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# Asymptotics for varying discrete Sobolev orthogonal polynomials 

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#### Abstract

We consider a varying discrete Sobolev inner product such as $$
(f, g)_{S}=\int f(x) g(x) d \mu+M_{n} f^{(j)}(c) g^{(j)}(c)
$$ where $\mu$ is a finite positive Borel measure supported on an infinite subset of the real line, $c$ is adequately located on the real axis, $j \geq 0$, and $\left\{M_{n}\right\}_{n \geq 0}$ is a sequence of nonnegative real numbers satisfying a very general condition. Our aim is to study asymptotic properties of the sequence of orthonormal polynomials with respect to this Sobolev inner product. In this way, we focus our attention on Mehler-Heine type formulae as they describe in detail the asymptotic behavior of these polynomials around $c$, just the point where we have located the perturbation of the standard inner product. Moreover, we pay attention to the asymptotic behavior of the (scaled) zeros of these varying Sobolev polynomials and some numerical experiments are shown. Finally, we provide other asymptotic results which strengthen the idea that Mehler-Heine asymptotics describe in a precise way the differences between Sobolev orthogonal polynomials and standard ones.


Keywords: Sobolev orthogonal polynomials • Mehler-Heine formulae • Asymptotics • Zeros. Mathematics Subject Classification (2010): 33C47 • 42C05

## 1 Introduction

Orthogonal polynomials with respect to the varying inner product

$$
\begin{equation*}
(f, g)_{S}=\int f(x) g(x) d \mu+M_{n} f^{(j)}(c) g^{(j)}(c), \quad j \geq 0 \tag{1}
\end{equation*}
$$

where $c$ is adequately located on the real axis, have been considered in some papers (see [6] and [7] and the references therein) recently. In these papers the authors focus their attention on MehlerHeine asymptotics given the relevance of this type of asymptotics for describing the differences between the sequences of orthogonal polynomials with respect to (1) and those with respect to $\mu$. The main goal of this paper is to give a final and global vision of the Mehler-Heine asymptotics for the orthogonal polynomials with respect to (1). In fact, wheather $\mu$ has bounded or unbounded support will not be relevant for the results that we will provide. Therefore, all the previous results about this type of asymptotics for varying Sobolev orthogonal polynomials in the aforementioned papers are particular cases of Theorem 1 (or of its symmetric version).

Mehler-Heine asymptotics were introduced for Legendre polynomials by H. E. Heine and G. F. Mehler in the 19th century. In Szegő's book [16, Sect. 8.1] we can find the corresponding MehlerHeine formulae for classical continuous orthogonal polynomials: Jacobi, Laguerre, and Hermite. As a consequence, using Hurwitz's theorem, the asymptotic behavior of the scaled zeros of these families of polynomials is deduced. Along this century, several authors have paid attention to this type of asymptotics in different contexts such as multiple orthogonal polynomials [18], Sobolev orthogonal polynomials (partially cited in the surveys [8] and [10]), generalized Freud polynomials [3], exceptional orthogonal polynomials [5], among others.

Coming back to the varying Sobolev inner product (1), Mehler-Heine asymptotics of the corresponding orthogonal polynomials have been studied for measures $\mu$ related to the Jacobi and Laguerre weight functions in [6] and [7]. In both papers the techniques used involve particular properties of Jacobi/Laguerre orthogonal polynomials. However, in [15] a general approach is given for the non-varying case. With that new technique the authors obtain asymptotic results for general measures which can have either bounded or unbounded support. Therefore, our aim in this paper is to apply this method to the varying case. With this paper, we conclude the study of the asymptotics for these varying orthogonal polynomials.

Thus, we consider the inner product (1) where $\left\{M_{n}\right\}_{n \geq 0}$ is a sequence of nonnegative real numbers satisfying the following general condition

$$
\begin{equation*}
\lim _{n \rightarrow \infty} M_{n} K_{n-1}^{(j, j)}(c, c)=L \in[0,+\infty] \tag{2}
\end{equation*}
$$

where $K_{n-1}^{(j, k)}(x, y)$ denotes the partial derivatives of the $n$th kernel for the sequence of polynomials $\left\{p_{n}\right\}_{n \geq 0}$ orthonormal with respect to the finite positive Borel measure $\mu$, i.e.

$$
K_{n}^{(j, k)}(x, y)=\frac{\partial^{j+k}}{\partial x^{j} \partial y^{k}} K_{n}(x, y)=\sum_{i=0}^{n} p_{i}^{(j)}(x) p_{i}^{(k)}(y), \quad j, k \in \mathbb{N} \cup\{0\}
$$

Notice that condition (2) is even more general than the condition stated for $\left\{M_{n}\right\}_{n \geq 0}$ in [6] and [7]. In fact, we can observe $M_{n} K_{n-1}^{(j, j)}(c, c)$ is nonnegative for each $n$ (actually it is positive for
almost every $n$ ). Thus, condition (2) is very general since we admit that this sequence can be either convergent or divergent $(L=+\infty)$.

We are going to work with orthonormal polynomials, so we denote by $\left\{q_{n}\right\}_{n \geq 0}$ the sequence of orthonormal polynomials with respect to the inner product (1). In fact, for each $n$, we have a square tableau of orthonormal polynomials $\left\{q_{k}^{\left(M_{n}\right)}\right\}_{k \geq 0}$ but we deal with the diagonal of this tableau $\left\{q_{n}^{\left(M_{n}\right)}\right\}_{n \geq 0}=:\left\{q_{n}\right\}_{n \geq 0}$. Along the paper we will use the notation $f_{n} \simeq g_{n}$ to indicate that $\lim _{n \rightarrow \infty} f_{n} / g_{n}=1$.

We will prove that for the varying case we obtain three different Mehler-Heine formulae depending on the value of $L$, or equivalently, on the size of the sequence $\left\{M_{n}\right\}_{n \geq 0}$. That is relevant since, on one hand, we will show that the term $M_{n} f^{(j)}(c) g^{(j)}(c)$ influences on the local asymptotic behavior of $q_{n}$ and, on the other hand, it is limited by the size of $L$ (for example, when $L=0$ there is no influence, and the case $L=+\infty$ includes, as a very particular situation, the constant case $M_{n}=M>0$, for all $n$ ).

The structure of the paper is the following. In Section 2 we establish the necessary background about the sequence of varying orthonormal polynomials $q_{n}$. In Section 3 we provide the MehlerHeine asymptotics for the sequence $\left\{q_{n}\right\}_{n \geq 0}$ distinguishing two cases: either $\mu$ is symmetric or $\mu$ is nonsymmetric. In Section 4 the consequences of the Mehler-Heine formulae on the asymptotic behavior of the zeros of $q_{n}$ are shown. In Section 5, we obtain the outer relative asymptotics between the families of polynomials $\left\{q_{n}\right\}_{n \geq 0}$ and $\left\{p_{n}\right\}_{n \geq 0}$ when $\mu$ is a measure that belongs to Szegö's class, as well as the Plancherel-Rotach asymptotics when $\mu$ has an unbounded support. Finally, we illustrate the asymptotic behavior of the zeros given in Section 4 with an example involving the Hermite weight.

