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Article

Preservation of Extreme Points

Juan Francisco Mena-Jurado [1] and Juan Carlos Navarro-Pascual [2,*]

1 Departamento de Análisis Matemático, Facultad de Ciencias, Universidad de Granada, 18071 Granada, Spain; jfmena@ugr.es
2 Departamento de Matemáticas, Facultad de Ciencias Experimentales, Universidad de Almería, 04120 Almería, Spain
* Correspondence: jcnav@ual.es

Abstract: We characterize the extreme points of the closed unit ball of the dual of a Banach space which are preserved by the adjoint of any extreme operator. The result is related to the structure topology introduced by Alfsen and Effros on the set of all extreme points in the dual of any Banach space. As a consequence, we prove that $c_0(I)$ is the only Banach space such that the adjoint of every extreme operator taking values into it preserves extreme points.

Keywords: Banach space; extreme operator; structure topology

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1. Introduction

In this paper, we consider only real Banach spaces. If X is such an space, B_X and S_X denote the closed unit ball of X and the unit sphere of X, respectively. Given a nonempty subset A of X, $\mathrm{co}(A)$ is the convex hull of A and $\mathrm{span}(A)$ is the linear span of A. If $e \in A$, it can be said that e is an extreme point of A if the equality $e = (1-t)a + tb$, with $t \in]0,1[$ and $a, b \in A$, is only possible for $e = a = b$. The symbol E_X stands for the set of extreme points of B_X.

On the other hand, if Y is another Banach space, $L(X,Y)$ is the space of all continuous linear operators from X into Y equipped with the operator norm. The elements of the set $E_{L(X,Y)}$ are called extreme operators. As usual, we write X^* instead of $L(X,\mathbb{R})$, and the adjoint of an operator T is represented by T^*. If B is any subset of X^*, we denote by $\overline{B}^{w^*}$ the closure of B in the w^* topology of X^*. If J is a subspace of X, then

$$J^{\perp} = \{x^* \in X^* : x^*(x) = 0 \text{ for all } x \in J\}.$$

Consider a compact Hausdorff space K and let $C(K)$ be the space of continuous real valued functions defined on K equipped with the supremum norm. It is well known and easy to prove that the adjoint of an extreme operator from any Banach space X into $C(K)$ maps the isolated points of K into extreme points of the unit ball of X^*. In fact, this property characterizes the isolated points of K, that is, if the adjoint of every extreme operator from any Banach space X into $C(K)$ takes a point t of K to an extreme point of the unit ball of X^*, then t is an isolated point of K. Indeed, if t is a cluster point of K, define $Y = \{y \in C(K) : y(t) = 0\}$. Then, it can be easily proven that the inclusion from Y into $C(K)$ is an extreme operator the adjoint of which maps t into zero.

Let X be a Banach space. The goal of this paper is to characterize the elements $e_0^* \in E_{X^*}$ such that for any Banach space Y and every extreme operator $T : Y \to X$, $T^*(e_0^*)$ belongs to E_{Y^*}. An element e_0^* in E_{X^*} satisfying the above condition is said to be *adjoint preserved*.

Given a nonempty set I, it is worth mentioning that $c_0(I)$ is the space of functions $x : I \to \mathbb{R}$ such that for every $\varepsilon \in \mathbb{R}^+$, the set

$$\{i \in I : |x(i)| \geq \varepsilon\}$$

is finite. This space is provided with its supremum norm.

The main result of this paper states that the adjoint preserved extreme points are simply the isolated points of E_{X^*} with respect to the structure topology defined by Alfsen and Effros in [1]. As a consequence, every nice Banach space, defined as that Banach space X in which every element of E_{X^*} is adjoint preserved, is isometrically isomorphic to $c_0(I)$ for some nonempty set I.

2. The Results

The structure topology is the main tool used here in order to achieve our results. This topology can be defined on the set of extreme points of the dual of any Banach space. We introduce below the necessary notions for its definition. For more information concerning these concepts and related results, see [2].

