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Discounted and Expected Utility from the Probability and Time Trade-Off Model

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Abstract: This paper shows the interaction between probabilistic and delayed rewards. In decision-making processes, the Expected Utility (EU) model has been employed to assess risky choices whereas the Discounted Utility (DU) model has been applied to intertemporal choices. Despite both models being different, they are based on the same theoretical principle: the rewards are assessed by taking into account the sum of their utilities and some similar anomalies have been revealed in both models. The aim of this paper is to characterize and consider particular cases of the Time Trade-Off (PTT) model and show that they correspond to the EU and DU models. Additionally, we will try to build a PTT model starting from a discounted and an expected utility model able to overcome the limitations pointed out by Baucells and Heukamp.

Keywords: risk; delay; decision-making process; probability; discount

JEL Classification: G12; D81; D9

1. Introduction

The main objective of this paper is to present the Probability and Time Trade-Off Model [1] as an accurate framework where risk and intertemporal decisions can be separately considered.

The methodology used in this paper consists of considering some particular cases of the PTT model and show that they correspond to EU and DU models. Moreover, the possibility of reversing the process is provided, i.e., obtaining a PTT model starting from an EU and a DU model.

The decision-making process is relatively simple when the alternatives differ only in one dimension (e.g., the amount, the probability of occurrence or the delay) while the rest of the variables remain constant [2]. However, decision-making problems frequently involve alternatives which differ in more than one dimension [3]. Along these lines, there are two traditional models to assess choices which differ in risk or time, in addition to the amount of their reward. Despite risk and delay initially appearing quite different, the individual behavior when facing risky and delayed outcomes is analogous [4]. Currently, due to the fact that most real-world decisions are made on alternatives which are both uncertain and delayed [4], there is a growing interest in understanding and modelling how risk and delay interact in the individual behavior. In this way, there are some scholars as Luckman [5] who show the attempts to explain the complex individual behaviors when facing alternatives which differ in more than two dimensions.

The decision-making process has been analyzed in psychophysics, neuroscience and social and behavioral sciences, such as economics or psychology. From an academic point of view, one of the most studied processes in decision-making is intertemporal choice which concerns those alternatives which differ in maturities (the choice between a smaller, sooner outcome, and a larger, later one) [3].

On the other hand, the decision-making under uncertainty involves alternatives whose rewards differ in relation to the probability of being received (the choice between “a smaller reward, to be received with greater probability, and a larger one, but less likely”) [3].

Both kinds of decisions have been traditionally analyzed by using two main systems of calculation: the Discounted Utility model and the Expected Utility theory which describe the present value of delayed rewards and the actuarial value of risky rewards, respectively. Both models are simple, widely accepted, and with a similar structure, since they use the same theoretical principle: the rewards are assessed by the sum of their utilities [6,7]. In the decision-making process, individuals choose the alternative which maximizes their current utility value (denoted by U_0).

As seen in Table 1, DU and EU are the basic models employed in the decision-making process which represent the rational choice over time and under risk, respectively.

Table 1. Classical models to obtain the utility value. Source: Own elaboration.

	Discounted Utility (DU)	Expected Utility (EU)
Pioneer work	Samuelson [8]	Von Neumann and Morgenstern [9]
Result	The present value of delayed rewards	The expected value of risky rewards
Formula	$U_0 = \sum_{t=0}^T \delta^t u_t$	$U_0 = \sum_{k=0}^n p_k u_k$
Parameters	u_t : utility from the reward at t t : reward maturity ($t = 0, 1, \dots, T$) δ : discount factor ($0 < \delta < 1$)	u_k : utility from the k -th reward p_k : probability of the k -th reward ($\sum_{k=0}^n p_k = 1$)

On the other hand, the Discounted Expected Utility (DEU) model (see Table 2) is employed in decision environments involving both intertemporal and risky decisions (Schoemaker [10] points out nine variants of the Expected Utility model). DEU model has been deeply analyzed in several recent works by Coble and Lusk [11] and Andreoni and Sprenger [12].

Table 2. DEU model. Source: Own elaboration.

Pioneer work	Jamison [13]
Result	The value of risky and delayed rewards
Formula	$V(c_0, \dots, c_r) = \sum_{t=0}^T D(t)u(c_t)$
Parameters	V : valuation of consumption in different periods t : reward maturity ($t = 0, 1, \dots, T$) $D(t)$: discount function $u(c_t)$: utility of consumption at t (c_t)

The accuracy of DU and EU models has been questioned to explain actual behaviors given that they are a simplification of reality [14]. In the same way, [15,16] show that DEU model fails as a predictor of intertemporal-risky choices. From an experimental point of view, Coble and Lusk [11] prove that the data does not support the DEU model assumption of a unique parameter that explain risk and time preferences. Usually, individuals’ preferences cannot be so easily determined and then several anomalies of these models must be taken into account when a real decision is analyzed. Some of these effects or anomalies are the consequence of psychophysical properties of time, probability and pay-out dimensions [17].

Indeed, several studies, such as [6,18], compare the analogies and the anomalies present in DU (intertemporal choices) and EU (risky choices) models, by observing that some risky choice inconsistencies are parallel to delay choice inconsistencies. In this line, it should be stressed the

contribution by Prelec and Loewenstein [19] where a one-to-one correspondence is shown between the behavioral anomalies of Expected and Discounted Utility models. Nevertheless, Green, Myerson and Ostażewski [20] findings suggest that processes involving delayed and probabilistic rewards, despite being similar, are not identical.

The main contribution of this paper is the demonstration of that both the Expected Utility (EU) and the Discounted Utility (DU) models can be embedded in the Probability and Time Trade-Off (PTT) model. Moreover, preliminary thoughts are provided on the possibility of reversing the process, i.e., obtaining a PTT starting from an EU or a DU model.

