

Electrostatic models for zeros of polynomials: old, new, and some open problems

F. Marcellán^{a,*}, A. Martínez-Finkelshtein^{b,c}
and P. Martínez-González^b

^a*Departamento de Matemáticas, Escuela Politécnica Superior, Universidad Carlos III, Avenida de la Universidad, 30, 28911 Leganés-Madrid, Spain*

^b*Departamento de Estadística y Matemática Aplicada, Universidad de Almería, La Cañada, 04120 Almería, Spain*

^c*Instituto Carlos I de Física Teórica y Computacional, Universidad de Granada, 18071 Granada, Spain*

DEDICATED TO NICO TEMME ON THE OCCASION OF HIS
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Abstract

We give a survey concerning both very classical and recent results on the electrostatic interpretation of the zeros of some well-known families of polynomials, and the interplay between these models and the asymptotic distribution of their zeros when the degree of the polynomials tends to infinity. The leading role is played by the differential equation satisfied by these polynomials. Some new developments, applications and open problems are presented.

Key words: Second order differential equation, orthogonal polynomials, zeros, asymptotics, electrostatic model, logarithmic potential

1 Introduction

The electrostatic interpretation of the zeros of the classical orthogonal polynomials is probably one of the most elegant results in the theory of special functions, linked in the first instance to such a distinguished name as Stieltjes (although studied also by Bôcher, Heine, Van Vleck, and Polya). Although this topic has remained “dormant”

* Corresponding author. E-mail: pacomarc@ing.uc3m.es

for almost a century, a renewed interest has appeared recently, partially motivated by the connection of this topic with modern powerful techniques from the theory of logarithmic potentials, as well as by an increasing interest on the study of new classes of special functions with roots in Physics, Combinatorics, Number Theory, etc.

This paper is a short and light survey on these topics. Our intention is to provide not a comprehensive list of results, but more to convey the “flavor” of techniques and ideas behind the electrostatic model and its direct implications in the asymptotic theory. With this purpose, in Section 2 we derive the classical electrostatic interpretation of the zeros of Jacobi polynomials, found originally by Stieltjes, and we describe some of its new and still-in-progress direct generalizations (Section 3). Next we discuss a recent progress by Ismail in connecting electrostatics with orthogonality in Section 4. Seeking for a model for complex zeros in Section 5 we formulate an alternative model, not constrained to the real line, as well as discuss some further generalizations and applications of the electrostatic model in Section 6. We conclude the paper with some examples of applications of these models to the study of the asymptotic distribution of zeros in the semiclassical limit.

2 Electrostatic model for classical orthogonal polynomials

Let us begin with a classical and very well-known result, found by Stieltjes in 1885, about the electrostatic interpretation of the zeros of Jacobi polynomials $P_n^{(\alpha,\beta)}$. The definitions and basic properties of these polynomials can be found in Chapter IV of the classical monograph by Szegő [63]; we will mention here only those that will be needed further.

Jacobi polynomials can be given explicitly by

$$P_n^{(\alpha,\beta)}(x) = 2^{-n} \sum_{k=0}^n \binom{n+\alpha}{n-k} \binom{n+\beta}{k} (x-1)^k (x+1)^{n-k}.$$

Nevertheless, this formula is not the most useful one, at least for what we are looking for. Another equivalent characterization is the Rodrigues formula

$$P_n^{(\alpha,\beta)}(x) = \frac{1}{2^n n!} (x-1)^{-\alpha} (x+1)^{-\beta} \left(\frac{d}{dx} \right)^n \left[(x-1)^{n+\alpha} (x+1)^{n+\beta} \right] \quad (2.1)$$

(cf. Section 4.3 of [63]). In particular, these expressions show that Jacobi polynomials $P_n^{(\alpha,\beta)}$ are analytic functions of the parameters $\alpha, \beta \in \mathbb{C}$ and that $\deg P_n^{(\alpha,\beta)} \leq n$.

A third characterization of these polynomials, which will play the leading role in the sequel, is that they are the only polynomial solutions (up to a constant factor) of the

linear differential equation

$$y''(x) + \left(\frac{\alpha + 1}{x - 1} + \frac{\beta + 1}{x + 1} \right) y'(x) - \frac{\lambda_n}{(x^2 - 1)} y(x) = 0, \quad (2.2)$$

where $\lambda_n = n(n + \alpha + \beta + 1)$. This is a second order linear differential equation of hypergeometric type.

The setting is considered “classical” when $\alpha > -1$ and $\beta > -1$. In this case, as we see from (2.2), the residues of the rational coefficient of y' in the differential equation are both positive. Moreover, we know that with this assumption the Jacobi polynomials are orthogonal on $[-1, 1]$ with respect the weight function $(1 - x)^\alpha(1 + x)^\beta$, i. e.

$$\int_{-1}^1 P_n^{(\alpha, \beta)}(x) x^k (1 - x)^\alpha (1 + x)^\beta dx = 0, \quad k = 0, 1, \dots, n - 1. \quad (2.3)$$

If both $\alpha > -1$ and $\beta > -1$, then the weight is integrable on $[-1, 1]$, and a well known consequence of (2.3) is that all the zeros of $P_n^{(\alpha, \beta)}$ are simple and lie on the interval $(-1, 1)$. Apparently, it was Stieltjes [61] who observed first that it is possible to give a nice interpretation of the location of these zeros as follows. Put two positive fixed charges of mass $(\beta + 1)/2$ and $(\alpha + 1)/2$ at -1 and $+1$, respectively, and allow n positive *unit* charges $X = \{x_1, \dots, x_m\}$ to move freely in $(-1, 1)$. If the interaction obeys the logarithmic potential law (that is, the force is inversely proportional to the relative distance), then in order to find the total energy $E(X)$ of this system we have to add to the energy of the mutual interaction of these charges,

$$E_{\text{mutual}}(X) = - \sum_{1 \leq k < j \leq n} \ln |x_k - x_j|,$$

the component given by the “external field” $\varphi(x)$ created by the fixed charges,

$$\varphi(x) = -\frac{\beta + 1}{2} \ln |x + 1| - \frac{\alpha + 1}{2} \ln |x - 1|. \quad (2.4)$$

In other words, the total energy is

$$E(X) = E_{\text{mutual}}(X) + \sum_{k=1}^n \varphi(x_k). \quad (2.5)$$

There is a unique configuration $X^* = \{x_1^*, \dots, x_m^*\}$, $-1 < x_1^* < x_2^* < \dots < x_n^* < 1$, providing the (strict) global minimum of $E(X)$ in $[-1, 1]^n$, corresponding to the unique equilibrium position for our free charges.