Definition 1. *Let X be a Banach space. A closed subspace J of X is called an L-summand (resp. M-summand) in X if there exists a closed subspace N of X such that, for every $x \in X$, there are $x_1 \in J$ and $x_2 \in N$ uniquely determined such that*

$$x = x_1 + x_2 \text{ and } \|x\| = \|x_1\| + \|x_2\| \text{ (resp. } \|x\| = \max\{\|x_1\|, \|x_2\|\}).$$

In short, $X = J \oplus_1 N$ (resp. $X = J \oplus_\infty N$). A closed subspace J of X is called an M-ideal in X if $J^\perp$ is an L-summand in X^*. Whether every M-summand is M-ideal can be easily checked.

L-summands and M-summands were introduced by Cunningham in [3,4], respectively. Alfsen and Effros introduced M-ideals in [1], where they defined a topology in E_{X^*} by means of M-ideals in X.

Definition 2. *The structure topology on E_{X^*} is that for which the closed sets are of the form $J^\perp \cap E_{X^*}$, where J stands for an M-ideal in X.*

The following technical statement plays a central role in determining the adjoint preserved extreme points.

Proposition 1. *Let X be a Banach space and let $e_0^* \in E_{X^*}$. The following conditions are equivalent:*

(i) *$\{\pm e_0^*\}$ is a structurally open subset of E_{X^*}*
(ii) *For every Banach space Y and for every $y_0^* \in Y^*$ there exists $S \in L(Y, X)$ such that $S^*(e_0^*) = y_0^*$ and $S^*(E_{X^*} \setminus \{\pm e_0^*\}) = \{0\}$*
(iii) *There exists a Banach space Y and $S \in L(Y, X)$, $S \neq 0$, such that $S^*(E_{X^*} \setminus \{\pm e_0^*\}) = \{0\}$.*

Proof. (i)⇒(ii) By assumption, there exists an M-ideal in X such that

$$E_{X^*} \cap M^\perp = E_{X^*} \setminus \{\pm e_0^*\}.$$

Then, there exists a closed subspace Z of X^* such that $X^* = M^\perp \oplus_1 Z$. It follows from [2], Lemma I.1.5, that $E_Z = E_{X^*} \cap Z = \{\pm e_0^*\}$. Because Z is a dual Banach space (in fact, it is isometrically isomorphic to $X^*/M^\perp \equiv M^*$), it follows from the Krein–Milman theorem that $Z = \mathbb{R}\, e_0^*$ and, as Z is isometric to M^*, $M = \mathbb{R}\, x_0$ for some $x_0 \in S_X$. Therefore,

$$X^* = M^\perp \oplus_1 \mathbb{R}\, e_0^* = (\mathbb{R}\, x_0)^\perp \oplus_1 \mathbb{R}\, e_0^*.$$

In accordance with [2], Lemma I.1.5, $E_{X^*} = E_{(\mathbb{R}\, x_0)^\perp} \cup \{\pm e_0^*\}$, and by [5], Fact 3.119, we can suppose that $e_0^*(x_0) = 1$. We thus define S from Y to X as $S(y) = y_0^*(y)x_0$. It can be easily checked that S fulfils all the required conditions.

(ii)⇒(iii) Let Y be any nontrivial Banach space and fix $y_0^* \in Y^* \backslash \{0\}$. From (ii), there exists $S \in L(Y, X)$ such that $S^*(E_{X^*} \setminus \{\pm e_0^*\}) = \{0\}$ and $S^*(e_0^*) = y_0^* \neq 0$; hence, $S \neq 0$.

(iii)⇒(i) Taking into account the w^*-continuity of S^*, it is clear that

$$\overline{\operatorname{span}(E_{X^*} \setminus \{\pm e_0^*\})}^{w^*} \subseteq Ker(S^*) \neq X^*.$$

Thus, the result follows from [6], Proposition 3.3. □

Theorem 1. *Let X be a Banach space and $e_0^* \in E_{X^*}$ such that $\{\pm e_0^*\}$ is a structurally open subset of E_{X^*}. Then, $T^*(e_0^*) \in E_{Y^*}$ for every Banach space Y and every extreme operator T from Y into X.*

Proof. Let us suppose that $T^*(e_0^*) \notin E_{Y^*}$ for some Banach space Y and some extreme operator T from Y into X. Then, there exists $y_0^* \in Y^* \setminus \{0\}$ such that

$$\|T^*(e_0^*) \pm y_0^*\| \leq 1.$$

According to the above theorem, there exists $S \in L(Y, X)$ such that $S^*(e_0^*) = y_0^*$ and

$$S^*(E_{X^*} \setminus \{\pm e_0^*\}) = \{0\}.$$

Therefore, $\|T^*(e^*) \pm S^*(e^*)\| \leq 1$ for every $e^* \in E_{X^*}$; that is, $(T^* \pm S^*)(E_{X^*}) \subseteq B_{Y^*}$. The Krein–Milman theorem allows us to conclude that $\|T \pm S\| = \|T^* \pm S^*\| \leq 1$. Because $S \neq 0$, we obtain a contradiction to the fact that T is an extreme operator. □

As usual, given a convex subset A of a vector space X, ∂A stands for the set of extreme points of A.