The relevance of this contribution is that the existence of the proved equivalence can be used to explain and relate behavioral inconsistencies in real choices. In effect, most existing literature presents the anomalies in intertemporal and probabilistic choices as separate inconsistencies [21] and, indeed, this paper can be used to clarify the equivalence between certain anomalies analyzed in the context of the DU and the EU models. A precedent can be found in the paper by Cruz Rambaud and Sánchez Pérez about the so-called peanuts effects [7]. This methodology could be applied to other anomalies present in both aforementioned models.

This paper has been organized as follows. After this Introduction, Section 2 presents an extensive revision of the existing literature on this topic, divided into three subsections to facilitate its reading. Section 3 has been structured into four subsections. Section 3.1 provides some basic definitions, properties and examples related to reward choices where both time and probability are involved. Sections 3.2 and 3.3 are devoted to obtaining the EU and DU from the PTT model, respectively. Section 3.4 is an essay to generate PTT starting from DU and EU models, and represents the framework in which further research is needed. In Section 4, the main obtained results are discussed regarding other works which aim to relate the properties of decisions under risk and under time. Finally, Section 5 summarizes and concludes.

2. Literature Review

Most of the previous research analyzing delay and risk parameters do so separately. Only a few papers study the individual preferences in decisions involving time-delayed and risky rewards. Initially, Prelec and Loewenstein [19] study uncertainty and delay from a common approach. In this sense, Weber and Chapman [4] and Gollier [22] analyze the relationship between choice under risk and time from a global perspective.

There have been several attempts to introduce a new model which explains individual preferences in decisions involving time-delayed and risky rewards [5,21]. Specifically, in some of these models, delayed rewards are studied from the perspective of the subjective probability [23,24].

These descriptive psychophysics models need to be empirically implemented. However, from a practical point of view, few publications, such as [25,26], have analyzed the risk and discount attitudes simultaneously. Specifically, some of them [11,27] explain the relationship between the risk and time in the decision-making process from an experimental point of view. From this perspective, it is necessary to highlight that uncertainty underlying time delays may be undermined when data is obtained through laboratory experiments [28].

Despite risk and delay being perceived as concepts psychologically distinct, in some contexts, decision-makers may identify both concepts as psychologically interchangeable since they influence preferences [17]. This is because behavioral patterns facing risk and delay are based on a common underlying dimension [17,23]. Even though time delay and uncertainty are interrelated, their interaction is controversial. Below, we can find some examples of studies that have analyzed the relation between time and risk preferences to assess delayed and probabilistic choices. In Section 2.1, uncertainty is treated as the fundamental concept and delays are transformed into risky terms. Next, in Section 2.2, by assuming that uncertainty is linked to delayed rewards, the reward probability is expressed as additional waiting time. Finally, in Section 2.3, the new models to assess risky intertemporal choices are pointed out.

2.1. Uncertainty as Central Concept. Transformation of Delays into Probability Terms

Keren and Roelofsma [28] defend that uncertainty is the psychologically central concept, given that decisions are affected by delay only if this delay entails uncertainty. They propose that the delay effect is actually the effect of the uncertainty inherent to delay. Indeed, they defend that the immediacy effect and the certainty effect are the same effects given that decision-makers are based on the implicit uncertainty, not the delay itself. In intertemporal choice, the immediate outcome entails no uncertainty while the delayed outcome is perceived as uncertain. In this way, “if delay implies risk, then risk should have the same effects as delay—the two are interchangeable” [4].

In this sense, Rachlin [23] translates reward delay into the probability of receiving it. Delays act like less-than-unit probabilities; longer delays correspond to lower probabilities, given that uncertainty increases as delay increases. In this way, intertemporal discount models may be translated into probabilistic discount functions (called “odds against” Θ , i.e., the average number of trials until a win). They are calculated as follows:

$$\Theta = \frac{1}{p} - 1,$$

where p is the probability of receiving an uncertain outcome.

2.2. Uncertainty is Inherent to Intertemporal Choice. Interpretation of Reward Probability as Waiting Time

When applying the DU model, sometimes future outcomes are modelled as though non-stochastic by ignoring its uncertainty. However, as stated by Fisher [29], “future income is always subject to some uncertainty, and this uncertainty must naturally have an influence on the rate of time preference, or degree of impatience, of its possessor”. In this vein, some recent studies consider the relation between behavior under risk and over time with the premise that uncertainty is inherent to intertemporal choices, given that any event may interfere in the process of acquiring the reward between the current and the promised date [30].

This implies that the “decision-maker’s valuation of delayed outcomes not only depends on her pure time preference, i.e., her preference for immediate utility over delayed utility, but also on her perception of the uncertainty and, consequently, on her risk preferences” [31]. Under the assumption that only present consumption is certain while any future consumption may be considered to be uncertain, risk preferences could influence intertemporal choice patterns. In this way, Takahashi [32] studies the aversion to subjective uncertainty associated with delay.

Soares dos Santos et al. [3] propose a generalized function for the probabilistic discount process by using time preferences. Specifically, probabilistic rewards are transformed into delayed rewards as follows: instead of using the probability of occurrence, they use the mean waiting time before a successful draw of the corresponding reward.

Probabilities are converted into comparable delays according to the constant of proportionality by [23] and through the examination of the indifference points of hypothetical rewards which are both delayed and risky. Once the probabilities are transformed into delays, this delay is added to the explicit delay, being this delay/probability combination the total delay used to assess delayed and risky rewards. The similar structure of time and risk shows that if the risk is interpreted as waiting time, both magnitudes may be combined into a single metric which is consistent with the hyperbolic discount function (better than exponential function given that as delay increases, there is a hyperbolic decay of probabilities of obtaining it). Furthermore, this metric may explain some of the observed behavior in choice under risk, such as the certainty effect [33].