There uniqueness of the global minimum is not obvious, but there is an elegant proof in [63], based on the inequality between the arithmetic and geometric means. However, it is also a consequence of the following statement (Stieltjes’ theorem): *points x_k^* are precisely the zeros of the polynomial $P_n^{(\alpha, \beta)}$* . The proof is quite straightforward and uses the differential equation (2.2). Indeed, since $E(X) \rightarrow +\infty$ as X approaches any

boundary point of $[-1, 1]^n$, we conclude that $X^* \subset (-1, 1)^n$. Then X^* is also a critical point for the energy functional $E(X)$, and as a consequence, every x_k^* is a critical point of E as a function of x_k (fixing the rest of x_j^* 's). Thus, the following necessary conditions must be satisfied:

$$\frac{\partial}{\partial x_k} E(X) \Big|_{X=X^*} = \frac{\partial}{\partial x_k} E_{mutual}(X) \Big|_{X=X^*} + \varphi'(x_k) = 0 \quad \text{for } k = 1, 2, \dots, n. \quad (2.6)$$

Let us consider the monic polynomial vanishing at x_k^* 's: $y(x) = \prod_{j=1}^n (x - x_j^*)$; it is easy to check that

$$\frac{\partial}{\partial x_k} E_{mutual}(X) \Big|_{X=X^*} = - \sum_{1 \leq j \leq n, j \neq k} \frac{1}{x_k^* - x_j^*} = - \frac{y''(x_k^*)}{2y'(x_k^*)},$$

so that (2.6) implies

$$y''(x) - 2\varphi'(x)y'(x) = 0 \quad \text{for } x \in X^*. \quad (2.7)$$

For the external field (2.4) it means that

$$y''(x) + \left(\frac{\beta + 1}{x + 1} + \frac{\alpha + 1}{x - 1} \right) y'(x) = 0 \quad \text{for } x \in X^*. \quad (2.8)$$

However, the left hand side in (2.8) is a rational function of the type $[n/2]$ with poles at ± 1 , so, up to a constant factor, it is equal to $y(x)/(x^2 - 1)$, which yields (2.2) and shows that $y(x) = \text{const } P_n^{(\alpha, \beta)}(x)$.

Observe that we started from the strongest requirement of the *global* minimum of $E(X)$ (that we refer to as a *stable equilibrium*) which implied that X^* is also a point of *local* minimum, and in consequence, we have a “*Nash-type*” *equilibrium* (each function $E(x_k) = E(x_1^*, \dots, x_{k-1}^*, x_k, x_{k+1}^*, \dots, x_n^*)$ attains its minimum at $x_k = x_k^*$), yielding that each x_k^* is a *critical point* of $E(x_1^*, \dots, x_{k-1}^*, x_k, x_{k+1}^*, \dots, x_n^*)$. Precisely this last statement turned out to be equivalent to the characterizing differential equation (2.2). The reader should take note of this clear hierarchy of equilibria in this problem.

Interesting enough, in the situation we are considering all the equilibria are equivalent. In order to prove it we can consider the Hessian matrix

$$H = (h_{ij}), \quad h_{ij} = \frac{\partial^2 E(X)}{\partial x_i \partial x_j} \quad (2.9)$$

of $E(X)$. Straightforward computations show that

$$h_{ij} = \begin{cases} -2(x_i - x_j)^{-2} & \text{if } i \neq j, \\ \frac{\beta + 1}{(x_i + 1)^2} + \frac{\alpha + 1}{(x_i - 1)^2} + 2 \sum_{1 \leq j \leq n; j \neq i} \frac{1}{(x_i - x_j)^2} & \text{if } i = j. \end{cases}$$

It is easy to observe that matrix H is real, symmetric, strictly diagonally dominant, and its diagonal entries are positive, from which we conclude that H is positive definite. In

particular, every critical point of E must be a point of a local minimum. Its uniqueness was observed above, hence it is in fact the point of the global minimum.

Stieltjes considered also similar electrostatic models for other two classical orthogonal polynomials: Laguerre and Hermite. Since in this situation the free charges can move on an unbounded set, what can't prevent them from escaping to infinity? Stieltjes found a clever solution in putting a constraint on either the first (Laguerre) or second (Hermite) moment of their zero counting measures (see e.g. [66]).

Namely, if for $\alpha > -1$ we fix one charge of mass $(\alpha + 1)/2$ at the origin, and allow n positive unit charges X to move in $[0, +\infty)$ with an additional constraint that their arithmetic mean is uniformly bounded, $\sum_{i=1}^n x_i/n \leq K$ for some positive constant K , then again the unique configuration providing the global minimum of the total energy of the system of interacting particles coincides with the rescaled zeros of the Laguerre polynomial $L_n^{(\alpha)}(r_n x)$, where $r_n = (n + \alpha)/K$. In a similar way, allowing X to move freely on the whole \mathbb{R} , but limiting the arithmetic mean of their *squares*, $\sum_{i=1}^n x_i^2/n \leq K$ for some constant K , the unique configuration providing the global minimum of the total energy of the system of interacting particles coincides with the rescaled zeros of the Hermite polynomial $H_n(s_n x)$, where $s_n = \sqrt{(n - 1)/(2K)}$.

The proof for these both cases is similar to the Jacobi case, except that now we are dealing with a constrained minimum, so that the Lagrange multipliers corresponding to the restrictions when looking for a necessary condition of a critical point will become part of the characterizing differential equation.

It is strange enough that Stieltjes himself did not realize that the unbounded cases (Laguerre and Hermite) could have been put in the same framework that the Jacobi one if we allow for an external field not necessarily generated by positive fixed charges. It was probably Ismail [30] who observed first that the zeros of these polynomials still provide the global minimum of the total energy (2.5) without any additional constraint or rescaling, if assuming for the Laguerre polynomials that φ is generated by a fixed charge of mass $(\alpha + 1)/2$ at the origin *plus* another term, equal to $x/2$; analogously, for the Hermite polynomials it is sufficient to take $\varphi(x) = x^2/2$.

