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Some classes of probabilistic normed spaces

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RIASSUNTO: *Recentemente Alsina, Schweizer e Sklar hanno introdotto una nuova definizione di spazio normato probabilistico (in breve spazio NP). Iniziamo qui lo studio di questi spazi dando diversi esempi; in particolare (a) presentiamo uno studio particolareggiato degli spazi α-semplici, (b) costruiamo uno spazio NP sullo spazio vettoriale delle classi di equivalenza delle variabili aleatorie definite sopra uno spazio di probabilità e (c) mostriamo che la norma probabilistica di quest'ultimo spazio genera da sola le norme di tutti gli spazi L^p e di Orlicz.*

ABSTRACT: *Probabilistic Normed Spaces (PN spaces) have recently been redefined by Alsina, Schweizer and Sklar. We begin the study of these spaces by giving several examples; in particular, we (a) present a detailed study of α-simple spaces, (b) construct a PN space on the vector space of (equivalence classes) of random variables and (c) show that its probabilistic norm alone generates the norms of all L^p- and Orlicz spaces.*

1 – Introduction

Probabilistic Normed (= PN) Spaces were introduced by ŠERSTNEV in [10] by means of a definition that was closely modeled on the theory of ordinary normed spaces. PN spaces are real linear spaces in which to each vector p there is assigned not a positive number $\|p\|$, its norm, but rather a probability distribution function ν_p. The value $\nu_p(x)$ of the

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probabilistic norm ν_p of p at $x \geq 0$ can be interpreted as the probability that the "classical" norm $\|p\|$ of p is smaller than x. The theory of PN spaces has been dormant after its initial applications until it was recently put on a new basis by ALSINA, SCHWEIZER and SKLAR [1]. Here we shall consistently adopt the new, and in our opinion convincing, definition of a PN space given by these authors. We believe that this new definition has great potential for future applications to various fields of mathematics.

In this paper we study special classes of PN spaces (simple, α-simple, EN spaces) that provide useful examples for the general theory and which are interesting in their own right. Their definitions are extensions of those known for Probabilistic Metric Spaces; although many of the PM space results can be extended to the new setting, some proofs need to be modified particularly in order to take into account axiom (N4) that is alien to the setting of PM spaces. In addition, we construct a PN space on the vector space of equivalence classes of random variables and show that its probabilistic norm *alone* generates the norms of all L^p- and Orlicz spaces.

We need some preliminaries.

A *distribution function*, briefly a *d.f.*, is a function F defined on the extended reals $\overline{\mathbb{R}} := [-\infty, +\infty]$ that is nondecreasing, left-continuous on $\mathbb{R}$ and such that $F(-\infty) = 0$ and $F(+\infty) = 1$. The set of all d.f.'s will be denoted by Δ; the subset of those d.f.'s such that $F(0) = 0$ will be denoted by Δ^+ and by $\mathcal{D}^+$ the subset of the d.f.'s in Δ^+ such that $\lim_{x\to+\infty} F(x) = 1$. For every $a \in \mathbb{R}$, ϵ_a is the d.f. defined by

$$\epsilon_a(x) := \begin{cases} 0, & x \leq a, \\ 1, & x > a. \end{cases}$$

The set Δ, as well as its subsets, can be partially ordered by the usual pointwise order; in this order, ϵ_0 is the maximal element in Δ^+. The topology of weak convergence in Δ, or in Δ^+ can be metrized by the *Sibley metric* [13] (among other ones).

A *triangle function* is a mapping $\tau : \Delta^+ \times \Delta^+ \to \Delta^+$ that is commutative, associative, nondecreasing in each variable, and which has ϵ_0 as identity. Typical continuous triangle functions are

$$\tau_T(F,G)(x) := \sup_{s+t=x} T\left(F(s), G(t)\right),$$

and

$$\tau_{T^*}(F,G)(x) := \inf_{s+t=x} T^*\left(F(s), G(t)\right).$$

Here T is a continuous *t-norm*, i.e. a continuous binary operation on $[0,1]$ that is commutative, associative, nondecreasing in each variable and has 1 as identity; T^* is a continuous *t-conorm*, namely a continuous binary operation on $[0,1]$ that is related to a continuous t-norm through

$$T^*(x,y) := 1 - T(1-x, 1-y).$$

A t-norm T is strict if and only if it can be represented in the form

$$\forall x, y \in [0,1] \quad T(x,y) = f^{-1}(f(x) + f(y))$$

where $f : [0,1] \to \overline{\mathbb{R}}_+$, the additive generator, is continuous, strictly decreasing and satisfies $f(0) = +\infty$ and $f(1) = 0$.