2.3. Models to Assess Risky Intertemporal Choices

Traditionally, risky intertemporal rewards have been evaluated by applying the EU and DU models separately. First, risky rewards are assessed by the EU model, and then their expected value at maturity is discounted by using a constant discount rate. More flexibility in the assessment of risky

delayed rewards is requested from a descriptive point of view by [34]. This flexibility is necessary given that the consequences of delayed rewards do not only affect the present utility but also the future utility. In the same way, the probability implementation red has an influence on the discount rates [35].

Luckman [5] has shown that there are three specific unifying models to deal with delayed and probabilistic choices: the Probability and Time Trade-off model [36], the Multiplicative Hyperboloid Discounting model [37], and the Hyperbolic Discounting model [38]. These three models have a common feature as they consider two special risky intertemporal choices: pure risky choices and pure intertemporal choices. These special risky intertemporal choices are consistent with the results from traditional models.

In the assessment of delayed but certain rewards, the DU model is employed. However, for those alternatives whose maturity is unknown or uncertain at the beginning, i.e., when the decision-maker does not know the exact realization time of the future outcome, the DEU model may be implemented [30]. Specifically, when there are different possible delays for a reward and their probabilities are known, its “timing risk” [16] is identified. Meanwhile, the term “timing uncertainty” is employed to denote those outcomes which possible delays are vaguely known or unknown [30]. Time lotteries which pay a specific prize at uncertain future dates are a clear example of this kind of rewards [39]. Onay and Öncüler [16] and Coble and Lusk [11] demonstrate that DEU model is not accurate enough to forecast intertemporal choice behavior under timing risk.

In 2012, Han and Takahashi [40] proposed that psychophysical time commonly explains anomalies in decision both over time and under risk. Moreover, they introduce the non-linear psychophysical time into the time discount function according to the Weber-Fechner law. Green and Myerson [2] show that a single process to assess risky and intertemporal operations is inconsistent. Apart from the fact that risk and time are not equivalent parameters, “the interaction between them is complex and not easily understood” [17]. Nevertheless, recent studies show that it is necessary to introduce a common framework to understand people’s perception of risk and delay when making decisions [41]. In this way, there is still room to improve the methodology and results.

Summarizing the introduction and the literature review, Figure 1 shows the existing methodologies to assess the delayed and uncertain rewards which differ in one, two or more dimensions.

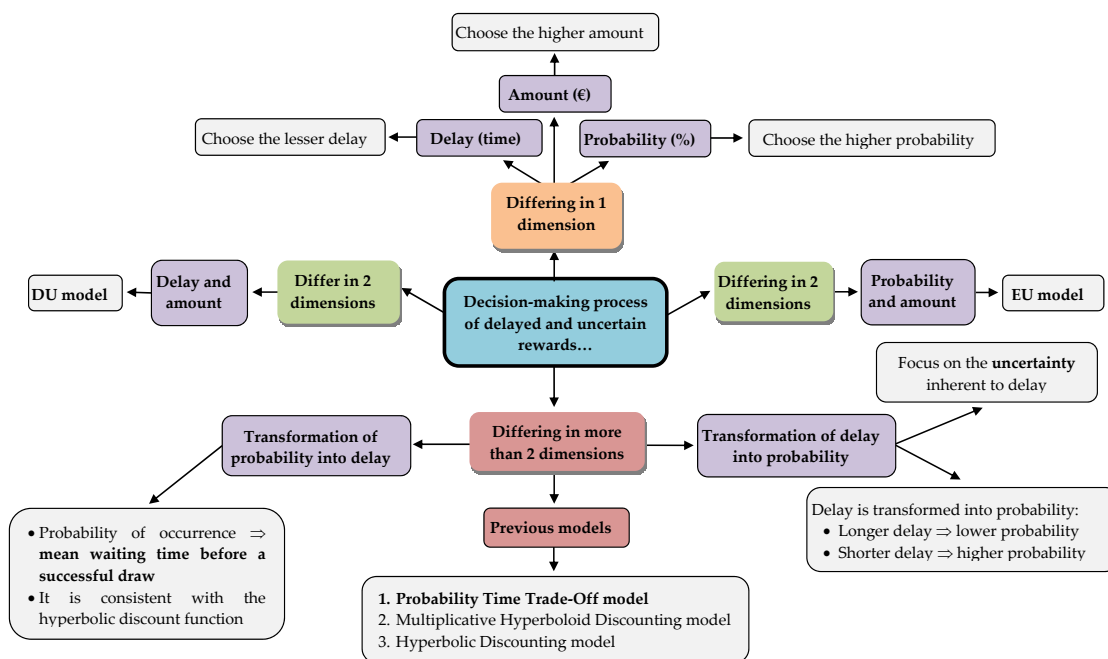


Figure 1. Decision-making process of delayed and uncertain rewards. Source: Own elaboration.

3. Deriving DU and EU from the Probability and Time Trade-Off Model

3.1. Introduction

It will prove useful to begin with the following definition in the ambit of the Probability and Time Trade-off (PTT) model [1].

Definition 1. Let \mathcal{M} be the set $X \times P \times T$, where $X = [0, +\infty)$, $P = [0, 1]$, and $T = [0, +\infty]$. A discount function in the context of the PTT model is a continuous real-valued map $V(x, p, t)$, defined on \mathcal{M} , which is strictly increasing with respect to the first and second components, and strictly decreasing according to the third. Moreover, it satisfies that $V(x, 1, 0) = x$, for every $x \in X$.

Example 1. $V(x, p, t) = (xp + 1)^{\exp\{-kt\}} - 1, k > 0$, is a discount function in the context of the PTT model.

