Negative numbers in the 18th and 19th centuries: phenomenology and representations

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Abstract

This article presents a categorization of the phenomena and representations used to introduce negative numbers in mathematics books published in Spain during the 18th and 19th centuries. Through a content analysis of fourteen texts which were selected for the study, we distinguished four phenomena typologies: physical, accounting, temporal and mathematical. Four types of representations are also identified: verbal, numeric, graphic and algebraic. These results reflect the same level of understanding and treatment of negative numbers in Spain as is found in other European countries during this time period.

Keywords. Negative numbers, phenomenology, representations, textbooks, mathematics history.

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Resumen

En este artículo se presenta una categorización sobre los fenómenos y las representaciones utilizados para presentar los números negativos en libros de texto de matemáticas publicados en España durante los siglos XVIII y XIX. Mediante un análisis de contenido en catorce textos seleccionados para el estudio, se distinguen cuatro tipologías de fenómenos: físicos, contables, temporales y matemáticos. También se identifican cuatro tipos de representaciones: verbales, numéricas, gráficas y algebraicas. Estos resultados reflejan en España un conocimiento y tratamiento de los números negativos al mismo nivel que el de otros países europeos de la época.

Palabras clave. Números negativos, fenomenología, representaciones, libros de texto, historia de las matemáticas.

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History of Mathematics and Mathematics Education

Mathematics Education studies how people learn and do mathematics at all levels, and also how this learning is influenced by teaching strategies (Dörfler, 2000; Schoenfeld, 2000). Given that Mathematics has a long history, it is also important to study how mathematical knowledge has evolved over time, until it has acquired the scientific meanings and formal development that are culturally accepted today. Wussing (1998) maintains that without the history of concepts, problems and special mathematic disciplines, our view of Mathematics development is incomplete, since all mathematical ideas and knowledge have been formed within a specific socio-historical situation.

Some mathematical concepts present such complexity and characteristics of abstraction, that not only was their acceptance, development and formalization slow in coming, but their emergence and consolidation took place during times of sharp epistemological differences between prestigious mathematicians of the era. A brief overview reveals that negative numbers are a clear example of one of these controversial historical concepts.

Epistemological difficulties that arise with negative numbers continue to be an issue in the daily practice of teaching Mathematics, when teachers find resistance or difficulty in students coming to understand, interpret and use them correctly. Researchers in Mathematics Education have taken an interest in negative numbers, since the difficulties students experience in understanding them also has repercussions on other concepts, on correctly interpreting certain situations and being able to solve problems that relate to them (Maz & Rico, 2007).

Some authors state that students' difficulties in understanding and operating with negative numbers reflect the historical stages of how these numbers developed (Glaeser, 1981; González-Marí, 1995), so it becomes necessary to understand a bit of history in order to properly teach this topic. According to Gómez (1993; p. 13): "*Understanding the historical evolution of numbering systems and the reasons that prompted the changes from one system to another help us give meaning to our existing knowledge*". On this matter, it is clear that the main purpose of history is "*the general study of change over time, highly relevant not only for our own present-day study of* *mathematics, but also for communicating mathematics at every level"* (Rogers, 1993; p. 107).

Old Mathematics books: the primary source for didactic inquiry

When constructing concepts within a scientific framework, particularly with mathematical knowledge, language takes on an important mediating role, where the referent is the literal text itself. According to Lizcano (1993, p. 30), text is where *"mathematics is actually produced"*, so the analysis of Mathematics textbooks takes on epistemological importance.

When working in Mathematics Education, and in the study of textbooks, one must consider that the texts being studied and analyzed are primary sources for understanding the state of scientific knowledge in a particular period, as well as for studying how this knowledge took shape in the prevailing educational structure at the time of their publication. Textbooks are didactic documents belonging to a determined curriculum framework, and therefore, analysis of their content should take into account the didactic nature of the documents. From this perspective, we emphasize that mathematics textbooks are not exclusively formal documents, but rather teaching material, with educational purposes, with the intent of conveying certain meanings to ensure a proper understanding of the formal concepts that are presented (Segovia & Rico, 2001).

These ideas indicate that studying textbooks, whether current or past, contributes toward our understanding of the development of specific knowledge within a scientific field, as well as the curriculum plans and educational systems within a country.

This need and interest in the history and development of concepts gives a certain relevance to our research in an important area of content analysis, that is, the study of the phenomenology and representations of negative numbers. This way we play our part in sustaining that *"knowledge of one's own history is one of the strongest incentives to consolidating one's own identity"* (Rico & Sierra, 1994; p. 100).