A *Probabilistic Metric Space* (briefly a PM space) is a triple $(S, \mathcal{F}, \tau)$ where S is a nonempty set, $\mathcal{F}$ is a mapping from $S \times S$ into Δ^+ and τ is a triangle function that satisfies the following conditions ($F_{p,q} := \mathcal{F}(p,q)$):

(M1) $F_{p,q} = \epsilon_0$ if, and only if, $p = q$;
(M2) $F_{p,q} = F_{q,p}$;
(M3) $F_{p,r} \geq \tau(F_{p,q}, F_{q,r})$ for all $p, q, r \in S$.

If $\mathcal{F}$ satisfies (M1) and (M2) then $(S, \mathcal{F}, \tau)$ will be said to be a *Probabilistic Semimetric Space*, briefly PSM space.

We are now ready to introduce the definition of PN space.

DEFINITION 1.1. A Probabilistic Normed space (briefly a PN space) is a quadruple (V, ν, τ, τ^*), where V is a vector space, τ and τ^* are continuous triangle functions with $\tau \leq \tau^*$ and ν is a mapping (the probabilistic norm) $\nu : V \to \Delta^+$ such that for every choice of p and q in V the following conditions hold:

(N1) $\nu_p = \epsilon_0$ if, and only if, $p = \theta$ (θ is the null vector in V);
(N2) $\nu_{-p} = \nu_p$;
(N3) $\nu_{p+q} \geq \tau(\nu_p, \nu_q)$;
(N4) $\nu_p \leq \tau^*(\nu_{\lambda p}, \nu_{(1-\lambda)p})$ for every $\lambda \in [0,1]$.

A Menger PN space under T is a PN space (V, ν, τ, τ^*), denoted by (V, ν, T), in which $\tau = \tau_T$ and $\tau^* = \tau_{T^*}$, for some continuous t-norm T and its t-conorm T^*.

If ν satisfies (N2), (N3), (N4) and $\nu_\theta = \epsilon_0$ (but not necessarily (N1)), then (V, ν, τ, τ^*) will be said to be a Probabilistic Pseudonormed Space (briefly, a PPN space). The pair (V, ν) is said to be a Probabilistic Seminormed Space (=PSN space) if $\nu : V \to \Delta^+$ satisfies (N1) and (N2).

A PN space is called a *Šerstnev space* if it satisfies (N1) and (N3) and the following condition, which implies (N2) and (N4) together

$$(\check{S}) \qquad \forall p \in V\ \forall \alpha \in \mathbb{R} - \{0\}\ \forall x > 0 \quad \nu_{\alpha p}(x) = \nu_p\left(\frac{x}{|\alpha|}\right).$$

2 – Simple PN spaces

A PSN space (V, ν) is said to be *equilateral* if there is a d.f. $F \in \Delta^+$, different from ϵ_0 and from ϵ_∞, such that, for every $p \neq \theta$, $\nu_p = F$. It is immediate that every equilateral PSN space (V, ν) is a PN space under $\tau = \mathbf{M}$ and $\tau^* = \mathbf{M}$, where $\mathbf{M}$ is the triangle function defined for all $G, H \in \Delta^+$ by

$$\mathbf{M}(G, H)(x) := \min\{G(x), H(x)\} \quad (x \in \mathbb{R}_+).$$

An equilateral PN space will be denoted by $(V, F, \mathbf{M})$.

DEFINITION 2.1. Let $G \in \Delta^+$ be different from ϵ_0 and from ϵ_∞, let $(V, \|\cdot\|)$ be a normed space and define $\nu : V \to \Delta^+$ by $\nu_\theta = \epsilon_0$ and, if $p \neq \theta$, by

$$\nu_p(t) := G\left(\frac{t}{\|p\|}\right) \qquad (t > 0).$$

The pair (V, ν) is called the simple space generated by $(V, \|\cdot\|)$ and by G.

THEOREM 2.1. *The simple space generated by $(V, \|\cdot\|)$ and by G is a Menger PN space under M, denoted by $(V, \|\cdot\|, G, M)$, and a Šerstnev space. Here $M(x, y) := \min\{x, y\}$.*

