COMPARISON PRINCIPLE FOR ELLIPTIC EQUATIONS IN DIVERGENCE WITH SINGULAR LOWER ORDER TERMS HAVING NATURAL GROWTH

DAVID ARCOYA, JOSÉ CARMONA, AND PEDRO J. MARTÍNEZ-APARICIO

ABSTRACT. In this paper we are concerned with the zero Dirichlet boundary value problem associated to the quasilinear elliptic equation

$$-\mathrm{div}(a(u)M(x)\nabla u) + H(x, u, \nabla u) = f(x), \quad x \in \Omega,$$

where Ω is an open and bounded set in $\mathbb{R}^N$ $(N \geq 3)$, a is a continuously differentiable real function in $(0, +\infty)$, $M(x)$ is an elliptic, bounded and symmetric matrix, $H(x, \cdot, \xi)$ is nonnegative and may be singular at zero and $f \in L^1(\Omega)$. We give sufficient conditions on H, M and a in order to have a comparison principle and, as a consequence, uniqueness of positive solutions being continuous up to the boundary.

1. INTRODUCTION

Let Ω be an open and bounded set in $\mathbb{R}^N$ $(N \geq 3)$ and $f \in L^1(\Omega)$. We consider the following boundary value problem

$$\begin{cases} -\mathrm{div}(a(u)M(x)\nabla u) + H(x, u, \nabla u) = f(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $M(x)$ is a symmetric matrix satisfying, for some $\alpha, \beta > 0$, that

$$\alpha|\xi|^2 \leq M(x)\xi \cdot \xi \leq \beta|\xi|^2, \forall \xi \in \mathbb{R}^N. \tag{1.2}$$

The function $a : (0, +\infty) \to \mathbb{R}$ is continuously differentiable and positive and $H : \Omega \times (0, +\infty) \times \mathbb{R}^N \to \mathbb{R}$ is a nonnegative Carathéodory function such that for a.e. $x \in \Omega$, $H(x, \cdot, \cdot)$ is continuously differentiable and

$$H(x, s, t\xi) = t^2 H(x, s, \xi), \quad \forall s > 0, \forall t \in \mathbb{R}, \forall \xi \in \mathbb{R}^+. \tag{1.3}$$

A comparison principle for general differential operators of the form

$$-\mathrm{div}(a(x, u, \nabla u)) + H(x, u, \nabla u)$$

is established in [6, Theorem 1.2, Theorem 2.1 and Theorem 2.3 for the bounded case]. In the case $a(x, s, \xi) = \xi$, conditions imposed to H in [6] imply in particular that $\partial_s H \geq 0$. Moreover, as it was observed in Remark 2.5 of that paper, the maximum principle still holds in various situations even when $\partial_s H < 0$ and it would be desirable to find convenient structure conditions on H including some particular cases where $\partial_s H < 0$. A slightly improvement of these conditions can be found in [5] for f small enough and, once again, it is required that $\partial_s H \geq 0$.

Research supported by MEC-FEDER (Spain) MTM2012- 31799 and Junta de Andalucia FQM-116 and FQM-194.

A different kind of comparison principle is proved in [4] where $M(x) = I$, the identity matrix, and $H(x, u, \nabla u) = g(u)|\nabla u|^2$ for some nonnegative continuous function g in $(0, +\infty)$. In this case, the authors imposed the integrability of $\frac{g(s)}{a(s)}$ at zero. This result handles the case that g is singular at zero (which necessarily implies that $\partial_s H \not\geq 0$). However, their techniques require strongly that the function H and the differential operator do not depend on x.

Some further extension, dealing with uniqueness, was done in [2] in the case $a(s) = 1$, $M(x) = I$ and $H(x, u, \nabla u) = -d(x)u - \mu(x)|\nabla u|^2 - h(x)$ for some $d, h \in L^p(\Omega)$, $p > N/2$, $d \leq 0$ and $\mu \in L^\infty(\Omega)$ (see [3] for an slightly improvement with general $M(x)$ and more general function H non decreasing on the variable s). It is once again imposed that $\partial_s H \geq 0$. Moreover, in some particular cases, with $\partial_s H < 0$ ($d(x) > 0$) they prove a multiplicity result (see Theorem 1.3 in [2]), that is, no uniqueness result is expected imposing only that $\partial_s H < 0$.

More recently, in [1] it is proved a comparison principle for (1.1) in the case $a(s) = 1$ and $M(x) = I$ for a particular class of functions $H(x, u, \nabla u)$ which are continuous at $u = 0$ and that may be decreasing on u.

The aim of this paper is to improve the above comparison principles in some directions: general matices $M(x)$, dependence on x and singularity at $u = 0$ on the gradient quadratic part.

