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$\mathcal{D}$-bounded sets in probabilistic normed spaces and in their products

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RIASSUNTO: *Si assegna una condizione sufficiente (ma non necessaria) affinché una famiglia di sottoinsiemi D-limitati di uno spazio probabilistico normato abbia la struttura di spazio lineare con operazioni di addizione e di moltiplicazione opportune. Si introduce poi la nozione nuova di τ-prodotto e si da una condizione sufficiente per la D-limitatezza del τ-prodotto.*

ABSTRACT: *The paper is divided into three sections, Section 1 being the introduction. In Section 2, a sufficient, but not necessary, condition is given for the family of $\mathcal{D}$-bounded subsets in a Probabilistic Normed space (briefly, PN space) to be a linear space when it is endowed with suitably defined operations of addition and multiplication by a scalar. Finally, in Section 3, τ-products of PN spaces are introduced and a sufficient condition is given for $\mathcal{D}$-boundedness in τ-products.*

1 – Introduction

The new concept of boundedness in PN spaces is motivated by the hope of extending the applications to Statistics that motivated Šerstnev in his original introduction of PN spaces. It may also turn out to be relevant to the study of "physical quantities", whenever one or more of

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them are not defined sharply. PN spaces may provide us with a set of tools suitable to study the geometry of nuclear physics, for instance.

By $\mathcal{D}$-bounded we shall mean "distributionally bounded" in the sense of [7].

Definition 1. A Probabilistic Normed Space, briefly a PN space, is a quadruple (V, ν, τ, τ^*) in which V is a linear space, τ and τ^* are continuous triangle functions with $\tau \leq \tau^*$ and ν, the probabilistic norm, is a map $\nu : V \to \Delta^+$ such that

(N1) $\nu_p = \epsilon_0$ if, and only if, $p = \theta$, θ being the null vector in V;
(N2) $\nu_{-p} = \nu_p$ for every $p \in V$;
(N3) $\nu_{p+q} \geq \tau(\nu_p, \nu_q)$ for all $p, q \in V$;
(N4) $\nu_p \leq \tau^*(\nu_{\alpha p}, \nu_{(1-\alpha)p})$ for every $\alpha \in [0, 1]$ and for every $p \in V$.

If, instead of (N1), we only have $\nu_\theta = \epsilon_0$, then we shall speak of a *Probabilistic Pseudo Normed Space*, briefly a PPN space. If the inequality (N4) is replaced by the equality $\nu_p = \tau_M(\nu_{\alpha p}, \nu_{(1-\alpha)p})$, then the PN space is called a *Šerstnev space* and, as a consequence, a condition stronger than (N2) holds, namely

$$\forall \lambda \neq 0 \; \forall p \in V \qquad \nu_{\lambda p} = \nu_p\Big(\frac{j}{|\lambda|}\Big)$$

Here j is the identity map on $\mathbb{R}$, i.e. $j(x) := x$ $(x \in \mathbb{R})$.

We recall that a set A in a PN space (V, ν, τ, τ^*) is said to be *bounded* if its probabilistic radius R_A belongs to $\mathcal{D}^+$, where

$$R_A = \begin{cases} l^- \inf\{\nu_p(x) : p \in A\}, & \text{if } x \in [0, +\infty[; \\ 1, & \text{if } x = +\infty\,. \end{cases}$$

Definition 2. By setting $F \leq G$ whenever $F(x) \leq G(x)$ for every $x \in \mathbb{R}^+$ and $F, G \in \Delta^+$, one introduces a natural ordering in Δ^+.

Definition 3. Let τ_1, τ_2 be two triangle functions. Then τ_1 dominates τ_2, and we write $\tau_1 \gg \tau_2$, if for all $F_1, F_2, G_1, G_2 \in \Delta^+$, one has

$$\tau_1(\tau_2(F_1, G_1), \tau_2(F_2, G_2)) \geq \tau_2(\tau_1(F_1, F_2), \tau_1(G_1, G_2))\,.$$

Notice that since τ_1 is associative one has $\tau_1 \gg \tau_1$, so that "dominates" is reflexive but its transitivity is still an open question.

