Short communication

Best-possible bounds on the set of copulas with a given value of Spearman's footrule

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Abstract

In this paper we find pointwise best-possible bounds on the set of copulas with a given value of the Spearman's footrule coefficient. We show that the lower bound is always a copula but, unlike the bounds on sets of copulas with a given value of other measures, such as Kendall's tau, Spearman's rho and Blonqvist's beta, the upper bound can be a copula or a proper quasi-copula. We characterised both of these cases.
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1. Introduction

Aggregation of pieces of information coming from different sources is an important task in expert and decision support systems, multi-criteria decision making, and group decision making. Aggregation operators are precisely the mathematical objects that allow this type of information fusion; and well-known operations in logic, probability theory, and statistics fit into this concept (for an overview, see [6,13]). Conjunctive aggregation operators, i.e., those aggregation operators that are bounded by the minimum, constitute an important class of operators that includes copulas and quasi-copulas.

(Bivariate) quasi-copulas were introduced in the field of probability (see [1]) and were characterized in [12]. They are also used in aggregation processes because they ensure that the aggregation is stable, in the sense that small error inputs correspond to small error outputs. In the last few years these functions have attracted an increasing interest by researchers in some topics of fuzzy sets theory, such as preference modeling, similarities and fuzzy logics. For a complete overview of quasi-copulas, we refer to [2,24].

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Copulas, (bivariate) probability distribution functions with uniform margins on [0, 1] restricted to the unit square, are a subclass of quasi-copulas. The importance of copulas in probability and statistics comes from *Sklar's theorem* [25], which states that the joint distribution H of a pair of random variables (X, Y) and the corresponding (univariate) marginal distributions F and G are linked by a copula C in the following manner:

$$H(x, y) = C(F(x), G(y)) \text{ for all } (x, y) \in [-\infty, \infty]^2.$$

If F and G are continuous, then the copula is unique; otherwise, the copula is uniquely determined on Range $F \times$ Range G ([8]). For a comprehensive review on copulas, we refer to the monographs [9,18].

The fundamental best-possible bounds inequality for the set of (quasi-)copulas is given by the Fréchet-Hoeffding bounds, i.e., for any quasi-copula Q we have

$$W(u, v) := (0 \vee u + v - 1) \leqslant Q(u, v) \leqslant (u \wedge v) =: M(u, v) \tag{1}$$

for all $(u, v) \in [0, 1]^2$, where $c \vee d := \max(c, d)$ and $c \wedge d := \min(c, d)$ for any two real numbers c and d. Furthermore, the bounds W and M are themselves copulas.

A procedure for finding pointwise best-possible bounds on sets of copulas and a given value of the population version of a measure of association, such as Kendall's tau, Spearman's rho or the population version of the medial correlation coefficient (or Blomqvist's beta) is illustrated in [19,21]. The bounds attained are evaluated –with the result that all the bounds are copulas– and compared. In [7] best-possible bounds on the set of (quasi-)copulas with given degree of non-exchangeability are established. In this paper, we focus on the Spearman's footrule coefficient and establish the best-possible bounds on the set of (quasi-)copulas with a given Spearman's footrule coefficient. We show some conditions under which the resulting bounds are copulas, which, unlike for some other measures of association, is not generally true.

After some preliminaries concerning (quasi-)copulas (Section 2), we present the main results in Section 3, where we find the best-possible bounds on the set of copulas with a given value of the Spearman's footrule coefficient and provide some of their salient properties. Section 4 is devoted to conclusions.

2. Preliminaries

A (bivariate) *copula* is a function $C \colon [0, 1]^2 \longrightarrow [0, 1]$ which satisfies

(C1) the boundary conditions $C(t, 0) = C(0, t) = 0$ and $C(t, 1) = C(1, t) = t$ for all t in [0, 1], and
(C2) the *2-increasing property*, i.e., $V_C([u_1, u_2] \times [v_1, v_2]) = C(u_2, v_2) - C(u_2, v_1) - C(u_1, v_2) + C(u_1, v_1) \geqslant 0$ for all u_1, u_2, v_1, v_2 in [0, 1] such that $u_1 \leqslant u_2$ and $v_1 \leqslant v_2$.

$V_C(R)$ is usually called as the *C-volume* of the rectangle R; and in the sequel we also consider the C-volume of a rectangle for real-valued functions on $[0, 1]^2$ which may not be copulas.

Let $\mathcal{B}([0, 1])$ and $\mathcal{B}([0, 1]^2)$ denote the respective Borel σ-algebras in [0, 1] and $[0, 1]^2$, and λ denotes the Lebesgue measure on [0, 1]. A measure μ on $\mathcal{B}([0, 1]^2)$ is *doubly stochastic* if $\mu(B \times [0, 1]) = \mu([0, 1] \times B) = \lambda(B)$ for every $B \in \mathcal{B}([0, 1])$. Each copula C induces a doubly stochastic measure μ_C by setting $\mu_C(R) = V_C(R)$ for every rectangle $R \subseteq [0, 1]^2$ and extending μ_C to $\mathcal{B}([0, 1]^2)$. The *support* of a copula C is the complement of the union of all open subsets of $[0, 1]^2$ with μ_C-measure zero.

Let $\mathcal{C}$ denote the set of all copulas.

The concept of a quasi-copula was introduced in [1] in order to characterize operations on distribution functions that can or cannot be derived from operations on random variables defined on the same probability space. A (bivariate) *quasi-copula* is a function $Q \colon [0, 1]^2 \longrightarrow [0, 1]$ which satisfies condition (C1) of copulas, but in place of (C2), the weaker conditions

(Q1) Q is non-decreasing in each variable, and
(Q2) the *Lipschitz* condition $|Q(u_1, v_1) - Q(u_2, v_2)| \leqslant |u_1 - u_2| + |v_1 - v_2|$ for all u_1, v_1, u_2, v_2 in $[0, 1]^2$.

While every copula is a quasi-copula, there exist *proper* quasi-copulas, i.e., quasi-copulas which are not copulas –distinctions concerning the mass distribution of copulas and proper quasi-copulas can be found, e.g., in [10,22].
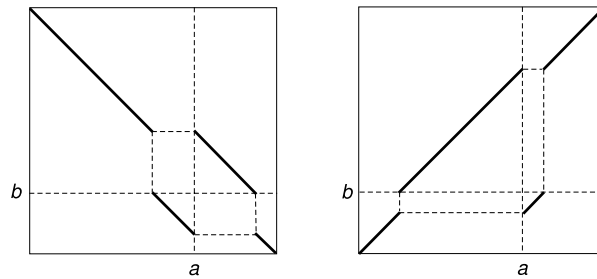
Fig. 1. The supports of $\underline{C}_{(a,b),\theta}$ (left) and $\overline{C}_{(a,b),\theta}$ (right) for a copula C with $C(a,b) = \theta$.

