



Use of Dynamic Geometry as a support to paper and pencil activities for comprehension of ratio and proportion topics

Elena Fabiola Ruiz¹ and José Luis Lupiáñez²

¹ Superior School of Computer Sciences, National Polytechnic Institute,
Mexico City

² Department of Mathematics Education. University of Granada

¹Mexico

²Spain

Correspondence: Elena Fabiola Ruiz Ledesma. Av. Juan de Dios Batiz, S/N. Esq. Miguel Othon de Mendizabal. Colonia Lindavista. Delegación Gustavo A Madero. México, D.F. 07738. E-mail: elen_fruiz@yahoo.com.mx

© Education & Psychology I+D+i and Editorial EOS (Spain)

Abstract

Introduction. The present paper shows the importance of a joint use of pencil and paper activities and of technology so that students may develop a complete understanding of ratio and proportion. A previous experience with strategy use when solving ratio and proportion problems provided background. Prompted by a recognition of the cognitive components in students, we decided to offer support through the observation and manipulation of representations made using dynamic geometry software.

Method. Application of three activities to be solved by a group of 29 sixth-grade pupils public, primary education in Mexico City. The activities were solved with paper and pencil and using dynamic geometry.

Results. Students built figures proportional to others they were given, using comparison by superimposing one figure over another, by dragging them, and by enlarging or reducing a figure by click-and-drag on one of its vertices. They also established numerically equivalent ratios by obtaining measurements for the figures given and making the comparison as a quotient. They also were able to recover data from the problems given and represent them using drawings, then they made use of these drawings to help them solve the assigned activity.

Discussion and Conclusion. Most of the students could establish figures that were proportional to the ones given by using registers of representation, such as using drawings with the data that they were able to extract from the story problems, using a table which they filled in with both external and internal ratios, as well as equivalence relationships between the ratios and the operations used to get the numeric values. They also showed development in their qualitative proportional thinking as a support to quantitative proportional thinking.

Keywords: Ratio, proportion, dynamic environment, geometry, representations.

Received: 11/10/09 *Initial Acceptance:* 11/23/09 *Final Acceptance:* 02/12/10

Resumen

Introducción. El presente artículo plantea la importancia de la conjunción de actividades de lápiz y papel y el uso de la tecnología con la finalidad de que los estudiantes complementen su aprendizaje de razón y proporción. Se toma como antecedente una experiencia previa sobre el uso de estrategias al resolver problemas de razón y proporción. Como respuesta al reconocimiento de los componentes cognitivos de los estudiantes se decidió tener un apoyo mediante la observación y manipulación de representaciones hechas por software de geometría dinámica.

Método. Aplicación de 3 actividades a resolver por un grupo de 29 estudiantes que cursaban sexto grado de educación primaria en una escuela pública de la Ciudad de México. Las actividades fueron resueltas con lápiz y papel y con el uso de la geometría dinámica.

Resultados. Los estudiantes construyeron figuras de forma proporcional a las dadas usando la comparación mediante la superposición de un figura en otra, a través del arrastre de estas, así como apoyándose en ampliar y reducir figuras con el arrastre de uno de los vértices de ellas. Por otra parte establecieron equivalencia de razones de forma numérica al obtener las medidas de las figuras dadas y hacer la comparación por cociente de ellas. También lograron rescatar los datos de problemas dados y los representaron mediante dibujos, después se apoyaron en ellos para dar solución a lo planteado.

Discusión y Conclusiones. En cuanto al aprendizaje que se detectó en los estudiantes, en su mayoría, lograron determinar figuras proporcionales a las dadas a través del empleo de registros de representación, como el empleo del dibujo con los datos que lograron extraer de los enunciados de los problemas, el uso de la tabla y la determinación en ella de razones tanto externas como internas, así como la relación de equivalencia entre las razones y las operaciones que emplearon para determinar valores numéricos. También mostraron un desarrollo en su pensamiento proporcional cualitativo como apoyo al pensamiento proporcional cuantitativo.

Palabras Clave: Razón, proporción, ambiente dinámico, geometría, representaciones

Recibido: 10/11/09 *Aceptación Inicial:* 23/11/09 *Aceptación Definitiva:* 12/02/10

Introduction

This paper is concerned with the value of the joint use of pencil and paper activities together with support from a dynamic geometry program in solving ratio and proportion problems (simple and direct). This study analyzes a series of tasks proposed to a group of Mexican primary school students, in terms of the knowledge and abilities involved in solving them.

We begin the study by drawing out the notions of proportionality as present in the Mexican curriculum for primary education. Next, we highlight some of the prior research on proportional thinking, from which we gain conceptual and methodological tools for our work. Finally, we close this first section with some reflections on the potentiality of new technology in dynamic geometry for designing proportionality tasks. The second part of the paper describes the main methodological aspects of the empirical study, as well as its principal results. Our main instrument is a model composed of several activities, some to be solved with paper and pencil, and others using dynamic geometry software. This analysis allows us to draw a number of conclusions pertaining to cognitive, conceptual, and language-usage aspects. For example, pupils used a drawing and a numerical representation for drawing rectangles, thus establishing a connection between the two symbolic systems; elsewhere, we perceived many students' progress in recognizing the multiplicative ideas behind proportional thinking. The perceptual and intuitive aspects, for several students, were a strong support to enlarging the figures in both dimensions. Observing the original drawing, and enlarging it by using the strategy of doubling or tripling its size in both dimensions, or reducing these to a third, a fourth or a fifth of their size, plus determining the ratios involved between two figures, show the transition that was made from qualitative ideas to their quantification. Being able to work using three representation registers (the drawing, the table and the numerical data) showed the conceptual progress that was achieved.