PROOF. Let $\nu_p = \epsilon_0$ and assume, if possible, $p \neq \theta$; therefore, for every $t > 0$, one has $G(t/\|p\|) = 1$. Since $\|p\| > 0$, this would imply $G = \epsilon_0$ contrary to the assumption. Therefore $p = \theta$. This proves (N1). (N2) is obvious. In order to prove (N3) we shall have recourse to duality (see [2] but also [8; Section 7.7]). Given a d.f. F, its quasi-inverse $F^\wedge$ is defined by

$$F^\wedge(x) := \sup\{t : F(t) < x\}.$$

Since $\nu_p^\wedge = \|p\|\, G^\wedge$, one has, for every $p, q \in V$,

$$\begin{aligned}[\tau_M(\nu_p, \nu_q)]^\wedge &= \nu_p^\wedge + \nu_q^\wedge = \|p\|\, G^\wedge + \|q\|\, G^\wedge \\ &= (\|p\| + \|q\|)\, G^\wedge \geq \|p + q\|\, G^\wedge = \nu_{p+q}^\wedge\end{aligned}$$

so that $\nu_{p+q} \geq \tau_M(\nu_p, \nu_q)$, i.e. (N3).

In order to prove (N4) we shall use the equality $\tau_{M^*} = \tau_M$ ([8; Corollary 7.5.8]). Thus, the argument we have just used yields

$$\begin{aligned}\left[\tau_{M^*}(\nu_{\lambda p}, \nu_{(1-\lambda)p})\right]^\wedge &= \left[\tau_M(\nu_{\lambda p}, \nu_{(1-\lambda)p})\right]^\wedge \\ &= \lambda\, \|p\|\, G^\wedge + (1 - \lambda)\, \|p\|\, G^\wedge = \|p\|\, G^\wedge = \nu_p^\wedge\end{aligned}$$

hence the assertion. By virtue of Theorem 2 in [1], a simple space is a Šerstnev space under τ_M. □

3 – α-Simple spaces

DEFINITION 3.1. Let $(V, \|\cdot\|)$ be a normed space and let $G \in \Delta^+$ be different from ϵ_0 and ϵ_∞; define $\nu : V \to \Delta^+$ by $\nu_\theta = \epsilon_0$ and

$$\nu_p(t) := G\left(\frac{t}{\|p\|^\alpha}\right) \qquad (p \neq \theta, t > 0),$$

where $\alpha \geq 0$. Then the pair (V, ν) will be called the α-simple space generated by $(V, \|\cdot\|)$ and by G.

The α-simple space generated by $(V, \|\cdot\|)$ and by G is, as is immediately checked, a PSN space; it will be denoted by $(V, \|\cdot\|, G; \alpha)$.

The PSM space associated with the PSN space $(V, \|\cdot\|, G; \alpha)$ is the α-simple PSM space $(V, d, G; \alpha)$ where d is the metric of the norm $\|\cdot\|$, i.e. $d(p, q) = \|p - q\|$.

For $\alpha = 0$ and $\alpha = 1$ one obtains the equilateral and simple spaces, respectively.

If $\alpha \in]0,1[$, then we know ([8; Section 8.6]) that d^α is a metric so that the simple PSM space with respect to the metric d^α coincides with the α-simple PSM space $(V, d, G; \alpha)$, which is a Menger space under M ([8; Theorem 8.4.2]). However, a new phenomenon arises in the case of PN spaces, for contrary to the above, in this case $\|\cdot\|^\alpha$ is not a norm if $\alpha \in]0,1[$ so that $(V, \|\cdot\|, G; \alpha)$ need not be a Menger PN space under M as the following example shows.

EXAMPLE 3.1. Let U be the d.f. of the uniform law on $(0,1)$ and consider the α-simple space $(V, \|\cdot\|, U; \alpha)$ with $\alpha \in]0,1[$. We shall show that the axiom (N4) does not hold for $\lambda = 1/2$. Now it is easy to evaluate $\nu_p(\|p\|^\alpha) = 1$. On the other hand, $\tau_{M^*} = \tau_M$ and $\tau_M(F,F)(x) = F(x/2)$ for every $F \in \Delta^+$ and for every $x \geq 0$. Therefore

$$\tau_{M^*}(\nu_{p/2}, \nu_{p/2})(\|p\|^\alpha) = U\left(\frac{\|p\|^\alpha}{2\,\|p/2\|^\alpha}\right) = \frac{2^\alpha}{\|p\|^\alpha}\,\frac{\|p\|^\alpha}{2} = 2^{\alpha-1} < 1\,.$$

Thus

$$\nu_p(\|p\|^\alpha) > \tau_{M^*}(\nu_{p/2}, \nu_{p/2})(\|p\|^\alpha).$$

It is not hard to deduce from the proof of Theorem 8.6.2 in [8], which holds with same proof, that an α-simple PSN space with $\alpha > 1$ need not be a Menger space under M. A large class of α-simple PSM spaces can be endowed with the structure of Menger spaces (see [6] or [8; Theorem 8.6.5]). The next theorems show that analogous results hold for PSN spaces; their proves are not trivial extensions of the respective ones for α-simple PSM spaces and, moreover, we shall also need to prove these results for the case $\alpha \in]0,1[$.