Reviewing the electrostatic models above several natural questions arise, such as:

- are there generalizations of these models to other families of polynomials?
- why necessarily the global minimum of the energy should be considered? Which other types of equilibria described above could be linked to the zeros of the polynomials?
- what is the appropriate model for the complex zeros (when they exist)?
- what kind of applications can these models have beyond their clear aesthetical value?

In the following sections we will try to give at least some partial answers or to formulate conjectures related to these questions.

3 A generalization of the electrostatic model: Lamé equation

In fact, Stieltjes studied a more general situation. The *generalized Lamé differential equation* (in an algebraic form) is

$$E''(x) + \left(\sum_{i=0}^p \frac{\rho_i}{x - a_i} \right) E'(x) - \frac{C(x)}{A(x)} E(x) = 0, \quad A(x) = \prod_{i=0}^p (x - a_i), \quad (3.1)$$

where C is a polynomial of degree $\leq p-1$ (in the sequel we use the notation $C \in \mathbb{P}_{p-1}$). The case $p = 1$ corresponds to the hypergeometric differential equation, such as (2.2), while for $p = 2$ we obtain the Heun's equation (see [51]).

Heine [28] proved that for every $n \in \mathbb{N}$ there exist at most

$$\sigma(n) = \binom{n+p-1}{n} \quad (3.2)$$

different polynomials C in (3.1) such that this equation has a polynomial solution of degree n . These coefficients C are called *Van Vleck polynomials* and the corresponding polynomial solutions E are known as *Heine-Stieltjes polynomials*.

Stieltjes studied the problem under the following two assumptions, generalizing the classical situation for the Jacobi polynomials: (i) the zeros a_i of A are assumed to be simple and real, so without loss of generality we can take

$$-1 = a_0 < a_1 < \dots < a_p = 1, \quad (3.3)$$

(ii) the residues of the coefficients of E' are all positive:

$$\rho_i > 0, \quad i = 0, \dots, p. \quad (3.4)$$

(cf. equation (2.2) with assumptions $\alpha > -1, \beta > -1$). Observe that zeros a_k 's define p intervals (a_{k-1}, a_k) of the real line, and the combinatorial number $\sigma(n)$ defined in (3.2) coincides with the number of different ways of placing n balls in p boxes. This is not a mere coincidence. According to Stieltjes [61], for each vector $\mathbf{n} = (n_1, \dots, n_p)$ such that $n_1 + \dots + n_p = n$ there exists a unique Heine-Stieltjes polynomial $E_{\mathbf{n}}$ and the unique corresponding Van Vleck polynomial $C_{\mathbf{n}} \in \mathbb{P}_{p-1}$ such that precisely n_k zeros of $E_{\mathbf{n}}$ belong to the interval (a_{k-1}, a_k) , $k = 1, \dots, p$. Moreover, these zeros provide the *unique global minimum* for the total energy (2.5) under the *additional restriction* of the number of zeros in each interval mentioned above, with the external field created by $p+1$ positive charges fixed at a_j 's:

$$\varphi(x) = - \sum_{j=0}^p \frac{\rho_j}{2} \ln |x - a_j|. \quad (3.5)$$

This electrostatic model has been used in [41] in order to study the semiclassical limit $n \rightarrow \infty$ (see Section 7), and in a number of papers [11,12] in describing the probabilistic distribution of these zeros as the number of intervals (that is, p) grows large together with n .

Further generalizations of the work by Heine and Stieltjes followed several paths; we will mention only some of them. First, under assumptions (3.3)–(3.4) Van Vleck [68] and Bôcher [9] proved that the zeros of C belong to $[a_0, a_p]$. A refinement of this result is due to several works of Shah [55,56,57,58]. Furthermore, Pólya [50] showed that, allowing complex zeros of A , condition (3.4) is sufficient to assure that the zeros of E are located in the convex hull of a_k 's. Marden [36] and later on, Alam, Al-Rashed, and Zaheer (see [1], [2], [69], [70]) established further results on location of the zeros of the Heine-Stieltjes polynomials under weaker conditions on the coefficients A and ρ_i of (3.1).

Going back to configuration (3.3), not much can be said if we drop the condition of positivity of the residues ρ_i in (3.4); the situation then becomes much more difficult to handle. The first attempt was made by Grünbaum [26,27], and later on by Dimitrov and Van Assche [18]. In particular, in [18] the case of $p = 3$ is analyzed when the positive and negative residues do not interlace: basically, the two cases considered are depicted in Figure 1.



Fig. 1. Configuration studied by Dimitrov and Van Assche.

Dimitrov and Van Assche proved that under certain restrictions on the degree of the polynomial and the residues ρ_i , for every $n \in \mathbb{N}$ there exists a unique pair (C_n, E_n) of respectively Van Vleck and Heine-Stieltjes polynomials with $\deg E_n = n$. All zeros of E_n belong to the interval enclosed by a_j 's with $\rho_j > 0$ (denoted by Δ in Figure 1), and they are in the equilibrium position, given by the global minimum of the discrete energy (2.5) with the external field (3.5). As far as we know, no further studies in this direction exist.

Interestingly enough, a similar situation appears when we study a special class of polynomials of multiple orthogonality (when the orthogonality conditions are distributed among several measures), which are object of an intensive study in the last years. Kaliaguine [32] (see also the more recent papers by Aptekarev [3] and Kaliaguine and Ronveaux [33]) considered the system of polynomials $\{p_{2n}\}$, $\deg p_{2n} = 2n$, satisfying

$$\int_{-1}^0 p_{2n}(x)x^k w(x) dx = \int_0^1 p_{2n}(x)x^k w(x) dx = 0, \quad k = 0, 1, \dots, n-1,$$

with $w(x) = x^\gamma(1-x)^\alpha(1+x)^\beta$, $\alpha, \beta, \gamma > -1$. It is a real case: p_{2n} has exactly n zeros in each interval $(-1, 0)$ and $(0, 1)$. The asymptotic analysis (using the tools of

the potential theory) suggests that a reasonable model for these zeros might be that each of these sets of n zeros is the equilibrium position of the positive unit charges, which besides the mutual interaction are subject to the external field generated by the three fixed charges $\alpha/2$, $\beta/2$, $\gamma/2$ at $1, -1, 0$, respectively, and by the n charges of the weight $1/2$ on the other subinterval. Nevertheless, this is an open problem.