Description of the study

In this document we focus our attention on Mathematics textbooks in Spain in the 18th and 19th centuries, especially in the content areas which we have mentioned. Although Spanish mathematicians in this period are not noted for their contributions to the development of Mathematics, there was a great deal of mathematics activity taking place, focused on teaching as well as on applying and disseminating knowledge through the publication of books (Rico & Maz, 2005).

The two guiding questions for this study are:

- What phenomena were used to exemplify and justify the introduction of negative numbers in mathematics books during the 18th and 19th centuries in Spain?
- How are negative numbers represented in the books selected?

During the 18th and 19th centuries in Spain, we find large contrasts in the type of Mathematics textbooks that were written, in the prototype of their authors, as well as in the social class of their target audiences (Maz, 2005). For this reason we have selected a few significant Arithmetic and Algebra texts, which are representative of this historical period.

Textbook selection criteria were as follows:

a) published in Spain between 1700 and 1900;

b) the author was Spanish;

c) written in Spanish.

64 textbooks were reviewed, and 15 were finally selected for the study (see appendix).

The representativity of authors whose Mathematics books were selected for the study is primarily cultural, and allows us an overview of the ideas that were present in mathematics production from this period .

Phenomenology and Representations

If we consider the curriculum organizing factors proposed by Rico (1997), we see that most of these can be found in mathematics textbooks and particularly in the older ones, since these are curriculum documents whose structure and organization are determined by their educational purpose. Let us focus on two of these organizing factors:

- phenomena that underlie the origin of concepts, and
- the ways that these concepts are expressed, that is, their representations.

The phenomenology of a mathematics structure studies the phenomena that underlie each of the concepts characterized therein. Phenomenology, or phenomenological analysis, considers the basic problems or questions which a certain mathematical concept addresses, as well as the usual situations where these questions are posed (Rico, Lupiáñez, Marín & Gómez, 2008). These phenomena, to a greater or lesser extent, underlie the examples and activities which are presented in a textbook. Therefore, if one wants to presents a mathematical topic in all its conceptual and procedural richness, showing the multiplicity of its uses and meanings, it should be examined through a diversity of contexts and situations, in connection with different problems, and it should be linked to other fields of knowledge (Rico & Lupiáñez, 2008).

Mathematics learning processes give an important place to phenomenology, since mathematical thinking arises from the phenomena which mathematical structures abstract and organize. These phenomena fall into large families from the natural world, and social and mental spheres (Rico et al., 2007).

Puig (2001) indicates that phenomenology is a means whereby to organize mathematical ideas, and when these ideas are related to the school systems where one is involved in teaching, we are looking at didactic phenomenology.

The authors examined in this study justify the introduction of negative numbers in different ways, from the interpretation of concrete situations such as movement (displacements), to the formal extension of subtraction, not to mention operational interpretations and rhetorical explanations typical of Arithmetic.

The situations used to exemplify and characterize negative amounts are organized globally into four groups:

- physical phenomena,
- accounting situations,
- temporal or chronological situations, and
- mathematical contexts.

We identify these situations below and present an example of each.

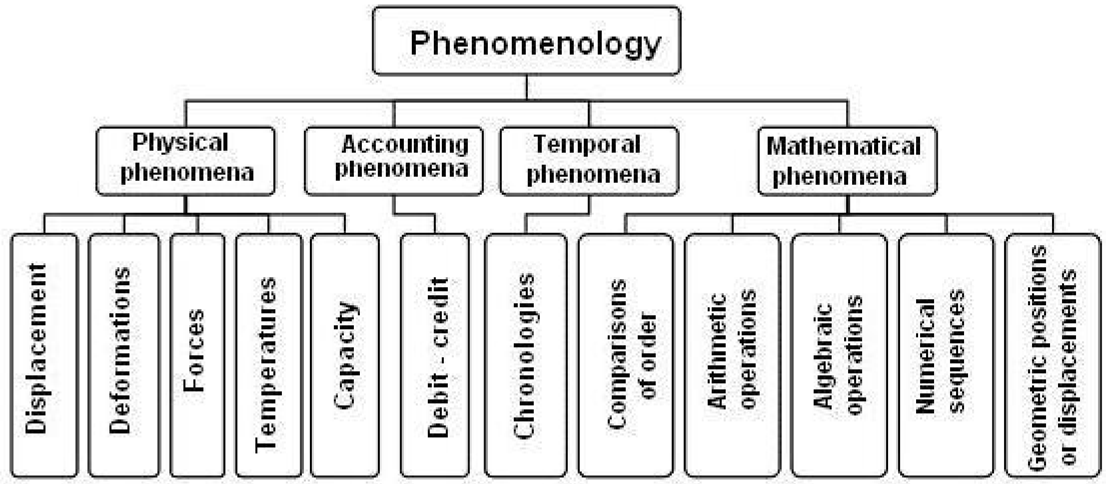


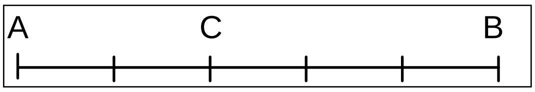
Figure 1. **Phenomenological typology**

Physical phenomena

Different authors often turn to natural phenomena which are explained by laws of physics; it appears that these phenomena facilitate the introduction of negative numbers in situations which are relatively familiar to the students. We find examples from five categories within this type.