One of the most important occurrences of quasi-copulas in statistics is due to the following observation ([20,23]): Every set $\mathcal{S}$ of (quasi-)copulas has the smallest upper bound and the greatest lower bound in the set of quasi-copulas (in the sense of pointwisely ordered functions). These bounds do not necessarily belong to the set $\mathcal{S}$, nor they are necessarily copulas if the set consists of copulas only.

In [27] best-possible bounds on the set of quasi-copulas that coincide on a given compact subset S of $[0, 1]^2$ are established and the author investigates sufficient conditions on S such that these bounds are also the best-possible bounds on the set of copulas that coincide on S. In [3–5] the monotonicity and the Lipschitz condition were used to establish tight upper and lower bounds on classes of functions in a more general setting, which include quasi-copulas as special cases, by a similar technique. The bounds in (1) can often be narrowed when we possess additional information about the copula C, e.g., in [18] the best-possible bounds on a set of copulas when a value at a single point is known are provided (we recall this result for our purposes, and in which $x^+ := 0 \vee x$ for any real number x).

Proposition 1. *Let C be a copula, and suppose $C(a, b) = \theta$, where $(a, b) \in [0, 1]^2$ and $W(a, b) \leqslant \theta \leqslant M(a, b)$. Then $\underline{C}_{(a,b),\theta}(u, v) \leqslant C(u, v) \leqslant \overline{C}_{(a,b),\theta}(u, v)$ for all $(u, v) \in [0, 1]^2$, where*

$$\underline{C}_{(a,b),\theta}(u, v) = \max\left(0, u + v - 1, \theta - (a - u)^+ - (b - v)^+\right)$$

and

$$\overline{C}_{(a,b),\theta}(u, v) = \min\left(u, v, \theta + (u - a)^+ + (v - b)^+\right).$$

Since $\underline{C}_{(a,b),\theta}(a, b) = \overline{C}_{(a,b),\theta}(a, b) = \theta$, the bounds are best-possible.

The bounds $\underline{C}_{(a,b),\theta}$ and $\overline{C}_{(a,b),\theta}$ in Proposition 1 are *shuffles of M*, a particular family of copulas (see [14,18] for more details). The supports of $\underline{C}_{(a,b),\theta}$ and $\overline{C}_{(a,b),\theta}$ (with thick line segments) are shown in Fig. 1.

3. Best-possible bounds when a given value of the Spearman's footrule coefficient is known

Let $(R_1, S_1), \ldots, (R_n, S_n)$ be ranks associated with a random sample $(X_1, Y_1), \ldots, (X_n, Y_n)$ from some continuous bivariate distribution $H(x, y) = \Pr(X \leqslant x, Y \leqslant y)$. The Spearman's footrule coefficient is a nonparametric measure of association, introduced by the psychologist C. Spearman in [26], given by the statistic

$$\varphi_n = 1 - \frac{3}{n^2 - 1} \sum_{i=1}^{n} |R_i - S_i|,$$

and which, in terms of the copula C associated with the continuous random vector (X, Y), can be expressed as [16,18]

$$\varphi_{X,Y} = \varphi(C) = 6 \int_0^1 C(u, u)\, du - 2.$$

We note that, for any copula C, we have $-1/2 = \varphi(W) \leqslant \varphi(C) \leqslant \varphi(M) = 1$. For a comprehensive review on the Spearman's footrule coefficient, we refer to [11].
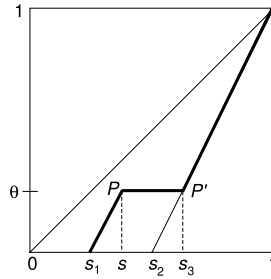
Fig. 2. Calculation of the integral in Lemma 2 as the area under the thick piecewise linear curve.

For any $t \in [-1/2, 1]$, let $\mathbf{F}_t$ denote the set of copulas with a common value t of the Spearman's footrule coefficient, i.e.,

$$\mathbf{F}_t = \{C \in \mathcal{C} \colon \varphi(C) = t\}.$$

Let $\underline{F}_t$ and $\overline{F}_t$ denote, respectively, the pointwise infimum and supremum of $\mathbf{F}_t$, i.e., for each (u, v) in $[0, 1]^2$,

$$\underline{F}_t(u, v) = \inf\{C(u, v) \colon C \in \mathbf{F}_t\} \text{ and } \overline{F}_t(u, v) = \sup\{C(u, v) \colon C \in \mathbf{F}_t\}. \tag{2}$$

Our main goal is to determine these bounds explicitly. In establishing these bounds we proceed as follows. First we determine the bounds on the diagonal $L_t(s) = \underline{F}_t(s, s)$ and $U_t(s) = \overline{F}_t(s, s)$ from the condition $\varphi(C) = t$ and the Lipschitz condition. Next we determine the bounds on the rest of the domain $\underline{F}_t$ and $\overline{F}_t$. Finally, we show that our bounds are active, that is $\underline{F}_t$ is not smaller than any other bound found from conditions $\varphi(C) = t$ and Lipschitzianity, and similarly that $\overline{F}_t$ is not larger than any other bound.

For this we need several preliminary results.

Lemma 2. *Let C be a copula, and let the value of $C(s, s) = \theta$ be given. Then the Spearman's footrule coefficient for the copula $\underline{C}_{(s,s),\theta}$ is*

$$\varphi\left(\underline{C}_{(s,s),\theta}\right) = 3\theta(\theta + 1 - 2s) - \frac{1}{2}.$$

Proof. Since

$$\int\limits_0^1 \underline{C}_{(s,s),\theta}(u, u)\, du = \int\limits_0^{s_1} 0\, du + \int\limits_{s_1}^s (2u - (2s - \theta))\, du + \int\limits_s^{s_3} \theta\, du + \int\limits_{s_3}^1 (2u - 1)\, du,$$

we can calculate the integral as the area under the curve geometrically (see Fig. 2): Note that the triangles $s_1 s P$ and $s_2 s_3 P'$ have the same area, and hence the area under the curve is $\frac{1}{4} + \theta(s_3 - s)$. The value s_3 is found from $\theta = 2s_3 - 1$ so $s_3 = \frac{\theta+1}{2}$ and hence we get

$$\int\limits_0^1 \underline{C}_{(s,s),\theta}(u, u)\, du = \frac{1}{4} + \theta\left(\frac{\theta + 1}{2} - s\right),$$