Let us illustrate our main result in the case $H(x, s, \xi) = h(x, s)|\xi|^2$, although we give a more general structure condition for H in section 2. More precisely, consider the boundary value problem:

$$
\begin{cases}
-\mathrm{div}(a(u)M(x)\nabla u) + h(x, u)|\nabla u|^2 = f(x) & \text{in } \Omega \\
u = 0 & \text{on } \partial\Omega,
\end{cases}
\tag{1.4}
$$

for a differentiable Carathéodory function h defined in $\Omega \times (0, +\infty)$ (i.e., a Carathéodory function such that $h(x, \cdot)$ is derivable for a.e. $x \in \Omega$).

We say that $u \in H_0^1(\Omega)$ with $u > 0$ is a subsolution (respectively, a supersolution) of (1.4) if $a(u)M(x)\nabla u \in L^2(\Omega)^N$, $h(x, u)|\nabla u|^2 \in L^1(\Omega)$ and

$$
\int_\Omega a(u)M(x)\nabla u \nabla \phi + \int_\Omega h(x, u)|\nabla u|^2 \phi \leq \int_\Omega f\phi\,, \quad \forall \phi \in H_0^1(\Omega) \cap L^\infty(\Omega)
$$

(respectively, if the reverse inequality holds). Thus, u is called a solution provided that it is both a subsolution and a super solution. We prove the following theorem.

Theorem 1.1. *Assume* (1.2) *and that for every $\nu > 0$ there exist $\theta \geq 0$ and a nonnegative function $g \in C^1((0, +\infty))$ with $a(s)e^{-\int_1^s \frac{g(t)}{a(t)}dt} \in L^1(0, 1)$ such that for a.e. $x \in \Omega$ and for every $0 < s < \nu$, the matrix*

$$
\begin{aligned}
\theta[a(s)\partial_s(h(x, s)I - g(s)M(x)) + (g(s) - 2a'(s))(h(x, s)I - g(s)M(x))] \\
-M^{-1}(x))(h(x, s)I - g(s)M(x))^2
\end{aligned}
\tag{1.5}
$$

is positive semidefinite. If $0 < v_1, v_2 \in H_0^1(\Omega) \cap C(\overline{\Omega})$ are respectively a sub and a supersolution for (1.4) *then $v_1 \leq v_2$. As a consequence, we have uniqueness of $C(\overline{\Omega})$ solutions of* (1.4). □

Observe that if we assume that $f \in L^q(\Omega)$ for some $q > N/2$, $\inf a > 0$ and $\partial\Omega$ is smooth enough, in the sense of condition (A) in [7, p. 6], we have (see [7, Theorems 6.1 and 7.1 of Chapter 2]) that any solution of (1.1) belongs to $C(\overline{\Omega})$.

Hence with this assumption on the smoothness of the boundary we would obtain from the above theorem uniqueness of solutions of (1.4).

We prove in Section 2 a more general result than the above theorem. In Section 3 some corollaries (see Corollary 3.1, 3.2 and 3.5) of Theorem 1.1 are obtained when the term $h(x,s)$ is not necessary increasing in s and either nonsingular or singular. In particular, special emphasis will be put in the singular case. Indeed, we show (Corollary 3.5), as an application of the theorem that if $a(s)=1$, $M(x)=I$ and $h(x,s)=\mu(x)/s^\gamma$ with $0<\mu_1\le\mu(x)\le\mu_2$ and $0<\gamma<1$, that the comparison principle holds. This improves [4] since dependence on x is allowed in $\mu(x)$. Even more, if $\mu(x)$ is a constant $m<1$, we also improve [4] since we can also handle (see Remark 3.4) the case $h(x,s)=m/s$ which was uncovered by [4].

2. Comparison principle

In this section we prove our main result. For the statement of our main result let us recall that $u\in H_0^1(\Omega)$ with $u>0$ is a subsolution (respectively, a supersolution) of (1.1) if $a(u)M(x)\nabla u\in L^2(\Omega)^N$, $H(x,u,\nabla u)\in L^1(\Omega)$ and

$$\int_\Omega a(u)M(x)\nabla u\nabla\phi+\int_\Omega H(x,u,\nabla u)\phi\le\int_\Omega f\phi\,,\quad \forall\phi\in H_0^1(\Omega)\cap L^\infty(\Omega)$$

(respectively, if the reverse inequality holds). If u is a subsolution and a supersolution then it is called a solution.