## 2 Varying Discrete Sobolev Orthonormal Polynomials

Let $\left\{p_{n}\right\}_{n \geq 0}\left(p_{n}(x)=\gamma_{n} x^{n}+\right.$ lower degree terms, and $\left.\gamma_{n}>0\right)$ be the sequence of orthonormal polynomials with respect to the measure $\mu$ and $\left\{q_{n}\right\}_{n \geq 0}\left(q_{n}(x)=\tilde{\gamma}_{n} x^{n}+\right.$ lower degree terms, and $\left.\tilde{\gamma}_{n}>0\right)$ the sequence of orthonormal polynomials with respect to the inner product (1). In addition, we denote by $\left\{p_{n}^{[2 i]}\right\}_{n \geq 0}$ the sequence of orthonormal polynomials with respect to the measure $d \mu_{2 i}(x)=(x-c)^{2 i} d \mu(x), i \geq 0$ and $c \in \mathbb{R} \backslash \operatorname{supp}(\mu)$. We notice that the leading coefficient of all the orthonormal polynomials considered in this paper are taken positive.

A useful connection formula between the families of polynomials $\left\{q_{n}\right\}_{n \geq 0}$ and $\left\{p_{n}^{[2 i]}\right\}_{n \geq 0}$ was given in [15, Th. 1] for non-varying discrete Sobolev orthogonal polynomials. The proof for the varying case is totally analogous, therefore we omit it. Thus, we have

Lemma 1 ([15]) Assuming that $p_{n}(c) p_{n-1}^{[2]}(c) \ldots p_{n-(j+1)}^{[2(j+1)]}(c) \neq 0$, there exists a family of coefficients $\left\{d_{i, n}\right\}_{i=0}^{j+1}$ not identically zero, such that the following connection formula holds

$$
\begin{equation*}
q_{n}(x)=\sum_{i=0}^{j+1} d_{i, n}(x-c)^{i} p_{n-i}^{[2 i]}(x), \quad n \geq j+1 \tag{3}
\end{equation*}
$$

The aim of this section is to establish the asymptotic behavior of the coefficients $d_{i, n}$ in (3) when $n \rightarrow \infty$.

Lemma 2 We assume that there exists a strictly increasing function $f$, with $2 f(0)+1>0$, such that the polynomials $\left\{p_{n}\right\}_{n \geq 0}$ satisfy the condition

$$
\begin{equation*}
p_{n}^{(k)}(c) \simeq C_{k, 0}(-1)^{n} n^{f(k)}, \quad 0 \leq k \leq n \tag{4}
\end{equation*}
$$

Then, for $k \geq 0$ we have

$$
\lim _{n \rightarrow \infty} \frac{q_{n}^{(k)}(c)}{p_{n}^{(k)}(c)}=\theta_{k, j, L}:=\frac{L(f(k)-f(j))+f(k)+f(j)+1}{(1+L)(f(k)+f(j)+1)}
$$

where $\theta_{k, j,+\infty}=\lim _{L \rightarrow \infty} \theta_{k, j, L}$.
Proof: We get (see, for example, [11, Sect. 2] for the non-varying case):

$$
\begin{equation*}
q_{n}(x)=\frac{\tilde{\gamma_{n}}}{\gamma_{n}}\left(p_{n}(x)-\frac{M_{n} p_{n}^{(j)}(c)}{1+M_{n} K_{n-1}^{(j, j)}(c, c)} K_{n-1}^{(0, j)}(x, c)\right) \tag{5}
\end{equation*}
$$

First, we claim that the following limit exists,

$$
\lim _{n \rightarrow \infty} \frac{K_{n-1}^{(k, j)}(c, c)}{n^{f(k)+f(j)+1}} \in \mathbb{R}
$$

Moreover, using (4) and Stolz's criterion, we have

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \frac{K_{n-1}^{(k, j)}(c, c)}{n^{f(k)+f(j)+1}}=\frac{C_{k, 0} C_{j, 0}}{f(k)+f(j)+1} \tag{6}
\end{equation*}
$$

and, therefore, we deduce in a straightforward way

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \frac{\tilde{\gamma_{n}}}{\gamma_{n}}=1 \tag{7}
\end{equation*}
$$

From (5) we get

$$
\begin{aligned}
\lim _{n \rightarrow \infty} \frac{q_{n}^{(k)}(c)}{p_{n}^{(k)}(c)} & =\lim _{n \rightarrow \infty} \frac{\tilde{\gamma_{n}}}{\gamma_{n}}\left(1-\frac{M_{n} K_{n-1}^{(k, j)}(c, c)}{1+M_{n} K_{n-1}^{(j, j)}(c, c)} \frac{p_{n}^{(j)}(c)}{p_{n}^{(k)}(c)}\right) \\
& =\lim _{n \rightarrow \infty} \frac{\tilde{\gamma_{n}}}{\gamma_{n}}\left(1-\frac{M_{n} K_{n-1}^{(j, j)}(c, c) \frac{K_{n}^{(k, j)}(c, c)}{K_{n-1}^{j(j)}(c, c)} \frac{n^{2 f(j)+1}}{n^{f(k)+f(j)+1}}}{1+M_{n} K_{n-1}^{(j, j)}(c, c)} \frac{\frac{p_{n}^{(j)}(c)}{n^{f(j)}}}{\frac{p_{n}^{(k)}(c)}{n^{f(k)}}}\right)
\end{aligned}
$$

It only remains to use (2) and (4) for different values of $L$. The case $L=0$ is trivial. If $L \in(0,+\infty)$ and taking into account (6), we deduce

$$
\begin{aligned}
\lim _{n \rightarrow \infty} \frac{q_{n}^{(k)}(c)}{p_{n}^{(k)}(c)} & =1-\frac{L \frac{C_{k, 0} C_{j, 0}}{f(k)+f(j)+1} \frac{2 f(j)+1}{C_{j, 0}^{2}} \frac{C_{j, 0}}{C_{k, 0}}}{1+L} \\
& =\frac{L(f(k)-f(j))+f(k)+f(j)+1}{(1+L)(f(k)+f(j)+1)}=\theta_{k, j, L} .
\end{aligned}
$$

In a similar way, we derive the result for the case $L=+\infty$.
Remark 1 The factor ( -1$)^{n}$ in the condition (4) may appear or not according to the type of measure that we are considering. In fact, this result and the next ones are true if we omit it. However, this factor is necessary in the Hermite case (when $\mu$ is the measure corresponding to Hermite weight) that we will use to illustrate our results in Section 6. So, we have decided to state the condition (4) and other similar conditions along the paper including this factor.