Theoretical Elements

Our research has three theoretical concerns. First, we are interested in how mathematical notions of proportionality are treated in the curriculum guidelines. Second, we look at different studies related to proportional thinking, and finally, we consider the role that new technologies can play in learning such mathematical notions.

Proportionality and the curriculum

Many of the needs seen in secondary students are rooted in elementary school. In this regard, Guerrero, Gil and Blanco (2006) indicate that learning depends on both the student's and the teacher's intervention. The way that some topics of mathematics are introduced in primary education, and how the intervening concepts are constructed, is what allows students to gain understanding and application of concepts that they will come across in later stages of education. Such is the case of ratio and proportion, since instruction in these topics begins in elementary school (SEP, 2001a and 2001b) and serves as a basis for later concepts. Lesh, Post and Behr (1988) point out that much critical knowledge consists fundamentally of recognizing similar patterns or structural similarity in two different situations.

Since proportional reasoning is one of the most common forms of structural similarity, it is often related to some of the more important elementary but profound concepts in early stages of many mathematical areas. Thus, proportional reasoning is involved in some of the first conceptual "problems" in the curriculum: equivalency of fractions, long division, value and percentage placement, measurement conversion and rates of exchange.

In the areas of fractions, percentages, measurement conversion and others, there are several generic types of problems related to proportions that appear naturally. These include unknown value problems, or comparison problems, as well as selection problems where the student identifies when a figure is reduced or enlarged proportionately to another.

Lesh, Post and Behr (1988) also indicate that not everyone who solves a problem referring to proportions necessarily uses proportional reasoning. In fact, one may note a simple numerical relation (if A is three times B , X must be three times D) or may use a mechanical algorithm such as cross multiplication. On the other hand, these same authors stress that proportional reasoning encompasses a broad, complex range of cognitive skills that include both the mathematical and the psychological dimension.

Studies related to proportional thinking

For Piaget and Inhelder (1978), the notion of proportion always begins qualitatively and logically, before being structured quantitatively. Piaget (1978) indicates that the child

acquires the qualitative identity before quantitative conservation, and he makes a distinction between qualitative comparisons and true quantification.

This author sustains that between 11 and 12 years of age, the subject becomes aware of the notion of proportion in different spheres, such as spatial proportions (similar figures), relationships between weights and the lengths of the arms on a beam balance, and probabilities. In the case of the beam balance, the subject can understand, through trial and error, that it is possible to maintain the balance by placing two equal weights at the same distance from the center, but that the balance is also maintained by decreasing one weight and placing it farther away, or by increasing a weight and bringing it closer to the center. Comprehension of this proportionality (both direct and inverse) is first reached qualitatively: “increasing the weight is the same as increasing the distance”, later being presented in simple metric forms: “decreasing the weight and extending the length is equivalent to increasing the weight and shortening the length”.

Other research studies have focused on instructional needs. For example, Hart (1988) indicates that proportional thinking is present in the adolescent and that it evolves once he or she has constructed certain concepts. For Hart, some levels of generalization, such as handling ratios or ways to generate equivalencies, occur when multiplicative strategies are being used.

Elsewhere, Freudenthal (1983) and Streefland (1984, 1990, 1991 and 1993) combined didactic aspects with mathematical reflection on ratio and proportion. Reports from these researchers allow us to consider their contributions to these fields. The studies mentioned above were the basis for creating the paper and pencil activities proposed here. We proceed from Freudenthal’s emphasis on working on both external and internal ratios, and on comparing figures through superposition, which, for this researcher, is the idea of measuring. With this as our reference point, we created the paper and pencil designs; in the football pitch problem, either internal or external ratios must be established in order to solve it. As for Streefland (1984, 1990, 1991 and 1993), he emphasizes using didactic resources for comprehension of ratio and proportion topics, as well as an early treatment of proportion using intuitive ideas when the student is very young. Furthermore, he supports the idea of a change in perspective; we feel that this idea supports the transferring of paper and pencil work to the use of a computer-assisted dynamic program, instruction designed from a constructivist approach.

The contribution of new technologies

An increase of educational projects that include the use of a technological component is verifiable in current research (English, 2009). This situation motivates teachers and researchers to include new technological equipment in their teaching activities. As Lupiáñez and Moreno (2001) point out, there is immediate applicability of such equipment in the teaching of notions such as ratio and proportion.

In this paper, we introduce certain activities in the Cabri-Géomètre¹ dynamic geometry environment. Activities were designed based on observations and results reported through two channels: (1) Ruiz and Lupiáñez (2009), regarding psychopedagogical obstacles detected when applying a questionnaire to primary school students in Mexico City, and (2) work by Ruiz (2002), Ruiz and Valdemoros (2002, 2004), using a didactic program based on tasks that are solved with paper and pencil.

The Cabri-Géomètre package provides a Geometry environment whose fundamental characteristic is the *drag* function: by dragging one of the elements of a figure, it is possible to generate proportional figures. This characteristic is important in didactic activities previously designed for paper and pencil (Santos & Espinosa, 2002). As Streefland (1991) points out, changing from the context of solving a problem with paper and pencil to solving it through use of a computer is a change of perspective. In other words, a model is created and exploited as much as possible using one idea; afterward, the model is picked up again and exploited in the light of another idea. The change in perspective is characterized by exchanging one part of the information in the problem situation.

Use of conservation of proportion in building micro worlds can be used when carrying out instructional activities, in at least three possible applications: i) in checking the students' calculations; ii) in generating new problems through dragging some suitable, available point; iii) in illustrating the transition between the possible configurations. From this point of view, Cabri-Géomètre becomes a resource for encouraging recognition and development of perceptual patterns. Some of the constructions will be aimed at providing greater conceptual

¹ www.cabri.com

and epistemic accuracy. Further on there will be a brief description of how such constructions can be carried out in this dynamic geometry environment.

