The semiclassical–Sobolev orthogonal polynomials: a general approach

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Abstract

We say that the polynomial sequence $(Q_n^{(\lambda)})$ is a semiclassical Sobolev polynomial sequence when it is orthogonal with respect to the inner product

$$\langle p, r\rangle_S = \langle \mathbf{u}, p\, r\rangle + \lambda \langle \mathbf{u}, \mathscr{D}p\, \mathscr{D}r\rangle ,$$

where $\mathbf{u}$ is a semiclassical linear functional, $\mathscr{D}$ is the differential, the difference or the q–difference operator, and λ is a positive constant.

In this paper we get algebraic and differential/difference properties for such polynomials as well as algebraic relations between them and the polynomial sequence orthogonal with respect to the semiclassical functional $\mathbf{u}$.

The main goal of this article is to give a general approach to the study of the polynomials orthogonal with respect to the above nonstandard inner product regardless of the type of operator $\mathscr{D}$ considered. Finally, we illustrate our results by applying them to some known families of Sobolev orthogonal polynomials as well as to some new ones introduced in this paper for the first time.

Key words: Orthogonal polynomials, Sobolev orthogonal polynomials, semiclassical orthogonal polynomials, operator theory, nonstandard inner product
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1 Introduction

In the pioneering work [9], D.C. Lewis introduced the inner products involving derivatives in order to obtain the least squares approximation of a function and its derivatives. Since

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that article, a large amount of papers have appeared on orthogonal polynomials with respect to inner products involving derivatives (and later differences or q–differences), the so–called Sobolev (or Δ–Sobolev or q–Sobolev) orthogonal polynomials.

We can see the evolution of this theory through different surveys from 1993 to 2006 (ordered by year: [10], [18], [15], [16], and [11]). In the most recent surveys some directions for the future are proposed.

The most studied cases correspond to the orthogonal polynomials with respect to an inner product of the form

$$\langle p, r\rangle_S = \langle \mathbf{u}_0, p\, r\rangle + \lambda\langle \mathbf{u}_1, \mathscr{D}p\, \mathscr{D}r\rangle, \qquad p, r \in \mathbb{P}, \tag{1}$$

where $\mathbf{u}_0$ and $\mathbf{u}_1$ are definite positive linear functionals, and $\mathscr{D}$ is the differential operator. Later, in [1] the discrete case was considered where $\mathscr{D} = \Delta$ was taken as the forward finite difference operator, and in [2] the q–difference operator $\mathscr{D} = \mathscr{D}_q$ was considered.

As we can observe in the surveys mentioned above, and in the references therein, all the results have been obtained by considering a fixed operator, that is, the differential operator in the continuous case, the forward operator in the discrete case and the q–difference operator in the q–Hahn case. Notice that a great part of the results have been obtained from knowing an explicit representation for the linear functionals $\mathbf{u}_0$ and $\mathbf{u}_1$, for example, when $(\mathbf{u}_0, \mathbf{u}_1)$ is a coherent (or symmetrically coherent) pair of functionals. The coherence is a case of special relevance from the numerical point of view (see [6]), although according to the classification of the coherent pairs (see [1], [2], and [19] for each case), the coherent functionals are close the classical ones; the complete classification of the coherent pairs in the continuous, discrete and q–Hahn cases can be found in [19], [1] and [2], respectively. Furthermore, a lot of papers consider the case $\mathbf{u} := \mathbf{u}_0 = \mathbf{u}_1$ when this functional is explicitly known, for example, if $\mathbf{u}$ is a very classical functional (Jacobi, Laguerre, Meixner, etc.) or if $\mathbf{u}$ is semiclassical (Freud–type). For the corresponding Sobolev orthogonal polynomials algebraic, differential/difference and asymptotic properties are obtained as well as a relation with the standard polynomials orthogonal with respect to $\mathbf{u}$.

In this paper we do not work in the framework of coherence (either Δ–coherence or q–coherence) in the sense of coherence worked with in [19] ([1] or [2]), although some families of coherent pairs of this type are included in our approach. In fact, we consider the case $\mathbf{u} := \mathbf{u}_0 = \mathbf{u}_1$ in (1) where $\mathbf{u}$ is a semiclassical linear functional and, therefore, our approach is not related to the coherence case.

We think that the theory of Sobolev orthogonal polynomials is a big puzzle in which many pieces have been connected during the last few decades but it lacks a general approach. Taking this into account, our main objective is to give a unified approach for the theory of Sobolev orthogonal polynomials regardless of the operator considered. More concretely, we consider a semiclassical linear functional $\mathbf{u}$ and the nonstandard inner product

$$\langle p, r\rangle_S = \langle \mathbf{u}, p\, r\rangle + \lambda\langle \mathbf{u}, \mathscr{D}p\, \mathscr{D}r\rangle, \qquad p, r \in \mathbb{P}, \tag{2}$$

and we obtain differential/difference properties of the orthogonal polynomials with respect to (2) in a unified way independently of the type of operator $\mathscr{D}$, i.e., the results hold for the continuous, discrete, and q–Hahn cases. We also find some relations between the semiclassical Sobolev orthogonal polynomials with respect to the inner product (2) and the semiclassical orthogonal polynomials with respect to $\mathbf{u}$. To do this, we use recent results, obtained in [4,7].

In the last section of the paper we illustrate our results, applying them to some known families of Sobolev orthogonal polynomials (Jacobi–Sobolev and Δ–Meixner–Sobolev ones) as well as to some new ones introduced in this paper for the first time such as q–Freud–Sobolev ones and another family related to a 1–singular semiclassical functional.

In conclusion, we show that it is possible to unify the continuous, the discrete and the q–semiclassical Sobolev orthogonal polynomials by using a suitable notation, which we believe could be used in the future for further research.

2 Basic definitions and notation. Semiclassical orthogonal polynomials

As we said in the introduction, the Sobolev polynomials, the Δ–Sobolev polynomials and the q–Sobolev polynomials have been considered in the literature with different approaches. Our main idea is to establish the main algebraic properties of the polynomials which are orthogonal with respect to the inner product $\langle\cdot,\cdot\rangle_S$ defined in (2) in a general way where the linear functional $\mathbf{u}$ satisfies a distributional equation with polynomial coefficients:

$$\mathscr{D}(\phi\mathbf{u}) = \psi\mathbf{u}, \tag{3}$$

where $\mathscr{D}$ is the differential or the difference or the q–difference operator. Notice that a functional $\mathbf{u}$ satisfying (3) with ϕ and ψ polynomials, $\deg\psi \geq 1$, is called a semiclassical functional.

Since, depending on the case, the polynomials are orthogonal with respect to different linear functionals, we need to introduce a rigorous notation to embed all of them.

(1) *The Lattice.* For the general case we use the variable z. Notice that in the q–Hahn case we need to replace z by $x(s) = c_1(q)q^s + c_2(q)q^{-s} + c_3(q)$, where $s = 0, 1, 2, \ldots,$ with $q \in \mathbb{C}$, $|q| \neq 0, 1$, and c_1, c_2 and c_3 could depend on q. However, in this paper we only consider the q–Hahn case, and so we work with the q–linear lattice

$$x(s) := c_1(q)q^s + c_3(q),$$

where $s = 0, 1, 2, \ldots,$ and $q \in \mathbb{C}$, $|q| \neq 0, 1$.

(2) *The operators.* For the general case we use five basic operators: $\mathscr{D}$, its dual $\mathscr{D}^*$, the identity $\mathscr{I}$, the shift operator $\mathscr{E}^+$, and $\mathscr{E}^-$. For a better reading of the paper we collect these operators acting on a polynomial $r \in \mathbb{P}$ in the following table:

	$(\mathscr{D}r)(x)$	$(\mathscr{D}^*r)(x)$	$(\mathscr{E}^+r)(x)$	$(\mathscr{E}^-r)(x)$
Continuous case	$\frac{dr(x)}{dx}$	$\frac{dr(x)}{dx}$	$r(x)$	$r(x)$
Discrete case	$(\Delta r)(x)$	$(\nabla r)(x)$	$r(x+1)$	$r(x-1)$
q–Hahn case	$(\mathscr{D}_q r)(x)$	$(\mathscr{D}_{1/q} r)(x)$	$r(qx)$	$r(x/q)$

where the backward and forward difference operators are defined as

$$(\Delta r)(x) := (\mathscr{E}^+ - \mathscr{I})(r)(x) = r(x+1) - r(x),$$
$$(\nabla r)(x) := (\mathscr{I} - \mathscr{E}^-)(r)(x) = r(x) - r(x-1),$$

and the q–difference operator $\mathscr{D}_q$ is (see, for example, [5, Eq. (2.3)])

$$(\mathscr{D}_q r)(x) = \begin{cases} \dfrac{r(qx)-r(x)}{(q-1)x}, & x \neq 0, \\ r'(0), & x = 0, \end{cases} \qquad r \in \mathbb{P}, \quad q \in \mathbb{C},\ |q| \neq 0, 1.$$

(3) *The constants.* Throughout the paper we use some constants: q, $[n]$, and $[n]^*$. In the continuous and discrete cases we set $q = 1$, and $[n] = [n]^* = n$, and in the q–Hahn case we set $q \in \mathbb{C}$, $|q| \neq 0, 1$, $[n] = (q^n - 1)/(q-1)$, and $[n]^* = (q^{-n} - 1)/(q^{-1} - 1)$.

Let $\mathbb{P}$ be the linear space of polynomials and let $\mathbb{P}'$ be its algebraic dual space. We denote by $\langle \mathbf{u}, p\rangle$ the duality bracket for $\mathbf{u} \in \mathbb{P}'$ and $p \in \mathbb{P}$, and by $(\mathbf{u})_n = \langle \mathbf{u}, x^n \rangle$, with $n \geq 0$, the canonical moments of $\mathbf{u}$.