Remark 2 For example, in [9] the authors consider the non-varying inner product

$$
(f, g)=\int_{-1}^{1} f(x) g(x) d \mu+\lambda f^{\prime}(c) g^{\prime}(c), \quad \lambda>0
$$

where $\mu$ belongs to Nevai class. They establish [9, f. (34)] that

$$
\begin{equation*}
\lim _{n \rightarrow \infty} q_{n}^{\prime}(c) p_{n}^{\prime}(c)=0 \tag{8}
\end{equation*}
$$

when $c \in[-1,1]$. In this context, the above inner product can be seen as the inner product (1) with $M_{n}=\lambda$ for every $n$ and $j=1$. This situation is a particular case of $L=+\infty$. Assuming (4) is satisfied, then from Lemma 2 we get

$$
\lim _{n \rightarrow \infty} \frac{q_{n}^{\prime}(c)}{p_{n}^{\prime}(c)}=0 .
$$

Furthermore, using (5), (6) and the fact that $L=+\infty$, we deduce easily

$$
\begin{equation*}
\frac{q_{n}^{\prime}(c)}{p_{n}^{\prime}(c)} \simeq \frac{K}{n^{2 f(1)+1}} \tag{9}
\end{equation*}
$$

where $K$ is a constant. Straightforward computations allow us to deduce (8) from (9) whenever (4) holds.

Now, we can proceed as in $\left[15\right.$, Th. 2] to establish that the coefficients $d_{i, n}$ for $i=0, \ldots, n$, converge. Moreover, we can give the explicit value of $\lim _{n \rightarrow \infty} d_{i, n}$ which is essential for numerical experiments.

Proposition 1 We suppose that there exists a strictly increasing function $f$, with $2 f(0)+1>0$, such that for all $i=0,1, \ldots, j+1$, the polynomials $\left\{p_{n}^{[2 i]}(x)\right\}_{n \geq 0}$ satisfy the condition

$$
\begin{equation*}
\left(p_{n}^{[2 i]}\right)^{(k)}(c) \simeq C_{k, i}(-1)^{n} n^{f(k+i)}, \quad 0 \leq k \leq n \tag{10}
\end{equation*}
$$

where $C_{k, i}$ is a nonzero constant independent of $n$. Then,

$$
\lim _{n \rightarrow \infty} d_{i, n}=d_{i}(L) \in \mathbb{R}, \quad i=0,1, \ldots, j+1
$$

where $\left\{d_{i, n}\right\}_{i=0}^{j+1}$ are the coefficients in the connection formula (3). Moreover, for $0 \leq i \leq j+1$,

$$
\begin{equation*}
d_{i}(L)=(-1)^{i} \frac{\theta_{i, j, L}-\sum_{m=0}^{i-1}(-1)^{m} d_{m}(L)\binom{i}{m} m!\frac{C_{i-m, m}}{C_{i, 0}}}{i!\frac{C_{0, i}}{C_{i, 0}}} \tag{11}
\end{equation*}
$$

where, by convention, we assume $\sum_{m=0}^{-1}=0$.
Proof: Taking the $k$ th derivative in (3) with $0 \leq k \leq j+1$, and evaluating the corresponding expression at $x=c$, we have

$$
q_{n}^{(k)}(c)=\sum_{i=0}^{k} d_{i, n}\binom{k}{i} i!\left(p_{n-i}^{[2 i]}\right)^{(k-i)}(c)
$$

From Lemma $2, \lim _{n \rightarrow \infty} \frac{q_{n}^{(k)}(c)}{p_{n}^{(k)}(c)}$ exists and it depends on the value of $L \in[0,+\infty]$. From the above expression, we have

$$
\frac{q_{n}^{(k)}(c)}{p_{n}^{(k)}(c)}=\sum_{i=0}^{k} d_{i, n}\binom{k}{i} i!A_{i}(k, n)
$$

where $A_{i}(k, n)=\frac{\left(p_{n-i}^{[2 i]}\right)^{(k-i)}(c)}{p_{n}^{(k)}(c)}$. Using (10) the sequence $\left\{A_{i}(k, n)\right\}_{n \geq 0}$ converges and

$$
\begin{equation*}
\lim _{n \rightarrow \infty} A_{i}(k, n)=(-1)^{i} \frac{C_{k-i, i}}{C_{k, 0}} \in \mathbb{R} \tag{12}
\end{equation*}
$$

Then, we deduce $\lim _{n \rightarrow \infty} d_{i, n}:=d_{i}(L) \in \mathbb{R}$ and we can apply a recursive algorithm to compute explicitly $d_{i}(L)$ with $0 \leq i \leq j+1$, i.e. for $k=0$ we get $d_{0, n}=q_{n}(c) / p_{n}(c)$, so $d_{0}(L)$ can be computed using Lemma 2. Thus

$$
d_{0}(0)=1, \quad d_{0}(L)=\theta_{0, j, L}, \quad 0<L<+\infty, \quad \text { and } \quad d_{0}(+\infty)=\frac{f(0)-f(j)}{f(0)+f(j)+1}
$$

Then, we take $k=1$ and, as a consequence,

$$
\frac{q_{n}^{\prime}(c)}{p_{n}^{\prime}(c)}=d_{0, n}(L)+d_{1, n}(L) A_{1}(1, n) .
$$

It is enough to take limits in the above expression to obtain $d_{1}(L)$. We can continue with the same recursive procedure, using (12) and Lemma 2 , so we deduce the value of $d_{i}(L)$ for $i=2, \ldots, j+1$.

Remark 3 The computation of the coefficients $d_{i}(L)$ will be essential for the numerical experiment in Section 6, so for a better understanding we write them depending on the three representative values of $L$.

$$
d_{i}(L)= \begin{cases}\delta_{i, 0}, & \text { if } L=0, \\ (-1)^{i} \frac{\theta_{i, j, L}-\sum_{m=0}^{i-1}(-1)^{m} d_{m}(L)\binom{i}{m} m!\frac{C_{i-m, m}}{C_{i, 0}}}{i!\frac{C_{0, i}}{C_{i, 0}},} & \text { if } L \in(0,+\infty), \\ (-1)^{i} \frac{\frac{f(i)-f(j)}{f(i)+f(j)+1}-\sum_{m=0}^{i-1}(-1)^{m} d_{m}(+\infty)\binom{i}{m} m!\frac{C_{i-m, m}}{C_{i, 0}}}{i!\frac{C_{0, i}}{C_{i, 0}}}, & \text { if } L=+\infty,\end{cases}
$$

where $\delta_{i, m}$ is the Kronecker's delta.

## 3 Mehler-Heine asymptotics

We focus our attention on Mehler-Heine type formulae for the sequence of varying discrete Sobolev orthonormal polynomials $\left\{q_{n}\right\}_{n \geq 0}=\left\{q_{n}^{\left(M_{n}\right)}\right\}_{n \geq 0}$ since we want to know how the discrete part in the inner product (1) influences the asymptotic behavior of the corresponding orthonormal polynomials.