In a recent paper Grinshpan [25] considered a configuration of zeros on the unit circle. He characterized the stable equilibrium of the n positive unit charges on $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ in the field generated by n negative unit charges in $\mathbb{C} \setminus (\mathbb{T} \cap \{0\})$ and showed that they are zeros of a Heine-Stieltjes polynomial satisfying a generalized Lamé differential equation with coefficients symmetric with respect to \mathbb{T} . Although it might give an impression that here we are dealing with complex zeros, the symmetry of the problem makes it essentially “real”. A more recent paper [40] extends the problem of stable equilibrium studied in [25] to the case of a field generated by $m = m(n) \in \mathbb{N}$ negative charges in $\mathbb{C} \setminus (\mathbb{T} \cap \{0\})$ with values $-\omega_{nk}$, where $\omega_{nk} > 0$ are the residues of R_n/Q_n in the zeros z_{nk} of Q_n that lie in the open unit disc, and Q_n, R_n are polynomials of degree $2m, 2m - 1$, respectively, such that, $A_n(z) = zQ_n(z)$ and $B_n = Q_n(z) - zR_n(z)$ are the coefficients of the corresponding generalized Lamé differential equation $A_n(z)y''(z) + B_n(z)y'(z) + C_n(z)y(z) = 0$.

4 Electrostatics and orthogonality

Stieltjes’ theorem is beautiful, and also useful for proving some monotonicity properties of the zeros with respect to the parameters of the system. But is the existence of such a description a mere accident? Why should an electrostatic interpretation of the zeros of general orthogonal polynomials exist at all? One fact (known from the ’80) that makes us expect further developments is that the appropriate description of the asymptotic behavior (as $n \rightarrow \infty$) of the zeros of orthogonal polynomials is given in terms of some *equilibrium measures*, that is, measures minimizing certain logarithmic potentials, eventually in presence of an external field (see [53]).

If we go back to the Ismail’s electrostatic model for the Laguerre and Hermite polynomials, then it is easy to recognize in the corresponding external field the influence of the weight of orthogonality of these polynomials. It was definitely an interesting problem to find out if this is a regular fact, tackled by Ismail in a series of papers [30,31]. We have noticed that so far the characterizing differential equation has been the cornerstone of the model, and it was a reasonable line of attack. Thus, generalizing some pioneering works of Bauldry [7], and Bonan and Clark [10], Ismail deduced a second order linear differential equation satisfied by a sequence of *general* orthogonal polynomials, after which he suggested a model for the zeros. Let us outline briefly his results.

Let $\{p_n(x)\}$ be polynomials orthonormal with respect to a unit weight function $w(x) =$

$\exp(-v(x))$ supported on an interval $[c, d] \subset \mathbb{R}$, finite or infinite:

$$\int_c^d p_m(x)p_n(x)w(x) dx = \delta_{m,n}.$$

Then sequence $\{p_n\}$ satisfies a three-term recurrence relation

$$xp_n(x) = a_{n+1}p_{n+1}(x) + b_np_n(x) + a_np_{n-1}(x), \quad n \geq 0, \quad a_n > 0,$$

with $p_{-1}(x) = 0$ and $p_0(x) = 1$. Moreover, it has been shown in [7], [10], and also [17], [29], that under certain assumptions on w the orthonormal polynomials p_n satisfy also the difference-differential relation

$$p'_n(x) = A_n(x)p_{n-1}(x) - B_n(x)p_n(x),$$

where the coefficients A_n and B_n are explicitly given in terms of w , the recurrence coefficients a_n , and the values of p_n at the endpoints c and d . A direct consequence of this is that p_n satisfies also the second order linear differential equation

$$p''_n(x) - 2R_n(x)p'_n(x) + S_n(x)p_n(x) = 0, \quad (4.1)$$

with

$$R_n(x) = \frac{v'(x)}{2} + \frac{A'_n(x)}{2A_n(x)},$$

$$S_n(x) = B'_n(x) - B_n(x)\frac{A'_n(x)}{A_n(x)} - B_n(x)[v'(x) + B_n(x)] + \frac{a_n}{a_{n-1}}A_n(x)A_{n-1}(x).$$

In particular, if $X^* \subset (\alpha, \beta)^n$ is the ordered set of zeros of p_n , then

$$p''_n(x) - 2R_n(x)p'_n(x) = 0 \quad \text{for } x \in X^*. \quad (4.2)$$

Comparing it with (2.7) can associate $\varphi'(x) = R_n(X)$, and it is natural to consider an electrostatic model for X^* with logarithmic interaction between particles and an external field

$$\varphi(x) = \frac{v(x)}{2} + \frac{\ln(k_n A_n(x))}{2} = \varphi_{\text{long}}(x) + \varphi_{\text{short}}(x) \quad (4.3)$$

(k_n is any appropriate normalization constant, taken in [30] equal to a_n^{-1}). Observe that this external field has two components: the first term in the right hand side of (4.3) has its origin in the orthogonality weight $w(x) = \exp(-v(x))$, and Ismail called it the *long range potential*. The second term, which received the name of the *short range potential*, is a bit mysterious, but explains several features of the classical models considered in Section 2, and at the same time, allows to give a further generalization of the electrostatic interpretation.

The main result of Ismail [30] states that, assuming $w(x) > 0$ on (c, d) , both v and $\ln(A_n)$ in $C^2(c, d)$, and the external field (4.3) convex, the total energy (2.5) has a

unique point of global minimum, which is precisely X^* (that is, the zeros of the orthogonal polynomial p_n). Obviously, what is used here is in fact the relation (4.2) equivalent to the fact that X^* is just a critical point for $E(X)$, and convexity assures that this is the unique point of minimum. The proof is in general very similar to the arguments for the Jacobi polynomials from Section 2.

Beside some restrictions on the weight of orthogonality, one drawback of this model in application to general weights is that usually the short range potential cannot be explicitly computed in terms of w . This is not the case of the classical polynomials and some of their generalizations.

For instance, for Jacobi polynomials,

$$w(x) = \frac{(1-x)^\alpha(1+x)^\beta\Gamma(\alpha+\beta+2)}{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)}, \quad \frac{A_n(x)}{a_n} = \frac{2n+\alpha+\beta+1}{1-x^2}.$$

In other words, w is responsible for the fields generated by two positive charges of mass $\beta/2$ and $\alpha/2$ at -1 and $+1$, respectively, while the short range potential adds the missing charges of size $1/2$ at ± 1 .

