Asymptotics of orthogonal polynomials with respect to an analytic weight with algebraic singularities on the circle

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Abstract

Strong asymptotics of polynomials orthogonal on the unit circle with respect to a weight of the form

$$W(z) = w(z) \prod_{k=1}^{m} |z-a_k|^{2\beta_k}, \quad |z|=1, \quad |a_k|=1, \quad \beta_k > -1/2, \quad k=1,\ldots,m,$$

where $w(z) > 0$ for $|z| = 1$ and can be extended as a holomorphic and non-vanishing function to an annulus containing the unit circle. The formulas obtained are valid uniformly in the whole complex plane. As a consequence, we obtain some results about the distribution of zeros of these polynomials, the behavior of their leading and Verblunsky coefficients, as well as give an alternative proof of the Fisher-Hartwig conjecture about the asymptotics of Toeplitz determinants for such type of weights. The main technique is the steepest descent analysis of Deift and Zhou, based on the matrix Riemann-Hilbert characterization proposed by Fokas, Its and Kitaev.

1 Introduction and statement of the main results

Let us first set some notation that will be widely used in what follows. We denote by $\mathbb{T}$ the unit circle on the complex plane $\mathbb{C}$ (circle of radius 1 centered at the origin), and $\mathbb{D} \overset{\text{def}}{=} \{z \in \mathbb{C} : |z| < 1\}$ is the open unit disc. If $r < 1$, let $\mathbb{A}_r \overset{\text{def}}{=} \{z \in \mathbb{C} : r < |z| < 1/r\}$. Every oriented Jordan curve or arc γ induces naturally the notions of left (or "+") and right (or "−") sides of γ. We also denote by a subindex "+" (respectively, "−") the left (respectively, right) boundary values of functions on γ.

An integrable non-negative function W defined on $\mathbb{T}$ is called a *weight* if

$$\int_{\mathbb{T}} W(z)\, |dz| > 0\,. \tag{1}$$

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For each weight W there exists a unique sequence of polynomials φ_n (called *Szegő polynomials*), orthonormal with respect to W, satisfying $\varphi_n(z) = \kappa_n z^n +$ lower degree terms, $\kappa_n > 0$, and

$$\oint_{\mathbb{T}} \varphi_n(z)\overline{\varphi_m(z)}\, W(z)|dz| = \delta_{mn}\,. \tag{2}$$

We denote by $\Phi_n(z) \stackrel{\text{def}}{=} \varphi_n(z)/\kappa_n$ the corresponding monic orthogonal polynomials. It is well known that they satisfy the Szegő recurrence

$$\Phi_{n+1}(z) = z\Phi_n(z) - \overline{\alpha_n}\,\Phi_n^*(z)\,, \quad \Phi_0(z) \equiv 1\,, \tag{3}$$

where we use the standard notation $\Phi_n^*(z) \stackrel{\text{def}}{=} z^n\overline{\Phi_n(1/\overline{z})}$. The parameters $\alpha_n = -\overline{\Phi_{n+1}(0)}$ are called *Verblunsky coefficients* (also *reflection coefficients* or *Schur parameters*) and satisfy $\alpha_n \in \mathbb{D}$ for $n = 0, 1, 2, \ldots$ (see [15] for details).

If the weight W satisfies the condition

$$\int_{\mathbb{T}} \log W(z)\,|dz| > -\infty\,, \tag{4}$$

then the *Szegő function* of W (see e.g. [17, Ch. X, §10.2]),

$$D(W; z) \stackrel{\text{def}}{=} \exp\left(\frac{1}{4\pi}\int_0^{2\pi} \log W(e^{i\theta})\,\frac{e^{i\theta}+z}{e^{i\theta}-z}\,d\theta\right)\,, \tag{5}$$

can be defined. This function is piecewise analytic and non-vanishing, defined for $z \notin \mathbb{T}$, and we will denote by D_{i} and D_{e} its values for $|z| < 1$ and $|z| > 1$, respectively. It is easy to verify that

$$\overline{D_{\mathrm{i}}\left(W; \frac{1}{\overline{z}}\right)} = \frac{1}{D_{\mathrm{e}}(W; z)}\,, \quad |z| > 1\,, \tag{6}$$

and in particular, $W(z) = |D_{\mathrm{e}}(W; z)|^{-2}$ for $z \in \mathbb{T}$. The role of the Szegő function in the description of the asymptotic behavior of the orthogonal polynomials is well known; for instance,

$$\lim_{n\to\infty} \frac{\Phi_n(z)}{z^n} = \frac{D_{\mathrm{e}}(W; z)}{D_{\mathrm{e}}(W; \infty)}$$

uniformly on each compact set in the exterior of the unit disk.

In this paper we focus on weights of a specific form. Assume that w is a strictly positive function defined on the unit circle, which can be extended as a holomorphic and non-vanishing function to an annulus $\mathbb{A}_\rho$ ($0 < \rho < 1$); in fact, a "canonical" form for such a function is $w(z) = |f(z)|^2$, $z \in \mathbb{T}$, where f is holomorphic and non-vanishing in $\mathbb{A}_\rho$. For points $a_k \in \mathbb{T}$ and values $\beta_k > -1/2$, $k = 1, \ldots, m$, we define the following weight:

$$W(z) \stackrel{\text{def}}{=} w(z)\prod_{k=1}^m |z - a_k|^{2\beta_k}\,, \quad z \in \mathbb{T}\,. \tag{7}$$

It can have zeros or blow up at a_k's, but still conditions (1) and (4) are satisfied. We are interested in the asymptotic behavior of the corresponding sequences $\{\Phi_n(z)\}$, $\{\kappa_n\}$ and $\{\alpha_n\}$ when $n \to \infty$ and $z \in \mathbb{C}$.

The case when all $\beta_k = 0$ (that is, when W is a positive analytic weight on $\mathbb{T}$) has been studied in [13], where a canonical representation of the corresponding orthogonal polynomials in terms of iterates of the Cauchy transform of the *scattering function* of W,

$$\mathcal{S}(W; z) \stackrel{\text{def}}{=} D_{\rm i}(W; z) D_{\rm e}(W; z)\,, \tag{8}$$

was derived. Also, detailed asymptotic formulas were obtained. A characterizing feature of this case is that the zeros of Φ_n's stay away from $\mathbb{T}$, clustering (with a possible exception of a $o(n)$ number of them) at an inner circle determined by the analytic continuation of $D_{\rm e}$.

When any $\beta_k \neq 0$, W is no longer a positive and analytic weight on $\mathbb{T}$, and in this situation the majority of the zeros of the Szegő polynomials are attracted by the unit circle. One of the main goals of the paper is to provide asymptotic formulas for Φ_n's valid uniformly on the whole complex plane, and in particular, in a neighborhood of a_k's. For partial results in the case when all $\beta_k = 1$ see [2].

In order to state the main results we need to introduce a new piece of notation.

Let $\mathcal{A} \stackrel{\text{def}}{=} \{a_1, \dots, a_m\}$; for the sake of brevity hereafter we call these points generically as "singularities" of the weight, although W is regular at a_k's for integer β_k's. We fix for what follows $0 < \delta < 1 - \rho$, such that additionally $\delta < \frac{1}{3} \min_{i \neq j} |a_i - a_j|$, and denote

$$\mathcal{B}_k \stackrel{\text{def}}{=} \{z \in \mathbb{C} : |z - a_k| < \delta\}\,, \quad \mathcal{C}_k \stackrel{\text{def}}{=} \{z \in \mathbb{C} : |z - a_k| = \delta\}\,, \quad k = 1, \dots, m\,, \tag{9}$$

as well as $B \stackrel{\text{def}}{=} \cup_{k=1}^m \mathcal{B}_k$. Furthermore, given a subset $X \subset \mathbb{C}$ and a value $a \in \mathbb{C}$ we will use the standard notation $a \cdot X = \{ax : x \in X\}$; consistently, $\mathcal{A} \cdot X \stackrel{\text{def}}{=} \cup_{k=1}^m (a_k \cdot X)$.

Under assumptions on W in (7), both $D_{\rm i}(W; z)$ and $D_{\rm e}(W; z)$ admit an analytic extension across $\mathbb{T} \setminus \mathcal{A}$; we keep the same notation for these analytic continuations. More precisely, $D_{\rm i}(W; z)$ is holomorphic in $\{z \in \mathbb{C} : |z| < 1/\rho\} \setminus (\mathcal{A} \cdot [1, 1/\rho))$, and $D_{\rm e}(W; z)$ is holomorphic in $\{z \in \mathbb{C} : |z| > \rho\} \setminus (\mathcal{A} \cdot (\rho, 1])$; see Section 2 for a detailed discussion.

First we describe the asymptotic behavior of Φ_n's away from the singular points of the weight:

Theorem 1 *For monic orthogonal polynomials Φ_n corresponding to the weight W given in* (7) *there exist a complete asymptotic expansion*

$$\Phi_n(z) = \sum_{k=0}^{\infty} \frac{\mathfrak{f}_k(z)}{n^k} \tag{10}$$

valid uniformly in $\mathbb{C}$. Each $\mathfrak{f}_k(z)$ is a piecewise analytic function, holomorphic in each domain specified below. In particular, there exist constants $\vartheta_k \in \mathbb{T}$, $k = 1, \dots, m$, defined by formula (39) *below, such that:*

(i) formula

$$\Phi_n(z) = \frac{D_{\rm i}(W; 0)}{D_{\rm i}(W; z)} \, \frac{1}{n} \left(\sum_{k=1}^{m} \frac{\beta_k \vartheta_k}{a_k - z} \, a_k^{n+1} + O\left(\frac{1}{n}\right) \right)$$

holds uniformly on every compact subset of $\mathbb{D}$;

(ii) formula

$$\Phi_n(z) = z^n \frac{D_e(W;z)}{D_e(W;\infty)} \left(1 + \frac{1}{n}\sum_{k=1}^{m} \frac{a_k \beta_k^2}{a_k - z} + \mathcal{O}\left(\frac{1}{n^2}\right)\right) + \frac{D_i(W;0)}{D_i(W;z)} \left(\frac{1}{n}\sum_{k=1}^{m} \frac{\beta_k \vartheta_k}{a_k - z}\, a_k^{n+1} + \mathcal{O}\left(\frac{1}{n^2}\right)\right)$$

holds uniformly on every compact subset of $\mathbb{A}_\rho \setminus (B \cup (\mathcal{A} \cdot (\rho, 1/\rho))$*;*

(iii) formula

$$\Phi_n(z) = z^n \frac{D_e(W;z)}{D_e(W;\infty)} \left(1 + \frac{1}{n}\sum_{k=1}^{m} \frac{a_k \beta_k^2}{a_k - z} + \mathcal{O}\left(\frac{1}{n^2}\right)\right).$$

holds uniformly on every compact subset of $\mathbb{C} \setminus \overline{\mathbb{D}}$*.*

Remark 1 It is well known (see [7] as well as Section 3 below) that further terms $\mathfrak{f}_k$ of the expansion (10) can be obtained by nested contour integration and the calculus of residues. However, the difficulty of the computation increases with k, and we limit our attention to the leading nontrivial terms of (10).

Corollary 1 *For every compact set* $K \subset \mathbb{D}$ *there exists* $N = N(K) \in \mathbb{N}$ *such that for every* $n \geq N$*, every* Φ_n *has at most* $m-1$ *zeros on* K*.*

However, the global behavior of these "spurious" $m-1$ zeros can be complicated; we describe their limiting set below (Theorem 6).

In order to formulate the asymptotic behavior of the Szegő polynomials in a neighborhood of each singular point a_k we define an auxiliary function: if $\zeta^{1/2}$ stands for the main branch of the square root in $\mathbb{C} \setminus (-\infty, 0]$ (that is, $\zeta^{1/2} > 0$ for $\zeta > 0$), and J_ν are the Bessel functions of the first kind, then for $\beta > -1/2$ set

$$\mathcal{H}(\beta;\zeta) \stackrel{\text{def}}{=} \begin{cases} e^{-2\pi i\beta}\zeta^{1/2}\left(iJ_{\beta+1/2}(\zeta) + J_{\beta-1/2}(\zeta)\right), & \text{if } \zeta \text{ is in the second quadrant,} \\ \zeta^{1/2}\left(iJ_{\beta+1/2}(\zeta) + J_{\beta-1/2}(\zeta)\right), & \text{otherwise.} \end{cases} \tag{11}$$

With the notations introduced in (9) we have the following result about local behavior of the polynomials at the singularities of the weight:

Theorem 2 *Let* $k \in \{1, \ldots, m\}$*,* $|z - a_k| \leq \delta$*. For* $z \in \mathcal{B}_k$ *define*

$$\zeta_n(z) \stackrel{def}{=} -i\,\frac{n}{2}\,\log\left(\frac{z}{a_k}\right), \tag{12}$$

where we take the principal branch of the logarithm. Then for $z \in \mathcal{B}_k$*,*

$$\Phi_n(z) = \sqrt{\frac{\pi}{2}}\, e^{\pi i\beta_k/2}\, \frac{D_e(W;z)}{D_e(W;\infty)}\, (a_k\, z)^{n/2}\, \mathcal{H}\left(\beta_k; \zeta_n(z)\right) \left(1 + \mathcal{O}\left(\frac{1}{n}\right)\right), \tag{13}$$

where we take the principal branch of the square root in the described neighborhood of $z = a_k$*. The* $\mathcal{O}(1/n)$ *term in* (13) *is uniform in the closed disk* $\overline{\mathcal{B}_k}$*.*

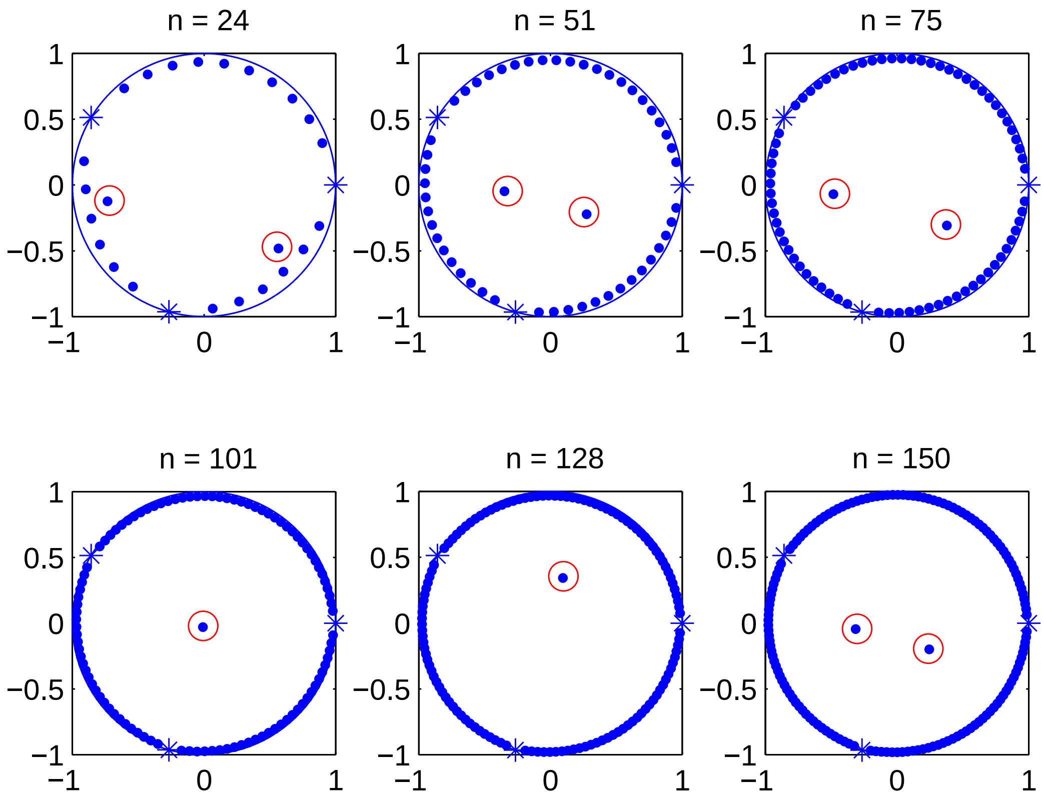

Figure 1: Zeros of Φ_n for several values of n with $W(z) = |(z-1)(z-a)(z-a^2)|^4$, $z \in \mathbb{T}$, where $a = \exp(\pi i \sqrt{2})$. Centers of the circles are the zeros of $\sum_{k=1}^{m} \beta_k \vartheta_k a_k^{n+1}/(a_k - z)$.