Definition 2.1 *For $\mathbf{u} \in \mathbb{P}'$, $\pi \in \mathbb{P}$, and $c \in \mathbb{C}$, let $\pi\mathbf{u}$, $(x-c)^{-1}\mathbf{u}$, and $\mathscr{D}^*\mathbf{u}$ be the linear functional defined by*

$$\langle \pi\, \mathbf{u}, p \rangle := \langle \mathbf{u}, \pi\, p \rangle, \quad p \in \mathbb{P},$$
$$\left\langle (x-c)^{-1}\mathbf{u}, p \right\rangle := \left\langle \mathbf{u}, \frac{p(x)-p(c)}{x-c} \right\rangle, \quad p \in \mathbb{P},$$
$$\langle \mathscr{D}^*\mathbf{u}, p \rangle := -q \langle \mathbf{u}, \mathscr{D}p \rangle, \quad p \in \mathbb{P}, \tag{4}$$

and thus $\langle \mathscr{D}\mathbf{u}, p\rangle = -q^{-1} \langle \mathbf{u}, \mathscr{D}^* p\rangle$.

Notice that for any $\mathbf{u} \in \mathbb{P}'$,

$$(x-c)((x-c)^{-1}\mathbf{u}) = \mathbf{u},$$

and

$$(x-c)^{-1}((x-c)\mathbf{u}) = \mathbf{u} - (\mathbf{u})_0 \delta_c,$$

where δ_c is the Dirac delta functional defined by $\langle \delta_c, p \rangle := p(c)$, $p \in \mathbb{P}$.

Taking these definitions into account we get for any $\mathbf{u} \in \mathbb{P}'$ and any polynomial π

$$\mathscr{D}(\pi\mathbf{u}) = (\mathscr{E}^+\pi)\mathscr{D}\mathbf{u} + (\mathscr{D}\pi)\mathbf{u}, \tag{5}$$
$$\mathscr{D}^*(\pi\mathbf{u}) = (\mathscr{E}^-\pi)\mathscr{D}^*\mathbf{u} + (\mathscr{D}^*\pi)\mathbf{u}. \tag{6}$$

Furthermore, for any two polynomials, π and ξ, the following relations are fulfilled:

$$\mathscr{D}(\pi\xi) = (\mathscr{E}^{+}\pi)\mathscr{D}\xi + (\mathscr{D}\pi)\xi, \tag{7}$$
$$\mathscr{D}^{*}(\pi\xi) = (\mathscr{E}^{-}\pi)\mathscr{D}^{*}\xi + (\mathscr{D}^{*}\pi)\xi. \tag{8}$$

Definition 2.2 *Given an integer δ, a linear functional $\mathbf{u}$, and a family of polynomials (p_n), we say that the polynomial sequence (p_n) is* quasi–orthogonal of order δ with respect to $\mathbf{u}$ *when the following properties are fulfilled:*

- *If $|n-m| \geq \delta + 1$, then $\langle \mathbf{u}, p_n p_m \rangle = 0$.*
- *If $|n-m| = \delta$, then $\langle \mathbf{u}, p_n p_m \rangle \neq 0$.*

Notice that if $\mathbf{u}$ is a semiclassical functional and (p_n) is orthogonal with respect to $\mathbf{u}$, i.e. $\delta = 0$, then (p_n) is called semiclassical orthogonal polynomial sequence.

Now we introduce some results regarding semiclassical orthogonal polynomials which allow us to obtain several results concerning Sobolev orthogonal polynomials in Sections 3 and 4. There is a lot of literature on semiclassical polynomials and an authoritative paper on this topic is [14].

First of all, we define the concept of an admissible pair which appears frequently linked with the concept of the semiclassical linear functional.

Definition 2.3 *We say that the pair of polynomials (ϕ, ψ) is an* admissible pair *if one of the following conditions is satisfied:*

- $\deg \psi \neq \deg \phi - 1$.
- $\deg \psi = \deg \phi - 1$, *with*
$$a_p + q^{-1}[n]^{*} b_t \neq 0, \qquad n \geq 0,$$
where a_p and b_t are the leading coefficients of ψ and ϕ, respectively.

We denote by $\sigma := \max\{\deg \phi - 2, \deg \psi - 1\}$ the order of the linear functional $\mathbf{u}$ *with respect to the admissible pair (ϕ, ψ). The* class *of such functional is the minimum of the order from among all the admissible pairs (see [7,13]).*

Remark 2.1 *Taking into account the two possible situations for the admissibility we use the following notation to unify them. We write*

$$\phi(z) = b_{\sigma+2} z^{\sigma+2} + \cdots, \qquad \psi(z) = a_{\sigma+1} z^{\sigma+1} + \cdots,$$

where:

- *If $t = \sigma + 2$ and $p < \sigma + 1$, then we set $b_{\sigma+2} = b_t$ and $a_{\sigma+1} = a_\sigma = \cdots = a_{p+1} = 0$.*
- *If $p = \sigma + 1$ and $t < \sigma + 2$, then we set $a_{\sigma+1} = a_p$ and $b_{\sigma+2} = b_{\sigma+1} = \cdots = b_{t+1} = 0$.*

Definition 2.4 *We say that the distributional equation (3) has an n_0–*singularity *if there exists a non–negative integer n_0 such that $a_{\sigma+1} + q^{-1}[n_0]^{*} b_{\sigma+2} = 0$. Otherwise we say that the distributional equation is* regular.

The following result will be very useful for our purposes. In the next result and in Lemma 2.1 we give some details in their proofs for each case, namely the continuous, discrete and q–Hahn cases, with the objective of making them more readable. It is also important to point out that all the results presented throughout the paper are valid for all the cases with no additional restrictions.

Theorem 2.1 *Let* $\mathbf{u}$ *be a linear functional satisfying the distributional equation*

$$\mathscr{D}(\phi\mathbf{u}) = \psi\mathbf{u},$$

where ϕ *and* ψ *are polynomials; then* $\mathbf{u}$ *also fulfills the distributional equation*

$$\mathscr{D}^*(\widetilde{\phi}\mathbf{u}) = \widetilde{\psi}\mathbf{u},$$

where $\widetilde{\phi}$ *and* $\widetilde{\psi}$ *are polynomials, with* $\deg\widetilde{\phi} \leq \sigma+2$*, and* $\deg\widetilde{\psi} \leq \sigma+1$*, with* $\sigma = \max\{\deg\phi-2, \deg\psi-1\}$.

Proof: The continuous case is trivial since $\mathscr{D} = \mathscr{D}^* = d/dx$.

In the discrete case $\mathscr{D} = \Delta$ and $\mathscr{D}^* = \nabla$, so

$$\mathscr{D}(\phi\mathbf{u}) = \psi\mathbf{u} \iff \Delta(\phi\mathbf{u}) = \mathscr{E}^+(\phi\mathbf{u}) - \phi\mathbf{u} = \psi\mathbf{u} \iff \mathscr{E}^+(\phi\mathbf{u}) = (\psi+\phi)\mathbf{u}.$$

Observe that, due to the property of the shift operator $\mathscr{E}^+$, the latter expression is also equivalent to $\phi\mathbf{u} = \mathscr{E}^-((\psi+\phi)\mathbf{u})$. Thus, setting $\widetilde{\phi} := \phi+\psi$ and $\widetilde{\psi} = \psi$, we get

$$\mathscr{D}^*(\widetilde{\phi}\mathbf{u}) = \nabla(\widetilde{\phi}\mathbf{u}) = (\phi+\psi)\mathbf{u} - \mathscr{E}^-((\phi+\psi)\mathbf{u}) = \psi\mathbf{u} = \widetilde{\psi}\mathbf{u}.$$

The q–Hahn case is a little more complicated, due to the notation, but the proof is almost identical.
In this case $\mathscr{D} = \Delta/\Delta x(s)$, and $\mathscr{D}^* = \nabla/\nabla x(s)$; thus the distributional equation for $\mathscr{D}$ can be written as

$$\frac{\mathscr{E}^+(\phi\mathbf{u})}{\Delta x} - \frac{\phi\mathbf{u}}{\Delta x} = \frac{(\Delta x)\psi\mathbf{u}}{\Delta x} \qquad \iff \qquad \frac{\mathscr{E}^+(\phi\mathbf{u})}{\Delta x} = \frac{(\phi+(\Delta x)\psi)}{\Delta x}\mathbf{u}.$$

Then,

$$\begin{aligned}\frac{(\mathscr{E}^-(\phi+(\Delta x)\psi))}{\nabla x}\nabla\mathbf{u} &= \frac{(\mathscr{E}^-(\phi+(\Delta x)\psi))}{\nabla x}\mathbf{u} - \frac{(\mathscr{E}^-(\phi+(\Delta x)\psi))}{\nabla x}\mathscr{E}^-\mathbf{u} \\ &= \frac{(\mathscr{E}^-(\phi+(\Delta x)\psi))}{\nabla x}\mathbf{u} - \frac{\phi\mathbf{u}}{\nabla x} = \left(\mathscr{E}^-\psi - \frac{\nabla\phi}{\nabla x}\right)\mathbf{u}.\end{aligned}$$

Thus, using the above expression, (6), and setting $\widetilde{\phi} := \phi+(\Delta x)\psi$, we get

$$\mathscr{D}^*(\widetilde{\phi}\mathbf{u}) = \frac{\nabla\Big((\phi+(\Delta x)\psi)\mathbf{u}\Big)}{\nabla x} = \left(\mathscr{E}^-\psi - \frac{\nabla\phi}{\nabla x}\right)\mathbf{u} + \frac{\nabla(\phi+(\Delta x)\psi)}{\nabla x}\mathbf{u} = q\psi\mathbf{u} = \widetilde{\psi}\mathbf{u}.$$

Therefore, it is a straightforward computation to check that both $\widetilde{\phi}$ and $\widetilde{\psi}$ are polynomials, with $\deg\widetilde{\phi} \leq \sigma+2$, and $\deg\widetilde{\psi} \leq \sigma+1$ where $\sigma = \max\{\deg\phi-2, \deg\psi-1\}$. $\square$

