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Probabilistic Norms for Linear Operators

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1. INTRODUCTION

We recall the definition of a Probabilistic Normed Space, PN space briefly, as given in [1], together with the notation that will be needed (see [10]).

A *distribution function* (= d.f.) is a function $F:\overline{\mathbf{R}} \to [0,1]$ that is nondecreasing and left-continuous on $\mathbf{R}$; moreover, $F(-\infty) = 0$ and $F(+\infty) = 1$. Here $\overline{\mathbf{R}} := \mathbf{R} \cup \{-\infty, +\infty\}$. The set of all the d.f.'s will be denoted by Δ and the subset of those d.f.'s, called *distance d.f.'s*, such that $F(0) = 0$, by Δ^+. We shall also consider $\mathscr{D}$ and $\mathscr{D}^+$, the subsets of Δ and Δ^+, respectively, formed by the *proper* d.f.'s, i.e., by those d.f.'s $F \in \Delta$ that

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satisfy the conditions

$$\lim_{x \to -\infty} F(x) = 0 \qquad \text{and} \qquad \lim_{x \to +\infty} F(x) = 1.$$

The first of these is obviously satisfied in all of Δ^+ since, in it, $F(0) = 0$. By setting $F \leq G$ whenever $F(x) \leq G(x)$ for every $x \in \mathbf{R}$, one introduces a natural ordering in Δ and in Δ^+. The maximal element for Δ^+ in this order is the d.f. given by

$$\epsilon_0(x) := \begin{cases} 0, & \text{if } x \leq 0, \\ 1, & \text{if } x > 0. \end{cases}$$

The space Δ can be metrized in several ways [12, 9, 11, 14], but we shall here adopt the Sibley metric d_S. If F and G are d.f.'s and h is in $]0,1]$, let $(F,G;h)$ denote the condition

$$F(x-h) - h \leq G(x) \leq F(x+h) + h \qquad \text{for all } x \in \left] -\frac{1}{h}, \frac{1}{h} \right[.$$

Then the Sibley metric d_S is defined by

$$d_S(F,G) := \inf\{h \in]0,1]: \text{both } (F,G;h) \text{ and } (G,F;h) \text{ hold}\}.$$

A *triangle function* is a binary operation on Δ^+, namely a function $\tau : \Delta^+ \times \Delta^+ \to \Delta^+$ that is associative, commutative, nondecreasing in each place and which has ϵ_0 as unit, viz. for all $F, G, H \in \Delta^+$

$$\tau(\tau(F,G),H) = \tau(F,\tau(G,H))$$
$$\tau(F,G) = \tau(G,F)$$
$$\tau(F,H) \leq \tau(G,H) \qquad \text{if } F \leq G$$
$$\tau(F,\epsilon_0) = F.$$

DEFINITION 1.1. A *Probabilistic Normed Space*, briefly a PN space, is a quadruple (V, ν, τ, τ^*), in which V is a linear space, τ and τ^* are continuous triangle functions with $\tau \leq \tau^*$ and ν, and the probabilistic norm is a map $\nu : V \to \Delta^+$ such that

(N1) $\nu_p = \epsilon_0$ if, and only if, $p = \theta$, θ being the null vector in V;

(N2) $\nu_{-p} = \nu_p$ for every $p \in V$;

(N3) $\nu_{p+q} \geq \tau(\nu_p, \nu_q)$ for all $p, q \in V$;

(N4) $\nu_p \leq \tau^*(\nu_{\alpha p}, \nu_{(1-\alpha)p})$ for every $\alpha \in [0,1]$ and for every $p \in V$.

If, instead of (N1), we only have $\nu_\theta = \epsilon_0$, then we shall speak of a *Probabilistic Pseudo Normed Space*, briefly a PPN space. If the inequality (N4) is replaced by the equality $\nu_p = \tau_M(\nu_{\alpha p}, \nu_{(1-\alpha)p})$, then the PN space is called a *Šerstnev space*; in a Šerstnev space, a condition stronger than (N2) holds, namely

$$\forall \lambda \neq 0 \, \forall p \in V, \qquad \nu_{\lambda p} = \nu_p\left(\frac{j}{|\lambda|}\right). \tag{Š}$$

Here j is the identity map on $\mathbf{R}$, i.e., $j(x) := x \ (x \in \mathbf{R})$.

There is a natural topology in a PN space (V, ν, τ, τ^*), called the *strong topology*; it is defined, for $t > 0$, by the neighbourhoods

$$\mathscr{I}_p(t) := \{q \in V : \nu_{q-p}(t) > 1 - t\} = \{q \in V : d_s(\nu_{q-p}, \epsilon_0) < t\}.$$

In [5], the present authors have introduced different concepts of boundedness for linear operators between two PN spaces $(V_1, \nu, \tau_1, \tau_1^*)$ and $(V_2, \nu', \tau_2, \tau_2^*)$ and studied their relationship with the property of continuity. We recall that a set A in a PN space (V, ν, τ, τ^*) is said to be *bounded* if its probabilistic radius R_A belongs to $\mathscr{D}^+$, where

$$R_A(x) := \begin{cases} l^- \inf\{\nu_p(x) : p \in A\}, & x \in [0, +\infty[, \\ 1, & x = +\infty. \end{cases}$$

Here $l^- f(x)$ denotes the left limit of the function f at the point x, $l^- f(x) := \lim_{t \to x^-} f(t)$.