Remark 2 It will be shown in Section 4 that $D_{\mathrm{e}}(W;z)\mathcal{H}(\beta_k;\zeta_n(z))$ is a holomorphic function in a neighborhood of $z = a_k$.

Remark 3 In the particular case of $\beta_k = 0$ (a removable singularity) we have

$$\mathcal{H}(0;\zeta) = \zeta^{1/2}\left(iJ_{\frac{1}{2}}(\zeta) + J_{-\frac{1}{2}}(\zeta)\right) = \sqrt{\frac{\pi}{2}}\, e^{i\zeta}\,,$$

and (13) takes the form

$$\Phi_n(z) = \frac{z^n D_{\mathrm{e}}(W;z)}{D_{\mathrm{e}}(W;\infty)} \left(1 + \mathcal{O}\left(\frac{1}{n}\right)\right),$$

(cf. Theorem 1, *(ii)*–*(iii)*).

Remark 4 The steepest descent analysis used for the proof of the theorems above allows to find further terms of the asymptotic expansion for the polynomials. However, for the sake of simplicity we decided to restrict our attention to the leading terms bearing already non-trivial information about the main parameters and the zeros, as it will be shown next.

For a weight W on $\mathbb{T}$ let us define the geometric mean

$$G[W] \stackrel{\text{def}}{=} \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \log\left(W\left(e^{i\theta}\right) d\theta\right)\right) . \tag{14}$$

If W is given by (7), then $G[W] = G[w] = D_{\mathrm{i}}^2(W;0)$ (see formulas (33)–(34) below).

With respect to the Verblunsky and leading coefficients of the orthonormal polynomials we have

Theorem 3 *For weight W given in* (7) *there exist constants $\vartheta_k \in \mathbb{T}$, $k = 1, \dots, m$, introduced in* (39) *below, such that the Verblunsky coefficients α_n defined by the recurrence* (3) *satisfy*

$$\alpha_n = -\frac{1}{n}\sum_{k=1}^{m} \frac{\beta_k}{\vartheta_k}\frac{1}{a_k^{n+1}} + \mathcal{O}\left(\frac{1}{n^2}\right) , \quad n \to \infty . \tag{15}$$

For the leading coefficients κ_n of the orthonormal polynomials φ_n the following asymptotic formula holds:

$$\kappa_{n-1}^2 = \frac{1}{G[2\pi w]}\left(1 - \frac{1}{n}\sum_{k=1}^{m}\beta_k^2 + \mathcal{O}\left(\frac{1}{n^2}\right)\right) , \quad n \to \infty. \tag{16}$$

Remark 5 Obviously formula (16) makes sense for $n \geq \sum_{k=1}^{m}\beta_k^2$, and exhibits, at least asymptotically, the growing character of the leading coefficients κ_n.

A direct consequence of (16) is the asymptotic behavior of the Toeplitz determinants related to the weight W. If we define the moments

$$d_k \stackrel{\text{def}}{=} \oint_{z\in\mathbb{T}} z^{-n} W(z)|dz| ,$$

then the Toeplitz determinants are

$$\mathcal{D}_n(W) \stackrel{\text{def}}{=} \det\left[(d_{j-i})_{i,j=0}^n\right] . \tag{17}$$

Theorem 4 *Under the assumption above there exists a constant $\varkappa$ depending on W such that*

$$\mathcal{D}_n(W) = \varkappa \left(G[2\pi w]\right)^n n^{\sum_{k=1}^m \beta_k^2} \left(1 + o(1)\right) , \quad n \to \infty . \tag{18}$$

This formula is in accordance with the well known Fisher-Hartwig conjecture (see e.g. [4]).

Remark 6 As it was mentioned above, in the case considered, $G[w] = G[W]$, so we may replace w by W in the right hand side of both formulas (16) and (18).

Let us discuss now how Theorems 1–2 reveal the behavior of the zeros of the polynomials Φ_n. Qualitatively we can describe the picture as follows: the vast majority of the zeros will approach the circle $\mathbb{T}$ regularly and radially uniformly along level curves Γ_n defined below, that are asymptotically close to circles centered at the origin with radius $n^{-1/n}$. Singularities of the weight exert the following influence on the zeros of Φ_n: those closest to points a_k converge to $\mathbb{T}$ faster than the rest (their absolute value behaves like $c^{-1/n}$, with $c > 1$ depending on β_k), and either leaving a "gap" around a_k (if $\beta_k > 0$) or approaching it radially (if $\beta_k < 0$). Furthermore, a bounded number of "spurious" zeros may wander inside the unit disk; each Φ_n will have at most $m-1$ of these zeros (Corollary 1), and their global behavior will depend in particular on the relative positions of points a_k.

Let

$$\Gamma_n \stackrel{\text{def}}{=} \left\{ z \in \mathbb{C} : \, |z|^n |\mathcal{S}(W;z)| = \frac{1}{n} \left| \sum_{k=1}^{m} \frac{\beta_k \vartheta_k a_k^{n+1}}{z - a_k} \right| \right\} , \tag{19}$$

where $\mathcal{S}(W;z)$ is the scattering function defined in (8). These curves are well defined: although $\mathcal{S}(W;z)$ is multivalued in a neighborhood of $\mathbb{T}$, by (37) below, $|\mathcal{S}(W;z)|$ is single-valued and positive in $\mathbb{A}_\rho$. Furthermore, given an analytic function f and $\varepsilon > 0$ denote

$$\mathcal{Z}(f) \stackrel{\text{def}}{=} \{ z : \, f(z) = 0 \} , \quad \mathcal{Z}_\varepsilon(f) \stackrel{\text{def}}{=} \left\{ z \in \mathbb{C} : \, \min_{t \in \mathcal{Z}(f)} |t - z| < \varepsilon \right\} , \tag{20}$$

and let

$$\Gamma_n(\varepsilon) \stackrel{\text{def}}{=} \left\{ z \in \mathbb{C} : \, \min_{t \in \Gamma_n} |t - z| < \varepsilon \right\} ,$$

be the ε-neighborhood of Γ_n.

Let also

$$\theta_1 = 1, \quad \text{and} \quad \theta_k = \frac{1}{2\pi} \left(\arg a_k - \arg a_1 \right) , \qquad k = 2, \ldots, m , \tag{21}$$

and

$$\mathcal{R}_n(z) \stackrel{\text{def}}{=} \sum_{k=1}^{m} \frac{\beta_k \vartheta_k}{z - a_k} \, e^{2\pi i (n+1) \theta_k} . \tag{22}$$

Theorem 5 *There exists $0 < \varepsilon < \delta$ such that for all sufficiently large $n \in \mathbb{N}$ every "pie slice" of the form*

$$\left\{ z \in \Gamma_n(\varepsilon) : \, \frac{\alpha + 2k_1\pi}{n} < \arg(z) < \frac{\alpha + 2k_2\pi}{n} \right\} \subset \mathbb{A}_\rho \setminus \left(B \cup Z_\varepsilon(\mathcal{R}_n) \right) ,$$

with appropriately chosen $\alpha \in \mathbb{R}$, contains exactly $k = k_2 - k_1$ zeros of Φ_n, $z_1^{(n)}, \ldots, z_k^{(n)}$, satisfying

$$|z_i^{(n)}| = 1 - \frac{\log(n)}{n} + \mathcal{O}\left(\frac{1}{n} \right) , \tag{23}$$

and

$$\arg\left(z_{i+j}^{(n)}\right) - \arg\left(z_i^{(n)}\right) = \frac{2\pi j}{n} + \mathcal{O}\left(\frac{1}{n^2}\right). \tag{24}$$

See Figure 2 for an illustration of the statements of Theorem 5.

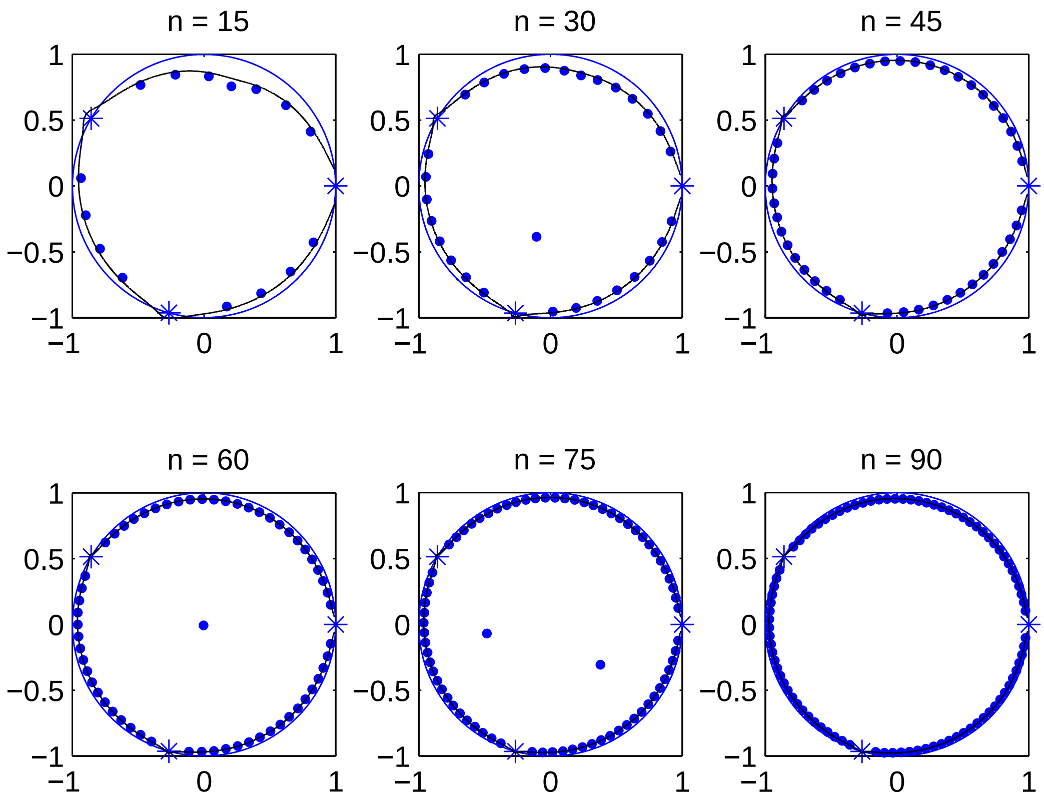

Figure 2: Zeros of Φ_n for $n = 15, 30, \ldots, 90$ with $W(z) = |(z-1)(z-a)(z-a^2)|^4$, $z \in \mathbb{T}$, where $a = \exp(\pi i\sqrt{2})$. Points 1, a and a^2 are indicated with asterisks. For comparison with the prediction of Theorem 5 we plot in each case the level curve Γ_n defined in (19).

We can also be more precise about the accumulation set of zeros of Φ_n's,

$$Z \stackrel{\text{def}}{=} \bigcap_{k\geq 1} \overline{\bigcup_{n\geq k} \mathcal{Z}(\Phi_n)}, \tag{25}$$

on compact sets of $\mathbb{D}$. Assertion *(i)* of Theorem 1 shows that the structure of $Z \cap \mathbb{D}$ depends on the relative positions of $a_1, \ldots, a_m$ on $\mathbb{T}$. Without loss of generality we may assume that $\theta_1 = 1, \theta_2, \ldots, \theta_v$ is the maximal subset of $\{\theta_1, \ldots, \theta_m\}$ linearly independent over the rational numbers $\mathbb{Q}$. Then there exist unique $r_{kj} \in \mathbb{Q}$, $k = 1, \dots, m$, $j = 1, \dots, v$, such that

$$\theta_k = \sum_{j=1}^{v} r_{kj}\, \theta_j\,, \qquad k = 1, \dots, m\,,$$

(obviously, $r_{kj} = \delta_{kj}$, for $1 \le k \le v$). Let $q_1 = 1$, and if $m \ge 2$, $0 \le p_k < q_k \in \mathbb{Z}$, $k = 2, \dots, m$ be such that

$$r_{k1} \equiv \frac{p_k}{q_k} \mod \mathbb{Z}, \quad k = 2, \dots, m\,.$$

Theorem 6 *Let $t \in Z \cap \mathbb{D}$. If $v = 1$ (that is, if all $\theta_k \in \mathbb{Q}$), then there exist $0 \le s_k < q_k$, $s_k \in \mathbb{Z}$, $k = 1, \dots, m$, such that*

$$\sum_{k=1}^{m} \frac{\beta_k \vartheta_k}{a_k - t}\, e^{2\pi i s_k / q_k} = 0\,. \tag{26}$$

If $v \ge 2$, then additionally there exist $X_2, \dots, X_v \in \mathbb{R}$ such that

$$\sum_{k=1}^{m} \frac{\beta_k \vartheta_k}{a_k - t}\, e^{2\pi i \left(s_k / q_k + \sum_{j=2}^{v} r_{kj}\, X_j \right)} = 0\,. \tag{27}$$

Moreover, every point in $\mathbb{D}$ that belongs to the manifold given by solutions of (27) when $X_2, \dots, X_v$ vary in $\mathbb{R}$, is an accumulation point of the zeros of $\{\Phi_n\}$.

Corollary 2 *(i) If $v = 1$, then $Z \cap \mathbb{D}$ is a discrete set of a finite number of points.*

(ii) If $v = 2$, then $Z \cap \mathbb{D}$ is an algebraic curve of degree $\le m$. In particular, if $v = m = 2$, then $Z \cap \mathbb{D}$ is either a circular arc (if $|\beta_1| \ne |\beta_2|$), or a diameter in $\mathbb{D}$ formed by the perpendicular bisector of the segment joining a_1 with a_2 (if $|\beta_1| = |\beta_2|$).

(iii) If $v > 2$, then $Z \cap \mathbb{D}$ is a two-dimensional domain bounded by algebraic curves.

Remark 7 From the method of proof it follows in fact that the zeros of the orthogonal polynomials will be equidistributed on the manifolds described above.

Compare these statements with the numerical results depicted in Figure 3. Take note that we plot *all* the zeros of Φ_n for $n = 1, \dots, 150$, in order to reveal the structure of Z inside $\mathbb{D}$.

Finally, in order to describe the behavior of the zeros that are closest to the singularity of the weight we must consider again function $\mathcal{H}(\beta; \zeta)$ introduced in (11); let us denote by $h(\beta)$ its zero of smallest absolute value such that $\mathrm{Re}(h(\beta)) \ge 0$ and $\mathrm{Im}(h(\beta)) > 0$. According to (85), $z = -\overline{h(\beta)}$ is also a zero of $\mathcal{H}(\beta; \zeta)$.