Definition 1 guarantees that $V(x, p, t) \leq x$. In effect, $x = V(x, 1, 0) \geq V(x, p, 0) \geq V(x, p, t)$. The triple (x, p, t) denotes the prospect of receiving a reward x at time t with probability p . Obviously,

1. If $p = 1$, then the concept of a discount function in the context of the DU model arises: $F(x, t) := V(x, 1, t)$ is a continuous real-valued map defined on $X \times T$, which is strictly increasing in the first component, strictly decreasing in the second component, and satisfies $F(x, 0) = x$, for every $x \in X$.
2. If $t = t_0$, then the concept of a discount function in the context of the EU model arises: $V(x, p) := V(x, p, t_0)$ is a continuous real-valued map defined on $X \times P$, which is strictly increasing in the two components, and satisfies $V(x, 1) = x$, for every $x \in X$.

Definition 2. Given a discount function $V(x, p, t)$ in the context of the PTT model, the domain of V is the maximum subset, \mathcal{D} , of \mathcal{M} where V satisfies all conditions of Definition 1.

Example 2. The domain of the discount function $V(x, p, t) = (xp + 1)^{\exp\{-kt\}} - 1, k > 0$, is $\mathcal{D} = \mathcal{D}$. On the other hand, the domain of the discount function

$$V(x, p, t) = \frac{xp + it}{1 + jt},$$

where $i > 0$ and $j > 0$, is:

$$\mathcal{D} = \left\{ (x, p) \in X \times P : xp > \frac{i}{j} \right\} \times T.$$

Baucells and Heukamp [1] require that V converges to zero when xpe^{-t} converges to zero. However, in the present paper this restriction will be removed, and we will allow that V tends to zero only when $x \rightarrow 0$, or $p \rightarrow 0$:

$$\lim_{t \rightarrow +\infty} V(x, p, t) := L(x, p) > 0,$$

provided that $T \subseteq \text{proy}_3(\mathcal{D})$.

Definition 3. A discount function in the context of the PTT model, V , is said to be regular if $L(x, p) = 0$, while V is said to be singular if $L(x, p) > 0$.

Example 3. The discount function $V(x, p, t) = (xp + 1)^{\exp\{-kt\}} - 1$ is regular since

$$L(x, p) = (xp + 1)^0 - 1 = 0.$$

On the other hand, the discount functions $V(x, p, t) = \frac{xp+it}{1+jt}$ and $V(x, p, t) = \frac{xp+x^2p^2it}{1+jt}$, where $i > 0$ and $j > 0$, are singular as

$$L(x, p) = \frac{i}{j} > 0$$

and

$$L(x, p) = \frac{x^2p^2i}{j} > 0,$$

respectively.

Thus, the paper by Baucells and Heukamp [1] implicitly assumes the regularity of V . However, in this work, this requirement will not necessarily hold.

Now, we are going to provide an interpretation of probability in the context of a prospect (x, p, t) . In effect, let $V(x, p, t)$ be a discount function in the context of the PTT model and consider a given value, x , of the amount. Therefore, $V_x(p, t) := V(x, p, t)$ is now a two-variable function. Consider the indifference line given by $V_x(p, t) = k$ ($0 < k \leq x$) (observe that $k = V_x(p_0, 0)$, for some p_0), which eventually can give rise to an explicit function, denoted by $p_{k,x}(t)$ (see Figure 2). Obviously, $p_x(t)$ is an increasing function which has 1 as upper bound and 0 as lower bound. Moreover, $p_{k,x}(0) = p_0 = k$. Let

$$t_{k,x}^{\min} = \min\{t : V_x(p, t) = k, \text{ for some } p\}$$

and

$$t_{k,x}^{\max} = \max\{t : V_x(p, t) = k, \text{ for some } p\}.$$

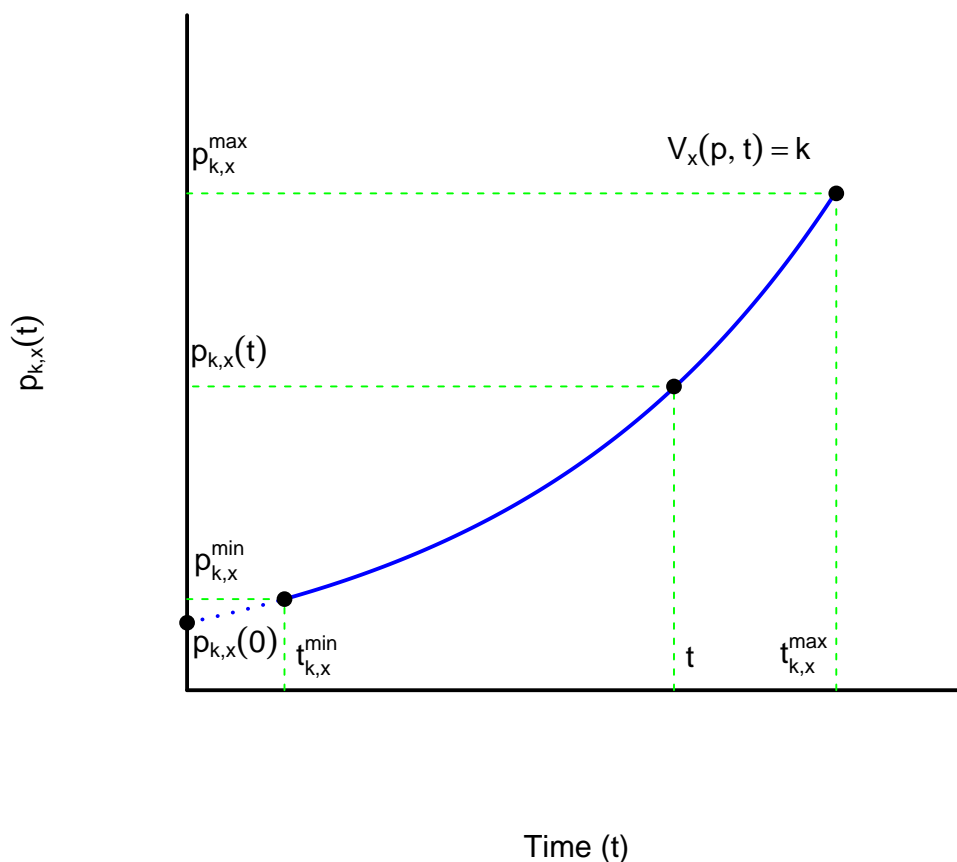


Figure 2. Indifference line $V_x(p, t) = k$ ($0 < k \leq x$). Source: Own elaboration.