In the following we shall investigate the properties of different spaces of linear operators between PN spaces; in so doing we shall also extend and make precise the results by Bocşan and Radu [3, 7, 8] who worked only in the special Šerstnev spaces in which the triangle function τ is of the form $\tau = \tau_T$ where T is a continuous t-norm [10]. We shall also refer to our paper [4].

2. CLASSES OF LINEAR OPERATORS

Let $(V_1, \nu, \tau_1, \tau_1^*)$ and $(V_2, \nu', \tau_2, \tau_2^*)$ be two PN spaces and let $L = L(V_1, V_2)$ be the vector space of linear operators $T : V_1 \to V_2$. Also let us denote by

—$L_b = L_b(V_1, V_2)$ the subset of L formed by the linear bounded operators from V_1 to V_2,

—$L_c = L_c(V_1, V_2)$ the subset of L formed by the linear continuous operators from V_1 to V_2.

—$L_{bc} = L_{bc}(V_1, V_2)$ the subset of L formed by the linear continuous *and* bounded operators from V_1 to V_2.

Let $(V_1, \nu, \tau_1, \tau_1^*)$ and $(V_2, \nu', \tau_2, \tau_2^*)$ be two PN spaces. As was shown in [2], PN spaces are not necessarily topological linear spaces. Therefore, that the subsets L_b, L_c, and L_{bc} are linear subspaces of L has to be proved. This is quite easy in the case of L_c, where the usual proof supplemented by the results in [2] leads to the result that we state as a theorem.

THEOREM 2.1. *$L_c(V_1, V_2)$ is a vector subspace of L.*

However, the sets L_b and L_{bc} are not necessarily linear subspaces of L. A sufficient condition for this is given by the following theorem.

THEOREM 2.2. *If the triangle function τ_2 maps $\mathscr{D}^+ \times \mathscr{D}^+$ into $\mathscr{D}^+$, i.e., if $\tau_2(\mathscr{D}^+, \mathscr{D}^+) \subset \mathscr{D}^+$, then $L_b(V_1, V_2)$ and $L_{bc}(V_1, V_2)$ are vector subspaces of $L(V_1, V_2)$.*

Proof. It suffices to show that $L_b(V_1, V_2)$ is a vector space. In this proof, we shall always denote a bounded subset of V_1 by A.

Let T_1 and T_2 be two bounded linear maps from $(V_1, \nu, \tau_1, \tau_1^*)$ into $(V_2, \nu', \tau_2, \tau_2^*)$. Then, by definition of boundedness, both R'_{T_1A} and R'_{T_2A} are in $\mathscr{D}^+$. Since, for every $p \in A$, one has

$$\nu'_{T_1p + T_2p} \geq \tau_2(\nu'_{T_1p}, \nu'_{T_2p}) \geq \tau_2(R'_{T_1A}, R'_{T_2A})$$

which belongs to $\mathscr{D}^+$, also $R'_{(T_1+T_2)A}$ belongs to $\mathscr{D}^+$ and $T_1 + T_2$ is bounded.

Now let $\alpha \in \mathbf{R}$ and $T \in L_b(V_1, V_2)$. Because of (N2), it suffices to consider the case $\alpha \geq 0$. If either $\alpha = 0$ or $\alpha = 1$, then αT is bounded. Proceeding by induction, assume that αT is bounded, i.e., that $R'_{\alpha TA} \in \mathscr{D}^+$ for $\alpha = 0, 1, \ldots, n-1$ with $n \in \mathbf{N}$. Then, for every $p \in A$,

$$\nu'_{nTp} \geq \tau_2(\nu'_{(n-1)Tp}, \nu'_{Tp})$$

and hence

$$R'_{nTA} \geq \tau_2(R'_{(n-1)TA}, R'_{TA})$$

so that $R'_{nTA} \in \mathscr{D}^+$ and nT is bounded. Therefore nT is bounded for every positive integer n. If α is not a positive integer, there is $n \in \mathbf{Z}_+$ such that $n - 1 < \alpha < n$; therefore by Lemma 2 in [6], for every $p \in A$ one has

$$\nu'_{nTP} \leq \nu'_{\alpha Tp}$$

whence

$$R'_{nTA} \leq R'_{\alpha TA},$$

which means that αT is bounded. ∎

3. PROBABILISTIC NORMS FOR OPERATORS

The following result is crucial for our purposes.

THEOREM 3.1. *If A is a subset of V_1 and $\nu^A(T) := R'_{TA}$, then the quadruple $(L, \nu^A, \tau_2, \tau_2^*)$ is a* PPN *space. Convergence in the probabilistic pseudonorm ν^A is equivalent to uniform convergence of operators on A.*

Proof. For (N1), if Θ is the null operator (i.e., $\Theta p = \theta_2$ for every $p \in V_1$, θ_2 being the null vector of V_2) then $R'_{\Theta(A)} = \epsilon_0$.