2 – The structure of the set of all $\mathcal{D}$-bounded subsets in a PN space

If A is a bounded set then αA need not be bounded set, but this will hold under suitable conditions, as is shown in the next theorem.

THEOREM 1. *Let (V, ν, τ, τ^*) and A be a PN space and a nonempty $\mathcal{D}$-bounded subset respectively. The set $\alpha A := \{\alpha p : p \in A\}$ is also $\mathcal{D}$-bounded for every fixed $\alpha \in \mathbb{R}$ if $\mathcal{D}^+$ is a closed set under τ, i.e. $\tau(\mathcal{D}^+ \times \mathcal{D}^+) \subseteq \mathcal{D}^+$.*

PROOF. Because of (N2), it suffices to consider the case $\alpha \geq 0$. If either $\alpha = 0$ or $\alpha = 1$, then αA is $\mathcal{D}$-bounded. If $\alpha \in (0, 1)$, then, for every $p \in A$ one has (see [5])

$$\nu_{\alpha p} \geq \nu_p \,.$$

Since $\nu_p \geq R_A$, it follows that

$$\nu_{\alpha p} \geq R_A \,.$$

If $\alpha > 1$, let $k = [\alpha] + 1$. We know that (see [2]) that

$$\nu_{\alpha p} \geq \nu_{kp} \,.$$

Now, let us denote by G_α the function $\tau^{k-1}(R_A, \ldots, R_A)$; one has by induction

$$\begin{aligned}\nu_{kp}(x) &\geq \tau(\nu_{(k-1)p}, \nu_p)(x) \geq \tau(\tau(\nu_{(k-2)p}, \nu_p), \nu_p)(x) \geq \cdots \geq \\ &\geq \tau^{k-1}(\nu_p, \ldots, \nu_p)(x) \geq \tau^{k-1}(R_A, \ldots, R_A)(x)\end{aligned}$$

and hence $\nu_{\alpha p} \geq G_\alpha$. Finally, one can say that

$$R_{\alpha A} \geq G_\alpha$$

for every $p \in A$ and for every fixed real scalar α, and since G_α belongs to $\mathcal{D}^+$, then αA is $\mathcal{D}$-bounded. $\square$

EXAMPLE 1. In particular, if A is a $\mathcal{D}$-bounded set in the Menger PN space $(V, \nu, \tau_T, \tau_{T^*})$, then the set αA is $\mathcal{D}$-bounded for every previously fixed $\alpha \in \mathbb{R}$.

EXAMPLE 2. Let us recall that among the triangle functions one has the function defined via

$$\mathbf{T}(F, G)(x) := T(F(x), G(x)),$$

for all F, G in Δ^+ and for all left-continuous t-norm T.

Now, if (V, ν, τ, τ^*) is a PN space with $\tau \equiv \mathbf{T}$, then one has also the same result because of

$$\mathbf{T}(\mathcal{D}^+ \times \mathcal{D}^+) \subseteq \mathcal{D}^+.$$

Analogously, in the following theorem one shows the condition under which the sum of two bounded sets can also be bounded.

THEOREM 2. *Let (V, ν, τ, τ^*) and A, B be a PN space and two nonempty $\mathcal{D}$-bounded subsets of V respectively. Then the set given by*

$$A + B := \{p + q : p \in A, q \in B\}$$

is $\mathcal{D}$-bounded if $\mathcal{D}^+$ is a closed set under τ.