Lemma 1. *The following statements hold:*

- (i) $t_{k,x}^{\min}$ is the solution of $V_x(0, t) = k$ if, and only if, $0 \in \text{proy}_2(\mathcal{D})$.
- (ii) $t_{k,x}^{\max}$ is the solution of $V_x(1, t) = k$ if, and only if, $1 \in \text{proy}_2(\mathcal{D})$.

Proof. Take into account that $V_x(0, t)$ and $V_x(1, t)$ are lower and upper bounded functions in time. This guarantees the existence of both a minimum and a maximum time. \square

Proposition 1. *If V is regular or V is singular and $0 < L(x, p) < k$, then $1 \in \text{proy}_2(\mathcal{D})$.*

Proof. In effect, by the definition of a discount function in the context of the PTT model, one has $V_x(1, 0) = x$. Moreover, if V is regular or V is singular and $0 < L(x, p) < k$, the following chain of inequalities holds:

$$0 \leq L(x, 1) < k < x.$$

Therefore, as V is continuous and decreasing with respect to time, by the Intermediate Value Theorem, there exists t_1 , such that $V_x(1, t_1) = k$. Consequently, $1 \in \text{proy}_2(\mathcal{D})$. \square

Example 4. *Let us consider the following discount function in the context of the PTT model:*

$$V(x, p, t) = \frac{xp}{1 + ixt},$$

where $i > 0$. If $V(x, p, t) = k$ ($0 < k \leq x$), then

$$p_{k,x}(t) = \frac{k}{x - ikt} = \frac{p_{k,x}(0)}{1 - ip_{k,x}(0)t},$$

which is obviously increasing with respect to t . The minimum value of t is $t_{k,x}^{\min} = 0$, and the maximum value of t is

$$t_{k,x}^{\max} = \frac{x - k}{ikx} = \frac{1 - p_{k,x}(0)}{ip_{k,x}(0)x}.$$

The corresponding minimum and maximum values of p are $p_{k,x}^{\min} = p_{k,x}(0) = k$ and $p_{k,x}^{\max} = 1$, respectively.

Finally, the map of indifference curves $V_x(p, t) = k, 0 < k \leq x$, gives rise to a family of distribution functions, denoted by $p_{k,x}(t)$, corresponding to a stochastic process, satisfying:

$$V(x, p_{k,x}(t), t) = k.$$

More, specifically, Figure 2 represents all couples (t, p) such that $(x, p, t) \in X \times P \times T$, for given values of x and $0 < k \leq x$. Moreover, it shows that there exists a functional relationship between p and t , given by $p_{k,x}(t)$, with $0 < k \leq x$. Indeed, $p_{k,x}(t)$ is a distribution function. In effect, let us consider the random variable $T_{k,x}$: "Time period in which the reward x can be delivered, at a discounting level k ". In other words, each random variable $T_{k,x}$ is a stopping time depending on k and x . Specifically, $p_{k,x}(t)$ is the distribution function of $T_{k,x}$:

$$p_{k,x}(t) = \Pr(T_{k,x} \leq t).$$

Consequently, starting from X , a continuous stochastic process has been obtained for every $x \in X$, giving rise to the following family of stochastic processes:

$$\{ \{ T_{k,x} \}_{0 < k \leq x} \}_{x \in X}.$$

Assume that $x_1 < x_2$. Then $T_{k,x_1} < T_{k,x_2}$, $0 < k \leq x_1 < x_2$, i.e., the stochastic process $\{T_{k,x}\}_{x \in X}$ is increasing with respect to x . Moreover, if $0 < k_1 < k_2 \leq x$, then $T_{k_1,x} < T_{k_2,x}$ and so $\{T_{k,x}\}_{0 < k \leq x}$ is increasing with respect to k . In this case, we can define a discount function where time is stochastic in the following way.

Definition 4. A discount function with stochastic time is a real function

$$F : \{ \{(x, T_{k,x})\}_{0 < k \leq x} \}_{x \in X} \longrightarrow \mathbb{R}$$

such that

$$(x, T_{k,x}) \mapsto F(x, T_{k,x}),$$

defined by:

$$F(x, T_{k,x}) = V(x, p_0, t_0),$$

such that $p_{k,x}(t_0) = p_0$.

Obviously, function F is well defined. Therefore, we can state the following theorem.

Theorem 1. A discount function $V(x, p, t)$ is equivalent to the discount function with stochastic time $F(x, T_{k,x})$, where $T_{k,x}$ is the random variable: "Time period in which the reward x can be delivered, at a discounting level k ".

Remark 1. Observe that for every $x \in X$, the random variables $T_{k,x}$ could be indexed, instead of by k , by another parameter in one-to-one correspondence with k . For example, for a given value p of probability, there is a biunivocal correspondence between k -values and calendar times. Thus, by denoting the calendar time as τ , the random variable $T_{k,x}$ becomes $T_{\tau,x}$, in whose case we will say that the discount function of Definition 4 is time-dependent (in the particular case in which time is age, the discount function is said to be age-dependent [42]).

In the same way, for a given value pt of time, there is a bijective correspondence between k -values and probabilities. Thus, by denoting the probability as q , the random variable $T_{k,x}$ becomes $T_{q,x}$, in whose case we will say that the discount function of Definition 4 is risk-dependent.