By a straightforward calculation one gets

LEMMA 3.1. *Let $(V, \|\cdot\|)$ be a normed space, $G \in \mathcal{D}^+$ a strictly increasing continuous d.f. and T a strict t-norm with additive generator f. Then, for every $\alpha > 0$ with $\alpha \neq 1$, $(V, \|\cdot\|, G; \alpha)$ is a Menger* PN *space under T if, and only if, the following inequalities hold for every $u, v \in]0, +\infty[$, for every $\lambda \in [0,1]$ and for every pair of points p and q in V with $p \neq \theta$, $q \neq \theta$ and $p + q \neq \theta$*

$$(1) \qquad (f \circ G)\left(\frac{u+v}{\|p+q\|^\alpha}\right) \leq (f \circ G)\left(\frac{u}{\|p\|^\alpha}\right) + (f \circ G)\left(\frac{v}{\|q\|^\alpha}\right)$$

and

$$(2) \quad (f \circ G^*)\left(\frac{u+v}{\|p\|^\alpha}\right) \leq (f \circ G^*)\left(\frac{u}{\lambda^\alpha\|p\|^\alpha}\right) + (f \circ G^*)\left(\frac{v}{(1-\lambda)^\alpha\|p\|^\alpha}\right),$$

where $G^*(x) := 1 - G(x)$.

Setting $h := f \circ G$ and $h^* := f \circ G^*$, one gets

(a) $h, h^* : [0, +\infty] \to [0, +\infty]$, $h(0) = h^*(+\infty) = +\infty$, $h(+\infty) = h^*(0) = 0$;

(b) both h and h^* are continuous;

(c) h is strictly decreasing and h^* is strictly increasing.

Therefore their inverses h^{-1} and $(h^*)^{-1}$ satisfy the same properties as h and h^*, respectively.

Let p and q be in V with $p \neq \theta$, $q \neq \theta$, $p + q \neq \theta$ and let $\lambda \in]0,1[$. For $u, v > 0$ let

$$s := h\left(\frac{u}{\|p\|^\alpha}\right) \qquad t := h\left(\frac{v}{\|q\|^\alpha}\right);$$

thus $h^{-1}(s) = u/\|p\|^\alpha$ and $h^{-1}(t) = v/\|q\|^\alpha$. Now an easy calculation shows that (1) is equivalent to

$$\|p+q\|^\alpha\, h^{-1}(s+t) \leq \|p\|^\alpha\, h^{-1}(s) + \|q\|^\alpha\, h^{-1}(t).$$

In a similar way one shows that (2) is equivalent to

$$\lambda^\alpha\,(h^*)^{-1}(s) + (1-\lambda)^\alpha\,(h^*)^{-1}(t) \leq (h^*)^{-1}(s+t).$$

Thus Lemma 3.1 takes the form.

LEMMA 3.2. Let $(V, \|\cdot\|)$ be a normed space, $G \in \mathcal{D}^+$ a strictly increasing *continuous d.f.* and T a strict t-norm with additive generator f and let $G^*(x) := 1 - G(x)$. Then, for every $\alpha > 0$ with $\alpha \neq 1$, $(V, \|\cdot\|, G; \alpha)$ is a Menger PN *space under T if, and only if, the following inequalities hold for every $s,t \in]0,+\infty[$, for every $\lambda \in [0,1]$ and for every pair of points p and q in V with $p \neq \theta$, $q \neq \theta$ and $p+q \neq \theta$,*

$$(3)\quad \|p+q\|^\alpha\,(f\circ G)^{-1}(s+t) \le \|p\|^\alpha\,(f\circ G)^{-1}(s) + \|q\|^\alpha\,(f\circ G)^{-1}(t)$$

$$(4)\quad (f\circ G^*)^{-1}(s+t) \ge \lambda^\alpha\,(f\circ G^*)^{-1}(s) + (1-\lambda)^\alpha\,(f\circ G^*)^{-1}(t)\,.$$

Notice that, if $\alpha < 1$ then inequality (3) is trivially satisfied, because $\|p+q\|^\alpha \le \|p\|^\alpha + \|q\|^\alpha$ and since $(f\circ G)^{-1}$ is strictly decreasing.

On the other hand, if $\alpha > 1$ then inequality (4) is trivial since $f\circ G^*$ is strictly increasing, and for every $\lambda \in]0,1[$ and for every $\alpha > 1$, $\lambda^\alpha + (1-\lambda)^\alpha < 1$. Thus one can rephrase Lemma 3.2 in the following form.