Theorem 2.1. *Assume* (1.2), (1.3) *and that for every $\nu>0$ there exist $\theta\ge 0$ and a nonnegative function $g\in C^1((0,+\infty))$, with $a(s)e^{-\int_1^s\frac{g(t)}{a(t)}dt}\in L^1(0,1)$, such that for almost everywhere $x\in\Omega$ and for every $0<s<\nu$ and $\xi\in\mathbb{R}^N$*

$$\begin{aligned}(2.1)\quad & a(s)(\partial_s H(x,s,\xi)-g'(s)M(x)\xi\cdot\xi)\\ &+(g(s)-2a'(s))(H(x,s,\xi)-g(s)M(x)\xi\cdot\xi)-\frac{1}{\theta}\Theta(x,s,\xi)\ge 0\end{aligned}$$

where

$$\Theta(x,s,\xi):=\frac14 M^{-1}(x)(\partial_\xi H(x,s,\xi)-2g(s)M(x)\xi)\cdot(\partial_\xi H(x,s,\xi)-2g(s)M(x)\xi).$$

If $0<v_1,v_2\in H_0^1(\Omega)\cap C(\overline{\Omega})$ are respectively a sub and a supersolution for (1.1), *then $v_1\le v_2$. As a consequence, we have uniqueness of $C(\overline{\Omega})$-solution v of* (1.1).

Proof. For every $s\ge 0$, we use the following notation $\gamma(s)=\int_1^s\frac{g(t)}{a(t)}dt$, $\psi(s)=\int_0^s a(t)e^{-\gamma(t)}dt$, and $G_\varepsilon(s)=(s-\varepsilon)^+$ $(\varepsilon>0)$. If $v_1,v_2\in H_0^1(\Omega)\cap C(\overline{\Omega})$ are respectively a sub and a supersolution for (1.1), we define $w=\psi(v_1)-\psi(v_2)$.

We observe that, since $\psi\in C([0,+\infty))$ and v_1,v_2 are continuous up to the boundary, the function $G_\varepsilon(w)$ has compact support $\Omega_\varepsilon:=\operatorname{supp}G_\varepsilon(w)$ in Ω, for every $\varepsilon>0$. Moreover, $e^{-\gamma(v_i)},\gamma'(v_i),\psi'(v_i)\in L^\infty(\Omega_\varepsilon)$ for $i=1,2$ and, even more, $G_\varepsilon(w)\in L^\infty(\Omega)$. Fix $\nu>\max\{\|v_1\|_{C(\Omega)},\|v_2\|_{C(\Omega)}\}$ and θ such that (2.1) is satisfied for every $0<s<\nu$. Thus, if n is the integer part of $\theta+1$, we can take $e^{-\gamma(v_1)}G_\varepsilon(w)^n$ as test function in the inequality satisfied by v_1 and $e^{-\gamma(v_2)}G_\varepsilon(w)^n$

in the inequality satisfied by v_2. Substracting and taking into account (1.2) we have

$$\begin{aligned}0\geq&\int_\Omega\left(-\gamma'(v_1)\psi'(v_1)M(x)\nabla v_1\cdot\nabla v_1+H(x,v_1,\nabla v_1)\frac{\psi'(v_1)}{a(v_1)}\right)G_\varepsilon(w)^n\\&-\int_\Omega\left(-\gamma'(v_2)\psi'(v_2)M(x)\nabla v_2\cdot\nabla v_2+H(x,v_2,\nabla v_2)\frac{\psi'(v_2)}{a(v_2)}\right)G_\varepsilon(w)^n\\&+n\int_\Omega G_\varepsilon(w)^{n-1}M(x)(\psi'(v_1)\nabla v_1-\psi'(v_2)\nabla v_2)\cdot\nabla w.\end{aligned}$$

By the homogeneity condition (1.3), if $s=\psi^{-1}(t\psi(v_1)+(1-t)\psi(v_2))$ and $\xi=t\nabla\psi(v_1)+(1-t)\nabla\psi(v_2)$, this means that

$$\begin{aligned}0\geq&\int_{\{w>\varepsilon\}}G_\varepsilon(w)^n\int_0^1\frac{d}{dt}\left(\frac{-\gamma'(s)}{\psi'(s)}M(x)\xi\cdot\xi+\frac{H(x,s,\xi)}{a(s)\psi'(s)}\right)dt\\&+n\int_{\{w>\varepsilon\}}G_\varepsilon(w)^{n-1}M(x)\nabla w\cdot\nabla w.\end{aligned}$$

After deriving we get

$$\begin{aligned}0\geq&\int_{\{w>\varepsilon\}}wG_\varepsilon(w)^n\int_0^1\frac{-g'(s)a(s)+2g(s)a'(s)-g(s)^2}{a(s)^2\psi'(s)^2}M(x)\xi\cdot\xi\,dt\\&+\int_{\{w>\varepsilon\}}wG_\varepsilon(w)^n\int_0^1\frac{\partial_sH(x,s,\xi)a(s)-H(x,s,\xi)\left(2a'(s)-g(s)\right)}{a(s)^2\psi'(s)^2}\,dt\\&+\int_{\{w>\varepsilon\}}G_\varepsilon(w)^n\int_0^1\left[\frac{-2g(s)M(x)\xi+\partial_\xi H(x,s,\xi)}{a(s)\psi'(s)}\right]\cdot\nabla w dt\\&+n\int_{\{w>\varepsilon\}}G_\varepsilon(w)^{n-1}M(x)\nabla w\cdot\nabla w.\end{aligned}$$