Theorem 2. *Let X be a Banach space and $e_0^* \in E_{X^*}$. The following conditions are equivalent:*

(i) $\{\pm e_0^*\}$ *is a structurally open subset of* E_{X^*}

(ii) $T^*(e_0^*) \in E_{Y^*}$ *for every Banach space Y and every extreme operator T from Y into X.*

Proof. Taking into account the above theorem, we only need to prove that (ii)⇒(i). Suppose that $\{\pm e_0^*\}$ is not a structurally open subset of E_{X^*}. In the vector space $Y = X \times \mathbb{R}$, we define the norm

$$\|(x, \alpha)\| := \max\{\|x\|, |\alpha|, |e_0^*(x) \pm \alpha|\}.$$

It is clear that

$$\operatorname{co}(B_{X^*} \times \{0\} \cup \{0\} \times [-1, 1] \cup \{(\pm e_0^*, \pm 1)\}) \subseteq B_{Y^*}.$$

Suppose $(x^*, \alpha) \notin \operatorname{co}(B_{X^*} \times \{0\} \cup \{0\} \times [-1, 1] \cup \{(\pm e_0^*, \pm 1)\})$; because

$$\operatorname{co}(B_{X^*} \times \{0\} \cup \{0\} \times [-1, 1] \cup \{(\pm e_0^*, \pm 1)\})$$

is weak*-compact, the Hahn–Banach Theorem yields an element $(x, \beta) \in Y$ such that $(x^*, \alpha)(x, \beta) > (y^*, \gamma)(x, \beta)$ for every $(y^*, \gamma) \in \operatorname{co}(B_{X^*} \times \{0\} \cup \{0\} \times [-1, 1] \cup \{(\pm e_0^*, \pm 1)\})$. From here, we have $(x^*, \alpha)(x, \beta) > \|(x, \beta)\|$; that is, $(x^*, \alpha) \notin B_{Y^*}$. This proves that

$$B_{Y^*} = \operatorname{co}(B_{X^*} \times \{0\} \cup \{0\} \times [-1, 1] \cup \{(\pm e_0^*, \pm 1)\}).$$

We define $T : Y \to X$ by $T(x, \alpha) = x$ for all $(x, \alpha) \in Y$. It is clear that $T \in B_{L(X,Y)}$. We now prove that T is an extreme operator and $T^*(e_0^*) \notin E_{Y^*}$. It can be easily checked that $T^*(x^*) = (x^*, 0)$ for all $x^* \in X^*$. Because $(e_0^*, 0) = \frac{1}{2}((e_0^*, 1) + (e_0^*, -1))$, it is clear that $T^*(e_0^*) = (e_0^*, 0) \notin E_{Y^*}$. Let e^* be an element in $E_{X^*} \setminus \{\pm e_0^*\}$. Because $B_{Y^*} \subseteq B_{X^*} \times [-1, 1]$, the set $A := \{(x^*, \alpha) \in B_{Y^*} : x^* = e^*\}$ is a (nonempty) weak*-closed face of B_{Y^*}. Then, $\partial A = E_{Y^*} \cap A$. The reversed Krein–Milman Theorem and the Krein–Milman Theorem guarantee that

$$E_{Y^*} \subseteq E_{X^*} \times \{0\} \cup \{0\} \times \{-1, 1\} \cup \{(\pm e_0^*, \pm 1)\}.$$