Analogously, for Laguerre polynomials we have

$$w(x) = \frac{x^\alpha \exp(-x)}{\Gamma(\alpha+1)}, \quad \frac{A_n(x)}{a_n} = \frac{1}{x},$$

so that the x^α factor in w plus the short range potential are responsible for the fixed charge of size $(\alpha+1)/2$ at $z=0$, and the exponential factor of the weight w generates the remaining component $x/2$ of the external field φ , as described at the end of Section 2.

So far the only role played by the short range component was adding some extra weight to the charges fixed by the weight. However, A_n can be responsible for a creation of “ghost” movable charges (either positive or negative) in the picture, as it was shown first by Grünbaum [26] (work that in fact predates [30]) or by Ismail himself when analyzing the case of the so-called Freud weight, $w(x) = \exp(-x^4)$.

Let us consider here a little bit more general situation. A generalized weight function (or measure) is *semiclassical* (see e.g. [37,59]) if it satisfies the *Pearson equation*

$$D(\phi w) = \psi w,$$

where ϕ, ψ are polynomials, with degree of $\psi \geq 1$, and D is the “derivative” operator (in the usual, but also possibly in a distributional sense). It is well known that for such a weight the corresponding orthogonal polynomials (called also *semiclassical*) satisfy a differential equation of the type (4.1), where the coefficients R_n and S_n are rational functions. The classical-type orthogonal polynomials considered by Grünbaum [26] are an example of a semiclassical family, but there are many more (for instance, the so-called sieved orthogonal polynomials [29], among others).

The systematic study of the semiclassical orthogonal polynomials from the point of view of the electrostatic interpretation of their zeros has started in the works of Garrido and collaborators [23], [24]. For instance, they consider a perturbation of the Freud weight function ($w(x) = \exp(-x^4)$) by the addition of a fixed charged point of mass λ at the origin; the corresponding orthogonal polynomials are known as Freud-type polynomials. The resulting orthogonality measure is semiclassical, and it was proved in [23] that these polynomials satisfy a second order linear differential equation of the form (4.1), and the electrostatic model is in sight. In fact, in the situation considered, and following Ismail's terminology, the long range potential is, as expected, $\varphi_{\text{long}}(x) = x^4/2$, while the short range potential depends on the degree n and has the following structure:

$$\varphi_{\text{short}}(x) = \frac{1}{2} \ln \left| \frac{Q_4(x, n)}{x^2} \right| \quad (4.4)$$

where $Q_4(x, n)$ has two real roots $r_1(n), r_2(n)$ and two simple conjugate complex roots $r_3(n), r_4(n)$. Their asymptotic behavior is

$$\lim_n r_1(n) = \lim_n r_2(n) = 0, \quad \lim_n r_3(n) = \lim_n r_4(n) = \infty. \quad (4.5)$$

In fact, in the odd case, both real zeros coalesce at the origin:

$$r_1(2n + 1) = r_2(2n + 1) = 0,$$

canceling out the denominator in the argument of the logarithm in (4.4). In other words, the short range potential is given by one unit positive charge fixed at the origin, plus 4 floating negative charges of weight $-1/2$ on \mathbb{C} situated symmetrically with respect to the origin. However, in the odd case two of these negative charges annihilate with the fixed positive one, paving the way to a zero of the orthogonal polynomial to take the origin, position it must occupy due to the symmetry of the problem. This annihilation has as a consequence that for odd n 's the Freud and the Freud-type polynomials are identical. Moreover, (4.5) shows that also for the even case they are asymptotically identical (as $n \rightarrow \infty$).

One of the main results of [23], [24] is that the zeros of the Freud-type polynomials provide a *critical configuration* for the total energy $E(X)$ (in other words, that for these zeros X^* equations (2.6) are satisfied). But can we assure that they are in a stable equilibrium, that is, that $E(X^*) = \min E(X)$? The complete answer is not clear yet. By computing the Hessian matrix H (2.9) it was proved that for small values of the correction charge λ this is really the case, but there are some evidences that make it possible to conjecture the existence of a critical value λ_0 such that for $0 \leq \lambda < \lambda_0(n)$ the zeros of the orthogonal polynomials attain the minimum of the total energy, but for $\lambda > \lambda_0(n)$ the equilibrium is no longer stable. Is still any other type of equilibria discussed in Section 2 preserved in this case (beside the critical point of the energy)? This is also an open question.

5 A max-min problem

From potential theory it is well known that many of energy minimization problems have in fact a min-max (or max-min) characterization. Let us analyze again, but from this point of view, the classical models of Stieltjes, and in passing, address the issue why the real line should be present in the model.

This question is not trivial: let us go back to the Jacobi polynomials $P_n^{(\alpha,\beta)}$ with $\alpha, \beta > -1$. Why their zeros should be real? A standard explanation is that these polynomials satisfy the orthogonality conditions (2.3), but where the real line is present in that integral? Observe that the integrand in (2.3) is an analytic function, so we perfectly can obtain the same result integrating along any reasonable curve joining -1 with $+1$.

The real line is also built in the electrostatic model of Stieltjes as an a priori constraint. However, we may put forward an alternative model free from this restriction. As in the Stieltjes' setting we fix two positive charges of mass $(\beta + 1)/2$ and $(\alpha + 1)/2$ at -1 and $+1$, respectively, and denote again by $E(X)$ the total energy of a discrete system $X \subset \mathbb{C}^n$, given by formulas (2.4)–(2.5) (assuming that in case of coincidence of any two points, $E(X) = +\infty$). Denote by \mathcal{F} the family of compact continua on \mathbb{C} containing both -1 and $+1$. For any $K \in \mathcal{F}$ set

$$m(K) = \inf\{E(X) : X = \{x_1, \dots, x_N\} \subset K\},$$

and consider the following extremal problem:

$$E^* = \sup\{m(K) : K \in \mathcal{F}\}.$$

Theorem 1 $E^* < +\infty$, and this value is attained at a unique n -tuple $X^* = \{x_1^*, \dots, x_n^*\}$, characterized by the fact that x_i^* are precisely the zeros of the Jacobi polynomial $P_n^{(\alpha,\beta)}$.