Theorem 7 *Among the zeros of $\Phi_n(z)$, the closest to the singularity a_k of the weight are those, given asymptotically by*

$$z^+ = a_k\, e^{2ih(\beta_k)/n}\, (1 + o(1)) \quad \text{and} \quad z^- = a_k\, e^{-2i\,\overline{h(\beta_k)}/n}\, (1 + o(1))\,,$$

approaching a_k symmetrically with respect to the radius $a_k \cdot (0, 1)$.

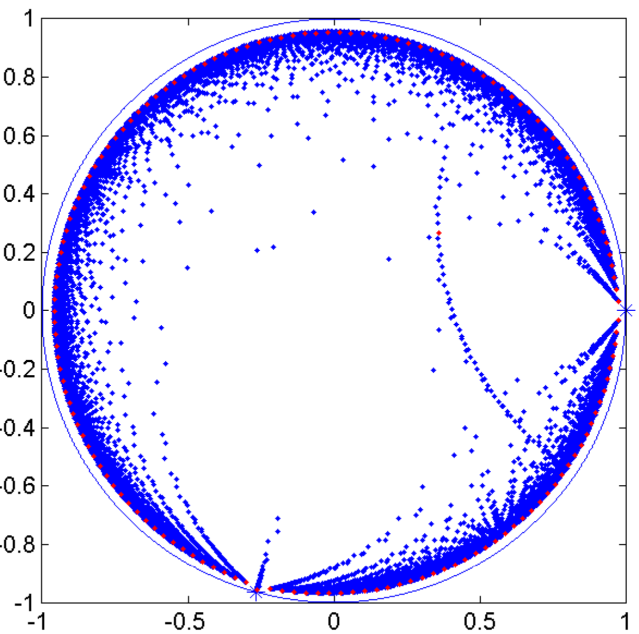

Figure 3: Zeros of Φ_n for $n = 1, 2, \ldots, 150$ with $W(z) = |z-1|^{1/3}|z-a|^{-2/3}$, $z \in \mathbb{T}$, where $a = \exp(\pi i\sqrt{2})$. Points 1 and a are indicated with asterisks. Observe different features of the behavior of the zeros described in the text: the zero of the weight at $z = 1$ repels the zeros of the polynomial, while the singularity at $z = a$ attracts a zero of $\Phi_n(z)$ that approaches $z = a$ radially (cf. Remark 9). Also the arc of the circle along which the zeros of Φ_n remaining inside $\mathbb{D}$ cluster is clearly visible (cf. Corollary 2).

Remark 8 Using formulas (9.6.3) of [1] we see that $\zeta = -ih(\beta)$ must satisfy the following equation:

$$I_{\beta+1/2}(\zeta) - I_{\beta-1/2}(\zeta) = 0\,, \quad \mathrm{Re}(\zeta) > 0\,, \quad \mathrm{Im}(\zeta) \geq 0\,, \tag{28}$$

where I_ν is the modified Bessel function. Some facts about the zeros of the function in the left hand side of (28) have been kindly provided to us by M. Muldoon [14]. For instance, for $\beta > 0$ this function is strictly negative in $(0, +\infty)$ (see [16]) and thus has no real positive zeros. Furthermore, for $-1/2 < \beta < 0$ this function has apparently a unique real positive zero, which is a monotonically decreasing function of β. Complementing these results with numerical experiments (see e.g. Remark 12 in Section 4), we claim that:

- for $\beta > 0$, $\mathrm{Re}(h(\beta)) > 0$ and $|h(\beta)| > \pi$;
- for $-1/2 < \beta < 0$, $\mathrm{Re}(h(\beta)) = 0$ and $|h(\beta)| < \pi$.

In particular, for $-1/2 < \beta < 0$, $z^+ = z^-$, lying (at least, asymptotically) on the radius $a_k \cdot (0, 1)$.

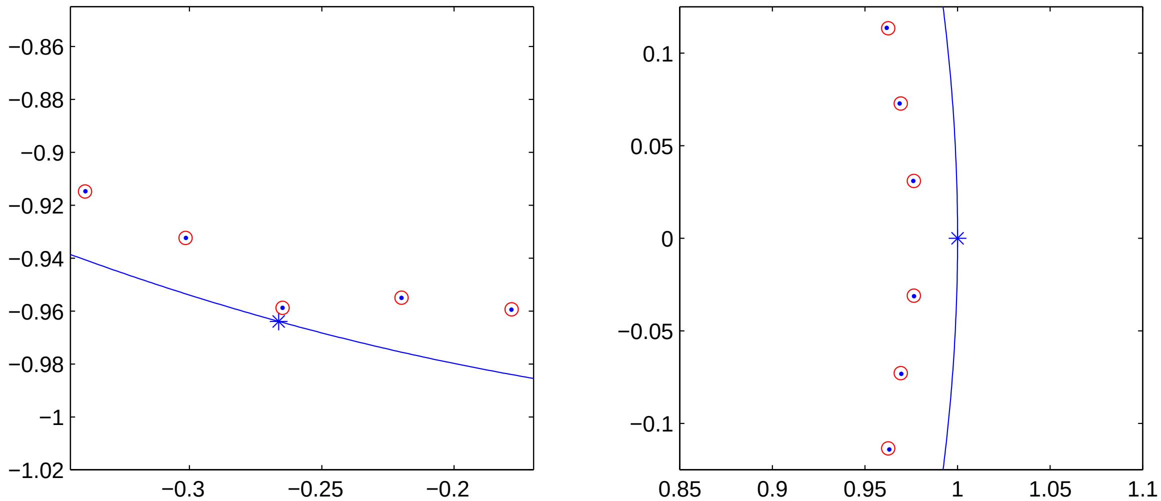

Figure 4: Zeros of Φ_{150} (dots) in a neighborhood of $z = a$ (left) and $z = 1$ (right) for $W(z) = |z-1|^{1/3}|z-a|^{-2/3}$, $z \in \mathbb{T}$, where $a = \exp(\pi i\sqrt{2})$. Points 1 and a are indicated with asterisks. Centers of the circles are the zeros of $\mathcal{H}(-1/3; \zeta_{150}(z))$ and $\mathcal{H}(1/6; \zeta_{150}(z))$, respectively.

Remark 9 We can compare the rate of convergence of the zeros of Φ_n to the unit circle. While, according to Theorem 5, the bulk of the zeros tend to $\mathbb{T}$ with a speed whose leading term is $n^{-1/n}$ (Theorem 5), the zeros closest to the singularity a_k are attracted by $\mathbb{T}$ with a rate, roughly speaking, equal to $c^{-1/n}$, where $c = e^{2\,\mathrm{Im}\, h(\beta_k)} > 1$. Taking into account also that $|z^{\pm} - a_k| = \frac{2}{n}|h(\beta)|\,(1 + o(1))$, Remark 8 and (24), it explains the influence that a_k's exert on the neighboring zeros of Φ_n: both pushing them to $\mathbb{T}$ and "repelling" them from (if $\beta > 0$) or attracting them to (if $\beta < 0$) the singularity (cf. Figure 3 and zoom in Figure 4).

Remark 10 The method of proof of the statements above can actually handle a more general situation, when the weight of orthogonality depends on n,

$$W_n(z) \stackrel{\text{def}}{=} w(z) \prod_{k=1}^{m} |z - a_k|^{2\beta_{k,n}}\,, \quad z \in \mathbb{T}\,, \quad (a_k \in \mathbb{T} \text{ for } k = 1, \ldots, m)\,,$$

assuming that for values $\beta_k(n) > -1/2$, $k = 1, \ldots, m$, the limits

$$\lim_{n\to\infty} \frac{\beta_{k,n}}{n} = \beta_k \geq 0\,, \quad k = 1, \ldots, m\,,$$

exist.

The structure of the rest of the paper is as follows. In the next section we study in more detail the properties of the Szegő and the scattering functions of W. The Riemann-Hilbert characterization and the nonlinear steepest descent methods of Deift and Zhou (see e.g. [6, 7], as well as the monograph [8]), are carried out in Section 3, which allows to prove the announced results in Section 4.

2 Szegő and scattering function for W

Following the notation introduced above, let $D_{\rm i}(w;z)$ and $D_{\rm e}(w;z)$ be, respectively, the interior and exterior values of the Szegő function defined by (5) for the analytic component w in (7). Since w is holomorphic and non-vanishing in $\mathbb{A}_\rho$, then both $D_{\rm i}(w;z)$ and $D_{\rm e}(w;z)$ admit a holomorphic extension across $\mathbb{T}$, and maintaining the same notation for these analytic continuations we have

$$\frac{D_{\rm i}(w;z)}{D_{\rm e}(w;z)} = w(z)\,.$$

This formula gives in a certain way a canonical analytic extension of w from $\mathbb{T}$ to the annulus $\mathbb{A}_\rho$.

It is convenient to construct explicitly the Szegő function for the modified weight W; with this purpose we select the singe-valued branches of the corresponding functions as follows: the generalized polynomial

$$q(z) \stackrel{\text{def}}{=} \prod_{k=1}^{m} (z-a_k)^{\beta_k/2} \tag{29}$$

is a single-valued analytic function in $\mathbb{C}\setminus(\cup_{k=1}^m a_k\cdot[1,+\infty))$, for which we fix the value of $q(0)$. Consequently, $\overline{q(1/\bar z)}$ is single-valued and analytic in $\mathbb{C}\setminus(\cup_{k=1}^m a_k\cdot[0,1])$, and

$$\overline{q(1/\bar z)}\,\big|_{z=\infty} = \overline{q(0)} = \frac{1}{q(0)}.$$

If we consider each ray $a_k\cdot[1,+\infty)$ oriented towards infinity, then

$$q_+(z) = e^{-\pi i\beta_k} q_-(z)\,, \quad z\in a_k\cdot[1,+\infty)\,, \quad k=1,\dots,m\,. \tag{30}$$

In the same fashion, if $a_k\cdot[0,1]$ also have the natural orientation from the origin to a_k, then

$$\left[\overline{q(1/\bar z)}\right]_+ = e^{\pi i\beta_k}\left[\overline{q(1/\bar z)}\right]_-\,, \quad z\in a_k\cdot(0,1)\,, \quad k=1,\dots,m\,. \tag{31}$$

With this convention we can write the Szegő functions for the modified weight W:

$$D_{\rm i}(W;z) = \frac{q^2(z)}{q^2(0)}\,D_{\rm i}(w;z)\,, \quad D_{\rm e}(W;z) = \frac{D_{\rm e}(w;z)}{q^2(0)\,\left(\overline{q(1/\bar z)}\right)^2}\,. \tag{32}$$

Then $D_{\rm i}(W;z)$ is holomorphic in $\mathbb{D}_{1/\rho}\setminus(\cup_{k=1}^m a_k\cdot[1,1/\rho))$, $D_{\rm e}(W;z)$ is holomorphic in $\{z\in\mathbb{C}:\,|z|>\rho\}\setminus(\cup_{k=1}^m a_k\cdot(\rho,1])$, and

$$\tau \stackrel{\text{def}}{=} \frac{1}{D_{\rm i}(W;0)} = \frac{1}{D_{\rm i}(w;0)} = D_{\rm e}(W;\infty) = D_{\rm e}(w;\infty) > 0\,. \tag{33}$$

Using the definition in (5) we see that τ is related with the geometric mean defined in (14) by

$$\tau^{-2} = G[W] = G[w]\,. \tag{34}$$

Furthermore, formula

$$W(z) = q^2(z)\overline{q(1/\bar{z})}^2 w(z) = \frac{D_\mathrm{i}(W;z)}{D_\mathrm{e}(W;z)} \tag{35}$$

is valid and provides an analytic continuation of the weight W to the cut annulus $\mathbb{A}_\rho \setminus (\cup_{k=1}^m a_k \cdot (\rho, 1/\rho))$. By (30)–(31), with the orientation of the cuts toward infinity we have

$$\begin{aligned} [D_\mathrm{i}(W;z)]_+ &= e^{-2\pi i\beta_k}\,[D_\mathrm{i}(W;z)]_-\,, \quad z \in a_k \cdot (1, 1/\rho)\,, \quad k = 1,\ldots,m\,, \\ [D_\mathrm{e}(W;z)]_+ &= e^{-2\pi i\beta_k}\,[D_\mathrm{e}(W;z)]_-\,, \quad z \in a_k \cdot (\rho, 1)\,, \quad k = 1,\ldots,m\,. \end{aligned} \tag{36}$$

Recall that the scattering function for w,

$$\mathcal{S}(w;z) = D_\mathrm{i}(w;z) D_\mathrm{e}(w;z)\,,$$

is holomorphic in the annulus $\mathbb{A}_\rho$. By (6),

$$\overline{\mathcal{S}\left(w;\frac{1}{\bar{z}}\right)} = \frac{1}{\mathcal{S}(w;z)}\,, \quad \text{for } z \in \mathbb{A}_\rho\,, \qquad |\mathcal{S}(w;z)| = 1 \text{ on } \mathbb{T}\,.$$

With the definition (8) and formulas (32) we have

$$\mathcal{S}(W;z) = D_\mathrm{i}(W;z) D_\mathrm{e}(W;z) = \left(\frac{q(z)}{q^2(0)\,\overline{q(1/\bar{z})}}\right)^2 \mathcal{S}(w;z)\,,$$

that is also analytic and single-valued in the cut annulus $\mathbb{A}_\rho \setminus (\cup_{k=1}^m a_k \cdot (\rho, 1/\rho))$. By (36),

$$\mathcal{S}_+(W;z) = e^{-2\pi i\beta_k}\mathcal{S}_-(W;z)\,, \quad z \in a_k \cdot (\rho, 1/\rho) \setminus \{a_k\}\,, \quad k = 1,\ldots,m\,. \tag{37}$$

It is straightforward also to check that $\mathcal{S}(W;z)$ is bounded in a neighborhood of each a_k.

Recall the definition in (9). With our assumptions on $\delta > 0$ the closed disks $\mathcal{B}_k \cup \mathcal{C}_k$ are disjoint, and $\mathcal{S}(w;z)$ is analytic in each $\mathcal{B}_k$. Furthermore, motivated by (37) we define

$$\widehat{\mathcal{S}}_k(W;z) \stackrel{\text{def}}{=} \begin{cases} e^{\pi i\beta_k}\mathcal{S}(W;z)\,, & \text{if } z \in \mathcal{B}_k \text{ and } \arg(z) > \arg(a_k)\,, \\ e^{-\pi i\beta_k}\mathcal{S}(W;z)\,, & \text{if } z \in \mathcal{B}_k \text{ and } \arg(z) < \arg(a_k)\,, \end{cases} \quad k = 1,\ldots,m\,. \tag{38}$$

Then $\widehat{\mathcal{S}}_k(W;z)$ is holomorphic in $\mathcal{B}_k$, $k = 1,\ldots,m$, and we define

$$\vartheta_k \stackrel{\text{def}}{=} \widehat{\mathcal{S}}_k(W;a_k) \in \mathbb{T}\,, \quad k = 1,\ldots,m\,. \tag{39}$$

These constants have been used in the formulation of several results in the previous section.

Remark 11 Formally

$$\vartheta_k = \prod_{j\neq k}\left(-\frac{a_k}{a_j}\right)^{\beta_k} \mathcal{S}(w;a_k)\,,$$

but the selection of the right branch in each case should be specified. So the definition by (39) is preferred in order to avoid ambiguity.