Remark 2.2 • *Although the reciprocal of this result is also true, and it can be proved in a straightforward way, we have decided not to include it since it is not used in this paper.*

• *In the sequel, given a pair* (ϕ, ψ) *we use the notation* $\widetilde{\phi} := \phi + \psi$ *and* $\widetilde{\psi} := \psi$ *for the discrete case, and* $\widetilde{\phi} := \phi + (\Delta x)\psi$ *and* $\widetilde{\psi} := q\psi$ *for the q–Hahn case.*

The inner product associated with the semiclassical functional $\mathbf{u}$,

$$\langle p, r\rangle := \langle \mathbf{u}, p\, r\rangle\,,$$

is Hankel, i.e.,

$$\langle zp, r\rangle = \langle p, zr\rangle, \tag{9}$$

and then the corresponding monic semiclassical orthogonal polynomial sequence (p_n) satisfies the recurrence relation

$$zp_n(z) = p_{n+1}(z) + B_n p_n(z) + C_n p_{n-1}(z), \quad n \geq 0, \tag{10}$$

with initial conditions $p_0 = 1$, $p_{-1} = 0$. Moreover the square of the norm of p_n, $d_n^2 = \langle \mathbf{u}, p_n^2\rangle$, satisfies the relation

$$d_n^2 = C_n d_{n-1}^2, \qquad n \geq 1. \tag{11}$$

It is well known that there are several ways to characterize semiclassical orthogonal polynomials. In our framework, we use a characterization of these polynomials that is usually called the first structure relation.

Theorem 2.2 *[12] & [7, Prop. 3.2] Let* (p_n) *be a sequence of monic orthogonal polynomials with respect to* $\mathbf{u}$*, and* ϕ *a polynomial of degree* t*. Then, the following statements are equivalent:*

(i) *There exists a non-negative integer* σ *such that*

$$\phi(z)\mathscr{D}p_{n+1}(z) = \sum_{\nu=n-\sigma}^{n+t} \lambda_{n,\nu} p_\nu(z), \quad n \geq \sigma,$$
$$\lambda_{n,n-\sigma} \neq 0, \quad n \geq \sigma + 1.$$

(ii) *There exists a polynomial* ψ*, with* $\deg \psi = p \geq 1$*, such that*

$$\mathscr{D}(\phi\mathbf{u}) = \psi\mathbf{u},$$

where the pair (ϕ, ψ) *is an admissible pair.*

We also use a recent result that establishes a nice relation between the semiclassical orthogonal polynomials (p_n) and the polynomials $(\mathscr{D}p_{n+1})$.

Theorem 2.3 *[4, Th. 3.2] & [21, Th. 3.5] Let* (p_n) *be a sequence of monic orthogonal polynomials with respect to* $\mathbf{u}$*, and* ϕ *a polynomial of degree* t*. Then, the following statements are equivalent:*

(i) *There exist three non–negative integers, σ, p, and r, with $p \geq 1$, $r \geq \sigma + t + 1$, and $\sigma = \max\{t-2, p-1\}$, such that*

$$\sum_{\nu=n-\sigma}^{n+\sigma} \xi_{n,\nu} p_\nu(z) = \sum_{\nu=n-t}^{n+\sigma} \varsigma_{n,\nu} p_\nu^{[1]}(z),$$

where $p_n^{[1]}(z) := [n+1]^{-1}(\mathscr{D}p_{n+1})(z)$, $\xi_{n,n+\sigma} = \varsigma_{n,n+\sigma} = 1$, $n \geq \max\{\sigma, t+1\}$, $\xi_{r,r-\sigma}\varsigma_{r,r-t} \neq 0$,

$$\langle \mathscr{D}(\phi\mathbf{u}), p_m\rangle = 0, \quad p+1 \leq m \leq 2\sigma+t+1, \quad \langle \mathscr{D}(\phi\mathbf{u}), p_p\rangle \neq 0,$$

and if $p = t-1$, *then* $\lim_{q\uparrow 1}\langle \mathbf{u}, p_p^2\rangle^{-1}\langle \mathbf{u}, \phi\mathscr{D}p_p\rangle \neq m b_t$, $m \in \mathbb{N}^*$, *and where* b_t *is the leading coefficient of* ϕ *(admissibility condition).*

(ii) *There exists a polynomial* ψ, *with* $\deg\psi = p \geq 1$, *such that*

$$\mathscr{D}(\phi\mathbf{u}) = \psi\mathbf{u},$$

where the pair (ϕ, ψ) *is an admissible pair.*

In this paper we use some linear functionals related to the semiclassical functional $\mathbf{u}$. Now, we introduce them and give some properties that they satisfy.

Lemma 2.1 *Let* $\mathbf{u}$ *be a linear functional satisfying* (3)*; then the linear functional* $\mathscr{E}^+(\phi\mathbf{u})$ *satisfies the following distributional equation:*

$$\mathscr{D}(\phi\,\mathscr{E}^+(\phi\mathbf{u})) = \Big(\mathscr{D}\phi + q\mathscr{E}^+\psi\Big)\mathscr{E}^+(\phi\mathbf{u}),$$

setting $q=1$ *in both cases, continuous and discrete.*

Proof: The result follows by applying (5). In fact, in the continuous case, since $\mathscr{E}^+$ is the identity, we get

$$\mathscr{D}(\phi\,\mathscr{E}^+(\phi\mathbf{u})) = \frac{d}{dx}(\phi\phi\mathbf{u}) = \frac{d}{dx}(\phi)\phi\mathbf{u} + \phi\frac{d}{dx}(\phi\mathbf{u}) = \Big(\mathscr{D}\phi + \mathscr{E}^+\psi\Big)\mathscr{E}^+(\phi\mathbf{u}).$$

In the discrete case, $\mathscr{E}^+$ is the shift operator, so applying (5) we get

$$\begin{aligned}
\mathscr{D}(\phi\,\mathscr{E}^+(\phi\mathbf{u})) &= \Delta(\phi\,\mathscr{E}^+(\phi\mathbf{u})) = \mathscr{E}^+\phi\,\Delta(\mathscr{E}^+(\phi\mathbf{u})) + \Delta\phi\,\mathscr{E}^+(\phi\mathbf{u}) \\
&= \mathscr{E}^+\phi\,\mathscr{E}^+(\psi\mathbf{u}) + \Delta\phi\,\mathscr{E}^+(\phi\mathbf{u}) \\
&= (\mathscr{E}^+\psi + \Delta\phi)\,\mathscr{E}^+(\phi\mathbf{u}).
\end{aligned}$$

In the q–Hahn case, $\mathscr{E}^+$ is also the shift operator. Thus applying (5) we get

$$\mathscr{D}(\phi\,\mathscr{E}^+(\phi\mathbf{u})) = \frac{\Delta(\phi\,\mathscr{E}^+(\phi\mathbf{u}))}{\Delta x(s)} = \frac{1}{\Delta x(s)}\Big(\Delta\phi\,\mathscr{E}^+(\phi\mathbf{u}) + \mathscr{E}^+\phi\,\Delta(\mathscr{E}^+(\phi\mathbf{u}))\Big).$$

Taking into account that in this case $\mathscr{D}\mathscr{E}^+ = q\mathscr{E}^+\mathscr{D}$, since the lattice $x(s)$ is q–linear, we get

$$\mathscr{D}(\phi\,\mathscr{E}^+(\phi\mathbf{u})) = \Big(\mathscr{D}\phi\,\mathscr{E}^+(\phi\mathbf{u}) + q\mathscr{E}^+\phi\,\mathscr{E}^+\mathscr{D}(\phi\mathbf{u})\Big) = \Big(\mathscr{D}\phi + q\mathscr{E}^+\psi\Big)\mathscr{E}^+(\phi\mathbf{u}).$$

$\square$

Lemma 2.2 *Let* $\mathbf{u}$ *be a linear functional satisfying* (3)*, and let* $\mathbf{v}$ *be a linear functional such that* $\mathbf{u} = \mathscr{E}^{+}(\phi\mathbf{v})$*; then* $\mathbf{v}$ *satisfies an analogous distributional equation.*

Proof: It follows from making computations in (3) where it is necessary to take into account how the operators $\mathscr{D}$ and $\mathscr{E}^{+}$ act, that is, $\mathscr{D}\mathscr{E}^{+} = q\mathscr{E}^{+}\mathscr{D}$. In fact, we obtain

$$\mathscr{D}(q(\mathscr{E}^{-}\phi)\phi\,\mathbf{v}) = (\mathscr{E}^{-}\psi)\phi\,\mathbf{v}.$$

$\square$

So, the previous lemmas provide motivation for defining a family of linear functionals which will be used throughout the paper.

