivity domain, or domain of subprojectivity, of a module M, $\underline{\mathfrak{Pr}}^{-1}(M)$, is defined to be the class

$$\underline{\mathfrak{Pr}}^{-1}(M) := \{N \in R\text{-Mod} : M \textit{ is } N\textit{-subprojective}\}.$$

One can see from the definition that a module M is projective if and only if its subprojectivity domain consists of all modules, that is, $\underline{\mathfrak{Pr}}^{-1}(M) = R\text{-Mod}$ (see [23, Proposition 2.4]). Thus, subprojectivity domains measure projectivity of modules.

On the other hand, the study of flatness was approached in [9] and [16] from two different but slightly similar alternative perspectives as both use the tensor product. Indeed, relative flatness studied in [9] is defined as follows:

Given a right R-module N, a left R-module M is said to be flat relative to N, relatively flat to N, or N-flat if the canonical morphism $K \otimes_R M \to N \otimes_R M$ is a monomorphism for every submodule K of N (see [9, Definition 2.1]). The flat domain of a module M, $\mathcal{F}^{-1}(M)$, is defined to be the class

$$\mathcal{F}^{-1}(M) := \{N \in \text{Mod-}R \ : M \text{ est } N\text{-flat}\}.$$

Clearly, a module M is flat if and only if $\mathcal{F}^{-1}(M) = \text{Mod-}R$. It is clear from the definition that a module is flat if and only if its domain of flatness is equal to the whole category of modules.

Durğun in [16], modifies in a subtle way the notion of relative flatness domains and defines absolutely pure domains. This time, the definition is based on the notion purity. Indeed, given a left module M and a right module N, N is said to be absolutely M-pure if $N \otimes_R M \to B \otimes_R M$ is a monomorphism for every extension

B of N (see [16, Definition 2.2]). For a module M, the absolutely pure domain of M, $\mathcal{A}p(M)$, is defined to be the class

$$\mathcal{A}p(M) := \{N \in \text{Mod-}R\ : N \text{ is absolutely } M\text{-pure}\}.$$

Clearly, a module M is flat if and only if $\mathcal{A}p(M) = \text{Mod-}R$.

In this thesis, we introduce a new and a fresh perspective on flatness of modules without the tensor product. But first, we set several domains under one general framework and introduce domains relative to a precovering class $\mathcal{X}$. We call these domains $\mathcal{X}$-precover completing domains because a diagram involving precovers can be completed.

The thesis is divided into three chapters.

Chapter 1 is dedicated to preliminaries. We set the terminology and introduce some results needed for the rest of the chapters.

In **Chapter 2**, we define a new alternative perspective on flatness of modules inspired by similar ideas studied in several papers about subprojectivity domains. In this process, projective modules should in general be replaced by flat modules. However, the nature of projectivity and flatness are so curiously different that each domain can be remarkably unique. We start by investigating a general context by introducing domains relative to a precovering class $\mathcal{X}$.

This chapter is organized as follows:

In *Section 2.1*, we define the $\mathcal{X}$-precover completing domain $\mathcal{X}^{-1}(\mathcal{L})$ for a class of modules $\mathcal{L}$ (see Definition 2.1.1) and we give the basic properties. Our emphasis is on the $\mathcal{X}$-precover completing domain $\mathcal{X}^{-1}(\mathcal{L})$ of a class of modules $\mathcal{L}$ instead of $\mathcal{X}^{-1}(M)$ for a module M. This approach, also previously adopted in [4], not only allows us to recover several known results but also leads to more applications.

When $\mathcal{X}$ is the class of projective modules, $\mathcal{X}$-precover completing domains will be nothing but subprojectivity domains. Moreover, it is easy to show that a module $M \in \mathcal{X}$ if and only if its $\mathcal{X}$-precover completing domain consists of the entire class of modules R-Mod (Proposition 2.1.3). And if $N \in \mathcal{X}$, then M is vacuously (N,$\mathcal{X}$)-precover completing.

We extend the study done in [4] and [23] to this relative case and gather different closeness properties that $\mathcal{X}$-precover completing domains verify (see Propositions 2.1.5, 2.1.9 and 2.1.10). Then, we investigate the $\mathcal{X}$-precover completing domain of a module embedded in a module in the class $\mathcal{X}$. Namely, we show that given a short exact sequence of the form $0 \to M \to X \to M' \to 0$, we have $M'^{\perp} \subseteq \mathcal{X}^{-1}(M)$ (Proposition 2.1.12) and as a consequence, $\mathcal{X}^{-1}(M)$ contains the class of injective modules. This brings us to Propositions 2.1.13 and 2.1.14 where we establish equivalent conditions for $\mathcal{X}$-precover completing domains to contain the particular class of injective modules $\mathcal{I}$ and that of pure-injective modules $\mathcal{PI}$, respectively. Finally, in Proposition 2.1.15, we compare the domains relative to two precovering classes $\mathcal{X}$ and $\mathcal{Y}$ such that $\mathcal{X} \subseteq \mathcal{Y}$.

In *Section 2.2*, we shed light on flat-precover completing domains, obtained by taking $\mathcal{X}$ to be the class of flat modules. This leads to new characterizations of known notions. For instance, in Proposition 2.2.10, we show that for any ring R, we have the following assertions:

1. $\mathrm{wdim}(R) \leq 1$ if and only if the flat-precover completing domain of any module is closed under submodules.
2. The ring R is right coherent if and only if the flat-precover completing domain of any module is closed under direct products.
3. The ring R is left semihereditary if and only if the flat-precover completing domain of any right R-module is closed under arbitrary direct products and the flat-precover completing domain of any left module is closed under submodules.

Futhermore, we were able to compute the flat-precover completing domains of some interesting kind of modules. Indeed, in Example 2.2.2, we show that

1. If M is Ding projective then there exists a Ding projective module M' such that the flat precover completing domain of M is $M'^{\perp}$.
2. If M is a finitely presented and strongly Gorenstein flat module then the flat precover completing domain of M is $M^{\perp}$.

Then, as in the classical homological context where the relation between flat and projective modules is extensively studied, we investigate the relation between flat-precover completing domains and subprojectivity domains (see Proposition 2.2.4). This shows us a new side to well-known notions. Recall that a ring R is called perfect if every flat module is projective. Here, we give a new characterization of perfect rings in terms of our domains (Corollary 2.2.5).

In Proposition 2.2.9, we provide a new perspective on the following known-result: A module is flat if and only if its character module is injective (see [26, Theorem]). Or equivalently, a module is flat if and only if its character modules is absolutely pure. In this context, we obtain the following result in terms of absolutely pure domains and flat-precover completing domains (Proposition 2.2.9).

Finally, in [11], coherent rings are characterized by the equivalence of the absolutely purity of modules and the flatness of their character modules. Here we show the counterpart result in our context. Namely, we characterize right coherent rings by means of flat-precover completing domains and absolutely pure domains (see Proposition 2.2.11).

In *Section 2.3*, we focus on the relation between flat-precover completing domains and subinjectivity domains. We give another new insight to [26, Theorem] that shows that a module is flat if and only if its character module is injective. In our new context, we use flat-precover completing domains and subinjectivity domains (Proposition 2.3.1).

One of the classical characterizations of quasi-Frobenius rings is that they are those rings for which any flat module is injective. In this work, we characterize quasi-Frobenius rings in terms of flat-precover completing domains and subinjectivity domains (Proposition 2.3.3). It is also well known that when the class of flat modules and the class of injective modules coincide, then the ring is nothing but a quasi-Frobenius ring. Here, we investigate the counterpart of this result in our context.So we consider the following question:

Q1. What is the structure of a ring over which the flat-precover completing domains and subinjectivity domains coincide?

We prove that a ring satisfies this condition if and only if every factor ring of R is QF if and only if the ring R is isomorphic to a direct product of full matrix rings over Artinian chain rings. (see Theorem 2.3.7). Then, we give equivalent characterizations for flat-precover completing domains to contain the class of injective modules (Proposition 2.3.9). As a consequence, this result allows us to give a straightforward proof to characterizations of IF-rings established in [12, Theorem 1] by Colby (Corollary 2.3.10).

In [5], a module with the smallest possible subinjectivity domain is said to be indigent. Here we introduce the opposite concept to that of flat modules; that is, modules for which the flat-precover completing domains are as small as possible. We call these new modules f-rugged modules. We show that there are f-rugged modules for any arbitrary ring and finally, we establish a connection between f-rugged modules and indigent modules (Proposition 2.3.11).

In **Chapter 3**, we continue the investigation of $\mathcal{X}$-precover completing domains. One of the classical problems in the context of precovers and preenvelopes is to characterize when every module has a preenvelope and even more when that preenvelope is of a specific type such as an epimorphic or a monomorphic preenvelope. In this chapter, we ask the following main question:

Q2. When does every module of a class $\mathcal{L}$ have an $\mathcal{X}^{-1}(\mathcal{L})$-preenvelope?

This yields a new insight into notions not involving precover completing domains. For example, taking $\mathcal{L}$ to be the class of finitely generated modules and $\mathcal{X}$ that of pure-projective modules, we determine when does every finitely generated module have an R-Mittag-Leffler preenvelope. This will be developed in the last section.

Moreover, the studies done in [28] and [31] also serve as nice examples of our work. Indeed, Parra and Rada in [31] defined $\mathcal{S}$-proj, for a class of finitely generated modules $\mathcal{S}$, as the class of modules N such that every morphism $f : S \to N$, where $S \in \mathcal{S}$, factors through a free module. Then, they investigated when every module of $\mathcal{S}$ has an $\mathcal{S}$-proj preenvelope. It turns out that the class $\mathcal{S}$-proj is precisely the subprojectivity domain $\underline{\mathfrak{Pr}}^{-1}(\mathcal{S})$ (see [4, Proposition 2.7]).

And in [28], one can see that Mao investigated when every simple module has an $\underline{\mathfrak{Pr}}^{-1}(\mathfrak{S})$-preenvelope, where $\mathfrak{S}$ denotes the class of simple modules.
Consequently, this present work provides a uniform background for questions relative to the existence of preenvelopes for a class modules. And such a study covers several possible applications.
We recall that a module M is said to be f-projective if, for every finitely generated submodule C of M, the inclusion map factors through a finitely generated free module. Now, we know by [4, Proposition 2.22] that the class $\underline{\mathfrak{Pr}}^{-1}(\mathcal{S})$ for a class $\mathcal{S}$ of finitely generated modules generalizes the concept of f-projective modules. This inspires us to define a natural extension of the concept of R-Mittag-Leffler modules. Here, we rely on $\mathcal{PP}$-precover completing domains, where $\mathcal{PP}$ denotes the class of pure-projective modules. We use a suitable characterization of R-Mittag-Leffler modules due to Goodearl in [21]. We call these modules $\mathcal{M}$-R-Mittag-Leffler modules, relative to various classes of finitely generated modules $\mathcal{M}$. With this approach, we recover known results and give a new perspective on R-Mittag-Leffler modules.

Chapter 3 is divided into two section.

In *Section 3.1*, we investigate $\mathcal{X}$-precover completing domains as preenveloping classes. An important tool in characterizing when every module of a class of modules $\mathcal{L}$ has an $\mathcal{X}^{-1}(\mathcal{L})$-preenvelope is the concept of relative locally initially small classes that we introduce as an extension of the classical notion of locally initially small classes (see [32, Definition 2.1]). We show one of the main result of this section that answers Question **Q2** (Theorem 3.1.3).

Then, we investigate when every module of $\mathcal{L}$ has an epic $\mathcal{X}^{-1}(\mathcal{L})$-preenvelope and a monic $\mathcal{X}^{-1}(\mathcal{L})$-preenvelope (see Theorems 3.1.7 and 3.1.9).

Section 3.2 is dedicated to applications. Several results for significant classes can be obtained as particular instances from previous results. Here, for a class of finitely generated modules $\mathcal{M}$, we define $\mathcal{M}$-R-Mittag-Leffler modules (Definition 3.2.1). We give some equivalent characterizations of $\mathcal{M}$-R-Mittag-Leffler modules (Proposition 3.2.3) and then gather and show their closeness properties (Propositions 3.2.5, 3.2.7 and 3.2.9). We show that $R\text{-mod}$-R-Mittag-Leffler modules are just R-Mittag-Leffler modules, where $R\text{-mod}$ denotes the class of finitely generated modules. We recall that in [6], a module is called singly pure-projective if the inclusion map from each cyclic submodule factors through a finitely presented module.

We show that $\mathcal{C}$-R-Mittag-Leffler modules are exactly singly pure-projective modules, where $\mathcal{C}$ denotes the class of cyclic modules (see Remark 3.2.2 and Example 3.2.4).

In the classical homological algebra, the relation between R-Mittag-Leffler modules, f-projective modules, and flat modules has been studied and characterized ([25, Proposition 1.2]). In our new context, we investigate the counterpart of this result by relating the notion of $\mathcal{M}$-R-Mittag-Leffler modules to that of $\mathcal{M}$-proj (see Proposition 3.2.12). Finally, based on results proved in the previous section, we give characterizations for when every module in $\mathcal{M}$ has an $\mathcal{M}$-R-Mittag-Leffler preenvelope (Proposition 3.2.13). Similarly, the existence of epic and monic $\mathcal{M}$-R-Mittag-Leffler preenvelopes for modules in the class $\mathcal{M}$ is characterized (Propositions 3.2.15 and 3.2.16). We end this section with a characterization of $\mathcal{M}$-R-Mittag-Leffler envelopes for modules in $\mathcal{M}$ (Proposition 3.2.17).

Preliminaries

CHAPTER 1

PRELIMINARIES

In this chapter we introduce the basic terminology for rings and modules that we use, as well as the fundamental results needed for this thesis. We assume that the fundamentals of module, ring theory and homological algebra are already known. All definitions which are not given here can be found in [17] and [34] for example.

1.1 Notation and terminology

Throughout this thesis, we will consider the following notation:

R-Mod	The class of all left R-modules.
Mod-R	The class of all right R-modules.
R-mod	The class of all finitely generated modules.
$\mathcal{P}$	The class of projective modules.
$\mathcal{I}$	The class of injective modules.
$\mathcal{PP}$	The class of pure-projective modules.
R-$\mathcal{ML}$	The class of R-Mittag-Leffler modules.
$E(M)$	The injective envelope of a module M
$PE(M)$	The pure-injective preenvelope of a module M.
M^+	The character module $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$.

$Sum(\mathcal{L})$	The class of all left R-modules.
$Summ(\mathcal{L})$	The class of all modules which are isomorphic to direct summands of modules of $\mathcal{L}$.
$Add(\mathcal{L})$	The class $Summ(Sum(\mathcal{L}))$.

To any given class of modules $\mathcal{L}$ we associate its right Ext-orthogonal class,

$$\mathcal{L}^{\perp} = \{M \in R\text{-Mod} \mid \text{Ext}^1(L, M) = 0, L \in \mathcal{L}\},$$

and its left Ext-orthogonal class,

$$^{\perp}\mathcal{L} = \{M \in R\text{-Mod} \mid \text{Ext}^1(M, L) = 0, L \in \mathcal{L}\}.$$

In particular, if $\mathcal{L} = \{M\}$ then we simply write $^{\perp}\mathcal{L} = {}^{\perp}M$ and $\mathcal{L}^{\perp} = M^{\perp}$.

1.2 Absolutely pure domains

In this section, we introduce the work done by Durğun in [16] where he evaluates the flatness of a module. The class of absolutely pure modules plays a crucial role in doing so. We start by recalling some basic definitions about purity.

Definition 1.2.1 ([17], Definition 5.3.6)**.** *A submodule T of a module N is said to be a pure submodule if $0 \to A \otimes T \to A \otimes N$ is exact for all right R-modules A, or equivalently, if $\text{Hom}(A, N) \to \text{Hom}(A, N/T) \to 0$ is exact for all finitely presented modules A.*
An exact sequence $0 \to T \to N \to N/T \to 0$ is said to be pure exact if T is a pure submodule of N.
A module M is said to be pure injective if for every pure exact sequence $0 \to T \to N$ of modules, $\text{Hom}(N, M) \to \text{Hom}(T, M) \to 0$ is exact. Clearly, every injective module is pure injective.

Recall that a module N is said to be absolutely pure if every extension of N is pure exact. The following proposition is useful to characterize the absolute purity of a given module.

Proposition 1.2.2 ([16], Proposition 2.1)**.** *Let N be a module. The following statements are equivalent:*

1. *N is absolutely pure.*

2. $M \otimes_R N \to M \otimes_R E(N)$ *is a monomorphism for each finitely presented right R-module M.*

3. $M \otimes_R N \to M \otimes_R E(N)$ *is a monomorphism for each right R-module M.*

4. $\mathrm{Ext}_R(F, N) = 0$ *for each finitely presented module F.*

Durğun in [16], introduce an alternative perspective on relative flatness domains and defines absolutely pure domains as follows.

Definition 1.2.3 ([16], Definition 2.2)**.** *Given a left module M and a right module N, N is said to be absolutely M-pure if $N \otimes_R M \to B \otimes_R M$ is a monomorphism for every extension B of N. For a module M, the absolutely pure domain of M, $\mathcal{A}p(M)$, is defined to be the class*

$$\mathcal{A}p(M) := \{N \in \text{Mod-}R \ : N \textit{ is absolutely } M\textit{-pure}\}.$$

Clearly, a module M is flat if and only if $\mathcal{A}p(M) = \text{Mod-}R$.

The next proposition turns out to be quite useful in showing that a module N is absolutely pure relative to another module M.

Proposition 1.2.4 ([16], Proposition 2.2)**.** *Let M be a right module and N be a left module. The following assertions are equivalent:*

1. *N is absolutely M-pure.*

2. $M \otimes_R N \to M \otimes_R E(N)$ *is a monomorphism.*

3. *There exists an absolutely pure extension E of N such that $M \otimes_R N \to M \otimes_R E$ is a monomorphism.*

1.3 Subprojectivity domains

In [23], an alternative perspective on the projectivity of a module was introduced. Indeed, the authors define subprojectivity domains as a tool to measure the extent of projectivity of any module. In this section, we gather some results needed for this thesis that can be found in [3] and [23]. We start with the definition of the subprojectivity domain of a module.

Definition 1.3.1 ([23], Definition 2.1)**.** *Given modules M and N, M is said to be N-subprojective if for every epimorphism $g : B \to N$ and every homomorphism $f : M \to N$, there exists a homomorphism $h : M \to B$ such that $gh = f$. The subprojectivity domain of a module M, or domain of subprojectivity, $\underline{\mathfrak{Pr}}^{-1}(M)$, is defined to be the class*

$$\underline{\mathfrak{Pr}}^{-1}(M) := \{N \in R\text{-Mod} : M \textit{ is } N\textit{-subprojective}\}.$$

Then, the authors in [23] proceed to simplify the definition of subprojectivity by showing that in order for a module M to be N-subprojective, it is sufficient to complete a diagram involving projective covers, free covers or even a single projective cover.