Property (N2) is obvious. As for (N3), if S and T belong to L, then, by definition of ν^A,

$$\tau_2\big(\nu^A(S), \nu^A(T)\big) \leq \tau_2(\nu'_{Sp}, \nu'_{Tp}) \leq \nu'_{(S+T)p}$$

for every $p \in A$ so that

$$\tau_2\big(\nu^A(S), \nu^A(T)\big) \leq R'_{(S+T)A} = \nu^A(S+T).$$

For (N4), if $\alpha \in [0,1]$ and $T \in L$, then, for every $p \in A$,

$$\nu^A(T) = R'_{TA} \leq \nu'_{Tp} \leq \tau_2^*(\nu'_{\alpha Tp}, \nu'_{(1-\alpha)Tp}).$$

Therefore, since τ_2^* is nondecreasing in each variable,

$$\nu^A(T) \leq \tau_2^*\Big(l^- \inf_{p\in A} \nu'_{\alpha Tp}, l^- \inf_{p\in A} \nu'_{(1-\alpha)Tp}\Big)$$
$$= \tau_2^*(\nu^A(\alpha T), \nu^A((1-\alpha)T)).$$

This proves that $(L, \nu^A, \tau_2, \tau_2^*)$ is a PN space.

Assume $T_n \to T$ in the topology of $(L, \nu^A, \tau_2, \tau_2^*)$; since $\nu^A(T_n - T) \leq \nu'_{T_n p - Tp}$ for every $p \in A$, then, for every $p \in A$,

$$d_S(\nu'_{T_n p - Tp}, \epsilon_0) \leq d_S\big(\nu^A(T_n - T), \epsilon_0\big)$$

which implies $T_n p \to Tp$ uniformly in $p \in A$.

Conversely, assume $T_n \to T$ uniformly on A, namely for every $\eta > 0$, there exists $n_0 = n_0(\eta) \in \mathbf{N}$ such that, for every $n \geq n_0$ and for all $p \in A$

$$d_S(\nu'_{T_n p - Tp}, \epsilon_0) < \frac{\eta}{2}$$

or, equivalently,

$$\nu'_{T_n p - Tp}\left(\frac{\eta}{2}\right) > 1 - \frac{\eta}{2}.$$

Therefore, for every $n \geq n_0$,

$$\nu^A(T_n - T)(\eta) \geq \nu^A(T_n - T)\left(\frac{\eta}{2}\right) \geq 1 - \frac{\eta}{2} > 1 - \eta,$$

i.e.,

$$d_S\left(\nu^A(T_n - T), \epsilon_0\right) < \eta. \quad \blacksquare$$

We give a condition that ensures that $(L, \nu^A, \tau_2, \tau_2^*)$ is a PN space. We shall assume that A contains a *Hamel* (or algebraic) basis for V_1 (see, e.g., [13]).

THEOREM 3.2. *If $A \subset V_1$ contains a Hamel basis for V_1, then the quadruple $(L, \nu^A, \tau_2, \tau_2^*)$ is a* PN *space whose topology is stronger than that of simple convergence for operators, i.e.,*

$$\nu^A(T_n - T) \to \epsilon_0 \Rightarrow \forall p \in V_1 \ \nu'_{T_n p - Tp} \to \epsilon_0.$$

Proof. One knows from Theorem 3.1 that $(L, \nu^A, \tau_2, \tau_2^*)$ is a PPN space and that $\nu^A(T) = \epsilon_0$ implies $Tp = \theta_2$ for every $p \in A$. If p does not belong to A, then there exist $n(p) \in \mathbf{N}$, $\alpha_j \in \mathbf{R}$, $p_j \in A$ $(j = 1, 2, \ldots, n(p))$ such that $p = \sum_{j=1}^{n(p)} \alpha_j p_j$. Therefore

$$Tp = T\left(\sum_{j=1}^{n(p)} \alpha_j p_j\right) = \sum_{j=1}^{n(p)} \alpha_j T p_j = \sum_{j=1}^{n(p)} \alpha_j \theta_2 = \theta_2.$$

Thus $Tp = \theta_2$ for every $p \in V_1$, i.e., $T = \Theta$.

If $T_n \to T$ in the topology of $(L, \nu^A, \tau_2, \tau_2^*)$ then, as in the proof of Theorem 3.1, $T_n p \to Tp$ for every $p \in A$. If p does not belong to A, write $p = \sum_{j=1}^{n(p)} \alpha_j p_j$. Since the operations of vector addition and multiplication by a fixed scalar are continuous in a PN space [2], then we obtain

$$\begin{aligned} T_n p &= T_n\left(\sum_{j=1}^{n(p)} \alpha_j p_j\right) \\ &= \sum_{j=1}^{n(p)} \alpha_j T_n p_j \xrightarrow[n \to +\infty]{} \sum_{j=1}^{n(p)} \alpha_j T p_j = T\left(\sum_{j=1}^{n(p)} \alpha_j p_j\right) = Tp. \quad \blacksquare \end{aligned}$$

Theorems 3.1 and 3.2 still hold when the first space is any space endowed with a topology.