Definition 2.5 *Given a semiclassical functional* $\mathbf{u}$ *satisfying* (3)*, for* $k \in \mathbb{Z}$ *we define the linear functional* $\mathbf{u}_k$ *in a constructive way as*

$$\mathbf{u}_k = \mathscr{E}^{+}(\phi_{k-1}\mathbf{u}_{k-1}),$$
$$\mathbf{u}_0 = \mathbf{u}, \quad \phi_0 = \phi,$$

where ϕ_k *is a multiple of* ϕ_{k-1}*.*

Remark 2.3 *Notice that, by Lemma 2.1 and Lemma 2.2, when* $\mathbf{u}$ *is a semiclassical functional then* $\mathbf{u}_k$ *is a semiclassical functional for every integer* k*. On the other hand, using Lemma 3.3 in [13] for the continuous case and Lemma 2.2 in [7] for discrete and q–Hahn cases, the statement of* $\mathbf{u}_k$ *in terms of* $\mathbf{u}_{k-1}$ *is well defined since if* $\mathbf{u}$ *is any semiclassical functional satisfying the distributional equations*

$$\mathscr{D}(\hat{\phi}_1\mathbf{u}) = \hat{\psi}_1\mathbf{u} \quad \text{and} \quad \mathscr{D}(\hat{\phi}_2\mathbf{u}) = \hat{\psi}_2\mathbf{u},$$

then for $\hat{\phi} := \gcd(\hat{\phi}_1, \hat{\phi}_2)$ *there exists a polynomial* $\hat{\psi}$ *with* $\deg\hat{\psi} \geq 1$ *such that* $\mathscr{D}(\hat{\phi}\mathbf{u}) = \hat{\psi}\mathbf{u}$*. Thus, observe that* ϕ_k *is not determined in a unique way.*

As a consequence of Lemmas 2.1 and 2.2 and some straightforward computations we can deduce:

Theorem 2.4 *Say we are given a linear functional* $\mathbf{u}$ *satisfying* (3)*, that is,*

$$\mathscr{D}(\phi\mathbf{u}) = \psi\mathbf{u},$$

where ψ *and* ϕ *are polynomials in the conditions of Definition 2.3. Then, for any integer* k *and any polynomial* π *we get*

$$\mathscr{D}(\pi\phi_k\mathbf{u}_k) = \widetilde{\pi}\mathbf{u}_k,$$

where $\widetilde{\pi}$ *is a polynomial of degree, at most,* $\deg\pi + \sigma_k + 1$*,* σ_k *being the order of the linear functional* $\mathbf{u}_k$ *with respect to the pair* (ϕ_k, ψ_k)*. Furthermore, if the pair* (ϕ_k, ψ_k) *is admissible, then* $\deg\widetilde{\pi} = \deg\pi + \sigma_k + 1$*.*

Finally, we obtain a useful result for our purposes.

Proposition 2.1 *If (p_n) is a sequence of semiclassical polynomials orthogonal with respect to the linear functional $\mathbf{u}$ of order σ, then the sequence of polynomials $(\mathscr{D}p_{n+1})$ is quasi–orthogonal of order σ with respect to the semiclassical functional $\mathbf{u}_1$.*

Proof: Using (3), (4) and (8), we have

$$\begin{aligned}\langle \mathbf{u}_1, \mathscr{D}p_{n+1}x^m\rangle &= \langle \mathscr{E}^+(\phi\mathbf{u}), \mathscr{D}p_{n+1}x^m\rangle = \langle \phi\mathbf{u}, \mathscr{E}^-(\mathscr{D}p_{n+1}x^m)\rangle \\ &= \langle \phi\mathbf{u}, \mathscr{D}^*(p_{n+1}x^m)\rangle - \langle \phi\mathbf{u}, p_{n+1}\mathscr{D}^*x^m\rangle \\ &= -\langle \mathbf{u}, p_{n+1}(q\psi x^m + \phi\mathscr{D}^*x^m)\rangle = 0,\end{aligned}$$

when $m + \sigma < n$. □

Remark 2.4 *As far as we know, the reciprocal of Proposition 2.1 is no longer true without the assumption that $\mathbf{v}$ is quasi-definite where $\mathbf{v}$ is the operator with respect to which $(\mathscr{D}p_{n+1})$ is quasi–orthogonal.*

For every integer k we denote by $(p_n^{\{k\}})$ the sequence of monic orthogonal polynomials with respect to the linear functional $\mathbf{u}_k$. Thus, using Theorem 2.4 and Proposition 2.1, we have that the $\mathscr{D}p_n^{\{k\}}$ are quasi–orthogonal of order σ_k with respect to $\mathbf{u}_{k+1}$. Therefore, we get for every integer k,

$$\mathscr{D}p_{n+1}^{\{k\}} = \sum_{\nu=n-\sigma_k}^{n} \alpha_{n,k,\nu}p_\nu^{\{k+1\}}, \qquad k \in \mathbb{Z}. \tag{12}$$

Then, using (12) and Theorem 2.3, we deduce for $k = -1$ the following relation:

$$\sum_{\nu=n-\sigma_{-1}}^{n+\sigma_{-1}} \xi^*_{n,\nu}p_\nu^{\{-1\}}(z) = \sum_{\nu=n-t_{-1}-\sigma_{-1}}^{n+\sigma_{-1}} \varsigma^*_{n,\nu}p_\nu(z), \tag{13}$$

where $\xi^*_{n,n+\sigma_{-1}-1} = \varsigma^*_{n,n+\sigma_{-1}} = 1$, $t_{-1} := \deg\phi_{-1}$, and σ_{-1} is the order of the functional $\mathbf{u}_{-1}$ associated with the admissible pair (ϕ_{-1}, ψ_{-1}).

Observe that in the classical case, that is, $\sigma = 0$, using Theorem 2.4 we have $\sigma_{-1} = 0$.

3 The semiclassical Sobolev orthogonal polynomials

Let us consider the $\mathscr{D}$–Sobolev inner product defined on $\mathbb{P}\times\mathbb{P}$ by

$$\langle p, r\rangle_S = \langle \mathbf{u}, p\,r\rangle + \lambda\langle \mathbf{u}, \mathscr{D}p\,\mathscr{D}r\rangle, \tag{14}$$

where $\mathbf{u}$ is a semiclassical functional of order σ and $\lambda \geq 0$. We denote by (p_n) the monic orthogonal polynomial sequence with respect to $\mathbf{u}$ and by $(Q_n^{(\lambda)})$ the OPS associated with the ($\mathscr{D}$–Sobolev) inner product $\langle\cdot,\cdot\rangle_S$ which we call the semiclassical Sobolev orthogonal polynomial sequence.

Proposition 3.1 *Let (p_n) be a monic semiclassical orthogonal polynomial sequence and let $(Q_n^{(\lambda)})$ be the semiclassical Sobolev orthogonal polynomial sequence. The following relation holds:*

$$\sum_{\nu=n-\sigma_{-1}}^{n+\sigma_{-1}} \xi_{n,\nu}^{*} p_{\nu}^{\{-1\}}(z) = Q_{n+\sigma_{-1}}^{(\lambda)}(z) + \sum_{\nu=n-\sigma_{-1}-H^*}^{n-1+\sigma_{-1}} \theta_{n,\nu}\, Q_{\nu}^{(\lambda)}(z), \quad n \geq \sigma_{-1} + H^*, \tag{15}$$

where $H^* := \max\{t_{-1}, \sigma_{-1}\}$,

$$\begin{aligned}\mathbf{d}_{n-\sigma_{-1}-H^*}^{2}\theta_{n,n-\sigma_{-1}-H^*}(\lambda) =& \varsigma_{n,n-t_{-1}-\sigma_{-1}}^{*} d_{n-t_{-1}-\sigma_{-1}}^{2} \delta_{H^*,t_{-1}} \\ &+ \lambda[n-2\sigma_{-1}]\hat{\xi}_{n,n-2\sigma_{-1}}^{*} d_{n-2\sigma_{-1}-1}^{2} \delta_{H^*,\sigma_{-1}},\end{aligned} \tag{16}$$

with $\mathbf{d}_n^2 := \left\langle Q_n^{(\lambda)}(z) Q_n^{(\lambda)}(z) \right\rangle_S$, *and the other coefficients $\theta_{n,\nu}$ in (15) can be computed recursively.*

Proof: If we apply (13) and expand it we get

$$\sum_{\nu=n-\sigma_{-1}}^{n+\sigma_{-1}} \xi_{n,\nu}^{*} p_{\nu}^{\{-1\}}(z) = \sum_{\nu=n-t_{-1}-\sigma_{-1}}^{n+\sigma_{-1}} \varsigma_{n,\nu}^{*} p_{\nu}(z) = Q_{n+\sigma_{-1}}^{(\lambda)}(z) + \sum_{i=0}^{n+\sigma_{-1}-1} \theta_{n,i} Q_i^{(\lambda)}(z),$$

where

$$\theta_{n,i} = \frac{\left\langle \sum_{\nu=n-\sigma_{-1}}^{n+\sigma_{-1}} \xi_{n,\nu}^{*} p_{\nu}^{\{-1\}}(z), Q_i^{(\lambda)}(z) \right\rangle_S}{\left\langle Q_i^{(\lambda)}(z), Q_i^{(\lambda)}(z) \right\rangle_S} = \frac{\left\langle \sum_{\nu=n-t_{-1}-\sigma_{-1}}^{n+\sigma_{-1}} \varsigma_{n,\nu}^{*} p_{\nu}(z), Q_i^{(\lambda)}(z) \right\rangle_S}{\mathbf{d}_i^2}.$$

Thus, using (12) and the orthogonality property we get for $0 \leq i < n - \sigma_{-1} - H^*$

$$\begin{aligned}\mathbf{d}_i^2\, \theta_{n,i} &= \left\langle \mathbf{u}, \left(\sum_{\nu=n-t_{-1}-\sigma_{-1}}^{n+\sigma_{-1}} \varsigma_{n,\nu}^{*} p_{\nu}(z) \right) Q_i^{(\lambda)}(z) \right\rangle \\ &\quad + \lambda \left\langle \mathbf{u}, \mathscr{D} \left(\sum_{\nu=n-\sigma_{-1}}^{n+\sigma_{-1}} \xi_{n,\nu}^{*} p_{\nu}^{\{-1\}}(z) \right) (\mathscr{D} Q_i^{(\lambda)})(z) \right\rangle \\ &= \lambda \left\langle \mathbf{u}, \left(\sum_{\nu=n-2\sigma_{-1}}^{n+\sigma_{-1}} \hat{\xi}_{n,\nu}^{*} p_{\nu-1}(z) \right) (\mathscr{D} Q_i^{(\lambda)})(z) \right\rangle = 0,\end{aligned}$$

and therefore (15) follows. To obtain (16) we only need to take $i = n - \sigma_{-1} - H^*$ in the above expression.