Lemma 1.3.2 ([23], Lemma 2.2). *Let M and N be modules. Then the following conditions are equivalent:*

1. *M is N-subprojective.*
2. *For every morphism $f : M \to N$ and every epimorphism $g : P \to N$ with P projective, there exists $h : M \to P$ such that $f = gh$.*
3. *For every morphism $f : M \to N$ and every epimorphism $g : F \to N$ with F free, there exists $h : M \to F$ such that $f = gh$.*
4. *For every morphism $f : M \to N$ there exists an epimorphism $g : P \to N$ with P projective and a morphism $h : M \to P$ such that $f = gh$.*

An easy observation shows that the subprojectivity domain of a projective module P is the whole category of modules, so $\underline{\mathfrak{Pr}}^{-1}(P) = R\text{-Mod}$. In this thesis, we also need to know the subprojectivity domain of a strongly Gorenstein projective module. We start first by recalling the definition of strongly Gorenstein projective modules.

First, recall that a module M is said to be strongly Gorenstein projective if there exists an exact sequence of projective modules

$$\cdots \xrightarrow{f} P \xrightarrow{f} P \xrightarrow{f} P \xrightarrow{f} \cdots$$

such that $M \cong \mathrm{Im} f$ and such that $Hom_R(-, Q)$ leaves the above sequence exact whenever Q is a projective module.

Similarly, a module M is said to be strongly Gorenstein flat if there exists an exact sequence of flat R-modules

$$\cdots \xrightarrow{f} F \xrightarrow{f} F \xrightarrow{f} F \xrightarrow{f} \cdots$$

such that $M \cong \mathrm{Im} f$ and such that $I \otimes_R -$ leaves the above sequence exact whenever I is an injective right R-module (see [8]). In fact, the relationship between the class of strongly Gorenstein projective and flat modules was investigated in the following result.

Proposition 1.3.3 ([8], Proposition 3.9). *A module is finitely generated strongly Gorenstein projective if and only if it is finitely presented and strongly Gorenstein flat.*

The next result determines the subprojectivity domain of a strongly Gorenstein projective module.

Proposition 1.3.4 ([4], Corollary 2.9)**.** *If M is a strongly Gorenstein projective module, then $\underline{\mathfrak{Pr}}_{\mathscr{A}}{}^{-1}(M) = M^{\perp}$.*

In [4], the authors develop a new treatment of the subprojectivity in the categorical context. Subprojectivity domains are introduced for classes of objects instead of just single objects. Let $\mathscr{A}$ denote an abelian category with enough projectives. Then, the definition is as follows.

Definition 1.3.5 ([4], Definition 2.14)**.** *The subprojectivity domain, or domain of subprojectivity, of a class of objects $\mathcal{M}$ of $\mathscr{A}$ is defined as*

$$\underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(\mathcal{M}) := \{N \in \mathscr{A} : M \text{ is } N\text{-subprojective for every } M \in \mathcal{M}\}.$$

Therefore, if $\mathcal{M} := \{M\}$ then $\underline{\mathfrak{Pr}}_{\mathscr{A}}{}^{-1}(\mathcal{M}) = \underline{\mathfrak{Pr}}_{\mathscr{A}}{}^{-1}(M)$.

The authors in [4] provide a useful way to use subprojectivity domains by characterizing subprojectivity domains through a factorization property. This turns out to be the key to many results in [4].

Proposition 1.3.6 ([4], Proposition 2.7)**.** *Let M and N be objects of $\mathscr{A}$. Then, M is N-subprojective if and only if every morphism $M \to N$ factors through a projective object.*

One easily sees that a module is projective precisely when its subprojectivity domain is the whole category R-Mod. Thus, the notion of subprojectivity domains introduced in [23], somehow, measures the projectivity of modules. This can also be seen in the following proposition in the categorical context.

Proposition 1.3.7 ([4], Proposition 2.4)**.** *Let M be an object of $\mathscr{A}$. Then the following conditions are equivalent:*

1. *$\underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(M)$ is the whole abelian category $\mathscr{A}$.*
2. *M is projective.*
3. *$M \in \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(M)$.*

In the next results, the subprojectivity domains of some known classes of objects are determined. Recall that an object F is said to be flat if every short exact sequence $0 \to A \to B \to F \to 0$ is pure, that is, if for every finitely presented object P, $\mathrm{Hom}_{\mathscr{A}}(P, -)$ makes this sequence exact (see [36]). Then, the following result can be obtained as an immediate consequence of the definition of subprojectivity domains (see also [15, Proposition 2.1]).

Proposition 1.3.8 ([4], Proposition 2.18)**.** *The subprojectivity domain of the class of finitely presented objects is the class of flat objects.*

The following result is useful to determine the subprojectivity domain of other known classes.

Proposition 1.3.9 ([4], Proposition 2.17)**.** *Let $\mathcal{L}$ be a class of objects of $\mathscr{A}$. Then*

$$\underline{\mathfrak{Pr}}_{\mathscr{A}}{}^{-1}(Add(\mathcal{L})) = \underline{\mathfrak{Pr}}_{\mathscr{A}}{}^{-1}(Sum(\mathcal{L})) = \underline{\mathfrak{Pr}}_{\mathscr{A}}{}^{-1}(Summ(\mathcal{L})) = \underline{\mathfrak{Pr}}_{\mathscr{A}}{}^{-1}(\mathcal{L}).$$

If $\mathcal{L}$ is a set, then all these classes coincide with the class $\underline{\mathfrak{Pr}}_{\mathscr{A}}{}^{-1}(\oplus_{L\in\mathcal{L}}L)$.

Recall now that an object is said to be pure-projective if it is projective with respect to every pure short exact sequence. One can show that in a locally finitely presented category, an object is pure-projective if and only if it is a direct summand of a direct sum of finitely presented objects. As a direct consequence of Proposition 1.3.9, we have the following result.

Proposition 1.3.10 ([4], Corollary 2.20)**.** *If the category is locally finitely presented then the subprojectivity domain of the class of all pure-projective objects is precisely the class of all flat objects.*

Another important class of modules is that of f-projective modules. Recall first that a module M is R-Mittag-Leffler if for every finitely generated submodule C of M, the inclusion map factors through a finitely presented module (see [25]). The relationship between f-projective modules and R-Mittag-Leffler is characterized as follows.

Proposition 1.3.11 ([25], Proposition 1.2)**.** *A module M is f-projective if and only if it is flat and R-Mittag-Leffler.*

Finally, the last result of this section determines the subprojectivity domain of the class of finitely generated modules.

Proposition 1.3.12 ([4], Proposition 2.22)**.** *The subprojectivity domain of the class of finitely generated modules is the class of f-projective modules.*

1.4 Subinjectivity domains

In contrast to the notion of subprojectivity domains, the authors in [5] introduce the notion of subinjectivity domains as an alternative way on the analysis of the injectivity of modules. We start with the definition.

Definition 1.4.1 ([5], Definition 2.1)**.** *We say that a module M is N-subinjective if for every extension K of N and every morphism $f : N \to M$, there exists a morphism $g : K \to M$ such that $g_{/N} = f$. The subinjectivity domain of a module M, or domain of subinjectivity $\underline{\mathfrak{In}}^{-1}(M)$, is defined to be the class*

$$\underline{\mathfrak{In}}^{-1}(M) := \{N \in R\text{-Mod} : M \textit{ is } N\textit{-subinjective}\}.$$

In the same way that a module is projective if and only if its subprojectivity domain consists of the entire class R-Mod, it is clear that a module is injective if and only if its subinjectivity domain equals R-Mod.

The next proposition shows that for M to be N-subinjective, one only needs to extend maps to $E(N)$.

Proposition 1.4.2 ([5], Lemma 2.2)**.** *The following statements are equivalent for any modules two M and N:*

1. *M is N-subinjective.*
2. *For each morphism $\phi : N \to M$ and for every essential extension K of N, there exists a homomorphism $\psi : K \to M$ such that $\psi|_N = \phi$.*
3. *For each morphism $\phi : N \to M$ there exists a homomorphism $\psi : E(N) \to M$ such that $\psi|_N = \phi$.*
4. *For each morphism $\phi : N \to M$ there exists an injective extension E of N and a homomorphism $\psi : E \to M$ such that $\psi|_N = \phi$.*

In classical homological algebra, the relationship between the classes of injective and projective modules was extensively studied. For instance, it is known that a ring R is quasi-Frobenius if and only if the classes of injective and projective modules coincide. In the relative context, the counterpart result is being investigated. The next result tells us when subinjectivity and subprojectivity domains coincide.
For convenience, we will define the following conditions for a ring R:
(P) : The subinjective domain and the subprojective domain coincide for every module, i.e. $\underline{\mathfrak{In}}^{-1}(M) = \underline{\mathfrak{Pr}}^{-1}(M)$ for all modules M.

Theorem 1.4.3 ([15], Theorem 4.1)**.** *The following conditions are equivalent:*

1. *R satisfies **(P)**.*
2. *Every factor ring of R satisfies **(P)**.*
3. *Every factor ring of R is quasi-Frobenius.*
4. *The ring R is isomorphic to a direct product of full matrix rings over Artinian chain rings.*

Condition (3) in Theorem 1.4.3 was extensively studied separately by Faith in [18]. Namely, we have the following result.
Recall that a ring R is called left FGF in case every finitely generated module embeds in a free module.

Proposition 1.4.4 ([18], Theorem 6.1)**.** *The following conditions are equivalent:*

1. *Every factor ring of R is quasi-Frobenius.*

2. *Every factor ring of R is right FGF.*

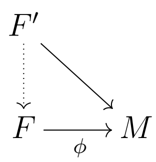
3. *The ring R is isomorphic to a direct product of full matrix rings over Artinian chain rings.*

1.5 Locally initially small and preenveloping classes

Several works were done throughout the years on the notion of injective envelopes; among them is the work of Matlis [29]. Simultaneously, Bass [7] introduced the projective cover as a dual notion. This work has enriched the theory of rings, namely by providing a new class of rings called perfect rings (recall that a ring is said to be perfect if and only if every module admits a cover projective; which is equivalent to saying that any flat module is projective). Ever since, there has been a great interest in the study of precovers and preenvelopes by many authors throughout the years (see [17] and [37]).
It should be noted that in this section, we will only discuss useful results which are needed for this thesis. We start with the definition of a cover of a module.

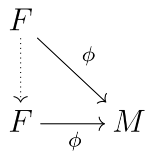
Definition 1.5.1 ([17], Definition 5.1.1)**.** *Let R be a ring and let $\mathcal{F}$ be a class of modules. Then for a module M, a morphism $\phi : F \to M$ is called an $\mathcal{F}$-cover of M if $F \in \mathcal{F}$ and the following two conditions hold:*

1. *Any diagram*



   with $F' \in \mathcal{F}$ can be completed to a commutative diagram.

2. *The diagram*



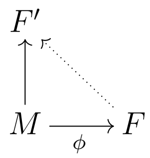
can be completed only by automorphisms of F.

If $\phi : F \to M$ *satisfies (1) but maybe not (2), then* ϕ *is called an* $\mathcal{F}$ *-precover of* M.
If every module has an $\mathcal{X}$*-(pre)cover,* $\mathcal{X}$ *is said to be (pre)covering.*

Having introduced covers, we now define the notion of preenvelopes and envelopes dually.

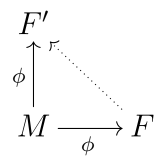
Definition 1.5.2 ([17], Definition 6.1.1)**.** *Let* $\mathcal{F}$ *be a class of modules. Then for a module* M*, a morphism* $\phi : M \to F$ *with* $F \in \mathcal{F}$ *is called an* $\mathcal{F}$*-envelope of* M *if the following two conditions hold:*

1. *Any diagram*



with $F \in \mathcal{F}$ *can be completed to a commutative diagram.*

2. *The diagram*



can be completed only by automorphisms of F.

If $\phi : M \to F$ *satisfies the condition (1) but maybe not (2), then* ϕ *is called an* $\mathcal{F}$*-preenvelope of* M.
If every module has an $\mathcal{F}$*-(pre)envelope, we say that* $\mathcal{F}$ *is (pre)enveloping.*

In order to characterize a ring R by means of the existence of $\mathcal{F}$-preenvelopes for all modules, the authors in [32] introduce the concept of locally initially small classes.

Definition 1.5.3 ([32], Definition 2.1)**.** *Let $\mathcal{F}$ be an arbitrary class of modules. We say that $\mathcal{F}$ is locally initially small if, for every module M, there exists a set $\mathcal{F}_M \subseteq \mathcal{F}$ such that every homomorphism $M \to F$, where $F \in \mathcal{F}$, factors through a direct product of modules in $\mathcal{F}_M$.*

The next proposition shows that the class of flat modules is locally initially small.

Proposition 1.5.4 ([32], Proposition 2.8)**.** *Every class of modules which is closed under pure submodules is a locally initially small class. In particular, the class* Flat *is a locally initially small.*

Other examples of locally initially small classes are given in the following proposition.

Proposition 1.5.5 ([32], Proposition 2.9)**.** *Let $\mathcal{F}$ be a set of modules. Then* $\mathrm{Sum}(\mathcal{F})$ *is a locally initially small class. In particular, the classes $\mathcal{P}$ and $\mathcal{PP}$ are locally initially small.*

With these tools in hand, Rada and Saorín proceeded in [32] to characterize those rings for which every module has an $\mathcal{F}$-preenvelope.

Theorem 1.5.6 ([32], Theorem 3.3)**.** *Let $\mathcal{F}$ be an arbitrary class. The following assertions are equivalent:*

1. *Every module has an $\mathcal{F}$-preenvelope.*
2. *Every module has a* $\mathrm{Summ}(\mathcal{F})$*-preenvelope.*
3. *$\mathcal{F}$ is locally initially small and the class* $\mathrm{Summ}(\mathcal{F})$ *is closed under direct products.*
4. *$\mathcal{F}$ is locally initially small and every product of modules in $\mathcal{F}$ is a direct summand of a module in $\mathcal{F}$.*

Given a class of modules $\mathcal{F}$, the authors in [32] then investigated under which conditions it is possible to ensure that every module has an epic $\mathcal{F}$-preenvelope. Recall that $\mathcal{F}$ is a pretorsion-free class provided that $\mathcal{F}$ is closed under direct products and submodules.

Proposition 1.5.7 ([32], Proposition 4.1)**.** *Let $\mathcal{F}$ be an arbitrary class of modules. Every module has an epic $\mathcal{F}$-preenvelope if and only if $\mathcal{F}$ is a pretorsion-free class.*

1.6 Natural identities, character modules and purity

There is a remarkable relationship between the Hom functor and the tensor product $\otimes$ coming from the adjunction of the two functors. In this section, we elaborate this relationship through natural isomorphisms and characterize some classes of rings by means of it.

We start with the following result.

Proposition 1.6.1 ([17], Theorem 3.2.11)**.** *Let R and S be rings. If A is a finitely presented module, B an (R,S)-bimodule, and C an injective right S-module, then*

$$\tau_{A,B,C} : \mathrm{Hom}_S(B,C) \underset{R}{\otimes} A \cong \mathrm{Hom}_S(Hom_R(A,B),C)$$

where the isomorphism is given by $\tau(f\otimes a)(g) = f(g(a))$.

The following proposition will also be useful.

Proposition 1.6.2 ([34], Theorem 2.76)**.** *Let R and S be rings. Let A be a left R-module, B an (S,R)-bimodule, and C a left R-module. There is a natural isomorphism:*

$$\tau'_{A,B,C} : \mathrm{Hom}_S(B \underset{R}{\otimes} A, C) \cong \mathrm{Hom}_R(A, Hom_S(B,C))$$

given by: for a morphism $f : B\otimes_R A \to C$, $a \in A$, $b \in B$,

$$\tau'_{A,B,C}(f)(a) : b \mapsto f(b\otimes a).$$

We now characterize coherent rings. This is a result of studying the connection between flat modules and absolutely pure modules by means of the character module.

Proposition 1.6.3 ([11], Theorem 1)**.** *The following statements are equivalent:*

1. *The ring R is a left coherent ring.*
2. *For any left module M, M is absolutely pure if and only if M^+ is a flat right module.*
3. *For any left module M, M is absolutely pure if and only if M^{++} is injective.*
4. *For any right module M, M is flat if and only if M^{++} is a flat right module.*

Using Propositions 1.6.2 and 1.6.3, one can show the following proposition that helps determine the flat precover of a module contained in a injective module.

Proposition 1.6.4 ([17], Proposition 5.3.5)**.** *Let R be a right coherent ring, M be a right R-module, and E be an injective right R-module containing M. Then $E^+ \to M^+$ is a flat precover.*

In fact, any character module is pure injective as shown in the next result.

Proposition 1.6.5 ([17], Proposition 5.3.7)**.** *For any module M, the character module M^+ is a pure injective right module.*

One can easily see that every injective module is absolutely pure. The next proposition shows that the converse happens if and only if the ring Noetherian.

Proposition 1.6.6 ([30], Theorem 3)**.** *A ring R is Noetherian if and only if every absolutely pure module is injective.*

Zimmerman [38] explores a stronger notion of purity consisting of monomorphisms called strongly pure monomorphisms. This strong notion of purity appears in the work of several authors (see for example [10, Proposition 2.2]). We recall the definition.

Definition 1.6.7 ([38], Definition 1.1)**.** *A submodule M of a module N is called strongly pure, s-pure for short, if for every finite tuple $x_1, \ldots, x_n$ of elements in M there is a map $t \in \mathrm{Hom}(N, M)$ such that $t(x_i) = x_i$, $1 \leq i \leq n$.*

The following result gives an example of a strongly pure monomorphism.

Proposition 1.6.8 ([38], Proposition 1.4)**.** *If M is a pure submodule of a pure-projective module N then M is s-pure in N.*

Flat precover completing domains

CHAPTER 2

FLAT PRECOVER COMPLETING DOMAINS

Recently, many authors have embraced the study of certain properties of modules such as projectivity, injectivity and flatness from an alternative point of view. Rather than saying a module has a certain property or not, each module is assigned a relative domain which, somehow, measures to which extent it has this particular property. In this chapter, we introduce a new and fresh perspective on flatness of modules. However, we will first investigate a more general context by introducing domains relative to a precovering class $\mathcal{X}$. We call these domains $\mathcal{X}$-precover completing domains. In particular, when $\mathcal{X}$ is the class of flat modules, we call them flat-precover completing domains. This approach allows us to provide a common frame for a number of classical notions. Moreover, some known results are generalized and some classical rings are characterized in terms of these domains.

2.1 Relative domains: Basic results

In this section we define $\mathcal{X}$-precover completing domains and state some general results needed for the rest of the thesis. Here, for an even more general context and to obtain more applications, we fix our attention on $\mathcal{X}$-precover completing domains of a class of modules $\mathcal{L}$, as also done in [4, Definition 2.13].

For the rest of this thesis, we will denote by $\mathcal{X}$ a *precovering class* of modules which satisfies the following conditions:

- $\mathcal{X}$ is closed under isomorphisms, i.e., if $M \in \mathcal{X}$ and $N \cong M$, then $N \in \mathcal{X}$;

- $\mathcal{X}$ is closed under taking finite direct sums, i.e., $M_1, ..., M_t \in \mathcal{X}$ then $M_1 \oplus \cdots \oplus M_t \in \mathcal{X}$;
- $\mathcal{X}$ is closed under taking direct summands, i.e., if $M = N \oplus L \in \mathcal{X}$ then $N, L \in \mathcal{X}$.