To establish the Mehler-Heine asymptotics we take $c$ as an endpoint of $I$, being $I$ the interval where $\mu$ is supported. We assume that $c=\inf (I) \in \mathbb{R}$. Next results can be also obtained with
$c=\sup (I) \in \mathbb{R}$ making some changes. Moreover, we need that the sequences $\left\{p_{n}^{[2 i]}\right\}_{n \geq 0}$ satisfy the Mehler-Heine type formulae

$$
\begin{equation*}
\lim _{n \rightarrow \infty}(-1)^{n} \frac{a_{n}^{1 / 2}}{b_{n}^{i}} p_{n}^{[2 i]}\left(c+\frac{z^{2}}{b_{n}}\right)=z^{-(\alpha+2 i)} J_{\alpha+2 i}(2 z), \quad i=0,1, \ldots, j+1 \tag{13}
\end{equation*}
$$

uniformly on compact subsets of $\mathbb{C}$, where

$$
\begin{equation*}
a_{n}^{-1 / 2} \simeq A n^{a}, \quad b_{n} \simeq B n^{b}, \quad A, B, b>0, \quad \alpha>-1, \quad \text { and } \quad 2 a+1=b(\alpha+1) . \tag{14}
\end{equation*}
$$

The assumptions (13-14) hold for remarkable families of measures such as measures in Nevai class or measures related to a Laguerre weight or to a generalized Jacobi weight (see [15]).

Theorem 1 Let $c=\inf (I)$ and assume that the sequences of orthonormal polynomials $\left\{p_{n}^{[2 i]}\right\}_{n \geq 0}$ satisfy (10) and (13-14). Then,

$$
\lim _{n \rightarrow \infty}(-1)^{n} a_{n}^{1 / 2} q_{n}\left(c+\frac{z^{2}}{b_{n}}\right)=\sum_{i=0}^{j+1}(-1)^{i} d_{i}(L) z^{-\alpha} J_{\alpha+2 i}(2 z):=\varphi_{\alpha, j, L}(z),
$$

uniformly on compact subsets of $\mathbb{C}$, where the coefficients $d_{i}(L)$ are given in (11).
Proof: Taking into account (13), it is enough to scale, take limits in (3), and pay attention to the proof of Corollary 1 in [3] to obtain

$$
\begin{aligned}
\lim _{n \rightarrow \infty}(-1)^{n} a_{n}^{1 / 2} q_{n}\left(c+\frac{z^{2}}{b_{n}}\right) & =\lim _{n \rightarrow \infty} \sum_{i=0}^{j+1}(-1)^{n} a_{n}^{1 / 2} d_{i, n} \frac{z^{2 i}}{b_{n}^{i}} p_{n-i}^{[2 i]}\left(c+\frac{z^{2}}{b_{n}}\right) \\
& =\sum_{i=0}^{j+1}(-1)^{i} \lim _{n \rightarrow \infty} d_{i, n} \cdot z^{2 i} \cdot \lim _{n \rightarrow \infty}(-1)^{n-i} \frac{a_{n}^{1 / 2}}{b_{n}^{i}} p_{n-i}^{[2 i]}\left(c+\frac{z^{2}}{b_{n}}\right) \\
& =\sum_{i=0}^{j+1}(-1)^{i} d_{i}(L) z^{-\alpha} J_{\alpha+2 i}(2 z),
\end{aligned}
$$

where $d_{i}(L)$ are given in (11) depending on $L$.
Remark 4 Notice that the Mehler-Heine formula obtained in this Theorem is qualitatively different from the one obtained in [15, Th. 5]. Despite the obvious difference because we have three different formulae in the varying case, when $L=+\infty$ (which includes the situation $M_{n}=M$, for all $n$ ) we have a linear combination of $j+2$ functions as the limit function whereas in Theorem 5 in [15] the limit function only involves a unique function. The reason for this is that we have a gap of order $j$, i.e., the terms $f^{(i)}(c) g^{(i)}(c)$ for $i=0, \ldots, j-1$, do not appear in the inner product (1). In [15, Th. 5], the authors establish their results for orthonormal polynomials with respect to an inner product where the terms $f^{(i)}(c) g^{(i)}(c)$ for $i=0, \ldots, j$, are multiplied by positive constants.

Remark 5 In the case $L=0$, or equivalently when the size of the sequence $\left\{M_{n}\right\}_{n \geq 0}$ is negligible, then

$$
\varphi_{\alpha, j, 0}(z)=z^{-\alpha} J_{\alpha}(2 z)
$$

Therefore, both families of orthogonal polynomials, Sobolev and standard ones, have the same Mehler-Heine asymptotics. This behavior is a consequence of the fact that the perturbation $M_{n} f^{(j)}(c) g^{(j)}(c)$ that we have introduced into the standard inner product is asymptotically insignificant, even in this type of local asymptotics.

We illustrate Theorem 1 with an example known in the literature (see [7]). So, we can show that the conditions posed to obtain this result are natural. We consider the varying Laguerre-Sobolev inner product

$$
\begin{equation*}
(f, g)_{S}=\int_{0}^{\infty} f(x) g(x) x^{\alpha} e^{-x} d x+M_{n} f^{(j)}(0) g^{(j)}(0), \quad \alpha>-1, \quad j \geq 0 \tag{15}
\end{equation*}
$$

where $\left\{M_{n}\right\}_{n \geq 0}$ is a sequence satisfying (2). Obviously, the above inner product is a particular case of (1) with $c=0$. From [16] we can deduce the properties of Laguerre orthonormal polynomials, $l_{n}^{\alpha}$, with positive leading coefficients. Thus,

$$
\left(l_{n}^{\alpha}\right)^{(k)}(0) \simeq \frac{(-1)^{k}}{\Gamma(\alpha+k+1)}(-1)^{n} n^{k+\alpha / 2}
$$

so, (4) holds with

$$
C_{k, 0}=\frac{(-1)^{k}}{\Gamma(\alpha+k+1)} \quad \text { and } \quad f(k)=k+\alpha / 2
$$

We observe that now the polynomials $p_{n}^{[2 i]}$ are orthonormal with respect to the weight function $x^{\alpha+2 i} e^{-x}$, i.e, $p_{n}^{[2 i]}=l_{n}^{\alpha+2 i}$. Thus, (10) holds with

$$
C_{k, i}=\frac{(-1)^{k}}{\Gamma(\alpha+k+2 i+1)}
$$

Finally, Mehler-Heine formula for Laguerre polynomials becomes

$$
\lim _{n \rightarrow \infty} \frac{(-1)^{n}}{n^{\alpha / 2+i}} l_{n}^{\alpha+2 i}\left(z^{2} / n\right)=z^{-(\alpha+2 i)} J_{\alpha+2 i}(2 z)
$$

uniformly on compact subsets of $\mathbb{C}$. Thus, we deduce that $b_{n}=n$ and $a_{n}=n^{-\alpha}$. Then (13-14) hold. Therefore, Theorem 1 yields the following Mehler-Heine asymptotics for the orthonormal polynomials with respect (15). Now we compare our result with the one obtained in [7]. For example, we choose $\alpha=5, j=2$, and $L=+\infty$. Then, we have

$$
\lim _{n \rightarrow \infty}(-1)^{n} n^{-\alpha / 2} q_{n}\left(z^{2} / n\right)=\sum_{i=0}^{3}(-1)^{i} d_{i}(+\infty) z^{-5} J_{5+2 i}(2 z):=\varphi_{5,2,+\infty}(z)
$$