Finally, if we apply this process for $n - \sigma_{-1} - H^* \leq j \leq n + \sigma_{-1} - 1$ we get an upper triangular linear system such that we can compute the coefficients $\theta_{n,j}$ in a recursive way and hence the result holds. □

Next, we particularize the above result for the classical case, that is, we name as the classical Sobolev orthogonal polynomial sequence the semiclassical Sobolev orthogonal polynomial sequence where the linear functional involved has order $\sigma = 0$, i.e., it is a classical functional.

Corollary 3.1 *Let (p_n) be a classical orthogonal polynomial sequence and let $(Q_n^{(\lambda)})$ be the classical Sobolev orthogonal polynomial sequence. The following relation holds:*

$$p_n^{\{-1\}}(z) = Q_n^{(\lambda)}(z) + f_{n-1}(\lambda)Q_{n-1}^{(\lambda)}(z) + e_{n-2}(\lambda)Q_{n-2}^{(\lambda)}(z), \quad n \geq 2, \tag{17}$$

with $Q_i^{(\lambda)}(z) = p_i(z)$, $i = 0, 1$, and

$$e_{n-2}(\lambda) = \tilde{\epsilon}_n \frac{d_{n-2}^2}{\mathbf{d}_{n-2}^2}, \quad e_0(\lambda) = \tilde{\epsilon}_2, \tag{18}$$

$$f_{n-1}(\lambda) = \frac{1}{\mathbf{d}_{n-1}^2} \left\{ \tilde{\delta}_n d_{n-1}^2 + \tilde{\epsilon}_n (\tilde{\delta}_{n-1} - f_{n-2}(\lambda)) d_{n-2}^2 \right\}, \quad f_0(\lambda) = \tilde{\delta}_1,$$

where

$$p_n^{\{-1\}}(z) = p_n(z) + \tilde{\delta}_n p_{n-1}(z) + \tilde{\epsilon}_n p_{n-2}(z). \tag{19}$$

Furthermore, in the classical case we can give nonlinear recurrence relations for the square of the norms of the Sobolev polynomials and for the coefficients $e_n(\lambda)$ appearing in (17).

Corollary 3.2 *For $n \geq 2$, we have*

$$\mathbf{d}_n^2 = d_n^2 + \Big(\lambda[n]^2 + \tilde{\delta}_n(\tilde{\delta}_n - f_{n-1}(\lambda))\Big) d_{n-1}^2 + \tilde{\epsilon}_n(\tilde{\epsilon}_n - e_{n-2}(\lambda) - f_{n-1}(\lambda)(\tilde{\delta}_{n-1} - f_{n-2}(\lambda))) d_{n-2}^2, \tag{20}$$

with $\mathbf{d}_0^2 = d_0^2$, and $\mathbf{d}_1^2 = d_1^2 + \lambda d_0^2$, and

$$e_n(\lambda) = \frac{C_n \tilde{\epsilon}_{n+2}}{\lambda[n]^2 + C_n + \tilde{\delta}_n(\tilde{\delta}_n - f_{n-1}(\lambda)) + \tilde{\epsilon}_n(\tilde{\epsilon}_n - e_{n-2}(\lambda) - f_{n-1}(\lambda)(\tilde{\delta}_{n-1} - f_{n-2}(\lambda)))C_{n-1}^{-1}}, \tag{21}$$

with the initial conditions

$$e_0(\lambda) = \tilde{\epsilon}_2, \qquad e_1(\lambda) = \frac{C_1 \tilde{\epsilon}_3}{\lambda + C_1}. \tag{22}$$

Proof: Using (17) and (19), we get

$$\begin{aligned}
\mathbf{d}_n^2 =& \Big\langle Q_n^{(\lambda)}(z), Q_n^{(\lambda)}(z) \Big\rangle_S = \Big\langle Q_n^{(\lambda)}(z), p_n^{\{-1\}}(z) \Big\rangle_S = d_n^2 + \lambda \left\langle \mathbf{u}, (\mathscr{D}Q_n^{(\lambda)})(z)\, (\mathscr{D}p_n^{\{-1\}})(z) \right\rangle \\
=& d_n^2 + \lambda \left\langle \mathbf{u}, (\mathscr{D}Q_n^{(\lambda)})(z) \Big([n]p_{n-1}(z) - \tilde{\delta}_n(\mathscr{D}p_{n-1})(z) - \tilde{\epsilon}_n(\mathscr{D}p_{n-2})(z) \Big) \right\rangle \\
=& d_n^2 + \lambda[n]^2 d_{n-1}^2 + \tilde{\delta}_n \left\langle \mathbf{u}, Q_n^{(\lambda)}(z)\, p_{n-1}(z) \right\rangle + \tilde{\epsilon}_n \left\langle \mathbf{u}, Q_n^{(\lambda)}(z)\, p_{n-2}(z) \right\rangle \\
=& d_n^2 + \lambda[n]^2 d_{n-1}^2 + \tilde{\delta}_n(\tilde{\delta}_n - f_{n-1}(\lambda)) d_{n-1}^2 + \tilde{\epsilon}_n(\tilde{\epsilon}_n - e_{n-2}(\lambda) - f_{n-1}(\lambda)(\tilde{\delta}_{n-1} - f_{n-2}(\lambda))) d_{n-2}^2,
\end{aligned}$$

which proves (20). Finally, (21) follows from (20) using (11), and the initial conditions (22) can be deduced from (18). $\square$

The inner product (14) does not satisfy the Hankel property $\langle zp, r \rangle_S = \langle p, zr \rangle_S$, and therefore the polynomial sequence $(Q_n^{(\lambda)})$ does not fulfill in general a three–term recurrence relation. However, we find an operator $\mathscr{J}$ which is symmetric with respect to the inner product (14), that is, $\langle \mathscr{J} p, r\rangle_S = \langle p, \mathscr{J} r\rangle_S$, where $p,\, r \in \mathbb{P}$. Thus, we generalize some results in this direction obtained for the continuous and discrete cases (see the surveys [10], [11], [15], [16], [18], and the references therein, and also [20]).

Proposition 3.2 *Let $\mathscr{J}$ be the linear operator*

$$\mathscr{J} := (\mathscr{E}^{-}\widetilde{\phi})\mathscr{I} + \frac{\lambda}{q}(\mathscr{D}^{*}\widetilde{\phi} - \widetilde{\psi})\mathscr{D}^{*} - \lambda(\mathscr{E}^{-}\widetilde{\phi})\mathscr{D}\,\mathscr{D}^{*}, \tag{23}$$

where $\mathscr{I}$ is the identity operator. Then,

$$\langle (\mathscr{E}^{-}\widetilde{\phi})p, r\rangle_S = \langle \mathbf{u}, p\,\mathscr{J} r\rangle, \quad p,\, r \in \mathbb{P}. \tag{24}$$

Proof: According to Theorem 2.1 the linear functional $\mathbf{u}$ satisfies the distributional equation $\mathscr{D}^{*}(\widetilde{\phi}\mathbf{u}) = \widetilde{\psi}\mathbf{u}$. From (14), we have

$$\langle (\mathscr{E}^{-}\widetilde{\phi})p, r\rangle_S = \langle \mathbf{u}, (\mathscr{E}^{-}\widetilde{\phi})pr\rangle + \lambda\langle \mathbf{u}, \mathscr{D}\Big((\mathscr{E}^{-}\widetilde{\phi})p\Big)\,\mathscr{D} r\rangle.$$

Now, using relation (7) with $\pi = \mathscr{D}^{*}(r)$ we get

$$\mathscr{D}\Big((\mathscr{E}^{-}\widetilde{\phi})p\,\mathscr{D}^{*}r\Big) = \mathscr{D}\Big((\mathscr{E}^{-}\widetilde{\phi})p\Big)\,\mathscr{D} r + \Big((\mathscr{E}^{-}\widetilde{\phi})p\Big)\mathscr{D}\Big(\mathscr{D}^{*}r\Big),$$

and so,

$$\langle (\mathscr{E}^{-}\widetilde{\phi})p, r\rangle_S = \langle \mathbf{u}, p\,\Big((\mathscr{E}^{-}\widetilde{\phi})r - \lambda(\mathscr{E}^{-}\widetilde{\phi})\mathscr{D}(\mathscr{D}^{*}r)\Big)\rangle + \lambda\langle \mathbf{u}, \mathscr{D}\Big((\mathscr{E}^{-}\widetilde{\phi})p\mathscr{D}^{*}r\Big)\rangle.$$

On the other hand, if we apply the relation (6) to the distributional equation with $\pi = \widetilde{\phi}$ we get

$$\widetilde{\psi}\mathbf{u} = (\mathscr{E}^{-}\widetilde{\phi})\mathscr{D}^{*}\mathbf{u} + (\mathscr{D}^{*}\widetilde{\phi})\mathbf{u},$$

and therefore

$$\Big\langle (\mathscr{E}^{-}\widetilde{\phi})p, r\Big\rangle_S = \Big\langle \mathbf{u}, p\,\Big((\mathscr{E}^{-}\widetilde{\phi})r - \lambda(\mathscr{E}^{-}\widetilde{\phi})\mathscr{D}(\mathscr{D}^{*}r)\Big)\Big\rangle - \frac{\lambda}{q}\langle \mathbf{u}, p\,(\widetilde{\psi} - \mathscr{D}^{*}\widetilde{\phi})\mathscr{D}^{*}r\rangle = \langle \mathbf{u}, p\mathscr{J} r\rangle.$$

□

Corollary 3.3 *The following identity holds:*

$$\langle (\widetilde{\psi} - \mathscr{D}^{*}\widetilde{\phi})p, r\rangle_S = \langle \mathscr{D}^{*}\mathbf{u}, p\,\mathscr{J} r\rangle, \quad p,\, r \in \mathbb{P}.$$

Proof: Using the same properties as in the proof of the previous result, we get

$$\begin{aligned}
\langle (\widetilde{\psi} - \mathscr{D}^{*}\widetilde{\phi})p, r\rangle_S &= \langle \mathbf{u}, (\widetilde{\psi} - \mathscr{D}^{*}\widetilde{\phi})pr\rangle + \lambda\Big\langle \mathbf{u}, \mathscr{D}\Big((\widetilde{\psi} - \mathscr{D}^{*}\widetilde{\phi})p\Big)\,\mathscr{D} r\Big\rangle \\
&= \Bigg\langle \mathscr{D}^{*}\mathbf{u}, p\,\Bigg((\mathscr{E}^{-}\widetilde{\phi})r - \frac{\lambda}{q}(\widetilde{\psi} - \mathscr{D}^{*}\widetilde{\phi})\mathscr{D}^{*}r\Bigg)\Bigg\rangle - \lambda\langle \mathscr{D}^{*}\mathbf{u}, (\mathscr{E}^{-}\widetilde{\phi})p\mathscr{D}(\mathscr{D}^{*}r)\rangle \\
&= \langle \mathscr{D}^{*}\mathbf{u}, p\mathscr{J} r\rangle.
\end{aligned}$$

$\square$

In the next result we prove that the linear functional $\mathscr{J}$ plays for the inner product (14) a role equivalent to the one played by the multiplication operator by z for standard inner products (see (9)).