Definition 2.1.1. *Given modules M and N, M is said to be (N,$\mathcal{X}$)-precover completing if for every morphism $f : M \to N$, and every $\mathcal{X}$-precover $g : X \to N$, there exists a morphism $h : M \to X$ such that $gh = f$. When no confusion arises, we will omit the name of the precovering class and say simply that M is N-precover completing. The $\mathcal{X}$-precover completing domain of a class of modules $\mathcal{L}$ is defined as the class of modules holding in the $\mathcal{X}$-precover completing domain of each module of $\mathcal{L}$.*

$$\mathcal{X}^{-1}(\mathcal{L}) := \{N \in R\text{-Mod} : M \textit{ is } (N, \mathcal{X})\textit{-precover completing for every } M \in \mathcal{L}\}.$$

In particular, if $\mathcal{L} := \{M\}$, then we write $\mathcal{X}^{-1}(\mathcal{L}) = \mathcal{X}^{-1}(M)$.

From the definition, one can see that for any class of modules $\mathcal{L}$, $\mathcal{X} \subseteq \mathcal{X}^{-1}(\mathcal{L})$. And if we take $\mathcal{X}$ to be the class of projective modules, the $\mathcal{X}$-precover completing domains are simply subprojectivity domains as defined in [23].

In the following proposition, we give a simple characterization of the notion of $\mathcal{X}$-precover completing domains.

Proposition 2.1.2. *Let $\mathcal{L}$ be a class of modules and N be a module. Then the following assertions are equivalent:*

1. *$N \in \mathcal{X}^{-1}(\mathcal{L})$.*
2. *There exists an $\mathcal{X}$-precover $g : X \to N$ such that $\operatorname{Hom}(M, g)$ is an epimorphism for every $M \in \mathcal{L}$.*
3. *Every morphism $M \to N$ with $M \in \mathcal{L}$ factors through a module in $\mathcal{X}$.*
4. *Every morphism $M \to N$ with $M \in \mathcal{L}$ factors through a module in $\mathcal{X}^{-1}(\mathcal{L})$.*

Proof. $(1) \Rightarrow (2) \Rightarrow (3)$ Clear.
$(3) \Rightarrow (4)$ Follows from the fact that $\mathcal{X} \subseteq \mathcal{X}^{-1}(\mathcal{L})$.
$(4) \Rightarrow (1)$ Let $M \in \mathcal{L}$ and consider a morphism $f : M \to N$. We denote by $g : X \to N$ an $\mathcal{X}$-precover. By hypothesis, there exist two morphisms $k : K \to N$ and $h : M \to K$ with $K \in \mathcal{X}^{-1}(\mathcal{L})$ such that $f = kh$. If $t : X' \to K$ is an $\mathcal{X}$-precover, then there exists $l : M \to X'$ such that $h = tl$. Since $g : X \to N$ is an $\mathcal{X}$-precover, there exists $m : X' \to X$ such that $gm = kt$. Therefere, $f = kh = ktl = gml$ and so $N \in \mathcal{X}^{-1}(\mathcal{L})$. □

As the subprojectivity domain does, the $\mathcal{X}$-precover completing domain measures when a module belongs to the class $\mathcal{X}$. This can be seen from the following result.

Proposition 2.1.3. *Let $\mathcal{L}$ be a class of modules. Consider the following conditions:*

1. $\mathcal{X}^{-1}(\mathcal{L}) = R\text{-Mod}$.

2. $\mathcal{L} \subseteq \mathcal{X}^{-1}(\mathcal{L})$.

3. $\mathcal{L} \subseteq \mathcal{X}$.

4. $\mathcal{X}^{-1}(\mathcal{L})$ *is closed under quotients.*

Then $1 \Leftrightarrow 2 \Leftrightarrow 3 \Rightarrow 4$*. If $\mathcal{X}$ contains the class of projective modules, then* $4 \Rightarrow 1$.

Proof. $(1) \Rightarrow (2)$ Clear.
$(2) \Rightarrow (3)$ Let $M \in \mathcal{L}$ and consider the identity morphism $1_M : M \to M$. By Proposition 2.1.2, 1_M factors through a module in $\mathcal{X}$ and so $M \in \mathcal{X}$.
$(3) \Rightarrow (1)$ Let $M \in \mathcal{L}$, N be any module and consider a morphism $f : M \to N$. Since $M \in \mathcal{X}$, we can see that f factors through a module in $\mathcal{X}$ and by Proposition 2.1.2, $N \in \mathcal{X}^{-1}(\mathcal{L})$.
$(1) \Rightarrow (4)$ Clear.
$(4) \Rightarrow (1)$ Let N be any module and let $g : X \to N$ be an $\mathcal{X}$-precover. If $\mathcal{X}$ contains the class of projective modules, then g is an epimorphism. Since $\mathcal{X} \subseteq \mathcal{X}^{-1}(\mathcal{L})$, we deduce that $N \in \mathcal{X}^{-1}(\mathcal{L})$ by hypothesis. □

Proposition 2.1.4. *Let $\{M_i; i \in I\}$ be a family of modules. Then, $\mathcal{X}^{-1}(\oplus_{i\in I} M_i) = \mathcal{X}^{-1}(\{M_i\}_{i\in I})$.*

Proof. Let $g : X \to N$ be an $\mathcal{X}$-precover. The following diagram is commutative

$$\begin{array}{ccc} \mathrm{Hom}(\oplus_{i\in I} M_i, X) & \xrightarrow{\mathrm{Hom}(\oplus_{i\in I} M_i, g)} & \mathrm{Hom}(\oplus_{i\in I} M_i, N) \\ \downarrow{\scriptstyle \psi^X} & & \downarrow{\scriptstyle \psi^N} \\ \prod_{i\in I} \mathrm{Hom}(M_i, X) & \xrightarrow{\prod_{i\in I} \mathrm{Hom}(M_i, g)} & \prod_{i\in I} \mathrm{Hom}(M_i, N) \end{array}$$

where ψ^X and ψ^N are isomorphisms. Hence the morphism $\mathrm{Hom}(\oplus_{i\in I} M_i, g)$ is epic if and only if $\prod_{i\in I} \mathrm{Hom}(M_i, g)$ is epic. Therefore, $N \in \mathcal{X}^{-1}(\oplus_{i\in I} M_i)$ if and only if $N \in \mathcal{X}^{-1}(M_i)$ for every $i \in I$. □

In what follows, we study different closeness properties that $\mathcal{X}$-precover completing domains verify. We start with the following result.

Proposition 2.1.5. *Let $\mathcal{L}$ be a class of modules. The following statements hold:*

1. *Given a short exact sequence of modules* $0 \to A \to B \to C \to 0$ *which is* $\mathrm{Hom}(\mathcal{X}, -)$ *exact, if* A *and* C *are in* $\mathcal{X}^{-1}(\mathcal{L})$*, then* B *is in* $\mathcal{X}^{-1}(\mathcal{L})$.

2. *For a finite family of modules* $\{N_i; 1, ..., m\}$*,* $N_i \in \mathcal{X}^{-1}(\mathcal{L})$ *for every* $i \in \{1, ..., m\}$*, if and only if* $\oplus_{i=1}^{m} N_i \in \mathcal{X}^{-1}(\mathcal{L})$.

Proof. It suffices to prove the result for precover completing domains of modules. Let us consider a module $M \in \mathcal{L}$.

1. Let $0 \to A \to B \to C \to 0$ be a short exact sequence of modules which is $\mathrm{Hom}(\mathcal{X}, -)$ exact and suppose that A and C are in $\mathcal{X}^{-1}(M)$. Consider the $\mathcal{X}$-precovers $X_A \to A$ and $X_C \to C$. We get the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X_A & \longrightarrow & X_A \oplus X_C & \longrightarrow & X_C & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \end{array}$$

We apply the functor $\mathrm{Hom}(M, -)$ and we obtain the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Hom}(M, X_A) & \longrightarrow & \mathrm{Hom}(M, X_A \oplus X_C) & \longrightarrow & \mathrm{Hom}(M, X_C) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{Hom}(M, A) & \longrightarrow & \mathrm{Hom}(M, B) & \longrightarrow & \mathrm{Hom}(M, C) & & \end{array}$$

with exact rows.
Since A and C hold in $\mathcal{X}^{-1}(M)$, the two morphisms $\mathrm{Hom}(M, X_A) \to \mathrm{Hom}(M, A)$ and $\mathrm{Hom}(M, X_C) \to \mathrm{Hom}(M, C)$ are epimorphisms. Thus, $\mathrm{Hom}(M, X_A \oplus X_C) \to \mathrm{Hom}(M, B)$ is also an epimorphism and since $X_A \oplus X_C \in \mathcal{X}$, by Proposition 2.1.2 we get $B \in \mathcal{X}^{-1}(M)$.

2. The closure of $\mathcal{X}^{-1}(M)$ under finite direct sums is a consequence of 1. Conversely, let $N \in \mathcal{X}^{-1}(M)$ and K be a direct summand of N. If $p : N \to K$ is the canonical projection then $\mathrm{Hom}(M, p)$ is epic and then we deduce that $K \in \mathcal{X}^{-1}(M)$ by Proposition 2.1.2. □

Recall that M is said to be a small module if $\mathrm{Hom}(M, -)$ preserves direct sums.

Proposition 2.1.6. *Let* $\mathcal{L}$ *be a class of small modules. The following assertions are equivalent:*

1. *The* $\mathcal{X}$*-precover completing domain of any module in* $\mathcal{L}$ *is closed under arbitrary direct sums.*

2. *The* $\mathcal{X}$*-precover completing domain of* $\mathcal{L}$ *is closed under arbitrary direct sums.*

3. *For any family of modules* $\{X_i\}_{i\in I}$ *in* $\mathcal{X}$, $\oplus_{i\in I}X_i \in \mathcal{X}^{-1}(\mathcal{L})$.

Proof. $(1) \Rightarrow (2)$ This follows from the definition of the $\mathcal{X}$-precover completing domain of $\mathcal{L}$.
$(2) \Rightarrow (3)$ Clear because $\mathcal{X} \subseteq \mathcal{X}^{-1}(\mathcal{L})$.
$(3) \Rightarrow (1)$ Let M be a small module of $\mathcal{L}$ and let $\{g_i : X_i \to N_i\}_{i\in I}$ be a family of $\mathcal{X}$-precovers where $\{N_i\}_{i\in I}$ is a family of modules in $\mathcal{X}^{-1}(M)$. The following diagram

$$\begin{array}{ccc} \mathrm{Hom}(M, \oplus_{i\in I}X_i) & \longrightarrow & \mathrm{Hom}(M, \oplus_{i\in I}N_i) \\ \cong\downarrow & & \downarrow\cong \\ \oplus_{i\in I}\mathrm{Hom}(M, X_i) & \longrightarrow & \oplus_{i\in I}\mathrm{Hom}(M, N_i) \end{array}$$

is commutative, and since each N_i is in $\mathcal{X}^{-1}(M)$, each $\mathrm{Hom}(M, X_i) \to \mathrm{Hom}(M, N_i)$ is epic and so the morphism $\oplus_{i\in I}\mathrm{Hom}(M, X_i) \to \oplus_{i\in I}\mathrm{Hom}(M, N_i)$ is epic. Then, we see that the morphism $\mathrm{Hom}(M, \oplus_{i\in I}X_i) \to \mathrm{Hom}(M, \oplus_{i\in I}N_i)$ is epic. Letting $f : M \to \oplus_{i\in I}N_i$ be any morphism, we see that f factors through $\oplus_{i\in I}X_i$. By assumption, we have $\oplus_{i\in I}X_i \in \mathcal{X}^{-1}(\mathcal{L})$, and clearly $\mathcal{X}^{-1}(\mathcal{L}) \subseteq \mathcal{X}^{-1}(M)$, so $\oplus_{i\in I}N_i \in \mathcal{X}^{-1}(M)$ by Proposition 2.1.2. □

In the next proposition, we give equivalent characterizations for $\mathcal{X}$-precover completing domains to be closed under direct products.

Proposition 2.1.7. *Let* $\mathcal{L}$ *be a class of modules. The following assertions are equivalent:*

1. *The* $\mathcal{X}$*-precover completing domain of any module in* $\mathcal{L}$ *is closed under direct products.*

2. *The* $\mathcal{X}$*-precover completing domain of* $\mathcal{L}$ *is closed under direct products.*

3. *For any family of modules* $\{X_i\}_{i\in I}$ *in* $\mathcal{X}$, $\prod_{i\in I} X_i \in \mathcal{X}^{-1}(\mathcal{L})$.

Proof. $(1) \Rightarrow (2)$ This follows from the definition of the $\mathcal{X}$-precover completing domain of $\mathcal{L}$.
$(2) \Rightarrow (3)$ Clear because $\mathcal{X} \subseteq \mathcal{X}^{-1}(\mathcal{L})$.
$(3) \Rightarrow (1)$ Let M be a module of $\mathcal{L}$ and $\{g_i : X_i \to N_i\}_{i\in I}$ be a family of $\mathcal{X}$-precovers where $\{N_i\}_{i\in I}$ is a family of modules in $\mathcal{X}^{-1}(M)$. The following diagram

$$\begin{array}{ccc} \mathrm{Hom}(M, \prod_{i\in I} X_i) & \longrightarrow & \mathrm{Hom}(M, \prod_{i\in I} N_i) \\ \cong\downarrow & & \downarrow\cong \\ \prod_{i\in I} \mathrm{Hom}(M, X_i) & \longrightarrow & \prod_{i\in I} \mathrm{Hom}(M, N_i) \end{array}$$

commutes. Since each N_i is in $\mathcal{X}^{-1}(M)$, each $\mathrm{Hom}(M, X_i) \to \mathrm{Hom}(M, N_i)$ is epic and so the morphism $\prod_{i\in I} \mathrm{Hom}(M, X_i) \to \prod_{i\in I} \mathrm{Hom}(M, N_i)$ is epic. Then, we see that the morphism $\mathrm{Hom}(M, \prod_{i\in I} X_i) \to \mathrm{Hom}(M, \prod_{i\in I} N_i)$ is epic. Thus, any morphism $f : M \to \prod_{i\in I} N_i$ factors through $\prod_{i\in I} X_i$. By assumption we have $\prod_{i\in I} X_i \in \mathcal{X}^{-1}(\mathcal{L})$ so $\prod_{i\in I} N_i \in \mathcal{X}^{-1}(M)$ by Proposition 2.1.2. □

In the next proposition, we give equivalent characterizations for $\mathcal{X}$-precover completing domains to be closed under submodules.

Proposition 2.1.8. *Let $\mathcal{L}$ be a class of modules. The following assertions are equivalent:*

1. *The $\mathcal{X}$-precover completing domain of any module in $\mathcal{L}$ is closed under submodules.*

2. *The $\mathcal{X}$-precover completing domain of $\mathcal{L}$ is closed under submodules.*

3. *For any submodule K of a module $X \in \mathcal{X}$, $K \in \mathcal{X}^{-1}(\mathcal{L})$.*

Proof. (1) $\Rightarrow$ (2) This follows from the definition of the $\mathcal{X}$-precover completing domain of $\mathcal{L}$.
(2) $\Rightarrow$ (3) Clear because $\mathcal{X} \subseteq \mathcal{X}^{-1}(\mathcal{L})$.
(3) $\Rightarrow$ (1) Let $M \in \mathcal{L}$, $N \in \mathcal{X}^{-1}(M)$ and D be a submodule of N. Consider a morphism $f : M \to D$. Letting $i : D \to N$ denote the injection map, if factors through a module in $\mathcal{X}$ by Proposition 2.1.2, that is, there exist two morphisms $h : M \to X$ and $g : X \to N$ such that $gh = if$ and $X \in \mathcal{X}$. Let $K = \mathrm{Im}h$ and consider $\bar{h} : M \to K$ such that $\iota\bar{h} = h$ with $\iota : K \to X$ the injection map. We have that $\mathrm{Ker}\bar{h} \subseteq \mathrm{Ker}f$ and so we can define a morphism $\phi : K \to D$ such that $\phi(\bar{h}(m)) = f(m)$ for every $m \in M$. We have $f = \phi\bar{h}$ and by assumption, $K \in \mathcal{X}^{-1}(M)$ and so by Proposition 2.1.2, $\bar{h} : M \to K$ factors through a module in the class $\mathcal{X}$. Thus, it is easy to see that f also factors through a module in $\mathcal{X}$ and so by Proposition 2.1.2, $D \in \mathcal{X}^{-1}(M)$. □

It is not known whether or not the subprojectivity domains are closed under kernels of epimorphisms. In [4, Proposition 3.2], a weak equivalent condition for this property was provided. In the relative case we need to assume the additional condition that the short exact sequences are $\mathrm{Hom}(\mathcal{X}, -)$ exact.

Proposition 2.1.9. *Suppose that $\mathcal{X}$ contains the class of projective modules and let $\mathcal{L}$ be a class of modules. Then the following conditions are equivalent:*

1. *For every short and $\mathrm{Hom}(\mathcal{X}, -)$ exact sequence $0 \to A \to B \to C \to 0$, if $B, C \in \mathcal{X}^{-1}(\mathcal{L})$ then $A \in \mathcal{X}^{-1}(\mathcal{L})$.*

2. *For every short exact sequence* $0 \to K \to X \to C \to 0$ *where* $X \to C$ *is an* $\mathcal{X}$*-precover, if* $C \in \mathcal{X}^{-1}(\mathcal{L})$ *then* $K \in \mathcal{X}^{-1}(\mathcal{L})$.

3. *For every* $\mathcal{X}$*-precover* $X \to C$ *with* $C \in \mathcal{X}^{-1}(\mathcal{L})$*, the pullback of* X *over* C *holds in* $\mathcal{X}^{-1}(\mathcal{L})$.