a) ***Displacements:*** These are based on situations where objects move forward or backward. They have to do with comparing measurements in opposite directions. Negative quantities are used to model the phenomenon of backward movement:

> *"It is easy to form ideas about these quantities which are less than nothing. Suppose that there are three leagues from C to B and from C to A there are two.*



If a traveler is at C with the intention of getting to B, and in effect leaves C and arrives at B, it is truthful to say that he progressed and that his progress is greater than nothing and that the measure of this progress is three leagues. If, despite his intent, he finds himself detained at C, he has made no progress, or, his progress is equal to nothing. If instead of walking toward B, he goes from C to A, in common language we would say he went backward, and in order to express that he did the opposite of what he intended to do, it may be said that he progressed less than nothing, and that his progress is -2 leagues, since in this case, two leagues are less than nothing." (Ulloa, 1705; pp. 20).

This example presents a comparison of situations that lead to measuring distances using different, opposing directions. The origin, or starting point at C, is used as a point of reference for the two movements, and for the intent to move. Since the choice of direction is arbitrary, the measurement obtained as negative could be either -2 or -3, at the convenience of the problem solver. This corresponds to a daily phenomenon based on movements (displacements) where one kind of action is assigned a negative quantity; it considers a quantity negative due to a transformation or change.

b) ***Deformations:*** In these contexts, objects are subjected to a certain action in one direction or another.

> "… finally, if *a* expresses how much a line has been lengthened, -*a* represents how much it has been shortened" (Verdejo, 1794; p. 37).

Here one takes recourse to a situation where the object changes from its initial state and undergoes some type of variation in terms of its length.

c) ***Forces:*** Analogies are presented between Newton's third law of action and reaction, and positive and negative quantities.

> "... if *a* is the value of a force that operates from right to left, -*a* is the same force operating from left to right" (Juan Justo García, 1782; p. 54).

It is interesting to see how the author uses Newton's third law (action and reaction in Dynamic Physics) in order to illustrate the nature of negative quantities, even though strictly speaking he refers to directed magnitudes more than integers.

d) ***Temperatures:*** Variations in temperature as measured on a thermometer with reference to the value of zero are used to compare a move from positive to negative numbers on the number line.

> "... in order to understand degrees of temperature, one needs to know not only how many degrees, but also whether the degrees are above or below zero on the thermometer" (Fernández & Cardín, 1858; p. 10).

The situation of the thermometer and temperatures is a situation which is represented perfectly in an integer model, since there is a unique zero and it is coherent to pass from one side of zero to the other: values on both sides still indicate the same type of quantity, that is, temperatures.

e) ***Capacity:*** Adding or removing products or substances to a recipient or a location can also serve as a simile with negative and positive numbers when a mathematical operation is performed with these quantities.

> "… if we want to figure out how long it will take to fill a water reservoir, where water enters on one end and goes out on another end, we must look not only at the water that enters, but also at the water that goes out. Since the water that enters is contributing to the purpose that we propose, this will be the positive amount; the water that is going out is emptying the reservoir, the opposite of filling it, so it will be the negative" (Vallejo, 1813; p. 163, second edition).

Accounting phenomena

These phenomena have to do with managing capital in the debit-credit relationship, or debts and profits; these are helpful both for giving meaning to negative situations and for illustrating what the authors understand by a quantity less than nothing.

> "… Let us turn now to another subject and to his accounts. We find that this person incurs a debt of 300 ducats every year; if we want to compare the situation of these two subjects, in order to see who is more indebted, we will say it is the latter, because 300 ducats is much more than the 150 ducats that we figured before. But if we pose the question resolved by Algebra, where our purpose is to determine this subject's savings, we would find that his annual savings is -300 ducats; and if we compare this savings to the earlier case, which was -150, we would not say that -300 is a greater savings than -150, quite the contrary; because in an absolute sense, if we want to know which of the two saves more, and we find that neither of them saves, the one who is in better standing is the one who incurs the least debt" (Vallejo, 1813; pp. 167-168).

