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p-Compactness of Bloch maps

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Abstract

Influenced by the concept of a p-compact operator due to Sinha and Karn (Stud Math 150(1): 17–33, 2002), we introduce p-compact Bloch maps of the open unit disk $\mathbb{D} \subseteq \mathbb{C}$ to a complex Banach space X, and obtain its most outstanding properties: surjective Banach ideal property, Möbius invariance, linearisation on the Bloch-free Banach space over $\mathbb{D}$, inclusion properties, factorisation of their derivatives, and transposition on the normalized Bloch space. We also present right p-nuclear Bloch maps of $\mathbb{D}$ to X and study its relation with p-compact Bloch maps.

Keywords Vector-valued holomorphic function · Bloch function · p-Compact operator · p-Compact Bloch function

Mathematics Subject Classification 47B07 · 30H30 · 47B10 · 46E15 · 46E40

Introduction

Grothendieck proved that a subset of a Banach space is relatively compact if and only if it is included in the closed convex hull of a norm null sequence. Motivated by this result, Sinha and Karn [18] introduced the property of p-compactness in Banach spaces for $p \in [0, \infty]$. Associated with the notion of p-compact set, they initiated the study of p-compact operators between Banach spaces.

From then, p-compact sets and p-compact operators have been covered by various authors as, for example, Choi and Kim [5], Delgado et al. [7] and with Oja [6], Lassalle and Turco [12] and with Galicer [9], and Pietsch [17], among many others.

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The extension of the theory of p-compact operators to the non-linear context was developed by other authors, for instance, by Achour et al. [1] to the Lipschitz setting, and by Aron et al. [3] and Aron et al. [4] to both polynomial and holomorphic frames.

Our aim in this note is to address this theory in the Bloch setting. Our approach is also motivated by the introduction in [11] of the concept of compact Bloch map from the open unit disk $\mathbb{D} \subseteq \mathbb{C}$ into a complex Banach space X. A good reference for the theory of Bloch functions is the book [19] by Zhu. Let $\widehat{\mathcal{B}}(\mathbb{D}, X)$ be the Banach space of all Bloch maps f from $\mathbb{D}$ into X with $f(0) = 0$, under the Bloch norm $\rho_{\mathcal{B}}$.

We have divided this paper into two sections. After reviewing in Sect. 1 some notions on p-compact operators, Sect. 2 gathers the main properties of p-compact Bloch maps. If $\widehat{\mathcal{B}}_{\mathcal{K}_p}(\mathbb{D}, X)$ denotes the Banach space of all p-compact Bloch maps from $\mathbb{D}$ into X for which $f(0) = 0$, equipped with a suitable norm $k_p^{\mathcal{B}}$, we prove that $[\widehat{\mathcal{B}}_{\mathcal{K}_p}, k_p^{\mathcal{B}}]$ is a surjective Banach normalized Bloch ideal which becomes regular whenever X is reflexive. Moreover, $\widehat{\mathcal{B}}_{\mathcal{K}_\infty}(\mathbb{D}, X)$ coincides with $\widehat{\mathcal{B}}_{\mathcal{K}}(\mathbb{D}, X)$ (the space of all zero-preserving compact Bloch maps from $\mathbb{D}$ into X) and its norm $k_\infty^{\mathcal{B}}$ is equal to the Bloch norm $\rho_{\mathcal{B}}$, and so we extend here some results stated in [11].

Another striking property is the invariance by Möbius transformations of $\mathbb{D}$ of the p-compact Bloch maps from $\mathbb{D}$ into X. We refer to the paper [2] by Arazy, Fisher and Peetre for a first introduction to Möbius-invariant function spaces.

If $\mathcal{G}(\mathbb{D})$ denotes the Bloch-free Banach space over $\mathbb{D}$ presented in [11], we prove that a holomorphic map $f : \mathbb{D} \to X$ with $f(0) = 0$ is p-compact Bloch if and only if its linearisation $S_f : \mathcal{G}(\mathbb{D}) \to X$ is a p-compact operator. This fact will allow us to extend to the Bloch setting some similar results on p-compact operators. For instance, we prove that the derivative of every map $f \in \widehat{\mathcal{B}}_{\mathcal{K}_p}(\mathbb{D}, X)$ admits a factorization $f' = T \circ g'$, with $g \in \widehat{\mathcal{B}}(\mathbb{D}, Y)$ and $T \in \mathcal{K}_p(Y, X)$ for some complex Banach space Y. Furthermore, $k_p^{\mathcal{B}}(f)$ is equal to $\inf\{k_p(T)\rho_{\mathcal{B}}(g)\}$, being the infimum taken over all such representations of f' and, surprisingly, it is a maximum at the decomposition $S_f \circ \Gamma$ got in [11] (see Theorem 1.1 below).

In addition, we establish some inclusion relations of such spaces, factorize such derivatives through a quotient space of ℓ_{p^*} and characterize Bloch p-compact maps as those Bloch maps whose Bloch transposes are quasi p-nuclear operators (respectively, factor through a subspace of ℓ_p). We also introduce the term of right p-nuclear Bloch map from $\mathbb{D}$ into X, establish its Banach ideal structure and analyse its relation with p-compact Bloch maps.

1 Preliminaries

We first fix some notation and recall the basic concepts of the theory of p-compact sets and p-compact operators.

From now on, X and Y will denote complex Banach spaces. As usual, we denote the closed unit ball of X by B_X, the dual space of X by X^*, and the Banach space of all bounded linear operators from X into Y endowed with the operator canonical norm by $\mathcal{L}(X, Y)$. The subspaces of $\mathcal{L}(X, Y)$ formed by all compact operators and all finite-rank bounded operators from X into Y will be represented by $\mathcal{K}(X, Y)$ and

$\mathcal{F}(X, Y)$, respectively. The canonical isometric linear embedding of X into X^{**} is denoted by κ_X. Given a set $A \subseteq X$, $\overline{\text{aco}}(A)$ stands for the norm-closed absolutely convex hull of A.

Given $p \in [1, \infty)$, $\ell_p(X)$ denotes the Banach space of all absolutely p-summable sequences (x_n) in X, endowed with the norm

$$\|(x_n)\|_p = \left(\sum_{n=1}^{\infty} \|x_n\|^p \right)^{\frac{1}{p}},$$

and $c_0(X)$ is the Banach space of all norm null sequences in X, equipped with the norm

$$\|(x_n)\|_\infty = \sup \{\|x_n\| : n \in \mathbb{N}\}.$$

In the case of complex-valued sequences, we will just write ℓ_p and c_0, respectively.

For $p \in (1, \infty)$ and $p^* = p/(p-1)$, the *p-convex hull* of a sequence $(x_n) \in \ell_p(X)$ is defined by

$$p\text{-conv}(x_n) = \left\{ \sum_{n=1}^{\infty} a_n x_n : (a_n) \in B_{\ell_{p^*}} \right\}.$$

Moreover, the *1-convex hull* of $(x_n) \in \ell_1(X)$ is given by

$$1\text{-conv}(x_n) = \left\{ \sum_{n=1}^{\infty} a_n x_n : (a_n) \in B_{c_0} \right\},$$

and the *∞-convex hull* of $(x_n) \in c_0(X)$ by

$$\infty\text{-conv}(x_n) = \left\{ \sum_{n=1}^{\infty} a_n x_n : (a_n) \in B_{\ell_1} \right\}.$$

Note that $\infty\text{-conv}(x_n) = \overline{\text{aco}}(\{x_n : n \in \mathbb{N}\})$ is compact by [13, Lemma 3.4.29].

Let $p \in [1, \infty]$ and let X be a Banach space. Following [18], a subset K of X is said to be *relatively p-compact* if there is a sequence $(x_n) \in \ell_p(X)$ ($(x_n) \in c_0(X)$ if $p = \infty$) such that $K \subseteq p\text{-conv}(x_n)$. Such a sequence is not unique but Lassalle and Turco [12] (see also [7, p. 297]) defined the *measure of the size of p-compactness of K* as

$$m_p(K, X) = \begin{cases} \inf\{\|(x_n)\|_p : (x_n) \in \ell_p(X),\ K \subseteq p\text{-conv}(x_n)\} & \text{if } 1 \le p < \infty, \\ \inf\{\|(x_n)\|_\infty : (x_n) \in c_0(X),\ K \subseteq p\text{-conv}(x_n)\} & \text{if } p = \infty. \end{cases}$$

If there is no confusion, we will simply write $m_p(K)$ instead of $m_p(K, X)$.

An operator $T \in \mathcal{L}(X, Y)$ is said to be *p-compact* if $T(B_X)$ is a relatively p-compact set in Y. The space of all p-compact linear operators from X into Y is denoted by $\mathcal{K}_p(X, Y)$ and it is a Banach operator ideal endowed with the norm $k_p(T) = m_p(T(B_X))$.

A classical result of Grothendieck [10, Chap. I, p. 112] assures that a subset K of X is relatively compact if and only for every $\varepsilon > 0$, there is a sequence $(x_n) \in c_0(X)$ with $\|(x_n)\|_\infty \leq \sup_{x\in K} \|x\| + \varepsilon$ such that $K \subseteq \infty\text{-conv}(x_n)$. Hence, we can consider compact sets as ∞-compact sets. In this form, $\mathcal{K}_\infty$ coincides with the compact operator ideal $\mathcal{K}$ and k_∞ is the usual operator norm.