On this question, we will return later in Section 3.3. Observe that now there is an extra force of discount given by the probability p . This statement can be shown in the following proposition.

Proposition 2. Assume that V is differentiable. Then, the instantaneous discount rate of V , denoted by δ_V , is greater than the instantaneous discount, δ_F , of the discount function in the context of the DU model, $F(x, t) := V(x, p, t)$, where p is constant.

Proof. In effect, the derivative of the implicit function $V(x, p_{k,x}(t), t) = k$ ($0 < k \leq x$), results in:

$$dV = \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial p} \frac{\partial p_{k,x}(t)}{\partial x} dx + \frac{\partial V}{\partial p} \frac{\partial p_{k,x}(t)}{\partial t} dt + \frac{\partial V}{\partial t} dt = 0,$$

from where the instantaneous discount rate of V at (x, p, t) is (see [43]):

$$\delta_V(x, p, t) := \frac{dx}{xdt} = -\frac{1}{x} \frac{\frac{\partial V}{\partial t} + \frac{\partial V}{\partial p} \frac{\partial p_{k,x}(t)}{\partial t}}{\frac{\partial V}{\partial x} + \frac{\partial V}{\partial p} \frac{\partial p_{k,x}(t)}{\partial x}}.$$

As

$$\frac{\partial V}{\partial p} \frac{\partial p_{k,x}(t)}{\partial t} > 0$$

and

$$\frac{\partial V}{\partial p} \frac{\partial p_{k,x}(t)}{\partial x} < 0,$$

then, for every $(x, p, t) \in \mathcal{M}$,

$$\delta_V(x, p, t) > -\frac{1}{x} \frac{\frac{\partial V}{\partial t}}{\frac{\partial V}{\partial x}},$$

which is the instantaneous discount rate when p is constant. Therefore,

$$\delta_V(x, p, t) > \delta_F(x, t),$$

as expected. \square

3.2. Deriving EU from PTT Model

The following result has been inspired in [7].

Lemma 2. Given $x \in X$, let us consider the real function $V_x : P \times T \rightarrow \mathbb{R}$, defined as $V_x(p, t) := V(x, p, t)$. Then, for every $(p, t) \in P \times T$ and every $s < t$, there exists $k = k(x, p, s, t)$ ($0 < k < 1$) such that $V_x(kp, s) = V_x(p, t)$.

Proof. In effect, given $x \in X$ and $(p, t) \in P \times T$, for every $s (s < t)$, let us consider the following real-valued function:

$$V_{x,s} : P \rightarrow \mathbb{R}$$

defined as:

$$V_{x,s}(q) := V_x(q, s).$$

By the definition of V , the inequality

$$V_x(p, t) < V_{x,s}(p) \tag{1}$$

holds. Moreover, when $q \rightarrow 0$, $V_{x,s}(q) \rightarrow 0$. Therefore, there exists q_0 , small enough, such that

$$V_{x,s}(q_0) \leq V_x(p, t). \tag{2}$$

Putting together inequalities (1) and (2), one has:

$$V_{x,s}(q_0) \leq V_x(p, t) < V_{x,s}(p).$$

As V is continuous and increasing in probability, by the Intermediate Value Theorem, there exists a value $k = k(x, p, t, s)$ ($0 < k < 1$), such that $V_{x,s}(kp) = V_x(p, t)$, i.e., $V_x(kp, s) = V_x(p, t)$.

Time s is well defined. In effect, starting now from another couple (p', t') , such that $V_x(p, t) = V_x(p', t')$, for every s under t and t' , one has:

$$V_x(kp, s) = V_x(p, t)$$

and

$$V_x(k'p', s) = V_x(p', t'),$$

from where $kp = k'p'$. \square

Analogously, it could be shown that under the same conditions as Lemma 2, for every $(p, t) \in P \times T$ and every $u > t$, there exists $k (k > 1)$ such that $V_x(kp, u) = V_x(p, t)$. However, it is possible that

this result is restricted for some values of p and t whereby, for every $x \in X$, we are going to consider the values p_x^{\max} and t_x^{\max} such that

$$V(x, p, t) = V(x, p_x^{\max}, t_x^{\max}).$$

Example 5. With the discount function of Example 1:

$$V(x, p, t) = \frac{xp}{1 + ixt}$$

where $i > 0$, for every (p, t) and s , the equation in q :

$$V(x, p, t) = V(x, q, s)$$

gives the following solution:

$$q = \frac{p}{1 + ixp(t - s)}.$$

Observe that q makes sense for every $s \leq t$, and even for every s such that

$$1 + ixp(t - s) \geq p,$$

from where:

$$t < s \leq t + \frac{1 - p}{ixp}.$$

However, we will assume that always $t_x^{\min} = 0$, for every $x \in X$. To derive EU model, let

$$(x_1, p_1, t_1), (x_2, p_2, t_2), \dots, (x_n, p_n, t_n)$$

be a sequence of n outcomes in the context of the PTT model. According to Lemma 1, there exist $p_0^1, p_0^2, \dots, p_0^n$ ($p_0^k \leq p_k, k = 1, 2, \dots, n$), such that:

$$V(x_k, p_k, t_k) = V(x_k, p_0^k, 0),$$

for every $k = 1, 2, \dots, n$. Consequently, the present value of these outcomes is:

$$V_0 := \sum_{k=1}^n V(x_k, p_k, t_k) = \sum_{k=1}^n V(x_k, p_0^k, 0).$$

Independently of the shape of function V , the PTT model has been transformed into an EU model because time has been removed from the prospect (x, p, t) . Observe that probabilities are not linear with respect to the discounted amounts.