PROOF. By (N3) one has, for all $p \in A$, $q \in B$,

$$\nu_{p+q} \geq \tau(\nu_p, \nu_q) \geq \tau(\nu_p, R_B) \geq \tau(R_A, R_B)$$

and hence

$$R_{A+B} \geq \tau(R_A, R_B),$$

and the assertion follows from the fact that $\mathcal{D}^+$ is closed under τ. □

Under the same conditions for the sets A, B, if $(V, \nu, \tau_T, \tau_{T^*})$ is Menger PN space for some t-norm T, and its t-conorm T^*, then the set $A + B$ is in $\mathcal{D}^+$.

We denote the set of all $\mathcal{D}$-bounded sets in a PN space $(V, \nu, \tau_T, \tau_{T^*})$ by $\mathcal{P}_{\mathcal{D}^+}(V)$.

THEOREM 3. *Let (V, ν, τ, τ^*) be a PN space. The triple $(\mathcal{P}_{\mathcal{D}^+}(V), +, \cdot)$ is a real linear space if $\tau(\mathcal{D}^+ \times \mathcal{D}^+) \subseteq \mathcal{D}^+$.*

PROOF. It suffices to apply the previous two theorems. ∎

The following results will be needed later in Theorem 6. In both of them it is not assumed that $\tau(\epsilon_c, \epsilon_d)$ is in $\mathcal{D}^+$; in fact and in general it is not in $\mathcal{D}^+$.

THEOREM 4. *The quadruple $(V, \nu, \tau, \mathbf{M})$ where V is a normed linear space, $\mathbf{M}$ the maximal triangle function, and ν, the probabilistic norm, is a map $\nu : V \to \Delta^+$ such that $\nu_\theta = \epsilon_0$, $\nu_p := \epsilon_{\frac{a+\|p\|}{a}}$ if $p \neq \theta$, $(a > 0)$, and $\tau(\epsilon_c, \epsilon_d) \leq \epsilon_{c+d}$, $(c > 0, d > 0)$ is a PN space that is neither a topological vector space (briefly, TVS) nor a Šerstnev space.*

PROOF. (N1) and (N2) are obvious.

(N3)
$$\begin{aligned}\nu_{p+q} &= \epsilon_{\frac{a+\|p+q\|}{a}} \geq \epsilon_{\frac{a+\|p\|+\|q\|}{a}} \geq \\ &\geq \epsilon_{\frac{a+\|p\|}{a}+\frac{a+\|q\|}{a}} \geq \tau(\epsilon_{\frac{a+\|p\|}{a}}, \epsilon_{\frac{a+\|q\|}{a}}) = \tau(\nu_p, \nu_q)\,.\end{aligned}$$

(N4) For every $\alpha \in]0,1[$,

$$\begin{aligned}\mathrm{Min}(\epsilon_{\frac{a+\alpha\|p\|}{a}}(x), \epsilon_{\frac{a+(1-\alpha)\|p\|}{a}}(x)) \geq \\ \geq \epsilon_{\frac{a+\|p\|}{a}}(x) = \nu_p(x)\,.\end{aligned}$$

Besides, the PN space in this theorem is not a TVS because, if α_n is a sequence in $\mathbb{R}$, with $\lim_{n\to+\infty} \alpha_n = 0$, then one has

$$\lim_{n\to+\infty} \nu_{\alpha_n p} = \lim_{n\to+\infty} \epsilon_{\frac{a+\alpha_n\|p\|}{a}} = \epsilon_1 \neq \epsilon_0\,.$$

Furthermore,

$$\begin{aligned}\tau_M(\nu_{\alpha p}, \nu_{(1-\alpha)p}) &= \tau_M(\epsilon_{\frac{a+|\alpha|\|p\|}{a}}, \epsilon_{\frac{a+(1-\alpha)\|p\|}{a}}) = \\ &= \epsilon_{\frac{2a+\|p\|}{a}} \leq \epsilon_{\frac{a+\|p\|}{a}} = \nu_p\,.\end{aligned}$$