LEMMA 3.3. *Let $(V,\|\cdot\|)$ be a normed space, $G \in \mathcal{D}^+$ a strictly increasing continuous d.f., T a strict t-norm with additive generator f and let $G^*(x) := 1-G(x)$.*

(a) *When $\alpha \in]0,1[$, then $(V,\|\cdot\|,G;\alpha)$ is a Menger PN space under T if, and only if, for every $\lambda \in]0,1[$ and for all $s,t \in]0,+\infty[$ inequality (4) holds;*

(b) *when $\alpha \in]1,+\infty[$, then $(V,\|\cdot\|,G;\alpha)$ is a Menger PN space under T if, and only if, for all $s,t \in]0,+\infty[$ and for all $p,q \in V$ such that $p \neq \theta$, $q \neq \theta$, $p+q \neq \theta$, inequality (3) holds.*

LEMMA 3.4. *If, beside the conditions of the previous Lemma, one has for $\alpha > 1$,*

$$(5)\qquad \forall x \in [0,+\infty] \qquad (f\circ G)(x) = x^{1/(1-\alpha)},$$

or, for $\alpha \in]0,1[$,

$$(6)\qquad \forall x \in [0,+\infty] \qquad (f\circ G^*)(x) = x^{1/(1-\alpha)},$$

then $(V,\|\cdot\|,G;\alpha)$ is a Menger PN space under T.

PROOF. In view of (5), if $\alpha > 1$ and $s,t \in]0,+\infty[$, then inequality (3), for all $p \neq \theta$, $q \neq \theta$ with $p+q \neq \theta$, is implied by the following inequality

$$(s+t)^{1-\alpha} \le x^\alpha s^{1-\alpha} + (1-x)^\alpha t^{1-\alpha},$$

for all $x \in]0,1[$, which can be proved in a straightforward manner. Similarly, in view of (6), if $\alpha \in]0,1[$, one can prove

$$\forall x \in]0,1[\ \forall s,t \in]0,+\infty[\quad (s+t)^{1-\alpha} \ge x^\alpha s^{1-\alpha} + (1-x)^\alpha t^{1-\alpha},$$

from which inequality (4) follows by setting $x = \lambda$. □

We are now ready to state the main results of this section.

THEOREM 3.1. *Let $(V,\|\cdot\|)$ be a normed space and let $\alpha > 1$.*

(a) *If the d.f. $G\in\mathcal{D}^+$ is continuous and strictly increasing, then $(V,\|\cdot\|,G;\alpha)$ is a Menger PN space under the strict t-norm defined for all $x,y \in [0,+\infty]$ by*

$$T_G(x,y) := G\left(\left\{[G^{-1}(x)]^{1/(1-\alpha)} + [G^{-1}(y)]^{1/(1-\alpha)}\right\}^{1-\alpha}\right);$$

(b) *if T is a strict t-norm with additive generator f, then the function $G : [0,+\infty] \to [0,1]$ defined by $G(x) := f^{-1}\left(x^{1/(1-\alpha)}\right)$ is a continuous, strictly increasing d.f. of $\mathcal{D}^+$ and $(V,\|\cdot\|,G;\alpha)$ is a Menger PN space under T.*

PROOF. (a) Given G define, for $x \in [0,1]$, $f(x) := [G^{-1}(x)]^{1/(1-\alpha)}$. It is immediate to check that f is the additive generator of a strict t-norm and that this latter is exactly T_G defined above. Then the assertion follows from Lemma 3.4.

(b) Given the strict t-norm T with f as additive generator, define $G(x) := f^{-1}(x^{1/(1-\alpha)})$ for every $x \in]0,+\infty[$. Then it is immediate to check that G thus defined is a continuous strictly increasing d.f. of $\mathcal{D}^+$. Again the assertion follows from Lemma 3.4. □

THEOREM 3.2. *Let $(V,\|\cdot\|)$ be a normed space and let $\alpha \in]0,1[$.*

(a) *If the d.f. $G\in\mathcal{D}^+$ is continuous and strictly increasing, then $(V,\|\cdot\|,G;\alpha)$ is a Menger PN space under the strict t-norm defined for all $x,y \in [0,+\infty]$ by*

$$T_{G^*}(x,y) := G^*\left(\left\{[(G^*)^{-1}(x)]^{1/(1-\alpha)} + [(G^*)^{-1}(y)]^{1/(1-\alpha)}\right\}^{1-\alpha}\right),$$

where $G^*(x) := 1-G(x)$;

(b) *if T is a strict t-norm with additive generator f, then the function $G : [0,+\infty] \to [0,1]$ defined by $G(x) := 1 - f^{-1}\left(x^{1/(1-\alpha)}\right)$ is a continuous,* *strictly increasing d.f. of* $\mathcal{D}^+$ *and* $(V, \|\cdot\|, G; \alpha)$ *is a Menger* PN *space under* T.