Multiplying by $\frac{\theta}{n}$ and taking into account that, by Young's inequality,

$$\begin{aligned}&\frac{\theta}{n}\left|G_\varepsilon(w)^n\left[\frac{-2g(s)M(x)\xi+\partial_\xi H(x,s,\xi)}{a(s)\psi'(s)}\right]\cdot\nabla w\right|\\&\qquad\leq\frac{\theta^2}{n}G_\varepsilon(w)^{n-1}M(x)\nabla w\cdot\nabla w+\frac{G_\varepsilon(w)^{n+1}}{n}\frac{\Theta(x,s,\xi)}{a(s)^2\psi'(s)^2},\end{aligned}$$

it follows by (1.2) that

$$\begin{aligned}0\geq&\alpha\theta\left(1-\frac{\theta}{n}\right)\int_{\{w>\varepsilon\}}G_\varepsilon(w)^{n-1}|\nabla w|^2\\&+\int_{\{w>\varepsilon\}}\int_0^1\frac{wG_\varepsilon(w)^n\theta}{na^2(s)\psi'(s)^2}\Big[\partial_sH(x,s,\xi)a(s)-H(x,s,\xi)(2a'(s)-g(s))\\&+(-g'(s)a(s)+2g(s)a'(s)-g(s)^2)M(x)\xi\cdot\xi-\frac{G_\varepsilon(w)}{\theta w}\Theta(x,s,\xi)\Big]dt.\end{aligned}$$

Since $G_\varepsilon(w)/w\leq 1$, the integrand in the second integral is greater than zero by (2.1), and we deduce that the first integral is zero, which implies that $G_\varepsilon(w)=0$ for every $\varepsilon>0$, i.e., $w^+=0$, concluding the proof. □

Remark 2.2. Since we consider the case of functions $H(x,\cdot,\xi)$ and $a(\cdot)$ that may be singular at zero, we have imposed the $C(\overline{\Omega})$-regularity of the subsolution v_1 and the supersolution v_2 in the previous theorem. This regularity is just used to guarantee that the function $e^{-\gamma(v_2)}(G_\varepsilon((\psi(v_1)-\psi(v_2))^+))^n$ has compact support (γ, ψ and n are introduced in the proof).

We observe that, when we only have that $v_1 \in H_0^1(\Omega)\cap L^\infty(\Omega)$ and $v_2 \in H_0^1(\Omega)$, the same proof works provided that the functions $e^{-\gamma(v_i)}((\psi(v_1)-\psi(v_2))^+)^n, \psi(v_i) \in H_0^1(\Omega)$, $i=1,2$. This is true, for example, if in (1.1) does not appear any singular term. Moreover, if $\frac{g(s)}{a(s)}$ and $a(s)$ are integrable at zero, a slightly modification can be performed in the proof by taking $e^{-\gamma(v_i+\varepsilon)}((\psi(v_1)-\psi(v_2))^+)^n \in H_0^1(\Omega)$ and passing to the limit as ε tends to zero to state the comparison principle for bounded sub and supersolutions.

Remark 2.3. As it has been observed just after Theorem 1.1, if $f \in L^q(\Omega)$ for some $q > N/2$, $\inf a > 0$ and $\partial\Omega$ is smooth then any solution of (1.1) belongs to $C(\overline{\Omega})$.

3. Consequences

In this section we use Theorem 2.1 to prove a comparison principle for some model problems in which $H(x,s,\xi) = h(x,s)|\xi|^2$.

In this case, as it has been mentioned in the Introduction, Theorem 1.1 correspond to rewrite Theorem 2.1 into this context. As a first particular case of Theorem 1.1, we study the case in which $h(x,s)$ does not depends on s, i.e., $H(x,s,\xi) = \mu(x)|\xi|^2$ with $0 < \mu_1 \le \mu(x) \le \mu_2$ (notice that $\partial_s H = 0$ in this case).

Corollary 3.1. *Assume* (1.2) *and that* $0<\mu_1\le\mu(x)\le\mu_2$, *a.e.* $x\in\Omega$. *Suppose also that there exist positive real numbers* a_1, a_2, a_3 *such that*

$$0 < a_1 \le a(s), \quad -a_3 \le a'(s) \le a_2 < \frac{\mu_1}{2\beta}, \quad \forall s > 0. \tag{3.1}$$

If $0 < v_1, v_2 \in H_0^1(\Omega)\cap C(\overline{\Omega})$ *are respectively a sub and a supersolution for*

$$\begin{cases} -\mathrm{div}(a(u)M(x)\nabla u) + \mu(x)|\nabla u|^2 = f(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

then $v_1 \le v_2$.