Now, for every $x > 0$ such that $x \in]1 + \frac{\|p\|}{a}, 2 + \frac{\|p\|}{a}[$ one has

$$\tau_M(\nu_{\alpha p}, \nu_{(1-\alpha)p})(x) < \nu_p(x)$$

so that the space considered is not a Šerstnev space. ∎

THEOREM 5. *Let τ be a triangle function such that $\tau(\epsilon_c, \epsilon_d) \leq \epsilon_{c+d}$, $(c > 0, d > 0)$. The quadruple $(V, \nu, \tau, \mathbf{M})$ where V is a normed linear space, $\mathbf{M}$ the maximal triangle function, and ν, the probabilistic norm, is a map $\nu : V \to \Delta^+$ defined by*

$$\nu_p := \epsilon_{\frac{\|p\|}{a+\|p\|}}$$

for every $p \in V$ and for a fixed $a > 0$, with $\nu_p(+\infty) = 1$, is a PN space, a TVS but not a Šerstnev space.

PROOF. (N1) and (N2) are obvious.
(N3) For every $p, q \in V$ since the function $x \mapsto \frac{x}{a+x}$, $x > 0$ is nondecreasing, then one has

$$\nu_{p+q} = \epsilon_{\frac{\|p+q\|}{a+\|p+q\|}} \geq \epsilon_{\frac{\|p\|+\|q\|}{a+\|p\|+\|q\|}} \geq$$
$$\geq \tau(\epsilon_{\frac{\|p\|}{a+\|p\|+\|q\|}}, \epsilon_{\frac{\|q\|}{a+\|p\|+\|q\|}}) \geq \tau\left(\epsilon_{\frac{\|p\|}{a+\|p\|}}, \epsilon_{\frac{\|q\|}{a+\|q\|}}\right) = \tau(\nu_p, \nu_q)\,.$$

(N4) If $\alpha \in]0,1[$, then

$$\mathrm{Min}(\epsilon_{\frac{\alpha\|p\|}{a+\alpha\|p\|}}(x), \epsilon_{\frac{(1-\alpha)\|p\|}{a+(1-\alpha)\|p\|}}(x)) \geq$$
$$\geq \epsilon_{\frac{\|p\|}{a+\|p\|}}(x) = \nu_p(x)\,.$$

Now, if $\{\lambda_n\} \subset \mathbb{R}^+$ is such that $\lim_{n\to+\infty}(\lambda_n) = 0$, then

$$\lim_{n\to+\infty} \nu_{\lambda_n p} = \lim_{n\to+\infty} \epsilon_{\frac{\lambda_n\|p\|}{a+\lambda_n\|p\|}} = \epsilon_0$$

and hence the PN space of this theorem is a TVS.

Furthermore, for all $p \in V$ and for all $\lambda \in]0,1[$,

$$\tau_M(\nu_{\lambda p}, \nu_{(1-\lambda)p})(x) =$$
$$= \sup\{\epsilon_{(\frac{\lambda\|p\|}{a+\lambda\|p\|})}(x-u) \wedge \epsilon_{(\frac{(1-\lambda)\|p\|}{a+(a+(1-\lambda)\|p\|)})}(u) : u > 0\}\,.$$

Let us assume $u > \frac{(1-\lambda)\|p\|}{a+(1-\lambda)\|p\|}$; since $\tau_M(\epsilon_a, \epsilon_b) = \epsilon_{a+b}$, then

$$\tau_M(\nu_{\lambda p}, \nu_{(1-\lambda)p}) = \epsilon_{\frac{a\|p\|+2\lambda(1-\lambda)\|p\|^2}{a^2+a\|p\|+\lambda(1-\lambda)\|p\|^2}}\,.$$

Now, one has

$$\tau_M(\nu_{\lambda p}, \nu_{(1-\lambda)p}) \leq \nu_p \,.$$

If

$$x \in \left]\frac{\|p\|}{a+\|p\|}, \frac{a\|p\| + 2\lambda(1-\lambda)\|p\|^2}{a^2 + a\|p\| + \lambda(1-\lambda)\|p\|^2}\right[,$$

then $\nu_p(x) = 1 > \tau_M(\nu_{\lambda p}, \nu - (1-\lambda)p)(x)$ so that $(V, \nu, \tau, \mathbf{M})$ is not a Šerstnev space. □

The following example shows that it is not necessary for $\mathcal{D}^+$ to be closed under τ for the set $(\mathcal{P}_{\mathcal{D}}(V), +, \cdot)$ to be a linear space.