PROOF. (a) Given G, define, for $x \in [0,1]$, $f(x) := [(G^*)^{-1}(x)]^{1/(1-\alpha)}$. As is easy to check, the function f thus defined is the additive generator of the t-norm T_{G^*}. Moreover $(f \circ G^*)(x) = x^{1/(1-\alpha)}$ so that the assertion is now a consequence of Lemma 3.4,

(b) Given f, let $G(x) := 1 - f^{-1}(x^{1/(1-\alpha)})$; then G is a continuous, strictly increasing d.f. in $\mathcal{D}^+$, $G^*(x) = f^{-1}(x^{1/(1-\alpha)})$ and, as a consequence, $(f \circ G^*)(x) = x^{1/(1-\alpha)}$ so that again Lemma 3.4 yields the assertion. □

The following two results of this section are the analogue of Theorem 3 in [6] and show the relevance of the t-norms T_G and T_{G^*} of Theorems 3.1 and 3.2.

THEOREM 3.3. *If* $\alpha > 1$, *then there exist normed spaces* $(V, \|\cdot\|)$ *with the following properties:*

(a) *if* $G \in \mathcal{D}^+$ *is continuous and strictly increasing, then the t-norm* T_G *is the strongest continuous t-norm under which* $(V, \|\cdot\|, G; \alpha)$ *is a Menger* PN *space (in the sense that, if* T *is any other continuous t-norm that makes it a Menger* PN *space, then* $T \le T_G$*);*

(b) *if* T *is a strict t-norm with additive generator* f, *then* T *is the strongest t-norm under which* $(V, \|\cdot\|, G; \alpha)$, *with*

$$G(x) := f^{-1}(x^{1/(1-\alpha)}),$$

is a Menger PN *space.*

PROOF. (a) Let $(V, \|\cdot\|) = (\mathbf{R}, |\cdot|)$. Assume that $(\mathbf{R}, |\cdot|, G; \alpha)$ is a Menger PN space under T, where T is a continuous t-norm. We wish to prove that for every $(s,t) \in [0,1]^2$ one has $T(s,t) \le T_G(s,t)$. In fact, we only need to prove this inequality in the interior of $[0,1]^2$. Now let $(s,t) \in]0,1[^2$ be fixed and set

$$p = \left[G^{-1}(s)\right]^{1/(1-\alpha)} \qquad q = \left[G^{-1}(t)\right]^{1/(1-\alpha)} \in]0,+\infty[.$$

Then

$$\nu_p(p) = G\left(\frac{p}{|p|^\alpha}\right) = G\left(p^{1-\alpha}\right) = G\left(G^{-1}(s)\right) = s;$$

similarly $\nu_q(q) = t$, so that

$$T(s,t) = T\left(\nu_p(p), \nu_q(q)\right) \le \sup_{u+v=p+q} \left\{T\left(\nu_p(u), \nu_q(v)\right)\right\} =$$
$$= \tau_T(\nu_p, \nu_q)(p+q) \le \nu_{p+q}(p+q) = G\left(\frac{p+q}{|p+q|^\alpha}\right).$$

On the other hand

$$T_G(s,t) = T\left(\nu_p(p), \nu_q(q)\right) =$$
$$= G\left\{\left(\left[G^{-1}(\nu_p(p))\right]^{1/(1-\alpha)} + \left[G^{-1}(\nu_q(q))\right]^{1/(1-\alpha)}\right)^{1-\alpha}\right\} =$$
$$= G\left((p+q)^{1-\alpha}\right) = G\left(\frac{p+q}{|p+q|^\alpha}\right).$$

This completes the proof of (a). The same example establishes (b). □

THEOREM 3.4. *If* $\alpha \in]0,1[$, *then there exist a normed space* $(V, \|\cdot\|)$ *with the following properties:*

(a) *if* $G \in \mathcal{D}^+$ *is continuous and strictly increasing, then the t-norm* T_{G^*} *is the strongest continuous t-norm under which* $(V, \|\cdot\|, G; \alpha)$ *is a Menger* PN *space;*