then the result follows. $\square$

Lemma 3. *Let the value of $C(s, s) = \theta$ be given. Then the Spearman's footrule coefficient for the copula $\overline{C}_{(s,s),\theta}$ is*

$$\varphi\left(\overline{C}_{(s,s),\theta}\right) = 1 - 6(s - \theta)^2.$$

Proof. We compute the integral $\int_0^1 \overline{C}_{(s,s),\theta}(u, u)\, du$ as the area under the curve geometrically again. Note that it is equal to $\frac{1}{2}$ minus the area of the shaded region on Fig. 3. In the shaded triangle, s is the midpoint between s_1 and s_2 and $s_2 - s_1 = 2(s - \theta)$. The area of the small triangle $PP'P''$ is $(s - \theta)^2$ and hence the area of the shaded triangle is also $(s - \theta)^2$. So the integral is $\frac{1}{2} - (s - \theta)^2$ and we obtain the result. $\square$
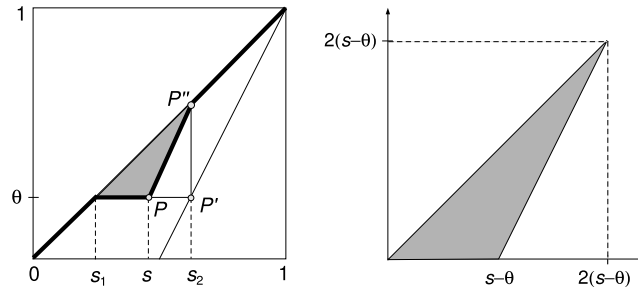
Fig. 3. Calculation of the integral in Lemma 3 as the area under the thick piecewise linear curve (left). It is equal to $\frac{1}{2}$ minus the area of the shaded triangle. The latter one (right) is done by schoolbook formula.

Now, we resolve the equations

$$\varphi\left(\overline{C}_{(s,s),\theta}\right) = 1 - 6(s - \theta)^2 = t \tag{3}$$

and

$$\varphi\left(\underline{C}_{(s,s),\theta}\right) = 3\theta(\theta + 1 - 2s) - \frac{1}{2} = t \tag{4}$$

for θ, when s is fixed. In the former case (3) we obtain

$$\theta = s \pm \sqrt{\frac{1 - t}{6}},$$

and in the latter case (4) we have

$$\theta = \frac{2s - 1}{2} \pm \frac{1}{2}\sqrt{(2s - 1)^2 + \frac{4}{3}\left(t + \frac{1}{2}\right)}.$$

Hence for a given $t \in [-1/2, 1]$ and fixed s, since $t = \varphi(C) \leqslant \varphi\left(\overline{C}_{(s,s),\theta}\right)$, then we have $t \leqslant 1 - 6(a - \theta)^2$, so θ must be no smaller than $s - \sqrt{\frac{1-t}{6}}$. Hence

$$\theta = C(s, s) \geqslant s - \sqrt{\frac{1 - t}{6}} = \underline{\theta} \text{ for every } s \in [0, 1].$$

Now on the other side, for a given $t \in [-1/2, 1]$ and any fixed s, since $t = \varphi(C) \geqslant \varphi\left(\underline{C}_{(s,s),\theta}\right)$, then we have $t \geqslant 3\theta(\theta + 1 - 2s) - \frac{1}{2}$. Therefore

$$\theta = C(s, s) \leqslant s - \frac{1}{2} + \frac{1}{2}\sqrt{(2s - 1)^2 + \frac{4}{3}\left(t + \frac{1}{2}\right)} = \overline{\theta} \text{ for every } s \in [0, 1].$$

Thus we now have two bounds on the values of $C(s, s)$ on the diagonal,

$$L_t(s) = \max\left(0, 2s - 1, s - \sqrt{\frac{1 - t}{6}}\right) \leqslant C(s, s) \leqslant \min\left(s, s - \frac{1}{2} + \frac{1}{2}\sqrt{(2s - 1)^2 + \frac{4}{3}\left(t + \frac{1}{2}\right)}\right) = U_t(s), \tag{5}$$

which also imply the bounds on the rest of the unit square by Proposition 1.

Lemma 4. *The functions*

$$\overline{C}_{(v,v),\overline{\theta}}(u, v) = \min\left(u, v, \overline{\theta} + (u - v)^+ + (v - v)^+\right) = \min(u, v, \overline{\theta} + (u - v))$$

and

$$\underline{C}_{(u,u),\underline{\theta}}(u,v) = \max\left(0, u + v - 1, \underline{\theta} - (u-u)^+ - (u-v)^+\right) = \max\left(0, u + v - 1, \underline{\theta} - (u-v)\right)$$

are upper and lower bounds on $\mathbf{F}_t$.

Proof. Since we have tight bounds (5), take the point (u, u) for $u \in [0, 1]$, where there is a quasi-copula C' such that $C'(u, u) = \max(0, 2u - 1, \underline{\theta})$. By Lipschitz condition and monotonicity we get the lower bound $\underline{C}_{(u,u),\underline{\theta}}(u,v)$. Similarly we get the upper bound. $\quad\square$

We note that these bounds are not necessarily tight. To establish tighter bounds we look at the range of values at a point (a, b) in $[0, 1]^2$ which permit the value $\varphi(C) = t$.

We begin with the pointwise infimum.

Lemma 5. *Let C be a copula and let the value $C(a, b) = \theta$ be given, where $(0 \vee a + b - 1) \leqslant \theta \leqslant (u \wedge v)$. Then the Spearman's footrule coefficient for the copula $\overline{C}_{(a,b),\theta}$ is*

$$\varphi\left(\overline{C}_{(a,b),\theta}\right) = 1 - 6(a - \theta)(b - \theta).$$

Proof. Suppose we have a point $(a, b) \in [0, 1]^2$ such that $a \leqslant b$ (the case $a \geqslant b$, by symmetry, is similar, and we omit it). We compute the Spearman's footrule coefficient for the copula $\overline{C}_{(a,b),\theta}$. Since

$$\int_0^1 \overline{C}_{(a,b),\theta}(u,u)\,du = \int_0^\theta u\,du + \int_\theta^a \theta\,du + \int_a^b (u - a + \theta)\,du + \int_b^{a+b-\theta} (2u - a - b + \theta)\,du + \int_{a+b-\theta}^1 u\,du$$

$$= \frac{1}{2} - (a - \theta)(b - \theta),$$

we have

$$\varphi\left(\overline{C}_{(a,b),\theta}\right) = 1 - 6(a - \theta)(b - \theta),$$

which completes the proof. $\quad\square$

And now we show the corresponding result for the pointwise supremum.