Remark 2 (On non-linear probabilities). Chew and Epstein [44] proposed some alternative theories with non-linear probabilities which may explain many behavioral paradoxes while holding normatively properties; for instance, consistency with stochastic dominance and risk aversion. In this vein, Halevy [45] provided a function for the discounted utility based on the non-linear probability weighting to evaluate the diminishing impatience related with the uncertainty of delayed rewards. These theories are useful, analytical tools in the decision-making process which allow separating the risk aversion from the elasticity of substitution.

To explain the non-linear psychological distance, Baucells and Heukamp [1] stated that it is necessary a non-linear probability weighting bonded with non-exponential discounting.

With the aim of illustrating the non-linear probability weighting, the mathematical description by Brandstätter, Kühberger and Schneider [46] is shown to reflect the elation and disappointment in probabilities.

In effect, being $1 - p$ the utility after a success and $-p$ the disutility after a failure, the expected success is calculated as follows:

$$\text{utility after a success} \times \text{probability of a success} = p(1 - p),$$

meanwhile the expected failure is:

$$\text{disutility after a failure} \times \text{probability of a failure} = -p(1 - p),$$

To take into account the non-linearity of the utility after a success (that is, elation) and the disutility after a failure (that is, disappointment), new utilities may be implemented to obtain the expected success and failure. In this way, the utility after a success and the disutility after a failure are $c_e s(1 - p)$ and $c_d s p$, respectively, where c_e and c_d are constants and s is a non-linear surprise function. Since disappointment is an aversive emotional state, c_d is expected to be negative. A steeper slope for disappointment than for elation is assumed (this means that $c_d > c_e$).

3.3. Deriving DU from PTT Model

Analogously to Lemma 2, we can show the following statement.

Lemma 3. Given an $x \in X$, let us consider the real function $V_x : P \times T \rightarrow \mathbb{R}$, defined as $V_x(p, t) := V(x, p, t)$. If V is regular, then, for every $(p, t) \in P \times T$ and every $q < p$, there exists $k = k(x, p, q, t)$ ($0 < k < 1$) such that $V_x(q, kt) = V_x(p, t)$.

Analogously, it could be shown that under the same conditions as Lemma 3, for every $(p, t) \in P \times T$ and every $q > p$, there exists $k = k(x, p, q, t)$ ($k > 1$) such that $V_x(q, kt) = V_x(p, t)$. However, as in Section 3.2, it is possible that this result is restricted for some values of p and t . In this section, we will always assume that $p_x^{\max} = 1$, for every $x \in X$.

To derive DU model, let

$$(x_1, p_1, t_1), (x_2, p_2, t_2), \dots, (x_n, p_n, t_n)$$

be a sequence of n outcomes in the context of the PTT model. If V is regular, according to Lemma 3, there exist $t_1^1, t_1^2, \dots, t_1^n$ ($t_1^k \geq t_k, k = 1, 2, \dots, n$), such that:

$$V(x_k, p_k, t_k) = V(x_k, 1, t_1^k),$$

for every $k = 1, 2, \dots, n$. Consequently, the present value of these outcomes is:

$$V_0 := \sum_{k=1}^n V(x_k, p_k, t_k) = \sum_{k=1}^n V(x_k, 1, t_1^k).$$

Independently of the shape of function V , the PTT model has been transformed into a DU model because probability is constant and equal to 1 whereby all outcomes are sure.

Corollary 1. A specific PTT model of the form $V(x, p, t) := V(xp, t)$ gives rise to both a DU and an EU model.

Proof. In effect, the equation:

$$V(xp, t) = V(xq, s)$$

implies:

1. For $q = 1, V(xp, t) = V(x1, t_x^{\max}) = V(x, t_x^{\max})$, which is the DU model.
2. For $t = 0, V(xp, t) = V(xp_x^{\min}, 0) = xp_x^{\min}$, which is the EU model.

□

3.4. Deriving PTT from DU or EU Model

In this context, it could be interesting to think about the converse construction, i.e., to generate a function $V(x, p, t)$ starting from $V(x, t)$ and $V(x, p)$ coming from a DU and an EU model, respectively. In effect,

- (A) Given a DU model, $V(x, t)$, it can be assumed that all prospects (x, t) have probability $p = 1$. In this case, we can construct a PTT model which coincides with the DU model when $p = 1$:

$$V(x, p, t) := V(x, t)p$$

or

$$V(x, p, t) := V(xp, t).$$

- (B) Analogously, given an EU model, $V(x, p)$, it can be understood that all prospects (x, p) expire at the same instant $t = t_0$. In this case, we can construct a PTT model which coincides with the EU model when $t = t_0$:

$$V(x, p, t) := x \left[\frac{V(x, p)}{x} \right]^{\frac{t}{t_0}}.$$

Observe that for the sake of generality, neither of the two PTT proposed models are of the form $V(x, p, t) = w(p)f(t)v(x)$ nor $V(x, p, t) = g(p, t)v(x)$ pointed out by [1].

- (C) Given a DU model, $V(x, t)$, and an EU model, $V(x, p)$, if $V(x, 0) = x$ and $V(x, 1) = x$, respectively, we can construct a PTT model which coincides with both models, at $t = 0$ and $p = 1$, respectively:

$$V(0, p, t) = 0$$

and

$$V(x, p, t) = \frac{1}{x} V(x, t) V(x, p),$$

otherwise.

4. Discussion

In this paper, it has been mathematically demonstrated that the PTT model is general enough to explain intertemporal and risky decisions separately. The shown association between PTT model and DU and EU models allows explaining and relating the behavioral properties and inconsistencies in real choices.

It must be mentioned that most previous literature studies anomalies in intertemporal and probabilistic choices separately. In this way, the analysis of the equivalence between certain anomalies in the context of the DU and EU models may be analyzed under the wide setting provided by the PTT model.