Proof. Let X be any configuration of n distinct points on the interval $(-1, 1)$ (the case of coalescence of some points is trivial), and let $K \in \mathcal{F}$ be any other continuum. Since $\pm 1 \in K$, we can assure that there exists a discrete set of points $Z \subset K$ of the form

$$Z = \{z_j = x_j + iy_j : y_j \in \mathbb{R}, j = 1, \dots, n\}$$

(that is, any vertical line passing through an x_j must intersect K). It is straightforward to check that

$$|x_k - x_j| \leq |z_k - z_j|, \quad k \neq j, \quad \text{and} \quad |1 \pm x_j| \leq |1 \pm z_j|, \quad j = 1, \dots, n.$$

Taking into account the expression of the total energy and the positivity of all the charges involved in the system we immediately conclude that $E(X) \geq E(Z)$. Hence,

$$m([-1, 1]) \geq m(K),$$

and since K was arbitrary, we obtain that $E^* = m([-1, 1])$. But from Stieltjes' theorem it follows that there exists a unique $X^* \subset (-1, 1)^n$ such that $E(X^*) = m([-1, 1])$, and the points of X^* are the zeros of $P_N^{(\alpha, \beta)}$, which concludes the proof. \square

Obviously, a similar result is possible to formulate for Laguerre and Hermite polynomials.

The value of the min-max model we have described is that it allows to consider easily complex zeros of these classical families, which appear when the parameters involved become no longer “classical” (see for instance Figure 2 with two examples of zeros of Jacobi polynomials). In fact, as it follows from the Rodrigues formula (2.1), if we allow for arbitrary real α and β , the Jacobi polynomials constitute a very rich object exhibiting several types of orthogonalities: hermitian, non-hermitian, and even multiple, depending on the values of the parameters and the degree (see the classification of all cases obtained [34]). We conjecture that Theorem 1 is valid in all these situations after a suitable modification of the family of continua \mathcal{F} . For instance, if $2n + \alpha + \beta < -1$ (see Figure 2, left), the appropriate class \mathcal{F} would include all continua extending to infinity, and *separating* -1 and $+1$. From the results of [34] a rule of the thumb follows: if $\alpha + n > 0$ (resp., if $\beta + n > 0$) then the continua from \mathcal{F} must contain $+1$ (resp., -1). Moreover, if $n + \alpha + \beta + 1 < 0$, then the continua from \mathcal{F} must include the infinity.

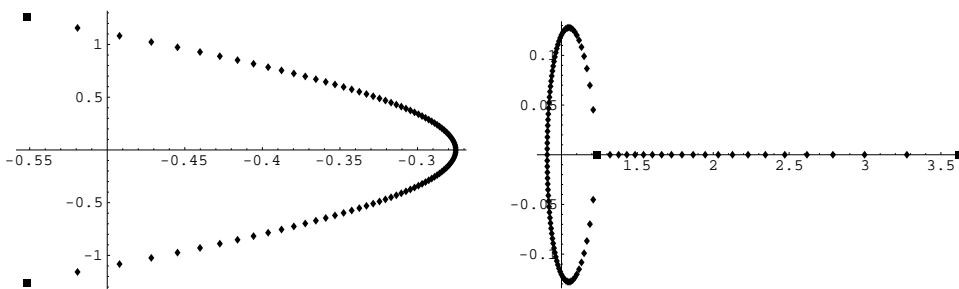


Fig. 2. Zeros of $P_{100}^{(-300, -200)}$ and $P_{100}^{(-80, -314)}$.

All these facts have been conjectured by one of this authors some time ago, but their proof still remains an open problem. A general approach to such kind of theorems would allow also to tackle other kind of problems not linked directly to the real line or exhibiting non-real zeros.

Typical examples are those related to the multiple (or Hermite-Padé) orthogonality, when the orthogonality conditions are distributed among several measures, or of non-hermitian orthogonality, when the path of integration is, in fact, not fixed a priori. In many situations these polynomials exhibit complex zeros, whose *asymptotic* distribution has been studied by several authors. From the pioneering works of Stahl [60] it is known that the limiting location of the zeros is usually described by some trajectories of a quadratic differential, which in turn have a min-max description. This gives an additional evidence to our conjecture, that in fact says that this behavior is not exclusive of large n 's (a kind of “discrete version” of Stahl's theorem).

6 Further generalizations and applications of the electrostatic model

Many new interesting problems related to the electrostatic models have their origin either in some non-standard orthogonality conditions (such as non-hermitian or multiple orthogonality mentioned above), or in physical applications. In this Section we will briefly review some of them.

Exactly solvable or quasi-exactly solvable multi-particle quantum mechanical systems have many remarkable properties. Especially, those of the Calogero-Sutherland-Moser (CSM) systems [13,14,45,62] and their integrable deformation called the Ruijsenaars-Schneider-van Diejen (RSvD) systems [52,67] have been well studied. A classical result is that the equilibrium positions of the CSM systems are described by the zeros of the classical orthogonal polynomials; the Hermite, Laguerre, and Jacobi polynomials [15,16] (see also [48] for a comprehensive review and bibliography therein). Following this analogy the authors of [48] proved recently that the equilibrium positions of the RSvD systems with rational and trigonometric potentials coincide again with zeros of polynomials from the Askey tableau of hypergeometric orthogonal polynomials; namely, they found connections with the Meixner-Pollaczek, continuous Hahn, continuous dual Hahn, and the Askey-Wilson polynomials. Hence, an electrostatic model for the zeros of hypergeometric polynomials in all their generality could have several interesting applications for exactly solvable quantum-mechanical systems.

Another interesting model was considered by Loutsenko [35], who studied a system $(X, Y) \subset \mathbb{C}^{n+m}$ of n positive and m negative *moving* charges in \mathbb{C} of masses 1 and $-\Lambda$, respectively (with $\Lambda \in \{1/2, 1, 2\}$), interacting again according to the logarithmic law. He proved in particular that if (X^*, Y^*) is a critical configuration of the mutual energy $E_{\text{mutual}}(X, Y)$, p is the monic polynomial vanishing at the points of X^* , and q is the monic polynomial vanishing at those of Y^* , then p, q are solutions of a *bilinear* differential equation

$$p''q - 2\Lambda p'q' + \Lambda^2 p q'' = 0.$$

Regarding the Lamé equation, once we depart from the classical setting considered by Stieltjes, the description of the Heine-Stieltjes polynomials is not apparent. It is clear however that we can hardly expect that their zeros provide a stable equilibrium; instead, we must concentrate on the least restrictive condition of configuration providing critical values of the energy functional. This study has again some physical applications. The BC_n elliptic Inozemtsev model is a quantum integrable system with n -particles whose potential is given by elliptic functions. For the case $n = 1$, finding eigenstates of its Hamiltonian is equivalent to solving the Heun equation (see e.g. [64,65]). In this sense, the BC_n Inozemtsev model is a generalization of the Heun equation.