Problem and objectives

Prior studies have verified that sixth-graders present various difficulties when facing ratio and proportion tasks (Ruiz, 2002). For this reason we propose activities in two environments, paper and pencil and a dynamic geometry environment for helping students overcome these difficulties. Thus, we can clearly outline the two objectives of this research:

- Examine the strategies used by a group of sixth-grade students when solving paper and pencil activities on ratio and proportion.
- Examine the benefits gained by using a dynamic program in solving the same ratio and proportion activities by a group of sixth-graders.

Method

Participants

For the empirical part of our research we worked with a group of 29 sixth-grade primary education students (11 year-olds) from a public school in Mexico City.

Instruments

We designed and carried out two educational sequences on ratio and proportion, one of them based on activities with pencil and paper and the other on activities in the Cabri-Géomètre software environment.

Design of the pencil-and-paper educational sequence for primary school students

The educational sequence for the teaching experiment included three of the nine learning activities used by Ruiz (2002), implemented at various times as required to perform

this study. These three activities were chosen because they allowed the student to more easily reproduce and resolve the figures in the computer software environment.

Table 1 shows the names of the three activities, their purposes and content, and how each one was to be used with the students.

Table 1. Structure of the Teaching Proposal

Activity	Purpose(s)	Included in each session	Work Dynamic
Football tournament	Work on proportion using different ratios	Determine the ratios of various dimensions of three football pitches.	Group / Collective / Individual
Construct your own pitch	Use different forms of representation	Draw the largest possible pitch in the school playground that is proportional to the one shown.	Collective / Individual
Your team picture	Work with decimal ratios, if necessary, to determine proportions.	Elicit the procedures the student uses while working on proportions.	Group / Collective / Individual

Procedure

In the three activities we worked on, we paid special attention to the various forms of representation that students used when they were confused about the situation they were presented with. We tried to allow students to use any of the following representation registers interchangeably: drawings, tables or numerical data.

It is important to note, as Franco points out (2008), that a child is able to act once he or she has clearly perceived reality: in order to make use of different representation registers, the pupil must have clearly perceived the situation being presented.

Shown below are the activities of the three teaching activity sequences that were implemented with a group of sixth-graders, starting with the use of pencil and paper and then using the dynamic geometry program.

Activities performed with pencil and paper

Activity No. 1: Football Tournament

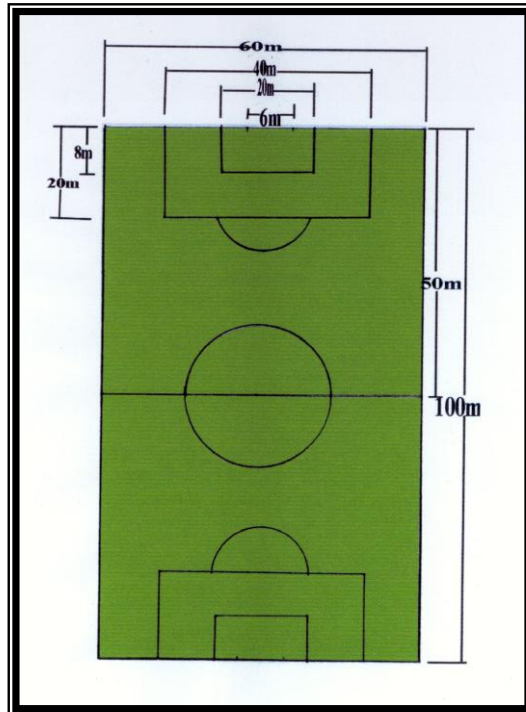


Figure 1. Field "A" (with official measurements).

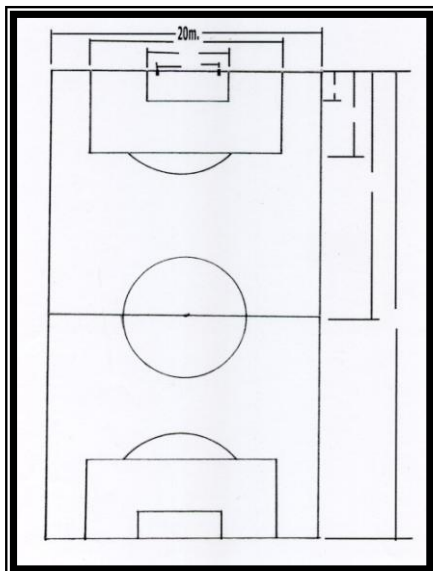


Figure 2. Field "B"

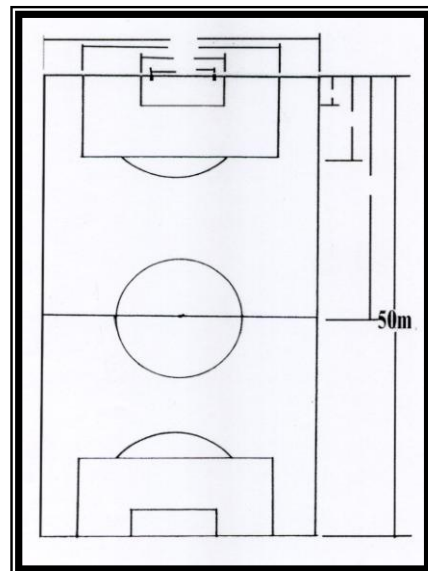


Figure 3. Field "C"

The purpose of this activity was for the students to use different ratios when working with proportion: first, in an intuitive manner by comparing and determining whether something fits either two or three times, then in an explicit manner by using quantities. To this end, they worked individually to fill in the requested data, extracting it from the drawing of the official field. Once the values of the different dimensions were known, they determined those required for the “B” field, where the fifth and sixth grade students were to play, as well as for the “C” field, which would be assigned to the smaller children. Figures 1, 2 and 3 depict the three fields:

Activity No. 2: Construct your own pitch

In this case the goal was for students to work on a problem where they would use ratios to determine the dimensions as well as make use of different representation registers to work with proportion.