A way to study the similarity between time delay and uncertainty in the decision-making process is through the analysis of the immediacy and certainty effects. These effects reflect the tendency of individuals to overestimate the significance of immediacy or certainty, relative to delayed or probable outcomes, respectively. The relation between both effects allows glimpsing the analogies and the influence of time and delay on the decision-making process. Some of the main findings are listed below:

- In the seminal paper by Allais [47], the disproportionate preference for present outcomes of the immediacy effect is shown, consequently not only of the intrinsic temptation but the certainty on the payment.
- Keren and Roelofsma [28] analyze the effect of risk on the immediacy effect and the effect of time delay on the certainty effect. They suggest that time distance makes outcomes seem more uncertain by eliminating the certainty advantage of the immediate outcome. Thereof, they reveal that the introduction of uncertainty reduces the importance of time delay (if two certain rewards

- are transformed to be equally probable (i.e., $p = 0.50$), the delayed one is generally preferred). In the same way, time distance decreases the influence of the probability on preferences.
- Chapman and Weber [17] prove that when the delay is introduced to sure outcomes, the certainty effect is almost eliminated just as when uncertainty is added. On the other hand, in a similar way, when explicit risk is introduced to immediate rewards, the immediacy effect is almost eliminated just as if time delay is added. It is necessary to clarify that presently there is no consensus on this topic, while Pennesi [48] confirms that when the immediate payoff becomes uncertain, the immediacy effect disappears, Abdellaoui et al. [35] claim that the immediacy effect persists under risk.
 - Epper et al. [31] stress that previous papers, in most cases, determine that there are interaction effects between time and risk, such as risk tolerance increases with delay.
 - Andreoni and Sprenger [49] conclude that risk preference is not time preference: “subjects exhibit a preference for certainty when it is available, but behave largely as discounted expected utility maximizers away from certainty”.

Another way to conclude that delay and risk choices have non-parallel decision mechanisms has been proved by Cruz Rambaud and Sánchez Pérez [7] and Chapman and Weber [17]. The effect of the reward size on the treatment of delayed and probabilistic outcomes moves in the opposite direction. On one hand, in intertemporal choices, as the reward amount increases, decision-makers prefer the delayed but greater reward (this is called the “magnitude effect”). On the other hand, in risky choices, the reward size increase implies a decreasing sensitivity; decision-makers prefer the more probable but lower reward (this is called the “peanut effect”).

Despite some previous papers claiming that DU and EU models anomalies move in the opposite direction, the findings of our research are in line with most part of the previous literature linking the anomalies exhibited by both models. Given that the PTT model considers a unique framework to deal with uncertainty and delayed rewards, it may imply that risk and time anomalies can be captured by a unique model. Thus, the results of this research contribute to clearly understanding the aforementioned relationship by considering risk and time anomalies inside the same framework.

Specifically, Leland and Schneider [14] pointed that “the PTT model accounts for three systematic interaction effects between risk and time preferences when choices involve both risk and time delays as well as some other fundamental behaviors such as the common ratio and common difference effects”.

In effect, Schneider [21] introduces the two following dual concepts:

- *Time interacts with risk preference* if, for every $x \in (0, z)$, $\alpha \in (0, 1)$, and $s > t$,

$$(x, p, t) \sim (z, \alpha p, t) \text{ implies } (x, p, s) \prec (z, \alpha p, s).$$

- *Risk interacts with time preference* if, for every $x \in (0, z)$, $t, \Delta > 0$, and $q < p$,

$$(x, p, t) \sim (z, p, t + \Delta) \text{ implies } (x, q, t) \prec (z, q, t + \Delta).$$

By applying Lemmas 2 and 3, it can be shown that the former definitions are equivalent in the presence of the so-called *reverse sub-endurance* (see [7]):

For every $x \in (0, z)$, $t, \Delta > 0$, and $\alpha \in (0, 1)$,

$$(z, p, t + \Delta) \sim (z, \alpha p, t) \text{ implies } (x, p, t + \Delta) \succ (x, \alpha p, t).$$

In this way, our objective will be to investigate the possible equivalence between other dual concepts in the ambit of risk and time preferences. However, this issue will be left for further research in the context of the PTT model.

5. Conclusions

In this paper, an extensive review of the previous methodologies which assess risk and delayed rewards in a unique framework has been made. It reveals that there is still room to improve the existing methodologies to reach this goal. Specifically, this paper has dealt with the classical problem of the possible relationship between the DU and the EU models but treated from the joint perspective of the PTT model.

In this paper, the equivalence of the PTT model with the DU is demonstrated and the EU models separately considered. In this sense, this paper's main findings are three-fold:

- On the one hand, the DU model has been derived from the PTT model, by taking a specific value of probability. Specifically, we have found that the PTT model is equivalent to the discount function with stochastic time $F(x, T_{k,x})$, where $T_{k,x}$ is the random variable: "Time period in which the reward x can be delivered, at a discounting level k ".
- Analogously, given a concrete value of time, the EU model can be derived from the PTT model.
- Finally, this paper provides some insights into the construction of a PTT model starting from a DU and an EU model. However, more future research is needed on this topic.

Thus, this paper shows the validity of the PTT model to assess risky and delayed rewards separately. As pointed out, a limitation of the paper is the difficulty of building a complete PTT model starting from DU and EU models in a same framework. However, a solution to do this is provided: the construction of the PTT model starting from DU and EU models by using specific models able to satisfy the appropriate patterns of time and probability pointed out by Baucells and Heukamp [1].

As a future line of research, we stress that it is necessary to continue analyzing the equivalence between the PTT and DU and EU models from the perspective of the different effects or anomalies when considering delay and time individually (in DU and EU models, respectively) and jointly (PTT model). A second future research line is to check the validity of the PTT model and its relationship with DU and EU models from an experimental point of view.

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Abbreviations

The following abbreviations are used in this manuscript:

DU	Discounted Utility
EU	Expected Utility
DEU	Discounted Expected Utility
PTT	Probability and Time Trade-Off

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