There are also connections of some problems in physics and representation theory with multiple orthogonal polynomials. It is known that zeros of the Jacobi polynomial $P_n^{(\alpha, \beta)}$ satisfy a system of algebraic equations, which is known as the *Bethe Ansatz equation* of

the Gaudin model associated to sl_2 and two irreducible modules with highest weights $-(\alpha+1), -(\beta+1)$. In [47] the authors generalized this connection, studying sequences of r polynomials whose zeros constitute the unique solution of the Bethe Ansatz equation associated with two highest weight sl_{r+1} irreducible modules, with the restriction that the highest weight of one of the modules is a multiple of the first fundamental weight. As a result, they show that the first polynomial in the sequence coincides with the well known Jacobi-Piñeiro multiple orthogonal polynomial, and others are given by Wronskian type determinants of Jacobi-Piñeiro polynomials. In [46] they derived a linear differential equation (alas, of order r) for these polynomials, paving a way to the electrostatic interpretation of their zeros.

Finally, we can mention the matrix orthogonal polynomials, whose non-trivial connections with (matrix) differential equations is being studied. Duran and Grünbaum gave in [19,20] some examples of matrix orthogonal polynomials $\{P_n\}$ satisfying the second order linear ODE of the form

$$P_n''(x)A_2(x) + P_n'(x)A_1(x) + P_n(x)A_0(x) = \Gamma_n P_n(x), \quad n \geq 0,$$

where (as in the scalar hypergeometric case) coefficients A_j are matrix polynomials that do not depend on n , of degrees $\leq 2, 1$, and 0 , respectively, and Γ_n are Hermitian matrices. These $\{P_n\}$ look like a natural generalization of the families of classical polynomials in the scalar case; likewise, they exhibit a rich variety of structural properties (see [21]). Furthermore, they might have some relevance in the analysis of the Dirac equation (relativistic analogue of the Schrodinger equation). However, a study of the electrostatic interpretation of the zeros of the orthogonal matrix polynomials remains completely open.

7 Asymptotic distribution of zeros

Assume we have an electrostatic model of n positive charges moving in an external field, and we are interested to analyze what happens when $n \rightarrow \infty$ (the so-called thermodynamic or semiclassical limit). In many situations this information can be extracted directly from the characterizing second order linear differential equation, using either the WKB approach (cf. [4]–[6], [42]–[44], [71]–[74]), or reducing the ODE to a Riccati form, a method that we outline below. But first we should introduce some notation and agree in the meaning of the “global” asymptotics of the zeros.

In the sequel, $\text{supp}(\mu)$ denotes the support of a measure μ , and

$$\hat{\mu}(z) = \int \frac{d\mu(t)}{z-t}, \quad z \in \mathbb{C} \setminus \text{supp}(\mu),$$

is its Stieltjes (or Cauchy) transform. With a function $y : \mathbb{C} \rightarrow \mathbb{C}$ we can associate its

zero counting measure ν_y , defined by

$$\nu_y = \sum_{y(x)=0} \delta_x,$$

where δ_x is the unit mass (Dirac delta) at x , and the sum goes along all the zeros of y taking into account their multiplicity. Equivalently, for each Borel set $\Delta \subseteq \mathbb{C}$ the number of zeros of y in Δ is

$$\nu_y(\Delta) = \int_{\Delta} d\nu_y(x).$$

An important observation is that if p is a polynomial and ν_p is the associated zero-counting measure, then the logarithmic derivative of p is the Stieltjes transform of ν :

$$\frac{p'(z)}{p(z)} = \sum_{k=1}^n \frac{1}{z - a_k} = \int \frac{d\nu(t)}{z - t} = \hat{\nu}(z), \quad (7.1)$$

which can be evaluated away from the zeros of p .

The “global” behavior of the zeros of polynomials $\{p_n\}$, $\deg(p_n) = n$, is described by the limit of the sequence of *normalized* measures $\mu_n = \nu_{p_n}/n$ as $n \rightarrow \infty$, in the sense of the weak-* topology. Recall (cf. [8, Section 2]) that a sequence of Borel measures $\{\mu_n\}$ converges to a measure μ in the weak-* topology (which we denote as $\mu_n \xrightarrow{*} \mu$) if

$$\lim_n \int f(x) d\mu_n(x) = \int f(x) d\mu(x), \quad \forall f \text{ continuous and compactly supported on } \mathbb{C}.$$

Another important fact is that the set of unit measures with uniformly bounded supports is compact in the weak-* topology. Hence, if we know for instance that all the zeros of the sequence of polynomials $\{p_n\}$, $\deg(p_n) = n$, belong to the same compact set $K \subset \mathbb{C}$, then there always exists a unit measure μ supported on K and a subsequence $\Lambda \subset \mathbb{N}$ such that $\mu_n \xrightarrow{*} \mu$ for $n \in \Lambda$ (where μ_n are the normalized zero counting measures defined above). In consequence, and taking into account (7.1), we have that

$$\lim_{n \in \Lambda} \frac{1}{n} \frac{p'_n(z)}{p_n(z)} = \hat{\mu}(z), \quad z \in \mathbb{C} \setminus \text{supp}(\mu). \quad (7.2)$$

The method of reduction of the original second order ODE to the Riccati form is based on the previous observation. This idea appears in a work of Saff, Ullman, and Varga [54], although its roots can be traced back to the famous Perron’s monograph [49], who in turn gives credit to some original works of Euler. Recently it has been successfully applied in a variety of problems (see, e. g. [22,38,39,44]).