3 Riemann-Hilbert analysis for orthogonal polynomials

We assume the unit circle $\mathbb{T}$ oriented counterclockwise. The starting point of all the analysis is the fact that under assumptions above conditions (2) can be rewritten in terms of a non-hermitian orthogonality for φ_n and φ_n^*:

$$\oint_{\mathbb{T}} \varphi_n(z) z^{n-k-1} \frac{W(z)}{z^n}\, dz = 0, \quad \text{for } k = 0, 1, \ldots, n-1\,,$$
$$\oint_{\mathbb{T}} \varphi_{n-1}^*(z) z^k \frac{W(z)}{z^n}\, dz = \begin{cases} 0, & k = 0, 1, \ldots, n-2, \\ i/\kappa_{n-1}, & k = n-1\,. \end{cases}$$

By standard arguments (see e.g. [3] or [8], as well as the seminal paper [9] where the Riemann-Hilbert approach to orthogonal polynomials started),

$$Y(z) = \begin{pmatrix} \Phi_n(z) & \dfrac{1}{2\pi i} \displaystyle\oint_{\mathbb{T}} \frac{\Phi_n(t) W(t)\, dt}{t^n (t-z)} \\ -2\pi \kappa_{n-1} \varphi_{n-1}^*(z) & -\dfrac{\kappa_{n-1}}{i} \displaystyle\oint_{\mathbb{T}} \frac{\varphi_{n-1}^*(t) W(t)\, dt}{t^n (t-z)} \end{pmatrix} \tag{40}$$

is a unique solution of the following Riemann-Hilbert problem: Y is holomorphic in $\mathbb{C} \setminus \mathbb{T}$,

$$Y_+(z) = Y_-(z) \begin{pmatrix} 1 & W(z)/z^n \\ 0 & 1 \end{pmatrix}, \quad z \in \mathbb{T}\,, \qquad \lim_{z \to \infty} Y(z) \begin{pmatrix} z^{-n} & 0 \\ 0 & z^n \end{pmatrix} = I\,,$$

where I is the 2×2 identity matrix, and

$$Y(z) = \begin{cases} \mathcal{O}\begin{pmatrix} 1 & |z-a_k|^{2\beta_k} \\ 1 & |z-a_k|^{2\beta_k} \end{pmatrix}, & \text{if } \beta_k < 0\,, \\ \mathcal{O}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, & \text{if } \beta_k \geq 0\,, \end{cases}$$

as $z \to a_k$, $z \in \mathbb{C} \setminus \mathbb{T}$, and $k = 1, \ldots, m$.

In order to perform the steepest descent analysis as described in [8] (see also [10]) we build a series of *explicit* and *reversible* steps in order to arrive at an equivalent problem, which is solvable, at least in an asymptotic sense. We will use the following notation: $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is the Pauli matrix, and for any non-zero x and integer m, $x^{\sigma_3} = \begin{pmatrix} x & 0 \\ 0 & 1/x \end{pmatrix}$.

3.1 Global analysis

If we define

$$H(z) \stackrel{\text{def}}{=} \begin{cases} z^{-n\sigma_3}, & \text{if } |z| > 1, \\ I, & \text{if } |z| < 1, \end{cases} \tag{41}$$

and put $T(z) \stackrel{\text{def}}{=} Y(z)\,H(z)$, then T becomes holomorphic in $\mathbb{C}\setminus\mathbb{T}$, with $\lim_{z\to\infty} T(z) = I$, and satisfying the jump condition

$$T_+(z) = T_-(z)\begin{pmatrix} z^n & W(z) \\ 0 & z^{-n}\end{pmatrix}$$

on $\mathbb{T}$. Obviously, T exhibits the same local behavior at $\mathcal{A}$ as Y.

Let $\gamma_{\rm i}$ be a closed Jordan contour, piecewise analytic, entirely contained in the cut annulus $\{z\in\mathbb{C}:\ \rho<|z|<1\}\setminus\cup_{k=1}^m a_k\cdot(\rho,1)$, except for points a_k: $\gamma_{\rm i}\cap\mathbb{T}=\mathcal{A}$. Let also $\gamma_{\rm e}\stackrel{\text{def}}{=}\{1/\bar z:\ z\in\gamma_{\rm i}\}$ the contour symmetric to $\gamma_{\rm i}$ with respect to $\mathbb{T}$. We take both $\gamma_{\rm i}$ and $\gamma_{\rm e}$ oriented counterclockwise. Let Ω_0 be the connected component of $\mathbb{C}\setminus\gamma_{\rm i}$ containing the origin, and Ω_∞ the corresponding unbounded component of $\mathbb{C}\setminus\gamma_{\rm e}$. Furthermore, we denote by

$$\Omega_{\rm i}\stackrel{\text{def}}{=}\mathbb{D}\setminus\overline{\Omega_0}\,,\qquad \Omega_{\rm e}\stackrel{\text{def}}{=}\{1/\bar z: z\in\Omega_{\rm i}\}$$

(see Fig. 5).

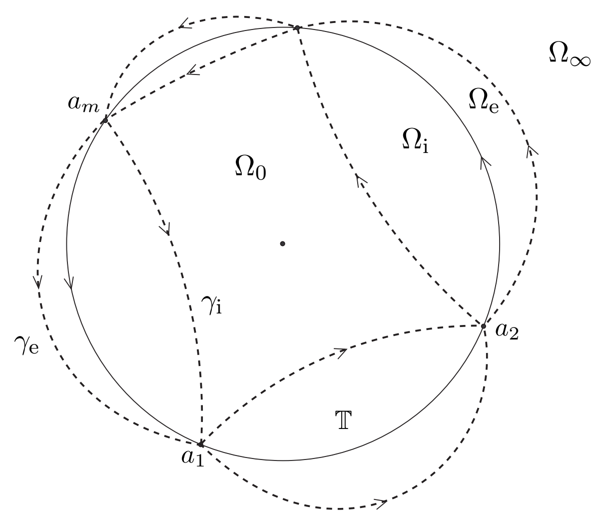

Figure 5: Opening lenses.

Then W is correctly defined in both $\Omega_{\rm i}$ and $\Omega_{\rm e}$, and we can set

$$K(z)\stackrel{\text{def}}{=}\begin{cases} I, & \text{if } z\in\Omega_0\cup\Omega_\infty,\\ \begin{pmatrix} 1 & 0\\ z^n/W(z) & 1\end{pmatrix}^{-1}, & \text{if } z\in\Omega_{\rm i},\\ \begin{pmatrix} 1 & 0\\ 1/(z^nW(z)) & 1\end{pmatrix}, & \text{if } z\in\Omega_{\rm e}.\end{cases} \tag{42}$$

Using K we make a new transformation: $U(z) \stackrel{\text{def}}{=} T(z)K(z)$. Matrix valued function U is holomorphic in $\mathbb{C} \setminus (\mathbb{T} \cup \gamma_{\rm i} \cup \gamma_{\rm e})$, $\lim_{z\to\infty} U(z) = I$, and

$$U_+(z) = U_-(z)\, J_U(z), \quad z \in \mathbb{T} \cup \gamma_{\rm i} \cup \gamma_{\rm e},$$

where

$$J_U(z) = \begin{cases} \begin{pmatrix} 0 & W(z) \\ -1/W(z) & 0 \end{pmatrix}, & \text{if } z \in \mathbb{T} \setminus \mathcal{A}, \\ \begin{pmatrix} 1 & 0 \\ z^n/W(z) & 1 \end{pmatrix}, & \text{if } z \in \gamma_{\rm i} \setminus \mathcal{A}, \\ \begin{pmatrix} 1 & 0 \\ 1/(z^n W(z)) & 1 \end{pmatrix}, & \text{if } z \in \gamma_{\rm e} \setminus \mathcal{A}. \end{cases} \tag{43}$$

Moreover, the local behavior for $U(z)$ as $z \to a_k$ from $\Omega_{\rm e} \cup \Omega_{\rm i}$ is now

$$U(z) = \begin{cases} \mathcal{O}\begin{pmatrix} 1 & |z-a_k|^{2\beta_k} \\ 1 & |z-a_k|^{2\beta_k} \end{pmatrix}, & \text{if } \beta_k < 0\,, \\ \mathcal{O}\begin{pmatrix} |z-a_k|^{-2\beta_k} & 1 \\ |z-a_k|^{-2\beta_k} & 1 \end{pmatrix}, & \text{if } \beta_k \geq 0\,. \end{cases}$$

With $D_{\rm e}(W;\cdot)$ and $D_{\rm i}(W;\cdot)$ defined in (32) and constant τ introduced in (33), let

$$N(z) \stackrel{\text{def}}{=} \begin{cases} \left(\dfrac{D_{\rm e}(W;z)}{\tau}\right)^{\sigma_3}, & \text{if } |z| > 1\,, \\ \begin{pmatrix} 0 & D_{\rm i}(W;z)/\tau \\ -\tau/D_{\rm i}(W;z) & 0 \end{pmatrix} = \left(\dfrac{D_{\rm i}(W;z)}{\tau}\right)^{\sigma_3} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & \text{if } |z| < 1\,. \end{cases} \tag{44}$$

By (35), N has the same jump on $\mathbb{T}$ as U, and by (33), it exhibits the same behavior at infinity. Hence, $U(z)N^{-1}(z)$ tends to I as $z \to \infty$, is holomorphic in $\mathbb{C} \setminus (\mathbb{T} \cup \gamma_{\rm e} \cup \gamma_{\rm i})$, and has jumps across these curves asymptotically close to I, except when we approach the singular set $\mathcal{A}$. We have to handle the behavior at each individual singular point a_k by means of the local analysis.

3.2 Local analysis

Let us pick a singular point $a_\ell \in \mathcal{A}$. For the sake of brevity along this subsection we use the following shortcuts for the notation: $a \stackrel{\text{def}}{=} a_\ell$, $\beta \stackrel{\text{def}}{=} \beta_\ell$, $\mathcal{B} \stackrel{\text{def}}{=} \mathcal{B}_\ell$, $\mathcal{C} \stackrel{\text{def}}{=} \mathcal{C}_\ell$ (where $\mathcal{B}_\ell$, $\mathcal{C}_\ell$ and δ were defined in (9)), and $\mathcal{B}^+ \stackrel{\text{def}}{=} \{z \in \mathcal{B} : \arg(z) > \arg(a)\}$, $\mathcal{B}^- \stackrel{\text{def}}{=} \{z \in \mathcal{B} : \arg(z) < \arg(a)\}$. We also write $\widehat{\Omega}_j \stackrel{\text{def}}{=} \Omega_j \cap \mathcal{B}$, where $j \in \{\mathrm{i}, \mathrm{e}, 0, \infty\}$, and analogous notation for curves: $\widehat{\mathbb{T}} \stackrel{\text{def}}{=} \mathbb{T} \cap \mathcal{B}$, etc. Finally, $\widehat{\mathcal{S}}(W;z) \stackrel{\text{def}}{=} \widehat{\mathcal{S}}_\ell(W;z)$ (see the definition in (38)).

Our goal is to build a matrix $P(z) \stackrel{\text{def}}{=} P(a,\beta;z)$ meeting the following requirements:

(P1) P is holomorphic in $\mathcal{B} \setminus (\mathbb{T} \cup \gamma_{\rm i} \cup \gamma_{\rm e})$ and satisfies across $\widehat{\mathbb{T}} \cup \widehat{\gamma}_{\rm i} \cup \widehat{\gamma}_{\rm e}$ the jump relation $P_+(t) = P_-(t) J_U(t)$, with J_U defined in (43).

(P2) $P(z)$ has the following local behavior as $z \to a$: if $\beta < 0$, then

$$P(z) = \mathcal{O}\begin{pmatrix} 1 & |z-a|^{2\beta} \\ 1 & |z-a|^{2\beta} \end{pmatrix},$$

and if $\beta \geq 0$,

$$P(z) = \begin{cases} \mathcal{O}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, & \text{from } \widehat{\Omega}_0 \cup \widehat{\Omega}_\infty\,, \\ \mathcal{O}\begin{pmatrix} |z-a|^{-2\beta} & 1 \\ |z-a|^{-2\beta} & 1 \end{pmatrix}, & \text{from } \widehat{\Omega}_{\mathrm{e}} \cup \widehat{\Omega}_{\mathrm{i}}\,. \end{cases}$$

(P3) $P(z)$ matches $N(z)$ on $\mathcal{C}$, in the sense $P(z)N^{-1}(z) = I + \mathcal{O}(n^{-1})$ for $z \in \mathcal{C}$.

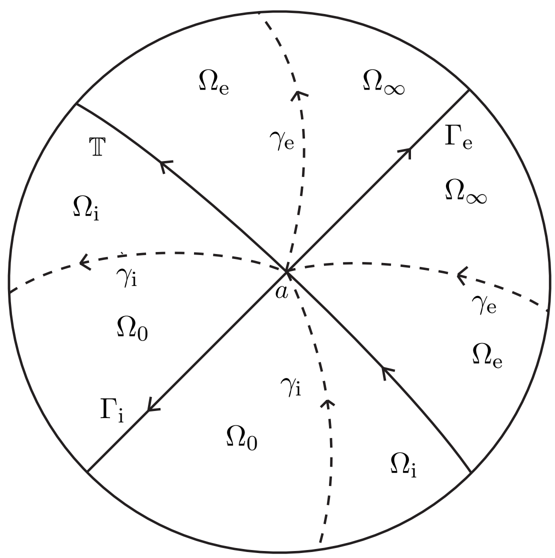

Figure 6: Local analysis in $\mathcal{B}$.