COROLLARY 3.1. *If A is an absorbing subset of V_1, then $(L, \nu^A, \tau_2, \tau_2^*)$ is a* PN *space; convergence in the probabilistic norm ν^A is equivalent to uniform convergence of operators on A.*

Proof. As the second statement has the same proof as in Theorem 3.1, we shall only prove the first one. To this end we shall show that an absorbing set A contains a Hamel basis for V_1.

Let B be a Hamel basis for V_1 and let p belong to B; since A is absorbing, there exists a scalar α_p such that $\alpha_p p$ belongs to A. Then $B' := \{\alpha_p p : p \in B\}$ is a Hamel basis for V_1. ∎

The probabilistic norm ν^{V_1} is the analogue of the usual operator norm.

COROLLARY 3.2. *The topology of the* PN *space* $(L, \nu^{V_1}, \tau_2, \tau_2^*)$ *is equivalent to that of uniform convergence of operators.*

It ought to be noticed that the results we have just presented are stronger than the analogous ones given by Radu [7] in the special case of those Šerstnev spaces in which $\tau = \tau_T$, in that in the present note the operators of L are only assumed to be linear and not also continuous.

In general, $(L_{bc}(V_1, V_2), \nu_F, \tau_2, \tau_2^*)$ need not be a PN space since the condition $\nu_F(T) = \epsilon_0$ is equivalent to $\nu'_{Tp} = \epsilon_0$ for every $p \in \sigma(F)$, i.e., $Tp = \theta_2$ for every $p \in \sigma(F)$. This latter condition is satisfied by every $T \in L_{bc}(V_1, V_2)$ different from the null element Θ and whose kernel contains $\sigma(F)$. In this direction an extreme example is provided below.

EXAMPLE. Let F and G be two d.f.'s belonging to Δ^+ both different from ϵ_0 and ϵ_∞ and such that the relationship $F \leq G$ does not hold. Consider (as in [4]) the PN spaces (V_1, G, M) and $(V_2, \nu', \tau_2, \tau_2^*)$, the first of which is equilateral; then consider the equilateral space $(L_{bc}(V_1, V_2), \nu_F, \tau_2, \tau_2^*)$, where, for every $T \in L_{bc}(V_1, V_2)$,

$$\nu_F(T) = l^- \inf\{\nu'_{Tp} : \nu_p \geq F\} = \epsilon_0.$$

Since $\nu_p = G$ for every $p \neq \theta$, $(L_{bc}(V_1, V_2), \nu_F, \tau_2, \tau_2^*)$ is a PN space if, and only if, $L_{bc}(V_1, V_2)$ consists only of the null operator Θ.

In the following we shall consider maps $\psi : \Delta^+ \to \Delta^+$ that satisfy some of the properties:

$$\psi(\epsilon_0) = \epsilon_0; \tag{1}$$

$$\psi(F_1) \leq \psi(F_2) \qquad \text{if } F_1 \leq F_2 \; (F_1, F_2 \in \Delta^+); \tag{2}$$

if $(V_1, \nu, \tau_1, \tau_1^*)$ and $(V_2, \nu', \tau_2, \tau_2^*)$ are two PN spaces and if T belongs to $L(V_1, V_1)$, then

$$\psi(\nu_p) \leq \nu_{Tp} \qquad \text{for all } p \in V_1; \tag{3}$$

ψ is continuous in ϵ_0 with respect to the weak topology, i.e., (4)

$$d_S(F_n, \epsilon_0) \to 0 \Rightarrow d_S(\psi(F_n), \psi(\epsilon_0)) \to 0;$$

$$\psi(\mathscr{D}^+) \subset \mathscr{D}^+. \tag{5}$$

Also we shall need the following classes of mappings $\psi : \Delta^+ \to \Delta^+$

$$\Omega_T := \{\psi : \Delta^+ \to \Delta^+ \text{ satisfies properties (1), (2), and (3)}\};$$
$$\Omega_T^c := \{\psi : \Delta^+ \to \Delta^+ \text{ satisfies properties (1), (2), (3), and (4)}\};$$
$$\Omega_T^b := \{\psi : \Delta^+ \to \Delta^+ \text{ satisfies properties (1), (2), (3), and (5)}\};$$
$$\Omega_T^{bc} := \{\psi : \Delta^+ \to \Delta^+ \text{ satisfies properties (1) through (5)}\}.$$

Clearly $\Omega_T^{bc} = \Omega_T^c \cap \Omega_T^b \subset \Omega_T^c \cup \Omega_T^b \subset \Omega_T$.

For $F \in \Delta^+$, let $\sigma(F)$ denote the subset of the PN space $(V_1, \nu, \tau_1, \tau_1^*)$ bounded by F, viz.

$$\sigma(F) := \{p \in V_1 : \nu_p \geq F\}.$$

If T is in $L(V_1, V_2)$ define $\phi_T : \Delta^+ \to \Delta^+$ via

$$\phi_T(F) := \nu^{\sigma(F)}(T) = R'_{T\sigma(F)}.$$

Starting from the probabilistic pseudonorm introduced in Theorem 3.1, in the next two theorems we provide characterizations of the classes of linear operators studied in the previous section.