Here is an everyday situation with relative quantities (having/owing) that gives meaning to negative quantities. In this case the comparison of capital is the phenomenon which is modeled by the use of negative quantities.

Temporal phenomena

These refer to comparing a time period with respect to a date fixed by a certain event, such as the birth of Christ or the French Revolution.

> "… the time before and after a certain era … are directly opposing quantities, and their different meanings, though they depend largely on the wishes of the reckoner, must be indicated with signs that represent positive or negative, indicating their opposite value" (Fernández Vallín y Bustillo, 1857; p. 233).

Mathematical phenomena

Objects from the mathematical world are used as a resource to illustrate negative numbers. Five types of such cases are found:

a) ***Comparisons of order:*** Through comparing numeric values.

> "... Also, given that $-5<-3$, it is therefore true that $-5+2<-3+2$ and $-5-2<-3-2$. The same occurs in $-3<2$ and $-3+4<2+4$ and $-3-4<2-4$." (Odriozola, 1827, p. 444).

b) ***Arithmetic operations:*** Through addition or subtraction.

> "If a positive +3, is summed with a negative, either equal in value, -3, or greater than it, -4. or less than it, -2, the sum will be +3-3 or +3-4 or +3-2: because the sum will be +3+-3 or +3+-4 or +3+-2 ..." (Ulloa, 1705; p. 21).

c) ***Algebraic operations:*** One recurs to solving for the roots of algebraic expressions and to problem solving.

> "Let yy + 5y + 6 Ω 0. Find its solution. The divisors of the last term as the same as in the previous example; because all the roots must be negative, since there is no alternation of signs, I will try to break it down by y + 2 Ω 0, by y + 3 Ω 0, etc. And since I find it comes out right by y + 2 Ω 0, I say that one of the roots is –2, and the other –3" (Tosca, 1707; p. 189).

d) ***Numerical sequences*****:** Successions and series make it possible to observe the positioning of numerical values according to their sign and value.

> "A progression is said to be increasing when the ratio is positive and the terms are becoming larger; and decreasing in the opposite case … 16, 12, 8, 4, 0, –4, –8, …" (Fernández y Cardín, 1858; p. 113).

e) ***Geometric positions and displacements*****:** The number line, and displacements on this line, as well as movements and rotation of segments around an axis point or around a circumference, help some authors "explain" the change of sign in numbers. Thus the segment changes its position without varying in magnitude.

> "Nor is there sufficient reason to deduce from the natural arithmetic series
>
> $$\ldots -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, \ldots$$
>
> what Mr. Coyteux erroneously calls increasing, when actually it is ascending, where

its uniform ascent is explained by the quantitative decrease in negative terms until it reaches zero, and the increase of the positives. This progressive ascent could be represented geometrically, by marking perpendicular absolute or numeric values of all the terms above or below the horizontal axis: an oblique line that passes through the point of origin, this would show the geometric place of all these lengths."

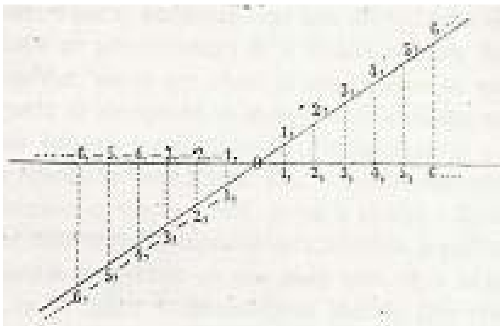


***Figure 2*.**
(Rey y Heredia, 1865; p. 33)

Representation systems

The concepts are shown through different types of symbols, graphs and signs, and each one constitutes a representation of the concept in question (Castro & Castro, 1997). Representation systems allow the author to present and communicate to the readers or students the mathematical ideas that he wishes to convey; in our case we have access to the written representations.

An analysis of the Mathematics textbooks we selected reveals four types of representations used by the authors:

a) ***Verbal:*** the author relies on giving explanations about negative numbers through verbal descriptions with a high rhetorical load.

> "Let us suppose that one man has no assets, and that he owes 1000 escudos; another man likewise has no assets, but he owes nothing; it is true that the former has less wealth than the latter; but the latter has nothing: therefore the former has less than nothing. Moreover, if 1000 escudos are given to the one who has nothing but owes 1000 escudos, he is able to pay his debt, and thereby increases his assets; but even after this increase his assets are nothing; therefore before the increase, his assets were less than nothing" (Tosca, 1709; p. 92-93).

b) ***Numeric:*** only combinations of numbers and signs are used in order to give the idea and explain negative quantities. All authors used numeric notations to present negatives.