From here, we have $E_{Y^*} \cap A \subseteq \{(e^*, 0)\}$. By again using the Krein–Milman Theorem, the set ∂A is not empty, and we can deduce that $T^*(e^*) = (e^*, 0)$ belongs to E_{Y^*}. Let $S \in L(Y, X)$ such that $\|T \pm S\| = \|T^* \pm S^*\| \leq 1$. Then, $\|T^*(e^*) \pm S^*(e^*)\| \leq 1$ for all e^* in $E_{X^*} \setminus \{\pm e_0^*\}$. We can thus conclude that $S^*(E_{X^*} \setminus \{\pm e_0^*\}) = \{0\}$. By [6], Proposition 3.3, we have $\overline{\text{span}(E_{X^*} \setminus \{\pm e_0^*\})}^{w^*} = X^*$, while the weak*-continuity of S^* allows us to obtain $S^* = 0$; hence, $S = 0$, proving that T is an extreme operator. This ends the proof. □

Definition 3. *Let X be a Banach space. It can be stated that an element $e_0^* \in E_{X^*}$ is adjoint preserved if $T^*(e_0^*) \in E_{Y^*}$ for any Banach space Y and every extreme operator T from Y into X.*

Taking into account [2], Example 1.4 (a), it is easy to see that for any compact Hausdorff space K, the elements of $E_{C(K)^*}$ which are structurally open are $\{\pm\delta_t\}$, with t an isolated point in K. The above theorem enables us to find that adjoint preserved points in $E_{C(K)^*}$ are just the isolated points in K, as we pointed out in the introduction.

Here, it is worth mentioning an application of Theorem 2 to the spaces of affine functions. To this end, we introduce several notations and concepts concerning this kind of spaces; see [7] for more information.

Let K be a compact convex subset of some (real) locally convex Hausdorff space. For $F \subseteq K$, the *complementary set* F' is the union of all faces of K disjoint from F. A face F of K is said to be a *split face* if F' is convex and every point in $K \setminus (F \cup F')$ can be uniquely represented as a convex combination of a point in F and a point in F'. It can be easily proven that F' is a split face whenever F is a split face. The sets of the form $F \cap \partial K$ in which F is a closed split face of K are the closed sets of a topology in ∂K which is called the *facial topology* of ∂K. This topological space is always compact, although it is non-Hausdorff in general (see [7], Proposition II.6.21 and Theorem II.7.8). The symbol $\mathcal{A}_0(K)$ denotes the space of all (real) continuous affine functions on K vanishing at a fixed extreme point of K which can be supposed to be zero. This last space is endowed with the supremum norm.

Corollary 1. *Let K be a compact convex set and $t \in \partial K \setminus \{0\}$; then, the following conditions are equivalent:*

(i) *$\{t\}$ is facially open in ∂K*
(ii) *$F = \{t\}$ is a split face of K and F' is closed in K*
(iii) *δ_t is adjoint preserved as an element of $E_{\mathcal{A}_0(K)^*}$*

Proof. According to [8], Proposition 2.3, (i)⇔(ii) and (ii) is equivalent to

$$\overline{\text{span}(E_{\mathcal{A}_0(K)^*} \setminus \{\pm\delta_t\})}^{w^*} \neq \mathcal{A}_0(K)^*.$$

By [6], Proposition 3.3, this last condition is equivalent to the fact that $\{\pm\delta_t\}$ is structurally open. Finally, Theorem 2 applies to find that (ii)⇔(iii). □

The notion of adjoint preserved elements is closely related to nice operators and nice Banach spaces, which we introduce below.

Definition 4. *Let X, Y be Banach spaces and $T \in L(X, Y)$. It can be stated that T is a "nice operator" if $T^*(E_{Y^*}) \subseteq E_{X^*}$. A Banach space X is said to be "nice" if, for any Banach space Y, every extreme operator in $L(Y, X)$ is a nice operator.*

It can be easily deduced from Krein–Milman theorem that nice operators are extreme operators. Nice operators appeared for the first time in [9]. It is clear that a Banach space X is nice if and only if every element in E_{X^*} is adjoint preserved.

Corollary 2. *A Banach space X is nice if and only if X is isometrically isomorphic to $c_0(I)$ for some nonempty set I.*

Proof. It is easy to check that $c_0(I)$ is nice for any nonempty set I. On the other hand, if X is nice, $\{\pm e^*\}$ is a structurally open subset of E_{X^*} for all e^* in E_{X^*} (Theorem 2). The result then follows from [10], Proposition 2. □

Finally, we point out that the previous corollary is an extension of other results already known for certain specific classes of Banach spaces (see [6,8,10]).

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