(b) *if* T *is a strict t-norm with additive generator* f, *then* T *is the strongest t-norm under which* $(V, \|\cdot\|, G; \alpha)$, *with*

$$G(x) := 1 - f^{-1}(x^{1/(1-\alpha)}),$$

is a Menger PN *space.*

PROOF. (a) As in the previous proof, assume that $(\mathbf{R}, |\cdot|, G; \alpha)$ is a Menger PN space under T, where T is a t-norm. For $(s,t) \in]0,1[^2$ set

$$p = \left[(G^*)^{-1}(s)\right]^{1/(1-\alpha)} \qquad q = \left[(G^*)^{-1}(t)\right]^{1/(1-\alpha)} \in]0,+\infty[$$

$$\lambda = \frac{p}{p+q} \in]0,1[.$$

Then

$$\begin{aligned}\nu_{\lambda(p+q)}(\lambda(p+q)) &= G\left(\frac{\lambda(p+q)}{(\lambda(p+q))^{\alpha}}\right) = G\left((\lambda(p+q))^{1-\alpha}\right) = \\ &= G\left(p^{1-\alpha}\right) = G\left((G^*)^{-1}(s)\right) = \\ &= 1 - G^*\left((G^*)^{-1}(s)\right) = 1 - s.\end{aligned}$$

Similarly $\nu_{(1-\lambda)(p+q)}\left((1-\lambda)(p+q)\right) = 1 - t$. Therefore

$$\begin{aligned}T(s,t) &= 1 - T^*(1-s, 1-t) = \\ &= 1 - T^*\left(\nu_{\lambda(p+q)}(\lambda(p+q)), \nu_{(1-\lambda)(p+q)}\left((1-\lambda)(p+q)\right)\right) \le \\ &\le 1 - \inf_{u+v=p+q}\left\{T^*\left(\nu_{\lambda(p+q)}(u), \nu_{(1-\lambda)(p+q)}(v)\right)\right\} = \\ &= 1 - \tau_{T^*}\left(\nu_{\lambda(p+q)}, \nu_{(1-\lambda)(p+q)}\right)(p+q) \le 1 - \nu_{p+q}(p+q) = \\ &= 1 - G\left(\frac{p+q}{(p+q)^{\alpha}}\right) = G^*\left((p+q)^{1-\alpha}\right).\end{aligned}$$

On the other hand

$$T_{G^*}(s,t) = T_{G^*}\left(G^*(p^{1-\alpha}), G^*(q^{1-\alpha})\right) = G^*\left((p+q)^{1-\alpha}\right).$$

Part (b) is proved in the same way. □

The following result shows that an α-simple PN space with $\alpha \ne 1$ is not, in general, a Šerstnev space.

THEOREM 3.5. *Let $\alpha > 0$ with $\alpha \ne 1$. The following statements are equivalent for an α-simple space $(V, \|\cdot\|, G; \alpha)$:*
(a) *$(V, \|\cdot\|, G; \alpha)$ is a Šerstnev* PSN *space under some triangle function τ;*
(b) *G is constant on $]0, +\infty[$.*

PROOF. Since the implication (b) $\Longrightarrow$ (a) is immediate, we need only deal with the other one (a) $\Longrightarrow$ (b).

In our case, axiom (Š) is equivalent to

$$\forall p \ne \theta,\ \forall \lambda \ne 0,\ \forall x > 0 \quad G\left(\frac{x}{\|\lambda p\|^{\alpha}}\right) = G\left(\frac{x}{\lambda\|p\|^{\alpha}}\right)$$

which is equivalent to the equality $G(t) = G\left(|\lambda|^{1-\alpha} t\right)$ for all $t > 0$ and $\lambda \ne 0$; this implies that G is constant on $]0, +\infty[$. □

4 – EN, L^p- and Orlicz spaces

An important class of PN spaces is that of E-normed spaces.

DEFINITION 4.1. Let $(\Omega, \mathcal{A}, P)$ be a probability space, $(V, \|\cdot\|)$ a normed space and S a linear space of V-valued random variables (possibly, the entire space). For every $p \in S$ and for every $x \in \overline{\mathbf{R}}_+$, let $\nu : S \to \Delta^+$ be defined by $\nu_p(x) := P\{\omega \in \Omega : \|p(\omega)\| < x\}$; then (S, ν) is an E-normed space (briefly, EN space) with base $(\Omega, \mathcal{A}, P)$ and target $(V, \|\cdot\|)$.