Proof. It suffices to prove the result for precover completing domains of modules. So let us consider a module M.
$(1) \Rightarrow (2)$ Clear.
$(2) \Rightarrow (1)$ Consider an exact sequence

$$0 \to A \to B \to C \to 0$$

which is $\mathrm{Hom}(\mathcal{X}, -)$ exact with $B,\ C \in \mathcal{X}^{-1}(M)$. We have the following pullback diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & K & = & K & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & D & \longrightarrow & X & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

where $X \to C$ is an $\mathcal{X}$-precover. Then, $K \in \mathcal{X}^{-1}(M)$ by assumption because $C \in \mathcal{X}^{-1}(M)$. D being the pullback of $X \to C$ and $B \to C$ and since $X \to C$ is an $\mathcal{X}$-precover, we can easily see that the sequence $0 \to K \to D \to B \to 0$ is $\mathrm{Hom}(\mathcal{X}, -)$ exact. Then, by assertion 1 in Proposition 2.1.5, $D \in \mathcal{X}^{-1}(M)$. And since $0 \to A \to B \to C \to 0$ is $\mathrm{Hom}(\mathcal{X}, -)$ exact and D is the pullback of $X \to C$ and $B \to C$, we see that $0 \to A \to D \to X \to 0$ splits. Thus, A is a direct summand of D. Using assertion 2 in Proposition 2.1.5 we deduce that $A \in \mathcal{X}^{-1}(M)$.
$(2) \Leftrightarrow (3)$ Consider the following diagram where D is the pullback of X over C

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K & \longrightarrow & D & \longrightarrow & X & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K & \longrightarrow & X & \longrightarrow & C & \longrightarrow & 0
\end{array}$$

Then, the short exact sequence $0 \to K \to D \to X \to 0$ is $\mathrm{Hom}(\mathcal{X}, -)$ exact. If $C \in \mathcal{X}^{-1}(M)$ then $K \in \mathcal{X}^{-1}(M)$ and so by assertion 1 in Proposition 2.1.5,

$D \in \mathcal{X}^{-1}(M)$. Conversely, if $D \in \mathcal{X}^{-1}(M)$ then, by assertion 2 in Proposition 2.1.5 $K \in \mathcal{X}^{-1}(M)$ (because the short exact sequence $0 \to K \to D \to X \to 0$ splits). □

It is easy to see that if the $\mathcal{X}$-precover completing domain of every class of modules is closed under kernels of epimorphisms, then $\mathcal{X}$ is also closed under kernels of epimorphisms. For the converse, the following result shows that we have a partial positive answer when we consider some special epimorphisms.

Proposition 2.1.10. *The class $\mathcal{X}$ is closed under kernels of epimorphisms if and only if for every class of modules $\mathcal{L}$ and every short exact sequence $0 \to A \to B \to X \to 0$ with $X \in \mathcal{X}$, if $B \in \mathcal{X}^{-1}(\mathcal{L})$ then $A \in \mathcal{X}^{-1}(\mathcal{L})$.*

Proof. It suffices to show this result for precover completing domains of modules. We first show that $\mathcal{X}$ is closed under kernels of epimorphisms. Consider a short exact sequence $0 \to K \to X_1 \to X_2 \to 0$ with $X_1, X_2 \in \mathcal{X}$. Clearly, we have $X_1, X_2 \in \mathcal{X}^{-1}(K)$ so $K \in \mathcal{X}^{-1}(K)$ and by Proposition 2.1.3 we deduce that $K \in \mathcal{X}$.
For the converse, let $X' \to B$ be an $\mathcal{X}$-precover and consider the following pullback diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & D & \longrightarrow & X' & \longrightarrow & X & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & X & \longrightarrow & 0 \end{array}$$

Since $\mathcal{X}$ is closed under kernels of epimorphisms, we have $D \in \mathcal{X}$. Applying the functor $\mathrm{Hom}(M, -)$ for a module $M \in \mathcal{L}$, we obtain the commutative diagrams with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{Hom}(M, D) & \longrightarrow & \mathrm{Hom}(M, X') & \longrightarrow & \mathrm{Hom}(M, X) \\ & & \downarrow & & \downarrow & & \| \\ 0 & \longrightarrow & \mathrm{Hom}(M, A) & \longrightarrow & \mathrm{Hom}(M, B) & \longrightarrow & \mathrm{Hom}(M, X) \end{array}$$

Since $B \in \mathcal{X}^{-1}(\mathcal{L})$, $\mathrm{Hom}(M, X') \to \mathrm{Hom}(M, B)$ is epic for any $M \in \mathcal{L}$ and so $\mathrm{Hom}(M, D) \to \mathrm{Hom}(M, A)$ is epic too. Therefore, $A \in \mathcal{X}^{-1}(\mathcal{L})$. □

We now give an extension of [4, Proposition 2.17].

Proposition 2.1.11. *Let $\mathcal{L}$ be a class of modules. Then*

$$\mathcal{X}^{-1}(Add(\mathcal{L})) = \mathcal{X}^{-1}(Sum(\mathcal{L})) = \mathcal{X}^{-1}(Summ(\mathcal{L})) = \mathcal{X}^{-1}(\mathcal{L}).$$

If $\mathcal{L}$ is a set, then all these classes coincide with the class $\mathcal{X}^{-1}(\oplus_{L \in \mathcal{L}} L)$.

Proof. Since $\mathcal{L}$ holds inside $Add(\mathcal{L})$, we see easily that $\mathcal{X}^{-1}(Add(\mathcal{L}))$ holds inside $\mathcal{X}^{-1}(\mathcal{L})$.

For the converse, let N be in $\mathcal{X}^{-1}(\mathcal{L})$ and M in $Add(\mathcal{L})$. Then, we know that there exists a module M' in $Add(\mathcal{L})$ and a family of modules $\{L_i\}$ in $\mathcal{L}$ such that $M \oplus M' = \oplus_i L_i$. By Proposition 2.1.4, we see that $N \in \mathcal{X}^{-1}(M)$. Consequently, $N \in \mathcal{X}^{-1}(Add(\mathcal{L}))$. We deduce that $\mathcal{X}^{-1}(Add(\mathcal{L})) = \mathcal{X}^{-1}(\mathcal{L})$.

We have clearly $\mathcal{L} \subseteq Summ(\mathcal{L}) \subseteq Add(\mathcal{L})$ and $\mathcal{L} \subseteq Sum(\mathcal{L}) \subseteq Add(\mathcal{L})$. Computing the $\mathcal{X}$-precover completing domain of each class, we have $\mathcal{X}^{-1}(Add(\mathcal{L})) \subseteq \mathcal{X}^{-1}(Summ(\mathcal{L})) \subseteq \mathcal{X}^{-1}(\mathcal{L})$ and $\mathcal{X}^{-1}(Add(\mathcal{L})) \subseteq \mathcal{X}^{-1}(Sum(\mathcal{L})) \subseteq \mathcal{X}^{-1}(\mathcal{L})$. We conclude that $\mathcal{X}^{-1}(Add(\mathcal{L})) = \mathcal{X}^{-1}(Sum(\mathcal{L})) = \mathcal{X}^{-1}(Summ(\mathcal{L})) = \mathcal{X}^{-1}(\mathcal{L})$.

If $\mathcal{L}$ is a set then, by Proposition 2.1.4, $\mathcal{X}^{-1}(\mathcal{L}) = \mathcal{X}^{-1}(\oplus_{L\in\mathcal{L}} L)$. □

In the rest of this section, we are interested in the precover completing domain of a single module M. We start by investigating the $\mathcal{X}$-precover completing domain of a module embedded in a module of the class $\mathcal{X}$.

Proposition 2.1.12. *Let* $0 \to M \overset{\alpha}{\to} X \to M' \to 0$ *be a short exact sequence with* $X \in \mathcal{X}$. *Then,* ${M'}^{\perp} \subseteq \mathcal{X}^{-1}(M)$. *Moreover, if* $\alpha : M \to X$ *is an* $\mathcal{X}$*-preenvelope, then* $\mathcal{X}^{-1}(M) \cap X^{\perp} \subseteq {M'}^{\perp}$.

Proof. Let $N \in {M'}^{\perp}$. Consider the following long exact sequence

$$\longrightarrow \mathrm{Hom}(X,N) \longrightarrow \mathrm{Hom}(M,N) \longrightarrow \mathrm{Ext}^1(M',N) \longrightarrow \mathrm{Ext}^1(X,N) \longrightarrow$$

Since $\mathrm{Ext}^1(M',N) = 0$, $\mathrm{Hom}(X,N) \to \mathrm{Hom}(M,N)$ is epic. Thus, $N \in \mathcal{X}^{-1}(M)$ by Proposition 2.1.2.

Now, let $N \in \mathcal{X}^{-1}(M) \cap X^{\perp}$ and $X' \to N$ be an $\mathcal{X}$-precover. We obtain the following commutative diagram by applying the functors $\mathrm{Hom}(-,X')$ and $\mathrm{Hom}(-,N)$ to $M \to X$

$$\begin{array}{ccccc}
\mathrm{Hom}(X,X') & \longrightarrow & \mathrm{Hom}(M,X') & & \\
\downarrow & & \downarrow & & \\
\mathrm{Hom}(X,N) & \longrightarrow & \mathrm{Hom}(M,N) & \longrightarrow & \mathrm{Ext}^1(M',N)
\end{array}$$

Since $N \in X^{\perp}$, $\mathrm{Ext}^1(X,N) = 0$. Then, from the long exact sequence above, we see that we only need to prove that $\mathrm{Hom}(X,N) \to \mathrm{Hom}(M,N)$ is epic to deduce that $\mathrm{Ext}^1(M',N) = 0$. Since $N \in \mathcal{X}^{-1}(M)$ we have $\mathrm{Hom}(M,X') \to \mathrm{Hom}(M,N)$ is an epimorphism. Moreover $\mathrm{Hom}(X,X') \to \mathrm{Hom}(M,X')$ is epic because $\alpha : M \to X$ is an $\mathcal{X}$-preenvelope. Therefore, $\mathrm{Hom}(X,N) \to \mathrm{Hom}(M,N)$ is epic. This completes the proof. □

From Proposition 2.1.12, we see that if a module M embeds in a module in $\mathcal{X}$ then $\mathcal{X}^{-1}(M)$ contains the class of injective modules. In the next result, we prove that this is in fact an equivalence and also establish some other equivalences similarly to [15, Lemma 2.2]

Proposition 2.1.13. *The following conditions are equivalent for a module M:*

1. *M embeds in a module in $\mathcal{X}$.*

2. *There exists a module M' such that $M'^{\perp} \subseteq \mathcal{X}^{-1}(M)$.*

3. *$\mathcal{I} \subseteq \mathcal{X}^{-1}(M)$.*

4. *$E(M) \in \mathcal{X}^{-1}(M)$.*

5. *For any flat right R-module F, $F^+ \in \mathcal{X}^{-1}(M)$.*

Proof. $(1) \Rightarrow (2)$ This follows from Proposition 2.1.12.
$(2) \Rightarrow (3) \Rightarrow (4)$ Clear.
$(4) \Rightarrow (1)$ Since $E(M) \in \mathcal{X}^{-1}(M)$, the inclusion map $f : M \to E(M)$ factors through a module in $\mathcal{X}$, that is, there exist two morphisms $h : M \to X$ and $g : X \to N$ with $X \in \mathcal{X}$ such that $f = gh$. But gh is a monomorphism and so $h : M \to X$ is monomorphism. We conclude that M can be embedded in a module in $\mathcal{X}$.
$(3) \Rightarrow (5)$ Clear.
$(5) \Rightarrow (3)$ Let E be an injective module and let $F \to E^+ \to 0$ be a flat precover. Then, we have $0 \to E^{++} \to F^+$. And since there exists a canonical monomorphism $E \to E^{++}$, we obtain a morphism $0 \to E \to F^+$ which splits. Now, $F^+ \in \mathcal{X}^{-1}(M)$, so by assertion 2 in Proposition 2.1.5 we obtain $E \in \mathcal{X}^{-1}(M)$. $\square$

Now we investigate when the domains contain the class of pure-injective modules $\mathcal{PI}$.

Proposition 2.1.14. *The following conditions are equivalent for a module M:*

1. *M is a pure submodule of a module in $\mathcal{X}$.*

2. *$\mathcal{PI} \subseteq \mathcal{X}^{-1}(M)$.*

3. *$PE(M) \in \mathcal{X}^{-1}(M)$.*

4. *For any right R-module N, $N^+ \in \mathcal{X}^{-1}(M)$.*

5. *For any pure-projective right R-module P, $P^+ \in \mathcal{X}^{-1}(M)$.*

Proof. (1) $\Rightarrow$ (2) Let E be a pure-injective module. Let $f : M \to E$ be a morphism. We have that M is a pure submodule of a module $X \in \mathcal{X}$. We denote by $i : M \to X$ the pure monomorphism. As E is pure-injective, there exists a morphism $h : X \to E$ such that $f = hi$ and so $f : M \to E$ factors through a module in $\mathcal{X}$. Thus, $E \in \mathcal{X}^{-1}(M)$.
(2) $\Rightarrow$ (3) Clear.
(3) $\Rightarrow$ (1) Since $PE(M) \in \mathcal{X}^{-1}(M)$, the inclusion map $f : M \to PE(M)$ factors through a module in $\mathcal{X}$, that is, there exist two morphisms $h : M \to X$ and $g : X \to N$ with $X \in \mathcal{X}$ such that $f = gh$. But gh is a pure monomorphism and so $h : M \to X$ is a pure monomorphism.
(2) $\Rightarrow$ (4) Clear because for every module N, N^+ is pure-injective.
(4) $\Rightarrow$ (5) Clear.
(5) $\Rightarrow$ (2) Let E be a pure-injective module and let $P \to E^+ \to 0$ be an epimorphism with P pure-projective. Then, we have a monomorphism $0 \to E^{++} \to P^+$, and composing with the canonical monomorphism $E \to E^{++}$, we get a splitting monomorphism $0 \to E \to P^+$. Consequently, E is a direct summand of P^+ and, by assumption, $P^+ \in \mathcal{X}^{-1}(M)$. Thus, by assertion 2 in Proposition 2.1.5, we obtain $E \in \mathcal{X}^{-1}(M)$. □

We end this section with the next proposition, where we establish a connection between domains relative to two precovering classes $\mathcal{X}$ and $\mathcal{Y}$ such that $\mathcal{X} \subseteq \mathcal{Y}$.

Proposition 2.1.15. *Let $\mathcal{X}$ and $\mathcal{Y}$ be two precovering classes such that $\mathcal{X} \subseteq \mathcal{Y}$ and let M be a module. Then, $\mathcal{X}^{-1}(M) \subseteq \mathcal{Y}^{-1}(M)$. Furthermore, $\mathcal{X}^{-1}(M) = \mathcal{Y}^{-1}(M)$ if and only if $\mathcal{Y} \subseteq \mathcal{X}^{-1}(M)$.*

Proof. Let M be a module and let $N \in \mathcal{X}^{-1}(M)$. Then any morphism $M \to N$ factors through a module in $\mathcal{X}$ by Proposition 2.1.2. Since $\mathcal{X} \subseteq \mathcal{Y}$, we deduce that any morphism $M \to N$ factors through a module in $\mathcal{Y}$. Thus, $N \in \mathcal{Y}^{-1}(M)$ by Proposition 2.1.2.
Suppose now that $\mathcal{Y} \subseteq \mathcal{X}^{-1}(M)$. Let $N \in \mathcal{Y}^{-1}(M)$ and $f : M \to N$ be a morphism. Then, there exists $g : Y \to N$ and $h : M \to Y$ such that $Y \in \mathcal{Y}$ and $gh = f$. Since $\mathcal{Y} \subseteq \mathcal{X}^{-1}(M)$, the morphism $h : M \to Y$ factors through a module in $\mathcal{X}$. Hence, the morphism $f : M \to N$ also factors through a module in $\mathcal{X}$. Therefore, $N \in \mathcal{X}^{-1}(M)$. Finally, it is clear that if $\mathcal{X}^{-1}(M) = \mathcal{Y}^{-1}(M)$ then $\mathcal{Y} \subseteq \mathcal{X}^{-1}(M)$. □

2.2 Flat-precover completing domains

Now we focus our attention on the aim of this chapter, which is when taking the class $\mathcal{X}$ to be that of flat modules. In that case, we refer to $\mathcal{X}$-precover completing domains as flat-precover completing domains and we denote the flat-precover completing domain of a module M by $\mathfrak{F}^{-1}(M)$. Clearly, all the properties proved

in Section 2.1 remain valid for flat-precover completing domains so we will omit repeating them here. The purpose of this section is twofold: to prove the utility of flat-precover completing domains by characterizing some classical rings in terms of these newly defined domains, and to study their relationship with subprojectivity domains and absolutely pure domains (see [16] and [23]).
First, let us begin with the following extension of the characterizations of a flat module in terms of flat-precover completing domains. Notice that, following [23], a module is projective if and only if $\underline{\mathfrak{Pr}}^{-1}(M) = R\text{-Mod}$. Thus, M is projective if and only if $\mathcal{P}^{\perp} \subseteq \underline{\mathfrak{Pr}}^{-1}(M)$. In the following proposition we show that the analogue result also holds for flat-precover completing domains.

Recall that a module C is called cotorsion if $\mathrm{Ext}^1(F, C) = 0$ for any flat module F. We denote the class of cotorsion modules by $\mathcal{C}$. A monomorphism $\alpha : M \to C$ with C cotorsion is said to be a special cotorsion preenvelope of M if $\mathrm{Coker}\alpha$ is flat. Recall, from [17, Section 7.4], that every module M has a special cotorsion preenvelope that we denote by $C(M)$.

Proposition 2.2.1. *Let M be a module. Then the following conditions are equivalent:*

1. *M is flat.*
2. $\mathfrak{F}^{-1}(M) = R\text{-Mod}$.
3. $M \in \mathfrak{F}^{-1}(M)$.
4. $\mathcal{C} \subseteq \mathfrak{F}^{-1}(M)$.
5. $C(M) \in \mathfrak{F}^{-1}(M)$.
6. $\mathcal{PI} \subseteq \mathfrak{F}^{-1}(M)$.
7. $PE(M) \in \mathfrak{F}^{-1}(M)$.
8. *For any module N, $N^+ \in \mathfrak{F}^{-1}(M)$.*
9. *For any pure-projective module P, $P^+ \in \mathfrak{F}^{-1}(M)$.*
10. *For any module U, $M \in \mathfrak{F}^{-1}(U)$.*
11. *$\mathfrak{F}^{-1}(M)$ is closed under quotients.*

Proof. $(1) \Leftrightarrow (2) \Leftrightarrow (3) \Leftrightarrow (11)$ This follow from Proposition 2.1.3.
$(2) \Rightarrow (4) \Rightarrow (5)$ Clear.
$(5) \Rightarrow (3)$ Consider the short exact sequence $0 \to M \overset{\alpha}{\to} C(M) \to \mathrm{Coker}\alpha \to 0$ where α is a special preenvelope (so $\mathrm{Coker}(\alpha)$ is flat). Since $C(M) \in \mathfrak{F}^{-1}(M)$, Proposition 2.1.10 implies that $M \in \mathfrak{F}^{-1}(M)$.
$(1) \Leftrightarrow (6) \Leftrightarrow (7) \Leftrightarrow (8) \Leftrightarrow (9)$ This follow from Proposition 2.1.14.
$(1) \Leftrightarrow (10)$ Clear. □

From Proposition 2.2.1 we see that if P is a projective module then M is flat if and only if $\mathfrak{F}^{-1}(M) = P^{\perp}$. But there are other examples of M such that $\mathfrak{F}^{-1}(M) = N^{\perp}$ for some module N (without $N^{\perp}$ being the whole category of modules).
Recall first that a module M is said to be strongly Gorenstein flat if there is an exact sequence $\cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$ of projective modules with $M \cong \mathrm{Ker}(P_0 \to P^0)$ such that $\mathrm{Hom}(-, \mathrm{Flat})$ leaves the sequence exact (see [14, Definition 2.1]). Gillespie in [20] renamed these modules as Ding projective modules since there exists an alternative definition of a strongly Gorenstein flat module given in [8, Definition 3.1] and that we recalled in Section 1.3.