Using (11) we can compute $d_{i}(+\infty)$ in a recursive way obtaining

$$
d_{0}(+\infty)=-\frac{1}{4}, \quad d_{1}(+\infty)=\frac{35}{36}, \quad d_{2}(+\infty)=\frac{1}{4}, \quad d_{3}(+\infty)=\frac{1}{36}
$$

To compare $\varphi_{5,2,+\infty}(z)$ with $d_{5}(z)$ in [7, Th. 1] it is necessary to use several times the well-known relation between Bessel functions

$$
J_{\nu-1}(2 z)+J_{\nu+1}(2 z)=\frac{\nu}{z} J_{\nu}(2 z)
$$

After some computations we get that both limit functions are the same. With respect to the Sobolev polynomials, it is convenient to bear in mind in [7, Th. 1] another standardization is used, although evidently this fact does not influence on the limit function.

To conclude this example, we highlight the fact that for theoretical results the value of the numbers $d_{i}(L)$ and the constants $C_{k, i}$ are not relevant, although as we have seen, if we want to obtain a concrete result, then these values are very important and (11) plays an important role.

Now, we consider a symmetric positive Borel measure $\mu$ supported on an interval $I=(-\rho, \rho)$, for example, related to the Hermite weight function or to the generalized Freud one, both on the real line. Then, we can also obtain the Mehler-Heine asymptotics being now the formal details technically more complicated.

It is convenient for Section 6 to give a quick overview about the well-known symmetrization process (see, for example, [2]). Let $\left\{p_{n}\right\}_{n \geq 0}$ be the sequence of orthonormal polynomials with respect to a symmetric measure $\mu$ supported on $I=(-\rho, \rho)$ with $0<\rho \leq \infty$. Obviously, these polynomials are symmetric. Thus, we take $v$ as the measure such that $\int_{0}^{\rho^{2}} p(x) d v=\int_{-\rho}^{\rho} p\left(x^{2}\right) d \mu$. Then, it is possible to rewrite the subsequences $\left\{p_{2 n}\right\}_{n \geq 0}$ and $\left\{p_{2 n+1}\right\}_{n \geq 0}$ as

$$
p_{2 n}(x)=\ell_{n}\left(x^{2}\right), \quad p_{2 n+1}(x)=x \ell_{n}^{*}\left(x^{2}\right)
$$

where $\left\{\ell_{n}\right\}_{n \geq 0}$ and $\left\{\ell_{n}^{*}\right\}_{n \geq 0}$ are the sequences of orthonormal polynomials with respect to the measures $d v(x)$ and $x d v(\bar{x})$, respectively. We also need the sequence of symmetric polynomials $\left\{p_{n}^{[2 i]}(x)\right\}_{n \geq 0}$. Since $c=0$ they can be rewritten as

$$
p_{2 n}^{[2 i]}(x)=l_{n}^{[i]}\left(x^{2}\right), \quad p_{2 n+1}^{[2 i]}(x)=x\left(l_{n}^{*}\right)^{[i]}\left(x^{2}\right)
$$

where $\left\{l_{n}^{[i]}\right\}_{n \geq 0}$ and $\left\{\left(l_{n}^{*}\right)^{[i]}\right\}_{n \geq 0}$ are the sequences of orthonormal polynomials with respect to the measures $x^{i} d v(x)$ and $x^{i+1} d v(x)$, respectively.

Then, the natural varying discrete Sobolev inner product is

$$
\begin{equation*}
(p, q)_{S}=\int p(x) q(x) d \mu+M_{n} p^{(j)}(0) q^{(j)}(0) \tag{16}
\end{equation*}
$$

where $\left\{M_{n}\right\}_{n \geq 0}$ verifies the condition (2). Again, we denote by $\left\{q_{n}\right\}_{n \geq 0}$ the sequence of orthonormal polynomials with respect to (16).

When $j=2 r$, applying the symmetrization process (see [2, Th. 2]), we get

$$
\begin{equation*}
(p, q)_{S_{1}}=\int_{0}^{\rho^{2}} p(x) q(x) d v+M_{2 n}\left((r+1)_{r}\right)^{2} p^{(r)}(0) q^{(r)}(0) \tag{17}
\end{equation*}
$$

where $(r)_{k}$ denotes the Pochhammer's symbol, i.e. $(r)_{k}=r(r+1) \cdots(r+k-1), k \geq 1,(r)_{0}=1$, as well as

$$
(p, q)_{S_{2}}=\int_{0}^{\rho^{2}} p(x) q(x) x d v
$$

Then,

$$
q_{2 n}(x)=s_{n}\left(x^{2}\right), \quad q_{2 n+1}(x)=x s_{n}^{*}\left(x^{2}\right),
$$

with $\left\{s_{n}\right\}_{n \geq 0}$ and $\left\{s_{n}^{*}\right\}_{n \geq 0}$ being the sequences of orthonormal polynomials with respect to $(\cdot, \cdot)_{S_{1}}$ and $(\cdot, \cdot)_{S_{2}}$, respectively.

When $j=2 r+1$, we get

$$
\begin{gathered}
(p, q)_{S_{3}}=\int_{0}^{\rho^{2}} p(x) q(x) d v \\
(p, q)_{S_{4}}=\int_{0}^{\rho^{2}} p(x) q(x) x d v+M_{2 n+1}\left((r+1)_{r+1}\right)^{2} p^{(r)}(0) q^{(r)}(0)
\end{gathered}
$$

Then,

$$
q_{2 n}(x)=t_{n}\left(x^{2}\right), \quad q_{2 n+1}(x)=x t_{n}^{*}\left(x^{2}\right),
$$

with $\left\{t_{n}\right\}_{n \geq 0}$ and $\left\{t_{n}^{*}\right\}_{n \geq 0}$ being the sequences of orthonormal polynomials with respect to $(\cdot, \cdot)_{S_{3}}$ and $(\cdot, \cdot)_{S_{4}}$, respectively.

In this way, we can apply our results to the orthonormal polynomials with respect to the inner products $(\cdot, \cdot)_{S_{i}}$ with $i=1, \ldots, 4$. Then, we can deduce the corresponding ones for the orthonormal polynomials with respect to (16). Notice that $(\cdot, \cdot)_{S_{i}}$ with $i=2,3$ are not Sobolev inner products but standard ones.