We now recall some notions and results on Bloch spaces that we will need later. If $\mathcal{H}(\mathbb{D}, X)$ stands for the space of all holomorphic maps from $\mathbb{D}$ into X, the *normalized Bloch space* $\widehat{\mathcal{B}}(\mathbb{D}, X)$ is the Banach space of all maps $f \in \mathcal{H}(\mathbb{D}, X)$ with $f(0) = 0$ so that

$$\rho_{\mathcal{B}}(f) = \sup\left\{(1 - |z|^2)\,\|f'(z)\| : z \in \mathbb{D}\right\} < \infty,$$

equipped with the norm $\rho_{\mathcal{B}}$. When $X = \mathbb{C}$, we will put $\widehat{\mathcal{B}}(\mathbb{D})$ in place of $\widehat{\mathcal{B}}(\mathbb{D}, \mathbb{C})$. We denote by $\widehat{\mathcal{H}}(\mathbb{D}, \mathbb{D})$ the set of all holomorphic functions h from $\mathbb{D}$ into itself such that $h(0) = 0$.

The *Bloch-free Banach space over* $\mathbb{D}$ is the space

$$\mathcal{G}(\mathbb{D}) := \overline{\text{span}}\,(\{\gamma_z : z \in \mathbb{D}\}) \subseteq \widehat{\mathcal{B}}(\mathbb{D})^*,$$

where $\gamma_z(f) = f'(z)$ for all $f \in \widehat{\mathcal{B}}(\mathbb{D})$.

We next collect the basic results on $\mathcal{G}(\mathbb{D})$.

Theorem 1.1 *[11]*

1. *The map $\Gamma : \mathbb{D} \to \mathcal{G}(\mathbb{D})$, given by $\Gamma(z) = \gamma_z$ for all $z \in \mathbb{D}$, is holomorphic and $\|\gamma_z\| = 1/(1 - |z|^2)$.*
2. *The map $\Lambda : \widehat{\mathcal{B}}(\mathbb{D}) \to \mathcal{G}(\mathbb{D})^*$, defined by $\Lambda(f)(\gamma) = \sum_{k=1}^n \lambda_k f'(z_k)$ if $f \in \widehat{\mathcal{B}}(\mathbb{D})$ and $\gamma = \sum_{k=1}^n \lambda_k \gamma_{z_k} \in \text{span}(\Gamma(\mathbb{D}))$, is a linear isometry of $\widehat{\mathcal{B}}(\mathbb{D})$ onto $\mathcal{G}(\mathbb{D})^*$.*
3. *$B_{\mathcal{G}(\mathbb{D})} = \overline{\text{aco}}(\mathcal{M}_{\mathcal{B}}(\mathbb{D})) \subseteq \widehat{\mathcal{B}}(\mathbb{D})^*$, where $\mathcal{M}_{\mathcal{B}}(\mathbb{D}) := \{(1 - |z|^2)\gamma_z : z \in \mathbb{D}\}$.*
4. *Given $h \in \widehat{\mathcal{H}}(\mathbb{D}, \mathbb{D})$, the map $C_h : f \in \widehat{\mathcal{B}}(\mathbb{D}) \mapsto f \circ h \in \widehat{\mathcal{B}}(\mathbb{D})$ is a nonexpansive linear operator.*
5. *For each $h \in \widehat{\mathcal{H}}(\mathbb{D}, \mathbb{D})$, there is a unique $\widehat{h} \in \mathcal{L}(\mathcal{G}(\mathbb{D}), \mathcal{G}(\mathbb{D}))$ satisfying $\widehat{h} \circ \Gamma = h' \cdot (\Gamma \circ h)$. Further, $\left(\widehat{h}\right)^* = C_h$.*
6. *For each map $f \in \widehat{\mathcal{B}}(\mathbb{D}, X)$, there is a unique $S_f \in \mathcal{L}(\mathcal{G}(\mathbb{D}), X)$ satisfying $S_f \circ \Gamma = f'$. Further, $||S_f|| = \rho_{\mathcal{B}}(f)$.*
7. *The map $f \mapsto S_f$ is a linear isometry of $\widehat{\mathcal{B}}(\mathbb{D}, X)$ onto $\mathcal{L}(\mathcal{G}(\mathbb{D}), X)$.*
8. *Given $f \in \widehat{\mathcal{B}}(\mathbb{D}, X)$, the map $f^t : x^* \in X^* \mapsto x^* \circ f \in \widehat{\mathcal{B}}(\mathbb{D})$ is a bounded linear operator and $\|f^t\| = \rho_{\mathcal{B}}(f)$. Moreover, $f^t = \Lambda^{-1} \circ (S_f)^*$.* □

2 *p*-Compact Bloch maps and their properties

We present and analyse the Bloch analogue of a p-compact linear operator between Banach spaces.

For any $f \in \mathcal{H}(\mathbb{D}, X)$, denote

$$\operatorname{rang}_{\mathcal{B}}(f) := \left\{(1-|z|^2)f'(z) \in X : z \in \mathbb{D}\right\},$$

and notice that f is Bloch if $\operatorname{rang}_{\mathcal{B}}(f)$ is bounded in X. According to [11, Definition 5.1], a map $f \in \mathcal{H}(\mathbb{D}, X)$ is called *compact Bloch* if $\operatorname{rang}_{\mathcal{B}}(f)$ is a relatively compact set in X. If $\widehat{\mathcal{B}}_{\mathcal{K}}(\mathbb{D}, X)$ denotes the space of all compact Bloch maps f of $\mathbb{D}$ into X for which $f(0) = 0$, then $[\widehat{\mathcal{B}}_{\mathcal{K}}, \rho_{\mathcal{B}}]$ is a Banach normalized Bloch ideal (see [11, Proposition 5.14]).

We may extend this concept as follows.

Definition 2.1 A map $f \in \mathcal{H}(\mathbb{D}, X)$ is called *p-compact Bloch* with $p \in [1, \infty]$ if $\operatorname{rang}_{\mathcal{B}}(f)$ is a relatively p-compact set in X. We denote by $\mathcal{B}_{\mathcal{K}_p}(\mathbb{D}, X)$ the linear space of all p-compact Bloch maps $f : \mathbb{D} \to X$, and by $\widehat{\mathcal{B}}_{\mathcal{K}_p}(\mathbb{D}, X)$ its vector subspace formed by all those f such that $f(0) = 0$. For each $f \in \mathcal{B}_{\mathcal{K}_p}(\mathbb{D}, X)$, we define

$$k_p^{\mathcal{B}}(f) = m_p(\operatorname{rang}_{\mathcal{B}}(f)).$$

In view of the following fact, we will only focus on the case $1 \leq p < \infty$.

Proposition 2.2 *$\mathcal{B}_{\mathcal{K}_\infty}(\mathbb{D}, X) = \mathcal{B}_{\mathcal{K}}(\mathbb{D}, X)$ and $k_\infty^{\mathcal{B}}(f) = \rho_{\mathcal{B}}(f)$ if $f \in \mathcal{B}_{\mathcal{K}_\infty}(\mathbb{D}, X)$.*

Proof Let f in $\mathcal{B}_{\mathcal{K}_\infty}(\mathbb{D}, X)$ and let (x_n) be in $c_0(X)$ such that $\operatorname{rang}_{\mathcal{B}}(f) \subseteq \infty\text{-conv}(x_n)$. Since $\infty\text{-conv}(x_n)$ is compact, $f \in \mathcal{B}_{\mathcal{K}}(\mathbb{D}, X)$. Moreover, for each $z \in \mathbb{D}$, there is a sequence $(a_n^{(z)}) \in B_{\ell_1}$ such that $(1-|z|^2)f'(z) = \sum_{n=1}^{\infty} a_n^{(z)} x_n$, and thus we have

$$(1-|z|^2)\left\|f'(z)\right\| \leq (1-|z|^2)\sum_{n=1}^{\infty}\left|a_n^{(z)}\right|\|x_n\| \leq \|(x_n)\|_\infty .$$

Taking supremum on all $z \in \mathbb{D}$ produces $\rho_{\mathcal{B}}(f) \leq \|(x_n)\|_\infty$, and passing to the infimum on all such sequences (x_n), we obtain $\rho_{\mathcal{B}}(f) \leq k_\infty^{\mathcal{B}}(f)$.

Conversely, let f in $\mathcal{B}_{\mathcal{K}}(\mathbb{D}, X)$, that is, $\operatorname{rang}_{\mathcal{B}}(f)$ is relatively compact in X. Hence, for every $\varepsilon > 0$, we can find a $(x_n) \in c_0(X)$ with $\|(x_n)\|_\infty \leq \rho_{\mathcal{B}}(f) + \varepsilon$ so that $\operatorname{rang}_{\mathcal{B}}(f) \subseteq \infty\text{-conv}(x_n)$. Thus f is in $\mathcal{B}_{\mathcal{K}_\infty}(\mathbb{D}, X)$ and $k_\infty^{\mathcal{B}}(f) \leq \rho_{\mathcal{B}}(f)$. □

2.1 Banach ideal property

Influenced by the concept of Banach operator ideal [16], the class of (Banach) normed normalized Bloch ideals on $\mathbb{D}$ was presented in [11, Definition 5.11].

For the next result, we only need to introduce the property of regularity. A normed ideal of normalized Bloch maps $[\mathcal{I}^{\widehat{\mathcal{B}}}, \|\cdot\|_{\mathcal{I}^{\widehat{\mathcal{B}}}}]$ is called

(R) *regular* if for every $f \in \widehat{\mathcal{B}}(\mathbb{D}, X)$, one has that f is in $\mathcal{I}^{\widehat{\mathcal{B}}}(\mathbb{D}, X)$ and $\|f\|_{\mathcal{I}^{\widehat{\mathcal{B}}}} = \|\kappa_X \circ f\|_{\mathcal{I}^{\widehat{\mathcal{B}}}}$ whenever $\kappa_X \circ f$ is in $\mathcal{I}^{\widehat{\mathcal{B}}}(\mathbb{D}, X^{**})$.