Example 3. Let (V, ν, τ, τ^*) be the same PN space as in Theorem 4 (or 5). Then the following statements hold:

(1) The bounded subsets in V with regard to the classical norm of V coincide with the certainly bounded subsets in V for the probabilistic norm, ν;

(2) In both cases — Theorems 4 and 5 — $(\mathcal{P}_{\mathcal{D}}(V), +, \cdot)$ is a linear space.

Example 4. If we take the probabilistic norm as in Theorem 5 and assume $V = \mathbb{R}$ and that A is a nonempty classically bounded set in $\mathbb{R}$ such that $\|p\| \leq s$, this is, A is bounded with respect to the classical norm, then $R_A = \epsilon_{(\frac{s}{a+s})} \in \mathcal{D}^+$, and hence, A is $\mathcal{D}$-bounded for the probabilistic norm. Now, if $A = \mathbb{R}$ one has

$$R_{\mathbb{R}} = l^- \inf\{\epsilon_{(\frac{\|p\|}{a+\|p\|})}\} = \epsilon_1 \in \mathcal{D}^+ \,,$$

and consequently, $\mathbb{R}$ is $\mathcal{D}$-bounded.

In a similar way, the sets in $\mathbb{R}$ given by $[0, +\infty]$, $\{2n : n \in \mathcal{Z}\}$ or any finite interval are $\mathcal{D}$-bounded.

We must say, however, that depending on the probabilistic norm, we can have $\mathbb{R}$ as a $\mathcal{D}$-unbounded set. Taking the probabilistic norm as in Theorem 4, one has

$$R_{\mathbb{R}} = l^- \inf\{\epsilon_{(\frac{a+\|p\|}{a})}\} = \epsilon_{+\infty} \notin \mathcal{D}^+ \,,$$

from what it follows that $\mathbb{R}$ is not $\mathcal{D}$-bounded.

Theorem 6. *Let (V, ν, τ, τ^*) be a PN space. Then, the triple*

$$(\mathcal{P}_{\mathcal{D}}(V), \cup, \cap)$$

is a net if $\mathcal{D}^+$ is closed under τ.

Proof. It suffices to consider that $R_{A\cup B} \geq \tau(R_A, R_B)$ (see [6], p. 188). Hence, $R_{A\cup B}$ is in $\mathcal{D}^+$. If moreover, we recall that $R_{A\cap B} \geq R_A$ (or $R_{A\cap B} \geq R_B$), one has that $A \cap B$ is a bounded set, and the proof is ended. □

Lemma 1. *The finite intersection of subsets of V is $\mathcal{D}$-bounded if, and only if, at least one is $\mathcal{D}$-bounded.*

Lemma 2. *For all equilateral PN space $(V, F, \mathbf{M})$, with $F \in \Delta^+$ the following equality holds*

$$\text{card}(\mathcal{P}_{\mathcal{D}}(V)) = \text{card}(\mathcal{P}(V)).$$

Lemma 3. *Let (V, ν, τ, τ^*) be a general PN space. Then the inequality*

$$\text{card}(\mathcal{P}_{\mathcal{D}}(V)) \leq \text{card}(\mathcal{P}(V))$$

holds.