As a first step we reduce the problem to the one with constant jumps. Let us denote $\Gamma_{\mathrm{i}} \stackrel{\text{def}}{=} a \cdot (1-\delta, 1)$ and $\Gamma_{\mathrm{e}} \stackrel{\text{def}}{=} a \cdot (1, 1+\delta)$, oriented both from $z = a$ to infinity (see Fig. 6). Let $w^{1/2}(z)$ and $z^{1/2}$ denote the principal holomorphic branches of these functions in b, and (cf. (35)),

$$W^{1/2}(z) \stackrel{\text{def}}{=} q(z)\overline{q(1/\overline{z})}\, w^{1/2}(z)$$

with q defined in (29). Then $W^{1/2}$ is holomorphic in $\mathcal{B} \setminus a \cdot (1-\delta, 1+\delta)$, and according to (30)–(31),

$$\frac{W_+^{1/2}(z)}{W_-^{1/2}(z)} = e^{-\pi i \beta} \quad \text{on } \Gamma_{\mathrm{i}}\,, \quad \text{and} \quad \frac{W_+^{1/2}(z)}{W_-^{1/2}(z)} = e^{\pi i \beta} \quad \text{on } \Gamma_{\mathrm{e}}\,.$$

Thus, if we define

$$\lambda(\beta;z) \stackrel{\text{def}}{=} \begin{cases} e^{\pi i\beta}W^{1/2}(z)z^{n/2}, & z\in(\widehat{\Omega}_{\text{e}}\cup\widehat{\Omega}_{\infty})\cap\mathcal{B}^{+}, \\ e^{-\pi i\beta}W^{1/2}(z)z^{n/2}, & z\in(\widehat{\Omega}_{\text{e}}\cup\widehat{\Omega}_{\infty})\cap\mathcal{B}^{-}, \\ e^{-\pi i\beta}W^{1/2}(z)z^{-n/2}, & z\in(\widehat{\Omega}_{\text{i}}\cup\widehat{\Omega}_{0})\cap\mathcal{B}^{+}, \\ e^{\pi i\beta}W^{1/2}(z)z^{-n/2}, & z\in(\widehat{\Omega}_{\text{i}}\cup\widehat{\Omega}_{0})\cap\mathcal{B}^{-}, \end{cases} \tag{45}$$

and set

$$R(z) \stackrel{\text{def}}{=} P(z)\,\lambda(\beta;z)^{\sigma_3}\,, \quad z\in b\setminus(\Gamma_{\text{i}}\cup\Gamma_{\text{e}}\cup\mathbb{T}\cup\gamma_{\text{i}}\cup\gamma_{\text{e}})\,, \tag{46}$$

we get for R the following problem: R is holomorphic in $b\setminus(\Gamma_{\text{i}}\cup\Gamma_{\text{e}}\cup\mathbb{T}\cup\gamma_{\text{i}}\cup\gamma_{\text{e}})$, and satisfies the jump relation $R_+(z)=R_-(z)J_R(z)$, with

$$J_R(z)=\begin{cases} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & \text{if } z\in\widehat{\mathbb{T}}, \\ \begin{pmatrix} 1 & 0 \\ e^{-2\pi i\beta} & 1 \end{pmatrix}, & \text{if } z\in(\widehat{\gamma}_{\text{i}}\cap\mathcal{B}^{+})\cup(\widehat{\gamma}_{\text{e}}\cap\mathcal{B}^{-}), \\ \begin{pmatrix} 1 & 0 \\ e^{2\pi i\beta} & 1 \end{pmatrix}, & \text{if } z\in(\widehat{\gamma}_{\text{i}}\cap\mathcal{B}^{-})\cup(\widehat{\gamma}_{\text{e}}\cap\mathcal{B}^{+}), \\ \begin{pmatrix} e^{\pi i\beta} & 0 \\ 0 & e^{-\pi i\beta} \end{pmatrix}, & \text{if } z\in\Gamma_{\text{i}}\cup\Gamma_{\text{e}}\,. \end{cases}$$

Moreover, R has the following local behavior as $z\to a$: if $\beta\geq 0$,

$$R(z)=\begin{cases} \mathcal{O}\begin{pmatrix} |z-a|^{\beta} & |z-a|^{-\beta} \\ |z-a|^{\beta} & |z-a|^{-\beta} \end{pmatrix}, & \text{if } z\in\widehat{\Omega}_0\cup\widehat{\Omega}_{\infty}, \\ \mathcal{O}\begin{pmatrix} |z-a|^{-\beta} & |z-a|^{-\beta} \\ |z-a|^{-\beta} & |z-a|^{-\beta} \end{pmatrix}, & \text{if } z\in\widehat{\Omega}_{\text{e}}\cup\widehat{\Omega}_{\text{i}}, \end{cases}$$

and if $\beta<0$, then

$$R(z)=\mathcal{O}\begin{pmatrix} |z-a|^{\beta} & |z-a|^{\beta} \\ |z-a|^{\beta} & |z-a|^{\beta} \end{pmatrix}.$$

As it could be expected, this problem resembles very much the one we face during the local analysis for the generalized Jacobi weight on the real line (see [18, Theorem 4.2], [11] and [12]). We take advantage of the results proved therein in order to abbreviate the exposition.

Let us define $\Sigma_i \stackrel{\text{def}}{=} \{xe^{i\pi/4} : x\geq 0\}$, $i=1,\ldots,8$; we take Σ_i oriented towards the origin for $i=3,4,5$, and the rest towards infinity (see Fig. 7). This splits the plane into eight regions, as indicated there, marked with Roman numbers I to $VIII$. Using Hankel functions $H_\nu^{(i)}$ and modified Bessel functions K_ν and I_ν we build a piece-wise analytic matrix-valued function $\Psi=\Psi(\beta;\cdot)$, $\beta>-1/2$, in the following manner (where $\zeta^{1/2}$ denotes the main branch in $(-\infty,0]$):

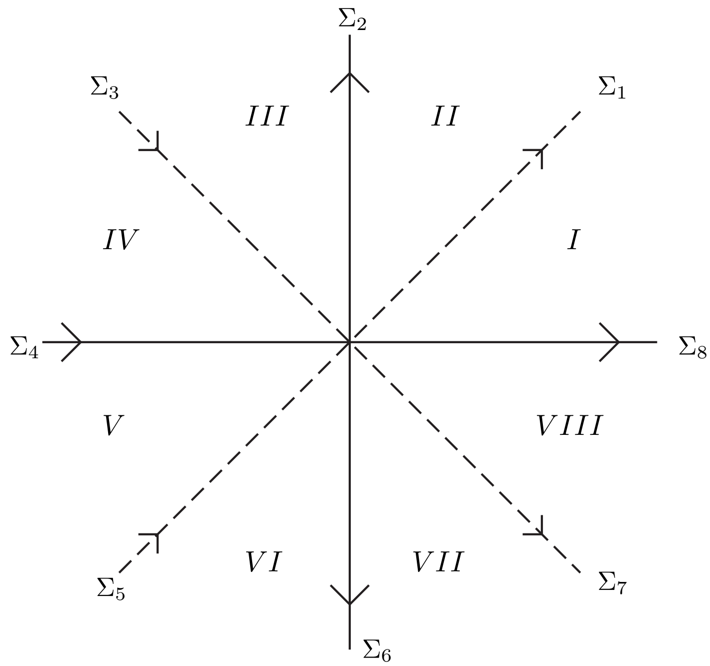

Figure 7: Local parametrix.

For $\zeta \in$ I,

$$\Psi(\beta;\zeta) \overset{\text{def}}{=} \frac{1}{2}\sqrt{\pi}\zeta^{1/2}\begin{pmatrix} H^{(2)}_{\beta+\frac{1}{2}}(\zeta) & -iH^{(1)}_{\beta+\frac{1}{2}}(\zeta) \\ H^{(2)}_{\beta-\frac{1}{2}}(\zeta) & -iH^{(1)}_{\beta-\frac{1}{2}}(\zeta) \end{pmatrix} e^{-(\beta+\frac{1}{4})\pi i\sigma_3}. \tag{47}$$

For $\zeta \in$ II,

$$\Psi(\beta;\zeta) \overset{\text{def}}{=} \begin{pmatrix} \sqrt{\pi}\zeta^{1/2}I_{\beta+\frac{1}{2}}(\zeta e^{-\frac{\pi i}{2}}) & -\frac{1}{\sqrt{\pi}}\zeta^{1/2}K_{\beta+\frac{1}{2}}(\zeta e^{-\frac{\pi i}{2}}) \\ -i\sqrt{\pi}\zeta^{1/2}I_{\beta-\frac{1}{2}}(\zeta e^{-\frac{\pi i}{2}}) & -\frac{i}{\sqrt{\pi}}\zeta^{1/2}K_{\beta-\frac{1}{2}}(\zeta e^{-\frac{\pi i}{2}}) \end{pmatrix} e^{-\frac{1}{2}\beta\pi i\sigma_3}. \tag{48}$$

For $\zeta \in$ III,

$$\Psi(\beta;\zeta) \overset{\text{def}}{=} \begin{pmatrix} \sqrt{\pi}\zeta^{1/2}I_{\beta+\frac{1}{2}}(\zeta e^{-\frac{\pi i}{2}}) & -\frac{1}{\sqrt{\pi}}\zeta^{1/2}K_{\beta+\frac{1}{2}}(\zeta e^{-\frac{\pi i}{2}}) \\ -i\sqrt{\pi}\zeta^{1/2}I_{\beta-\frac{1}{2}}(\zeta e^{-\frac{\pi i}{2}}) & -\frac{i}{\sqrt{\pi}}\zeta^{1/2}K_{\beta-\frac{1}{2}}(\zeta e^{-\frac{\pi i}{2}}) \end{pmatrix} e^{\frac{1}{2}\beta\pi i\sigma_3}. \tag{49}$$

For $\zeta \in$ IV,

$$\Psi(\beta;\zeta) \overset{\text{def}}{=} \frac{1}{2}\sqrt{\pi}(-\zeta)^{1/2}\begin{pmatrix} iH^{(1)}_{\beta+\frac{1}{2}}(-\zeta) & -H^{(2)}_{\beta+\frac{1}{2}}(-\zeta) \\ -iH^{(1)}_{\beta-\frac{1}{2}}(-\zeta) & H^{(2)}_{\beta-\frac{1}{2}}(-\zeta) \end{pmatrix} e^{(\beta+\frac{1}{4})\pi i\sigma_3}. \tag{50}$$

For $\zeta \in$ V,

$$\Psi(\beta;\zeta) \stackrel{\text{def}}{=} \frac{1}{2}\sqrt{\pi}(-\zeta)^{1/2} \begin{pmatrix} -H^{(2)}_{\beta+\frac{1}{2}}(-\zeta) & -iH^{(1)}_{\beta+\frac{1}{2}}(-\zeta) \\ H^{(2)}_{\beta-\frac{1}{2}}(-\zeta) & iH^{(1)}_{\beta-\frac{1}{2}}(-\zeta) \end{pmatrix} e^{-(\beta+\frac{1}{4})\pi i\sigma_3}. \tag{51}$$

For $\zeta \in$ VI,

$$\Psi(\beta;\zeta) \stackrel{\text{def}}{=} \begin{pmatrix} -i\sqrt{\pi}\zeta^{1/2} I_{\beta+\frac{1}{2}}(\zeta e^{\frac{\pi i}{2}}) & -\frac{i}{\sqrt{\pi}}\zeta^{1/2} K_{\beta+\frac{1}{2}}(\zeta e^{\frac{\pi i}{2}}) \\ \sqrt{\pi}\zeta^{1/2} I_{\beta-\frac{1}{2}}(\zeta e^{\frac{\pi i}{2}}) & -\frac{1}{\sqrt{\pi}}\zeta^{1/2} K_{\beta-\frac{1}{2}}(\zeta e^{\frac{\pi i}{2}}) \end{pmatrix} e^{-\frac{1}{2}\beta\pi i\sigma_3}. \tag{52}$$

For $\zeta \in$ VII,

$$\Psi(\beta;\zeta) \stackrel{\text{def}}{=} \begin{pmatrix} -i\sqrt{\pi}\zeta^{1/2} I_{\beta+\frac{1}{2}}(\zeta e^{\frac{\pi i}{2}}) & -\frac{i}{\sqrt{\pi}}\zeta^{1/2} K_{\beta+\frac{1}{2}}(\zeta e^{\frac{\pi i}{2}}) \\ \sqrt{\pi}\zeta^{1/2} I_{\beta-\frac{1}{2}}(\zeta e^{\frac{\pi i}{2}}) & -\frac{1}{\sqrt{\pi}}\zeta^{1/2} K_{\beta-\frac{1}{2}}(\zeta e^{\frac{\pi i}{2}}) \end{pmatrix} e^{\frac{1}{2}\beta\pi i\sigma_3}. \tag{53}$$

And finally, for $\zeta \in$ VIII,

$$\Psi(\beta;\zeta) \stackrel{\text{def}}{=} \frac{1}{2}\sqrt{\pi}\zeta^{1/2} \begin{pmatrix} -iH^{(1)}_{\beta+\frac{1}{2}}(\zeta) & -H^{(2)}_{\beta+\frac{1}{2}}(\zeta) \\ -iH^{(1)}_{\beta-\frac{1}{2}}(\zeta) & -H^{(2)}_{\beta-\frac{1}{2}}(\zeta) \end{pmatrix} e^{(\beta+\frac{1}{4})\pi i\sigma_3}. \tag{54}$$

Proposition 1 ([18], Theorem 4.2) *Function $\Psi = \Psi(\beta;\cdot)$ defined above is holomorphic in $\mathbb{C} \setminus \bigcup_{i=1}^{8} \Sigma_i$, and exhibits the following jumps:*

$$\Psi_+(\zeta) = \Psi_-(\zeta) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \text{for } \zeta \in \Sigma_4 \cup \Sigma_8, \tag{55}$$

$$\Psi_+(\zeta) = \Psi_-(\zeta) \begin{pmatrix} 1 & 0 \\ e^{-2\pi i\beta} & 1 \end{pmatrix}, \quad \text{for } \zeta \in \Sigma_1 \cup \Sigma_5, \tag{56}$$

$$\Psi_+(\zeta) = \Psi_-(\zeta) \begin{pmatrix} 1 & 0 \\ e^{2\pi i\beta} & 1 \end{pmatrix}, \quad \text{for } \zeta \in \Sigma_3 \cup \Sigma_7, \tag{57}$$

$$\Psi_+(\zeta) = \Psi_-(\zeta) \begin{pmatrix} e^{\pi i\beta} & 0 \\ 0 & e^{-\pi i\beta} \end{pmatrix}, \quad \text{for } \zeta \in \Sigma_2 \cup \Sigma_6. \tag{58}$$

Additionally, if $\beta \geq 0$ then as $\zeta \to 0$,

$$\Psi(\zeta) = \begin{cases} \mathcal{O}\begin{pmatrix} |\zeta|^{\beta} & |\zeta|^{-\beta} \\ |\zeta|^{\beta} & |\zeta|^{-\beta} \end{pmatrix}, & \text{if } \zeta \text{ is in domains II, III, VI, VII,} \\ \mathcal{O}\begin{pmatrix} |\zeta|^{-\beta} & |\zeta|^{-\beta} \\ |\zeta|^{-\beta} & |\zeta|^{-\beta} \end{pmatrix}, & \text{otherwise,} \end{cases}$$

and if $\beta < 0$ then

$$\Psi(\zeta) = \mathcal{O}\begin{pmatrix} |\zeta|^{\beta} & |\zeta|^{\beta} \\ |\zeta|^{\beta} & |\zeta|^{\beta} \end{pmatrix}.$$

Consider in $\mathbb{C} \setminus (-\infty, 0)$ the transformation

$$f(z) = -i \log(z/a),$$

where we take the principal branch of the logarithm. Then f is a conformal 1-1 map of $\mathcal{B}$ onto a neighborhood of the origin. Moreover, $\mathbb{T}$ is mapped onto $\mathbb{R}$ oriented positively, and we may use the freedom in the selection of the contours deforming them in such a way that $f(\widehat{\gamma}_{\mathrm{i}})$ and $f(\widehat{\gamma}_{\mathrm{e}})$ follow the rays Σ_i in $\mathcal{B}$ with odd indices i (dashed lines, see Figs. 6 and 7). By construction, matrix

$$\Psi\left(\beta; \frac{n}{2} f(z)\right)$$

matches the jumps and the local behavior of R in $\mathcal{B}$. Since a left multiplication by a holomorphic function has no influence on the jumps, and taking into account (46), we see that matrix P can be built of the form

$$P(z) = E(z)\, \Psi\left(\beta; \frac{n}{2} f(z)\right) \lambda(\beta; z)^{-\sigma_3}, \tag{59}$$

where E is any holomorphic function in $\mathcal{B}$. An adequate selection of E is motivated by the matching requirement $P(z)N^{-1}(z) = I + \mathcal{O}(n^{-1})$ on the boundary $\mathcal{C}$.