Theorem 3.1 *The linear operator* $\mathscr{J}$ *defined in* (23) *satisfies*

$$\langle \mathscr{J} p, r\rangle_S = \langle p, \mathscr{J} r\rangle_S \qquad p,\, r \in \mathbb{P}.$$

Proof: Applying Proposition 3.2 and Corollary 3.3 together with (7) and the fact that $\mathscr{E}^+ \mathscr{D}^* = \mathscr{D}$, then we get

$$\begin{aligned}
\langle \mathscr{J} p, r\rangle_S &= \langle (\mathscr{E}^- \widetilde{\phi})p, r\rangle_S - \frac{\lambda}{q}\langle (\widetilde{\psi} - \mathscr{D}^*\widetilde{\phi})(\mathscr{D}^* p), r\rangle_S - \lambda\langle (\mathscr{E}^-\widetilde{\phi})\mathscr{D}(\mathscr{D}^* p), r\rangle_S \\
&= \langle \mathbf{u}, p\mathscr{J} r\rangle - \frac{\lambda}{q}\langle \mathscr{D}^*\mathbf{u}, (\mathscr{D}^* p)\mathscr{J} r\rangle - \lambda\langle \mathbf{u}, (\mathscr{D}(\mathscr{D}^* p))\mathscr{J} r\rangle \\
&= \langle \mathbf{u}, p\mathscr{J} r\rangle + \lambda\langle \mathbf{u}, (\mathscr{D}p)(\mathscr{D}\mathscr{J} r)\rangle \\
&= \langle p, \mathscr{J} r\rangle_S.
\end{aligned}$$

$\square$

Remark 3.1 *Taking into account the proof of the last results, if* $\gcd(\mathscr{E}^-\widetilde{\phi}, \mathscr{D}^*\widetilde{\phi} - \widetilde{\psi}) = d > 1$*, then we can consider* $(\mathscr{E}^-\widetilde{\phi})/d$ *and* $(\mathscr{D}^*\widetilde{\phi} - \widetilde{\psi})/d$ *in the definition (23) of the operator* $\mathscr{J}$ *and these results also hold.*

To end this section, we give some algebraic differential/difference results. One of them allows us to give an expression for the polynomial $(\mathscr{E}^-\widetilde{\phi})(z)p_n(z)$ in terms of a finite number of $\mathscr{D}$–Sobolev orthogonal polynomials. Another one gives the *second–order* $\mathscr{D}$*–equation* (differential or difference or q–difference equation) that is satisfied by the polynomials $(Q_n^{(\lambda)})$.

Corollary 3.4 *The following relations hold*

$$\begin{aligned}
(\mathscr{E}^-\widetilde{\phi})(z)p_n(z) &= \sum_{\nu=n-H}^{\nu=n+\deg\widetilde{\phi}} \mu_{n,\nu} Q_\nu^{(\lambda)}(z), \quad n \geq H, \\
\mathscr{J} Q_n^{(\lambda)}(z) &= \sum_{\nu=n-\deg\widetilde{\phi}}^{n+H} \vartheta_{n,\nu} p_\nu(z), \quad n \geq \deg\widetilde{\phi}, \\
\mathscr{J} Q_n^{(\lambda)}(z) &= \sum_{\nu=n-H}^{n+H} \varpi_{n,\nu} Q_\nu^{(\lambda)}(z), \quad n \geq H,
\end{aligned}$$

where $H := \max\{\deg\widetilde{\psi} - 1, \deg\widetilde{\phi}\}$.

Proof: To prove the first relation, we write

$$(\mathscr{E}^{-}\widetilde{\phi})(z)p_n(z) = \sum_{\nu=0}^{n+\deg\widetilde{\phi}} \mu_{n,\nu} Q_\nu^{(\lambda)}(z),$$

where $\mu_{n,\nu}\mathbf{d}_\nu^2 = \langle (\mathscr{E}^{-}\widetilde{\phi})p_n, Q_\nu^{(\lambda)}\rangle_S = \langle \mathbf{u}, p_n \mathscr{J} Q_\nu^{(\lambda)}\rangle$ which, using Proposition 3.2, vanishes when $\nu + H < n$ since $\deg \mathscr{J}\pi = H + \deg\pi$.

To prove the second relation it is enough to take (24) into account to get

$$\mathscr{J} Q_n^{(\lambda)}(z) = \sum_{\nu=0}^{n+H} \vartheta_{n,\nu} p_\nu(z),$$

where $\vartheta_{n,\nu} d_\nu^2 = \langle \mathbf{u}, p_\nu \mathscr{J} Q_n^{(\lambda)}\rangle = \langle (\mathscr{E}^{-}\widetilde{\phi})p_\nu, Q_n^{(\lambda)}\rangle_S$ which vanishes when $\nu + \deg\widetilde{\phi} < n$.

Finally, using Theorem 3.1 we obtain

$$\mathscr{J} Q_n^{(\lambda)}(z) = \sum_{\nu=0}^{n+H} \varpi_{n,\nu} Q_\nu^{(\lambda)}(z),$$

where $\varpi_{n,\nu}\mathbf{d}_\nu^2 = \langle \mathscr{J} Q_n^{(\lambda)}, Q_\nu^{(\lambda)}\rangle_S = \langle Q_n^{(\lambda)}, \mathscr{J} Q_\nu^{(\lambda)}\rangle_S$ which vanishes when $\nu + H < n$, and thus the third relation is proved. □

4 The examples

We illustrate the results obtained in this paper with several examples covering continuous, discrete and q–Hahn cases. Other examples appearing in the literature are also included in this general approach and their properties can be deduced from the results introduced here. We also remark that the nonstandard inner product associated with the q–Freud linear functional and another one related to a 1–singular semiclassical functional are new and they have not been considered before.

4.1 A continuous example: the Jacobi–Sobolev Polynomials

The family of monic Jacobi orthogonal polynomials $(P_n^{\alpha,\beta})$ is at the top of the continuous classical polynomials in the Askey scheme and they can be written as (see, for example, [8])

$$P_n^{\alpha,\beta}(x) = \frac{2^n(\alpha+1)_n}{(\alpha+\beta+n+1)_n}\, {}_2F_1\left(\left. \begin{matrix} -n,\ \alpha+\beta+n+1 \\ \alpha+1 \end{matrix} \right| \frac{1-x}{2} \right), \qquad \alpha,\ \beta > -1.$$

In fact, this family satisfies the following orthogonality property:

$$\left\langle \mathbf{u}^{\alpha,\beta}, P_n^{\alpha,\beta} P_m^{\alpha,\beta} \right\rangle = \frac{2^{2n+\alpha+\beta+1}\Gamma(n+1)\Gamma(\alpha+n+1)\Gamma(\beta+n+1)\Gamma(\alpha+\beta+n+1)}{(\alpha+\beta+2n+1)(\Gamma(\alpha+\beta+2n+1))^2}\delta_{n,m},$$

where the linear functional $\mathbf{u}^{\alpha,\beta}$ has the following integral representation:

$$\left\langle \mathbf{u}^{\alpha,\beta}, P \right\rangle = \int_{-1}^{1} P(x)(1-x)^{\alpha}(1+x)^{\beta} dx,$$

and satisfies the distributional equation:

$$\mathscr{D}((1-x^2)\mathbf{u}^{\alpha,\beta}) = (\beta - \alpha - x(\alpha+\beta+2))\mathbf{u}^{\alpha,\beta},$$

where, as we pointed out in Section 2, $\mathscr{D} = \mathscr{D}^* = \frac{d}{dx}$, so $t = 2$, $p = 1$, $\sigma = 0$, and $H = 2$.

Then, we can consider the Sobolev inner product defined by

$$\langle f, g \rangle_S = \int_{-1}^{1} f(x)g(x)(1-x)^{\alpha}(1+x)^{\beta} dx + \lambda \int_{-1}^{1} f'(x)g'(x)(1-x)^{\alpha}(1+x)^{\beta} dx,$$

where α, $\beta > -1$, and $\lambda \geq 0$. This nonstandard inner product has been considered in a lot of articles, and we refer the reader to the surveys mentioned several times during the paper for more details. We denote by $(Q_n^{\alpha,\beta})$ the sequence of monic polynomials orthogonal with respect to the inner product $(f, g)_S$, which are called monic Jacobi–Sobolev orthogonal polynomials.