THEOREM 3.3. *Let $(V_1, \nu, \tau_1, \tau_1^*)$ and $(V_2, \nu', \tau_2, \tau_2^*)$ be two PN spaces and let T be in $L(V_1, V_2)$. Then*

(a) *ϕ_T belongs to Ω_T;*

(b) *T is in $L_c(V_1, V_2)$ if, and only if, ϕ_T belongs to Ω_T^c;*

(c) *T is in $L_b(V_1, V_2)$ if, and only if, ϕ_T belongs to Ω_T^b;*

(d) *T is in $L_{bc}(V_1, V_2)$ if, and only if, ϕ_T belongs to Ω_T^{bc}.*

Proof. (a) (1) $\phi_T(\epsilon_0) = \nu^{\sigma(\epsilon_0)}(T) = \nu^{\{\theta_1\}}(T) = \epsilon_0$.

(2) Let $F_1 \leq F_2$. Then $p \in \sigma(F_2)$ implies $\nu_p \geq F_2 \geq F_1$ and hence $p \in \sigma(F_1)$, so that $\sigma(F_2) \subset \sigma(F_1)$. Thus

$$\phi_T(F_2) = \nu^{\sigma(F_2)}(T) = R'_{T\sigma(F_2)} \geq R'_{T\sigma(F_1)} = \nu^{\sigma(F_1)}(T) = \phi_T(F_1).$$

(3) For every $p \in V_1$ one has $p \in \sigma(\nu_p)$, whence, by definition,

$$\phi_T(\nu_p) = l^- \inf_{q \in \sigma(\nu_p)} \nu'_{Tq} \leq \nu'_{Tp}.$$

(b) Assume that ϕ_T satisfies (4) and let $\eta > 0$; then there exists $\delta = \delta(\eta) > 0$ such that $d_S(\phi_T(F), \epsilon_0) < \eta$ whenever $d_S(F, \epsilon_0) < \delta$. On the other hand, it follows from (a) that ϕ_T satisfies (3) so that one has, for every $p \in V_1$,

$$d_S(\nu'_{Tp}, \epsilon_0) \leq d_S(\phi_T(\nu_p), \epsilon_0).$$

Therefore, if $d_S(\nu_p, \epsilon_0) < \delta$ then $d_S(\nu'_{Tp}, \epsilon_0) < \eta$, in other words, T is continuous.

Conversely, let T be continuous; then, for every $\eta > 0$, there exists $\delta = \delta(\eta) > 0$ such that $d_S(\nu'_{Tp}, \epsilon_0) < \eta/2$ whenever $d_S(\nu_p, \epsilon_0) < \delta$. Assume now $F_n \to \epsilon_0$ in the weak topology, i.e., $d_S(F_n, \epsilon_0) \to 0$. Because of the definition of $\phi_T(F_n)$, for all $x > 0$ there exists $p_{\eta/2} \in \sigma(F_n)$ such that

$$\phi_T(F_n)(x) \geq \nu'_{Tp_{\eta/2}}(x) - \frac{\eta}{2}. \tag{6}$$

Since $F_n \to \epsilon_0$, one has $d_S(F_n, \epsilon_0) < \delta$ provided n is large enough, say $n \geq n_0$ for a suitable $n_0 = n_0(\delta) \in \mathbf{N}$. Therefore, for every $n \geq n_0$ and for every $p \in \sigma(F_n)$,

$$d_S(\nu_p, \epsilon_0) \leq d_S(F_n, \epsilon_0) < \delta,$$

and hence $d_S(\nu'_{Tp}, \epsilon_0) < \eta/2$. As a consequence (see [10, (4.3.4)]), for $n \geq n_0$,

$$\nu'_{Tp}\left(\frac{\eta}{2}\right) > 1 - \frac{\eta}{2}$$

for every $p \in \sigma(F_N)$; in particular, from (6), one has

$$\phi_T(F_n)(\eta) \geq \phi_T(F_n)\left(\frac{\eta}{2}\right) \geq \nu'_{Tp_{\eta/2}}\left(\frac{\eta}{2}\right) - \frac{\eta}{2} > 1 - \eta,$$

viz, $d_S(\phi_T(F_n), \epsilon_0) < \eta$ for every $n \geq n_0$.

(c) Let T be bounded and let F be in $\mathscr{D}^+$. Then $\sigma(F)$ is bounded and so is $T\sigma(F)$; therefore $\phi_T(F) = R'_{T\sigma(F)}$ is in $\mathscr{D}^+$. Conversely, if A is a nonempty bounded set of V_1, then R_A belongs to $\mathscr{D}^+$ and $\nu_p \geq R_A$ for every $p \in A$, so that $A \subset \sigma(R_A)$. Therefore $R'_{TA} \geq R'_{T\sigma(R_A)} = \phi_T(R_A) \in \mathscr{D}^+$, whence T is bounded.

(d) This now follows from (b) and (c). ∎

The following result can be proved in a similar manner; therefore its proof will not be given.