We now study the structure of $\widehat{\mathcal{B}}_{\mathcal{K}_p}$ as a normalized Bloch ideal.

Theorem 2.3 *Let $p \in [1, \infty)$. Then $[\widehat{\mathcal{B}}_{\mathcal{K}_p}, k_p^{\mathcal{B}}]$ is a Banach normalized Bloch ideal. Moreover, the ideal $[\widehat{\mathcal{B}}_{\mathcal{K}_p}, k_p^{\mathcal{B}}]$ is regular for the components $\widehat{\mathcal{B}}_{\mathcal{K}_p}(\mathbb{D}, X)$ whenever X is reflexive.*

Proof We first will prove that $(\widehat{\mathcal{B}}_{\mathcal{K}_p}(\mathbb{D}, X), k_p^{\mathcal{B}})$ satisfies the required properties whenever $p \in (1, \infty)$. The another case follows similarly.

(N1) Let $f \in \widehat{\mathcal{B}}_{\mathcal{K}_p}(\mathbb{D}, X)$ and let (x_n) be a sequence in $\ell_p(X)$ such that $\text{rang}_{\mathcal{B}}(f) \subseteq p\text{-conv}(x_n)$. It is clear that $(1 - |z|^2)\left\|f'(z)\right\| \leq \|(x_n)\|_p$ for all $z \in \mathbb{D}$, and thus $f \in \widehat{\mathcal{B}}(\mathbb{D}, X)$ with $\rho_{\mathcal{B}}(f) \leq \|(x_n)\|_p$. Taking infimum over all such sequences (x_n), we deduce that $\rho_{\mathcal{B}}(f) \leq k_p^{\mathcal{B}}(f)$.

We now claim that $(\widehat{\mathcal{B}}_{\mathcal{K}_p}(\mathbb{D}, X), k_p^{\mathcal{B}})$ is a normed space. Let $f \in \widehat{\mathcal{B}}_{\mathcal{K}_p}(\mathbb{D}, X)$. Clearly, $k_p^{\mathcal{B}}(f) \geq 0$. Suppose $k_p^{\mathcal{B}}(f) = 0$. Since $\rho_{\mathcal{B}}(f) \leq k_p^{\mathcal{B}}(f)$ and $\rho_{\mathcal{B}}$ is a norm on $\widehat{\mathcal{B}}(\mathbb{D}, X)$, it follows that $f = 0$.

Let $\lambda \in \mathbb{C}$. It is clear that $\text{rang}_{\mathcal{B}}(\lambda f) \subseteq p\text{-conv}(\lambda x_n)$ and, therefore, $\lambda f \in \widehat{\mathcal{B}}_{\mathcal{K}_p}(\mathbb{D}, X)$ with $k_p^{\mathcal{B}}(\lambda f) \leq |\lambda| k_p^{\mathcal{B}}(f)$. This implies that $k_p^{\mathcal{B}}(\lambda f) = 0 = |\lambda|\, k_p^{\mathcal{B}}(f)$ for $\lambda = 0$. If $\lambda \neq 0$, one has $k_p^{\mathcal{B}}(f) \leq |\lambda|^{-1}\, k_p^{\mathcal{B}}(\lambda f)$, therefore $|\lambda|\, k_p^{\mathcal{B}}(f) \leq k_p^{\mathcal{B}}(\lambda f)$, and so $k_p^{\mathcal{B}}(\lambda f) = |\lambda|\, k_p^{\mathcal{B}}(f)$.

Let $f_i \in \widehat{\mathcal{B}}_{\mathcal{K}_p}(\mathbb{D}, X)$ for $i = 1, 2$. Taking $K_i = \text{rang}_{\mathcal{B}}(f_i)$ with $i = 1, 2$ in [12, Lemma 3.1], we deduce that the set

$$K = \left\{(1 - |z|^2) f_1'(z) + (1 - |w|^2) f_2'(w) \colon z, w \in \mathbb{D}\right\}$$

is relatively p-compact in X with $m_p(K) \leq m_p(K_1) + m_p(K_2)$. Since $\text{rang}_{\mathcal{B}}(f_1 + f_2) \subseteq K$, it follows that $f_1 + f_2 \in \widehat{\mathcal{B}}_{\mathcal{K}_p}(\mathbb{D}, X)$ with $k_p^{\mathcal{B}}(f_1 + f_2) \leq k_p^{\mathcal{B}}(f_1) + k_p^{\mathcal{B}}(f_2)$.

To show that the norm $k_p^{\mathcal{B}}$ is complete on $\widehat{\mathcal{B}}_{\mathcal{K}_p}(\mathbb{D}, X)$, we will prove that if (f_n) is a sequence in $\widehat{\mathcal{B}}_{\mathcal{K}_p}(\mathbb{D}, X)$ such that $\sum k_p^{\mathcal{B}}(f_n)$ converges, then $\sum f_n$ is convergent in $(\widehat{\mathcal{B}}_{\mathcal{K}_p}(\mathbb{D}, X), k_p^{\mathcal{B}})$. Since $\rho_{\mathcal{B}}(f_n) \leq k_p^{\mathcal{B}}(f_n)$ if $n \in \mathbb{N}$ and $(\widehat{\mathcal{B}}(\mathbb{D}, X), \rho_{\mathcal{B}})$ is complete, we can find $f \in \widehat{\mathcal{B}}(\mathbb{D}, X)$ for which $\rho_{\mathcal{B}}(\sum_{k=1}^{n} f_k - f)$ converges to 0 if $n \to \infty$. We claim that $f \in \widehat{\mathcal{B}}_{\mathcal{K}_p}(\mathbb{D}, X)$ and $k_p^{\mathcal{B}}(f) \leq \sum_{n=1}^{\infty} k_p^{\mathcal{B}}(f_n)$. Indeed, since the sequence $(\text{rang}_{\mathcal{B}}(f_n))$ consists of relatively p-compact subsets of X such that $\sum m_p(\text{rang}_{\mathcal{B}}(f_n)) = \sum k_p^{\mathcal{B}}(f_n)$ converges, Lemma 3.1 in [12] assures that the series $\sum_{n\geq 1}(1 - |z_n|^2) f_n(z_m)$ is absolutely convergent for any choice of points $z_m \in \mathbb{D}$ with $m \in \mathbb{N}$, and the set

$$K = \left\{\sum_{n=1}^{\infty}(1 - |z_m|^2) f_n'(z_m) \colon z_m \in \mathbb{D},\ m \in \mathbb{N}\right\}$$

is relatively p-compact in X with $m_p(K) \leq \sum_{n=1}^{\infty} m_p(\text{rang}_{\mathcal{B}}(f_n))$. Clearly, $\text{rang}_{\mathcal{B}}(f) \subseteq K$ and this proves our claim. The previous proof can be applied to show that for every $m \in \mathbb{N}$, $\sum_{n=m+1}^{\infty} f_n \in \widehat{\mathcal{B}}_{\mathcal{K}_p}(\mathbb{D}, X)$ with $k_p^{\mathcal{B}}\left(\sum_{n=m+1}^{\infty} f_k\right) \leq \sum_{n=m+1}^{\infty} k_p^{\mathcal{B}}(f_n)$. Hence,

$$k_p^{\mathcal{B}}\left(f - \sum_{n=1}^{m} f_n\right) \leq \sum_{n=m+1}^{\infty} k_p^{\mathcal{B}}(f_n)$$

for every $m \in \mathbb{N}$, and thus $k_p^{\mathcal{B}}(f - \sum_{n=1}^{m} f_n) \to 0$ as $m \to \infty$.

(N2) Let g in $\widehat{\mathcal{B}}(\mathbb{D})$ and x in X. Assume $g \neq 0$ and $x \neq 0$ (otherwise, there is nothing to prove). Clearly, the sequence (x_n), given by $x_1 = \rho_{\mathcal{B}}(g)x$ and $x_n = 0$ for all $n \geq 2$, is in $\ell_p(X)$ and $\text{rang}_{\mathcal{B}}(g \cdot x) \subseteq p\text{-conv}(x_n)$. Therefore, $g \cdot x \in \widehat{\mathcal{B}}_{\mathcal{K}_p}(\mathbb{D}, X)$ and $k_p^{\mathcal{B}}(g \cdot x) \leq \|(x_n)\|_p = \rho_{\mathcal{B}}(g)\,\|x\|$. The reverse inequality follows immediately from (N1).

