Journal of Mathematical Analysis and Applications **232**, 183–196 (1999)
Article ID jmaa.1998.6261, available online at http://www.idealibrary.com on IDEAL®

A Study of Boundedness in Probabilistic Normed Spaces

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Submitted by Ulrich Höhle

Received October 7, 1996

Probabilistic normed spaces have been redefined by Alsina, Schweizer, and Sklar. We give a detailed analysis of various boundedness notions for linear operators between such spaces and we study the relationship among them and also with the notion of continuity.

1. INTRODUCTION

Probabilistic normed spaces (PN spaces henceforth) were introduced by Šerstnev in [13] by means of a definition that was closely modeled on the theory of normed spaces. Here we consistently adopt the new, and in our opinion convincing, definition of PN space given in the paper by Alsina, Schweizer, and Sklar [1]. We recall it. The notation and the concepts used are those of [12, 1, and 2].

DEFINITION 1.1. A *probabilistic normed space* (briefly a PN space) is a quadruple (V, ν, τ, τ^*), where V is a real vector space, τ and τ^* are continuous triangle functions with $\tau \leq \tau^*$ and ν is a mapping (the

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0022-247X/99 $30.00

probablistic norm) $\nu\colon V \to \Delta^+$ such that for every choice of p and q in V the following conditions hold:

(N1) $\nu_p = \epsilon_0$ if, and only if, $p = \theta$ (θ is the null vector in V);

(N2) $\nu_{-p} = \nu_p$;

(N3) $\nu_{p+q} \geq \tau(\nu_p, \nu_q)$;

(N4) $\nu_p \leq \tau^*(\nu_{\lambda p}, \nu_{(1-\lambda)p})$ for every $\lambda \in [0,1]$.

A *Menger* PN *space* under T is a PN space (V, ν, τ, τ^*) in which $\tau = \tau_T$ and $\tau^* = \tau_{T^*}$, for some continuous t-norm T and its t-conorm T^*; it is denoted by (V, ν, T).

A PN space is called a *Šerstnev space* if it satisfies (N1) and (N3) and the following condition,

$$\nu_{\alpha p}(x) = \nu_p\left(\frac{x}{|\alpha|}\right), \quad \text{for every } \alpha \in \mathbf{R} - \{0\} \text{ and for every } x > 0, \tag{Š}$$

which clearly implies (N2) and also (see [1]) (N4) in the strengthened form,

$$\nu_p = \tau_M(\nu_{\lambda p}, \nu_{(1-\lambda)p}), \quad \text{for every } \lambda \text{ in } [0,1].$$

There is a natural topology in a PN space (V, ν, τ, τ^*), called the *strong topology*; it is defined by the neighbourhoods,

$$\mathcal{N}_p(t) := \{q \in V\colon \nu_{q-p}(t) > 1 - t\} = \{q \in V\colon d_L(\nu_{q-p}, \epsilon_0) < t\},$$

where $t > 0$. Here d_L is the modified Lévy metric ([14]).

Adopting this definition of a PN space, we proved in [6] that every PN has a completion and studied in [7] special classes of PN spaces. Here we present a detailed analysis of various concepts of boundedness for subsets of PN spaces (Section 2) and we study the connections between continuity and boundedness, in its various versions, for linear operators between PN spaces (Section 3). Our standpoint is entirely new, because the authors (see [8–10]) who previously have studied continuity and boundedness for linear operators did so in the context of Šerstnev spaces and—a great restriction indeed—limited their attention to the operators that we call strongly bounded.

An ordinary normed space can always be regarded as a special PN space.

EXAMPLE 1.1. Let $(V, \|\cdot\|)$ be a normed space and define $n\colon V \to \Delta^+$ via

$$n_p := \epsilon_{\|p\|}. \tag{1}$$

Let τ be a triangle function such that

$$\tau(\epsilon_a, \epsilon_b) = \epsilon_{a+b}, \tag{2}$$

for all $a, b \geq 0$ and let τ^* be a second triangle function with $\tau \leq \tau^*$. For instance, it suffices to take $\tau = \tau_T$ and $\tau^* = \tau_{T^*}$, where T is a continuous t-norm and T^* is its t-conorm. Then (V, n, τ, τ^*) is a PN space.