Lemma 6. *Let C be a copula, let (a, b) be a point in $[0, 1]^2$ such that $b \leqslant a$, and let the value $C(a, b) = \theta$ be given, where $(0 \vee a + b - 1) \leqslant \theta \leqslant b$. Then the Spearman's footrule coefficient for the copula $\underline{C}_{(a,b),\theta}$ is*

$$\varphi\left(\underline{C}_{(a,b),\theta}\right) = \begin{cases} -\dfrac{1}{2}, & a \geqslant \dfrac{1}{2} + \theta \\[2mm] 6\left(a - \theta - \dfrac{1}{2}\right)^2 - \dfrac{1}{2}, & (b + \theta) \vee \dfrac{1+\theta}{2} \leqslant a \leqslant \dfrac{1}{2} + \theta \\[2mm] 3\theta(1 - 2a) + \dfrac{9}{2}\theta^2 - \dfrac{1}{2}, & b + \theta \leqslant a \leqslant \dfrac{1+\theta}{2} \\[2mm] \dfrac{3}{2}(\theta + 1 - a - b)(3\theta - 3a + b + 1) - \dfrac{1}{2}, & \dfrac{1+\theta}{2} \leqslant a \leqslant b + \theta \\[2mm] -\dfrac{3}{2}(a - b)^2 + 3(1 - a - b)\theta + 3\theta^2 - \dfrac{1}{2}, & a \leqslant (b + \theta) \wedge \dfrac{1+\theta}{2}. \end{cases} \tag{6}$$

Proof. First, note that for computing the integral
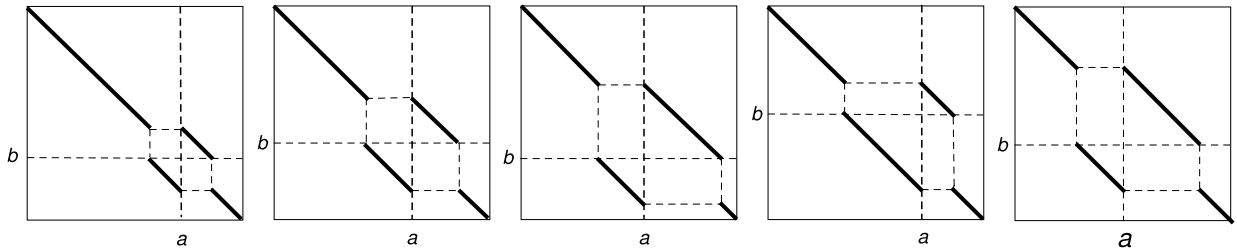
$$\int_0^1 \underline{C}_{(a,b),\theta}(u,u)\,du$$

Fig. 4. The respective cases 1, 2, 3, 4, and 5 (left to right) for computing $\varphi\left(\underline{C}_{(a,b),\theta}\right)$ in Lemma 6.

we have to consider five cases, which depend on the location of the point (a,b), with $b \leqslant a$, in the copula $\underline{C}_{(a,b),\theta}$. These cases are:

$$
\begin{cases}
a \geqslant \dfrac{1}{2} + \theta \\[2mm]
(b+\theta) \vee \dfrac{1+\theta}{2} \leqslant a \leqslant \dfrac{1}{2} + \theta \\[2mm]
b + \theta \leqslant a \leqslant \dfrac{1+\theta}{2} \\[2mm]
\dfrac{1+\theta}{2} \leqslant a \leqslant b + \theta \\[2mm]
a \leqslant (b+\theta) \wedge \dfrac{1+\theta}{2}.
\end{cases}
$$

Fig. 4 illustrates the supports of the copula $\underline{C}_{(a,b),\theta}$ for those five cases. Thus, we obtain:

Case 1.

$$
\int_{0}^{1} \underline{C}_{(a,b),\theta}(u,u)\,du = \int_{0}^{1/2} 0\,du + \int_{1/2}^{1} (2u-1)\,du = \frac{1}{4}.
$$

Case 2.

$$
\int_{0}^{1} \underline{C}_{(a,b),\theta}(u,u)\,du = \int_{0}^{a-\theta} 0\,du + \int_{a-\theta}^{1-a+\theta} (u-a+\theta)\,du + \int_{1-a+\theta}^{1} (2u-1)\,du
$$

$$
= (a-\theta)^2 - (a-\theta) + \frac{1}{2}.
$$

Case 3.

$$
\int_{0}^{1} \underline{C}_{(a,b),\theta}(u,u)\,du = \int_{0}^{a-\theta} 0\,du + \int_{a-\theta}^{a} (u-a+\theta)\,du + \int_{a}^{(1+\theta)/2} \theta\,du + \int_{(1+\theta)/2}^{1} (2u-1)\,du
$$

$$
= \frac{1}{4} + \frac{3}{4}\theta^2 + \theta\left(\frac{1}{2} - a\right).
$$

Case 4.

$$
\int_{0}^{1} \underline{C}_{(a,b),\theta}(u,u)\,du = \int_{0}^{(a+b-\theta)/2} 0\,du + \int_{(a+b-\theta)/2}^{b} (2u-a-b+\theta)\,du + \int_{b}^{1-a+\theta} (u-a+\theta)\,du
$$

$$+ \int\limits_{1-a+\theta}^{1} (2u-1)\, du$$

$$= \frac{3a^2 + 2a(b - 3\theta - 2) - b^2 - 2b\theta + 3\theta^2 + 4\theta + 2}{4}.$$

Case 5.

$$\int\limits_{0}^{1} \underline{C}_{(a,b),\theta}(u,u)\, du = \int\limits_{0}^{(a+b-\theta)/2} 0\, du + \int\limits_{(a+b-\theta)/2}^{b} (2u - a - b + \theta)\, du + \int\limits_{b}^{a} (u - a + \theta)\, du$$

$$+ \int\limits_{a}^{(1+\theta)/2} \theta\, du + \int\limits_{(1+\theta)/2}^{1} (2u-1)\, du$$

$$= \frac{1}{4} \left(1 - (a-b)^2\right) + \frac{1}{2}(1 - a - b)\theta + \frac{\theta^2}{2}.$$

The integrals above can be evaluated directly or geometrically in the same way it was done in Lemma 2. Thus we have

$$\int\limits_{0}^{1} \underline{C}_{(a,b),\theta}(u,u)\, du = \begin{cases} \dfrac{1}{4}, & a \geqslant \dfrac{1}{2} + \theta \\[2mm] \dfrac{1}{4} + \left(a - \theta - \dfrac{1}{2}\right)^2, & (b+\theta) \vee \dfrac{1+\theta}{2} \leqslant a \leqslant \dfrac{1}{2} + \theta \\[2mm] \dfrac{1}{4}(1 + 2\theta - 4a\theta + 3\theta^2), & b + \theta \leqslant a \leqslant \dfrac{1+\theta}{2} \\[2mm] \dfrac{1}{4}((\theta + 1 - a - b)(3\theta - 3a + b + 1) + 1), & \dfrac{1+\theta}{2} \leqslant a \leqslant b + \theta \\[2mm] \dfrac{1}{4}\left(1 - (a-b)^2\right) + \dfrac{1}{2}(1 - a - b)\theta + \dfrac{\theta^2}{2}, & a \leqslant (b+\theta) \wedge \dfrac{1+\theta}{2}. \end{cases}$$

Therefore, the values that the Spearman's footrule coefficient can take are given by (6), which completes the proof. □

The following technical lemma is crucial for our purposes. It involves four upper bounds on $C(u,v)$ obtained by resolving the conditions (6) for θ.