3 – The τ-product of two PN space and $\mathcal{D}$-boundedness defined in finite τ-products

Definition 4. Let $(V_1, \nu_1, \tau, \tau^*)$ and $(V_2, \nu_2, \tau, \tau^*)$ be two PN spaces under the same triangle functions τ and τ^*. Let τ_1 be a triangle function. The τ_1-product of the two PN spaces is the quadruple

$$(V_1 \times V_2, \nu_1\tau_1\nu_2, \tau, \tau^*)$$

where

$$\nu_1\tau_1\nu_2 : V_1 \times V_2 \longrightarrow \Delta^+$$

is a probabilistic seminorm given by

$$(\nu_1\tau_1\nu_2)(p,q) := \tau_1(\nu_1(p),\nu_2(q))$$

for all $(p,q) \in V_1 \times V_2$.

THEOREM 7. *Let (V_1,ν_1,τ,τ^*), (V_2,ν_2,τ,τ^*) and τ_1 be two PN spaces under the same triangle functions and a triangle function respectively. Assume that $\tau^* \gg \tau_1$ and $\tau_1 \gg \tau$, then the τ_1-product $(V_1 \times V_2, \nu_1\tau_1\nu_2)$ is a PN space under τ and τ^*.*

PROOF. We are going to check whether the above probabilistic seminorms satisfy the four axioms of a PN space.

(N1) Let θ and θ' the null vectors of V_1 and V_2, respectively; then

$$(\nu_1\tau_1\nu_2)(\theta,\theta') = \tau_1(\nu_1(\theta),\nu_2(\theta')) = \tau_1(\epsilon_0,\epsilon_0) = \epsilon_0,$$

and $(p,q) \neq (\theta,\theta') \Leftrightarrow p \neq \theta$ or $q \neq \theta' \Leftrightarrow (\nu_1(p),\nu_2(q)) \neq (\epsilon_0,\epsilon_0) \Leftrightarrow (\nu_1\tau_1\nu_2)(p,q) \neq \epsilon_0$.

(N2) is obvious.

$$\text{(N3)}\qquad \begin{aligned} (\nu_1\tau_1\nu_2)((p,q)+(p',q')) &= (\nu_1\tau_1\nu_2)(p+p',q+q') = \\ &= \tau_1(\nu_1(p+p'),\nu_2(q+q')) \geq \\ &\geq \tau_1(\tau(\nu_1(p),\nu_1(p')),\tau(\nu_2(q),\nu_2(q'))) \geq \\ &\geq \tau(\tau_1(\nu_1(p),\nu_2(q)),\tau_1(\nu_1(p'),\nu_2(q'))) = \\ &= \tau((\nu_1\tau_1\nu_2)(p,q),(\nu_1\tau_1\nu_2)(p',q')). \end{aligned}$$

$$\text{(N4)}\qquad \begin{aligned} &(\nu_1\tau_1\nu_2)(p,q) = (\tau_1(\nu_1(p),\nu_2(q)) \leq \\ &\leq \tau_1[\tau^*(\nu_1(\alpha p),\nu_1((1-\alpha)p)),\tau^*(\nu_2(\alpha q),\nu_2((1-\alpha)q))] \leq \\ &\leq \tau^*[\tau_1(\nu_1(\alpha p),\nu_2(\alpha q)),\tau_1(\nu_1((1-\alpha)p),\nu_2((1-\alpha)q))] = \\ &= \tau^*[(\nu_1\tau_1\nu_2)(\alpha(p,q)),(\nu_1\tau_1\nu_2)((1-\alpha)(p,q))], \end{aligned}$$

for every $\alpha \in [0,1]$. □

Corollary 1. *The* $\mathbf{T}$*-product* $(V_1 \times V_2, \nu_1 \mathbf{T} \nu_2)$ *of the two PN spaces* $(V_1, \nu_1, \tau_T, \mathbf{M})$ *and* $(V_2, \nu_2, \tau_T, \mathbf{M})$ *is a PN space under* τ_T *and* $\mathbf{M}$.