In the other direction, if (V, ν, τ, τ^*) is a PN space in which τ satisfies (2) and if there is a function $f: V \to \mathbf{R}_+$ such that $n_p := \epsilon_{f(p)}$ holds, then f satisfies the two properties

(i) $f(p) = 0$ if, and only if, $p = \theta$, where θ is the null vector of V;

(ii) $f(p + q) \leq f(p) + f(q)$ for all $p, q \in V$.

Note that such a function is not necessarily a norm. In order to see this it suffices to consider the case $V = \mathbf{R}$ and to choose

$$f(p) := \frac{p}{1 + p};$$

this latter function satisfies (i) and (ii), but it is not a norm, as is immediately seen. If, moreover, (V, n, τ, τ_M) is a Ŝerstnev space, then f is actually a norm on V.

2. BOUNDED SETS

Given a nonempty set A in a PN space (V, ν, τ, τ^*) its *probabilistic radius* R_A is defined by

$$R_A(x) := \begin{cases} l^- \phi_A(x), & x \in [0, +\infty[, \\ 1, & x = +\infty, \end{cases}$$

where $l^- f(x)$ denotes the left limit of the function f at the point x and $\phi_A(x) := \inf\{\nu_p(x): p \in A\}$.

The following definition sharpens that of [12, Section 12.4] as we detail in Section 4.

DEFINITION 2.1. A nonempty set A in a PN space (V, ν, τ, τ^*) is said to be:

(a) *certainly bounded*, if $R_A(x_0) = 1$ for some $x_0 \in]0, +\infty[$,

(b) *perhaps bounded*, if one has $R_A(x) < 1$, for every $x \in]0, +\infty[$, and $l^- R_A(+\infty) = 1$;

(c) *perhaps unbounded*, if $R_A(x_0) > 0$ for some $x_0 \in]0, +\infty[$ and $l^- R_A(+\infty) \in]0, 1[$;

(d) *certainly unbounded*, if $l^- R_A(+\infty) = 0$, i.e., if $R_A = \epsilon_\infty$.

Moreover, A will be said to be *D-bounded* if either (a) or (b) holds, i.e., if $R_A \in \mathscr{D}^+$; otherwise, i.e., if $R_A \in \Delta^+ \setminus \mathscr{D}^+$, A will said to be *D-unbounded.*

Note that in the previous definition we could have used ϕ_A instead of R_A.

The following lemma, whose proof is simple, is useful in the remainder of this paper.

LEMMA 2.1. *Let A be a nonempty set in a* PN *space* (V, ν, τ, τ^*). *Then*

(a) *A is certainly bounded if, and only if, $\phi_A(x_0) = 1$ for some $x_0 \in]0, +\infty[$;*

(b) *A is perhaps bounded if, and only if, $\phi_A(x_0) < 1$ for every $x_0 \in]0, +\infty[$ and $l^- \phi_A(+\infty) = 1$;*

(c) *A is perhaps unbounded if, and only if, $l^- \phi_A(+\infty) \in]0, 1[$;*

(d) *A is certainly unbounded if, and only if, $l^- \phi_A(+\infty) = 0$, i.e., $\phi_A = 0$;*

It is easy to provide an intuitive justification for the preceding definition, and, at the same time, to explain why we use two types of bounded sets, absolutely and perhaps bounded, in place of the traditional one ([12, Definition 12.4.3]). It suffices to think of the value at x of the probabilistic norm ν_p of p as the probability that the norm $\|p\|$ is smaller than x. Then a set A is certainly bounded if, and only if, there is $x_0 > 0$ such that, with probability 1, $\|p\| < x_0$ for every p in A; thus, almost certainly A is included in the open ball $B(x_0)$ centered at the origin θ and of radius x_0. This closely corresponds to the idea of what a bounded set is in probabilistic terms. If the set A is not certainly bounded, then it is perhaps bounded if, and only if, for every $\delta > 0$, there exists $x_0 = x_0(\delta) > 0$ such that every point p in A belong to $B(x_0)$ with probability greater than $1 - \delta$. The set A is certainly unbounded if, and only if, for every $\delta > 0$ and for every $x_0 > 0$, there exists some point p in A that lies outside $B(x_0)$ with probability greater than $1 - \delta$. Finally, if A is not certainly unbounded, then A is perhaps unbounded if, and only if, there exists $\delta \in]0, 1[$ such that, for every $x_0 > 0$, there is a point p in A that lies outside $B(x_0)$ with probability greater than δ.