THEOREM 3.4. *Let* $(V_1, \nu, \tau_1, \tau_1^*)$ *and* $(V_2, \nu', \tau_2, \tau_2^*)$ *be two* PN *spaces and let* T *be in* $L(V_1, V_2)$. *Then*

(a) *T is in $L_c(V_1, V_2)$ if, and only if,* $\Omega_T^c \neq \emptyset$;

(b) *T is in $L_b(V_1, V_2)$ if, and only if,* $\Omega_T^b \neq \emptyset$;

(c) *T is in $L_{bc}(V_1, V_2)$ if, and only if,* $\Omega_T^{bc} \neq \emptyset$.

THEOREM 3.5. *If F is in Δ^+ and T is in $L(V_1, V_2)$, then*

(a) $\phi_T(F) = \max\{\psi(F) : \psi \in \Omega_T\}$;
(b) *if T is in $L_c(V_1, V_2)$, then* $\phi_T(F) = \max\{\psi(F) : \psi \in \Omega_T^c\}$;
(c) *if T is in $L_b(V_1, V_2)$, then* $\phi_T(F) = \max\{\psi(F) : \psi \in \Omega_T^b\}$;
(d) *if T is in $L_{bc}(V_1, V_2)$, then* $\phi_T(F) = \max\{\psi(F) : \psi \in \Omega_T^{bc}\}$.

Proof. Let T be in $L(V_1, V_2)$ and set $\nu_F(T) := \sup\{\psi(F) : \psi \in \Omega_T\}$. By definition, $\nu_F(T) \geq \psi(F)$ for every $\psi \in \Omega_T$, so that, by Theorem 3.3. $\nu_F(T) \geq \phi_T(F)$.

On the other hand one has $\nu'_{Tp} \geq \psi(\nu_p)$ for every $p \in V_1$ and for every $\psi \in \Omega_T$, so that

$$\nu'_{Tp} \geq \psi(\nu_p) \geq \psi(F)$$

for every $p \in \sigma(F)$. Thus one has, for every $p \in \sigma(F)$,

$$\nu'_{Tp} \geq \sup\{\psi(F) : \psi \in \Omega_T\} = \nu_F(T)$$

and hence

$$\phi_T(F) = l^-\inf\{\nu'_{Tp} : p \in \sigma(F)\} \geq \nu_F(T).$$

The proof of the remaining assertion is similar. ∎

THEOREM 3.6. *Let $(V_1, \nu, \tau_1, \tau_1^*)$, $(V_2, \nu', \tau_2, \tau_2^*)$, and $(V_3, \nu'', \tau_3, \tau_3^*)$ be three* PN *spaces and let T_1 and T_2 be linear operators in $L(V_1, V_2)$ and $L(V_2, V_3)$, respectively. Then $T_2 \circ T_1$ belongs to $L(V_1, V_3)$ and*

$$\phi_{T_2 \circ T_1} \geq \phi_{T_1} \circ \phi_{T_2}. \tag{7}$$

Proof. We need only prove inequality (7), or, equivalently,

$$R''_{(T_2 \circ T_1)\sigma(F)} \geq R''_{T_2\sigma(R'_{T_1(\sigma(F))})} \tag{8}$$

for every $F \in \Delta^+$. Since $A \subset \sigma(R_A)$ for every set A, we have, in particular, $T_1(\sigma(F)) \subset \sigma(R'_{T_1(\sigma(F))})$, which implies

$$(T_2 \circ T_1)\sigma(F) = T_2[T_1(\sigma(F))] \subset T_2\sigma(R'_{T_1(\sigma(F))}),$$

an inclusion that immediately yields inequality (8). ∎

4. COMPLETENESS RESULTS

It is interesting to study when some of the PN spaces that we have introduced above are complete.

THEOREM 4.1. *Let A be a closed subset of the* PN *space* $(V_1, \nu, \tau_1, \tau_1^*)$ *that contains a Hamel basis for* V_1. *If the* PN *space* $(V_2, \nu', \tau_2, \tau_2^*)$ *is complete, then both* $(L(V_1, V_2), \nu^A, \tau_2, \tau_2^*)$ *and* $(L_c(V_1, V_2), \nu^A, \tau_2, \tau_2^*)$ *are complete.*

Proof. Let $\{T_n\}$ be a Cauchy sequence in $(L(V_1, V_2), \nu^A, \tau_2, \tau_2^*)$; in other words, for every $\delta > 0$ there exists $n_1 = n_1(\delta) \in \mathbf{N}$ such that for all $n, m \geq n_1$

$$d_S\big(\nu^A(T_n - T_m), \epsilon_0\big) < \delta.$$

Because of the definition of ν^A, one has, for every $p \in A$,

$$d_S(\nu'_{T_n p - T_m p}, \epsilon_0) \leq d_S\big(\nu^A(T_n - T_m), \epsilon_0\big) < \delta, \tag{9}$$

so that for every $p \in A$, $\{T_n p\}$ is a Cauchy sequence in $(V_2, \nu', \tau_2, \tau_2^*)$, which is complete. Therefore there exists $y_p \in V_2$ such that $T_n p \to y_p$ for every $p \in A$. Since A contains a Hamel basis for V_1, every $p \notin A$ can be represented in the form

$$p = \sum_{i=1}^{n(p)} \alpha_i p_i,$$

where the p_i's are in A and belong to a Hamel basis for V_1.