Lemma 7. *Let* $t \in [-1/2, 1]$ *be fixed, and for every* (u,v) *in* $[0,1]^2$ *consider the functions*

$$f_1(u,v;t) := \frac{6(u \vee v) - 3 + \sqrt{3(2t+1)}}{6}, \tag{7}$$

$$f_2(u,v;t) := \frac{2(u \vee v) - 1 + \sqrt{2\left(2(u \vee v)^2 - 2(u \vee v) + t + 1\right)}}{3}, \tag{8}$$

$$f_3(u,v;t) := \frac{3(u \vee v) + (u \wedge v) - 2 + \sqrt{2\left(2(u \wedge v)^2 - 2(u \wedge v) + t + 1\right)}}{3}, \tag{9}$$

$$f_4(u,v;t) := \frac{1}{2}\left(u + v - 1 + \sqrt{\frac{(3(u+v) - 1)^2 - 24uv + 4(1+t)}{3}}\right). \tag{10}$$

Then we have $\max\left(f_1(u,v;t), f_2(u,v;t), f_3(u,v;t), f_4(u,v;t)\right) = f_4(u,v;t)$ *for all* $(u,v) \in [0,1]^2$.

Proof. Firstly, we simplify notation. Let $\alpha = 2u - 1$, $\beta = 2v - 1$, and $k = 1 + 2t$ (note that $\alpha, \beta \in [-1, 1]$ and $k \in [0, 3]$), and assume $\alpha \geqslant \beta$ (the case $\alpha < \beta$ is similar and we omit it). Then we have

$$f_1(\alpha, \beta; k) = \frac{\alpha}{2} + \frac{\sqrt{3k}}{6},$$

$$f_2(\alpha, \beta; k) = \frac{\alpha}{3} + \frac{\sqrt{\alpha^2 + k}}{3},$$

$$f_3(\alpha, \beta; k) = \frac{3\alpha + \beta}{6} + \frac{\sqrt{\beta^2 + k}}{3},$$

$$f_4(\alpha, \beta; k) = \frac{\alpha + \beta}{4} + \sqrt{\frac{3(\alpha^2 + \beta^2)}{16} - \frac{\alpha\beta}{8} + \frac{k}{6}}.$$

We prove that $f_4(\alpha, \beta; k) \geqslant f_1(\alpha, \beta; k)$ for every $\alpha, \beta \in [-1, 1]$ and a fixed k. Denoting $\gamma := \alpha + \beta$ and $\delta := \alpha - \beta$, f_4 can be rewritten as

$$f_4(\gamma, \delta; k) = \frac{\gamma}{4} + \sqrt{\frac{\delta^2}{8} + \frac{\gamma^2}{16} + \frac{k}{6}}.$$

Since $\alpha \geqslant \beta$, we have $\delta \geqslant 0$ and

$$\frac{\partial f_4}{\partial \delta}(\gamma, \delta; k) = \frac{\delta}{8\sqrt{\frac{\delta^2}{8} + \frac{\gamma^2}{16} + \frac{k}{6}}} \geqslant 0,$$

whence $\min\limits_{\delta} f_4(\gamma, \delta; k)$ is reached at $\delta = 0$, i.e., $\alpha = \beta$; therefore

$$f_4(\alpha, \beta; k) \geqslant f_4(\alpha, \alpha; k) = \frac{\alpha}{2} + \sqrt{\frac{\alpha^2}{4} + \frac{k}{6}} \geqslant f_1(\alpha, \alpha; k) = f_1(\alpha, \beta; k).$$

To prove that $f_4(\alpha, \beta; k) \geqslant f_2(\alpha, \beta; k)$ for all $\alpha, \beta \in [-1, 1]$ such that $\alpha \geqslant \beta$, since $f_2(\alpha, \beta; k) = f_2(\alpha, \alpha; k)$, we only need to prove that $f_4(\alpha, \alpha; k) \geqslant f_2(\alpha, \alpha; k)$ for all $\alpha \in [-1, 1]$. Let f be the function defined by

$$f(\alpha; k) := f_4(\alpha, \alpha; k) - f_2(\alpha, \alpha; k) = \frac{\alpha}{6} + \sqrt{\frac{\alpha^2}{4} + \frac{k}{6}} - \sqrt{\frac{\alpha^2 + k}{9}}.$$

We consider two cases:

1. $k = 0$. Then we have $f(\alpha; 0) = \frac{|\alpha| + \alpha}{6} \geqslant 0$ for all $\alpha \in [-1, 1]$.
2. $0 < k \leqslant 3$. In this case, $f'(\alpha; k) = 0$ implies the roots

$$\alpha_1 = \frac{\sqrt{\left(2\sqrt{7} - 5\right)k}}{3} \quad \text{and} \quad \alpha_2 = -\frac{\sqrt{\left(2\sqrt{7} - 5\right)k}}{3}.$$

Since

$$f''(\alpha_1; k) = f''(\alpha_2; k) = \frac{5}{108} \cdot \sqrt{\frac{\left(4506\sqrt{7} - 11892\right)}{k}} > 0$$

for all $k \in]0, 3]$, then we have that α_1 and α_2 are minimum points, and

$$f(\alpha_1; k) = \frac{\sqrt{\left(6\sqrt{7} - 12\right)k}}{18} > 0 \quad \text{and} \quad f(\alpha_2; k) = \frac{\sqrt{\left(26\sqrt{7} - 68\right)k}}{18} > 0$$

for all $k \in]0, 3]$.

Thus, $f_4(\alpha, \beta; k) \geqslant f_2(\alpha, \beta; k)$ for all $\alpha, \beta \in [-1, 1]$ such that $\alpha \geqslant \beta$.