The proof of the following result is very simple.

THEOREM 2.1. *A set A in the* PN *space* (V, ν, τ, τ^*) *is D-bounded if, and only if, there exists a d.f. $G \in \mathscr{D}^+$ such that $\nu_p \geq G$ for every $p \in A$.*

It follows at once from Definition 2.1 that a set A is D-bounded if, and only if,

$$\lim_{x \to +\infty} \phi_A(x) = 1. \tag{3}$$

This latter condition implies that

$$\lim_{x \to +\infty} \nu_p(x) = 1, \quad \text{for every } p \in A. \tag{4}$$

The converse is not true as the following example shows

EXAMPLE 2.1. Let $(\mathbf{R}, |\cdot|)$ be the normed space of the reals $\mathbf{R}$ endowed with the usual norm. It can be made into a Menger PN space as in Example 1.1 by choosing any continuous t-norm T. Let A be any unbounded subset of $\mathbf{R}$. Then, for every $x \in]0, +\infty[$, there exists $p \in A$ such that $|p| \geq x$. Consequently,

$$\phi_A(x) = \inf_{p \in A} n_p(x) = \inf_{p \in A} \epsilon_{|p|}(x) = 0,$$

whence $\lim_{x \to +\infty} \phi_A(x) = 0$; and thus (3) is not satisfied. On the other hand, for every $p \in A$

$$\lim_{x \to +\infty} n_p(x) = \lim_{x \to +\infty} \epsilon_{|p|}(x) = 1,$$

so that (4) is satisfied.

For the definition of the special PN spaces in the following examples we refer to our paper [7].

EXAMPLE 2.2. Let $(V, F, \mathbf{M})$ be an equilateral PN space. If there is a $x_0 \in]0, +\infty[$ such that $F(x_0) = 1$, then every nonempty set of V is certainly bounded; otherwise, only the singleton $\{\theta\}$ is certainly bounded; for any subset A, one has $\phi_A = F$ so that A is perhaps bounded if, and only if, $l^- F(-\infty) = 1$; if $l^- F(+\infty) < 1$, then A is perhaps unbounded.

EXAMPLE 2.3. Let $(V, \|\cdot\|)$ be a normed space, and consider the simple Menger space $(V, \|\cdot\|, G, M)$. Then

(a) if there exists $x_0 \in]0, +\infty[$ such that $G(x_0) = 1$, then the certainly bounded sets of $(V, \|\cdot\|, G, M)$ coincide with the bounded sets of $(V, \|\cdot\|)$. Moreover, an unbounded set in $(V, \|\cdot\|)$ is either perhaps unbounded or certainly unbounded in $(V, \|\cdot\|, G, M)$ according to whether $l^+ G(0) := \lim_{x \to 0+} G(x)$ belongs to $]0, 1[$ or is equal to 0, respectively;

(b) If $l^- G(+\infty) = 1$ but, for every $x \in]0, +\infty[$, $G(x) < 1$, then the only certainly bounded set of $(V, \|\cdot\|, G, M)$ is the singleton $\{\theta\}$; the perhaps bounded sets of $(V, \|\cdot\|, G, M)$ coincide with the bounded sets of

$(V, \|\cdot\|)$, while the unbounded sets of $(V, \|\cdot\|)$ are either perhaps unbounded or certainly unbounded in $(V, \|\cdot\|, G, M)$ according to whether $l^+G(0) > 0$ or $l^+G(0) = 0$, respectively;

(c) If $l^-G(+\infty) \in]0,1[$, everything behaves as in the previous case, the only difference being that the bounded sets of $(V, \|\cdot\|)$ different from $\{\theta\}$ are perhaps unbounded in $(V, \|\cdot\|, G, M)$.

The same results hold for α-simple spaces.

Notice that for equilateral, simple, and α-simple PN spaces the nature of a set, as far as boundedness is concerned, depends only on the properties of the one distribution function that appears in the definition of those spaces.