(N3) Let $h \in \widehat{\mathcal{H}}(\mathbb{D}, \mathbb{D})$, $T \in \mathcal{L}(X, Y)$ and $f \in \widehat{\mathcal{B}}_{\mathcal{K}_p}(\mathbb{D}, X)$. Clearly, $T \circ f \circ h \in \mathcal{H}(\mathbb{D}, Y)$ and $(T \circ f \circ h)' = h' \cdot (T \circ f' \circ h)$. Let $(x_n) \in \ell_p(X)$ be for which $\text{rang}_{\mathcal{B}}(f) \subseteq p\text{-conv}(x_n)$. For each $z \in \mathbb{D}$, there is a sequence $(a_n^{(z)}) \in B_{\ell_{p^*}}$ such that $(1 - |z|^2)f'(z) = \sum_{n=1}^{\infty} a_n^{(z)} x_n$, and thus we have

$$\begin{aligned}(1 - |z|^2)(T \circ f \circ h)'(z) &= \frac{(1 - |z|^2)h'(z)}{1 - |h(z)|^2} T((1 - |h(z)|^2)f'(h(z))) \\ &= \frac{(1 - |z|^2)h'(z)}{1 - |h(z)|^2} T\left(\sum_{n=1}^{\infty} a_n^{(h(z))} x_n\right) \\ &= \sum_{n=1}^{\infty} b_n^{(z)} T(x_n),\end{aligned}$$

where

$$b_n^{(z)} = \frac{(1 - |z|^2)h'(z)}{1 - |h(z)|^2} a_n^{(h(z))} \qquad (n \in \mathbb{N}).$$

By applying Pick–Schwarz Lemma, notice that

$$\left\|(b_n^{(z)})\right\|_{p^*} = \frac{(1 - |z|^2)|h'(z)|}{1 - |h(z)|^2} \left\|(a_n^{(h(z))})\right\|_{p^*} \leq 1.$$

Therefore, $T \circ f \circ h \in \widehat{\mathcal{B}}_{\mathcal{K}_p}(\mathbb{D}, Y)$ and $k_p^{\mathcal{B}}(T \circ f \circ h) \leq \|(T(x_n))\|_p \leq \|T\|\,\|(x_n)\|_p$. Taking infimum over all such sequences (x_n), we arrive at $k_p^{\mathcal{B}}(T \circ f \circ h) \leq \|T\|\,k_p^{\mathcal{B}}(f)$.

(R) Assume that X is reflexive and thus $\ell_p(X^{**}) = \kappa_X(\ell_p(X))$. Take $f \in \widehat{\mathcal{B}}(\mathbb{D}, X)$ and assume that $\kappa_X \circ f \in \widehat{\mathcal{B}}_{\mathcal{K}_p}(\mathbb{D}, X^{**})$. Let $(x_n) \in \ell_p(X)$ be with $\text{rang}_{\mathcal{B}}(\kappa_X \circ f) \subseteq p\text{-conv}(\kappa_X(x_n))$. It is clear that $\text{rang}_{\mathcal{B}}(\kappa_X \circ f) = \kappa_X(\text{rang}_{\mathcal{B}}(f))$ and $p\text{-conv}(\kappa_X(x_n)) = \kappa_X(p\text{-conv}(x_n))$. Hence $\kappa_X(\text{rang}_{\mathcal{B}}(f)) \subseteq \kappa_X(p\text{-conv}(x_n))$ and the injectivity of κ_X gives us that $\text{rang}_{\mathcal{B}}(f) \subseteq p\text{-conv}(x_n)$. Hence, $f \in \widehat{\mathcal{B}}_{\mathcal{K}_p}(\mathbb{D}, X)$

with $k_p^{\mathcal{B}}(f) \leq \|(x_n)\|_p = \|(\kappa_X(x_n))\|_p$, and so $k_p^{\mathcal{B}}(f) \leq k_p^{\mathcal{B}}(\kappa_X \circ f)$ by taking infimum over all such sequences $(\kappa_X(x_n))$. The converse inequality follows from (N3). □

The surjectivity of the ideal $[\widehat{\mathcal{B}}_{\mathcal{K}_p}, k_p^{\mathcal{B}}]$ will be addressed later.

2.2 Möbius invariance

Let $\mathrm{Aut}(\mathbb{D})$ be the *Möbius group of* $\mathbb{D}$. Every $\psi \in \mathrm{Aut}(\mathbb{D})$ is of the form $\psi = \tau\psi_a$ with $\tau \in \mathbb{T}$ and $a \in \mathbb{D}$, where $\psi_a(z) = (a - z)/(1 - \overline{a}z)$ for all $z \in \mathbb{D}$.

A vector space $\mathcal{A}(\mathbb{D}, X)$ of Bloch maps of $\mathbb{D}$ to X, with a seminorm $\rho_{\mathcal{A}}$, is called *invariant by Möbius transformations* whenever:

(i) There is a constant c so that $\rho_{\mathcal{B}}(f) \leq c\rho_{\mathcal{A}}(f)$ for any $f \in \mathcal{A}(\mathbb{D}, X)$,
(ii) $f \circ \psi \in \mathcal{A}(\mathbb{D}, X)$ with $\rho_{\mathcal{A}}(f \circ \psi) = \rho_{\mathcal{A}}(f)$ for any $f \in \mathcal{A}(\mathbb{D}, X)$ and $\psi \in \mathrm{Aut}(\mathbb{D})$.

In the light of Theorem 2.3, $\mathcal{B}_{\mathcal{K}_p}(\mathbb{D}, X)$ satisfies the condition (i) above with $c = 1$ and $\rho_{\mathcal{A}} = k_p^{\mathcal{B}}$. In order to prove (ii), note first that if $f \in \mathcal{H}(\mathbb{D}, X)$ and $\psi \in \mathrm{Aut}(\mathbb{D})$, then $h = f \circ \psi$ holds that

$$(1 - |z|^2)h'(z) = (1 - |\psi(z)|^2) f'(\psi(z)) \frac{\psi'(z)}{|\psi'(z)|} \qquad (z \in \mathbb{D}).$$

Now, if $f \in \mathcal{B}_{\mathcal{K}_p}(\mathbb{D}, X)$, let (x_n) be a sequence in $\ell_p(X)$ so that $\mathrm{rang}_{\mathcal{B}}(f) \subseteq p\text{-conv}(x_n)$. Hence, for each $z \in \mathbb{D}$, we can find a sequence $(a_n^{(z)})$ in $B_{\ell_{p^*}}$ (in B_{c_0} if $p = 1$) for which $(1 - |z|^2) f'(z) = \sum_{n=1}^{\infty} a_n^{(z)} x_n$, and, consequently, one has

$$(1 - |z|^2)h'(z) = \frac{\psi'(z)}{|\psi'(z)|} \sum_{n=1}^{\infty} a_n^{(\psi(z))} x_n = \sum_{n=1}^{\infty} b_n^{(z)} x_n,$$

where

$$b_n^{(z)} = \frac{\psi'(z)}{|\psi'(z)|} a_n^{(\psi(z))} \qquad (n \in \mathbb{N}).$$

Consequently, $h \in \mathcal{B}_{\mathcal{K}_p}(\mathbb{D}, X)$ and $k_p^{\mathcal{B}}(h) \leq k_p^{\mathcal{B}}(f)$. Since $\psi^{-1} \in \mathrm{Aut}(\mathbb{D})$, the previous proof yields the converse inequality $k_p^{\mathcal{B}}(f) \leq k_p^{\mathcal{B}}(h)$. In this way, we have the following.

Theorem 2.4 *$\mathcal{B}_{\mathcal{K}_p}(\mathbb{D}, X)$ is Möbius-invariant for any $p \in [1, \infty)$.* □

2.3 Linearisation

Next result shows the good connection of the Bloch p-compactness of a map f in $\widehat{\mathcal{B}}(\mathbb{D}, X)$ and the p-compactness of its linearisation S_f in $\mathcal{L}(\mathcal{G}(\mathbb{D}), X)$.

Theorem 2.5 *If $p \in [1, \infty)$ and $f \in \widehat{\mathcal{B}}(\mathbb{D}, X)$, then f is p-compact Bloch if and only if $S_f : \mathcal{G}(\mathbb{D}) \to X$ is p-compact, which leads to $k_p^{\mathcal{B}}(f) = k_p(S_f)$. Further, the correspondence $f \mapsto S_f$ is a linear isometry of $(\widehat{\mathcal{B}}_{\mathcal{K}_p}(\mathbb{D}, X), k_p^{\mathcal{B}})$ onto $(\mathcal{K}_p(\mathcal{G}(\mathbb{D}), X), k_p)$.*

Proof If $f \in \widehat{\mathcal{B}}_{\mathcal{K}_p}(\mathbb{D}, X)$, then $S_f \in \mathcal{K}_p(\mathcal{G}(\mathbb{D}), X)$ and

$$k_p(S_f) = m_p(S_f(B_{\mathcal{G}(\mathbb{D})})) \leq m_p(\overline{\text{aco}}(\text{rang}_{\mathcal{B}}(f))) = m_p(\text{rang}_{\mathcal{B}}(f)) = k_p^{\mathcal{B}}(f),$$

by applying the inclusion

$$S_f(B_{\mathcal{G}(\mathbb{D})}) = S_f(\overline{\text{aco}}(\mathcal{M}_{\mathcal{B}}(\mathbb{D}))) \subseteq \overline{\text{aco}}(S_f(\mathcal{M}_{\mathcal{B}}(\mathbb{D}))) = \overline{\text{aco}}(\text{rang}_{\mathcal{B}}(f))$$

and that a set is p-compact in X if and only if its norm-closed absolutely convex hull is p-compact with the same measure under m_p (see [12, p. 1205]).

Conversely, if $S_f \in \mathcal{K}_p(\mathcal{G}(\mathbb{D}), X)$, then $f \in \widehat{\mathcal{B}}_{\mathcal{K}_p}(\mathbb{D}, X)$ and

$$k_p^{\mathcal{B}}(f) = m_p(\text{rang}_{\mathcal{B}}(f)) \leq m_p(S_f(B_{\mathcal{G}(\mathbb{D})})) = k_p(S_f),$$

in view of the inclusion

$$\text{rang}_{\mathcal{B}}(f) = S_f(\mathcal{M}_{\mathcal{B}}(\mathbb{D})) \subseteq S_f(B_{\mathcal{G}(\mathbb{D})}).$$

The final affirmation is obtained easily from Theorem 1.1. □

2.4 Factorization

We now prove that the derivatives of the members of the Bloch ideal $\widehat{\mathcal{B}}_{\mathcal{K}_p}$ can be produced composing with the Banach operator ideal $\mathcal{K}_p$.