Finally, to prove that $f_4(\alpha, \beta; k) \geqslant f_3(\alpha, \beta; k)$ for all $(\alpha, \beta) \in [-1, 1]^2$ such that $\alpha \geqslant \beta$, note that $f_3(\alpha, \alpha, k) = f_2(-\alpha, -\alpha, k) + \alpha$, and thus

$$f_4(\alpha, \beta; k) - f_3(\alpha, \beta; k) \geqslant f_4(\alpha, \alpha; k) - f_3(\alpha, \alpha; k) = f_4(\alpha, \alpha; k) - f_2(-\alpha, -\alpha; k) - \alpha$$
$$= f_4(-\alpha, -\alpha; k) - f_2(-\alpha, -\alpha; k) \geqslant 0,$$

which completes the proof. $\quad\square$

Now we are in position to provide the explicit expressions of $\underline{F}_t$ and $\overline{F}_t$.

Theorem 8. *Let $\underline{F}_t$ and $\overline{F}_t$ denote the pointwise infimum and supremum (2) of $\mathbf{F}_t$ for $t \in [-1/2, 1]$. Then, for any (u, v) in $[0, 1]^2$, we have*

$$\underline{F}_t(u, v) = \max\left(0, u + v - 1, \frac{1}{2}\left[u + v - \sqrt{(u-v)^2 + 2/3(1-t)}\right]\right) \tag{11}$$

and

$$\overline{F}_t(u, v) = \min\left(u, v, \frac{1}{2}\left(u + v - 1 + \sqrt{\frac{(3(u+v)-1)^2 - 24uv + 4(1+t)}{3}}\right)\right). \tag{12}$$

Proof. Let us deal with the lower bound first. We use the result of Lemma 5. If C is a copula such that $\varphi(C) = t$, for $t \in [-1/2, 1]$, since $\varphi(C) \leqslant \varphi\left(\overline{C}_{(a,b),\theta}\right)$. By solving this for θ and taking the smallest root we get

$$\lambda = \frac{1}{2}\left(a + b - \sqrt{(a-b)^2 + 2/3(1-t)}\right).$$

It follows that $\lambda \leqslant \theta$, so that $C(a, b) \geqslant \max(0, a + b - 1, \lambda)$, and hence $\underline{F}_t(a, b) \geqslant \max(0, a + b - 1, \lambda)$. To establish that this bound is tight we need to show that there exists a copula C in $\mathbf{F}_t$ such that $C(a, b) = \max(0, a + b - 1, \lambda)$. Assume $a + b - 1 \geqslant 0$ (the case $a + b - 1 < 0$ is similar). If $\lambda \geqslant a + b - 1$, then $\overline{C}_{(a,b),\lambda} \in \mathbf{F}_t$ (because it is a shuffle of the min function) and $\overline{C}_{(a,b),\lambda}(a, b) = \lambda$. If $\lambda < a + b - 1$, then $C_\alpha = (1 - \alpha)W + \alpha\overline{C}_{(a,b),a+b-1}$, for $\alpha \in [0, 1]$, is a family of copulas for which $C_\alpha(a, b) = a + b - 1$ and for which $\varphi(C_\alpha)$ is a continuous function of α —for details, see [17]— satisfying $\varphi(C_0) = \varphi(W) = -1/2 \leqslant t \leqslant \varphi\left(\overline{C}_{(a,b),a+b-1}\right) = \varphi(C_1)$. It follows from the intermediate value theorem that there is $\alpha \in [0, 1]$ such that $\varphi(C_\alpha) = t$.

Next we deal with the upper bound. We consider the five cases in Lemma 6 (and obtain the other five by symmetry). The first case is trivial, $t \geqslant -\frac{1}{2}$. In each other case we resolve the corresponding equation for θ and taking the largest roots of the quadratic equations, we obtain the bounds $\theta_1 := f_1(a, b; t)$, $\theta_2 := f_2(a, b; t)$, $\theta_3 := f_3(a, b; t)$ and $\theta_4 := f_4(a, b; t)$, where f_1, f_2, f_3 and f_4 are the functions given by (7), (8), (9) and (10), respectively. By taking their maximum, which is θ_4 –recall Lemma 7– we obtain the upper bound.

We need to show that there exists a copula $C \in \mathbf{F}_t$ such that $C(a, b) = \min(a, b, \theta_4)$ for any $t \in [-1/2, 1]$. Assume $a \leqslant b$ (the case $b \leqslant a$ is similar). If $\theta_4 \leqslant a$, then $\underline{C}_{(a,b),\theta_4} \in \mathbf{F}_t$ and $\underline{C}_{(a,b),\theta_4}(a, b) = \theta_4$. If $\theta_4 > a$, then $C_\beta = \beta M + (1 - \beta)\underline{C}_{(a,b),a}$, for $\beta \in [0, 1]$, is a family of copulas for which $C_\beta(a, b) = a$ and for which $\varphi(C_\beta)$ is a continuous function of β satisfying $\varphi(C_0) = \varphi\left(\underline{C}_{(a,b),a}\right) \leqslant t \leqslant 1 = \varphi(M) = \varphi(C_1)$. It now follows from the intermediate value theorem that there is $\beta \in [0, 1]$ such that $\varphi(C_\beta) = t$, which completes the proof. $\quad\square$

We know that the functions $\underline{F}_t$ and $\overline{F}_t$ given by (11) and (12), respectively, are quasi-copulas, based on the Lipschitz condition and monotonicity. The next two results show some salient properties of these functions (we recall that a copula C is *radially symmetric* if $(u, v) = u + v - 1 + C(1 - u, 1 - v)$ for every $(u, v) \in [0, 1]^2$: see [15,18]).

Theorem 9. *Let $\underline{F}_t$ be the quasi-copula given by (11). Then it holds that:*

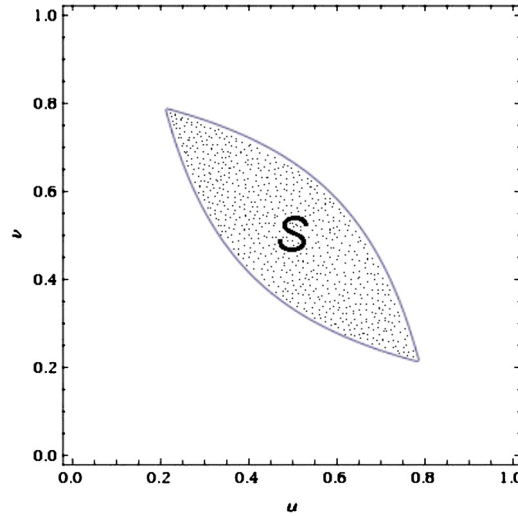(a) *$\underline{F}_t$ is a copula for every $t \in [-1/2, 1]$.*

Fig. 5. The set S in Theorem 9 for $t = 0$.