Example 2.2.2. *1. For any Ding projective module M, there exists a Ding projective module M' such that $\mathfrak{F}^{-1}(M) = M'^{\perp}$.*

2. For any strongly Gorenstein flat and finitely presented module M, $\mathfrak{F}^{-1}(M) = M^{\perp}$.

Proof. 1. Any Ding projective module M has a flat preenvelope $\alpha : M \to F$ with F projective. Moreover, $\mathrm{Coker}(\alpha)$ is Ding projective. We then apply Proposition 2.1.12 to deduce that $\mathfrak{F}^{-1}(M) = \mathrm{Coker}(\alpha)^{\perp}$.
2. By Proposition 1.3.3, a module is finitely generated and strongly Gorenstein projective if and only if it is finitely presented and strongly Gorenstein flat. By Corollary 2.2.6, $\mathfrak{F}^{-1}(M) = \underline{\mathfrak{Pr}}^{-1}(M)$ for any finitely presented module M and by Corollary 1.3.4, we can conclude that $\mathfrak{F}^{-1}(M) = M^{\perp}$. □

In the classical homological algebra, relations between flat modules, projective modules and absolutely pure modules have been extensively studied. In our new context, we investigate the counterpart results. Namely, we study the relation between flat-precover completing domains, subprojectivity domains and absolutely pure domains (see [4] and [16]). We start by investigating the relationship between flat-precover completing domains and subprojectivity domains.
Clearly, the subprojectivity domain $\underline{\mathfrak{Pr}}^{-1}(M)$ of a module M is contained in $\mathfrak{F}^{-1}(M)$. However, they are not necessary equal as it is shown by the following example.

Example 2.2.3. *The abelian group $\mathbb{Q}$ is a flat $\mathbb{Z}$-module, thus by Proposition 2.1.3, $\mathbb{Q} \in \mathfrak{F}^{-1}(\mathbb{Q})$. But $\mathbb{Q} \notin \underline{\mathfrak{Pr}}^{-1}(\mathbb{Q})$ for otherwise $\mathbb{Q}$ would be a projective $\mathbb{Z}$-module.*

From Proposition 2.1.15 we deduce the following result.

Proposition 2.2.4. *Let M be a module. Then, $\mathfrak{F}^{-1}(M) = \underline{\mathfrak{Pr}}^{-1}(M)$ if and only if* $\mathrm{Flat} \subseteq \underline{\mathfrak{Pr}}^{-1}(\mathrm{M})$.

Recall that a ring R is perfect if and only if any flat module is projective. In terms of flat-precover completing domains, we have the following result.

Corollary 2.2.5. *The following conditions are equivalent:*

1. *R is perfect.*
2. *$\mathfrak{F}^{-1}(M) = \underline{\mathfrak{Pr}}^{-1}(M)$ for any module M.*
3. *$\mathfrak{F}^{-1}(M) \subseteq \underline{\mathfrak{Pr}}^{-1}(M)$ for any module M.*

It is shown in Corollary 1.3.10 that the subprojectivity domain of any pure-projective module contains the class of flat modules. Thus, by Proposition 2.2.4, we deduce the following result.

Corollary 2.2.6. *For any pure-projective module M, $\mathfrak{F}^{-1}(M) = \underline{\mathfrak{Pr}}^{-1}(M)$.*

Remark 2.2.7. *Corollary 2.2.6 stands as a generalization of the very well-known result that says: any finitely presented and flat module is projective. Indeed, if N is a finitely presented and flat module, then by Proposition 2.2.1 $N \in \mathfrak{F}^{-1}(N)$ and so $N \in \underline{\mathfrak{Pr}}^{-1}(N)$ by Corollary 2.2.6. Therefore, we conclude that N is projective by Proposition 1.3.7.*

We now apply Proposition 2.2.4 to characterize when the flat-precover completing domains and subprojectivity domains of finitely generated modules coincide. We obtain a characterization of rings over which every flat module is f-projective in terms of flat-precover completing domains. An extensive study of rings over which every flat module is f-projective is done by Shenglin (see [35, Corollary 5]).

Proposition 2.2.8. *The following assertions are equivalent:*

1. *Any flat module is f-projective.*
2. *Any flat module is R-Mittag-Leffler.*
3. *$\mathfrak{F}^{-1}(M) = \underline{\mathfrak{Pr}}^{-1}(M)$ for any finitely generated module M.*
4. *The class of modules holding in $\mathfrak{F}^{-1}(M)$ for any finitely generated module M is precisely the class of f-projective modules.*

Proof. $(1) \Leftrightarrow (2)$ By Proposition 1.3.11, a module M is f-projective if and only if M is flat and R-Mittag-Leffler.
$(1) \Leftrightarrow (3)$ By Proposition 2.2.4 $\mathfrak{F}^{-1}(M) = \underline{\mathfrak{Pr}}^{-1}(M)$ for every finitely generated module M if and only if the class of flat modules is inside the subprojectivity domain of every finitely generated module. By Proposition 1.3.12, the class of modules holding in the subprojectivity domain of every finitely generated module is precisely the class of f-projective modules. Thus, $\mathfrak{F}^{-1}(M) = \underline{\mathfrak{Pr}}^{-1}(M)$ for every finitely generated module M if and only any flat module is f-projective.
$(3) \Rightarrow (4)$ Follows from Proposition 1.3.12.
$(4) \Rightarrow (1)$ Clear. □

Lambek [26] proved that, over any ring, a module is flat if and only if its character module is injective. By Proposition 1.6.5, for any module M, the character module M^+ is a pure injective right module. Thus, we see that a module is flat if and only if its character module is absolutely pure. The next proposition generalizes this fact while connecting flat-precover completing domains with absolutely pure domains.

Proposition 2.2.9. *Let M be a finitely presented module and N a module. Then, $N \in \mathfrak{F}^{-1}(M)$ if and only if $N^+ \in \mathcal{A}p(M)$.*

Proof. Let $F \to N \to 0$ be a flat precover of N. Since M is finitely presented, by Proposition 1.6.1 we have the following commutative diagram

$$\begin{array}{ccc} (\mathrm{Hom}(M,N))^+ & \longrightarrow & (\mathrm{Hom}(M,F))^+ \\ \cong\big\downarrow & & \big\downarrow\cong \\ N^+ \otimes_R M & \longrightarrow & F^+ \otimes_R M \end{array}$$

Hence, $N^+ \otimes_R M \to F^+ \otimes_R M$ is monic if and only if $\mathrm{Hom}(M,F) \to \mathrm{Hom}(M,N)$ is epic. Therefore, $N^+ \in \mathcal{A}p(M)$ if and only if $N \in \mathfrak{F}^{-1}(M)$ by Proposition 1.2.4. □

We end this section with some characterizations of coherent rings in terms of flat-precover completing domains.

First notice that from Propositions 2.1.7 and 2.1.8, we immediately get the following result. Recall that a ring R is left semihereditary if and only if R is left coherent and $\mathrm{wdim} R \leq 1$.

Proposition 2.2.10. *The following properties hold:*

1. *The ring R is right coherent if and only if the flat-precover completing domain of any module is closed under arbitrary direct products.*

2. $\mathrm{wdim}(R) \leq 1$ *if and only if the flat-precover completing domain of any module is closed under submodules.*

3. *The ring R is left semihereditary if and only if the flat-precover completing domain of any right R-module is closed under arbitrary direct products and the flat-precover completing domain of any left module is closed under submodules.*

Cheatham and Stone characterized in [11, Theorem 1] right coherent rings in terms of absolutely pure and flat modules. The next proposition gives the counterpart result in terms of flat-precover completing domains and absolutely pure domains.

Proposition 2.2.11. *The following assertions are equivalent:*

1. *The ring R is right coherent.*

2. *For any module M and any right module N, $N \in \mathcal{A}p(M)$ if and only if $N^+ \in \mathfrak{F}^{-1}(M)$.*

3. *For any finitely presented module M and any module N, $N \in \mathfrak{F}^{-1}(M)$ if and only if $N^{++} \in \mathfrak{F}^{-1}(M)$.*

Proof. $(1) \Rightarrow (2)$ Let $0 \to N \to E$ be an injective preenvelope of N. Since R is coherent, $E^+ \to N^+ \to 0$ is a flat precover by Proposition 1.6.4. We have the following commutative diagram by Proposition 1.6.2

$$\begin{array}{ccc} \mathrm{Hom}(M, E^+) & \longrightarrow & \mathrm{Hom}(M, N^+) \\ \cong\downarrow & & \downarrow\cong \\ (E \otimes_R M)^+ & \longrightarrow & (N \otimes_R M)^+ \end{array}$$

Hence, $\mathrm{Hom}(M, E^+) \to \mathrm{Hom}(M, N^+)$ is epic if and only if $N \otimes_R M \to E \otimes_R M$ is monic. Therefore, $N^+ \in \mathfrak{F}^{-1}(M)$ if and only if $N \in \mathcal{A}p(M)$.
$(2) \Rightarrow (3)$ Let M be a finitely presented module and N be a module. Then, $N \in \mathfrak{F}^{-1}(M)$ if and only if $N^+ \in \mathcal{A}p(M)$ by Proposition 2.2.9 and so $N \in \mathfrak{F}^{-1}(M)$ if and only if $N^{++} \in \mathfrak{F}^{-1}(M)$ by assumption.
$(3) \Rightarrow (1)$ Let N be a flat module. Then, by Proposition 2.2.1, $N \in \mathfrak{F}^{-1}(M)$ for any finitely presented module M. By assumption, we have $N^{++} \in \mathfrak{F}^{-1}(M)$ for any finitely presented module M and so N^{++} is flat by Propositions 1.3.8 and 2.2.6. We conclude by Proposition 1.6.3 that R is right coherent. □

2.3 Subinjectivity and flat-precover completing domains

The relation between flat and injective modules was widely studied by several authors throughout the years leading to a wide range of rich results. In this section, we study to what extent we can compare the degrees of flatness and injectivity of modules.

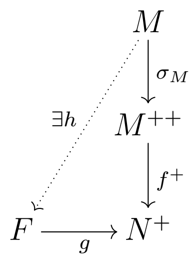
We start with the following result.

Proposition 2.3.1. *The following properties hold:*

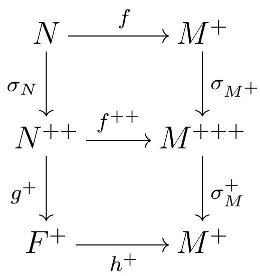
1. *For any module M and any right module N, if $N^+ \in \mathfrak{F}^{-1}(M)$ then $N \in \underline{\mathfrak{In}}^{-1}(M^+)$.*

2. *The ring R is right coherent if and only if for any module M and any right module N, if $N \in \underline{\mathfrak{In}}^{-1}(M^+)$ then $N^+ \in \mathfrak{F}^{-1}(M)$.*

Proof. For any module L, we denote by $\sigma_L : L \to L^{++}$ the evaluation morphism.
1. Let $N^+ \in \mathfrak{F}^{-1}(M)$ with $F \to N^+$ a flat precover and let $f : N \to M^+$ be any morphism. Then we have a morphism $f^+ : M^{++} \to N^+$. Since $N^+ \in \mathfrak{F}^{-1}(M)$, there exists a morphism $h : M \to F$ such that the following diagram commutes



Then, $gh = f^+\sigma_M$ and so $h^+g^+ = \sigma_M^+ f^{++}$.
We have the following diagram where each square is commutative



Now, $\sigma_M^+ \sigma_{M^+} = 1_{M^+}$. Therefore, $f = \sigma_M^+ \sigma_{M^+} f = h^+ g^+ \sigma_N$. Since F^+ is injective, we deduce by Proposition 1.4.2 that $N \in \underline{\mathfrak{In}}^{-1}(M^+)$.

2. Suppose first that R is coherent and let $N \in \underline{\mathfrak{In}}^{-1}(M^+)$ with $g : N \to E$ an injective preenvelope (notice that E^+ is flat). Then for any morphism $f : M \to N^+$, there exists a morphism $h : E \to M^+$ such that the following diagram commutes

$$\begin{array}{ccc} M^+ & & \\ \uparrow{\scriptstyle f^+} & \nwarrow{\scriptstyle \exists h} & \\ N^{++} & & \\ \uparrow{\scriptstyle \sigma_N} & & \\ N & \xrightarrow[g]{} & E \end{array}$$

Then, $hg = f^+\sigma_N$ and so $g^+h^+ = \sigma_N^+ f^{++}$. We have the following diagram with each square commutative

$$\begin{array}{ccc} M & \xrightarrow{f} & N^+ \\ {\scriptstyle \sigma_M}\downarrow & & \downarrow{\scriptstyle \sigma_{N^+}} \\ M^{++} & \xrightarrow{f^{++}} & N^{+++} \\ {\scriptstyle h^+}\downarrow & & \downarrow{\scriptstyle \sigma_N^+} \\ E^+ & \xrightarrow[g^+]{} & N^+ \end{array}$$

Again, $\sigma_N^+\sigma_{N^+} = 1_{N^+}$ so $f = g^+h^+\sigma_M$. Therefore, $N^+ \in \mathfrak{F}^{-1}(M)$ by Proposition 2.1.2.

For the converse, we note first that given a module N, if N^+ is flat then N is absolutely pure. This is an easy consequence from the fact that every character module is pure injective (see Proposition 1.6.5). Thus, to prove that R is right coherent it suffices to prove, by Proposition 1.6.3, that for any module N, if N is absolutely pure then N^+ is flat.

Let now N be an absolutely pure right module. Then, $N \in \underline{\mathfrak{In}}^{-1}(K)$ for any pure-injective module K. In particular, $N \in \underline{\mathfrak{In}}^{-1}(M^+)$ for any module M as every character module is pure-injective. By assumption, $N^+ \in \mathfrak{F}^{-1}(M)$ for any module M and so by Proposition 2.2.1, N^+ is flat. □

Remark 2.3.2. *One can see that the first assertion in Proposition 2.3.1 enables us to find again the well-known result: if a module M is flat then M^+ is injective. Indeed, if M is flat, then for any right R-module N, $N^+ \in \mathfrak{F}^{-1}(M)$. Thus, by Proposition 2.3.1, $N \in \underline{\mathfrak{In}}^{-1}(M^+)$ for any right R-module N and so M^+ is injective.*

One can also find again that for any module N, if N^+ is flat then N is absolutely pure. Indeed, since N^+ is flat, we have that $N^+ \in \mathfrak{F}^{-1}(M)$ for any right R-module M and so, by Proposition 2.3.1, $N \in \underline{\mathfrak{In}}^{-1}(M^+)$ for any right R-module M. Then we can easily show that N is absolutely pure. Consider $0 \to N \to E(N)$. Since

$N \in \underline{\mathfrak{In}}^{-1}(M^+)$, $\mathrm{Hom}(E(N), M^+) \to \mathrm{Hom}(N, M^+)$ *is epic. But there is a natural isomorphism* $\mathrm{Hom}(N, M^+) \cong (M \otimes N)^+$*, then* $M \otimes N \to M \otimes E(N)$ *is monic for any module* M *and so* N *is absolutely pure.*

One of the classical characterizations of QF rings is that they are those rings for which any flat module is injective. Now we characterize QF rings in terms of flat-precover completing domains and subinjectivity domains.

Proposition 2.3.3. *The following assertions are equivalent:*

1. *The ring* R *is QF.*
2. *For any two modules* M *and* N*,* $N \in \mathfrak{F}^{-1}(M)$ *if and only if* $M \in \underline{\mathfrak{In}}^{-1}(N)$*.*

Proof. $(1) \Rightarrow (2)$ Consider $0 \to M \xrightarrow{i} E(M)$ and let $F \xrightarrow{g} N$ be a flat precover of N. Suppose that $N \in \mathfrak{F}^{-1}(M)$. To prove that $M \in \underline{\mathfrak{In}}^{-1}(N)$, let $f : M \to N$ be any morphism. Since $N \in \mathfrak{F}^{-1}(M)$, there exists $h : M \to F$ such that $f = gh$. But R is QF, so F is injective. Thus, there exists $l : E(M) \to F$ such that $li = h$ so $f = gh = gli$ and by Proposition 1.4.2, $M \in \underline{\mathfrak{In}}^{-1}(N)$.
Now suppose that $M \in \underline{\mathfrak{In}}^{-1}(N)$ and let us prove that $N \in \mathfrak{F}^{-1}(M)$. For let $f : M \to N$ be any morphism. Since $M \in \underline{\mathfrak{In}}^{-1}(N)$, there exists $h : E(M) \to N$ such that $hi = f$. But since R is QF, $E(M)$ is flat and so by Proposition 2.1.2 $N \in \mathfrak{F}^{-1}(M)$.

$(2) \Rightarrow (1)$ Let F be a flat module. Then, $F \in \mathfrak{F}^{-1}(M)$, for every module M. By assumption we obtain that for every module M, $M \in \underline{\mathfrak{In}}^{-1}(F)$ and so F is injective. We conclude that R is QF. □

In [27], López-Permouth and Simental call a ring R super QF-ring if the relative projectivity and relative injectivity domains coincide. In their paper, they prove that a ring R is super QF-ring if and only if R is isomorphic to a direct product of full matrix rings over Artinian chain rings. Later, in [15, Section 4] Y. Durğun proved that the subprojectivity and subinjectivity domains coincide if and only if R is super QF-ring.

In this section, we investigate the coincidence of flat-precover completing domains and subinjectivity domains. We will say that the flat-precover completing domains and subinjectivity domains coincide over R if $\mathfrak{F}^{-1}(M) = \underline{\mathfrak{In}}^{-1}(M)$ for any module M.

By following the same reasoning adopted in [15] and [27], we show that a ring satisfies $\mathfrak{F}^{-1}(M) = \underline{\mathfrak{In}}^{-1}(M)$ for any module M if and only if the ring R is isomorphic to a direct product of full matrix rings over Artinian chain rings.