EXAMPLE 2.4. The description of the various type of sets of Definition 2.1 is particularly transparent in the case of EN spaces; and here the motivation behind the definitions also comes to the surface. Let A be a subset of an EN space (S, ν), i.e., a subset of V-valued random variables; then A is:

(a) certainly bounded if, and only if, it is P-a.s. bounded; i.e., the random variables of A are P-a.s. uniformly bounded;

(b) perhaps bounded, if, and only if, for every $\epsilon > 0$, there is a ball B_ϵ in $(V, \|\cdot\|)$ such that all the random variables in A take values in B_ϵ with probability greater than $1 - \epsilon$;

(c) perhaps unbounded if, and only if, there exists $\beta \in]0,1[$ such that, for every $x \in]0, +\infty[$, there is a random variable $p \in A$ such that $P\{\omega \in \Omega: \|p(\omega)\| \geq x\} \geq \beta > 0$ (in other words, with strictly positive probability, the radius of A is actually infinite);

(d) certainly unbounded, if, and only if, for every $\epsilon > 0$, for every $x \in]0, +\infty[$, there is $p \in A$ such that $P\{\omega \in \Omega: \|p(\omega)\| \geq x\} > 1 - \epsilon$.

Next we present two results concerning the probabilistic radius. The first one is just the analogue of a classical result, while the second one generalizes the well-known relationship $r_{A \cup B} \leq r_A + r_B$ valid for the radii of the sets A and B.

THEOREM 2.2. *In a* PN *space* (V, ν, τ, τ^*), *the probabilistic radius has the following properties*:

(a) *for every nonempty set* A, $R_A = R_{\overline{A}}$ *where* $\overline{A}$ *denotes the closure of* A *in the strong topology*;

(b) $R_{A \cup B} \geq \tau(R_A, R_B)$, *if* A *and* B *are nonempty.*

Proof. (a) Because $A \subset \overline{A}$, and, as a consequence, $R_A \geq R_{\overline{A}}$, one has only to show the converse inequality $R_A \leq R_{\overline{A}}$. When (V, ν, τ, τ^*) is endowed with the strong topology and Δ^+ is endowed with the topology of weak convergence, i.e., the topology of the modified Lévy metric d_L, the probabilistic norm $\nu: V \to \Delta^+$ is uniformly continuous ([2, Theorem 1]); in other words, for every $\eta \in]0,1[$ there exists $\delta = \delta(\eta) > 0$ such that $d_L(\nu_p, \nu_q) < \eta$ whenever $d_L(\nu_{p-q}, \epsilon_0) < \delta$.

Now, for every $p \in \overline{A}$, there exists $q(p) \in A$ such that

$$d_L(\nu_{p-q(p)}, \epsilon_0) < \delta;$$

therefore $d_L(\nu_p, \nu_{q(p)}) < \eta$. In particular, for every $t \in]0, 1/\eta[$, one has

$$\nu_p(t) \geq \nu_{q(p)}(t - \eta) - \eta.$$

Then, for $t \in]0, 1/\eta[$,

$$\begin{aligned}\phi_{\overline{A}}(t) &= \inf_{p \in \overline{A}} \nu_p(t) \geq \inf_{p \in \overline{A}} \nu_{q(p)}(t - \eta) - \eta \\ &= \inf_{p \in A} \nu_{q(p)}(t - \eta) - \eta \\ &\geq \inf_{p \in A} \nu_p(t - \eta) - \eta = \phi_A(t - \eta) - \eta.\end{aligned}$$

Therefore, if $t \in]0, 1/\eta[$, then $R_{\overline{A}}(t) \geq R_A(t - \eta) - \eta$. This latter inequality holds for every $\eta \in]0,1[$ and for every $t \in]0, 1/\eta[$. Thus, letting $\eta \to 0$ and using the left-continuity of R_A yields that, for every $t > 0$,

$$R_{\overline{A}}(t) \geq R_A(t).$$

(b) For every $p \in A \cup B$ and for every $q \in B$ we have that

$$\nu_p = \tau(\nu_p, \epsilon_0) \geq \tau(\nu_p, \nu_q) \geq \tau(\nu_p, R_B),$$

because $R_B \leq \nu_q$ for all $q \in B$. Therefore, if $p \in A$ we have $\nu_p \geq \tau(R_A, R_B)$.