(b) $\underline{F}_t = W$ if, and only if, $t = -1/2$; and $\underline{F}_t = M$ if, and only if, $t = 1$.
(c) $\underline{F}_t$ is radially symmetric.

Proof. To prove part (a), first note that the boundary conditions are trivially satisfied since $\underline{F}_t$ is a quasi-copula. Now, it is clear that

$$\underline{F}_t(u, v) = \begin{cases} \dfrac{1}{2}\left(u + v - \sqrt{(u-v)^2 + 2/3(1-t)}\right), & uv \wedge (1-u)(1-v) \geqslant \dfrac{1-t}{6} \\ W(u, v), & \text{otherwise.} \end{cases}$$

Note that the set

$$S = \left\{ (u, v) \in [0, 1]^2 : uv \wedge (1-u)(1-v) \geqslant \frac{1-t}{6} \right\}$$

is the region between two symmetric hyperbolic arcs with respect to the line $v = 1 - u$, and cut at the points $\big((1 + \sqrt{(1+2t)/3})/2, (1 - \sqrt{(1+2t)/3})/2\big)$ and $\big((1 - \sqrt{(1+2t)/3})/2, (1 + \sqrt{(1+2t)/3})/2\big)$ (see Fig. 5). Therefore, the only $\underline{F}_t$-volumes to be studied are those of the rectangles that have some vertex in the interior of S—denoted by int(S)—, and in turn, the study of these volumes can be reduced to the case in which the four vertices of the rectangle are in int(S). But these volumes are non-negative since, if $(u, v) \in$ int(S), then we have

$$\frac{\partial^2 \underline{F}_t}{\partial u \partial v}(u, v) = \frac{\sqrt{3}(1-t)}{\big(3(u-v)^2 + 2(1-t)\big)^{3/2}} \geqslant 0,$$

i.e., $\underline{F}_t$ is 2-increasing at such points, and hence $\underline{F}_t$ is a copula.

To prove $\underline{F}_t = W$ if, and only if, $t = -1/2$ in part (b), let $(u, v) \in [0, 1]^2$ and suppose $u + v \leqslant 1$. Then, by using the arithmetic-geometric mean inequality, we have $uv \leqslant 1/4$, which is equivalent to $(u+v)^2 - (u-v)^2 \leqslant 1$, hence it is easy to derive

$$\frac{1}{2}\left(u + v - \sqrt{(u-v)^2 + 1}\right) \leqslant 0.$$

If $u + v > 1$, or equivalently $(1 - u) + (1 - v) < 1$, then by using again the arithmetic-geometric mean inequality we have $(1 - u)(1 - v) < 1/4$, i.e. $(2 - 2u)(2 - 2v) < 1$, which is equivalent to $(2 - u - v)^2 < (u - v)^2 + 1$, i.e.

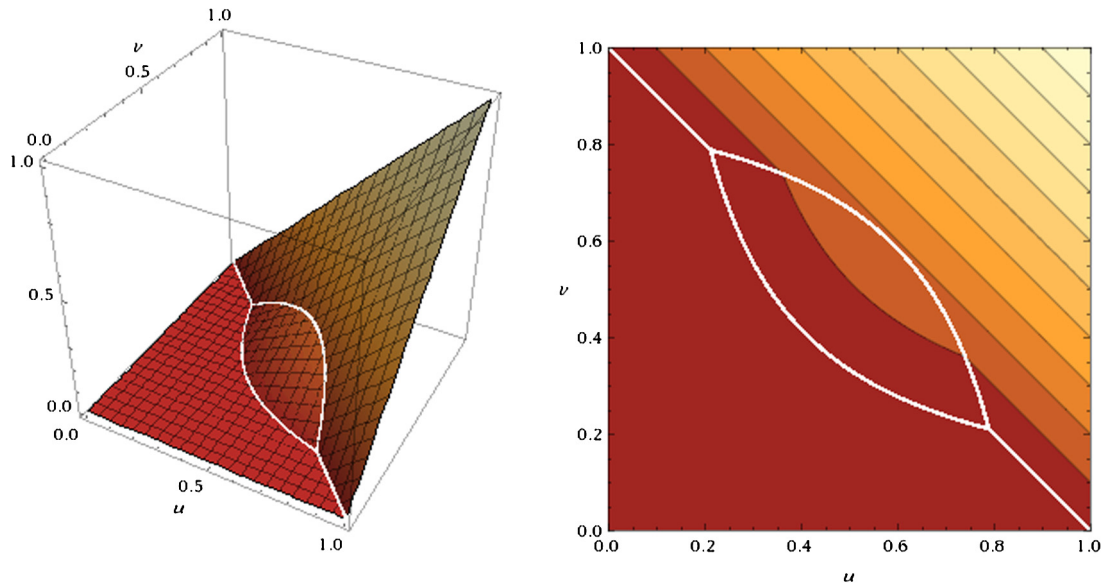$$\frac{1}{2}\left(u + v - \sqrt{(u-v)^2 + 1}\right) < u + v - 1;$$

Fig. 6. The graph (left) and the level curves (right) of the copula $\underline{F}_0$.

therefore, we obtain $\underline{F}_{-1/2} = W$. Now, note $\underline{F}_t(1/2, 1/2) = 1/2\left(1 - \sqrt{2/3(1-t)}\right) > 0$ if, and only if, $t > -1/2$; whence $\underline{F}_t = W$ if, and only if, $t = -1/2$.

On the other hand, for $u \in (0, 1)$, we have $\underline{F}_t(u, u) = u$ if, and only if, $1/2\left(2u - \sqrt{2/3(1-t)}\right) = u$, i.e. $t = 1$; whence $\underline{F}_t = M$ if, and only if, $t = 1$.

Finally, part (c) can be easily proved by using elementary algebra, hence the proof is complete. □

Fig. 6 shows the graph and the level curves of the copula $\underline{F}_0$.

Theorem 10. *The quasi-copula* $\overline{F}_t$ *given by* (12) *is the copula M for* $t \in [1/4, 1]$ *and a proper quasi-copula for* $t \in [-1/2, 1/4[$.

Proof. First let us establish $\overline{F}_t = M$ for $t = 1/4$. Notice that in this case the function f_4 given by (10) can be written as

$$f_4(u, v; 1/4) = \frac{u + v - 1 + \sqrt{u^2 + v^2 + (1-u)^2 + (1-v)^2 + (u-v)^2}}{2}.$$

The expression under the square root is a convex quadratic function with the minimum at the point $(1/2, 1/2)$ reaching 1. Therefore $f_4(u, v; 1/4) \geqslant (u + v)/2 \geqslant \min(u, v)$, and hence $\overline{F}_{1/4}(u, v) = \min(u, v)$. Since the function f_4 is monotone increasing with t, its maximum is increasing as well, and hence it remains larger than $\min(u, v)$ for $1/4 < t \leqslant 1$.