Since both addition and product by a fixed scalar are continuous [2], we can define a linear operator $T : V_1 \to V_2$ through

$$T_p := \begin{cases} y_p, & \text{if } p \in A, \\ \displaystyle\sum_{i=1}^{n(p)} \alpha_i y_{p_i}, & \text{if } p \notin A \text{ and } p = \displaystyle\sum_{i=1}^{n(p)} \alpha_i p_i. \end{cases}$$

Then $T_n p \to Tp$ uniformly on A, i.e., $T_n \to T$ in the strong topology of the PN space $(L(V_1, V_2), \nu^A, \tau_2, \tau_2^*)$.

In order to show that the PN space $(L_c(V_1, V_2), \nu^A, \tau_2, \tau_2^*)$ is complete it suffices to prove that the limit operator T just obtained is continuous if $\{T_n\}$ was a Cauchy sequence in $(L_c(V_1, V_2), \nu^A, \tau_2, \tau_2^*)$.

It follows from the uniform continuity of the probabilistic norm [2, Theorem 1] that, for every $\eta > 0$ there exists $\delta = \delta(\eta) > 0$ such that if p, q belong to V_2 and $d_S(\nu'_{p-q}, \epsilon_0) < \delta$, then $d_S(\nu'_p, \nu'_q) < \eta/2$. Now, since $T_n p$ converges uniformly to Tp, there is $n_0 = n_0(\eta) \in \mathbf{N}$ such that $d_S(\nu'_{T_n p - Tp}, \epsilon_0) < \delta$ for every $p \in V_1$ whenever $n \geq n_o$. Therefore $d_S(\nu'_{T_n p}, \nu'_{Tp}) < \eta/2$ for every $p \in V_1$ when $n \geq n_0$. Since T_{n_0} is continuous, there is $\rho = \rho(\eta) > 0$ such that $d_S(\nu'_{T_{n_0}p}, \epsilon_0) < \eta/2$ whenever $d_S(\nu_p, \epsilon_0) < \rho$. Thus

$$d_S(\nu'_{Tp}, \epsilon_0) \le d_S(\nu'_{Tp}, \nu'_{T_{n_0}p}) + d_S(\nu'_{T_{n_0}p}, \epsilon_0) < \eta$$

whenever $d_S(\nu_p, \epsilon_0) < \rho$, i.e., T is continuous. ∎

THEOREM 4.2. *If the* PN *space* $(V_2, \nu', \tau_2, \tau_2^*)$ *is complete and if the triangle function* τ_2 *maps* $\mathscr{D}^+ \times \mathscr{D}^+$ *into* $\mathscr{D}^+$, *then also the* PN *spaces* $(L_b(V_1, V_2), \nu^{V_1}, \tau_2, \tau_2^*)$ *and* $L_{bc}(V_1, V_2), \nu^{V_1}, \tau_2, \tau_2^*)$ *are complete.*

Proof. Let $\{T_n\}$ be a Cauchy sequence in $(L_b(V_1, V_2), \nu^{V_1}, \tau_2, \tau_2^*)$; since it is also a Cauchy sequence in $(L(V_1, V_2), \nu^{V_1}, \tau_2, \tau_2^*)$, it converges, by Theorem 4.1, to a linear operator T in this latter space. In order to show that T is bounded, let D be a bounded set of V_1, i.e., $R_D \in \mathscr{D}^+$; then one has to prove that there exists a d.f. G_D in $\mathscr{D}^+$, such that, for every $p \in D$, $\nu'_{Tp} \ge G_D$. Assume, if possible, that this is not so, namely that there exist $p_0 \in D$ and $\beta < 1$ such that $\nu'_{Tp_0}(x) \le \beta < 1$ for every $x > 0$. By the same argument as in the previous proof, for every $\eta < (1 - \beta)/2$, one has $d_S(\nu'_{T_np}, \nu'_{Tp}) < \eta$ for every $p \in V_1$ whenever $n \ge n_0(\eta)$. For every $x > 0$ there is η small enough to have $x < 1/\eta$; for every such value of η one has, in particular, for every $n \ge n_0$,

$$\nu'_{T_np_0}(x) < \nu'_{Tp_0}(x + \eta) + \eta < \beta + \eta < \frac{1 - \beta}{2} < 1$$

so that T_nD could not be bounded, a contradiction. As a consequence, TD is bounded. ∎

5. FAMILIES OF LINEAR OPERATORS

DEFINITION 5.1. A set of B linear operators, $B \subset L(V_1, V_2)$, is said to be *equicontinuous* if, for every $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$ such that, for every $T \in B$ and for every $p \in V_1$, one has

$$d_S(\nu'_{Tp}, \epsilon_0) < \epsilon \qquad \text{whenever } d_S(\nu_p, \epsilon_0) < \delta.$$

A set B of linear operators, $B \subset L(V_1, V_2)$, is said to be *uniformly bounded* if for every bounded subset A of V_1 there exists a d.f. G_A in $\mathscr{D}^+$ such that $R'_{TA} \ge G_A$ for every $T \in B$.