Corollary 2.6 *Let $p \in [1, \infty)$ and $f \in \widehat{\mathcal{B}}(\mathbb{D}, X)$. Then f is p-compact Bloch if and only if there exist a complex Banach space Y, $g \in \widehat{\mathcal{B}}(\mathbb{D}, Y)$ and $T \in \mathcal{K}_p(Y, X)$ such that $f' = T \circ g'$. In this case, $k_p^{\mathcal{B}}(f) = \inf\{k_p(T)\rho_{\mathcal{B}}(g) : f' = T \circ g'\}$, and it is a maximum for $T = S_f$ and $g = \Gamma$.*

Proof If $f \in \widehat{\mathcal{B}}_{\mathcal{K}_p}(\mathbb{D}, X)$, then $f' = S_f \circ \Gamma$, with $S_f \in \mathcal{K}_p(\mathcal{G}(\mathbb{D}), X)$ and $\Gamma \in \mathcal{H}(\mathbb{D}, \mathcal{G}(\mathbb{D}))$ by applying Theorems 1.1 and 2.5. Also, the function $h : \mathbb{D} \to \mathcal{G}(\mathbb{D})$ given by

$$h(z) = \int_{[0,z]} \Gamma(w)\, \mathrm{d}w \qquad (z \in \mathbb{D}),$$

is Bloch with $h'(z) = \Gamma(z)$ for all $z \in \mathbb{D}$, $h(0) = 0$ and $\rho_{\mathcal{B}}(h) = 1$. Thus $f' = S_f \circ h'$. Further, $\inf\left\{k_p(T)\rho_{\mathcal{B}}(g)\right\} \leq k_p(S_f)\rho_{\mathcal{B}}(h) = k_p^{\mathcal{B}}(f)$.

Conversely, assume that $f' = T \circ g'$ as in the statement. Since $g' = S_g \circ \Gamma$ by Theorem 1.1, we have $f' = T \circ S_g \circ \Gamma$ and this gives $S_f = T \circ S_g$, and hence

$S_f \in \mathcal{K}_p(\mathcal{G}(\mathbb{D}), X)$ since $[\mathcal{K}_p, k_p]$ is a ideal [18, Theorem 4.2]. By Theorem 2.5, we get that $f \in \widehat{\mathcal{B}}_{\mathcal{K}_p}(\mathbb{D}, X)$ and

$$k_p^{\mathcal{B}}(f) = k_p(S_f) \leq k_p(T) \left\| S_g \right\| = k_p(T)\rho_{\mathcal{B}}(g).$$

Passing to the infimum over all decompositions of f' gives $k_p^{\mathcal{B}}(f) \leq \inf\{k_p(T)\rho_{\mathcal{B}}(g)\}$. □

From the factorization of p-compact operators established in [9, Proposition 2.9], we next obtain that the derivative of a p-compact Bloch map can be represented as a composition of three maps: the derivative of a compact Bloch map, a p-compact operator from a quotient of ℓ_{p^*} to a separable space and a compact operator on this last space.

Corollary 2.7 *Let $p \in [1, \infty)$ and $f \in \widehat{\mathcal{B}}(\mathbb{D}, X)$. Then f is p-compact Bloch if and only if there exist a closed subspace M in ℓ_{p^*} (c_0 instead of ℓ_{p^*} if $p = 1$), a separable Banach space Z, an operator T in $\mathcal{K}_p(\ell_{p^*}/M, Z)$, a map g in $\widehat{\mathcal{B}}_{\mathcal{K}}(\mathbb{D}, \ell_{p^*}/M)$ and an operator $S \in \mathcal{K}(Z, X)$ such that $f' = S \circ T \circ g'$, in whose case $k_p^{\mathcal{B}}(f) = \inf\{\|S\| \, k_p(T)\rho_{\mathcal{B}}(g)\}$, where the infimum is extended over all factorizations of f' as above.*

Proof Assume $p \in (1, \infty)$. For $p = 1$, the proof is similar.

Suppose that f is in $\widehat{\mathcal{B}}_{\mathcal{K}_p}(\mathbb{D}, X)$. By Theorem 2.5, S_f is in $\mathcal{K}_p(\mathcal{G}(\mathbb{D}), X)$ with $k_p(S_f) = k_p^{\mathcal{B}}(f)$. Applying [9, Proposition 2.9], for each $\varepsilon > 0$, there exist a closed subspace $M \subseteq \ell_{p^*}$ (c_0 instead of ℓ_{p^*} if $p = 1$), a separable Banach space Z, an operator $T \in \mathcal{K}_p(\ell_{p^*}/M, Z)$, an operator $S \in \mathcal{K}(Z, X)$ and an operator $R \in \mathcal{K}(\mathcal{G}(\mathbb{D}), \ell_{p^*}/M)$ such that $S_f = S \circ T \circ R$ with $\|S\| \, k_p(T) \, \|R\| \leq k_p(S_f) + \varepsilon$. Moreover, there exists $g \in \widehat{\mathcal{B}}_{\mathcal{K}}(\mathbb{D}, \ell_{p^*}/M)$ so that $R = S_g$ with $\rho_{\mathcal{B}}(g) = \|R\|$ by Theorem 2.5. Thus we obtain

$$f' = S_f \circ \Gamma = S \circ T \circ R \circ \Gamma = S \circ T \circ S_g \circ \Gamma = S \circ T \circ g'$$

with

$$\|S\| \, k_p(T)\rho_{\mathcal{B}}(g) = \|S\| \, k_p(T) \, \|R\| \leq k_p(S_f) + \varepsilon = k_p^{\mathcal{B}}(f) + \varepsilon.$$

Since ε was arbitrary, we deduce that $\|S\| \, k_p(T)\rho_{\mathcal{B}}(g) \leq k_p^{\mathcal{B}}(f)$.

Conversely, suppose that $f' = S \circ T \circ g'$ is a factorization as in the statement. Since $S \circ T \in \mathcal{K}_p(\ell_{p^*}/M, X)$, an application of Corollary 2.6 yields that $f \in \widehat{\mathcal{B}}_{\mathcal{K}_p}(\mathbb{D}, X)$ with

$$k_p^{\mathcal{B}}(f) \leq k_p(S \circ T)\rho_{\mathcal{B}}(g) \leq \|S\| \, k_p(T)\rho_{\mathcal{B}}(g),$$

and from this we infer that $k_p^{\mathcal{B}}(f) \leq \inf\{\|S\| \, k_p(T)\rho_{\mathcal{B}}(g)\}$. □

2.5 Inclusion

Combining Theorem 2.5 with the fact that $\mathcal{K}_p \subseteq \mathcal{K}_q$ whenever $1 \leq p \leq q < \infty$ with $k_q(T) \leq k_p(T)$ for all $T \in \mathcal{K}_p$ (see [18, Proposition 4.3]), we get the following inclusions.

Corollary 2.8 *Let $p, q \in [1, \infty)$ with $p \leq q$. Then $\widehat{\mathcal{B}}_{\mathcal{K}_p}(\mathbb{D}, X) \subseteq \widehat{\mathcal{B}}_{\mathcal{K}_q}(\mathbb{D}, X)$ and $k_q^{\mathcal{B}}(f) \leq k_p^{\mathcal{B}}(f)$ for all $f \in \widehat{\mathcal{B}}_{\mathcal{K}_p}(\mathbb{D}, X)$.* □

According to [11, Definition 5.2], a map $f \in \widehat{\mathcal{B}}(\mathbb{D}, X)$ has *finite dimensional Bloch rank* if $\operatorname{span}(\operatorname{rang}_{\mathcal{B}}(f))$ is a finite dimensional subspace of X. We denote by $\widehat{\mathcal{B}}_{\mathcal{F}}(\mathbb{D}, X)$ the set of all finite-rank Bloch maps f from $\mathbb{D}$ into X for which $f(0) = 0$. Notice that $\widehat{\mathcal{B}}_{\mathcal{F}}(\mathbb{D}, X)$ is a vector subspace of $\widehat{\mathcal{B}}_{\mathcal{K}_p}(\mathbb{D}, X)$ (apply [11, Theorem 5.7], [18, Theorem 4.2] and Theorem 2.5). We can enlarge this subspace with the following class of Bloch maps.

Definition 2.9 A map $f \in \widehat{\mathcal{B}}(\mathbb{D}, X)$ is called *p-approximable* with $p \in [1, \infty)$ if we can find a (f_n) in $\widehat{\mathcal{B}}_{\mathcal{F}}(\mathbb{D}, X)$ for which $k_p^{\mathcal{B}}(f_n - f) \to 0$ as $n \to \infty$. Let $\widehat{\mathcal{B}}_{\overline{\mathcal{F}}_p}(\mathbb{D}, X)$ denote the space of all p-approximable Bloch maps of $\mathbb{D}$ into X for which $f(0) = 0$.