Therefore, we summarize the more relevant results which will be very useful in Section 6, although we omit the proofs. We assume that there is a strictly increasing function $f$, with $2 f(0)+$ $1>0$, satisfying the conditions

$$
\begin{align*}
\left(p_{2 n}^{[4 i]}\right)^{(2 k)}(0) & \simeq C_{k, i}(-1)^{n} n^{f(2 k+2 i)}=C_{k, i}(-1)^{n} n^{g(k+i)}, \quad 0 \leq k \leq n,  \tag{18}\\
\left(p_{2 n+1}^{[4 i]}\right)^{(2 k+1)}(0) & \simeq \widetilde{C}_{k, i}(-1)^{n} n^{f(2 k+2 i+1)}=\widetilde{C}_{k, i}(-1)^{n} n^{g^{*}(k+i)}, \quad 0 \leq k \leq n, \tag{19}
\end{align*}
$$

where $g$ and $g^{*}$ are strictly increasing functions satisfying

$$
\begin{array}{rll}
g(k):=f(2 k) & \text { with } & 2 g(0)+1>0, \\
g^{*}(k):=f(2 k+1) & \text { with } & 2 g^{*}(0)+1>0 .
\end{array}
$$

$C_{k, i}$ and $\widetilde{C}_{k, i}$ are nonzero constants independent of $n$. We also assume that for all $i \geq 0$,

$$
\begin{align*}
\lim _{n \rightarrow \infty}(-1)^{n} \frac{a_{n}^{1 / 2}}{b_{n}^{i}} p_{2 n}^{[2 i]}\left(\frac{z}{b_{n}}\right) & =z^{-(\alpha+i)} J_{\alpha+i}(2 z),  \tag{20}\\
\lim _{n \rightarrow \infty}(-1)^{n} \frac{a_{n}^{1 / 2}}{b_{n}^{i}} p_{2 n+1}^{[2 i]}\left(\frac{z}{b_{n}}\right) & =z^{-(\alpha+i)} J_{\alpha+i+1}(2 z), \tag{21}
\end{align*}
$$

uniformly on compact subsets of $\mathbb{C}$, where

$$
\begin{equation*}
a_{n}^{-1 / 2} \simeq A n^{a}, \quad b_{n} \simeq B n^{b}, \quad A, B, b>0, \quad \alpha>-1, \quad \text { and } \quad 2 a+1=2 b(\alpha+1) \tag{22}
\end{equation*}
$$

This assumption is natural and holds under general conditions (see [15, Lemma 3]).
Thus, according the steps given in the nonsymmetric case (Lemma 2) we can construct the following quantities for $k \geq 0, r \geq 0$ and $L \in[0,+\infty]$,

$$
\begin{align*}
\tau_{k, r, L} & =\frac{L(g(k)-g(r))+g(k)+g(r)+1}{(1+L)(g(k)+g(r)+1)}  \tag{23}\\
\varrho_{k, r, L} & =\frac{L\left(g^{*}(k)-g^{*}(r)\right)+g^{*}(k)+g^{*}(r)+1}{(1+L)\left(g^{*}(k)+g^{*}(r)+1\right)} \tag{24}
\end{align*}
$$

where

$$
\tau_{k, r,+\infty}=\lim _{n \rightarrow+\infty} \tau_{k, r, L}=\frac{g(k)-g(r)}{g(k)+g(r)+1}, \quad \varrho_{k, r,+\infty}=\lim _{n \rightarrow+\infty} \varrho_{k, r, L}=\frac{g^{*}(k)-g^{*}(r)}{g^{*}(k)+g^{*}(r)+1} .
$$

Now, we have all the ingredients to write down the Mehler-Heine asymptotics for this type of varying Sobolev orthonormal polynomials when the measure $\mu$ is symmetric.

Theorem 2 Assuming (18-22), we have the following Mehler-Heine formulae for the sequence of orthonormal polynomials, $\left\{q_{n}\right\}_{n \geq 0}$, with respect to (16):

- If $j=2 r$,

$$
\begin{align*}
\lim _{n \rightarrow \infty}(-1)^{n} a_{n}^{1 / 2} q_{2 n}\left(\frac{z}{b_{n}}\right) & =\sum_{i=0}^{r+1}(-1)^{i} d_{i, 1}(L) z^{-\alpha} J_{\alpha+2 i}(2 z):=\Phi_{\alpha, r, L}(z),  \tag{25}\\
\lim _{n \rightarrow \infty}(-1)^{n} a_{n}^{1 / 2} q_{2 n+1}\left(\frac{z}{b_{n}}\right) & =z^{-\alpha} J_{\alpha+1}(2 z) .
\end{align*}
$$

- If $j=2 r+1$,

$$
\begin{align*}
\lim _{n \rightarrow \infty}(-1)^{n} a_{n}^{1 / 2} q_{2 n}\left(\frac{z}{b_{n}}\right) & =z^{-\alpha} J_{\alpha}(2 z), \\
\lim _{n \rightarrow \infty}(-1)^{n} a_{n}^{1 / 2} q_{2 n+1}\left(\frac{z}{b_{n}}\right) & =\sum_{i=0}^{r+1}(-1)^{i} d_{i, 2}(L) z^{-\alpha} J_{\alpha+2 i+1}(2 z):=\Phi_{\alpha, r, L}^{*}(z) . \tag{26}
\end{align*}
$$

The coefficients $d_{i, 1}(L)$ and $d_{i, 2}(L)$ are given by

$$
\begin{align*}
& d_{i, 1}(L)=(-1)^{i} \frac{\tau_{i, r, L}-(2 i)!\sum_{m=0}^{i-1}(-1)^{i} d_{m, 1}(L) \frac{C_{i-m, m}}{(2(i-m))!C_{i, 0}}}{(2 i)!\frac{C_{0, i}}{C_{i, 0}}}  \tag{27}\\
& d_{i, 2}(L)=(-1)^{i} \frac{\rho_{i, r, L}-(2 i+1)!\sum_{m=0}^{i-1}(-1)^{i} d_{m, 2}(L) \frac{\widetilde{C}_{i-m, m}}{(2(i-m)+1)!\widetilde{C}_{i, 0}}}{(2 i+1)!\frac{\widetilde{C}_{0, i}}{\widetilde{C}_{i, 0}}} \tag{28}
\end{align*}
$$

where $\tau_{i, r, L}$ and $\varrho_{i, r, L}$ are obtained from (23) and (24), respectively. By convention, we assume $\sum_{m=0}^{-1}=0$. All the limits hold uniformly on compact subsets of $\mathbb{C}$.