In principle, students worked in teams, where they read the following instructions from the assignment sheet: *“The dimensions of the school playground are 40 meters long by 12 meters wide. You are to build the biggest possible football pitch that is proportional to the official field, without exceeding the dimensions given.”*

Activity No. 3: Your Team Picture

In the third activity, the goal was for students to use proportions and ratios based on natural operators and to work with the three representation registers: drawings, tables and numerical operations.

Each team was given an assignment sheet including a picture of one of the teams. After reading the rules of the game, we discussed their meaning together to make sure the students had understood what was asked of them.

The students were told that they could use drawings, tables, perform arithmetic operations, or do whatever they considered would help them determine the heights of the people in the picture. The worksheet is shown in Figure 4.

Nombre del alumno(a): _____
Grado: _____ Grupo: _____ Fecha: _____

SESIÓN 13: La fotografía de tu equipo

Vamos a jugar

Tienes que descubrir cuánto mide cada una de las personas que están en la foto, para ello debes seguir las siguientes reglas:

- 1) Debes medir sólo a uno de los que integran la foto.
- 2) A esa misma persona la debes medir en la foto.
- 3) Debes medir a las demás personas sólo en la fotografía. No puedes medirlas realmente
- 4) No puedes preguntarles cuánto miden.




Figure 4. Activity: Your Team Picture

Proportion Activities using Cabri-Géomètre

Students in the upper primary grades have the potential to acquire and use their understanding of ratio and proportion in contexts that are consistent with their prior mathematical knowledge. This potential has been clearly proven in light of results from implementing the Ruiz teaching proposal (2002). The present proposal provides adjustments to these activities as described in the following paragraphs, and the objective is for students to complement their learning through observing and handling the representations made using dynamic geometry in the Cabri-Géomètre environment. In any case, some of the assignments are better suited for continued work at the beginning of secondary education, and not for primary education (grades 1-6). We assume that students at either educational level have acquired the basic skills to operate this software.

(Activity No. 1a): Recognizing Patterns

We used this activity to familiarize students with the dynamic geometry program and with recognizing proportional figures. Also, students were to discover one of the advantages

of the program, dragging and superimposing figures, actions which they had previously done with pencil and paper, or dragging a figure's vertex and enlarging or reducing it, where in both cases the program maintains proportionality to the original. This is in line with indications from Freudenthal, (1983), Piaget (1978) and Streefland (1984, 1990, 1993).

The objective is for students to adjust the rectangles according to their visual perception, and after doing so, to have the program show the dimensions of the rectangles. Then the students can determine the ratios of the sides of the rectangles once they determine the number of times one side fits inside the other.

(Activity No. 1b): Building Proportional and Non-Proportional Figures

The purpose of this activity is to introduce a criterion for building rectangles that are proportional to each other. The rectangles are to be constructed in such a way that they share one vertex and diagonals on the same straight line.

In the early grades of secondary education, it is possible to introduce non-variation in similar figures, using the concept of homothetia. This aspect becomes especially easy and illustrative in the Cabri-Géomètre environment. It is very interesting to see how the mobile point "P" generates families of similar figures.

Activity No. 2: Constructing a football pitch

Recalling the most important dimensions of a real football pitch, we have the following: the proportion of the width to the length is 60 to 100, that is, 3 to 5; the ratio of the length of the penalty area to the length of the field is 20 to 100, that is, 1 to 5; the length of the goal area compared to the length of the field gives a ratio of 8 to 100, or 2 to 25; the width of the penalty area to the width of the field is 40 to 60, that is, 2 to 3; the width of the goal area to the width of the field is 20 to 60, or 1 to 3; the width of the goal to the width of the field is 6 to 60, which is 1 to 10. Also, there is a line at the middle of the field. All of these proportions are faithfully reproduced in the model.

There are other characteristic dimensions in the field, like the distance between the center of the goal and the penalty spot. Some of these were shown selectively to produce a closer resemblance to an actual field.

Activity No. 3: Your Team Picture

The measurement of segments and the ratio between them can be obtained using the appropriate commands in Cabri-Géomètre. The precision of these values can be adjusted to 1, 2, 3, 4, or more decimal places by changing the parameter in the preferences menu. The values obtained are updated while moving the free points of the construction (see Figure 5).

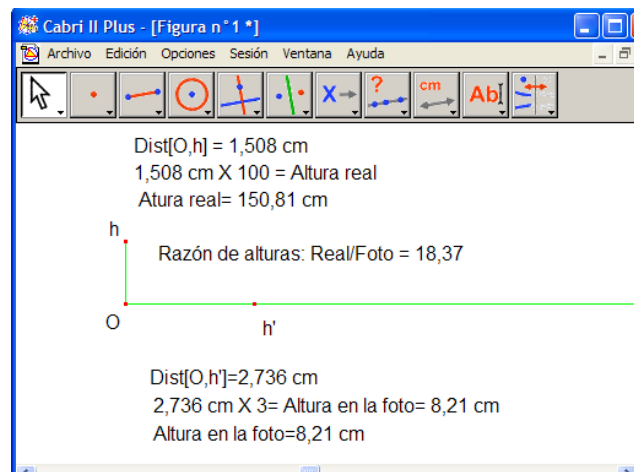


Figure 5. Constructing segments that represent the real height and the photograph height.

This construction has the characteristic that when point h is moved, the length of the segment Oh' is adjusted to preserve the ratio between the real height and the height in the picture. But when h' is moved, the ratio between these measurements is changed.