Corollary 2.10 $\widehat{\mathcal{B}}_{\overline{\mathcal{F}}_p}(\mathbb{D}, X) \subseteq \widehat{\mathcal{B}}_{\mathcal{K}_p}(\mathbb{D}, X)$ *for any $p \in [1, \infty)$.*

Proof If $f \in \widehat{\mathcal{B}}_{\overline{\mathcal{F}}_p}(\mathbb{D}, X)$, we have a (f_n) in $\widehat{\mathcal{B}}_{\mathcal{F}}(\mathbb{D}, X)$ for which $k_p^{\mathcal{B}}(f_n - f) \to 0$. As $S_{f_n} \in \mathcal{F}(G(\mathbb{D}), X)$ by [11, Theorem 5.7], $\mathcal{F}(G(\mathbb{D}), X) \subseteq \mathcal{K}_p(\mathcal{G}(\mathbb{D}), X)$ by [18, Theorem 4.2] and $k_p(S_{f_n} - S_f) = k_p^{\mathcal{B}}(f_n - f)$ if $n \in \mathbb{N}$ by Theorems 1.1 and 2.5, one obtains that $S_f \in \mathcal{K}_p(\mathcal{G}(\mathbb{D}), X)$ by [18, Theorem 4.2], thus $f \in \widehat{\mathcal{B}}_{\mathcal{K}_p}(\mathbb{D}, X)$ from Theorem 2.5. □

2.6 Transposition

We now characterize p-compact Bloch maps in terms of their Bloch transposes. Towards this end, let us recall (see [15]) that given $p \in [1, \infty)$, a map $T \in \mathcal{L}(X, Y)$ is *quasi p-nuclear* if we can find a $(x_n^*) \in \ell_p(X^*)$ for which

$$\|T(x)\| \leq \left(\sum_{n=1}^{\infty} \left| x_n^*(x) \right|^p \right)^{\frac{1}{p}} \qquad (x \in X).$$

The linear space of such operators, denoted $\mathcal{QN}_p(X, Y)$, is a Banach space with the norm

$$\nu_p^{\mathcal{Q}}(T) = \inf \left\{ \left\| (x_n^*) \right\|_p : \ \|T(x)\| \leq \left(\sum_{n=1}^{\infty} \left| x_n^*(x) \right|^p \right)^{\frac{1}{p}}, \ \forall x \in X \right\}.$$

Moreover, the pair $[\mathcal{QN}_p, \nu_p^{\mathcal{Q}}]$ is an operator Banach ideal. In [7, Proposition 3.8], it was stated that an operator $T \in \mathcal{K}_p(X, Y)$ if and only if its adjoint $T^* \in$

$\mathcal{QN}_p(Y^*, X^*)$. Moreover, $k_p(T) = \nu_p^{\mathcal{Q}}(T^*)$ by [9, Corollary 2.7]. The next result presents the analogue in the Bloch setting.

Corollary 2.11 *Let $p \in [1, \infty)$ and $f \in \widehat{\mathcal{B}}(\mathbb{D}, X)$. Then $f : \mathbb{D} \to X$ is p-compact Bloch if and only if $f^t : X^* \to \widehat{\mathcal{B}}(\mathbb{D})$ is quasi p-nuclear. In this case, $k_p^{\mathcal{B}}(f) = \nu_p^{\mathcal{Q}}(f^t)$.*

Proof Applying Theorem 2.5, [9, Corollary 2.7] and [15, p. 32], respectively, one has

$$\begin{aligned} f \in \widehat{\mathcal{B}}_{\mathcal{K}_p}(\mathbb{D}, X) &\Leftrightarrow S_f \in \mathcal{K}_p(\mathcal{G}(\mathbb{D}), X) \\ &\Leftrightarrow (S_f)^* \in \mathcal{QN}_p(X^*, \mathcal{G}(\mathbb{D})^*) \\ &\Leftrightarrow f^t \in \mathcal{QN}_p(X^*, \widehat{\mathcal{B}}(\mathbb{D})). \end{aligned}$$

Moreover, $k_p^{\mathcal{B}}(f) = k_p(S_f) = \nu_p^{\mathcal{Q}}((S_f)^*) = \nu_p^{\mathcal{Q}}(f^t)$. □

The Banach space of p-summing operators with $1 \leq p < \infty$, denoted by Π_p and equipped with a natural norm π_p, appears involved in the following result. A complete study of this Banach operator ideal may be found, for instance, in [16, 17.3].

Corollary 2.12 *Let $p \in [1, \infty)$, $f \in \widehat{\mathcal{B}}(\mathbb{D}, X)$ and $g \in \widehat{\mathcal{B}}(\mathbb{D}, X^*)$. Assume that S_f is p-summing and g is compact Bloch. Then $f^t \circ g$ is p-compact Bloch with $k_p^{\mathcal{B}}(f^t \circ g) \leq \pi_p(S_f)\rho_{\mathcal{B}}(g)$.*

Proof By Theorem 2.5, $S_g \in \mathcal{K}(\mathcal{G}(\mathbb{D}), X^*)$ with $||S_g|| = \rho_{\mathcal{B}}(g)$. Consequently, by [7, Proposition 3.13], $(S_f)^* \circ S_g \in \mathcal{K}_p(\mathcal{G}(\mathbb{D}), \mathcal{G}(\mathbb{D})^*)$ with $k_p((S_f)^* \circ S_g) \leq \pi_p(S_f)||S_g||$. In view of $f^t \circ S_g = \Lambda^{-1} \circ (S_f)^* \circ S_g$, the ideal property of $[\mathcal{K}_p, k_p]$ yields that $f^t \circ S_g \in \mathcal{K}_p(\mathcal{G}(\mathbb{D}), \widehat{\mathcal{B}}(\mathbb{D}))$ with $k_p(f^t \circ S_g) = k_p((S_f)^* \circ S_g)$. From the equality $f^t \circ S_g \circ \Gamma = f^t \circ g' = (f^t \circ g)'$, one infers $S_{f^t \circ g} = f^t \circ S_g$ by Theorem 1.1. So $f^t \circ g \in \widehat{\mathcal{B}}_{\mathcal{K}_p}(\mathbb{D}, \widehat{\mathcal{B}}(\mathbb{D}))$ with $k_p^{\mathcal{B}}(f^t \circ g) = k_p(S_{f^t \circ g})$ by Theorem 2.5. Furthermore,

$$k_p^{\mathcal{B}}(f^t \circ g) = k_p(S_{f^t \circ g}) = k_p((S_f)^* \circ S_g) \leq \pi_p(S_f) \left\| S_g \right\| = \pi_p(S_f)\rho_{\mathcal{B}}(g).$$

□

Theorem 3.2 in [18] assures that p-compact operators are exactly those for which their adjoints factor through a subspace of ℓ_p. We now have a similar decomposition for the Bloch transpose of a p-compact Bloch map (compare also to [7, Proposition 3.10]).

Corollary 2.13 *Let $p \in [1, \infty)$ and $f \in \widehat{\mathcal{B}}(\mathbb{D}, X)$. Then f is p-compact Bloch if and only if there exist a closed subspace $M \subseteq \ell_p$ and operators $R \in \mathcal{QN}_p(X^*, M)$ and $S \in \mathcal{L}(M, \widehat{\mathcal{B}}(\mathbb{D}))$ such that $f^t = S \circ R$.*

Proof If $f \in \widehat{\mathcal{B}}_{\mathcal{K}_p}(\mathbb{D}, X)$, we have $S_f \in \mathcal{K}_p(\mathcal{G}(\mathbb{D}), X)$ by Theorem 2.5. By [7, Proposition 3.10], there exist a closed subspace $M \subseteq \ell_p$ and operators $R \in \mathcal{QN}_p(X^*, M)$ and $S_0 \in \mathcal{L}(M, \mathcal{G}(\mathbb{D})^*)$ such that $(S_f)^* = S_0 \circ R$. Taking $S = \Lambda^{-1} \circ S_0 \in \mathcal{L}(M, \widehat{\mathcal{B}}(\mathbb{D}))$, we have $f^t = S \circ R$.

Conversely, assume $f^t = S \circ R$, being S and R as in the statement. It follows that $(S_f)^* = \Lambda \circ f^t = \Lambda \circ S \circ R$, and so $S_f \in \mathcal{K}_p(\mathcal{G}(\mathbb{D}), X)$ by [7, Proposition 3.10]. Hence $f \in \widehat{\mathcal{B}}_{\mathcal{K}_p}(\mathbb{D}, X)$ by Theorem 2.5 □

2.7 Ideal surjectivity

This section deals with the surjectivity of the ideal $[\widehat{\mathcal{B}}_{\mathcal{K}_p}, k_p^{\mathcal{B}}]$. We will first prove that this ideal is surjective.

In the setting of operator ideals, for Banach spaces X, Y, Z, a normed operator ideal $[\mathcal{I}, \|\cdot\|_I]$ is *surjective* if for every metric surjection $Q \in \mathcal{L}(Z, X)$ and every $T \in \mathcal{L}(X, Y)$, it follows from $T \circ Q \in \mathcal{I}(Z, Y)$ that $T \in \mathcal{I}(X, Y)$ with $\|T\|_{\mathcal{I}} = \|T \circ Q\|_{\mathcal{I}}$.