## 4 Asymptotic behavior of the zeros

Let $\left\{p_{n}\right\}_{n \geq 0}$ be the sequence of orthonormal polynomials with respect to a nonsymmetric measure $\mu$ supported on an interval $I$. We assume that they satisfy the following Mehler-Heine formula

$$
\lim _{n \rightarrow \infty}(-1)^{n} a_{n}^{1 / 2} p_{n}\left(c+\frac{z^{2}}{b_{n}}\right)=z^{-\alpha} J_{\alpha}(2 z), \quad \alpha>-1
$$

uniformly on compact subsets of $\mathbb{C}$, and the parameters involved in the above formula satisfy (14). We denote by $x_{n, 1}<x_{n, 2}<\cdots<x_{n, n}$ the zeros of the polynomial $p_{n}(x)$ in an increasing order. Then, applying Hurwitz's theorem (see [16, Th.1.91.3]) we get

$$
\lim _{n \rightarrow \infty} b_{n}\left(x_{n, k}-c\right)=\frac{j_{\alpha, k}^{2}}{4}, \quad 1 \leq k \leq n
$$

where $j_{\alpha, k}$ is the $k$ th positive zero of $J_{\alpha}$. This is a simple and nice consequence about the zeros which has been obtained using Mehler-Heine asymptotics.

Since in the previous section we have deduced Mehler-Heine formulae for the sequence of varying Sobolev orthonormal polynomials $\left\{q_{n}\right\}_{n \geq 0}$, we can expect to get similar results for the zeros of $q_{n}$. We assume that $c=\inf I \in \mathbb{R}$ but similar results can be obtained when $c=\sup I$ (with $I$ upper bounded). The following result was established for the non-varying case within a more general framework by H. G. Meijer in [12, Th. 4.1] (see also [1, Lemma 2]). Actually, that proof can be written in the same way for the varying case, so we omit it.

Proposition 2 The polynomial $q_{n}(x), n \geq 1$, orthonormal with respect to (1), has $n$ real and simple zeros and at most one of them is located outside $\operatorname{supp}(\mu)$.

We can give more information about the location of the zeros.
Proposition 3 Let $y_{n, 1}<y_{n, 2}<\cdots<y_{n, n}$ be the zeros of the polynomial $q_{n}(x)$ in an increasing order, then for $n$ large enough and $j>0$, we have

- If $L=0$, then all zeros of $q_{n}(x)$ are located inside $\operatorname{supp}(\mu)$.
- If $L=+\infty$, then $y_{n, 1}<c$.
- If $L \in(0,+\infty)$, then $y_{n, 1}<c$ if and only if

$$
L>\frac{f(0)+f(j)+1}{f(j)-f(0)} .
$$

Remark 6 When $j=0$, all the zeros are inside supp $(\mu)$ since, in this case, (1) becomes a varying standard inner product.

Proof. To prove the three cases we will use Lemma 2 with $k=0$, Proposition 2 and the following facts: the leading coefficient $\tilde{\gamma}_{n}$ of $q_{n}$ is positive and $p_{n}$ has all its zeros inside $\operatorname{supp}(\mu)$. Then,

- If $L=0$, then by Lemma $2, \frac{q_{n}(c)}{p_{n}(c)}>0$ for $n$ large enough, so $q_{n}$ could have 2 or 4 or $6, \ldots$ sign changes in $(-\infty, c)$, although taking into account Proposition 2 that is not possible.
- If $L=+\infty$, then $\frac{q_{n}(c)}{p_{n}(c)}<0$ for $n$ large enough, which implies that $q_{n}$ changes sign at least once in $(-\infty, c)$ and, by Proposition 2, it is the only one.
- If $L \in(0,+\infty)$, then $y_{n, 1}<c$ if and only if $\frac{q_{n}(c)}{p_{n}(c)}<0$ for $n$ large enough, and this holds if and only if

$$
L>\frac{f(0)+f(j)+1}{f(j)-f(0)}
$$

Finally, we obtain the asymptotic behavior of the scaled zeros of $q_{n}$ as a consequence of the Mehler-Heine asymptotics for $q_{n}$ given in Theorem 1.

Proposition 4 Let $y_{n, 1}<y_{n, 2}<\cdots<y_{n, n-1}<y_{n, n}$ be the zeros of $q_{n}$, the function $\varphi_{\alpha, j, L}$ is defined in Theorem 1, and $b_{n}$ is given in (14). Then,

1. If $L=0$, then

$$
\lim _{n \rightarrow \infty} b_{n}\left(y_{n, i}-c\right)=\frac{j_{\alpha, i}^{2}}{4}, \quad i \geq 1
$$

where $j_{\alpha, i}$ denotes the ith positive zero of the Bessel function.
2. If $L=+\infty$, then

$$
\lim _{n \rightarrow \infty} y_{n, 1}=c, \quad \lim _{n \rightarrow \infty} b_{n}\left(y_{n, i}-c\right)=\frac{t_{\alpha, i-1}^{2}}{4}, \quad i \geq 2,
$$

where $t_{\alpha, i}$ denotes the $i$ th positive zero of the function $\varphi_{\alpha, j,+\infty}(z)$.
3. If $L \in(0,+\infty)$, then we have two cases:
a) If $L<\frac{f(0)+f(j)+1}{f(j)-f(0)}$, then $y_{n, 1}>c$ for $n$ large enough, and

$$
\lim _{n \rightarrow \infty} b_{n}\left(y_{n, i}-c\right)=\frac{s_{\alpha, i}^{2}}{4}, \quad i \geq 1
$$

b) If $L>\frac{f(0)+f(j)+1}{f(j)-f(0)}$, then $y_{n, 1}<c$ for $n$ large enough, and

$$
\lim _{n \rightarrow \infty} y_{n, 1}=c, \quad \lim _{n \rightarrow \infty} b_{n}\left(y_{n, i}-c\right)=\frac{s_{\alpha, i-1}^{2}}{4}, \quad i \geq 2
$$

In both situations $s_{\alpha, i}$ denotes the ith positive zero of the function $\varphi_{\alpha, j, L}(z)$.
Proof. It follows from Theorem 1, Proposition 3, and Hurwitz's Theorem.

## 5 Other asymptotic results

As we have pointed out previously, one of our goals is to find out the differences between the asymptotic behavior of Sobolev orthogonal polynomials and the standard ones. These differences are clearly shown through Mehler-Heine asymptotics because we are just looking around the point where we have introduced the perturbation of the standard inner product. In this section, we assume $\mu$ is nonsymmetric and we will show that both families of polynomials have the same asymptotic behavior on compact subsets of the complex plane outside $\operatorname{supp}(\mu)$.

In the bounded case, we consider $\mu$ in a general framework, i.e., $\mu$ belongs to the Szegő's class. We denote this fact by $\mu \in \mathcal{S}$. Let's remind (see, for example, [14, p. 121]) that $\mu \in \mathcal{S}$ with $\operatorname{supp}(\mu)=[-1,1]$ if its absolutely continuous part, $w(x)$, satisfies Szegő's condition:

$$
\int_{-1}^{1} \frac{\log w(x)}{\sqrt{1-x^{2}}} d x>-\infty
$$

Theorem 3 We suppose $\mu \in \mathcal{S}$. Let $\left\{p_{n}\right\}_{n \geq 0}$ be the sequence of orthonormal polynomials with respect to $\mu$ satisfying (4) and let $\left\{q_{n}\right\}_{n \geq 0}$ be the sequence of orthonormal polynomials with respect to (1). Then,

$$
\lim _{n \rightarrow \infty} \frac{q_{n}(z)}{p_{n}(z)}=1
$$

uniformly on compact subsets of $\mathbb{C} \backslash[-1,1]$.