Proof. It is easy from the previous Theorem and Lemma 12.7.4 in [12]. □

We wonder whether the τ_1-product of two PN spaces characterized in Theorem 1 coincides with the τ_1-product of corresponding PM spaces. The following theorem gives an answer in the affirmative to this question.

Corollary 2. *Let* $(V_1, F, \mathbf{M})$ *and* $(V_2, G, \mathbf{M})$ *be two equilateral PN spaces with d.d.f.'s* F *and* G *respectively. Then, their* $\mathbf{M}$*-product is an equilateral PN space with a d.d.f. given by* $\mathbf{M}(F, G)$*. In particular, if* $F \equiv G$*, the* $\mathbf{M}$*-product is an equilateral PN space with the same d.d.f.* F.

Proof. It suffice to notice that

$$(\nu_1 \mathbf{M} \nu_2)(p, q) = \mathbf{M}(\nu_1(p), \nu_2(q)) = \mathbf{M}(F, G)(p, q)\,. \qquad \square$$

Theorem 8. *Let* $(V_1 \times V_2, \nu_1 \tau_1 \nu_2, \tau, \tau^*)$ *be the* τ_1*-product of the PN spaces* $(V_1, \nu_1, \tau, \tau^*)$ *and* $(V_2, \nu_2, \tau, \tau^*)$*, where* $\tau^* \gg \tau_1$ *and* $\tau_1 \gg \tau$*. Let* $\mathcal{P}_{\mathcal{D}}(V_1 \times V_2)$ *denote the set of all* $\mathcal{D}$*-bounded subsets in* $V_1 \times V_2$*. Then the following statements hold:*

(a) *If* A *and* B *are* $\mathcal{D}$*-bounded subsets in the PN spaces* $(V_1, \nu_1, \tau, \tau^*)$ *and* $(V_2, \nu_2, \tau, \tau^*)$ *respectively, then their cartesian product* $A \times B$ *is a* $\mathcal{D}$*-bounded subset of the* τ_1*-product* $(V_1 \times V_2, \nu_1 \tau_1 \nu_2, \tau, \tau^*)$*;*

(a) *The triple* $\mathcal{P}_{\mathcal{D}}(V_1 \times V_2, +, \cdot)$ *is a real linear space if* $\mathcal{D}^+$ *is closed under both* τ *and* τ_1.

Proof. Let A_1, A_2 and B_1, B_2 be $\mathcal{D}$-bounded subsets in V_1 and V_2 respectively. It follows from the monotonicity of τ_1 that

$$\begin{aligned}&\inf\{(\nu_1 \tau_1 \nu_2)(p, q) : (p, q) \in A_1 \times B_1\} = \\ &\quad = \inf\{\tau_1(\nu_1(p), \nu_2(q)) : (p, q) \in A_1 \times B_1\} = \tau_1(R_{A_1}, R_{B_1})\,.\end{aligned}$$

Therefore $R_{A_1 \times B_1} = \tau_1(R_{A_1}, R_{B_1}) \in \mathcal{D}^+$, and by theorem 1 we know that $\alpha(A_1 \times B_1)$ is $\mathcal{D}$-bounded for all fixed $\alpha \in \mathbb{R}$.

Analogously, $\tau(R_{A_1}, R_{A_2})$ and $\tau(R_{B_1}, R_{B_2})$ are in $\mathcal{D}^+$. Now,

$$(\nu_1\tau_1\nu_2)((p_1, q_1) + (p_2, q_2)) = (\nu_1\tau_1\nu_2)(p_1 + p_2, q_1 + q_2) =$$
$$= \tau_1(\nu_1(p_1 + p_2), \nu_2(q_1 + q_2)) \geq \tau_1(\tau(R_{A_1}, R_{A_2}), \tau(R_{B_1}, R_{B_2}))$$

is in $\mathcal{D}^+$ for every $(p_1, q_1), (p_2, q_2)$ in $A_1 \times B_1$, $A_2 \times B_2$ respectively. □

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