Corollary 2.14 *For $p \in [1, \infty)$, the Banach normalized Bloch ideal $[\widehat{\mathcal{B}}_{\mathcal{K}_p}, k_p^{\mathcal{B}}]$ is surjective.*

Proof (S) Let $f \in \widehat{\mathcal{B}}(\mathbb{D}, X)$ and assume that $f \circ \pi \in \widehat{\mathcal{B}}_{\mathcal{K}_p}(\mathbb{D}, X)$, where $\pi \in \widehat{\mathcal{H}}(\mathbb{D}, \mathbb{D})$ and $\widehat{\pi}$ is a metric surjection from $\mathcal{G}(\mathbb{D})$ into itself. By Theorem 1.1, $\widehat{\pi} \circ \Gamma = \pi' \cdot (\Gamma \circ \pi)$. As $S_f \circ \widehat{\pi} \in \mathcal{L}(\mathcal{G}(\mathbb{D}), X)$ with

$$(S_f \circ \widehat{\pi}) \circ \Gamma = S_f \circ [\pi' \cdot (\Gamma \circ \pi)] = \pi' \cdot [(S_f \circ \Gamma) \circ \pi] = \pi' \cdot (f' \circ \pi) = (f \circ \pi)',$$

one has $S_{f\circ\pi} = S_f \circ \widehat{\pi}$ by Theorem 1.1. Since $S_f \circ \widehat{\pi} = S_{f\circ\pi} \in \mathcal{K}_p(\mathcal{G}(\mathbb{D}), X)$ by Theorem 2.5 and the operator ideal $[\mathcal{K}_p, k_p]$ is surjective by [7, Proposition 3.11], one has that $S_f \in \mathcal{K}_p(\mathcal{G}(\mathbb{D}), X)$ and $k_p(S_f) = k_p(S_f \circ \widehat{\pi})$. Thus $f \in \widehat{\mathcal{B}}_{\mathcal{K}_p}(\mathbb{D}, X)$ and

$$k_p^{\mathcal{B}}(f) = k_p(S_f) = k_p(S_f \circ \widehat{\pi}) = k_p(S_{f\circ\pi}) = k_p^{\mathcal{B}}(f \circ \pi)$$

by Theorem 2.5. □

We will now try to give a description of the surjective normed normalized Bloch ideal $[\widehat{\mathcal{B}}_{\mathcal{K}_p}, k_p^{\mathcal{B}}]$.

Given a Banach space X and $p \in [1, \infty)$, $\ell_p^{\text{weak}}(X)$ denotes the Banach space of all weakly p-summable sequences (x_n) in X, endowed with the norm

$$\|(x_n)\|_p^{\text{weak}} = \sup\left\{\left(\sum_{n=1}^{\infty} |f(x_n)|^p\right)^{\frac{1}{p}} : f \in B_{X^*}\right\}.$$

For $p \in [1, \infty)$, $T \in \mathcal{L}(X, Y)$ is *right p-nuclear* if there are sequences $(x_n^*) \in \ell_{p^*}^{\text{weak}}(X^*)$ and $(y_n) \in \ell_p(Y)$ such that $T(x) = \sum_{n=1}^{\infty} x_n^*(x) y_n$ for all $x \in X$, where the series converges in $\mathcal{L}(X, Y)$ (see [14]). The *right p-nuclear norm of T* is defined by

$$\nu^p(T) = \inf\left\{\left\|(x_n^*)\right\|_{p^*}^{\text{weak}} \|(y_n)\|_p\right\},$$

where the infimum extends over all representations of T as above. The set of such operators, denoted $\mathcal{N}^p(X, Y)$, is a Banach space with the right p-nuclear norm.

The Bloch analogue of this class of operators can be introduced as follows.

Definition 2.15 A map $f \in \widehat{\mathcal{B}}(\mathbb{D}, X)$ is called *right p-nuclear Bloch* with $p \in [1, \infty)$ if there exist sequences (g_n) in $\ell_{p^*}^{\text{weak}}(\widehat{\mathcal{B}}(\mathbb{D}))$ and (x_n) in $\ell_p(X)$ so that $f = \sum_{n=1}^{\infty} g_n \cdot x_n$ in $(\widehat{\mathcal{B}}(\mathbb{D}, X), \rho_{\mathcal{B}})$. We will say that $\sum_{n\geq 1} g_n \cdot x_n$ is a *right p-nuclear Bloch representation of f*. Define

$$\nu^{p\mathcal{B}}(f) = \inf \left\{ \|(g_n)\|_{p^*}^{\text{weak}} \, \|(x_n)\|_p \right\},$$

with the infimum taken over all right p-nuclear Bloch representations of f. The set of all right p-nuclear Bloch maps of $\mathbb{D}$ into X for which $f(0) = 0$ will be denoted by $\widehat{\mathcal{B}}_{\mathcal{N}^p}(\mathbb{D}, X)$.

Theorem 2.16 *$[\widehat{\mathcal{B}}_{\mathcal{N}^p}, \nu^{p\mathcal{B}}]$ is a Banach normalized Bloch ideal for any $p \in [1, \infty)$.*

Proof (N1) Let $f \in \widehat{\mathcal{B}}_{\mathcal{N}^p}(\mathbb{D}, X)$ and let $\sum_{n\geq 1} g_n \cdot x_n$ be a right p-nuclear Bloch representation of f. It is clear that $f'(z) = \sum_{n=1}^{\infty} g_n'(z)x_n$ for all $z \in \mathbb{D}$. For each z in $\mathbb{D}$, we have

$$\begin{aligned}(1-|z|^2)\sum_{k=1}^{m}\left\|g_k'(z)x_k\right\| &\leq \left(\sum_{k=1}^{m}(1-|z|^2)^{p^*}\left|g_k'(z)\right|^{p^*}\right)^{\frac{1}{p^*}}\left(\sum_{k=1}^{m}\|x_k\|^p\right)^{\frac{1}{p}} \\ &= \left(\sum_{k=1}^{m}\left|(1-|z|^2)\gamma_z(g_k)\right|^{p^*}\right)^{\frac{1}{p^*}}\left(\sum_{k=1}^{m}\|x_k\|^p\right)^{\frac{1}{p}} \\ &\leq \|(g_n)\|_{p^*}^{\text{weak}}\,\|(x_n)\|_p\end{aligned}$$

for all $m \in \mathbb{N}$. Hence,

$$(1-|z|^2)\left\|f'(z)\right\| \leq (1-|z|^2)\sum_{n=1}^{\infty}\left\|g_n'(z)x_n\right\| \leq \|(g_n)\|_{p^*}^{\text{weak}}\,\|(x_n)\|_p$$

for all $z \in \mathbb{D}$, which gives $\rho_{\mathcal{B}}(f) \leq \|(g_n)\|_{p^*}^{\text{weak}}\,\|(x_n)\|_p$. Since the right p-nuclear Bloch representation of f was arbitrary, we deduce that $\rho_{\mathcal{B}}(f) \leq \nu^{p\mathcal{B}}(f)$. Mimicking the proof of Theorem 5.25 in [8], we can prove that $[\widehat{\mathcal{B}}_{\mathcal{N}^p}, \nu^{p\mathcal{B}}]$ is a Banach normalized Bloch ideal.

(N2) Take g in $\widehat{\mathcal{B}}(\mathbb{D})$ and x in X. Clearly, $g \cdot x \in \widehat{\mathcal{B}}_{\mathcal{N}^p}(\mathbb{D}, X)$ with $\nu^{p\mathcal{B}}(g \cdot x) \leq \rho_{\mathcal{B}}(g)\,\|x\|$. For the reverse inequality, apply that $\rho_{\mathcal{B}} \leq \nu^{p\mathcal{B}}$ on $\widehat{\mathcal{B}}_{\mathcal{N}^p}(\mathbb{D}, X)$ by (N1), and that $[\widehat{\mathcal{B}}, \rho_{\mathcal{B}}]$ is a normed normalized Bloch ideal by [11, Proposition 5.13].

(N3) Let $h \in \widehat{\mathcal{H}}(\mathbb{D}, \mathbb{D})$, $f \in \widehat{\mathcal{B}}_{\mathcal{N}^p}(\mathbb{D}, X)$ and $T \in \mathcal{L}(X, Y)$. Let $\sum_{n\geq 1} g_n \cdot x_n$ be a right p-nuclear Bloch representation of f. We have

$$\begin{aligned}&(1-|z|^2)\left\|\left(T\circ f\circ h-\sum_{k=1}^{n}(g_k\circ h)\cdot T(x_k)\right)'(z)\right\| \\ &\qquad = (1-|z|^2)\left|h'(z)\right|\left\|T\left(f'(h(z))-\sum_{k=1}^{n}g_k'(h(z))x_k\right)\right\|\end{aligned}$$

$$\leq (1-|h(z)|^2)\,\|T\|\left\|\left(f-\sum_{k=1}^{n} g_k\cdot x_k\right)'(h(z))\right\|$$
$$\leq \|T\|\,\rho_{\mathcal{B}}\left(f-\sum_{k=1}^{n} g_k\cdot x_k\right)$$

for any $z\in\mathbb{D}$ and $n\in\mathbb{N}$, by using Pick–Schwarz Lemma. Taking supremum over all $z\in\mathbb{D}$, we obtain

$$\rho_{\mathcal{B}}\left(T\circ f\circ h-\sum_{k=1}^{n}(g_k\circ h)\cdot T(x_k)\right)\leq \|T\|\,\rho_{\mathcal{B}}\left(f-\sum_{k=1}^{n} g_k\cdot x_k\right)$$

for all $n\in\mathbb{N}$. From this, $T\circ f\circ h=\sum_{n=1}^{\infty}(g_n\circ h)\cdot T(x_n)$ in $(\widehat{\mathcal{B}}(\mathbb{D},Y),\rho_{\mathcal{B}})$, where $(g_n\circ h)\in\ell_{p^*}^{\text{weak}}(\widehat{\mathcal{B}}(\mathbb{D}))$ with

$$\|(g_n\circ h)\|_{p^*}^{\text{weak}}=\sup_{\phi\in B_{\widehat{\mathcal{B}}(\mathbb{D})^*}}\left(\sum_{n=1}^{\infty}|\phi(g_n\circ h)|^{p^*}\right)^{\frac{1}{p^*}}$$
$$=\sup_{\phi\in B_{\widehat{\mathcal{B}}(\mathbb{D})^*}}\left(\sum_{n=1}^{\infty}|(\phi\circ C_h)(g_n)|^{p^*}\right)^{\frac{1}{p^*}}\leq \|(g_n)\|_{p^*}^{\text{weak}},$$

and $\|(T(x_n))\|_p\leq\|T\|\,\|(x_n)\|_p$. Hence, $T\circ f\circ h\in\widehat{\mathcal{B}}_{\mathcal{N}^p}(\mathbb{D},Y)$ with

$$\nu^{p\mathcal{B}}(T\circ f\circ h)\leq\|(g_n)\|_{p^*}^{\text{weak}}\,\|T\|\,\|(x_n)\|_p,$$

and so $\nu^{p\mathcal{B}}(T\circ f\circ h)\leq\|T\|\,\nu^{p\mathcal{B}}(f)$. □

A right p-nuclear Bloch map f of $\mathbb{D}$ into X with $f(0)=0$ and its associate linearisation S_f from $\mathcal{G}(\mathbb{D})$ into X are related as follows.