> "**Whole number series**. If all the positive and negative numbers are placed in a line, so as to be counted starting at zero, the series
>
> …. –5, –4, –3, –2, –1, 0, 1, 2, 3, 4, 5, …..
>
> would be the place or the domain of all the whole sums, such that the addends and the sums will be found in the same series" (García de Galdeano, 1883; p. 48).

c) ***Graphs***: taking recourse to the graph of a straight line, generally containing letters that indicate places or positions. Although the diagram used is the same for several authors, if we compare the graphic representation from Ulloa in 1706 with that of Cortázar in 1892, different arguments used to justify the negative quantities that arise from these nearly identical graphics.

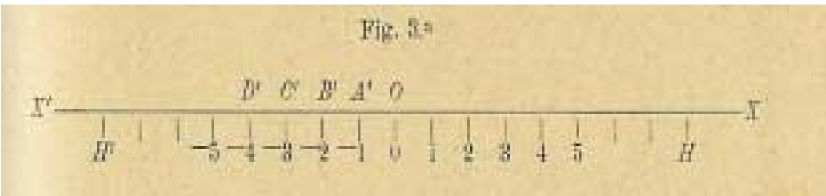


Figure 3. (Octavio de Toledo, 1900; p. 81).

d) ***Algebraic***: these representations combine numbers with signs and letters; equations are used as a recourse to show how negative quantities arise and how they are operated on.

"… we have supposed implicitly that the direct operation $\frown$ verified over a series of objects

$$a, b, c, d \ldots, \qquad (A)$$

was always possible, and yielded an object belonging to the same series, which we will call a direct series; we also supposed a new series would constitute the reciprocal objects of those in Series (A), the series

$$\bar{a}, \bar{b}, \bar{c}, \bar{d}, \ldots.., \quad (\bar{A})$$

which we will call a reciprocal series;" (Octavio de Toledo, 1900; pp. 52-53)

Conclusions

The phenomena which underlie the system of negative numbers may be based on the comparison of quantities, using relationships of “more than” and “less than”, or they rely on transformations in quantities, expressing an increase or decrease from the initial quantity, allowing a comparison between the initial and final situation using the terms “greater than” or “less than”.

In the books analyzed, we observe that the authors use phenomena drawn from contexts and situations of daily life as well as from Mathematics itself.

Our study identified four types of situations used by Spanish authors of mathematics textbooks in the 18th and 19th centuries. These situations, as exemplified earlier, correspond to: physical phenomena, accounting situations, temporal or chronological situations, and mathematical contexts.

The physical phenomena referred to have to do with displacements, deformations, resulting forces, temperatures and capacities. In all these cases there is a transformation of quantities that involve an increase or decrease from an initial quantity, making it possible to express the change using a positive or negative value. There are therefore two types of quantities: those that produce an increase (positive) and those that produce a decrease (negative). This is what Kant calls “qualified quantities”. In this case negative numbers express relative quantities, with a double, natural-number structure, that is, natural numbers with opposite directions, this situation being distinct from integers.

In the particular case of temperature, an initial point of equilibrum (point zero or the freezing point) is taken as the value of zero; quantities that raise this value are positive and those that lower it are negative, thereby establishing a continuity between negative and positive values. These are relative numbers, with an order relationship that allows for comparisons between the negative and the positive part.

Accounting phenomena show a use of negatives in the two-fold accounting of credit/debit, working with qualified quantities; as in the physical phenomena, these phenomena exemplify relative numbers.

Temporal phenomena, as in the case of temperature, allow for a fixed point of reference, where times that follow are positive, and preceding times are negative, with continuity between positive and negative quantitites.

In the mathematical phenomena, the algebraic structure of the equation $a+x = b$ comes into play, drawing out the implications of giving a numeric value to the solution where $b<a$. The negative numbers used in these situations are integers.

All these situations show applied contexts of negative numbers and expose the mathematical structure that underlies interpretation of this concept in each case. The variety of phenomena found reveals that textbook authors in the 18th and 19th centuries were interested and concerned with presenting, explaining and applying the concept of negative numbers; they also show different conceptions underlying the concept of negative number.

The evolution of representation systems reflects evolution in the understanding of negative numbers, and how its formalization progressively finds its way into Mathematics books, as seen in the representations found in this study: verbal, numeric, graphic and algebraic.

The analysis of old Mathematics manuals contributes valuable information about the long road that negative numbers have taken until they were formalized and accepted in the world of mathematics; results here can be contrasted with the phenomena and representations currently used in Mathematics school books, making their examination a matter of interest.

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