In particular, every operator in an equicontinuous family is continuous and every operator in a uniformly bounded family is bounded.

In the following we shall need mappings $\phi : \Delta^+ \to \Delta^+$ that satisfy some of the properties (1)–(5) of Section 3 and the other one:

$$\text{if } (V_1, \nu, \tau_1, \tau_1^*) \text{ and } (V_2, \nu', \tau_2, \tau_2^*) \text{ are PN spaces and } B \text{ is a set of linear operators from } V_1 \text{ into } V_2,\ B \subset L(V_1, V_2), \text{ then } \phi(\nu_p) \le \nu'_{Tp} \text{ for all } T \in B \text{ and for all } p \in V_1. \tag{10}$$

It is convenient to introduce the families

$$\Omega_B := \{\psi : \Delta^+ \to \Delta^+ \text{ satisfies properties (1), (2), and (10)}\};$$
$$\Omega_B^c := \{\psi \in \Omega_B : \text{satisfies property (4)}\};$$
$$\Omega_B^b := \{\psi \in \Omega_B : \text{satisfies property (5)}\};$$
$$\Omega_B^{bc} := \{\psi \in \Omega_B : \text{satisfies properties (4) and (5)}\}.$$

As above it is obvious that

$$\Omega_B^{bc} = \Omega_B^c \cap \Omega_B^b \subset \Omega_B^c \cup \Omega_B^b \subset \Omega_B.$$

We can now characterize equicontinuous families and uniformly bounded families of linear operators.

THEOREM 5.1. *Let $(V_1, \nu, \tau_1, \tau_1^*)$ and $(V_2, \nu', \tau_2, \tau_2^*)$ be two* PN *spaces, let B be a family of linear operators from V_1 into V_2, $B \subset L(V_1, V_2)$, and define a mapping $\phi_B : \Delta^+ \to \Delta^+$ through*

$$\phi_B(F) := l^- \inf\{\nu'_{Tp} : T \in B, p \in \sigma(F)\}.$$

Then

(a) $\phi_B \in \Omega_B$;

(b) *B is equicontinuous if, and only if, ϕ_B belongs to Ω_B^c;*

(c) *B is uniformly bounded if, and only if, ϕ_B belongs to Ω_B^b;*

(d) *B is both equicontinuous and uniformly bounded if, and only if, ϕ_B belongs to Ω_B^{bc}.*

Proof. (a) This is immediate, while the proof of (b) is a simple adaptation of that part (b) of Theorem 3.3.

(c) Let $B \subset L(V_1, V_2)$ be uniformly bounded and let F be any d.f. in $\mathscr{D}^+$. Since $\sigma(F)$ is bounded and hence $R'_{T\sigma(F)} \ge G_{T\sigma(F)}$, this latter being the d.f. of Definition 5.1, one has $\nu'_{Tp} \ge G_{\sigma(F)}$ which belongs to $\mathscr{D}^+$. Therefore $\phi_B(\Phi^+) \subset \mathscr{D}^+$.

Conversely, let A be a bounded subset of V_1 so that R_A is in $\mathscr{D}^+$; since $\nu_p \ge R_A$ for every $p \in A$, one has $A \subset \sigma(R_A)$ so that $R'_{TA} \ge \phi_B(R_A) \in \mathscr{D}^+$ for every $T \in B$, whence B is a uniformly bounded subset of $L(V_1, V_2)$. ∎

Now one can easily prove the analogues of Theorems 3.4 and 3.5.

THEOREM 5.2. *If* $(V_1, \nu, \tau_1, \tau_1^*)$ *and* $(V_2, \nu', \tau_2, \tau_2^*)$ *are two* PN *spaces, and if B is a family of linear operators from* V_1 *into* V_2, $B \subset L(V_1, V_2)$. *Then*

(a) *B is equicontinuous if, and only if,* $\Omega_B^c \neq \varnothing$;

(b) *B is uniformly bounded if, and only if,* $\Omega_B^b \neq \varnothing$;

(c) *B is both equicontinuous and uniformly bounded if, and only if,* $\Omega_B^{bc} \neq \varnothing$.

THEOREM 5.3. *Let* $(V_1, \nu, \tau_1, \tau_1^*)$ *and* $(V_2, \nu', \tau_2, \tau_2^*)$ *be two* PN *spaces and let B be a family of linear operators from* V_1 *into* V_2, $B \subset L(V_1, V_2)$; *then*

(a) $\phi_B = \max\{\phi \in \Omega_B\}$;

(b) $\phi_B = \max\{\phi \in \Omega_B^c\}$, *if B is equicontinuous;*

(c) $\phi_B = \max\{\phi \in \Omega_B^b\}$, *if B is uniformly bounded;*

(d) $\phi_B = \max\{\phi \in \Omega_B^{bc}\}$, *if B is both equicontinuous and uniformly bounded.*

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