Proposition 2.17 *Let $p\in[1,\infty)$ and $f\in\widehat{\mathcal{B}}(\mathbb{D},X)$. Then $f\colon\mathbb{D}\to X$ is right p-nuclear Bloch if and only if $S_f\colon\mathcal{G}(\mathbb{D})\to X$ is right p-nuclear, in whose case, $\nu^p(S_f)=\nu^{p\mathcal{B}}(f)$. Moreover, $f\mapsto S_f$ is a linear isometry from $(\widehat{\mathcal{B}}_{\mathcal{N}^p}(\mathbb{D},X),\nu^{p\mathcal{B}})$ onto $(\mathcal{N}^p(\mathcal{G}(\mathbb{D}),X),\nu^p)$.*

Proof Assume that $f\in\widehat{\mathcal{B}}_{\mathcal{N}^p}(\mathbb{D},X)$ and let $\sum_{n\geq1}g_n\cdot x_n$ be a right p-nuclear Bloch representation of f. By Theorem 1.1, there is a unique $S_f\in\mathcal{L}(\mathcal{G}(\mathbb{D}),X)$ for which $S_f\circ\Gamma=f'$. Analogously, for each $n\in\mathbb{N}$, we have a functional $S_{g_n}\in\mathcal{G}(\mathbb{D})^*$ with $||S_{g_n}||=\rho_{\mathcal{B}}(g_n)$ and $S_{g_n}\circ\Gamma=g_n'$. Notice that $\sum_{n=1}^{+\infty}S_{g_n}\cdot x_n\in\mathcal{L}(\mathcal{G}(\mathbb{D}),X)$. Indeed, given $m\in\mathbb{N}$, the Hahn–Banach Theorem guarantees that for each $k\in\{1,\ldots,m\}$, there exists a functional $\phi_k\in B_{\widehat{\mathcal{B}}(\mathbb{D})^*}$ such that $|\phi_k(g_k)|=\rho_{\mathcal{B}}(g_k)$ and, using the Hölder inequality, we have

$$\sum_{k=1}^{m}\left\|S_{g_k}\cdot x_k\right\| = \sum_{k=1}^{m}\left\|S_{g_k}\right\|\|x_k\| = \sum_{k=1}^{m}\rho_{\mathcal{B}}(g_k)\|x_k\|$$
$$\leq \left(\sum_{k=1}^{m}|\phi_k(g_k)|^{p^*}\right)^{\frac{1}{p^*}}\left(\sum_{k=1}^{m}\|x_k\|^p\right)^{\frac{1}{p}} \leq \|(g_n)\|_{p^*}^{\text{weak}}\|(x_n)\|_p .$$

We can write

$$f' = \sum_{n=1}^{\infty} g_n' \cdot x_n = \sum_{n=1}^{\infty}(S_{g_n}\circ\Gamma)\cdot x_n = \left(\sum_{n=1}^{\infty} S_{g_n}\cdot x_n\right)\circ\Gamma.$$

Hence, $S_f = \sum_{n=1}^{\infty} S_{g_n}\cdot x_n$ by Theorem 1.1, where $(S_{g_n}) \in \ell_{p^*}^{\text{weak}}(\mathcal{G}(\mathbb{D})^*)$ and also $\left\|(S_{g_n})\right\|_{p^*}^{\text{weak}} \leq \|(g_n)\|_{p^*}^{\text{weak}}$. Thus $S_f \in \mathcal{N}^p(\mathcal{G}(\mathbb{D}), X)$ with $\nu^p(S_f) \leq \|(g_n)\|_{p^*}^{\text{weak}}\|(x_n)\|_p$. Passing to the infimum over all right p-nuclear Bloch representation of f, we get that $\nu^p(S_f) \leq \nu^{p\mathcal{B}}(f)$.

Conversely, suppose that $S_f \in \mathcal{N}^p(\mathcal{G}(\mathbb{D}), X)$ and let $\sum_{n\geq 1}\phi_n\cdot x_n$ be a right p-nuclear representation of S_f. By Theorem 1.1, for a natural n, we can take a $g_n \in \widehat{\mathcal{B}}(\mathbb{D})$ for which $\Lambda(g_n) = \phi_n$ with $\rho_{\mathcal{B}}(g_n) = ||\phi_n||$. Therefore,

$$\begin{aligned}(1-|z|^2)\left\|\left(f-\sum_{k=1}^{n} g_k\cdot x_k\right)'(z)\right\| &= (1-|z|^2)\left\|f'(z)-\sum_{k=1}^{n} g_k'(z)x_k\right\| \\ &= (1-|z|^2)\left\|S_f(\gamma_z)-\sum_{k=1}^{n}\Lambda(g_k)(\gamma_z)x_k\right\| \\ &= (1-|z|^2)\left\|\left(S_f-\sum_{k=1}^{n}\phi_k\cdot x_k\right)(\gamma_z)\right\| \\ &\leq (1-|z|^2)\left\|S_f-\sum_{k=1}^{n}\phi_k\cdot x_k\right\|\|\gamma_z\| \\ &= \left\|S_f-\sum_{k=1}^{n}\phi_k\cdot x_k\right\|\end{aligned}$$

for all $z \in \mathbb{D}$ and $n \in \mathbb{N}$. Taking supremum over all $z \in \mathbb{D}$, we obtain

$$\rho_{\mathcal{B}}\left(f-\sum_{k=1}^{n} g_k\cdot x_k\right) \leq \left\|S_f-\sum_{k=1}^{n}\phi_k\cdot x_k\right\|$$

for all $n \in \mathbb{N}$. Hence, $f = \sum_{n=1}^{\infty} g_n\cdot x_n$ in $(\widehat{\mathcal{B}}(\mathbb{D}, X), \rho_{\mathcal{B}})$, where $(g_n) \in \ell_{p^*}^{\text{weak}}(\widehat{\mathcal{B}}(\mathbb{D}))$ with $\|(g_n)\|_{p^*}^{\text{weak}} \leq \|(\phi_n)\|_{p^*}^{\text{weak}}$. So $f \in \widehat{\mathcal{B}}_{\mathcal{N}^p}(\mathbb{D}, X)$ with $\nu^{p\mathcal{B}}(f) \leq \|(\phi_n)\|_{p^*}^{\text{weak}}\|(x_n)\|_p$, and thus $\nu^{p\mathcal{B}}(f) \leq \nu^p(S_f)$.

The last assertion in the statement follows easily from what was proved above and from Theorem 1.1. □

Corollary 2.18 *If* $p \in [1, \infty)$ *and* $f \in \widehat{\mathcal{B}}_{\mathcal{N}^p}(\mathbb{D}, X)$*, then* $f \in \widehat{\mathcal{B}}_{\mathcal{K}_p}(\mathbb{D}, X)$ *and* $k_p^{\mathcal{B}}(f) \leq \nu^{p\mathcal{B}}(f)$.

Proof From Proposition 2.17, one has $S_f \in \mathcal{N}^p(\mathcal{G}(\mathbb{D}), X)$ and $\nu^p(S_f) = \nu^{p\mathcal{B}}(f)$. Thus, $S_f \in \mathcal{K}_p(\mathcal{G}(\mathbb{D}), X)$ and $k_p(S_f) \leq \nu^p(S_f)$ (see [7, p. 295]). So $f \in \widehat{\mathcal{B}}_{\mathcal{K}_p}(\mathbb{D}, X)$ and $k_p^{\mathcal{B}}(f) \leq \nu^{p\mathcal{B}}(f)$ by Theorem 2.5. □

Inspired by operator ideal theory (see [16, Section 4.7]), we introduce:

Definition 2.19 Given a normed normalized Bloch ideal $\mathcal{I}^{\widehat{\mathcal{B}}}$, its *surjective hull* is the smallest surjective normed normalized Bloch ideal which contains $\mathcal{I}^{\widehat{\mathcal{B}}}$, and it is denoted by $(\mathcal{I}^{\widehat{\mathcal{B}}})^{\text{sur}}$.

We have seen above that the Banach normalized Bloch ideal $(\widehat{\mathcal{B}}_{\mathcal{K}_p}, k_p^{\mathcal{B}})$ is surjective and contains $\widehat{\mathcal{B}}_{\mathcal{N}^p}$. Therefore, $(\widehat{\mathcal{B}}_{\mathcal{N}^p})^{\text{sur}} \subseteq \widehat{\mathcal{B}}_{\mathcal{K}_p}$. It would be interesting to know if this inclusion becomes an equality as it occurs (see [7, Proposition 3.11]) in the linear setting.

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