Results

Results are presented in two sections: Section I presents the results of the three educational activities that were done with pencil and paper, and Section II shows results using Cabri-Géomètre. Each Section, in turn, has three parts, a, b and c, that contain the results and analysis of each activity.

Section I - Results from the pencil and paper sequence

Activity No. 1: Football Tournament

To determine the dimensions of the “B” field, students noted that its length was half that of the official field, and since they had been told it had to be proportional, they realized that all of the linear dimensions of the “B” field would have to measure half the value of those of the other field. To obtain half they divided by two.

a) Analysis of Activity No. 1

A very small number of students (10.24%) established halves by subtracting a fixed amount from the given dimension and verifying that the same value was left over. Since we were working with halves, it was not possible to ascertain whether these three students were thinking of multiplication as an abbreviated addition or whether they incorrectly used addition for a proportionality problem. In other words, we do not know for sure whether they were having difficulty in acknowledging multiplication as the precursor to proportion. The rest of the students (89.6%) obtained half by dividing by two.

It was necessary to do a few exercises that helped the children understand that adding an amount twice is like multiplying it by two and that they could add that same amount as many times as they wished and that the number of times would correspond to the factor by which they multiplied the amount. That is, they concluded that multiplication is abbreviated addition. Thus, adding a number three times is the same as multiplying it by three, but not the same as adding an amount in order to obtain its tripled value. Several exercises were done along these lines.

Regarding field “C”, some students realized that one of the dimensions given corresponded to one fourth its counterpart in the official field. They said, “*Since it’s supposed to be proportional, all the linear dimensions should be one fourth of those in the official pitch,*” and they determined the dimensions by dividing by four. (55.2% of 9 students performed this procedure.) The rest of the school children, 44.8%, noticed that one of the dimensions of the “C” field was half of the corresponding dimension of the “B” field, and

they obtained the remaining values by taking half of the rest of the dimensions, so as to make it proportional.

When some of the students were asked to share their results, they concluded that half of half a value is one fourth of the same.

We observed that several students (65.5%) prepared a table to record the data from fields “C” and “B” and then read the values of the other field, applying the ratios as required. Other children (34.5%) read the amounts directly from the field drawings. This time, the notation they used to write the ratios was fractional. With this, we noted that some students began to use and relate different representation registers to express the ratios: drawings, tables and numerical operations.

Activity No. 2: Football Tournament

In general, we observed that all the teams used drawings to represent both their school playground and the official field, where they noted the length and width measurements. The next step (taken by some students) was to establish what was half of the length and the width dimensions of the official field and to check whether these would fit within the measurements of their playground. Then they continued this process by establishing halves. This action was commonly observed among students according to Piaget (1978) and Piaget and Inhelder (1978).

b) Analysis of Activity No. 2

We detected that three students only paid attention to one of the dimensions when establishing halves, and if this one was less than its corresponding dimension on the official field, they immediately claimed to have found what had been asked of them.

A few students (13.8%) followed another action described in the previous study, focusing on a single dimension when drawing one figure proportional to another. However, most students in the present study were able to establish that if a linear measurement is reduced by half, all the others also have to be reduced in order for it to be proportional to the original. The remaining students, that is 86.2%, established halves in both dimensions.

We found another difficulty in a few students (24.1%), consisting of considering only one of the dimensions to see if it was smaller than the corresponding one from the school playground, and if so, these students concluded, “*The field would fit in it.*” When questioned about the other dimension, then they commented, “*This one is bigger than the playground,*” and they had to continue reducing, this time verifying that both the length and the width of the field were smaller than those of the school playground. By successively taking halves, they obtained a proportional field, but they realized “*It’s not the biggest one that could fit in the school playground.*”

After various attempts, that is, after following the process of taking halves, all 29 students decided to calculate one third of the dimensions as well as one fourth. Finally, they went on to establish one fifth of the official field dimensions and said, “*We have found a field that is proportional and at the same time it is the largest one that could fit in the school playground.*” (See Figure 6.)

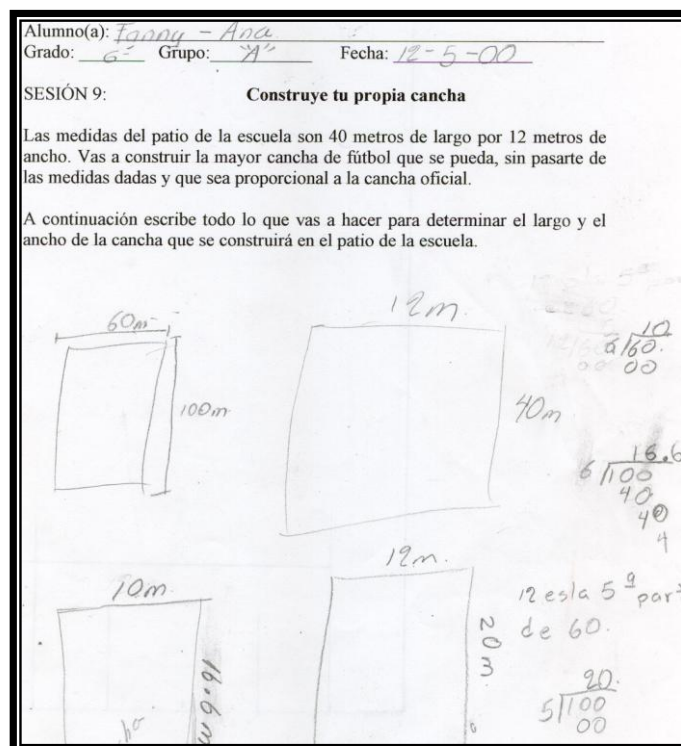


Figure 6. Teamwork

The students worked with natural numbers, although they obtained decimal numbers when doing divisions, and they decided to try for dimensions that corresponded to natural

numbers for the field they were to build on the playground. It is easy to see that this is a matter of optimization in calculating maximums, something that could be worked on at a higher level of education, once the function and corresponding calculation tool is known. But at this level, we worked with arithmetic, using drawings and tables as representation registers.