THEOREM 4.1. *An* EN *space (S, ν) is a* PPN *space under τ_W and τ_M. It is said to be canonical if it is a* PN *space under the same two triangle functions. In this latter case, it is a Šerstnev space.*

PROOF. The proof in [11] and [12] (but see also [8]) needs to be supplemented by the part regarding the property (N4) that was missing in the old definition. Actually, more is true, for it is not hard to show that $\nu_p = \tau_M(\nu_{\alpha p}, \nu_{(1-\alpha)p})$ for every $p \in S$ and for every $\alpha \in [0,1]$. Therefore, by virtue of Theorem 2 in [1], (S, ν) is a Šerstnev space under τ_W. □

The proof of the following result is immediate

THEOREM 4.2. *In an* EN *space (S, ν), let a relationship $\sim$ be defined through*

$$p \sim q \qquad \text{if and only if} \quad \nu_p = \nu_q .$$

Then $\sim$ is an equivalence relation on S. Moreover, if $\overline{S} = S/\sim$ is the quotient space and if $\overline{\nu} : \overline{S} \to \Delta^+$ is defined via

$$\overline{\nu}_{\overline{p}} := \nu_p \tag{7}$$

for every p in the equivalence class $\overline{p}$, then $(\overline{S}, \overline{\nu})$ is a canonical EN *space, called the quotient* EN *space of (S, ν).*

The previous theorem can be applied to $L^0 := L^0(\Omega, \mathcal{A}, P)$, the linear space of equivalence classes of random variables $f : \Omega \to \mathbb{R}$. In this case the quotient mapping (7) is given explicitly by

$$n_f(x) := P\{\omega \in \Omega : |f(\omega)| < x\}, \quad x > 0, \tag{8}$$

As is usual in probability theory, we shall write f even when we refer to $\overline{f}$ the equivalence class of f.

As a consequence of Theorems 4.1 and 4.2, (L^0, n, τ_W) is a Šerstnev space. Any linear subspace S of L^0 inherits the property of last theorem; in other words, (S, n, τ_W) is a Šerstnev space. (Here we still denote by n the restriction of n to S).

We are now ready to establish the result we have announced about L^p and Orlicz spaces.

It was proved by Schweizer and Sklar (see [7]) that all L^p metrics ($p \in]0, +\infty]$) could be derived from a single probabilistic metric. Later this result was extended, with a simplified proof, by one of the present authors (see [9]), to the case of Orlicz spaces. However, since both L^p spaces with $p \in [1, +\infty]$ and Orlicz spaces are normed —in fact, Banach— spaces, and therefore their metrics derive from a norm, it is more natural to show that a single probabilistic *norm* generates the norms of all these spaces.

Essentially by the change of variables formula (see, e.g., [4]) one can prove the next theorem. For the definition and the properties of Orlicz spaces see [3] or [5].

THEOREM 4.3. *Let* $L^p = L^p(\Omega, \mathcal{A}, P) := \{f \in L^0 : \int_\Omega |f|^p \, dP < +\infty\}$ *for* $p \in [1, +\infty[$ *and* $L^\infty := \{f \in L^0 : \|f\|_\infty := \operatorname{ess\,sup} |f| < +\infty\}$. *If the probabilistic norm* $n : L^0 \to \Delta^+$ *is defined by*

$$n_f(x) := P\{\omega \in \Omega : |f(\omega)| < x\}, \quad x > 0,$$

then

$$\forall f \in L^p \ (p \in [1, +\infty[) \quad \|f\|_p = \left(\int_{\mathbb{R}_+} x^p \, dn_f(x) \right)^{1/p},$$

$$\forall f \in L^\infty \qquad \|f\|_\infty = \sup\{x > 0 : n_f(x) < 1\}.$$

In the Orlicz space $L^\phi = L^\phi(\Omega, \mathcal{A}, P)$, *the Luxemburg norm*

$$\forall f \in L^\phi \quad \|f\|_\phi = \inf\left\{k > 0 : \int_\Omega \phi(f/k) \, dP \le 1\right\}.$$

is given by

$$\|f\|_\phi = \inf\left\{k > 0 : \int_{\mathbb{R}_+} \phi\left(\frac{x}{k}\right) \, dn_f(x) \le 1\right\}.$$

If $p \in [1, +\infty[$, then a sequence $\{f_n\}$ in L^p converges to $f \in L^p$ if, and only if the p-th moment of $n_{f_n - f}$, the probabilistic norm of $f_n - f$, tends to zero, viz.

$$\int_{\mathbb{R}_+} x^p \, dn_{f_n - f}(x) \xrightarrow[n \to +\infty]{} 0.$$

A measurable function f belongs to L^∞ if, and only if, there is a point $x > 0$ at which its probabilistic norm n_f takes the value 1; in this case, $\|f\|_\infty = x_0$ where

$$x_0 := \inf\{x > 0 : n_f(x) = 1\}.$$

In a forthcoming paper we shall give some results on the functional analysis of PN spaces.

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