Proposition 2.3.4. *Let* R *be a ring over which* $\mathfrak{F}^{-1}(M) \subseteq \underline{\mathfrak{In}}^{-1}(M)$ *for any module* M *and let* I *be an ideal of* R*. Then* $\mathfrak{F}^{-1}(M') \subseteq \underline{\mathfrak{In}}^{-1}(M')$ *for any* R/I*-module* M'*.*

Proof. We may identify R/I-Mod with the full subcategory of R-Mod consisting of modules annihilated by I. It is clear from the definition of subinjectivity domains that $\underline{\mathfrak{In}}^{-1}({}_RM) \cap R/I\text{-Mod} = \underline{\mathfrak{In}}^{-1}({}_{R/I}M)$. We also have $\mathfrak{F}^{-1}({}_{R/I}M) \subseteq \mathfrak{F}^{-1}({}_RM) \cap R/I\text{-Mod}$. But $\mathfrak{F}^{-1}({}_RM) \subseteq \underline{\mathfrak{In}}^{-1}({}_RM)$. Thus, we conclude that $\mathfrak{F}^{-1}({}_{R/I}M) \subseteq \underline{\mathfrak{In}}^{-1}({}_{R/I}M)$. □

Proposition 2.3.5. *Let R_1 and R_2 be rings over which the flat-precover completing domains and subinjectivity domains coincide. Then, the flat-precover completing domains and subinjectivity domains coincide over $R_1 \times R_2$.*

Proof. We have for any $R_1 \times R_2$-module M, $M = M_1 \oplus M_2$ with $M_1 \in R_1$-Mod and $M_2 \in R_2$-Mod. Thus, $\mathfrak{F}^{-1}(M) = \mathfrak{F}^{-1}(M_1) \times \mathfrak{F}^{-1}(M_2) = \underline{\mathfrak{In}}^{-1}(M_1) \times \underline{\mathfrak{In}}^{-1}(M_2) = \underline{\mathfrak{In}}^{-1}(M)$. We conclude that $R_1 \times R_2$ satisfies $\mathfrak{F}^{-1}(M) = \underline{\mathfrak{In}}^{-1}(M)$ for any module M. □

Proposition 2.3.6. *Let R be a ring over which the flat-precover completing domains and subinjectivity domains coincide and let S be Morita equivalent to R. Then the flat-precover completing domains and subinjectivity domains coincide over S.*

Proof. Since S is Morita equivalent to R, there exists an equivalence of categories $\phi : R\text{-Mod} \to S\text{-Mod}$. It is easy to see that $A \in \mathfrak{F}^{-1}(M)$ if and only if $\phi(A) \in \mathfrak{F}^{-1}(\phi(M))$. Similarly, $A \in \underline{\mathfrak{In}}^{-1}(M)$ if and only if $\phi(A) \in \underline{\mathfrak{In}}^{-1}(\phi(M))$. Thus, the result follows easily. □

Theorem 2.3.7. *The following conditions are equivalent:*

1. $\mathfrak{F}^{-1}(M) = \underline{\mathfrak{In}}^{-1}(M)$ *for any module M,*
2. $\mathfrak{F}^{-1}(M) \subseteq \underline{\mathfrak{In}}^{-1}(M)$ *for any module M,*
3. $\mathfrak{F}^{-1}(M) \subseteq \underline{\mathfrak{In}}^{-1}(M)$ *for any module over a factor ring of R,*
4. *Every factor ring of R is QF,*
5. *The ring R is isomorphic to a direct product of full matrix rings over Artinian chain rings.*

Proof. (1) ⇒ (2) Clear.

(2) ⇒ (3) This follows from Proposition 2.3.4.

(3) ⇒ (4) This follows from the fact that every flat module over a factor ring of R is injective. Indeed, let F be a flat R/I-module. Then $F \in \mathfrak{F}^{-1}(F) \subseteq \underline{\mathfrak{In}}^{-1}(F)$ and so F is injective.

(4) ⇒ (5) Follows from Proposition 1.4.4.

(5) ⇒ (1) We suppose that $R \cong \prod_{i=1}^{k_i} M_{n_i}(D_i)$ where the D_i's are Artinian chain rings. By Proposition 1.4.3, the subprojectivity and subinjectivity domains coincide

over each D_i. But Artinian rings are perfect so by Proposition 2.2.5 the subprojectivity and flat-precover completing domains coincide over each D_i. Thus, the flat-precover completing domains and subinjectivity domains coincide over each D_i. Then, by Proposition 2.3.6, the flat-precover completing domains and subinjectivity domains coincide over each $M_{n_i}(D_i)$ as it is Morita equivalent to D_i. We conclude by Proposition 2.3.5 that the subinjectivity and the flat-precover completing domains coincide over R. □

Remark 2.3.8. *It is worth noting that unlike the implication* $2 \Rightarrow 1$ *in Theorem 2.3.7, the implication* $\underline{\mathfrak{In}}^{-1}(M) \subseteq \mathfrak{F}^{-1}(M) \Rightarrow \mathfrak{F}^{-1}(M) = \underline{\mathfrak{In}}^{-1}(M)$ *for all modules* M *does not hold in general. For instance, if we consider a von Neumann regular ring which is not semisimple then the inclusion* $\underline{\mathfrak{In}}^{-1}(M) \subseteq \mathfrak{F}^{-1}(M)$ *holds for every module* M*. However, there is a flat module* F *which is not injective. It could be interesting to study rings which satisfy* $\underline{\mathfrak{In}}^{-1}(M) \subseteq \mathfrak{F}^{-1}(M)$ *for any module* M*. We do not have a complete characterization of rings over which this property holds but we can observe that when this happens,* $\mathcal{I} \subseteq \mathfrak{F}^{-1}(M)$ *for any module* M *and so* R *is an* IF*-ring.*

We have the following result.

Proposition 2.3.9. *The following conditions are equivalent for a module* M*:*

1. $\mathcal{I} \subseteq \mathfrak{F}^{-1}(M)$.
2. $E(M) \in \mathfrak{F}^{-1}(M)$.
3. *For any flat right* R*-module* F*,* $F^+ \in \mathfrak{F}^{-1}(M)$.
4. M *embeds in a flat module.*
5. *For any injective module* E *and any submodule* N *of* E*, if* $N \in M^{\perp}$ *then* $E/N \in \mathfrak{F}^{-1}(M)$.

Proof. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) This follow from Proposition 2.1.13.
(5) $\Rightarrow$ (1) Take $N = 0$.
(1) $\Rightarrow$ (5) Let E be an injective module. Consider the short exact sequence $0 \to N \to E \to E/N \to 0$ with $N \in M^{\perp}$. Applying the functor $\operatorname{Hom}(M, -)$, we obtain the exact sequence

$$\operatorname{Hom}(M, E) \to \operatorname{Hom}(M, E/N) \to \operatorname{Ext}^1(M, N)$$

Since $\operatorname{Ext}^1(M, N) = 0$, $\operatorname{Hom}(M, E) \to \operatorname{Hom}(M, E/N)$ is epic. Since $E \in \mathfrak{F}^{-1}(M)$, every map from M to E factors through a flat module and by Proposition 2.1.2, $E/N \in \mathfrak{F}^{-1}(M)$. □

Now, applying Proposition 2.3.9 to the class of finitely presented modules and to the class of pure-projective modules, we find characterizations of IF-rings in terms of flat-precover completing domains, some of which figure in [12, Theorem 1].

We recall first that modules for which the subinjectivity domain is as small as possible are called indigent modules (see [5, Definition 3.1]). Here, modules for which the flat-precover completing domain is as small as possible will be called f-rugged modules to distinguish them from rugged modules defined in [9]. The flat-precover completing domain of such modules will consist of only flat modules.

Notice that f-rugged modules exist for any ring R. Indeed, consider $X = \oplus_{M\in S} M$, where S is representative set of finitely presented modules. Then, by Proposition 1.3.8 and Corollary 2.2.6, we have $\mathfrak{F}^{-1}(X) = \mathrm{Flat}$.

Corollary 2.3.10. *The following assertions are equivalent:*

1. *R is an IF-ring.*
2. *Any module embeds in a flat module.*
3. *Any pure-projective module embeds in a flat module.*
4. *Any finitely presented module embeds in a flat module.*
5. *For any flat module F, F^+ is flat.*
6. *For any injective module E and any submodule N of E, if N is injective then E/N is flat.*
7. *For any injective module E and any submodule N of E, if N is absolutely pure then E/N is flat.*
8. *There exists an f-rugged module that embeds in a flat module.*

Proof. $1 \Leftrightarrow 2 \Leftrightarrow 3 \Leftrightarrow 4 \Leftrightarrow 5 \Leftrightarrow 6 \Leftrightarrow 7$ This follow from Proposition 2.3.9 by considering each time every module $M \in R$-Mod, every pure-projective module and every finitely presented module. Recall that a module N is flat if and only if $N \in \mathfrak{F}^{-1}(M)$ for every $M \in R\text{-}\mathrm{Mod}$ if and only $N \in \mathfrak{F}^{-1}(M)$ for every pure-projective module M if and only $N \in \mathfrak{F}^{-1}(M)$ for every finitely presented module M (Proposition 2.2.1).
$1 \Rightarrow 8$ Clear.
$8 \Rightarrow 1$ Suppose that there exists an f-rugged module M which embeds in a flat module. Hence, $\mathfrak{F}^{-1}(M) = \mathrm{Flat}$ and by Proposition 2.3.9, $\mathcal{I} \subseteq \mathfrak{F}^{-1}(M)$. Whence, any injective module is flat. Consequently, R is an IF-ring. □

The following proposition relates the existence of f-rugged modules to that of indigent modules. Notice that f-rugged modules are not necessarily finitely generated.

Proposition 2.3.11. *Let M be a module. If R is right Noetherian and M is f-rugged, then M^+ is indigent. Conversely, if there exists a finitely presented module M such that M^+ is indigent, then M is f-rugged and R is right Noetherian.*

Proof. Let $N \in \underline{\mathfrak{In}}^{-1}(M^+)$. By Proposition 2.3.1, $N^+ \in \mathfrak{F}^{-1}(M)$. But M is f-rugged so N^+ is flat and so N is injective. We conclude that M^+ is indigent.
Let now M be a finitely presented module such that M^+ is indigent and let us prove that R is right Noetherian.
We have M^+ is pure-injective and it is easy to see that the class of absolutely pure modules is inside the subinjectivity domain of any pure-injective module. Thus, every absolutely pure right module is injective and by Proposition 1.6.6, we deduce that R is right Noetherian.
It is left to prove that M is f-rugged. For let $N \in \mathfrak{F}^{-1}(M)$. Since M is finitely presented, By Proposition 2.2.11, we have $N^{++} \in \mathfrak{F}^{-1}(M)$ and applying Proposition 2.3.1 we deduce that $N^+ \in \underline{\mathfrak{In}}^{-1}(M^+)$. Since M^+ is indigent, N^+ is injective and thus N is flat. We conclude that M is f-rugged. □

Precover completing domains and approximations

CHAPTER 3

PRECOVER COMPLETING DOMAINS AND APPROXIMATIONS

In this chapter, we continue the investigation of $\mathcal{X}$-precover completing domain $\mathcal{X}^{-1}(\mathcal{L})$, with $\mathcal{L}$ being a class of modules and not necessarily a single module. Namely, we answer the following question: "When does every module in $\mathcal{L}$ has an $\mathcal{X}^{-1}(\mathcal{L})$-preenvelope?". Epic and monic $\mathcal{X}^{-1}(\mathcal{L})$-preenvelopes are also investigated. This study plays a key role in setting a general framework for several classical results. Then, for a class of finitely generated modules $\mathcal{M}$, we introduce the notion of $\mathcal{M}$-R-Mittag-Leffler modules as a natural extension of R-Mittag-Leffler modules. This enables us to find easier proofs of some known results and also establish new ones.

3.1 Preenveloping classes

There has been a great interest in the study of preenveloping classes. Many authors also were interested in studying epic and monic preenvelopes. This leads us to study when $\mathcal{X}$-precover completing domains are preenveloping. Epic and monic preenvelopes are also investigated.

Inspired by the paper of J. Rada and M. Saorín [32], we first focus on the question when the $\mathcal{X}$-precover completing domains are preenveloping classes. Here, we are interested in the question of when every module of a class has a preenvelope rather than when every module of the whole category of modules has a preenvelope. We give some interesting applications.

We need a relative notion of the locally initially small classes, previously defined in Definition 1.5.3, to describe when the $\mathcal{X}$-precover completing domains are preenveloping.

Definition 3.1.1. *A class of modules $\mathcal{F}$ is said to be locally initially small relative to $\mathcal{L}$ if, for every $M \in \mathcal{L}$, there exists a set $\mathcal{F}_M \subseteq \mathcal{F}$ such that every morphism $M \to F$, where $F \in \mathcal{F}$, factors through a direct product of modules in $\mathcal{F}_M$.*

A class of modules $\mathcal{F}$ being locally initially small relative to R-Mod simply means that $\mathcal{F}$ is locally initially small. And if $\mathcal{F}$ is locally initially small, then $\mathcal{F}$ is locally initially small relative to any class of modules $\mathcal{L}$. Moreover, any class of modules $\mathcal{F}$ is locally initially small relative to itself.

Proposition 3.1.2. *Let $\mathcal{L}$ be a class of modules. Then, $\mathcal{X}$ is locally initially small relative to $\mathcal{L}$ if and only if $\mathcal{X}^{-1}(\mathcal{L})$ is locally initially small relative to $\mathcal{L}$.*

Proof. Suppose that $\mathcal{X}$ is locally initially small relative to $\mathcal{L}$ and let $M \in \mathcal{L}$. Then, by definition, there exists a set $\mathcal{X}_M \subseteq \mathcal{X}$ such that every morphism $M \to X$, where $X \in \mathcal{X}$, factors through a direct product of modules in $\mathcal{X}_M$. Every morphism $f : M \to N$ with $N \in \mathcal{X}^{-1}(\mathcal{L})$ factors through a module $X \in \mathcal{X}$ by Proposition 2.1.2. And such a factorization $M \to X$ factors through a product of modules in the set $\mathcal{X}_M$ because $\mathcal{X}$ is supposed to be locally initially small relative to $\mathcal{L}$. So we have just seen that every morphism $f : M \to N$ with $N \in \mathcal{X}^{-1}(\mathcal{L})$ factors through a product of elements in $\mathcal{X}_M$ with $\mathcal{X}_M \subseteq \mathcal{X}^{-1}(\mathcal{L})$. Therefore, $\mathcal{X}^{-1}(\mathcal{L})$ is locally initially small relative to $\mathcal{L}$.

For the converse, suppose that $\mathcal{X}^{-1}(\mathcal{L})$ is locally initially small relative to $\mathcal{L}$ and let $M \in \mathcal{L}$. Then, there exists a set $\mathcal{N}_M \subseteq \mathcal{X}^{-1}(\mathcal{L})$ such that every morphism $M \to X$, where $X \in \mathcal{X}$, factors through a direct product of modules in $\mathcal{N}_M$. We construct a set $\mathcal{X}_M$ in the following way: for each $N \in \mathcal{N}_M$ we choose an $\mathcal{X}$-precover $X \to N$ of N and include X in $\mathcal{X}_M$. Then, $\mathcal{X}_M$ is clearly a set with $\mathcal{X}_M \subseteq \mathcal{X}$.

Let $f : M \to X$ be any morphism with $X \in \mathcal{X}$. Since $X \in \mathcal{X}^{-1}(\mathcal{L})$, there exist $g : \prod_{i\in I} N_i \to X$ and $h : M \to \prod_{i\in I} N_i$ with $N_i \in \mathcal{N}_M$ for every $i \in I$ such that $f = gh$. Let $g_i : X_i \to N_i$ be an $\mathcal{X}$-precover with $X_i \in \mathcal{X}_M$ for every $i \in I$. We denote by $\pi_j : \prod_{i\in I} X_i \to X_j$ and $p_j : \prod_{i\in I} N_i \to N_j$ the natural projections. By the universal property of the direct product, there exists a homomorphism $g' :$

$\prod_{i\in I} X_i \to \prod_{i\in I} N_i$ such that $g_i\pi_i = p_i g'$ for every $i \in I$.

$$\begin{array}{ccccc} & & & & M \\ & & & \swarrow^{h} & \downarrow f \\ \prod_{i\in I} X_i & \overset{g'}{\dashrightarrow} & \prod_{i\in I} N_i & \xrightarrow[g]{} & X \\ \pi_j \downarrow & & \downarrow p_j & & \\ X_j & \xrightarrow[g_j]{} & N_j & & \end{array}$$

Since $N_i \in \mathcal{X}^{-1}(\mathcal{L})$, there exists $h_i : M \to X_i$ such that $p_i h = g_i h_i$. By the universal property of the direct product, there exists $h' : M \to \prod_{i\in I} X_i$ such that for every $i \in I$, $\pi_i h' = h_i$. Then, for every $i \in I$ we have that $p_i g' h' = g_i \pi_i h' = g_i h_i = p_i h$. Thus, $g'h' = h$ and so $f = gh = gg'h'$. So, $f : M \to X$ factors through a product of element in the set $\mathcal{X}_M$. Therefore, $\mathcal{X}$ is locally initially small relative to $\mathcal{L}$. □

Now we are in a position to provide in the following result necessary and sufficient conditions for every module in a class of modules $\mathcal{L}$ to have an $\mathcal{X}^{-1}(\mathcal{L})$-preenvelope.

Theorem 3.1.3. *Let $\mathcal{L}$ be a class of modules. The following conditions are equivalent:*

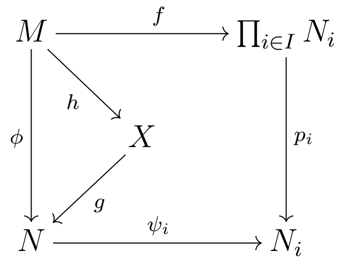
1. *Every module in $\mathcal{L}$ has an $\mathcal{X}$-preenvelope.*
2. *Every module in $\mathcal{L}$ has an $\mathcal{X}^{-1}(\mathcal{L})$-preenvelope.*
3. *The class $\mathcal{X}^{-1}(\mathcal{L})$ is locally initially small relative to $\mathcal{L}$ and closed under direct products.*
4. *The class $\mathcal{X}$ is locally initially small relative to $\mathcal{L}$ and $\mathcal{X}^{-1}(\mathcal{L})$ is closed under direct products.*

Proof. (1) ⇒ (2) Let $M \in \mathcal{L}$ and let $\phi : M \to X'$ be an $\mathcal{X}$-preenvelope. Let $N \in \mathcal{X}^{-1}(\mathcal{L})$ and let $f : M \to N$ be any morphism. Then, by Proposition 2.1.2, f factors though a module of $\mathcal{X}$, so there exist two morphisms $g : X \to N$ and $h : M \to X$ with $X \in \mathcal{X}$ such that $gh = f$. Since ϕ is an $\mathcal{X}$-preenvelope, there exists $\psi : X' \to X$ such that $\psi\phi = h$. Hence $f = gh = g\psi\phi$ and therefore ϕ is also an $\mathcal{X}^{-1}(\mathcal{L})$-preenvelope of M.

(2) ⇒ (1) Let $M \in \mathcal{L}$ and let $f : M \to N$ be an $\mathcal{X}^{-1}(\mathcal{L})$-preenvelope. We have that f factors though a module in $\mathcal{X}$, so there exist two morphisms $g : X \to N$ and $h : M \to X$ with $X \in \mathcal{X}$ such that $gh = f$. We claim that $h : M \to X$ is an $\mathcal{X}$-preenvelope.

Let $\phi : M \to X'$ be a morphism with $X' \in \mathcal{X}$. Since $X' \in \mathcal{X}^{-1}(\mathcal{L})$ and $f : M \to N$ is an $\mathcal{X}^{-1}(\mathcal{L})$-preenvelope, there exists $\psi : N \to X'$ such that $\psi f = \phi$. Hence $\phi = \psi f = \psi gh$. Therefore, $h : M \to X$ is an $\mathcal{X}$-preenvelope.