Proof. This is a direct consequence of Theorem 1.1 with $g(s) = m > 0$, for every $s>0$, where $2a_2 < m < \frac{\mu_1}{\beta}$. Indeed, observe that $a(s)e^{-\int_1^s \frac{g(t)}{a(t)}dt} \in L^1(0,1)$ by (3.1). In addition, from (1.2), there exists $\lambda > 0$ such that,

$$\begin{aligned} M^{-1}(x)(\mu(x)I - mM(x))^2\xi\cdot\xi \le& \beta|M^{-1}(x)(\mu(x)I-mM(x))\xi|^2 \\ \le& \beta(\mu_2^2|M^{-1}(x)|^2 + m^2 + 2m\mu_2|M^{-1}(x)|)|\xi|^2 \\ \le& \beta(\mu_2^2\lambda^2 + m^2 + 2m\mu_2\lambda)|\xi|^2. \end{aligned}$$

Moreover, it follows that $(\mu(x)I - mM(x))\xi\cdot\xi \ge (\mu_1 - m\beta)|\xi|^2$. We deduce that if

$$\theta > \beta\frac{\mu_2^2\lambda^2 + m^2 + 2m\mu_2\lambda}{(m-2a_2)(\mu_1 - m\beta)},$$

then

$$\theta(m-2a'(s))(\mu(x)I - mM(x))\xi\cdot\xi \ge M^{-1}(x)(\mu(x)I-mM(x))^2\xi\cdot\xi, \quad \forall\xi\in\mathbb{R}^N,$$

which means that the matrix given by (1.5) is positive semidefinite in this case. $\square$

Similarly, we consider the case in which $h(x,s)$ depends only in s, that is, when $H(x,s,\xi)=h(s)\xi^2$.

Corollary 3.2. *Let* (1.2) *be satisfied and assume that there exist positive functions* $a,h\in C^1((0,+\infty))$ *and* $c>0$ *such that* $a(s)e^{-c\int_1^s \frac{h(t)}{a(t)}dt}\in L^1(0,1)$ *and for some* $\tau>0$

$$\left(\frac{h'(s)}{h(s)^2}a(s)+c-\frac{2a'(s)}{h(s)}\right)(I-cM(x))\xi\cdot\xi\geq\tau|\xi|^2, \tag{3.2}$$

for every $s>0$, $\xi\in\mathbb{R}^N$ *and a.e.* $x\in\Omega$. *If* $0<v_1,v_2\in H_0^1(\Omega)\cap C(\overline{\Omega})$ *are respectively a sub and a supersolution for*

$$\begin{cases}-\mathrm{div}(a(u)M(x)\nabla u)+h(u)|\nabla u|^2=f(x) & \text{in } \Omega\\ u=0 & \text{on } \partial\Omega,\end{cases}$$

then $v_1\leq v_2$.

Proof. The result is a direct consequence of Theorem 1.1 with $g(s)=c\,h(s)$, observe that $a(s)e^{-c\int_1^s \frac{h(t)}{a(t)}dt}\in L^1(0,1)$. On the other hand, using (1.2), we deduce that there exists $\lambda>0$ with

$$\begin{aligned}M^{-1}(x)(I-cM(x))^2\xi\cdot\xi\leq&\beta|M^{-1}(x)(I-cM(x))\xi|^2\\ \leq&\beta(|M^{-1}(x)|^2+c^2+2c|M^{-1}(x)|)|\xi|^2\\ \leq&\beta(\lambda^2+c^2+2c\lambda)|\xi|^2.\end{aligned}$$

Thus, if

$$\theta>\frac{\beta(\lambda^2+c^2+2c\lambda)}{\tau},$$

then by (3.2) we have

$$\theta\left(\frac{h'(s)}{h(s)^2}a(s)+c-\frac{2a'(s)}{h(s)}\right)(I-cM(x))\xi\cdot\xi-M^{-1}(x)(I-cM(x))^2\xi\cdot\xi\geq0, \tag{3.3}$$

which implies that, in this case, the matrix given by (1.5) is positive semidefinite and the proof is finished. □