Repeating the same argument for $p \in A \cup B$ and $q \in A$ leads to the inequality $\nu_p \geq \tau(R_A, R_B)$ for every $p \in B$. Now the last two inequalities yield the assertion. ∎

As a consequence of Theorem 2.2(a), any boundedness property that holds for a set A holds also for its closure $\overline{A}$ and conversely.

Our definition of boundedness has a topological content. It has been shown (see, e.g., [3, 4, 15, 17]) that in a PM space the topological issues involved are delicate and, in general, do not follow traditional patterns. However, in the present setting, there is one topology that comes to the fore—the strong topology [12, Chap. 12]. In what follows we focus our attention exclusively on it.

If (V, ν, τ, τ^*) is a topological vector space with respect to the strong topology—which is the case if either (V, ν, τ) is a Šerstnev space or if τ^* is Archimedean (see [2])—then (see [11]) there is a unique translation invariant uniformity $\mathscr{U}$ that induces the strong topology and which makes $(V, \mathscr{U})$ into a uniform space (see, e.g., [5]). If $\{\mathscr{N}_\theta(1/n): n \in \mathbf{N}\}$ is a base of neighbourhoods of θ in the strong topology, then a base for the uniformity $\mathscr{U}$ is given by the sets

$$\left\{(x, y) \in V \times V: y - x \in \mathscr{N}_\theta\left(\frac{1}{n}\right), n \in \mathbf{N}\right\}.$$

In a uniform space, a concept of boundedness for sets is given; we call this type of boundedness "uniform" in order to distinguish it from those previously introduced.

We recall that a subset $A \subset V$ of a uniform space $(V, \mathscr{U})$ is uniformly bounded if, and only if, for every circled neighbourhood U of the origin θ there exists $k \in \mathbf{N}$ such that $A \subset kU$.

The following result shows that uniform boundedness and perhaps boundedness coincide in the case of Šerstnev spaces.

THEOREM 2.3. *For a subset $A \subset V$ of a Šerstnev space (V, ν, τ) the following are equivalent*:

(a) *A is uniformly bounded*;

(b) *the probabilistic radius R_A of A belongs to $\mathscr{D}^+$*.

Proof. (a) ⇒ (b) Let A be uniformly bounded and consider the neighbourhood of θ, $\mathscr{N}_\theta(1/n)$. Then, there exists $k \in \mathbf{N}$ such that, for every $p \in A$, $p = kq$ for some $q \in \mathscr{N}_\theta(1/n)$. If $x > k/n$, then, because of (Š),

$$\nu_p(x) = \nu_{kq}(x) = \nu_q\left(\frac{x}{k}\right) \geq \nu_q\left(\frac{1}{n}\right) > 1 - \frac{1}{n},$$

so that

$$R_A(x) \geq 1 - \frac{1}{n},$$

i.e., $R_A \in \mathscr{D}^+$.

(b) ⇒ (a) Let R_A belong to $\mathscr{D}^+$. Then, for every $n \in \mathbf{N}$, there exists

$x_n > 0$ such that $R_A(x_n) > 1 - 1/n$. Therefore, for every $p \in A$,

$$\nu_p(x_n) \geq R_A(x_n) > 1 - \frac{1}{n}.$$

Set $k := \min\{h \in \mathbf{N}: h/n \geq x_n\}$. Then

$$\nu_p\left(\frac{k}{n}\right) \geq \nu_p(x_n) > 1 - \frac{1}{n}.$$

Now (Š) yields $\nu_{p/k}(1/n) > 1 - 1/n$ so that p/k belongs to $\mathcal{N}_\theta(1/n)$, viz. there exists $q \in \mathcal{N}_\theta(1/n)$ such that $p = kq$; this means that $A \subset k \cdot \mathcal{N}_\theta(1/n)$. ∎

As was mentioned earlier, if τ^* is an Archimedean triangle function, then (V, ν, τ, τ^*) is a topological vector space ([2]). If the requirement that τ^* be Archimedean is dropped, then (V, ν, τ, τ^*) need not be a topological vector space and the condition characterizing uniform boundedness takes a more complicated form (see, for instance [5, p. 130]). But even if τ^* is Archimedean, the present state of our knowledge about PN spaces does not allow us to decide, one way or the other, whether a result similar to Theorem 2.3 holds.