(2) $\Rightarrow$ (3) Let $\{N_i\}_{i\in I}$ be a family of modules in $\mathcal{X}^{-1}(\mathcal{L})$ and let $f : M \to \prod_{i\in I} N_i$ be a morphism with $M \in \mathcal{L}$. For each $i \in I$ we denote by $p_i : \prod_{i\in I} N_i \to N_i$ the canonical projection. Let $\phi : M \to N$ be an $\mathcal{X}^{-1}(\mathcal{L})$-preenvelope. Then, ϕ factors through a module in $\mathcal{X}$ by Proposition 2.1.2. So there exist two morphisms $h : M \to X$ and $g : X \to N$ with $X \in \mathcal{X}$ such that $\phi = gh$. Since ϕ is an $\mathcal{X}^{-1}(\mathcal{L})$-prenvelope, for each $i \in I$ there exists a morphism $\psi_i : N \to N_i$ such that $p_i f = \psi_i \phi$.



Therefore, for each $i \in I$, we have a morphism $\psi_i g : X \to N_i$. By the universal property of the direct product, there exists a unique morphism $l : X \to \prod_{i\in I} N_i$ such that $\psi_i g = p_i l$ for every $i \in I$. Hence, $p_i lh = \psi_i gh = \psi_i \phi = p_i f$ for each $i \in I$. Therefore, $lh = f$. Thus, $\prod_{i\in I} N_i \in \mathcal{X}^{-1}(\mathcal{L})$ by Proposition 2.1.2.
It is left to prove that $\mathcal{X}^{-1}(\mathcal{L})$ is locally initially small relative to $\mathcal{L}$. For that, let $M \in \mathcal{L}$ and let $\phi : M \to N_M$ be an $\mathcal{X}^{-1}(\mathcal{L})$-prenvelope. Then, any morphism $M \to N$, where $N \in \mathcal{X}^{-1}(\mathcal{L})$, factors through N_M. Thus, $\mathcal{X}^{-1}(\mathcal{L})$ is locally initially small relative to $\mathcal{L}$.

(3) $\Rightarrow$ (4) We know from Proposition 3.1.2 that $\mathcal{X}$ is locally initially small relative to $\mathcal{L}$ if and only if $\mathcal{X}^{-1}(\mathcal{L})$ is locally initially small relative to $\mathcal{L}$. So the implication follows easily.

(4) $\Rightarrow$ (2) Let $M \in \mathcal{L}$. Since $\mathcal{X}$ is locally initially small relative to $\mathcal{L}$, one can see that there exists a set $\mathcal{X}_M \subseteq \mathcal{X}$ such that any morphism $M \to N$, with $N \in \mathcal{X}^{-1}(\mathcal{L})$, factors through a product of modules in $\mathcal{X}_M$. We let $F = \prod_{X\in\mathcal{X}_M} X^{\mathrm{Hom}(M,X)}$. Then $F \in \mathcal{X}^{-1}(\mathcal{L})$ because $\mathcal{X}^{-1}(\mathcal{L})$ is supposed to be closed under direct products. Now, for each $X \in \mathcal{X}_M$ there exists a canonical morphism $\lambda_X : M \to X^{\mathrm{Hom}(M,X)}$. Then, there exists a morphism $\lambda : M \to F$ such that $\pi_X \lambda = \lambda_X$ for every $X \in \mathcal{X}_M$, where π_X are the canonical projections. We claim that $\lambda : M \to F$ is an $\mathcal{X}^{-1}(\mathcal{L})$-preenvelope of M. Indeed, taking any morphism $f : M \to N$ with $N \in \mathcal{X}^{-1}(\mathcal{L})$, there exist $h : M \to \prod_{X'\in\mathcal{X}_M} X'$ and $g : \prod_{X'\in\mathcal{X}_M} X' \to N$ such that $f = gh$. Consider the projections $\pi_{X'} : F \to X'^{\mathrm{Hom}(M,X')}$ and $\pi_{p_{X'}h} : X'^{\mathrm{Hom}(M,X')} \to X'$ (the projection to the component $p_{X'}h$ where $p_{X'} : \prod_{X'\in\mathcal{X}_M} X' \to X'$ the canonical projection). Then, there exists a unique morphism $\eta : F \to \prod_{X'\in\mathcal{X}_M} X'$ such that

$\pi_{p_{X'}h}\pi_{X'} = p_{X'}\eta$.

$$\begin{array}{ccccc}
X'^{\mathrm{Hom}(M,X')} & \xleftarrow{\ \pi_{X'}\ } & F & \xleftarrow{\ \lambda\ } & M \\
\downarrow{\scriptstyle \pi_{p_{X'}h}} & & {\scriptstyle \eta}\downarrow & \swarrow{\scriptstyle h} & \downarrow{\scriptstyle f} \\
X' & \xleftarrow[p_{X'}]{} & \prod_{X' \in \mathcal{X}_M} X' & \xrightarrow[g]{} & N
\end{array}$$

Therefore, $p_{X'}\eta\lambda = \pi_{p_{X'}h}\pi_{X'}\lambda = \pi_{p_{X'}h}\lambda_{X'} = p_{X'}h$ for all $X' \in \mathcal{X}$, so $\eta\lambda = h$ and hence $g\eta\lambda = gh = f$. We deduce that $\lambda : M \to F$ is an $\mathcal{X}^{-1}(\mathcal{L})$-preenvelope of M. □

In practice, the precover completing domain of a class $\mathcal{L}$ is not easy to compute. For this, it would be nice to have a characterization of the existence of $\mathcal{X}$-preenvelopes for every module in $\mathcal{L}$ with conditions depending just on $\mathcal{X}$ and $\mathcal{L}$, that is, without using $\mathcal{X}^{-1}(\mathcal{L})$. Theorem 3.1.3 tells us that $\mathcal{X}^{-1}(\mathcal{L})$ has to be closed under direct products. By Proposition 2.1.7, we know that $\mathcal{X}^{-1}(R\text{-Mod}) = \mathcal{X}$ is closed under direct products if and only if $\mathcal{X}^{-1}(\mathcal{L})$ is closed under direct products for every class of modules $\mathcal{L}$. But having one $\mathcal{X}^{-1}(\mathcal{L})$ closed under direct products is not enough to ensure the closeness of $\mathcal{X}$ under direct products. Using Theorem 3.1.3, we can get the following sufficient condition.

Corollary 3.1.4. *Let $\mathcal{L}$ be a class of modules. If $\mathcal{X}$ is closed under direct products and is locally initially small relative to $\mathcal{L}$, then every module M in $\mathcal{L}$ has an $\mathcal{X}$-preenvelope.*

Remark 3.1.5. *Notice that the converse of the previous result is not true. Indeed, if we take $\mathcal{X} = \mathcal{L}$, then clearly every module in $\mathcal{L}$ has an $\mathcal{X}$-preenvelope and $\mathcal{X}$ is locally initially small relative to $\mathcal{L}$ but $\mathcal{X}$ can be not closed under direct products.*

We now give an application. Letting $\mathcal{X}$ be the class of projective modules $\mathcal{P}$, we see by Proposition 1.3.2 that $\mathcal{P}$-precover completing domains are just subprojectivity domains (see Definition 1.3.1).

For an arbitrary class $\mathcal{S}$ of finitely generated modules, the notion of $\mathcal{S}$-projective modules was introduced in [31] and is defined as the class of modules N such that every morphism $f : S \to N$, where $S \in \mathcal{S}$, factors through a free module. Using Proposition 1.3.6, we see that the class $\mathcal{S}$-Proj of $\mathcal{S}$-projective modules is the subprojectivity domain of the class of modules $\mathcal{S}$. When taking $\mathcal{S}$ to be the class of cyclic modules, $\mathcal{S}$-projective modules are called cyclic-projective modules in [31].

Corollary 3.1.6 ([31], Corollary 3.2)**.** *The following conditions are equivalent:*

1. *Every cyclic module has a projective preenvelope.*

2. *Every cyclic module has a cyclic-projective preenvelope.*

3. *The class of cyclic-projective modules is closed under direct products.*

Proof. We set $\mathcal{X}$ to be the class of projective modules and $\mathcal{L}$ that of cyclic modules. From the discussion above, the subprojectivity domain of the class of cyclic modules is the class of cyclic-projective modules. And by Proposition 1.5.5, the class of projective modules is locally initially small. The result then follows easily from Theorem 3.1.3. □

In [31], it is proved that a ring R verifies the conditions of Corollary 3.1.6 if and only if the left annihilator of every right ideal of R is finitely generated.

Next, in the following result, we investigate when every module of a class $\mathcal{L}$ admits an epic $\mathcal{X}^{-1}(\mathcal{L})$-preenvelope.

Theorem 3.1.7. *Let $\mathcal{L}$ be a class of modules. The following conditions are equivalent:*

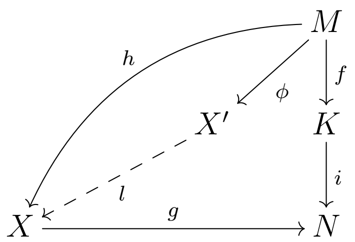
1. *Every module in $\mathcal{L}$ has an epic $\mathcal{X}$-preenvelope.*
2. *Every module in $\mathcal{L}$ has an epic $\mathcal{X}^{-1}(\mathcal{L})$-preenvelope.*

If, in addition, $\mathcal{L}$ is closed under quotients, then the conditions above are equivalent to

3. *$\mathcal{X}^{-1}(\mathcal{L})$ is closed under direct products and submodules.*
4. *$\mathcal{X}$ is locally initially small relative to $\mathcal{L}$, $\mathcal{X}^{-1}(\mathcal{L})$ is closed under direct products and for any submodule L of a module $N \in \mathcal{X}^{-1}(\mathcal{L})$ with $L \in \mathcal{L}$, we have that $L \in \mathcal{X}^{-1}(\mathcal{L})$.*

Proof. (1) $\Rightarrow$ (2) Let $M \in \mathcal{L}$ and let $\phi : M \to X'$ be an epic $\mathcal{X}$-preenvelope. Let $N \in \mathcal{X}^{-1}(\mathcal{L})$ and let $f : M \to N$ be any morphism. Then, by Proposition 2.1.2, f factors though a module in $\mathcal{X}$, so there exist two morphisms $g : X \to N$ and $h : M \to X$ with $X \in \mathcal{X}$ such that $gh = f$. Since ϕ is an $\mathcal{X}$-preenvelope, there exists $\psi : X' \to X$ such that $\psi\phi = h$. Hence $f = gh = g\psi\phi$ and therefore ϕ is also an epic $\mathcal{X}^{-1}(\mathcal{L})$-preenvelope of M.
(2) $\Rightarrow$ (1) Let $M \in \mathcal{L}$. If $f : M \to N$ is an epic $\mathcal{X}^{-1}(\mathcal{L})$-preenvelope, then $N \in \mathcal{X}$. Indeed, we have by Proposition 2.1.2 that f factors though a module in $\mathcal{X}$. So, there exist two morphisms $g : X \to N$ and $h : M \to X$ with $X \in \mathcal{X}$ such that $gh = f$. Since f is an $\mathcal{X}^{-1}(\mathcal{L})$-preenvelope, there exists a morphism $k : N \to X$ such that $kf = h$. Therefore, $gkf = gh = f$. But f is epic, so $gk = 1_N$ and thus $N \in \mathcal{X}$. Since $\mathcal{X} \subseteq \mathcal{X}^{-1}(\mathcal{L})$, we deduce that $f : M \to N$ is an epic $\mathcal{X}$-preenvelope of M.
(2) $\Rightarrow$ (3) It is clear from Theorem 3.1.3 that $\mathcal{X}^{-1}(\mathcal{L})$ is closed under direct products. We prove that $\mathcal{X}^{-1}(\mathcal{L})$ is closed under submodules.

Let K be a submodule of a module N in $\mathcal{X}^{-1}(\mathcal{L})$ and $f : M \to K$ be any morphism with $M \in \mathcal{L}$. Letting $i : K \to N$ denote the injection map, if factors through a module in $\mathcal{X}$ by Proposition 2.1.2, so there exist two morphisms $h : M \to X$ and $g : X \to N$ such that $if = gh$ and $X \in \mathcal{X}$. Let $\phi : M \to X'$ denote an epic $\mathcal{X}^{-1}(\mathcal{L})$-preenvelope. Then, there exists a morphism $l : X' \to X$ such that $l\phi = h$. Consider the following commutative diagram



Then, $if = gl\phi$ and so $\mathrm{Ker}\phi \subseteq \mathrm{Ker}f$. So, we can define a morphism $\theta : X' \to K$ such that $\theta(\phi(m)) = f(m)$ for every $m \in M$. Therefore, we see that f factors through a module in $\mathcal{X}$ and so $K \in \mathcal{X}^{-1}(\mathcal{L})$ by Proposition 2.1.2.
$(3) \Rightarrow (4)$ Since $\mathcal{X}^{-1}(\mathcal{L})$ is closed under submodules, $\mathcal{X}^{-1}(\mathcal{L})$ is locally initially small by Proposition 1.5.4. So, $\mathcal{X}^{-1}(\mathcal{L})$ is locally initially small relative to $\mathcal{L}$. We use Proposition 3.1.2 to deduce that $\mathcal{X}$ is locally initially small relative to $\mathcal{L}$. The rest of the proof is clear.
$(4) \Rightarrow (2)$ Let $M \in \mathcal{L}$. By Theorem 3.1.3, every module in $\mathcal{L}$ has an $\mathcal{X}^{-1}(\mathcal{L})$-preenvelope. So, we let $f : M \to N$ be an $\mathcal{X}^{-1}(\mathcal{L})$-preenvelope. It is easy to see that the morphism $M \to \mathrm{Im}f$ is an epic $\mathcal{X}^{-1}(\mathcal{L})$-preenvelope. □

When $\mathcal{L}$ is taken to be the whole category of modules R-Mod, we find again Proposition 1.5.7 which states that a class of modules $\mathcal{X}$ is epic preenveloping if and only if $\mathcal{X}$ is closed under direct products and submodules (because $\mathcal{X}^{-1}(R\text{-}\mathrm{Mod}) = \mathcal{X}$ by Proposition 2.1.3).

Corollary 3.1.8. *The following conditions are equivalent:*

1. *Every finitely generated module has an epic projective preenvelope.*
2. *Every finitely generated module has an epic f-projective preenvelope.*
3. *The class of f-projective modules is closed under direct products and submodules.*
4. *The class of f-projective modules is closed under direct products and any finitely generated submodule of an f-projective module is f-projective.*

Proof. We take $\mathcal{X}$ to be the class of projective modules and $\mathcal{L}$ that of finitely generated modules R-mod. Then, by Proposition 1.3.12, we know that $\underline{\mathfrak{Pr}}^{-1}(R\text{-}\mathrm{mod})$

is the class of f-projective modules. And we know that the class of projective modules is locally initially small by Proposition 1.5.5. Then, the result follows from Theorem 3.1.7. □

Recall that a ring R is called π-coherent in case every finitely generated torsionless right module is finitely presented. By [13, Corollary 5.3], it is proved that the assertions in Corollary 3.1.8 are equivalent to the ring R being right π-coherent and left semihereditary.

In the next result, we investigate when every module of a class $\mathcal{L}$ admits a monic $\mathcal{X}^{-1}(\mathcal{L})$-preenvelope.

Theorem 3.1.9. *Let $\mathcal{L}$ be a class of modules. The following conditions are equivalent:*

1. *Every module in $\mathcal{L}$ has a monic $\mathcal{X}$-preenvelope.*
2. *Every module in $\mathcal{L}$ has a monic $\mathcal{X}^{-1}(\mathcal{L})$-preenvelope.*
3. *Every module in $\mathcal{L}$ has an $\mathcal{X}^{-1}(\mathcal{L})$-preenvelope and $\mathcal{I} \subseteq \mathcal{X}^{-1}(\mathcal{L})$.*
4. *The class $\mathcal{X}$ is locally initially small relative to $\mathcal{L}$, $\mathcal{X}^{-1}(\mathcal{L})$ is closed under direct products and $R^+ \in \mathcal{X}^{-1}(\mathcal{L})$.*

Proof. $(1) \Rightarrow (2)$ Let $M \in \mathcal{L}$ and let $\phi : M \to X'$ be a monic $\mathcal{X}$-preenvelope. Let $N \in \mathcal{X}^{-1}(\mathcal{L})$ and let $f : M \to N$ be any morphism. Then, by Proposition 2.1.2, f factors though a module in $\mathcal{X}$, so there exist two morphisms $g : X \to N$ and $h : M \to X$ with $X \in \mathcal{X}$ such that $gh = f$. Since ϕ is an $\mathcal{X}$-preenvelope, there exists $\psi : X' \to X$ such that $\psi\phi = h$. Hence $f = gh = g\psi\phi$ and therefore ϕ is also a monic $\mathcal{X}^{-1}(\mathcal{L})$-preenvelope of M.

$(2) \Rightarrow (3)$ Let E be an injective module and let us prove that $E \in \mathcal{X}^{-1}(\mathcal{L})$. Let $M \in \mathcal{L}$ and $M \to N$ be a monic $\mathcal{X}^{-1}(\mathcal{L})$-preenvelope. Then, $\mathrm{Hom}(N, E) \to \mathrm{Hom}(M, E) \to 0$ is exact. Thus, any morphism $M \to E$ factors through $N \in \mathcal{X}^{-1}(\mathcal{L})$ and so it also factors through a module in $\mathcal{X}$. Therefore, $E \in \mathcal{X}^{-1}(M)$ by Proposition 2.1.2.

$(3) \Rightarrow (4)$ Clear by Theorem 3.1.3.

$(4) \Rightarrow (1)$ By Theorem 3.1.3, every module M in $\mathcal{L}$ has an $\mathcal{X}^{-1}(\mathcal{L})$-preenvelope. Let $M \to N$ be an $\mathcal{X}^{-1}(\mathcal{L})$-preenvelope. Since $R^+ \in \mathcal{X}^{-1}(\mathcal{L})$, $\mathrm{Hom}(N, R^+) \to \mathrm{Hom}(M, R^+) \to 0$ is exact. Thus, $N^+ \to M^+ \to 0$ is exact by Proposition 1.6.2 and therefore, $0 \to M \to N$ is exact. □

As an application of Theorem 3.1.9, we show the following corollary. The arguments are analogue to those of Corollary 3.1.8.

Corollary 3.1.10 ([32], Corollary 5.3)**.** *The following conditions are equivalent:*

1. *Every finitely generated module has a monic projective preenvelope.*

2. *Every finitely generated module has a monic f-projective preenvelope.*

3. *The class of f-projective modules is closed under direct products and every injective module is f-projective.*

4. *The class of f-projective modules is closed under direct products and R^+ is f-projective.*

The above conditions are equivalent to R being right π-coherent and left FGF as proved in [32, Corollary 5.3].

We now investigate when every module of a class $\mathcal{L}$ has an $\mathcal{X}^{-1}(\mathcal{L})$-envelope. We saw in Proposition 1.3.8 that the subprojectivity domain of the class of finitely presented modules is the class of flat modules. By [17, Proposition 6.5.3.], if $M \to F$ is a flat envelope and M is finitely presented, then F is projective. The following generalizes this result.