For the sake of brevity let us denote

$$\zeta = \frac{n}{2} f(z) = -i \frac{n}{2} \log(z/a), \tag{60}$$

(we omit the explicit reference to the dependence of ζ from z, a and n in the notation). Matching condition can be rewritten as

$$E(z) = \left[I + \mathcal{O}\left(\frac{1}{n}\right)\right] N(z)\lambda(\beta; z)^{\sigma_3}\, \Psi\left(\beta; \zeta\right)^{-1}, \quad z \in \mathcal{C}. \tag{61}$$

The key idea is to replace $\Psi(\beta; \zeta)$ by its leading asymptotic term as $\zeta \to \infty$. The complete expansion at infinity of the entries of $\Psi(\beta; \zeta)$ is well known (see e.g. [1, Chapter 9]), so we can insert it in formulas (47)–(54). This computation has been carried out in [18, Section 4.3]; we can formulate the result therein by defining the matrix-valued function $\mathcal{G}$:

$$\mathcal{G}(\beta; \zeta) \stackrel{\text{def}}{=} e^{\frac{\pi i}{4}\sigma_3} e^{-i\zeta\sigma_3} e^{-\frac{1}{2}\beta\pi i\sigma_3}, \quad \text{if } \zeta \text{ is in the first quadrant,} \tag{62}$$

$$\stackrel{\text{def}}{=} e^{\frac{\pi i}{4}\sigma_3} e^{-i\zeta\sigma_3} e^{\frac{1}{2}\beta\pi i\sigma_3}, \quad \text{if } \zeta \text{ is in the second quadrant,} \tag{63}$$

$$\stackrel{\text{def}}{=} e^{\frac{\pi i}{4}\sigma_3} e^{-i\zeta\sigma_3} e^{\frac{1}{2}\beta\pi i\sigma_3} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \text{if } \zeta \text{ is in the third quadrant,} \tag{64}$$

$$\stackrel{\text{def}}{=} e^{\frac{\pi i}{4}\sigma_3} e^{-i\zeta\sigma_3} e^{-\frac{1}{2}\beta\pi i\sigma_3} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \text{if } \zeta \text{ is in the fourth quadrant.} \tag{65}$$

It is easy to check that $\mathcal{G}$ is holomorphic in each quadrant and matches the jumps of Ψ on Σ_2, Σ_4, Σ_6 and Σ_8 (solid lines in Figure 7), given in (55) and (58).

Lemma 1 ([18]) *For function* $\Psi(\beta;\zeta)$ *defined in* (47)–(54),

$$\Psi(\beta;\zeta) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} \left[I + \sum_{k=1}^{\infty} \frac{i^k}{2^{k+1}\zeta^k} \begin{pmatrix} (-1)^k s_{\beta,k} & -it_{\beta,k} \\ i(-1)^k t_{\beta,k} & s_{\beta,k} \end{pmatrix} \right] \mathcal{G}(\beta;\zeta), \qquad (66)$$

as $\zeta \to \infty$, *uniformly for* ζ *in each quadrant. Here, the constants* $s_{\beta,k}$ *and* $t_{\beta,k}$ *are given by*

$$s_{\beta,k} = \left(\beta + \frac{1}{2}, k\right) + \left(\beta - \frac{1}{2}, k\right), \qquad t_{\beta,k} = \left(\beta + \frac{1}{2}, k\right) - \left(\beta - \frac{1}{2}, k\right), \qquad (67)$$

where

$$(\nu, k) = \frac{(4\nu^2 - 1)(4\nu^2 - 9)\dots(4\nu^2 - (2k-1)^2)}{2^{2k} k!}.$$

In particular,

$$\Psi(\beta;\zeta) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} \left[I - \frac{i\beta}{2\zeta} \begin{pmatrix} \beta & i \\ i & -\beta \end{pmatrix} + \mathcal{O}\left(\frac{1}{\zeta^2}\right) \right] \mathcal{G}(\beta;\zeta), \quad \zeta \to \infty, \qquad (68)$$

uniformly for ζ *in each sector.*

Taking into account (61) and (68), it is reasonable to set in (59)

$$E(z) \stackrel{\text{def}}{=} N(z)\lambda(\beta;z)^{\sigma_3} \left[\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} \mathcal{G}(\beta;\zeta) \right]^{-1}. \qquad (69)$$

Proposition 2 *Matrix valued function* E *defined in* (69) *is holomorphic in* $\mathcal{B}$ *and has there the following representation:*

$$E(z) = \left(\frac{\widehat{\mathcal{S}}(W;z)}{\tau^2} ia^n \right)^{\sigma_3/2} \frac{1}{\sqrt{2}} \begin{pmatrix} i & 1 \\ -1 & -i \end{pmatrix}, \qquad (70)$$

where we take the principal branch of the square root.

Proof. The verification of formula (70) is straightforward (and analyticity of E is a direct consequence of it). Under transformation (60) we have the following correspondence:

$$\begin{aligned} z \in (\widehat{\Omega}_{\mathrm{i}} \cup \widehat{\Omega}_0) \cap \mathcal{B}^+ &\quad\Leftrightarrow\quad \zeta \text{ is in the first quadrant,} \\ z \in (\widehat{\Omega}_{\mathrm{i}} \cup \widehat{\Omega}_0) \cap \mathcal{B}^- &\quad\Leftrightarrow\quad \zeta \text{ is in the second quadrant,} \\ z \in (\widehat{\Omega}_{\mathrm{e}} \cup \widehat{\Omega}_\infty) \cap \mathcal{B}^- &\quad\Leftrightarrow\quad \zeta \text{ is in the third quadrant,} \\ z \in (\widehat{\Omega}_{\mathrm{e}} \cup \widehat{\Omega}_\infty) \cap \mathcal{B}^+ &\quad\Leftrightarrow\quad \zeta \text{ is in the fourth quadrant.} \end{aligned}$$

Assume for instance $z \in (\widehat{\Omega}_{\mathrm{i}} \cup \widehat{\Omega}_0) \cap \mathcal{B}^+$, in which case, according to (44) and (45),

$$N(z) = \left(\frac{D_{\mathrm{e}}(W;z)}{\tau} \right)^{\sigma_3}, \quad \lambda(\beta;z) = e^{-\pi i \beta} W^{1/2}(z) z^{-n/2},$$

and by (62),

$$\mathcal{G}(\beta;\zeta) = e^{\frac{\pi i}{4}\sigma_3} e^{-i\zeta\sigma_3} e^{-\frac{1}{2}\beta\pi i\sigma_3} .$$

Gathering these elements in (69) we arrive at (70). The analysis in the rest of the quadrants is similar.

□

Corollary 3 *Matrix* $P(z) = P(a,\beta;z)$,

$$P(a,\beta;z) \stackrel{def}{=} \left(\frac{\widehat{\mathcal{S}}(W;z)}{\tau^2}\, i a^n\right)^{\sigma_3/2} \frac{1}{\sqrt{2}} \begin{pmatrix} i & 1 \\ -1 & -i \end{pmatrix} \Psi\left(\beta;\zeta\right)\, \lambda(\beta;z)^{-\sigma_3} , \tag{71}$$

with ζ *given by* (60), *solves the Riemann-Hilbert problem (P1)–(P2) defined at the beginning of this subsection.*

Thus, it remains to check the matching condition (P3).

Proposition 3 *Let* $P(z) = P(a,\beta;z)$ *be given by* (71). *Then for* $z \in \mathcal{C}$, *function* $P(z)N^{-1}(z)$ *has the following asymptotic expansion:*

$$\begin{aligned} P(z)N^{-1}(z) = I & \\ + \sum_{k=1}^{\infty} \frac{i^k}{2^{k+1}\zeta^k} & \begin{pmatrix} s_{\beta,k} & (-1)^k \tau^{-2}\, a^n \widehat{\mathcal{S}}(W;z)\, t_{\beta,k} \\ \tau^2\, a^{-n}\left(\widehat{\mathcal{S}}(W;z)\right)^{-1} t_{\beta,k} & (-1)^k s_{\beta,k} \end{pmatrix} , \end{aligned} \tag{72}$$

where constants $s_{\beta,k}$ *and* $t_{\beta,k}$ *were defined in* (67). *In particular,*

$$P(z)N^{-1}(z) = I + \frac{i\beta}{2\zeta} \begin{pmatrix} \beta & -\tau^{-2} a^n \widehat{\mathcal{S}}(W;z) \\ \tau^2 a^{-n}\left(\widehat{\mathcal{S}}(W;z)\right)^{-1} & -\beta \end{pmatrix} + \mathcal{O}\left(\frac{1}{n^2}\right) , \tag{73}$$

so that $P(z)N^{-1}(z) = I + \mathcal{O}(n^{-1})$ *for* $z \in \mathcal{C}$.

Proof. By (59),

$$P(z)N^{-1}(z) = E(z)\, \Psi\left(\beta;\zeta\right)\, \left[N(z)\lambda(\beta;z)^{\sigma_3}\right]^{-1} .$$

By definition of E in (69) we have

$$N(z)\lambda(\beta;z)^{\sigma_3} = E(z) \left[\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} \mathcal{G}\left(\beta;\zeta\right)\right] .$$

Thus,

$$P(z)N^{-1}(z) = E(z)\, \Psi\left(\beta;\zeta\right)\, \left[\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} \mathcal{G}\left(\beta;\zeta\right)\right]^{-1} E(z)^{-1}. \tag{74}$$

By (68),

$$\begin{aligned}&\Psi(\beta;\zeta)\left[\frac{1}{\sqrt{2}}\begin{pmatrix}1 & -i\\ -i & 1\end{pmatrix}\mathcal{G}(\beta;\zeta)\right]^{-1}\\ &=\frac{1}{\sqrt{2}}\begin{pmatrix}1 & -i\\ -i & 1\end{pmatrix}[I+V(\zeta)]\,\mathcal{G}(\beta;\zeta)\left[\frac{1}{\sqrt{2}}\begin{pmatrix}1 & -i\\ -i & 1\end{pmatrix}\mathcal{G}(\beta;\zeta)\right]^{-1}\\ &=\frac{1}{\sqrt{2}}\begin{pmatrix}1 & -i\\ -i & 1\end{pmatrix}[I+V(\zeta)]\left[\frac{1}{\sqrt{2}}\begin{pmatrix}1 & -i\\ -i & 1\end{pmatrix}\right]^{-1},\end{aligned}$$

where V is given by the asymptotic expansion

$$V(\zeta)\stackrel{\text{def}}{=}\sum_{k=1}^{\infty}\frac{i^k}{2^{k+1}\zeta^k}\begin{pmatrix}(-1)^k s_{\beta,k} & -it_{\beta,k}\\ i(-1)^k t_{\beta,k} & s_{\beta,k}\end{pmatrix},$$

and constants $s_{\beta,k}$ and $t_{\beta,k}$ were defined in (67). Using it in (74) and taking into account (70) we get

$$\begin{aligned}P(z)N^{-1}(z)&=\left(E(z)\frac{1}{\sqrt{2}}\begin{pmatrix}1 & -i\\ -i & 1\end{pmatrix}\right)[I+V(\zeta)]\left(E(z)\frac{1}{\sqrt{2}}\begin{pmatrix}1 & -i\\ -i & 1\end{pmatrix}\right)^{-1}\\ &=\left(\left(\frac{\widehat{\mathcal{S}}(W;z)}{\tau^2}\,ia^n\right)^{\sigma_3/2}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\right)[I+V(\zeta)]\left(\left(\frac{\widehat{\mathcal{S}}(W;z)}{\tau^2}\,ia^n\right)^{\sigma_3/2}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\right)^{-1},\end{aligned}$$

and formula (72) follows. Furthermore, since $s_{\beta,1}=2\beta^2$, and $t_{\beta,1}=2\beta$, we obtain (73).

□

3.3 Final transformation

With the notation introduced in (9) and with $P(a,\beta;z)$ defined by (71) let us take

$$P(z)\stackrel{\text{def}}{=}P(a_k,\beta_k;z)\quad\text{for } z\in\mathcal{B}_k\setminus(\mathbb{T}\cup\gamma_{\rm e}\cup\gamma_{\rm i}),\quad k=1,\dots,m,$$

and put

$$S(z)\stackrel{\text{def}}{=}\begin{cases}U(z)N^{-1}(z), & \text{for } z\in\mathbb{C}\setminus(B\cup\mathbb{T}\cup\gamma_{\rm e}\cup\gamma_{\rm i}),\\ U(z)P^{-1}(z), & \text{for } z\in B\setminus(\mathbb{T}\cup\gamma_{\rm e}\cup\gamma_{\rm i}).\end{cases}\tag{75}$$

It is easy to show that this transformation is well defined, since the inverses exist. Matrix S is holomorphic in the whole plane cut along $\gamma\cup C$, where

$$\gamma\stackrel{\text{def}}{=}(\gamma_{\rm e}\cup\gamma_{\rm i})\setminus B\quad\text{and}\quad C\stackrel{\text{def}}{=}\cup_{k=1}^m\mathcal{C}_k$$

(see Fig. 8), $S(z)\to I$ as $z\to\infty$, and if we orient all $\mathcal{C}_k$'s clockwise, $S_+(t)=S_-(t)J_S$, with

$$J_S(t)=\begin{cases}P(z)N^{-1}(z), & \text{if } z\in C,\\ \begin{pmatrix}1 & 0\\ \tau^2/(z^n\mathcal{S}(W;z)) & 1\end{pmatrix}, & \text{if } z\in\gamma_{\rm e}\setminus B,\\ \begin{pmatrix}1 & -z^n\mathcal{S}(W;z)/\tau^2\\ 0 & 1\end{pmatrix}, & \text{if } z\in\gamma_{\rm i}\setminus B.\end{cases}$$

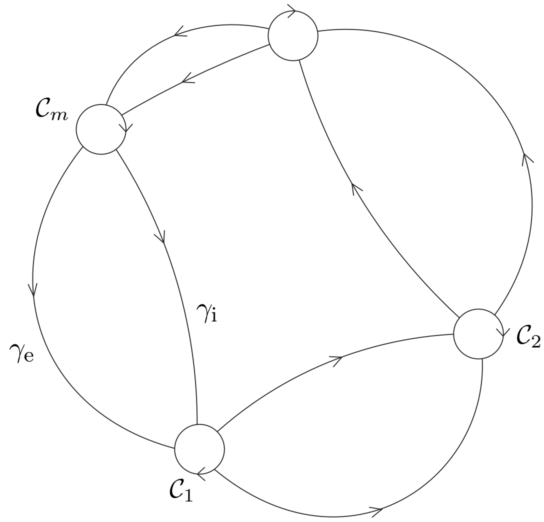

Figure 8: Jumps of S.

It is clear that the off-diagonal terms of J_S on $\gamma_{\rm i} \setminus B$ and $\gamma_{\rm e} \setminus B$ decay exponentially fast. On the other hand, by (73), $J_S(z) = I + \mathcal{O}(1/n)$ for $z \in C$. So the conclusion is that the jump matrix $J_S = I + O(1/n)$ uniformly for $z \in \gamma \cup C$. In fact, (72) gives us the complete asymptotic expansion of J_S in negative powers of n. Then arguments such as in [6, 7, 8] (see e.g. [7, Section 7.2]) allow to show that S itself has an asymptotic expansion in negative powers of n. The main observation is that if we define

$$\mathbf{K}(f)(z) \stackrel{\rm def}{=} \frac{1}{2\pi i} \int \frac{f(t)(J_S(t) - I)\, dt}{t - z}\,, \qquad \mathbf{M}(f)(t) = (\mathbf{K}(f))_-(t)\,, \tag{76}$$

where we integrate along contours $\gamma \cup C$ with the orientation shown in Fig. 8, then $\mathbf{M}$ defines a bounded linear operator acting in $L^2(\gamma \cup C)$ (with respect to the Lebesgue measure), with the operator norm

$$\|\mathbf{M}\|_{L^2 \to L^2} \leq \text{const} \|J_S - I\|_{L^2}\,,$$

where "const" depends on $\gamma \cup C$ only (see e.g. [8, Chapter 7] for details). In particular, for all sufficiently large n, operator $(1 - \mathbf{M})$ is invertible, and we have the following

Proposition 4 *Let n be such that $(1 - \mathbf{M})$ is invertible, and denote*

$$\mu = (1 - \mathbf{M})^{-1} I\,,$$

where I is the 2×2 identity matrix. Then matrix S in (75) can be expressed as

$$S = I + \mathbf{K}\left(\mu(J_S(t) - I)\right)\,, \tag{77}$$

where we integrate along contours $\gamma \cup C$ with the orientation shown in Fig. 8.