The results obtained in this paper allow us to recover some relations between Jacobi and Jacobi–Sobolev orthogonal polynomials:

$$P_n^{\alpha-1,\beta-1}(x) = Q_n^{\alpha,\beta}(x) + \theta_{n,n-1}^{\alpha,\beta} Q_{n-1}^{\alpha,\beta}(x) + \theta_{n,n-2}^{\alpha,\beta} Q_{n-2}^{\alpha,\beta}(x),$$

$$(x^2-1)P_n^{\alpha,\beta}(x) = Q_{n+2}^{\alpha,\beta}(x) + \sum_{\nu=n-2}^{n+1} \mu_{n,\nu}^{\alpha,\beta} Q_{\nu}^{\alpha,\beta}(x).$$

Moreover, according to (23) we can define the linear functional

$$\mathscr{J}^{\alpha,\beta} = (1-x^2)\mathscr{I} + \lambda(\alpha - \beta + x(\alpha+\beta))\frac{d}{dx} - \lambda(1-x^2)\frac{d^2}{dx^2},$$

and Corollary 3.4 yields

$$-\mathscr{J}^{\alpha,\beta} Q_n^{\alpha,\beta}(x) = P_{n+2}^{\alpha,\beta}(x) + \sum_{\nu=n-2}^{n+1} \vartheta_{n,\nu}^{\alpha,\beta} P_{\nu}^{\alpha,\beta}(x),$$

$$-\mathscr{J}^{\alpha,\beta} Q_n^{\alpha,\beta}(x) = Q_{n+2}^{\alpha,\beta}(x) + \sum_{\nu=n-2}^{n+1} \varpi_{n,\nu}^{\alpha,\beta} Q_{\nu}^{\alpha,\beta}(x).$$

Observe that the minus signs appear due to the factor $(1-x^2)\mathscr{I}$ in $\mathscr{J}$.

4.2 A discrete example: the Δ–Meixner–Sobolev polynomials

Monic Meixner orthogonal polynomials can be written as (see, for example, [8])

$$M_n(x;\beta,c) = \frac{(\beta)_n c^n}{(c-1)^n}\, {}_2F_1\left(\begin{array}{c} -n,\ -x \\ \beta \end{array} \middle| 1-\frac{1}{c} \right), \qquad \beta>0,\ 0<c<1.$$

In fact, this family satisfies the following orthogonality property:

$$\left\langle \mathbf{u}^M, M_n M_m \right\rangle = \frac{(\beta)_n c^n n!}{(1-c)^{2n+\beta}} \delta_{n,m},$$

where the linear functional $\mathbf{u}^M$ has the following representation:

$$\left\langle \mathbf{u}^M, P \right\rangle = \sum_{x=0}^{\infty} P(x)\, \frac{(\beta)_x}{\Gamma(x+1)}\, c^x,$$

and it satisfies the distributional equations

$$\mathscr{D}^*((x+\beta)\mathbf{u}^M) = \left(x\left(1-\frac{1}{c}\right)+\beta\right)\mathbf{u}^M,$$

$$\mathscr{D}(x\mathbf{u}^M) = (x(c-1)+\beta c)\,\mathbf{u}^M,$$

where $\mathscr{D}=\Delta$ and $\mathscr{D}^*=\nabla$, so $t=1$, $p=1$, $\sigma=0$, and $H=1$.

Now let us consider the nonstandard inner product defined by

$$\langle f,g\rangle_S = \sum_{x=0}^{\infty} f(x)g(x)\,\frac{(\beta)_x}{\Gamma(x+1)}\,c^x + \lambda\sum_{x=0}^{\infty}(\Delta f)(x)(\Delta g)(x)\,\frac{(\beta)_x}{\Gamma(x+1)}\,c^x, \qquad (25)$$

where $\beta>0$, $0<c<1$, and $\lambda>0$. This inner product is known in the literature as the Δ–Sobolev inner product. We denote by $(Q_n^{(\lambda)}(x;\beta,c))$ the sequence of monic polynomials orthogonal with respect to (25), which are called monic Δ–Meixner–Sobolev orthogonal polynomials.

Like in the Jacobi case, the results obtained in this paper allow us to recover some relations between the families of orthogonal polynomials $(M_n(x;\beta,c))$ and $(Q_n^{(\lambda)}(x;\beta,c))$:

$$M_n(x;\beta-1,c) = Q_n^{(\lambda)}(x;\beta,c) + f_n^M(\lambda;\beta,c)Q_{n-1}^{(\lambda)}(x;\beta,c),$$

$$(x-1)M_n(x;\beta,c) = Q_{n+1}^{(\lambda)}(x;\beta,c) + \mu_{n,n}^M Q_n^{(\lambda)}(x;\beta,c) + \mu_{n,n-1}^M Q_{n-1}^{\alpha,\beta}(x;\beta,c).$$

Moreover we define the operator $\mathscr{J}^M$ as

$$\mathscr{J}^M = (x-1)\mathscr{I} + \lambda\,(1+\beta c - x(c-1))\,\nabla - \lambda(x-1)\Delta\nabla,$$

and applying the results of Section 3 we obtain

$$\mathscr{J}^M Q_n^{(\lambda)}(x;\beta,c) = M_{n+1}(x;\beta,c) + \vartheta_{n,n}^M M_n(x;\beta,c) + \vartheta_{n,n-1}^M M_{n-1}(x;\beta,c),$$

$$\mathscr{J}^M Q_n^{(\lambda)}(x;\beta,c) = Q_{n+1}^{(\lambda)}(x;\beta,c) + \varpi_{n,n}^M Q_n^{(\lambda)}(x;\beta,c) + \varpi_{n,n-1}^M Q_{n-2}^{(\lambda)}(x;\beta,c).$$

4.3 A q example: q–Freud type polynomials

The family of monic q–Freud polynomials, (P_n), satisfies the relation [4]

$$(\mathscr{D}P_n)(x(s)) = [n]P_{n-1}(x(s)) + a_n P_{n-3}(x(s)), \qquad n \geq 0,$$

where $x(s) = q^s$, with $0 < q < 1$, $\mathscr{D} = \mathscr{D}_q$, $P_{-1} \equiv 0$, $P_0 \equiv 1$, and $P_1(x) = x$.

So by Theorem 2.2 we get $\phi(x) = 1$, $\sigma = 2$, and $t = 0$; hence (P_n) is orthogonal with respect to the linear functional $\mathbf{u}^{qF}$ of class 2 which fulfills the distributional equations

$$\mathscr{D}(\mathbf{u}^{qF}) = \psi\mathbf{u}^{qF}, \qquad \deg\psi = 3, \tag{26}$$

$$\mathscr{D}^*((1 + x(q-1)\psi)\mathbf{u}^{qF}) = q\psi\mathbf{u}^{qF}.$$

Furthermore, these polynomials are symmetric (see [4]) and satisfy the three–term recurrence relation

$$xP_n = P_{n+1} + c_n P_{n-1}, \qquad n \geq 1.$$

In fact, a straightforward computation shows $a_n = K(q)q^{-n}c_n c_{n-1} c_{n-2}$, and the sequence (c_n) satisfies the nonlinear recurrence relation

$$q[n]c_{n-1} + K(q)q^{-n+1}c_n c_{n-1} c_{n-2} = [n-1]c_n + K(q)q^{-n-1}c_{n+1}c_n c_{n-1}, \quad n \geq 2,$$

with initial condition $c_0 = 0$, and for the sake of simplicity we choose the initial conditions c_1 and c_2 in such a way that

$$c_1^2 + c_1 c_2 = 1,$$

and we have $\lim_{q\to 1^-} K(q) = 4$. Therefore, $\psi(x) = -K(q)q^{-3}x^3$ and the linear functional $\mathbf{u}^{qF}$ has the following integral representation:

$$\left\langle \mathbf{u}^{qF}, P \right\rangle = \int_{-1}^{1} P(x) \frac{1}{((q-1)K(q)q^{-3}q^{4x};q^4)_\infty} d_q(x),$$

where

$$\int_{-1}^{1} f(x)d_q(x) = (1-q)\sum_{k=0}^{\infty} f(q^k)q^k + (1-q)\sum_{k=0}^{\infty} f(-q^k)q^k.$$

In fact, this family satisfies the following orthogonality property:

$$\left\langle \mathbf{u}^{qF}, P_n P_m \right\rangle = 2c_1 c_2 \cdots c_n \delta_{n,m}.$$

Now, to illustrate our results we can introduce the nonstandard inner product defined by

$$\begin{aligned} \langle f,g\rangle_S = &\int_{-1}^{1} f(x)g(x)\frac{1}{((q-1)K(q)q^{-3}q^{4x};q^4)_\infty} d_q(x) \\ &+ \lambda\int_{-1}^{1} (\mathscr{D}_q f)(x)(\mathscr{D}_q f)(x)\frac{1}{((q-1)K(q)q^{-3}q^{4x};q^4)_\infty} d_q(x). \end{aligned} \tag{27}$$

We denote by (Q_n^{qF}) the sequence of monic polynomials orthogonal with respect to the inner product (27), which we call the monic q–Freud–Sobolev orthogonal polynomials.

The theory developed in the previous sections allows us to link the q–Freud–Sobolev polynomials with the q–Freud polynomials. Taking into account the distributional equation (26) and Proposition 3.2 we define

$$\mathscr{J}^{qF} = (1-(q-1)K(q)q^{-7}x^4)\mathscr{I} + \frac{\lambda}{q}K(q)q^{-6}x^3\mathscr{D}_{1/q} - \lambda(1-(q-1)K(q)q^{-7}x^4)\mathscr{D}_q\mathscr{D}_{1/q}.$$

In this case $H = \deg\widetilde{\phi} = 4$ and the results in Section 3 can be rewritten as

$$(1-(q-1)K(q)q^{-7}x^4)P_n(x) = \sum_{\nu=n-4}^{n+4} \mu_{n,\nu}^{qF} Q_\nu^{qF}(x),$$

$$\mathscr{J}^{qF} Q_n^{qF}(x) = \sum_{\nu=n-4}^{n+4} \vartheta_{n,\nu}^{qF} P_\nu(x),$$
$$\mathscr{J}^{qF} Q_n^{qF}(x) = \sum_{\nu=n-4}^{n+4} \varpi_{n,\nu}^{qF} Q_\nu^{qF}(x).$$

4.4 A 1–singular semiclassical polynomials of class 1

The family considered for this example was studied by J.C. Medem in [17]. Such a family of monic polynomials, (S_n), which we call Medem polynomials, is orthogonal with respect to the linear functional $\mathbf{w}$ which satisfies the distributional equation

$$\mathscr{D}(x^3\mathbf{w}) = (-x^2+4)\mathbf{w}, \tag{28}$$

where $\mathscr{D} = \mathscr{D}^* = \dfrac{d}{dx}$, $t=3$, $p=2$, and so $\sigma = 1$, with initial condition $(\mathbf{w})_1 = \langle \mathbf{w}, x\rangle = 0$.