3. LINEAR OPERATORS

THEOREM 3.1. *Let (V, ν, τ, τ^*) and $(V', \mu, \sigma, \sigma^*)$ be* PN *spaces. A linear map $T: V \to V'$ is either continuous at every point of V or at no point of V.*

We omit the proof because, except for a change of language and notation, it is the same as the usual one (see, e.g., [16]).

COROLLARY 3.1. *If $T: (V, \nu, \tau, \tau^*) \to (V', \mu, \sigma, \sigma^*)$ is linear, then T is continuous if, and only if, it is continuous at θ.*

We recall that, in general, an operator T from a metric or normed space V into another metric or normed space V' is said to be *bounded* if it maps every bounded set A of V into a bounded set TA of V'. This notion is translated in the next definition.

DEFINITION 3.1. A linear map $T: (V, \nu, \tau, \tau^*) \to (V', \mu, \sigma, \sigma^*)$ is said to be

(a) *certainly bounded* if, and only if, it maps every certainly bounded set A of the space (V, ν, τ, τ^*) into a certainly bounded set TA of the space $(V', \mu, \sigma, \sigma^*)$, i.e., if there exists $x_0 \in]0, +\infty[$ such that $\nu_p(x_0) = 1$ for every $p \in A$, then there exists $x_1 \in]0, +\infty[$ such that $\mu_{T_p}(x_1) = 1$ for every $p \in A$;

(b) *bounded* if it maps every D-bounded set of V into a D-bounded set of V', i.e., if, and only if, R_{TA} belongs to $\mathscr{D}^+$ for every D-bounded subset A of V. Equivalently, T is bounded if, and only if, it satisfies the implication,

$$\lim_{x \to +\infty} \phi_A(x) = 1 \quad \Rightarrow \quad \lim_{x \to +\infty} \phi_{TA}(x) = 1.$$

for every nonempty subset A of V;

(c) *strongly bounded* if there exists a constant $k > 0$ such that, for every $p \in V$ and for every $x > 0$,

$$\mu_{T_p}(x) \geq \nu_p\left(\frac{x}{k}\right),$$

or, equivalently, if there exists a constant $h > 0$ such that, for every $p \in V$ and for every $x > 0$,

$$\mu_{T_p}(hx) \geq \nu_p(x).$$

Notice that the definition of a strongly bounded operator in a PN space is naturally suggested by the classical definition of a bounded linear operator: an operator T from the normed space $(V, \|\cdot\|)$ into the normed space $(V', \|\cdot\|')$ is bounded if, and only if, there is a constant $k > 0$ such that, for every $x \in V$,

$$\|Tx\|' \leq k\|x\|. \tag{5}$$

For this reason these operators were the first to be studied ([8–10]) in the context of Šerstnev PN spaces.

Notice also that, as a consequence of (5), a continuous linear operator on an ordinary normed space is uniformly continuous. The same result holds in PN spaces as an immediate consequence of Corollary 3.1.

COROLLARY 3.2. *If* $T\colon (V_1, \nu, \tau_1, \tau_1^*) \to (V_2, \nu', \tau_2, \tau_2^*)$ *is linear and continuous then it is uniformly continuous.*

The identity map I between any PN space (V, ν, τ, τ^*) and itself is a strongly bounded operator with $k = 1$. Also, all linear contraction mappings, according to the definition of [12, Section 12.6], are strongly bounded. Another, nontrivial, example of a strongly bounded operator is provided in the following example.

EXAMPLE 3.1. Consider the spaces $C([0,1])$ and $C_1([0,1])$ of the functions that are, respectively, continuous and continuous together with their first derivatives on the interval $[0,1]$. They are Banach spaces with respect to the two norms $\|f\|_0 := \max_{x\in[0,1]}|f(x)|$ in $C([0,1])$ and $\|f\|_1 := \|f\|_0 + \|f'\|_0$ in $C_1([0,1])$. Choose any distribution function G from Δ^+ different from ϵ_0 and from ϵ_∞ and consider the derivative map D from $(C_1([0,1]), \|\cdot\|_1, G, M)$ into $(C([0,1]), \|\cdot\|_0, G, M)$ defined by $Df = f'$. Then, for every $x > 0$, one has $\nu'_{Df}(x) \geq \nu_f(x)$, whence D is strongly bounded.