Let us see how it works in the simplest case of Jacobi polynomials $p_n = P_n^{(\alpha,\beta)}$ with $\alpha, \beta > -1$. The first step is to rewrite equation (2.2) in terms of the normalized logarithmic derivative of the polynomial solution,

$$h_n(z) = \frac{1}{n} \frac{p'_n(z)}{p_n(z)},$$

which yields

$$\left(\frac{h'_n(z)}{n} + h_n^2(z)\right) + \left(\frac{\alpha+1}{z-1} + \frac{\beta+1}{z+1}\right) \frac{h_n(x)}{n} + \frac{n+\alpha+\beta+1}{n} = 0. \quad (7.3)$$

Since all the zeros of p_n are in $[-1, 1]$, by the argument explained above we can assure the existence of a unit measure μ supported on $[-1, 1]$ and a subsequence $\Lambda \subset \mathbb{N}$ such that (see (7.2)) $h_n(z) \rightarrow \hat{\mu}(z)$, $n \in \Lambda$, uniformly on compact subsets of $\mathbb{C} \setminus [-1, 1]$, where h'_n are also uniformly bounded. Taking limits in (7.3) when $n \in \Lambda$ we arrive at an algebraic equation for $h = \hat{\mu}$: $(1 - x^2)h^2(x) + 1 = 0$, so that

$$\hat{\mu}(z) = (z^2 - 1)^{-1/2},$$

and the appropriate branch of the square root in $\mathbb{C} \setminus [-1, 1]$ is fixed by the condition $\lim_{z \rightarrow \infty} z\hat{\mu}(z) = 1$. Furthermore, since $\hat{\mu}$ is independent on Λ , we can claim that μ is in fact the weak-* limit of $\{\mu_n\}$ along $n \rightarrow \infty$.

It remains only to recover μ from its Stieltjes transform. Since we know (important fact!) that $\text{supp}(\mu) \subset [-1, 1]$, this task is just a straightforward application of the Sokhotsky-Plemelj's formulas (that allow to find μ from the boundary values of $\hat{\mu}$ on $(-1, 1)$), that yield that the limit measure μ is absolutely continuous and its density is

$$\mu'(x) = \frac{1}{\pi} \frac{dx}{\sqrt{1-x^2}} > 0, \quad x \in (-1, 1),$$

(fact that we do not claim that was discovered by us; probably, it is at least 150 years old).

In some more difficult situations the described elementary method becomes technically involved, and we have to turn to other resources. One of the advantages of the electrostatic interpretation of the zeros of a sequence of polynomials is that it allows us to guess what happens in the thermodynamic limit. For instance, if for each n the configuration of zeros is a global minimizer of the total energy, then it is natural to expect (and can be proved) that the asymptotics in a “global sense” should be well described (at least, in the first approximation) by a continuous distribution which is solution of the corresponding extremal problem for the logarithmic potential energy in the class of all probability measures (including the absolutely continuous ones). This distribution is known as the equilibrium measure, probably, in an external field (see [53] for details and definitions).

Let us consider for example the generalized Lamé equation (3.1) under the assumptions (3.3)–(3.4). We are interested in the limit $n \rightarrow \infty$ with the extra assumptions

$$\lim_{n \rightarrow \infty} \frac{n_i}{n} = \theta_i, \quad i = 1, \dots, p$$

(see Section 3 for notation). Since for every \mathbf{n} the zeros minimize the total discrete energy of the system, we can expect that any weak-* limit μ of the zero-counting

unit measures of the Heine-Stieltjes polynomials E_n solves a similar, but continuous, extremal problem. Indeed, it was proved in [41] that if $\mathcal{M} = \mathcal{M}(\theta_1, \dots, \theta_p)$ denotes the class of all probability measures ν on $[-1, 1]$ such that

$$\int_{a_{i-1}}^{a_i} d\nu = \theta_i, \quad i = 1, \dots, p,$$

then μ is the minimizer of the logarithmic energy in the class \mathcal{M} . Existence and uniqueness of this minimizer is proved by standard methods of potential theory. At this stage we can apply the reduction to the Riccati form method described above to get the full description of μ . Let us summarize briefly some results from [41].

For any system of $p - 1$ points

$$-1 \leq \beta_1 \leq \dots \leq \beta_{p-1} \leq 1 \tag{7.4}$$

we define the functions

$$R(x) := \prod_{j=1}^{p-1} (x - \beta_j), \quad H(x) := \sqrt{\frac{R(x)}{A(x)}} \sim \frac{1}{x}, \quad x \rightarrow \infty.$$

Then there exist $p - 1$ points (7.4) uniquely determined by the following system of equations:

$$\operatorname{Im} \int_{a_{j-1}}^{a_j} H(x) dx = -\pi \theta_j, \quad j = 1, \dots, p - 1,$$

where we take the limit values of H from the upper half plane. If we introduce the counting function

$$Z(x) := [\nu_A - \nu_R] \left((-\infty, x] \right),$$

then $\operatorname{supp}(\mu) = \overline{\{x \in \mathbb{R} : Z(x) = 1\}}$ and it consists of at most $p - 1$ disjoint intervals in $[-1, 1]$.

Furthermore, μ is an absolutely continuous measure,

$$\mu'(x) = -\frac{1}{\pi i} H(x) = \frac{1}{\pi} |H(x)|, \quad x \in \operatorname{supp}(\mu),$$

and, for $z \notin \operatorname{supp}(\mu)$, $\hat{\mu}(z) = -H(z)$.

The methods described in this section yield many nice results, but can fail for two reasons:

- (a) If we have no a priori information on the location of the zeros, then the reduction of the ODE to the Riccati form will give us at most an expression of the Cauchy transform $\hat{\mu}$ of the limit distribution μ in the domains (unknown) disjoint with $\operatorname{supp}(\mu)$ (and $\operatorname{supp}(\mu)$ could be, eventually, a subset of \mathbb{C} of positive plane measure). Does this information determine μ ? It is not clear, although for some specific expressions of $\hat{\mu}$ this should be really the case.

- (b) If we are analyzing Heine-Stieltjes polynomials whose zeros provide a critical configuration for the total energy $E(X)$ (but not the global minimum!), then what measure μ should we expect in the semiclassical limit? Clearly, μ is not necessarily an equilibrium measure, but rather a *critical measure*, that on the real line can be characterized by the fact that its potential (plus the external field, if exists) is constant on each connected component of $\text{supp}(\mu)$, but unlike in the equilibrium case, these constants need not to be the same. However, in this description two open problems remain: (i) to prove that discrete critical measures converge in the thermodynamic limit to continuous critical measures, and that all continuous critical measures can be obtained this way; (ii) to find a feasible description of the multi-parametric family of continuous critical measures in the given class.

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