Moreover a straightforward computation shows that $a_{\sigma+1} + b_{\sigma+2} = 0$, i.e., $\mathbf{w}$ is 1–singular. Indeed, $\mathbf{w}$ is the symmetrized linear functional associated with the linear functional $\mathbf{b}^{(-\frac{5}{2})}$ (see [3] Chapter 1, Sections 8 and 9), i.e., $(\mathbf{w})_{2n+1} = 0$ and $(\mathbf{w})_{2n} = (\mathbf{b}^{(-\frac{5}{2})})_n$, for any $n \geq 0$.

Notice that the linear functional $\mathbf{b}^{(\alpha)}$ has the following integral representation:

$$\left\langle \mathbf{b}^{(\alpha)}, P \right\rangle = \frac{1}{2\pi i}\int_{\mathbb{T}} P(z) z^\alpha e^{-\frac{2}{z}} dz, \qquad \alpha > -2,$$

and thus, after a straightforward calculation, we get that the linear functional $\mathbf{w}$ has the following integral representation:

$$\langle \mathbf{w}, P\rangle = \frac{1}{2\pi i}\int_{\mathbb{T}} P(z) z^{-4} e^{-\frac{2}{z^2}} dz,$$

where $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ is the unit circle.

This family can be written in terms of the Bessel polynomials as follows [17]:

$$S_{2n}(x) = \frac{2^n}{(n-\frac{3}{2})_n} B_n^{(-\frac{5}{2})}(x^2) = r_{2n} x^{5-n} e^{\frac{2}{x^2}} (\mathscr{D}^*)^n \left(x^{4n-5} e^{-\frac{2}{x^2}} \right),$$
$$S_{2n+1}(x) = \frac{2^n}{(n-\frac{1}{2})_n} x B_n^{(-\frac{3}{2})}(x^2) = r_{2n+1} x^{3-n} e^{\frac{2}{x^2}} (\mathscr{D}^*)^n \left(x^{4n-3} e^{-\frac{2}{x^2}} \right),$$

where $B_n^{(\alpha)}$ is the Bessel polynomial of degree n with parameter α, and $r_n \neq 0$ are the corresponding normalization coefficients for $n \geq 0$.

We can introduce the nonstandard inner product defined by

$$(f,g)_S = \frac{1}{2\pi i} \int_{\mathbb{T}} f(z) \overline{g(z)} z^{-4} e^{-\frac{2}{z^2}} dz + \frac{\lambda}{2\pi i} \int_{\mathbb{T}} f'(z) \overline{g'(z)} z^{-4} e^{-\frac{2}{z^2}} dz. \tag{29}$$

We denote by (Q_n^S) the sequence of monic polynomials orthogonal with respect to the inner product (29), which we call the monic Medem–Sobolev orthogonal polynomials.

Taking into account the distributional equation (28) and Proposition 3.2 we define

$$\mathscr{J}^S = x^3 \mathscr{J} + \lambda(4x^2 - 4) \frac{d}{dx} - \lambda x^3 \frac{d^2}{dx^2}.$$

In this case $H = \deg \widetilde{\phi} = 3$ and the results in Section 3 can be rewritten as

$$x^3 S_n(x) = Q_{n+3}^S(x) + \sum_{\nu=n-3}^{n+2} \mu_{n,\nu}^S Q_\nu^S(x),$$
$$\mathscr{J}^S Q_n^S(x) = S_{n+3}(x) + \sum_{\nu=n-3}^{n+2} \vartheta_{n,\nu}^S S_\nu(x),$$
$$\mathscr{J}^S Q_n^S(x) = Q_{n+3}^S(x) + \sum_{\nu=n-3}^{n+2} \varpi_{n,\nu}^S Q_\nu^S(x).$$

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Appendix. Proof of Theorem 2.4

We are going to prove the result for $k \geq 0$ since the case $k < 0$ is totally analogous to this one. When $k = 0$ we know that ϕ and ψ are polynomials of degree, at most $\sigma + 2$ and $\sigma + 1$, respectively, where $\sigma = \sigma_0$ is the order of $\mathbf{u}$ with respect to the pair (ϕ, ψ). Then, using (5) we get for any monic polynomial π,

$$\mathscr{D}(\pi\phi\mathbf{u}_0) = (\mathscr{D}\pi)\phi\mathbf{u}_0 + (\mathscr{E}^+\pi)\mathscr{D}(\phi\mathbf{u}_0) = \Big((\mathscr{D}\pi)\phi + (\mathscr{E}^+\pi)\psi\Big)\mathbf{u}_0,$$

where $\tilde{\pi} := (\mathscr{D}\pi)\phi + (\mathscr{E}^+\pi)\psi$ is a polynomial of degree, at most, $\deg\pi + \sigma + 1$.
Moreover, if the pair (ϕ, ψ) is admissible, then the leading coefficient is

$$q^m(a_{\sigma+1} + q^{-1}[m]^* b_{\sigma+2}) \neq 0, \quad \text{with} \quad m = \deg\pi.$$

If we assume the result holds for every $0 \leq k < K$, let us prove it for $k = K$. Thus, the linear functionals $\mathbf{u}_k$ satisfy the distributional equation $\mathscr{D}(\phi_k\mathbf{u}_k) = \psi_k\mathbf{u}_k$ for $0 \leq k < K$, and then we can obtain

$$\begin{aligned}
\mathscr{D}(\phi_{K-1}\mathbf{u}_K) &= (\mathscr{D}\phi_{K-1})\mathbf{u}_K + (\mathscr{E}^+\phi_{K-1})\mathscr{D}\mathbf{u}_K \\
&= (\mathscr{D}\phi_{K-1})\mathbf{u}_K + q(\mathscr{E}^+\phi_{K-1})\mathscr{E}^+\left((\mathscr{D}\phi_{K-1})\,\mathbf{u}_{K-1}\right) \\
&= (\mathscr{D}\phi_{K-1})\mathbf{u}_K + q(\mathscr{E}^+\phi_{K-1})\mathscr{E}^+\Big(\psi_{K-1}\mathbf{u}_{K-1}\Big) \\
&= (\mathscr{D}\phi_{K-1})\mathbf{u}_K + q(\mathscr{E}^+\psi_{K-1})\mathscr{E}^+\Big(\phi_{K-1}\mathbf{u}_{K-1}\Big) \\
&= \Big(\mathscr{D}\phi_{K-1} + q(\mathscr{E}^+\psi_{K-1})\Big)\mathbf{u}_K = \widetilde{\psi}_K\,\mathbf{u}_K.
\end{aligned}$$

Therefore taking into account Definition 2.5 and Remark 2.3, there exist polynomials ϕ_K, ψ_K satisfying

$$\mathscr{D}(\phi_K\mathbf{u}_K) = \psi_K\mathbf{u}_K, \tag{30}$$

and a monic polynomial ξ_K such that $\phi_{K-1} = \xi_K\phi_K$, and now, taking (30) into account the polynomials ψ_K and ψ_{K-1} fulfill the relation

$$\mathscr{D}(\phi_{K-1}) + q(\mathscr{E}^+\psi_{K-1}) = \mathscr{D}(\xi_K)\phi_K + \mathscr{E}^+(\xi_K)\psi_K, \tag{31}$$

and we can check that ψ_K is a polynomial of degree, at most, $\sigma_K + 1$. Furthermore, using (31) and working in the same way as in the case $k = 0$, we get

$$\mathscr{D}(\pi\phi_K\mathbf{u}_K) = \Big((\mathscr{D}\pi)\phi_K + (\mathscr{E}^+\pi)\psi_K\Big)\mathbf{u}_K.$$

Clearly, $\tilde{\pi} := (\mathscr{D}\pi)\phi_K + (\mathscr{E}^+\pi)\psi_K$ is a polynomial of degree, at most, $m + \sigma_K + 1$.

In fact, if the pair (ϕ_K, ψ_K) is admissible the leading coefficient of $\widetilde{\pi}$ is

$$q^m(a_{\sigma_K+1} + q^{-1}[m]^* b_{\sigma_K+2}) \neq 0, \tag{32}$$

where the coefficients a_{σ_K+1} and b_{σ_K+2} can be obtained recursively

$$q^{\sigma_{K-1}+2}\left(a_{\sigma_{K-1}+1}+q^{-1}[\sigma_{K-1}+2]^*b_{\sigma_{K-1}+2}\right)=q^{\sigma_{K-1}-\sigma_K}\left(a_{\sigma_K+1}+q^{-1}[\sigma_{K-1}-\sigma_K]^*b_{\sigma_K+2}\right). \tag{33}$$

Notice that in the continuous and discrete cases $q=1$ and $[m]^*=m$. Then, using the admissibility condition of Definition 2.3, the above expression, and (32) in a recursive way, we get that the leading coefficient is equal to

$$a_{\sigma_K+1}+mb_{\sigma_K+2}=a_{\sigma+1}+\left(m+\sum_{\nu=1}^{k}(\sigma_\nu+2)\right)b_{\sigma+2}\neq 0, \quad k\geq 0.$$

Therefore in the q–Hahn case there are infinitely many values of q for which the expression (32) is different from zero for any nonnegative integers m and k. Hence the result follows for $k\geq 0$.

Observe that by (33) if there exists an integer k such that the pair (ϕ_k,ψ_k) is admissible, then the pair (ϕ_ℓ,ψ_ℓ) is admissible for every $\ell\geq k$.