The next result is immediate.

THEOREM 3.2. (a) *Every strongly bounded operator is also certainly bounded.*

(b) *Every strongly bounded operator is also perhaps bounded.*

However the converse need not be true.

EXAMPLE 3.2. Let $V = V' = \mathbf{R}$, $\nu_0 = \mu_0 = \epsilon_0$, while, if $p \neq 0$, then, for $x > 0$, let

$$\nu_p(x) = G\left(\frac{x}{|p|}\right), \qquad \mu_p(x) = U\left(\frac{x}{|p|}\right),$$

where

$$G(x) = \tfrac{1}{2}1_{]0,1]}(x) + 1_{]1,+\infty]}(x),$$

and U is the d.f. of the uniform law on $(0,1)$,

$$U(x) = x1_{]0,1]}(x) + 1_{]1,+\infty]}(x).$$

Consider now the identity map $I\colon (R, |\cdot|, G, M) \to (\mathbf{R}, |\cdot|, U, M)$. From Example 2.3, it is easy to prove that I is certainly bounded and bounded. But I is not strongly bounded, because for every $k > 0$ and for every $p \neq 0$, one has, for $x < |p|\min\{\frac{1}{2}, k\}$,

$$\mu_{Ip}(x) = \mu_p(x) = U\left(\frac{x}{|p|}\right) = \frac{x}{|p|} < \frac{1}{2} = G\left(\frac{x}{k|p|}\right) = \nu_p\left(\frac{x}{k}\right).$$

Moreover, the notions of certainly bounded and bounded operators do not imply each other.

EXAMPLE 3.3. Let $(V, \|\cdot\|)$ be a normed space. Let G and G' be in $\Delta^+ - \{\epsilon_0, \epsilon_\infty\}$ and consider the identity map I between $(V, \|\cdot\|, G, M)$ and $(V, \|\cdot\|, G', M)$. Now, with reference to Example 2.3

(a) if $G(x_0) = 1$ for some $x \in]0, +\infty[$ while $G'(x) < 1$ for every $x \in]0, +\infty[$, but $l^- G'(+\infty) = 1$, then I is bounded but not certainly bounded;

(b) if $G(x) < 1$ for every $x \in]0, +\infty[$, if $l^- G(+\infty) = 1$ and if $l^- G'(+\infty) < 1$, then I is certainly bounded but not bounded.

In the classical theory, condition (5) is necessary as well as sufficient for the continuity of a linear operator. In a PN space its analogue, namely, strong boundedness, is only sufficient as proved in the following theorem but not necessary as shown in Example 3.4.

THEOREM 3.3. *Every strongly bounded linear operator T is continuous with respect to the strong topologies in (V, ν, τ, τ^*) and $(V', \mu, \sigma, \sigma^*)$, respectively.*

Proof. Because of Corollary 3.1, it suffices to verify that T is continuous at θ. Let $\mathcal{N}'_{\theta'}(t)$, with $t > 0$, be an arbitrary neighbourhood of θ'. Take $s \leq \min\{t, t/k\}$; then, for every $p \in \mathcal{N}_\theta(s)$, one has

$$\mu_{T_p}(t) \geq \nu_p\left(\frac{t}{k}\right) \geq \nu_p(s) > 1 - s \geq 1 - t,$$

viz. $Tp \in \mathcal{N}'_{\theta'}(t)$; in other words, T is continuous. ∎

EXAMPLE 3.4. Consider again the simple spaces of Example 3.2, and the same linear map I between them. The map I is continuous. It is easy to check that, for every $t \in]0, 1[$, the neighbourhood $\mathcal{N}'_0(t)$ coincides with the set $\{p \in \mathbf{R}: |p| < t/(1 - t)\}$. On taking $s \leq \min\{t/(1 - t), \frac{1}{2}\}$, one has $\mathcal{N}_0(s) = \{p \in \mathbf{R}: |p| < s\}$. Thus, if $p \in \mathcal{N}_0(s)$, $|p| < s \leq t/(1 - t)$, so that $p \in \mathcal{N}'_0(t)$.