Remark 3.3. Observe that if $c<1/\beta$, then $(I-cM(x))\xi\cdot\xi\geq(1-c\beta)|\xi|^2\geq0$, while if $c>\frac{1}{\alpha}$ we have $(I-cM(x))\xi\cdot\xi\leq(1-c\alpha)|\xi|^2\leq0$. Thus, condition (3.2) is satisfied provided that there exists a positive constant τ such that either

$$\left(\frac{h'(s)}{h(s)^2}a(s)+c-\frac{2a'(s)}{h(s)}\right)=-\left(\frac{a(s)^2}{h(s)}\right)'\frac{1}{a(s)}+c\geq\tau,\ \text{if } c<\frac{1}{\beta},$$

or

$$\left(\frac{a(s)^2}{h(s)}\right)'\frac{1}{a(s)}-c\geq\tau,\ \text{if } c>\frac{1}{\alpha}.$$

In particular, in the case $0<c<1/\beta$, hypothesis (3.2) is satisfied if the function $\frac{a(s)^2}{h(s)}$ is non increasing.

Remark 3.4. Although (3.2) is not satisfied, we observe that (3.3) in the proof of Corollary 3.2 is clearly satisfied in the case $a(s)=1$, $M(x)=I$ with $c=1$. In particular we can deduce that, if $h\in C^1((0,+\infty))$ is a positive function such that

$e^{-\int_1^s h(t)dt} \in L^1(0,1)$ and $0 < v_1, v_2 \in H_0^1(\Omega) \cap C(\overline{\Omega})$ are respectively a sub and a supersolution for

$$\begin{cases} -\Delta u + h(u)|\nabla u|^2 = f(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

then $v_1 \leq v_2$. For example, we deduce the same comparison principle obtained in [4] for the case $h(s) = \frac{m}{s^\gamma}$ with $0 < \gamma < 1$ and $m > 0$. Even more, we can also deal with $h(s) = \frac{m}{s}$ with $0 < m < 1$.

The case in which the singularity is depending also on the x variable is particularly interesting. We obtain several improvements with respect to the singular problem studied in [4] where the authors assume that the quadratic part in ∇u does not depend on s. Specifically, we take $a(s) = 1$, $M(x) = I$, $H(x, s, \xi) = \mu(x)\frac{|\xi|^2}{s^\gamma}$ with $0 < \gamma < 1$, in order to study the problem

$$\begin{cases} -\Delta u + \mu(x)\dfrac{|\nabla u|^2}{u^\gamma} = f(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.4}$$

Corollary 3.5. *Assume that* $0 < \gamma < 1$, $0 < \mu(x) \in L^\infty(\Omega)$ *and* $0 \leq f \in L^1(\Omega)$. *If* $0 \leq v_1, v_2 \in H_0^1(\Omega) \cap C(\overline{\Omega})$ *are respectively a sub and a supersolution for* (3.4), *then* $v_1 \leq v_2$.

Proof. Choose $0 < \gamma < d < 1$ and $C > 0$ such that

$$\|\mu\|_\infty \leq \min\left\{ dC, C\left(\frac{d-\gamma}{1-\gamma}\right)^{1-\gamma} \right\}. \tag{3.5}$$

Consider the function g given by

$$g(s) = \begin{cases} \dfrac{dC}{s^\gamma}, & s < \left(\frac{\gamma}{C}\right)^{\frac{1}{1-\gamma}} \\ \dfrac{d\gamma}{\gamma s + \left(\frac{\gamma}{C}\right)^{\frac{1}{1-\gamma}}(1-\gamma)}, & s \geq \left(\frac{\gamma}{C}\right)^{\frac{1}{1-\gamma}}, \end{cases}$$

for every $s > 0$. Notice that $g \in C^1(0, +\infty)$ with

$$g'(s) = \begin{cases} -\dfrac{\gamma dC}{s^{\gamma+1}}, & s < \left(\frac{\gamma}{C}\right)^{\frac{1}{1-\gamma}} \\ -\dfrac{d\gamma^2}{\left(\gamma s + \left(\frac{\gamma}{C}\right)^{\frac{1}{1-\gamma}}(1-\gamma)\right)^2}, & s \geq \left(\frac{\gamma}{C}\right)^{\frac{1}{1-\gamma}}, \end{cases}$$

and $e^{-\int_1^s g(t)dt} \in L^1(0,1)$. Thus, in order to apply Theorem 1.1 with this choice of function g, we just have to prove that for every $\nu > 0$ there exists $\theta > 0$ for which the matrix given by (1.5) is positive semidefinite for every $0 < s < \nu$ and a.e. $x \in \Omega$, or equivalently, that

$$s^{2\gamma}g'(s) + s^{2\gamma}g(s)^2 - s^\gamma\mu(x)g(s) + \gamma s^{\gamma-1}\mu(x) + \frac{1}{\theta}(\mu(x) - s^\gamma g(s))^2 \leq 0, \tag{3.6}$$

for every $0 < s < \nu$ and a.e. $x \in \Omega$.