To prove that $\overline{F}_t$ is a proper quasi-copula for $t < 1/4$, note first that it can be written as

$$\overline{F}_t(u, v) = \begin{cases} \dfrac{1}{2}\left(u + v - 1 + \sqrt{\dfrac{(3(u+v) - 1)^2 - 24uv + 4(1+t)}{3}}\right), & u^2 + v^2 - 2(u \wedge v) \leqslant -\dfrac{1+2t}{3} \\ M(u, v), & \text{otherwise.} \end{cases}$$

Note that the set

$$T = \left\{(u, v) \in [0, 1]^2 : u^2 + v^2 - 2(u \wedge v) \leqslant -\frac{1+2t}{3}\right\}$$

is the region between two symmetric hyperbolic arcs with respect to the line $v = u$, and cut at the points $\left((1 + \sqrt{(1-4t)/3})/2, (1 + \sqrt{(1-4t)/3})/2\right)$ and $\left((1 - \sqrt{(1-4t)/3})/2, (1 - \sqrt{(1-4t)/3})/2\right)$ (see Fig. 7). Now,
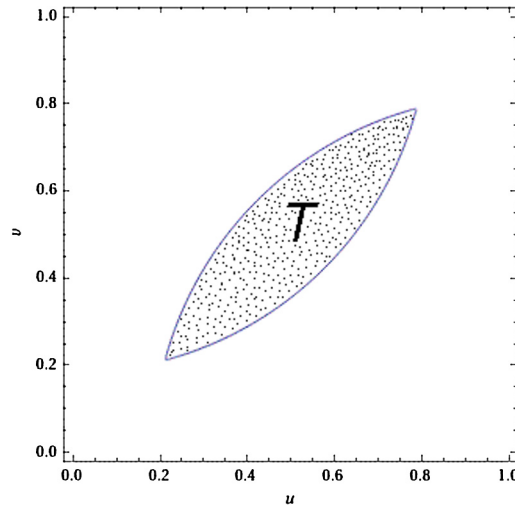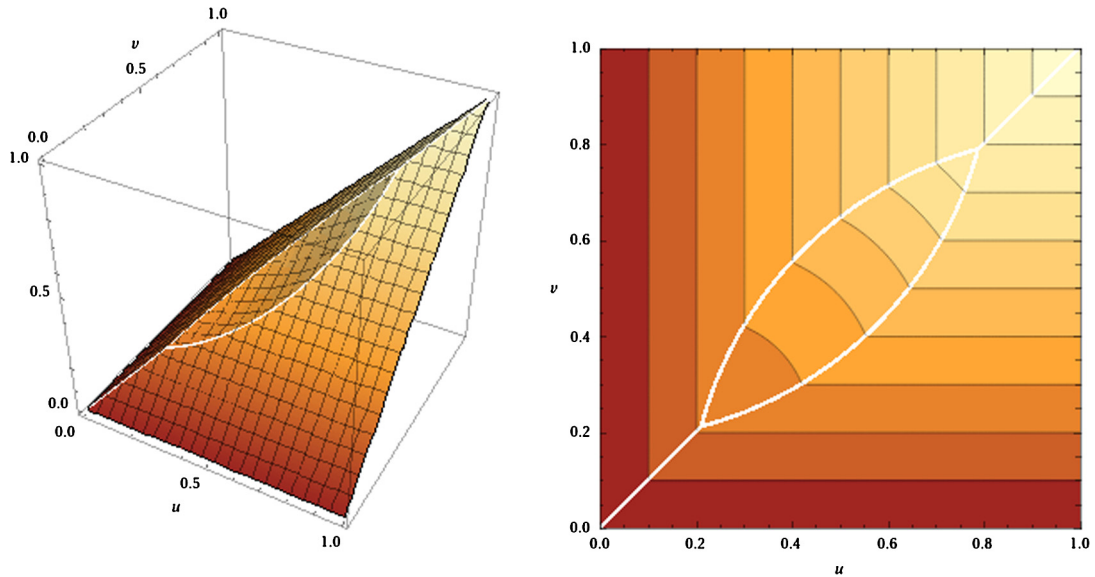
Fig. 7. The set T in Theorem 10 for $t = 0$.

Fig. 8. The graph (left) and the level curves (right) of the proper quasi-copula $\overline{F}_0$.

for $t = 1/4 - \delta$, with $0 < \delta \leqslant 3/4$, we only need to check the $\overline{F}_t$-volumes for rectangles in int(T). We consider the rectangle $R = [1/2 - \varepsilon, 1/2 + \varepsilon]^2$ with $\varepsilon > 0$ (we note that it is necessary $\varepsilon < \sqrt{(1-t)/3} - 1/2 = \sqrt{1/4 + \delta/3} - 1/2 \leqslant (\sqrt{2} - 1)/2 \approx 0.2071$). Then we have

$$V_{\overline{F}_t}(R) = \frac{\sqrt{6}}{3}\left(\sqrt{6\varepsilon^2 + 2t + 1} - \sqrt{12\varepsilon^2 + 2t + 1}\right) = \frac{\sqrt{3}}{3}\left(\sqrt{12\varepsilon^2 - 4\delta + 3} - \sqrt{24\varepsilon^2 - 4\delta + 3}\right).$$

Taking, for instance, $\delta = 6\varepsilon$ we obtain

$$V_{\overline{F}_t}(R) = \sqrt{4\varepsilon^2 - 8\varepsilon + 1} - \sqrt{8\varepsilon^2 - 8\varepsilon + 1} < 0,$$

i.e., for any $\delta \in]0, 3/4]$ we can always select such $\varepsilon = \delta/6$ (observe that $0 < \varepsilon < 1/8 = 0.125$) that the $\overline{F}_t$-volume is negative, which completes the proof. $\square$

Fig. 8 shows the graph and the level curves of the proper quasi-copula $\overline{F}_0$.

4. Conclusions

In this paper we have found best-possible bounds on the set of copulas with a given value of the Spearman's footrule coefficient. We have proved that the pointwise infimum $\underline{F}_t$ is always a copula; but the pointwise supremum $\overline{F}_t$ can be a copula (for $1/4 \leqslant t \leqslant 1$) or a proper quasi-copula ($-1/2 \leqslant t < 1/4$). We stress the fact that this is not the case of best-possible bounds given a value of other measures, such as Kendall's tau, Spearman's rho or Blomqvist's beta, for which these bounds are always copulas.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgements

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