Activity No. 3: Your Team Picture

Most of the teams made a real life measurement of one of the people in the photograph and also measured them on the photograph itself. Then they measured the rest of the people on the photograph and asked the question, “How much does each person measure in real life, based on these other known measurements?”

c) Analysis of Activity No. 3

Most of the students, that is 89.6%, represented the real and photograph heights using drawings, transforming the measured values from centimeters to millimeters.

The students realized they were looking for a figure that was proportional, so they established an external ratio, relating measurements on different scales (real height measurement and height measured in the photograph). They wrote the fraction Real Height 1 / Picture Height 1, and when dividing these numbers, they obtained a decimal number. There was a moment when the students stopped and did not know what to do with the obtained value. That is, we noted a degree of uncertainty about how to continue solving the problem, so we asked them to share out loud what they had done up to that moment and what they intended to do for the two figures they had drawn. The students answered, “They ought to be proportional.” When asked if they remembered what it meant for two figures to be proportional, the children said yes and that in that case they would obtain the same ratio of the real height of the other people in the picture to the height they had measured on the picture itself.

Next, they performed their operations and realized that the quotient from dividing the real height by that in the photo had to be the same in both cases (Person 1 and Person 2). Since they had to determine the value of the person’s size in real life, they found that they knew two of the values from the division, those represented by the divisor and the quotient,

and that by multiplying the two they would get the dividend, representing the measurement they were looking for. This was their method for solving the problem. After doing their multiplications with the established ratio and the photo height values, they filled out a table.

It is worth noting that they used three representation registers to solve the problem and that they were able to move data from one to another.

Section II – Results from the dynamic geometry program

Activity 1a: Pattern recognition

In this first activity, students were shown a family of rectangles. These rectangles were moved and superimposed on the screen without being distorted. Superimposing one rectangle over another is a form of measuring, as Freudenthal (1983) indicates.

a) Analysis of Activity 1

Students were asked to choose two rectangles that were proportional at first glance. Once students superimposed one rectangle over another they could drag the corner of the smaller rectangle to make it bigger and check whether it became the same size as the other rectangle. This way the 29 students verified whether the rectangle they had selected was proportional to the one given. This activity is clearly related to similarity of figures, as indicated by Lesh, Post and Behr (1988), even though the “similarity” term is not used.

After discussing whether there was a relationship in simple cases, such that one relationship is applicable to all cases, the first measurements and ratio computations appeared. The objective of this activity was for students to learn basic software tools in Cabri-Géomètre and use them to work with ratios. For this purpose, the 29 students drew linear segments and afterward measured them to discover the ratios between their dimensions (see Figure 7).

Beginning with these simple manipulations, students demonstrated important abilities: taking measurements, manipulating measurements and computation possibilities.

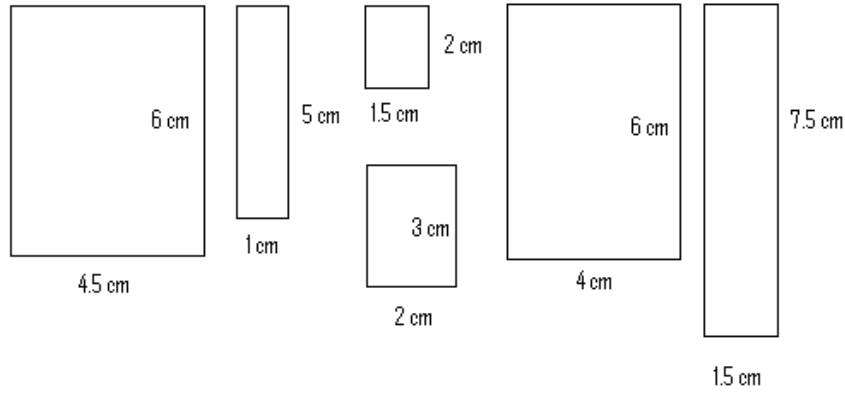


Figure 7. Activity 1 in a dynamic environment

b) Analysis of Activity 1b

The students first constructed rectangles in such a way that they shared one vertex, and their diagonals on the same straight line (Figures 8 and 9). The 29 students moved Vertex F on a straight line and maintained a record of the different measurements in the table. We can verify that the resulting proportion does not vary, so that a complete family of similar rectangles was obtained. Likewise, Figure 12 shows how students added the same constant to the dimensions of the rectangle sides, and in this case they did not obtain a proportional rectangle to the original. After carrying out these tasks, it was possible to take on the “football tournament” activity by implementing this same criterion.

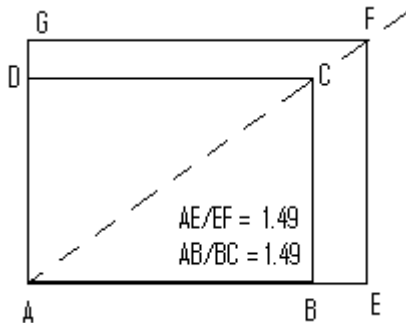


Figure 8.

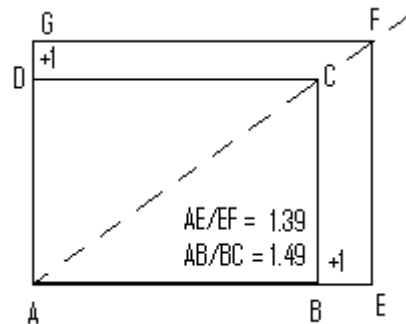


Figure 9.