To make it, we take for every fixed $\nu > 0$,

(3.7)
$$\theta > \max\left\{ \frac{dC + \|\mu\|_\infty}{C(1-d)}, \frac{2d\left(\|\mu\|_\infty^2 \nu^{2(1-\gamma)} + \gamma^2\right)}{(1-d)\gamma^2}, \frac{2\|\mu\|_\infty^2 \left(\frac{1-\gamma}{d-\gamma}\right)^{2(1-\gamma)} + 2d^2C^2}{d(1-d)C^2\left(1 - \frac{\|\mu\|_\infty}{C}\left(\frac{1-\gamma}{d-\gamma}\right)^{1-\gamma}\right)} \right\}$$

and we show that (3.6) is satisfied by dividing the verification in three cases for $s \in (0, \beta)$:

i) If $s < \left(\frac{\gamma}{C}\right)^{\frac{1}{1-\gamma}}$, then
$$\frac{\gamma}{s^{1-\gamma}} - dC + \frac{1}{\theta}(\mu(x) - dC) \geq C - dC + \frac{1}{\theta}(\mu(x) - dC).$$
By (3.7), $\theta > \frac{dC - \mu(x)}{C - dC}$ and we deduce that
$$\frac{\gamma}{s^{1-\gamma}} - dC + \frac{1}{\theta}(\mu(x) - dC) > 0.$$
and, since $\|\mu\|_\infty \leq dC$ (by (3.5)), that
$$(\mu(x) - dC)\left(\frac{\gamma}{s^{1-\gamma}} - dC + \frac{1}{\theta}(\mu(x) - dC)\right) \leq 0,$$
which is (3.6) for $s \in \left(0, \left(\frac{\gamma}{C}\right)^{\frac{1}{1-\gamma}}\right)$.

ii) If $\left(\frac{\gamma}{C}\right)^{\frac{1}{1-\gamma}} \leq s \leq \frac{1-\gamma}{d-\gamma}\left(\frac{\gamma}{C}\right)^{\frac{1}{1-\gamma}}$, using that $(1-\gamma)\left(\frac{\gamma}{C}\right)^{\frac{1}{1-\gamma}} s^{-1}$ is decreasing in s and that $\|\mu\|_\infty < C\left(\frac{d-\gamma}{1-\gamma}\right)^{1-\gamma}$, we have
$$(d-1)d\gamma^2 + \frac{\gamma\mu(x)}{s^\gamma}\left(\gamma s + (1-\gamma)\left(\frac{\gamma}{C}\right)^{\frac{1}{1-\gamma}}\right)\left(\gamma - d + (1-\gamma)\left(\frac{\gamma}{C}\right)^{\frac{1}{1-\gamma}} s^{-1}\right)$$
$$\leq (d-1)d\gamma^2\left(1 - \frac{\mu(x)}{C}\left(\frac{1-\gamma}{d-\gamma}\right)^{1-\gamma}\right) < 0.$$
Thus, using (3.7), we also obtain, for $\left(\frac{\gamma}{C}\right)^{\frac{1}{1-\gamma}} \leq s \leq \frac{1-\gamma}{d-\gamma}\left(\frac{\gamma}{C}\right)^{\frac{1}{1-\gamma}}$ that
$$(d-1)d\gamma^2 + \frac{\gamma\mu(x)}{s^\gamma}\left(\gamma s + (1-\gamma)\left(\frac{\gamma}{C}\right)^{\frac{1}{1-\gamma}}\right)\left(\gamma - d + (1-\gamma)\left(\frac{\gamma}{C}\right)^{\frac{1}{1-\gamma}} s^{-1}\right)$$
$$+ \frac{1}{\theta}\left\{\frac{\mu(x)}{s^\gamma}\left[\gamma s + (1-\gamma)\left(\frac{\gamma}{C}\right)^{\frac{1}{1-\gamma}}\right] - d\gamma\right\}^2 \leq 0.$$
Hence, multiplying by $s^{2\gamma}\left(\gamma s + (1-\gamma)\left(\frac{\gamma}{C}\right)^{\frac{1}{1-\gamma}}\right)^{-2}$, we get that (3.6) holds true in this case.