The following examples together with Example 3.3 prove that, in the class of linear operators, no two of the concepts of certain boundedness, boundedness and continuity imply each other.

EXAMPLE 3.5. (A continuous linear operator that is neither certainly bounded nor bounded). Let $(V, \|\cdot\|)$ be a normed space and let F and G be distribution functions in $\mathcal{D}^+$ with $F(x_0) = 1$ for some $x_0 \in]0, +\infty[$. Consider the identity map I from the equilateral space $(V, F, \mathbf{M})$ into the simple space $(V, \|\cdot\|, G, M)$. Let A be an unbounded set of $(V, \|\cdot\|)$. Then A is certainly bounded in $(V, F, \mathbf{M})$. But A is not D-bounded in $(V, \|\cdot\|, G, M)$. Therefore, I is neither certainly bounded nor bounded.

On the other hand, because the strong topology in an equilateral PM space is discrete (see [12, Section 12.3]), and the strong topology in $(V, \|\cdot\|, G, M)$ is the usual one in $(V, \|\cdot\|)$ because G belongs to $\mathcal{D}^+$, the identity I is continuous.

EXAMPLE 3.6. In the previous example, I^{-1} is both certainly bounded and bounded without being continuous, as is immediately checked.

THEOREM 3.4. *Let (V, ν, τ, τ^*) and $(V', \mu, \sigma, \sigma^*)$ be two* PN *spaces and let $T: (V, \nu, \tau, \tau^*) \to (V', \mu, \sigma, \sigma^*)$ be a linear map. If there exists a constant $h > 0$ such that, for every $x > 0$ and for every $p \in V$,*

$$\nu_p(x) \geq \mu_{Tp}(hx), \tag{6}$$

then T has a linear inverse T^{-1} defined on TV and T^{-1} is strongly bounded.

Proof. Take $Tp = \theta'$ in (6); then, for every $x > 0$, $\nu_p(x) \geq 1$, i.e., $\nu_p(x) = 1$, so that $p = \theta$. This yields the existence and the linearity of T^{-1}. Now (6) can be written in the form,

$$\nu_{T^{-1}q}(x) \geq \mu_p(hx),$$

where q is any element of TV. Therefore T^{-1} is strongly bounded. ∎

In particular, under the assumptions of the last theorem, the operator T^{-1} is continuous, bounded and certainly bounded. Moreover, it is not hard to check that T maps certainly unbounded sets of (V, ν, τ, τ^*) into certainly unbounded sets of $(V', \mu, \sigma, \sigma^*)$ and T maps D-unbounded sets of (V, ν, τ, τ^*) into D-unbounded sets of $(V', \mu, \sigma, \sigma^*)$.

The proofs of the next two results follow easily from what we have shown.

COROLLARY 3.2. *Let $T: (V, \nu, \tau, \tau^*) \to (V', \mu, \sigma, \sigma^*)$ be a linear onto map with an inverse T^{-1}. If both T and T^{-1} are strongly bounded, then T is a homeomorphism between the* PN *spaces (V, ν, τ, τ^*) and $(V', \mu, \sigma, \sigma^*)$.*

The identity I of Example 3.3(a) is a homeomorphism and its inverse is not strongly bounded; therefore the converse of Theorem 3.4 does not hold in general. The same example shows that also the converse of the next corollary may not hold.

COROLLARY 3.3. *Let (V, ν, τ, τ^*) and $(V, \mu, \sigma, \sigma^*)$ be two* PN *spaces having the same support V. If the identity and its inverse are both strongly bounded, then the strong topologies of the two* PN *spaces are equivalent.*

ACKNOWLEDGMENTS

It is a pleasure to thank C. Alsina, B. Schweizer, and A. Sklar for sending us reference [2] even before submission, and to thank B. Schweizer for conversations about the subject matter of this paper. The research of the first two authors was partially supported by grants from the Junta de Andalucía and the University of Almería.

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