Construction of a proportional rectangle in a dynamic environment.

Another interesting method that the 29 sixth-graders used to find a proportional rectangle was to use the center and the diagonals of the given rectangle (see Figure 10). All 29 students determined their new rectangle with the same center, and obtained diagonals that coincided with the diagonals drawn from the given rectangle. Students noted that since the rectangles were similar, their side measurements were proportional.

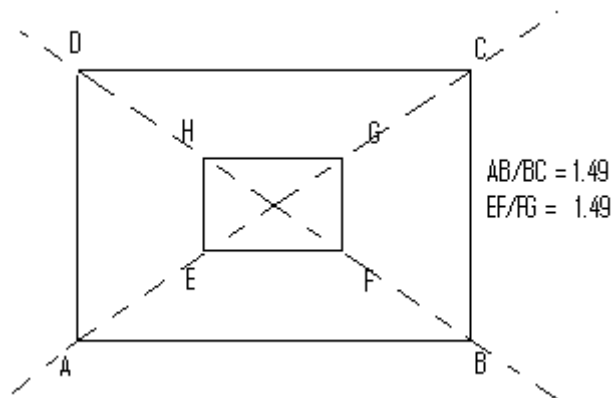


Figure 10. Construction of a proportional rectangle.

Activity 2: Construction of a football pitch

First, students drew a rectangle with a length of any amount and a width proportional to that indicated earlier. The rectangle played the role of a football field. Once the rectangle was drawn, they proceeded to fill in the areas and other objects, creating ratios based on the drawing, or with the combined use of calculator commands and transferral of measurements. Afterward, students continued by first drawing all that was necessary for one half of the field, and then mirroring all the objects on the opposite side using the half-way line.

c) Analysis of Activity 2

Direct manipulation of the length of the field made it possible to change the problem to suit the situation that we wished to present to the students. This is an advantage of using the dynamic geometry program in ratio and proportion activities, in addition to dragging a point in order to construct rectangles or other figures that are proportional to each other.

Finally, the 29 sixth-grade students were able to construct fields proportional to the one given (see Figure 11).

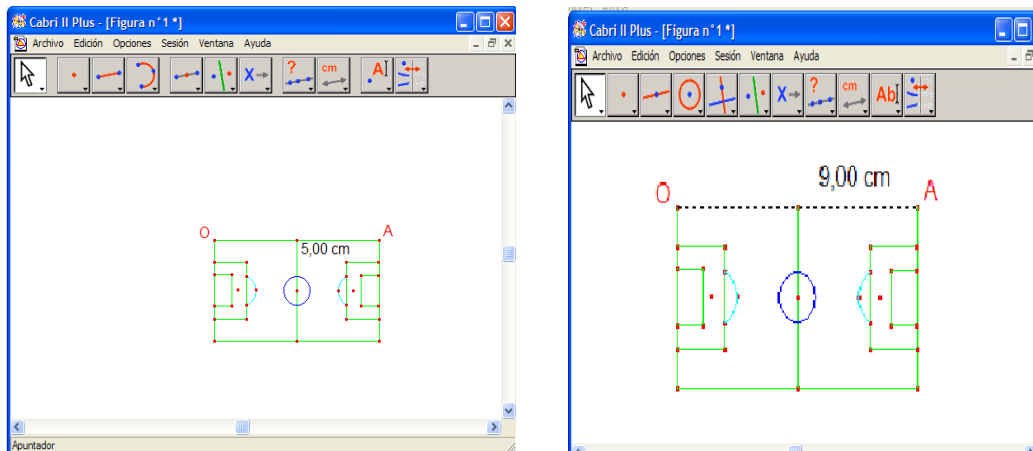


Figure 11. Construction of the football pitch.

Activity 3: Your team picture

Students used the drawing to represent the two heights of one team member: the real height and the height from the photo. On another segment they placed the value of the photograph height for another team member, this way they were able to construct similar figures.

d) Analysis of Activity 3

In order to find the real height of the other person, they worked with external ratios and established the corresponding proportion.

The unknown value in this case was the real height of the second person. In order to determine this height, they used the operation of dividing the real height of Person 1 by the photograph height of the same person. The value obtained was divided into the photograph height of the second person.

Discussion

Solving these activities shows that most students were able to determine the ratios involved between the figures and to establish equivalency relationships that enabled them to complete the activities successfully. Many students checked their answers with a visual assessment, meaning that the qualitative approach was not totally abandoned, but allowed the school children to corroborate their quantitative proportional thinking, as indicated by Kieren (1985).

Use of conservation of proportion in constructing micro worlds was verifiable in the process of completing the didactic activities, with the three applications indicated under theoretical elements: i) in order to check the calculations made; this was noted when they determined the proportional football pitches by establishing ratios numerically, giving values to the sides of the rectangles obtained, ii) as a generator of new problems by dragging some appropriate free point, this was observed when the school children dragged one vertex of the rectangle that exemplified the football pitch in order to make it proportional to the one given, or when they determined a person's height by using lines that they marked out vertically and horizontally, checking to see that they were in the same ratio, and iii) as an illustrator of the transition between possible configurations. What they observed when manipulating the points enabled them to preserve the ratio between real height and photograph height. Indications by Franco (2008) were corroborated when students were able to work in three representation registers (the drawing, the table and the numeric data), implying that they were able to make sense of and to understand the situations presented to them.

As for what the students were able to learn through use of the pencil and paper activities as well as the dynamic geometry program, we can affirm that there was a connection made between working in the two environments, as seen when they use the action of physically superimposing figures and also dragging them by using the mouse. This indicates that they compared the sides and established ratios implicitly, just as indicated by Freudenthal (1983).