Since for all n sufficiently large, $\|\mathbf{M}\|_{L^2 \to L^2} < 1$, the auxiliary function μ can be computed in terms of the Neumann series,

$$\mu = \sum_{k=0}^{\infty} \mathbf{M}^k(I)\,,$$

so that

$$S = \sum_{j=0}^{\infty} S^{(j)}\,, \quad \text{with} \quad S^{(0)} = I\,, \quad S^{(j+1)} = \mathbf{K}\left(S_-^{(j)}\right)\,, \quad j \in \mathbb{N}\,. \tag{78}$$

In other words, S can be recovered from J_S by nested contour integration.

Second observation is that, taking into account the exponential decay of the off-diagonal terms of J_S on $\gamma_{\mathrm{e}} \setminus B$ and $\gamma_{\mathrm{i}} \setminus B$, we can restrict the integration in (76) to contours $\mathcal{C}$, replacing then J_S by PN^{-1}; in this way, only exponentially small terms are neglected. Pluggin the asymptotic expansion of PN^{-1}, given by (72), into equation (78) allows to find successively the terms $\mathfrak{s}_k$. In particular, for terms $S^{(j)}$ in (78) we have

$$S^{(j)}(z) = \mathcal{O}\left(\frac{1}{n^j}\right)$$

locally uniformly in $\mathbb{C}$. Furthermore,

$$S(z) = I + \sum_{k=1}^{\infty} \frac{\mathfrak{s}_k(z)}{n^k}\,, \tag{79}$$

uniformly in z, where $\mathfrak{s}_k$'s are piece-wise analytic functions in $\mathbb{C} \setminus \mathcal{C}$. Let us determine explicitly the first nontrivial term $\mathfrak{s}_1$. By (76) and (78),

$$S^{(1)}(z) = \mathbf{K}\,(I) = -\sum_{k=1}^{m} \frac{1}{2\pi i} \oint_{\mathcal{C}_k} \frac{(P(t)N^{-1}(t) - I)\,dt}{t - z} + \text{exponentially small terms,}$$

where integrals are taken counterclockwise. Hence, we have

Corollary 4 *Matrix S satisfies*

$$S(z) = I - \sum_{k=1}^{m} \frac{1}{2\pi i} \oint_{\mathcal{C}_k} \frac{(P(t)N^{-1}(t) - I)\,dt}{t - z} + \mathcal{O}\left(\frac{1}{n^2}\right)$$

locally uniformly for $z \in \mathbb{C} \setminus (\gamma \cup C)$, where integrals along $\mathcal{C}_k$'s are taken counterclockwise.

Define for $z \in \mathcal{B}_k \cup \mathcal{C}_k$ $(k = 1, \ldots, m)$

$$\mathcal{F}_k(z) \stackrel{\text{def}}{=} -\frac{\beta_k}{n \log(z/a_k)} \begin{pmatrix} \beta_k & -\tau^{-2} a_k^n\, \widehat{\mathcal{S}}_k(W;z) \\ \tau^2 a_k^{-n}\, \widehat{\mathcal{S}}_k^{-1}(W;z) & -\beta_k \end{pmatrix}\,,$$

so that by (73),

$$P(z)N^{-1}(z) - I = \mathcal{F}_k(z) + \mathcal{O}\left(\frac{1}{n^2}\right), \quad z \in \mathcal{C}_k\,.$$

Since $\mathcal{F}_k(z)$ is a meromorphic function in $\mathcal{B}_k \cup \mathcal{C}_k$ with a simple pole at $z = a_k$, by the residue theorem, for $z \notin \mathcal{B}_k \cup \mathcal{C}_k$,

$$\frac{1}{2\pi i} \oint_{\mathcal{C}_k} \frac{\mathcal{F}_k(t)\, dt}{t - z} = \operatorname*{res}_{t=a_k} \frac{\mathcal{F}_k(t)\, dt}{t - z} = F_k(z)\,,$$

where

$$F_k(z) \stackrel{\text{def}}{=} -\frac{a_k \beta_k}{n(a_k - z)} \begin{pmatrix} \beta_k & -\tau^{-2} a_k^n \vartheta_k \\ \tau^2 a_k^{-n} \vartheta_k^{-1} & -\beta_k \end{pmatrix}.$$

Consequently,

$$\frac{1}{2\pi i} \oint_{\mathcal{C}_k} \frac{(P(t)N^{-1}(t) - I)\, dt}{t - z} = \begin{cases} F_k(z) + \mathcal{O}(1/n^2), & \text{if } z \in \mathbb{C} \setminus \overline{\mathcal{B}_k}\,, \\ (\mathcal{F}_k + F_k)(z) + \mathcal{O}(1/n^2), & \text{if } z \in \mathcal{B}_k\,. \end{cases}$$

In particular, in (79),

$$\mathfrak{s}_1(z) = \sum_{k=1}^{m} \frac{a_k \beta_k}{a_k - z} \begin{pmatrix} \beta_k & -\tau^{-2} a_k^n \vartheta_k \\ \tau^2 a_k^{-n} \vartheta_k^{-1} & -\beta_k \end{pmatrix} \quad \text{for } z \in \mathbb{C} \setminus (\gamma \cup B)\,. \tag{80}$$

Now we are ready for the asymptotic analysis of the original matrix Y (and in particular, of its entries $(1,1)$ and $(2,1)$), that we perform in the next section.

4 Asymptotic analysis

Unraveling our transformations we have

$$Y(z) = \begin{cases} S(z)N(z)K^{-1}(z)\, H^{-1}(z), & \text{if } z \in \mathbb{C} \setminus B, \\ S(z)P(z)K^{-1}(z)\, H^{-1}(z), & \text{if } z \in B. \end{cases} \tag{81}$$

Proof of Theorem 1 and Corollary 1. Using (79) in (81) we readily obtain (10). In order to prove the rest of the statements we must analyze the consequences of (81) in each domain (see Fig. 9).

In $\Omega_0 \setminus B$ we have (cf. (41), (42) and (44))

$$N(z) = \begin{pmatrix} 0 & D_{\mathrm{i}}(W; z)/\tau \\ -\tau / D_{\mathrm{i}}(W; z) & 0 \end{pmatrix}, \quad K(z) = H(z) = I\,.$$

Hence, $Y(z) = S(z)N(z)$, so that

$$Y_{11}(z) = -\frac{\tau}{D_{\mathrm{i}}(W; z)}\, S_{12}(z)\,, \quad Y_{21}(z) = -\frac{\tau}{D_{\mathrm{i}}(W; z)}\, S_{22}(z)\,.$$

Taking into account (80) and recalling that $\Phi_n = Y_{11}$ we obtain

$$\Phi_n(z) = \frac{1}{\tau\, D_{\mathrm{i}}(W; z)} \frac{1}{n} \left(\sum_{k=1}^{m} \frac{\beta_k \vartheta_k a_k^{n+1}}{a_k - z} + O\left(\frac{1}{n}\right) \right), \quad z \in \Omega_0 \setminus B\,,$$

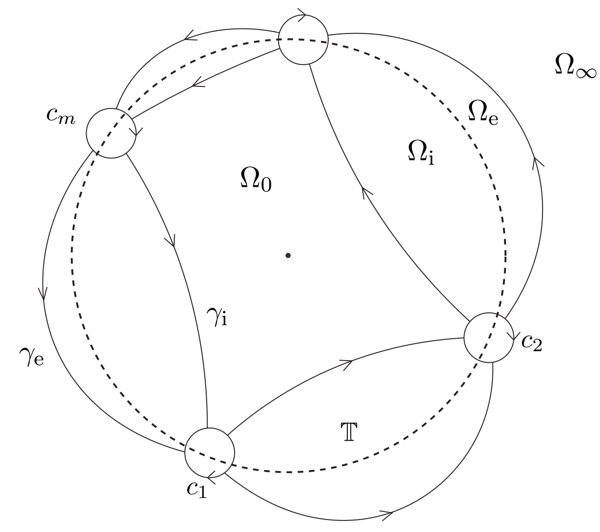

Figure 9: Domains for the asymptotic analysis.

which yields formula (i) in Theorem 1.

Observe that the leading term in the right hand side is a rational function with at most $m-1$ zeros. In consequence, for all sufficiently large n each $\Phi_n(z)$ can have at most $m-1$ zeros on a compact subset of the unit disk (Corollary 1).

Let us analyze the asymptotic behavior of the polynomials close to the inner boundary of the unit circle, but still away from the singular points $\mathcal{A}$. For $z \in \Omega_{\rm i} \setminus B$ we have

$$N(z) = \begin{pmatrix} 0 & D_{\rm i}(W;z)/\tau \\ -\tau/D_{\rm i}(W;z) & 0 \end{pmatrix}, \quad K^{-1}(z) = \begin{pmatrix} 1 & 0 \\ z^n/W(z) & 1 \end{pmatrix}, \quad H(z) = I\,. \tag{82}$$

Hence,

$$Y(z) = S(z) \begin{pmatrix} z^n D_{\rm i}(W;z)/(\tau W(z)) & D_{\rm i}(W;z)/\tau \\ -\tau/D_{\rm i}(W;z) & 0 \end{pmatrix} = S(z) \begin{pmatrix} z^n D_{\rm e}(W;z)/\tau & D_{\rm i}(W;z)/\tau \\ -\tau/D_{\rm i}(W;z) & 0 \end{pmatrix}.$$

Analyzing Y_{11} we obtain that

$$\begin{aligned} \Phi_n(z) =& \frac{z^n D_{\rm e}(W;z)}{\tau} \left(1 + \frac{1}{n} \sum_{k=1}^{m} \frac{a_k \beta_k^2}{a_k - z} + \mathcal{O}\left(\frac{1}{n^2}\right) \right) \\ & + \frac{1}{\tau\, D_{\rm i}(W;z)} \left(\frac{1}{n} \sum_{k=1}^{m} \frac{\beta_k \vartheta_k a_k^{n+1}}{a_k - z} + \mathcal{O}\left(\frac{1}{n^2}\right) \right), \end{aligned} \tag{83}$$

which coincides with the formula of statement (ii) of Theorem 1.

The exterior asymptotic is analyzed likewise. For $z \in \Omega_e \setminus B$ we have (cf. (41), (42) and (44))

$$N(z) = ((D_e(W;z)/\tau)^{\sigma_3}\,, \quad K^{-1}(z) = \begin{pmatrix} 1 & 0 \\ -1/(z^n W(z)) & 1 \end{pmatrix}, \quad H^{-1}(z) = z^{n\sigma_3}\,.$$

Unraveling the transformation we conclude that formula (83) is valid locally uniformly also in $\Omega_e \setminus B$, that is, holds in a neighborhood of $\mathbb{T}$ away from the singular set $\mathcal{A}$.

For $z \in \Omega_\infty \setminus B$ we have

$$N(z) = ((D_e(W;z)/\tau)^{\sigma_3}\,, \quad K = I, \quad H^{-1}(z) = z^{n\sigma_3}\,.$$

Hence, $Y(z) = S(z)\,((z^n D_e(W;z)/\tau)^{\sigma_3}$, and statement (iii) of Theorem 1 is a direct consequence of formula (80). □

Proof of Theorem 2. Now we turn to the asymptotic analysis for Φ_n's in a neighborhood of a singularity a_k, where we have to use the expression

$$Y(z) = S(z)P(z)K^{-1}(z)\,H^{-1}(z)$$

with $P(z) = P(a_k, \beta_k; z)$ for $z \in \mathcal{B}_k$.

Assume that $z \in \Omega_i \cap \mathcal{B}_k$ and $\arg(z) > \arg(a_k)$ in such a way that $\zeta = -i\,\frac{n}{2}\log(z/a_k)$ belongs to sector I in Figure 7. Then Ψ is given by (47), which along with (45) replaced in (71) yields

$$P(z) = \frac{\sqrt{\pi}}{2\sqrt{2}}\,\zeta^{1/2}\left(\frac{\widehat{S}_k(W;z)}{\tau^2}\,i a_k^n\right)^{\sigma_3/2} M(\zeta)\left(e^{-\pi i/4}W^{-1/2}(z)z^{n/2}\right)^{\sigma_3},$$

where

$$M(\zeta) \stackrel{\text{def}}{=} \begin{pmatrix} iH^{(2)}_{\beta_k+\frac12} + H^{(2)}_{\beta_k-\frac12} & H^{(1)}_{\beta_k+\frac12} - iH^{(1)}_{\beta_k-\frac12} \\ -H^{(2)}_{\beta_k+\frac12} - iH^{(2)}_{\beta_k-\frac12} & iH^{(1)}_{\beta_k+\frac12} - H^{(1)}_{\beta_k-\frac12} \end{pmatrix}(\zeta)\,.$$

Thus,

$$P_{11}(z) = \frac{\sqrt{\pi}}{2\sqrt{2}}\,\zeta^{1/2}\left(\frac{\widehat{S}_k(W;z)}{\tau^2}\,a_k^n\frac{z^n}{W(z)}\right)^{1/2} M_{11}(\zeta)\,,$$

$$P_{12}(z) = \frac{\sqrt{\pi}}{2\sqrt{2}}\,\zeta^{1/2}\left(-\frac{\widehat{S}_k(W;z)}{\tau^2}\,a_k^n\frac{W(z)}{z^n}\right)^{1/2} M_{12}(\zeta)\,,$$

$$P_{21}(z) = \frac{\sqrt{\pi}}{2\sqrt{2}}\,\zeta^{1/2}\left(-\frac{\tau^2}{\widehat{S}_k(W;z)}\,a_k^{-n}\frac{z^n}{W(z)}\right)^{1/2} M_{21}(\zeta)\,,$$

$$P_{22}(z) = \frac{\sqrt{\pi}}{2\sqrt{2}}\,\zeta^{1/2}\left(\frac{\tau^2}{\widehat{S}_k(W;z)}\,a_k^{-n}\frac{W(z)}{z^n}\right)^{1/2} M_{22}(\zeta)\,.$$

Taking into account that K and H are as in (82) we see that

$$
\begin{aligned}
(P(z)K^{-1}(z))_{11} &= P_{11}(z) + P_{12}(z)\,\frac{z^n}{W(z)} \\
&= \frac{\sqrt{\pi}}{2\sqrt{2}}\,\zeta^{1/2}\left(\frac{\widehat{S}_k(W;z)}{\tau^2}\,a_k^n\frac{z^n}{W(z)}\right)^{1/2}\,(M_{11}(\zeta)+iM_{12}(\zeta))\,, \\
(P(z)K^{-1}(z))_{21} &= P_{21}(z) + P_{22}(z)\,\frac{z^n}{W(z)} \\
&= \frac{\sqrt{\pi}}{2\sqrt{2}}\,\zeta^{1/2}\left(\frac{\tau^2}{\widehat{S}_k(W;z)}\,a_k^{-n}\frac{z^n}{W(z)}\right)^{1/2}\,(iM_{21}(\zeta)+M_{22}(\zeta))\,.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
Y_{11}(z) &= S_{11}(z)(P(z)K^{-1}(z))_{11} + S_{12}(z)(P(z)K^{-1}(z))_{21} \\
&= (P(z)K^{-1}(z))_{11}\,\left(1+\mathcal{O}\left(\frac{1}{n}\right)\right)\,.
\end{aligned}
$$

By [1, formulas 9.1.3–9.1.4],

$$
M_{11}(\zeta)+iM_{12}(\zeta) = 2\left(iJ_{\beta_k+\frac{1}{2}} + J_{\beta_k-\frac{1}{2}}\right)(\zeta)\,,
$$

so that for z in the domain considered

$$
\Phi_n(z) = \sqrt{\frac{\pi}{2}}\,\left(\frac{\widehat{S}_k(W;z)}{\tau^2}\,\frac{a_k^n z^n}{W(z)}\right)^{1/2}\mathcal{H}(\beta_k;\zeta)\,\left(1+\mathcal{O}\left(\frac{1}{n}\right)\right)\,, \tag{84}
$$

where $\zeta = -i\frac{n}{2}\log(z/a_k)$, and $\mathcal{H}(\beta;\zeta)$ has been defined in (11). Function $\mathcal{H}(\beta;\zeta)$ is holomorphic in the complex plane cut along the positive imaginary axis $i\mathbb{R}_+$, and by [1, formula 9.1.35],

$$
\mathcal{H}_+(\beta;\zeta) = e^{2\pi i\beta}\mathcal{H}_-(\beta;\zeta)\,,\quad \zeta\in i\mathbb{R}_+\,, \tag{85}
$$

if the positive imaginary axis is oriented towards the origin.