iii) If $\frac{1-\gamma}{d-\gamma}\left(\frac{\gamma}{C}\right)^{\frac{1}{1-\gamma}} < s < \nu$, then $ds \geq \gamma s + (1-\gamma)\left(\frac{\gamma}{C}\right)^{\frac{1}{1-\gamma}}$ and
$$-\frac{d\gamma s^\gamma \mu(x)}{\gamma s + (1-\gamma)\left(\frac{\gamma}{C}\right)^{\frac{1}{1-\gamma}}} + \gamma s^{\gamma-1}\mu(x) \leq 0. \tag{3.8}$$
Furthermore, since $(d-1)d\gamma^2 < 0$ and
$$\left|\frac{\mu(x)}{s^\gamma}\left[\gamma s + (1-\gamma)\left(\frac{\gamma}{C}\right)^{\frac{1}{1-\gamma}}\right]\right| \leq \|\mu\|_\infty d\nu^{1-\gamma},$$

for every $s \in \left(\frac{1-\gamma}{d-\gamma}\left(\frac{\gamma}{C}\right)^{\frac{1}{1-\gamma}}, \nu\right)$, we also obtain by (3.7) that

$$(d-1)d\gamma^2 + \frac{1}{\theta}\left\{\frac{\mu(x)}{s^\gamma}\left[\gamma s + (1-\gamma)\left(\frac{\gamma}{C}\right)^{\frac{1}{1-\gamma}}\right] - d\gamma\right\}^2 \leq 0,$$

and consequently, multiplying by $s^{2\gamma}\left(\gamma s + (1-\gamma)\left(\frac{\gamma}{C}\right)^{\frac{1}{1-\gamma}}\right)^{-2}$, we get

$$\frac{d(d-1)s^{2\gamma}}{\left(s+(1-\gamma)\gamma^{\frac{\gamma}{1-\gamma}}C^{\frac{-1}{1-\gamma}}\right)^2} + \frac{1}{\theta}\left(\mu(x) - \frac{ds^\gamma}{s+(1-\gamma)\gamma^{\frac{\gamma}{1-\gamma}}C^{\frac{-1}{1-\gamma}}}\right)^2 \leq 0.$$

This and (3.8) imply that

$$\frac{d(d-1)s^{2\gamma}}{\left(s+(1-\gamma)\gamma^{\frac{\gamma}{1-\gamma}}C^{\frac{-1}{1-\gamma}}\right)^2} - \frac{ds^\gamma\mu(x)}{s+(1-\gamma)\gamma^{\frac{\gamma}{1-\gamma}}C^{\frac{-1}{1-\gamma}}}$$
$$+ \gamma s^{\gamma-1}\mu(x) + \frac{1}{\theta}\left(\mu(x) - \frac{ds^\gamma}{s+(1-\gamma)\gamma^{\frac{\gamma}{1-\gamma}}C^{\frac{-1}{1-\gamma}}}\right)^2 \leq 0,$$

for every $s \in \left(\frac{1-\gamma}{d-\gamma}\left(\frac{\gamma}{C}\right)^{\frac{1}{1-\gamma}}, \nu\right)$, which means that (3.6) is satisfied in this case.

□

References

[1] D. Arcoya, J. Carmona and P. J. Martínez-Aparicio, *Gelfand type quasilinear elliptic problems with quadratic gradient terms*, *Ann. Inst. H. Poincarè Anal. Non Linéaire.*, **31** (2014) no. 2, 249–265.

[2] D. Arcoya, C. de Coster, L. JeanJean and K. Tanaka, *Continuum of solutions for an elliptic problem with critical growth in the gradient. J. Funct. Anal.*, **268** (2015) no. 8, 2298–2335.

[3] D. Arcoya, C. de Coster, L. JeanJean and K. Tanaka, *Remarks on the uniqueness for quasilinear elliptic equations with quadratic growth conditions. J. Math. Anal. Appl.*, **420** (2014) no. 1, 772–780.

[4] D. Arcoya and S. Segura de León, *Uniqueness of solutions for some elliptic equations with a quadratic gradient term. ESAIM Control Optim. Calc. Var.*, **16** (2010) no. 2, 327–336.

[5] G. Barles, A. P. Blanc, C. Georgelin and M. Kobylanski, *Remarks on the maximum principle for nonlinear elliptic PDEs with quadratic growth conditions. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* **28** (1999) no. 3, 381–404.

[6] G. Barles and F. Murat, *Uniqueness and the maximum principle for quasilinear elliptic equations with quadratic growth conditions. Arch. Rational Mech. Anal.* **133** (1995) no. 1, 77–101.

[7] O. Ladyzenskaya and N. Ural'tseva, *Linear and Quasilinear Elliptic Equations*, translated by Scripta Technica, Academic Press, New York, 1968.

(D. Arcoya) Departamento de Análisis Matemático, Campus Fuentenueva S/N, Universidad de Granada 18071 - Granada, Spain. e-mail: darcoya@ugr.es

(J. Carmona) Departamento de Matemáticas, Universidad de Almería, Ctra. Sacramento s/n, La Cañada de San Urbano, 04120 - Almería, Spain. e-mail: jcarmona@ual.es

(P. J. Martínez-Aparicio) Departamento de Matemática Aplicada y Estadística, Universidad Politécnica de Cartagena, 30202 - Murcia, Spain.
e-mail: pedroj.martinez@upct.es