Another action that the school children performed in both environments was obtaining halves or thirds, or the doubled amount, thus numerically determining ratios and their

equivalence. The three representation registers (drawing, table and numeric data) were used in both environments as mentioned in the Results section.

On the other hand, in order to propose solid experimental models where the same things can be done using paper and pencil or with technology, a broad theoretical framework is required, and one that has been tested in diverse investigations. This experimentation with the proposed sequences using technology will contribute data to help understand the problems that students have concerning notions of ratio and proportion, especially as these notions mature over the course of their education. It is worth noting that the most recent proposals have to do with the geometric aspect, which is often overshadowed by an emphasis given to algorithmic aspects.

The basic notions of ratio and proportion that the subject acquires are expected to evolve as these students progress through the curriculum. In order to study the students' mental explanations and constructions, instruments are needed that can identify and verify the maturity of these notions.

There are different activities for enriching and expanding students' knowledge of ratio and proportion. Representation and visual verification of the Thales theorem, the study of linear functions, computation and graphic representation of constants like π or the golden ratio can be used for these purposes. In addition, some of these activities can serve as a strong motivating element.

References

- English, L. (2009). Setting an agenda for international research in mathematics education. In L. English (Ed.), *Handbook of international research in mathematics education* (pp. 3-19). New York: Routledge.
- Franco, C. (2008). Creative relaxation, motor creativity, self-concept in a sample of children from Early Childhood Education. *Electronic Journal of Research in Educational Psychology*, 14, 6(1), 29-50.
- Freudenthal, H. (1983). *Didactical Phenomenology of Mathematical Structures*. Dordrecht: Reidel.
- Guerrero, E., Gil, N. & Blanco, L. (2006). The affective domain in learning mathematics. *Electronic Journal of Research in Educational Psychology*, 8, 4(1), 47-72.
- Hart, K. (1988). Ratio and proportion. In J. Hiebert & M. Behr (Eds.), *Concepts and operations in the Middle Grades*, 2 (198-219). Reston, VA: National Council of Teachers of Mathematics.
- Kieren, T., Nelson, D. & Smith, G. (1985). Graphical Algorithms in Partitioning Tasks. *The journal of mathematical behavior* 4, 25-36.
- Lesh, R., Post, T. & Behr., M. (1988). Proportional reasoning. In J. Hiebert & M. Behr (Eds.), *Concepts and operations in the Middle Grades*, 2 (pp. 93-139). Reston, VA: National Council of Teachers of Mathematics.
- Lupiáñez, J. L. & Moreno, L. (2001). Tecnología y Representaciones Semióticas en el Aprendizaje de las Matemáticas. [Technology and Semiotic Representations in Learning Mathematics.] In L. Rico & P. Gómez (Eds.), *Iniciación a la Investigación en Didáctica de la Matemática* (pp. 291-300). Granada: Editorial Universidad de Granada.
- Piaget, J. (1978). *Psicología del niño*. [Child Psychology.] Madrid: Morata.
- Piaget, J. & Inhelder, B. (1978). Las operaciones intelectuales y su desarrollo. [Intellectual operations and their development.] In J. Delval (Comp.), *Lecturas en Psicología del Niño I* (pp. 70-119). Madrid: Alianza Editorial.

- Ruiz, E. F. (2002). *Estudio de Estrategias de Solución y una Propuesta para la Enseñanza de razón y proporción. Tesis Doctoral.* [A study of solution strategies and a proposal for teaching ratio and proportion. Doctoral thesis.] Cinvestav, IPN. México.
- Ruiz, E. F. & Valdemoros, M. (2002). Concepts of ratio and proportion in basic level students: case study. In D. Mewborn, P. Sztajn, E. White, H. Wiegel, R. Bryant & K. Nooney (Eds.), *Proceedings of the 24th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, Vol. 4* (pp. 1651–1657). Columbus, OH: Eric Clearinghouse for Science, Mathematics, and Environmental Education.
- Ruiz, E. F. & Valdemoros, M. (2004). Connections between qualitative and quantitative thinking about proportion: The case of Paulina. In M. J. Hoines & A. B. Flugstad (Eds), *Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education. Vol. 1* (pp. 201-208). Bergen: Bergen University College.
- Santos, L. M. & Espinosa, H. (2002). Searching and exploring properties of geometric configurations using dynamic software. *International Journal of Mathematics Education, Science and Technology*, 33(1), 37-50.
- Secretaría de Educación Pública (2001a). *Plan de programas de estudio. Educación Básica. Primaria. Dirección General de Materiales y Métodos Educativos de la Subsecretaría de Educación Básica y Normal.* [Plan of study programs. Basic Education. Primary. General Administration of Educational Materials and Methods of the Deputy Secretary of Basic and Normal Education.] México D.F.: SEP.
- Secretaría de Educación Pública. (2001b). *Matemáticas. Sexto grado.* [Mathematics. Sixth grade.] México D. F.: Autor.
- Streefland, L. (1984). Search for the roots of ratio: Some tough on the long term learning process. Part I. *Educational Studies in Mathematics*, 15(3), 327-348.
- Streefland, L. (1990). Free Productions in Teaching and Learning Mathematics. In K. Gravemeijer, M. Van de Hual & L. Streefland (Eds.), *Context Free Production Tests and Geometry in Realistic Mathematics Education* (pp. 33-52). Utrecht: Research Group for Mathematical Education, Computer Center, State University of Utrecht.

Streefland, L. (1991). *Fractions in realistic mathematics education. Doctoral thesis.*
Dordrecht: Kluwer Academic Publishers.

Streefland, L. (1993). The design of a mathematics course. A theoretical reflection.
Educational Studies in Mathematics, 25(1-2), 109-135.

[This page intentionally left blank]