Taking into account that

$$
\left(\frac{\widehat{S}_k(W;z)}{W(z)}\right)^{1/2} = \begin{cases} e^{\pi i\beta_k/2}D_{\mathrm{e}}(W;z)\,, & \text{if } z\in\mathcal{B}_k \text{ and } \arg(z)>\arg(a_k)\,,\\ e^{-\pi i\beta_k/2}D_{\mathrm{e}}(W;z)\,, & \text{if } z\in\mathcal{B}_k \text{ and } \arg(z)<\arg(a_k)\,,\end{cases} \quad k=1,\dots,m\,,
$$

we see that (13) is valid for $z\in\Omega_{\mathrm{i}}\cap\mathcal{B}_k$ and $\arg(z)>\arg(a_k)$.

Since transformation $z\mapsto\zeta$ maps the ray $a_k\cdot(0,1)$ onto the positive imaginary axis oriented towards the origin, from formulas (36) and (85) it follows that $D_{\mathrm{e}}(W;z)\mathcal{H}(\beta_k;\zeta_n)$ is single-valued in a neighborhood of $z=a_k$. Hence, by uniqueness of the analytic continuation we must conclude that (13) is valid in fact in the whole $\mathcal{B}_k$. □

Proof of Theorem 3. For the Verblunsky coefficients we have also

$$\overline{\alpha_n} = -\Phi_{n+1}(0) = -\frac{1}{n}\sum_{k=1}^{m} \beta_k \vartheta_k a_k^{n+1} + O\left(\frac{1}{n^2}\right), \quad n \to \infty,$$

and it remains to take into account that $\beta_k \in \mathbb{R}$, $\vartheta_k \in \mathbb{T}$, and $a_k \in \mathbb{T}$ in order to arrive at formula (15).

Furthermore, observe from (40) that the leading coefficients κ_n of the orthonormal polynomials φ_n have the following representation in terms of matrix Y:

$$\kappa_{n-1}^2 = -\frac{1}{2\pi} Y_{21}(0).$$

Since in $\Omega_0 \setminus B$ we have $Y(z) = S(z)N(z)$, with

$$N(z) = \begin{pmatrix} 0 & D_{\mathrm{i}}(W;z)/\tau \\ -\tau/D_{\mathrm{i}}(W;z) & 0 \end{pmatrix},$$

then

$$Y_{21}(z) = -\frac{\tau}{D_{\mathrm{i}}(W;z)} S_{22}(z),$$

so that

$$\kappa_{n-1}^2 = \frac{\tau^2}{2\pi} S_{22}(0),$$

and formula (16) follows from (34) and (80).

□

Proof of Theorem 4. One of the main connections of Toeplitz determinants $\mathcal{D}_n(W)$ defined in (17) with the orthogonal polynomials on the unit circle is that they can be expressed in terms of the leading coefficients κ_n using the following formula (see e.g. [15, Theorem 1.5.11]):

$$\frac{\mathcal{D}_n(W)}{\mathcal{D}_{n-1}(W)} = \frac{1}{\kappa_n^2}.$$

Since $\mathcal{D}_0(W) = d_0 = \oint z^{-n} W(z)|dz|$, we obtain that

$$\mathcal{D}_n(W) = d_0 \prod_{j=1}^{n} \kappa_j^{-2}.$$

Fix $N \in \mathbb{N}$ such that $N > \sum_{k=1}^m \beta_k^2$. By formula (16), for $n > N$,

$$\begin{aligned} \mathcal{D}_n(W) &= d_0 \prod_{j=1}^{N-1} \kappa_j^{-2} \prod_{j=N}^{n} \kappa_j^{-2} \\ &= d_0 \prod_{j=N}^{n} \left[\frac{\tau^2}{2\pi}\left(1 - \frac{1}{j+1}\sum_{k=1}^{m} \beta_k^2 + \mathcal{O}\left(\frac{1}{j^2}\right)\right)\right]^{-1} \end{aligned}$$

where we denote $\mathcal{E}_1 \stackrel{\text{def}}{=} d_0 \prod_{j=1}^{N-1} \kappa_j^{-2}$. Hence,

$$\begin{aligned}\mathcal{D}_n(W) &= \mathcal{E}_1 \left(\frac{2\pi}{\tau^2}\right)^n \prod_{j=N}^{n} \left[1 - \frac{1}{j+1} \sum_{k=1}^{m} \beta_k^2 + \mathcal{O}\left(\frac{1}{j^2}\right)\right]^{-1} \\ &= \mathcal{E}_1 \left(\frac{2\pi}{\tau^2}\right)^n \prod_{j=N}^{n} \left[1 - \frac{1}{j+1} \sum_{k=1}^{m} \beta_k^2\right]^{-1} \prod_{j=N}^{n} \left[1 + \frac{\mathcal{O}\left(j^{-2}\right)}{1 - \frac{1}{j+1} \sum_{k=1}^{m} \beta_k^2 + \mathcal{O}\left(j^{-2}\right)}\right]^{-1} .\end{aligned}$$

This last infinite product is convergent to a constant, which we denote by $\mathcal{E}_2$. Hence,

$$\begin{aligned}\mathcal{D}_n(W) &= \mathcal{E}_1\mathcal{E}_2 \left(\frac{2\pi}{\tau^2}\right)^n \prod_{j=N}^{n} \left[1 - \frac{1}{j+1} \sum_{k=1}^{m} \beta_k^2\right]^{-1} \\ &= \mathcal{E}_1\mathcal{E}_2 \left(\frac{2\pi}{\tau^2}\right)^n \frac{\Gamma(n+2)\,\Gamma\left(N+1-\sum_{k=1}^{m}\beta_k^2\right)}{\Gamma(N+1)\,\Gamma\left(n+2-\sum_{k=1}^{m}\beta_k^2\right)} .\end{aligned}$$

Gathering all the constants in $\varkappa$ and using Stirling formula for the asymptotics of the Gamma function we obtain that

$$\mathcal{D}_n(W) = \varkappa \left(\frac{2\pi}{\tau^2}\right)^n n^{\sum_{k=1}^{m}\beta_k^2} \left(1 + \mathcal{O}(1)\right) , \quad n \to \infty .$$

It remains to use formula (34) in order to set the proof of Theorem 4. □

Proof of Theorem 5. Let $\varepsilon > 0$. We may rewrite formula *(iii)* of Theorem 1 using the notation (22):

$$\frac{D_{\mathrm{i}}(W;z)}{a_1^{n+1} D_{\mathrm{i}}(W;0)} \Phi_n(z) = \frac{D_{\mathrm{i}}(W;z)}{a_1^{n+1}} \left\{ z^n \mathcal{S}(W;z) - \frac{1}{n}\mathcal{R}_n(z)\right\} + H_n(z) . \tag{86}$$

If $z \in \Gamma_n(\varepsilon)$, then there exists a constant $C = C(\varepsilon)$ such that

$$|H_n(z)| \le \frac{C}{n^2} . \tag{87}$$

We conclude the proof using standard arguments involving Rouche's theorem (see e.g. proof of Theorem 4 in [13]). □

Proof of Theorem 6. Let $K \subset \mathbb{D}$ be a compact set. For n large enough and $z \in K$, we have $|z|^n \le n^{-2}$, so that by (86),

$$Z \cap K \subset \bigcap_{k\ge 1} \overline{\bigcup_{n \ge k} \mathcal{Z}(\mathcal{R}_n)} ,$$

with $\mathcal{R}_n$ defined in (22). Thus, it is sufficient to describe all the possible limit points of $\{\mathcal{G}_n\}$. Observe that with the notation introduced in Section 1, before the formulation of Theorem 6,

$$\mathcal{R}_n(z) = \sum_{k=1}^{m} \frac{\beta_k \vartheta_k}{z - a_k} \exp\left(2\pi i (n+1)\theta_k\right) = \sum_{k=1}^{m} \frac{\beta_k \vartheta_k}{z - a_k} \exp\left(2\pi i \sum_{j=1}^{v} r_{kj}\,(n-m+1)\theta_j\right) .$$

Let $v = 1$; this means that all $\theta_k \in \mathbb{Q}$, and $\theta_k = r_{k1} \equiv p_k/q_k \mod \mathbb{Z}$, $k = 2, \ldots, m$. Then clearly all possible limits of $\mathcal{R}_n(z)$ are given by equation (26).

Assume $v \geq 2$. By Kronecker's theorem (also known as Kronecker-Weyl theorem, see e.g. [5, Ch. III]), since $\theta_1 = 1, \theta_2, \ldots, \theta_v$ are rationally independent then (and in fact, if and only if) sequence

$$\left\{ \left(e^{2\pi i \theta_2 n}, \ldots, e^{2\pi i \theta_v n} \right) \right\}_{n \in \mathbb{N}} \subset \mathbb{T}^{v-1}$$

is dense and uniformly distributed in the $(v-1)$-dimensional torus $\mathbb{T}^{v-1}$. In particular, for any real numbers $X_2, \ldots, X_v$ there exists a sub-sequence $\Lambda \subset \mathbb{N}$ such that

$$\lim_{n \in \Lambda} e^{2\pi i \theta_j n} = e^{2\pi i X_j}, \quad j = 2, \ldots, v. \tag{88}$$

In fact, we can say more:

Lemma 2 *Let $\theta_1 = 1, \theta_2, \ldots, \theta_v$ be rationally independent, and let $r_{kj} \in \mathbb{Q}$, $j = 2, \ldots, v$, $k = 1, \ldots, m$. Then for any real numbers $X_2, \ldots, X_v$ there exists a sub-sequence $\Lambda \subset \mathbb{N}$ such that*

$$\lim_{n \in \Lambda} e^{2\pi i r_{kj} \theta_j n} = e^{2\pi i r_{kj} X_j}, \quad j = 2, \ldots, v, \quad k = 1, \ldots, m. \tag{89}$$

Indeed, for $r_{kj} \in \mathbb{Z}$ this statement follows trivially from (88). If $1/r_{kj} \in \mathbb{Z}$, then

$$e^{2\pi i r_{kj} \theta_j n} = \left(e^{2\pi i \theta_j n} \right)^{\frac{1}{1/r_{kj}}},$$

and we can specify the single-valued branch of the r_{kj}^{-1}-th root in the neighborhood of $\exp(2\pi i X_j)$ in such a way that the corresponding limit in (89) holds. Combining these two observations we obtain Lemma 2, which shows that the set of limit points of $\{\mathcal{R}_n\}$ is given by the left hand side of (27). Now the statement follows for $v \geq 2$. $\square$

Proof of Corollary 2. There is a finite number of numbers $s_k \in [0, q_k) \cap \mathbb{Z}$; for each possible combination of s_k's, the left hand side in (26) is a rational function with denominator of degree $\leq m - 1$. Now the statement *(i)* follows.

If $v = 2$, then $Z \cap \mathbb{D}$ is a manifold parameterized by a continuous parameter $X_2 \in \mathbb{R}$, which shows that it is a curve. It is easy to check that its equation is a polynomial in two real variables of degree $\leq m$. Furthermore, if $v = m = 2$, then equation (27) is equivalent to

$$\frac{\beta_1 \vartheta_1 (z - a_2)}{\beta_2 \vartheta_2 (a_1 - z)} = e^{2\pi i X_2} \quad \Longleftrightarrow \quad \left| \frac{\beta_1 \vartheta_1 (z - a_2)}{\beta_2 \vartheta_2 (a_1 - z)} \right| = 1,$$

which reduces to $|\beta_1| \, |z - a_1| = |\beta_2| \, |z - a_2|$. This proves *(ii)*.

Finally, for $v > 2$, the set $Z \cap \mathbb{D}$ is a manifold parameterized by at least two continuous parameter $X_i \in \mathbb{R}$, showing that generically it is a two-dimensional domain. Its boundary is again an algebraic curve or a union of algebraic curves. $\square$

Proof of Theorem 7. It is a straightforward consequence of the asymptotic formula (13), that shows that in the neighborhood $\mathcal{B}_k$ of a_k the zeros of Φ_n match (up to a $\mathcal{O}(1/n)$ term) those of $\mathcal{H}(\beta_k; \zeta_n)$. $\square$

Remark 12 We don't know any explicit formula for $h(\beta)$, although it can be easily computed numerically. In order to find a good initial value for $h(\beta)$ we can use the continued fraction expansion [1, Formula 9.1.73]:

$$\frac{J_\nu(z)}{J_{\nu-1}(z)} = \cfrac{1}{2\nu/z - \cfrac{1}{2(\nu+1)/z - \cfrac{1}{\ddots}}}.$$

In particular, truncating at the second term and equating to zero we can take

$$h_0(\beta) = \frac{\sqrt{(2\beta+3)(6\beta+1)} + i(2\beta+3)}{2}$$

as a reasonably good approximation for any iterative (say, Newton-type) zero-finding method of computation of $h(\beta)$. As an illustration, in the table below we compare the values of $h_0(\beta)$ and $h(\beta)$ for $\beta = -1/4$ and $\beta = 1, \dots, 5$:

β	-0.25	1	2	3	4	5
$h_0(\beta)$	$0.69i$	$2.96 + 2.5i$	$4.77 + 3.5i$	$6.54 + 4.5i$	$8.3 + 5.5i$	$10.04 + 6.5i$
$h(\beta)$	$0.68i$	$3.73 + 1.04i$	$5.08 + 0.87i$	$6.34 + 0.79i$	$7.56 + 0.74i$	$8.75 + 0.71i$

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