Proposition 3.1.11. *If $M \to N$ is an $\mathcal{X}^{-1}(\mathcal{L})$-envelope with M a module in $\mathcal{L}$, then $N \in \mathcal{X}$.*

Proof. Let $f : M \to N$ be an $\mathcal{X}^{-1}(\mathcal{L})$-envelope. Then, f factors though a module in $\mathcal{X}$, that is, there exist two morphisms $g : X \to N$ and $h : M \to X$ with $X \in \mathcal{X}$ such that $gh = f$. Since f is an $\mathcal{X}^{-1}(\mathcal{L})$-envelope, there exists a morphism $k : N \to X$ such that $kf = h$. So $f = gkf$ and then, gk is an automorphism and thus $N \in \mathcal{X}$. □

Proposition 3.1.12. *Let $\mathcal{L}$ be a class of modules. The following conditions are equivalent:*

1. *Every module in $\mathcal{L}$ has an $\mathcal{X}$-envelope.*

2. *Every module in $\mathcal{L}$ has an $\mathcal{X}^{-1}(\mathcal{L})$-envelope.*

Proof. (1) $\Rightarrow$ (2) Let $M \in \mathcal{L}$ and let $\phi : M \to X'$ be an $\mathcal{X}$-envelope. Let $N \in \mathcal{X}^{-1}(\mathcal{L})$ and let $f : M \to N$ be any morphism. Then, by Proposition 2.1.2, f factors though a module in $\mathcal{X}$, so there exist two morphisms $g : X \to N$ and $h : M \to X$ with $X \in \mathcal{X}$ such that $gh = f$. Since ϕ is an $\mathcal{X}$-preenvelope, there exists $\psi : X' \to X$ such that $\psi\phi = h$. Hence $f = gh = g\psi\phi$ and therefore ϕ is also an $\mathcal{X}^{-1}(\mathcal{L})$-envelope of M.

(2) $\Rightarrow$ (1) By Proposition 3.1.11, if $f : M \to N$ is an $\mathcal{X}^{-1}(\mathcal{L})$-envelope, then $N \in \mathcal{X}$. Since $\mathcal{X} \subseteq \mathcal{X}^{-1}(\mathcal{L})$, we deduce that $f : M \to N$ is an $\mathcal{X}$-envelope. □

3.2 Application: $\mathcal{M}$-R-Mittag-Leffler modules

Letting $\mathcal{X}$ be the class $\mathcal{PP}$ of pure-projective modules, we get $\mathcal{PP}$-precover completing domains. Recall that in [1, Definition 1], given modules M and N, M is said to be N-pure-subprojective if for every pure epimorphism $g : B \to N$ and morphism $f : M \to N$, there exists a homomorphism $h : M \to B$ such that $gh = f$. The pure subprojectivity domain of a module M, or domain of pure subprojectivity, $\underline{\mathfrak{PP}}^{-1}(M)$, is defined to be the class

$$\underline{\mathfrak{PP}}^{-1}(M) := \{N \in R\text{-Mod} : M \text{ is } N\text{-pure-subprojective}\}.$$

By [1, Lemma 2], we see that $\mathcal{PP}$-precover completing domains are precisely pure-subprojectivity domains.

The class of R-Mittag-Leffler modules were also studied in [6] and called finitely pure-projective modules. In [33], the author defines a relative version of R-Mittag-Leffler modules, using their characterization with the tensor product (see also [24]). Here, the key is the characterization of R-Mittag-Leffler modules in terms of factorization of mappings ([21, Theorem 1]). Indeed, we study R-Mittag-Leffler type conditions relative to an arbitrary class of finitely generated modules instead of restricting ourselves to the whole class of finitely generated modules. Therefore, from now on, $\mathcal{M}$ will denote a class of finitely generated modules.

Definition 3.2.1. *A module N is said to be $\mathcal{M}$-R-Mittag-Leffler if N is in the pure-subprojectivity domain of the class $\mathcal{M}$, $\underline{\mathfrak{PP}}^{-1}(\mathcal{M})$.*

Remark 3.2.2. *If we let R-mod be the class of all finitely generated modules, then $\underline{\mathfrak{PP}}^{-1}(R\text{-mod})$ is the class of all R-Mittag-Leffler modules, denoted R-$\mathcal{ML}$. Indeed, by Proposition 3.2.3, it is clear that $\underline{\mathfrak{PP}}^{-1}(R\text{-mod}) \subseteq$ R-$\mathcal{ML}$. Conversely, let N be an R-Mittag-Leffler module. If $f : M \to N$ is any morphism with M a finitely generated module, then the inclusion map $i : \mathrm{Im}(f) \to N$ factors through a finitely presented module. Thus, f factors through a finitely presented module and so $N \in \underline{\mathfrak{PP}}^{-1}(M)$.*

Let $\mathfrak{F} = \oplus_{M \in S} M$ where S is a representative set of finitely presented modules. We have the following proposition where we give alternative characterizations for the notion of relative R-Mittag-Leffler.

Proposition 3.2.3. *Let N be any module. The following assertions are equivalent:*

1. *N is $\mathcal{M}$-R-Mittag-Leffler.*
2. *For every module $M \in \mathcal{M}$, every homomorphism $f : M \to N$ factors through a finitely presented module.*

3. *For every module $M \in \mathcal{M}$, every homomorphism $f : M \to N$ factors through $\mathfrak{F}$.*

Proof. $(1) \Rightarrow (2)$ Let $f : M \to N$ be any morphism, where $M \in \mathcal{M}$. Since $N \in \underline{\mathfrak{PP}}^{-1}(\mathcal{M})$, then by Proposition 2.1.2 there exist homomorphisms $g : M \to P$ and $h : P \to N$, where P is a pure-projective module and such that $f = hg$. But it is clear from the definition that any pure-projective module is R-Mittag-Leffler. Therefore, the morphism g factors through a finitely presented module and thus, f does too.
$(2) \Rightarrow (3)$ Clear.
$(3) \Rightarrow (1)$ Nothing to prove because $\mathfrak{F}$ is a pure-projective module. □

In the next example, we compute the class of $\mathcal{FP}$-Mittag-Leffler modules and that of $\mathcal{C}$-Mittag-Leffler modules.

Example 3.2.4. *1. The class of $\mathcal{FP}$-R-Mittag-Leffler modules is the whole class of modules R-*Mod.

2. The class of $\mathcal{C}$-R-Mittag-Leffler modules is precisely the class of singly pure-projective modules.

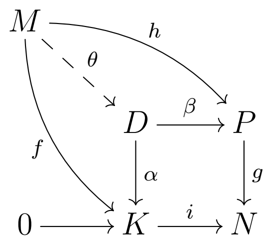
Proof. 1. By Proposition 2.1.11, $\underline{\mathfrak{PP}}^{-1}(\mathcal{FP}) = \underline{\mathfrak{PP}}^{-1}(\mathcal{PP})$ and by Proposition 2.1.3 we have that $\underline{\mathfrak{PP}}^{-1}(\mathcal{PP}) = R\text{-Mod}$.
2. By Proposition 3.2.3, it is clear that every $\mathcal{C}$-R-Mittag-Leffler modules is singly pure-projective modules. Conversely, let N be a singly pure-projective modules. If $f : M \to N$ is any morphism with M a finitely generated module, then the inclusion map $i : \mathrm{Im}(f) \to N$ factors through a finitely presented module. Thus, f factors through a finitely presented module and so $N \in \underline{\mathfrak{PP}}^{-1}(M)$. □

We will say that a class of modules $\mathcal{F}$ is closed under pure extensions if for any pure short exact sequence of modules $0 \to A \to B \to C \to 0$ such that A and C are in $\mathcal{F}$, B is also in $\mathcal{F}$.

Proposition 3.2.5. *The class of $\mathcal{M}$-R-Mittag-Leffler modules is closed under direct summands, direct sums, pure extensions, and pure submodules.*

Proof. The closeness under direct summands and pure extensions follow directly from Proposition 2.1.5. The closeness under direct sums follows from 2.1.6 because any finitely generated module is small.
We now show that the $\mathcal{M}$-R-Mittag-Leffler modules is closed under pure submodules.
Let $N \in \underline{\mathfrak{PP}}^{-1}(\mathcal{M})$ and K be a pure submodule of N. Let $M \in \mathcal{M}$ and

$f : M \to K$ be any morphism, $i : K \to N$ be the inclusion map and $g : P \to N$ be a pure-projective precover. Consider the following pullback diagram:



Since $N \in \underline{\mathfrak{PP}}^{-1}(\mathcal{M})$, there exists $h : M \to P$ such that $if = gh$.
By the universal property of pullbacks, there exists $\theta : M \to D$ such that $\beta\theta = h$. Note that β is pure. Then, by Proposition 1.6.8, there exists $k : P \to D$ such that $k\beta\theta = \theta$. Thus, $kh = \theta$ and $i\alpha kh = i\alpha\theta = if$ and since i is monic, $\alpha kh = f$ and so $K \in \underline{\mathfrak{PP}}^{-1}(\mathcal{M})$ by Proposition 2.1.2. □

By Proposition 1.5.4, any class of modules that is closed under pure submodules is locally initially small. Consequently, by Proposition 3.2.5, we have the following result.

Corollary 3.2.6. *The class of $\mathcal{M}$-R-Mittag-Leffler modules is locally initially small.*

The following proposition is obtained directly from Proposition 2.1.8

Proposition 3.2.7. *The following assertions are equivalent:*

1. *The class of $\mathcal{M}$-R-Mittag-Leffler modules is closed under submodules.*
2. *The pure-subprojectivity domain of any module in $\mathcal{M}$ is closed under submodules.*
3. *Any submodule of a pure-projective module is $\mathcal{M}$-R-Mittag-Leffler.*

And as a consequence, we have the following result which characterizes when the class of R-Mittag-Leffler modules is closed under submodules.

Corollary 3.2.8. *The following assertions are equivalent:*

1. *The class of R-Mittag-Leffler modules is closed under submodules.*
2. *The pure-subprojectivity domain of any finitely generated module is closed under submodules.*
3. *Any submodule of a pure-projective module is R-Mittag-Leffler.*

In the next result, we see when the class of $\mathcal{M}$-R-Mittag-Leffler modules is closed under quotients.

Proposition 3.2.9. *The following conditions are equivalent:*

1. *The class of $\mathcal{M}$-R-Mittag-Leffler modules is closed under quotients.*
2. *The class of $\mathcal{M}$-R-Mittag-Leffler modules is closed under direct limits.*
3. *Every module is $\mathcal{M}$-R-Mittag-Leffler.*
4. *Every module in $\mathcal{M}$ is pure-projective.*
5. *Every module in $\mathcal{M}$ is $\mathcal{M}$-R-Mittag-Leffler.*

Proof. $(1) \Leftrightarrow (3) \Leftrightarrow (4) \Leftrightarrow (5)$ Follow from Proposition 2.1.3.
$(3) \Rightarrow (2)$ Clear.
$(2) \Rightarrow (3)$ Since every module is a direct limit of finitely presented modules and finitely presented modules are $\mathcal{M}$-R-Mittag-Leffler, we deduce that any module is $\mathcal{M}$-R-Mittag-Leffler. □

Since every projective module is pure-projective, it is clear from Proposition 2.1.15 that for any class of modules $\mathcal{L}$, $\underline{\mathfrak{Pr}}^{-1}(\mathcal{L}) \subseteq \underline{\mathfrak{PP}}^{-1}(\mathcal{L})$. However, they need not be equal, as it is shown in [1, Example 1]. In the next result, we show that for a flat module F, if F belongs to one of the two domains, then F is also in the other.

Proposition 3.2.10. *Let $\mathcal{L}$ be any class of modules and F be a flat module. Then, $F \in \underline{\mathfrak{PP}}^{-1}(\mathcal{L})$ if and only if $F \in \underline{\mathfrak{Pr}}^{-1}(\mathcal{L})$. In particular,* $\mathrm{Flat} \subseteq \underline{\mathfrak{PP}}^{-1}(\mathcal{L})$ *if and only if* $\mathrm{Flat} \subseteq \underline{\mathfrak{Pr}}^{-1}(\mathcal{L})$.

Proof. The sufficient condition follows from the discussion above. Conversely now, let F be a flat module and consider an epimorphism $g : B \to F$. Since F is flat, g is a pure epimorphism and so by assumption, $\mathrm{Hom}(L, g)$ is epic for every $L \in \mathcal{L}$. Thus, $F \in \underline{\mathfrak{Pr}}^{-1}(\mathcal{L})$. □

Corollary 3.2.11. *Every flat module is $\mathcal{M}$-R-Mittag-Leffler if and only if every flat module is $\mathcal{M}$-projective (that is, it is M-subprojective for every $M \in \mathcal{M}$).*

Proposition 1.3.11 shows the relation between f-projective modules and R-Mittag-Leffler modules. In the following proposition, we investigate the counterpart result in our context.

Proposition 3.2.12. *Any $\mathcal{M}$-projective module N is $\mathcal{M}$-R-Mittag-Leffler and if $\mathcal{M}$ contains the class of finitely presented modules, then N is also flat.*

Proof. Since $\underline{\mathfrak{Pr}}^{-1}(\mathcal{M}) \subseteq \underline{\mathfrak{PP}}^{-1}(\mathcal{M})$, any $\mathcal{M}$-projective module N is $\mathcal{M}$-R-Mittag-Leffler. If now $\mathcal{M}$ contains the class of finitely presented modules, then any short exact sequence $0 \to K \to B \to N \to 0$ is $\mathrm{Hom}(M, -)$ exact for any finitely presented module M. Thus, N is flat. □

Theorem 3.1.3 allows us to characterize when every module of $\mathcal{M}$ has an $\mathcal{M}$-R-Mittag-Leffler preenvelope. We show that this is equivalent to the class of $\mathcal{M}$-R-Mittag-Leffler modules being closed under direct products. In particular, we get a characterization of the closure under direct products of the class of R-Mittag-Leffler modules (see for example [33, Theorem 5.12 and Corollary 2.7]).

Proposition 3.2.13. *The following conditions are equivalent:*

1. *Every module in $\mathcal{M}$ has a pure-projective preenvelope.*
2. *Every module in $\mathcal{M}$ has an $\mathcal{M}$-R-Mittag-Leffler preenvelope.*
3. *The class of $\mathcal{M}$-R-Mittag-Leffler modules is closed under direct products.*
4. *Every module has an $\mathcal{M}$-R-Mittag-Leffler preenvelope.*
5. *The pure-subprojectivity domain of any module in $\mathcal{M}$ is closed under direct products.*
6. *For any family of pure-projective modules $\{N_i\}_{i\in I}$, $\prod_{i\in I} N_i$ is $\mathcal{M}$-R-Mittag-Leffler.*
7. *$\mathfrak{F}^I$ is $\mathcal{M}$-R-Mittag-Leffler for every set I.*

Proof. $(1) \Leftrightarrow (2) \Rightarrow (3)$ This follow from Theorem 3.1.3.

$(3) \Rightarrow (4)$ We know by Theorem 1.5.6 that every module has an $\mathcal{M}$-R-Mittag-Leffler preenvelope if and only if $\mathrm{Summ}(\mathcal{M}$-$R$-Mittag-Leffler$)$ is closed under direct products and the class of $\mathcal{M}$-R-Mittag-Leffler modules is locally initially small. But we have $\mathrm{Summ}(\mathcal{M}$-$R$-Mittag-Leffler$) = \mathcal{M}$-$R$-Mittag-Leffler by Proposition 3.2.5. And by Corollary 3.2.6, the class of $\mathcal{M}$-R-Mittag-Leffler modules is locally initially small. Thus, every module has an $\mathcal{M}$-R-Mittag-Leffler preenvelope.

$(4) \Rightarrow (2)$ Clear.

$(3) \Leftrightarrow (5) \Leftrightarrow (6)$ This follow from Proposition 2.1.7.

$(6) \Rightarrow (7)$ Clear because $\mathfrak{F}$ is pure-projective.

$(7) \Rightarrow (6)$ Suppose that $N_i \in \underline{\mathfrak{PP}}^{-1}(\mathcal{M})$ for any $i \in I$. Then, for every $M \in \mathcal{M}$, any morphism $M \to N_i$ factors through $\mathfrak{F}$ by Proposition 3.2.3. Using similar arguments to those in 2.1.7 we get the result. □

Corollary 3.2.14. *The following conditions are equivalent:*

1. *Every finitely generated module has a pure-projective preenvelope.*
2. *Every finitely generated module has an R-Mittag-Leffler preenvelope.*
3. *The class of R-Mittag-Leffler modules is closed under direct products.*

4. *Every module has an R-Mittag-Leffler preenvelope.*

5. *The pure-subprojectivity domain of any finitely generated module is closed under direct products.*

6. *For any family of pure-projective modules $\{N_i\}_{i\in I}$, $\prod_{i\in I} N_i$ is R-Mittag-Leffler.*

7. *$\mathfrak{F}^I$ is R-Mittag-Leffler for every set I.*

Now, using Theorem 3.1.7, we characterize when every module in $\mathcal{M}$ has an epic $\mathcal{M}$-R-Mittag-Leffler preenvelope.

Proposition 3.2.15. *The following conditions are equivalent:*

1. *Every module in $\mathcal{M}$ has an epic pure-projective preenvelope.*

2. *Every module in $\mathcal{M}$ has an epic $\mathcal{M}$-R-Mittag-Leffler preenvelope.*

If, in addition, $\mathcal{M}$ is closed under quotients, then conditions above are equivalent to

3. *The class of $\mathcal{M}$-R-Mittag-Leffler modules is closed under direct products and submodules.*

4. *The class of $\mathcal{M}$-R-Mittag-Leffler modules is closed under direct products and any finitely generated submodule of an $\mathcal{M}$-R-Mittag-Leffler module is $\mathcal{M}$-R-Mittag-Leffler.*

Proof. Using Theorem 3.1.7, we only need to show that the class of pure-projective modules $\mathcal{PP}$ is locally initially small relative to $\mathcal{M}$. We know by Proposition 3.1.2 that $\mathcal{PP}$ is locally initially small relative to $\mathcal{M}$ if and only if $\underline{\mathfrak{PP}}^{-1}(\mathcal{M})$ is locally initially small relative to $\mathcal{M}$. But by Corollary 3.2.6, $\underline{\mathfrak{PP}}^{-1}(\mathcal{M})$ is locally initially small. □

Now we characterize with Theorem 3.1.9 when every module in $\mathcal{M}$ has a monic $\mathcal{M}$-R-Mittag-Leffler envelope. We use the same argument as in the proof of Proposition 3.2.15.

Proposition 3.2.16. *The following conditions are equivalent:*

1. *Every module in $\mathcal{M}$ has a monic pure-projective preenvelope.*

2. *Every module in $\mathcal{M}$ has a monic $\mathcal{M}$-R-Mittag-Leffler preenvelope.*

3. *The class of $\mathcal{M}$-R-Mittag-Leffler modules is closed under direct products and every injective module is $\mathcal{M}$-R-Mittag-Leffler.*

4. *The class of $\mathcal{M}$-R-Mittag-Leffler modules is closed under direct products and R^+ is an $\mathcal{M}$-R-Mittag-Leffler module.*

Finally, as an application of Proposition 3.1.12, we have the following result.

Proposition 3.2.17. *The following conditions are equivalent:*

1. *Every module in $\mathcal{M}$ has a pure-projective envelope.*
2. *Every module in $\mathcal{M}$ has an $\mathcal{M}$-R-Mittag-Leffler envelope.*

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