Proof: We know that when $\mu \in \mathcal{S}$, then (see for instance [13, p. 36]):

$$
\begin{align*}
\lim _{n \rightarrow \infty} \frac{p_{n-1}(z)}{p_{n}(z)} & =\frac{1}{z+\sqrt{z^{2}-1}}, \quad z \in \mathbb{C} \backslash[-1,1]  \tag{29}\\
\lim _{n \rightarrow \infty} \frac{\gamma_{n-1}}{\gamma_{n}} & =\frac{1}{2} \tag{30}
\end{align*}
$$

From (5), we get

$$
\lim _{n \rightarrow \infty} \frac{q_{n}(z)}{p_{n}(z)}=\lim _{n \rightarrow \infty} \frac{\tilde{\gamma}_{n}}{\gamma_{n}}\left(1-\frac{\frac{M_{n} p_{n}^{(j)}(c)}{1+M_{n} K_{n-1}^{(j, j)}(c, c)} K_{n-1}^{(0, j)}(z, c)}{p_{n}(z)}\right) .
$$

Since by (7) we have $\lim _{n} \tilde{\gamma}_{n} / \gamma_{n}=1$, then it is enough to prove that the expression in brackets tends to 1 when $n \rightarrow \infty$. Using Christoffel-Darboux's formula for orthonormal polynomials

$$
K_{n}(x, y)=\frac{\gamma_{n}}{\gamma_{n+1}} \frac{p_{n+1}(x) p_{n}(y)-p_{n}(x) p_{n+1}(y)}{x-y},
$$

we deduce in a straightforward way

$$
K_{n-1}^{(0, j)}(z, c)=\frac{\gamma_{n-1}}{\gamma_{n}}\left(p_{n}(z)\left(\sum_{i=0}^{j} \frac{p_{n-1}^{(i)}(c) j!}{(z-c)^{j-i+1} i!}\right)-p_{n-1}(z)\left(\sum_{i=0}^{j} \frac{p_{n}^{(i)}(c) j!}{(z-c)^{j-i+1} i!}\right)\right) .
$$

Let $z$ be fixed and belonging to an arbitrary compact subset of $\mathbb{C} \backslash[-1,1]$. Then, applying (4), (6), and (29-30) we obtain

$$
\begin{aligned}
& \lim _{n \rightarrow \infty} \frac{\frac{M_{n} p_{n}^{(j)}(c)}{1+M_{n} K_{n-1}^{(j, j)}(c, c)} K_{n-1}^{(0, j)}(z, c)}{p_{n}(z)} \\
= & \lim _{n \rightarrow \infty} \frac{M_{n} K_{n-1}^{(j, j)}(c, c) \frac{p_{n}^{(j)}(c)}{K_{n-1}^{(j, j)}(c, c)}}{1+M_{n} K_{n-1}^{(j, j)}(c, c)} \\
\times & \frac{\gamma_{n-1}}{\gamma_{n}}\left(\left(\sum_{i=0}^{j} \frac{p_{n-1}^{(i)}(c) j!}{(z-c)^{j-i+1} i!}\right)-\frac{p_{n-1}(z)}{p_{n}(z)}\left(\sum_{i=0}^{j} \frac{p_{n}^{(i)}(c) j!}{(z-c)^{j-i+1} i!}\right)\right) \\
= & \lim _{n \rightarrow \infty} \frac{M_{n} K_{n-1}^{(j, j)}(c, c) \frac{n^{2 f(j)+1}}{K_{n-1}^{(j, j}(c, c)} n^{-(2 f(j)+1)} \frac{p_{n}^{(j)}(c)}{n^{f(j)}} n^{f(j)}}{1+M_{n} K_{n-1}^{(j, j)}(c, c)} \\
\times & \frac{\gamma_{n-1}}{\gamma_{n}} n^{f(j)+1}\left(\left(\sum_{i=0}^{j} \frac{p_{n-1}^{(i)}(c) j!}{n^{f(j)+1}(z-c)^{j-i+1} i!}\right)-\frac{p_{n-1}(z)}{p_{n}(z)}\left(\sum_{i=0}^{j} \frac{p_{n}^{(i)}(c) j!}{n^{f(j)+1}(z-c)^{j-i+1} i!}\right)\right) \\
= & 0,
\end{aligned}
$$

so, the result is proved.
From now on, we assume that $\mu$ is unbounded. An analogous result to Theorem 3 can be established through Plancherel-Rotach asymptotics. Since $\left\{p_{n}\right\}_{n \geq 0}$ is a sequence of standard orthogonal polynomials, a three-term recurrence relation holds

$$
x p_{n}(x)=\lambda_{n+1} p_{n+1}(x)+\eta_{n} p_{n}(x)+\lambda_{n} p_{n-1}(x), \quad n=0,1,2, \ldots
$$

with $p_{-1}(x)=0$ and $p_{0}(x)=1$. The coefficients of this relation are given by

$$
\begin{cases}\lambda_{n}=\frac{\gamma_{n-1}}{\gamma_{n}}>0, & n=1,2, \ldots \\ \eta_{n}=\int_{-\infty}^{\infty} x p_{n}^{2}(x) d \mu(x) \in \mathbb{R}, & n=0,1,2 \ldots\end{cases}
$$

Following [17], we assume there exists a nondecreasing function $\varphi: \mathbb{R}^{+} \rightarrow \mathbb{R}^{+}$such that for every $t \in \mathbb{R}$

$$
\begin{equation*}
\lim _{x \rightarrow \infty} \frac{\varphi(x+t)}{\varphi(x)}=1 \tag{31}
\end{equation*}
$$

and suppose that there exist constants $\lambda$ and $\eta$ satisfying

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \frac{\lambda_{n}}{\varphi(n)}=\lambda \geq 0, \quad \lim _{n \rightarrow \infty} \frac{\eta_{n}}{\varphi(n)}=\eta \in \mathbb{R} \tag{32}
\end{equation*}
$$

Under these assumptions, we get the following ratio asymptotics ([17, Th. 4.10]),

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \frac{p_{n-1}(\varphi(n) z)}{p_{n}(\varphi(n) z)}=\frac{2 \lambda}{z-\eta+\sqrt{(z-\eta)^{2}-4 \lambda^{2}}} \tag{33}
\end{equation*}
$$

uniformly on compact subsets of $\mathbb{C} \backslash[A, B]$ where $[A, B]$ is the smallest interval containing $\{0\}$ and $[\eta-2 \lambda, \eta+2 \lambda]$.

Theorem 4 Let $\left\{p_{n}\right\}_{n \geq 0}$ be the sequence of orthonormal polynomials with respect to $\mu$ satisfying (4) and let $\left\{q_{n}\right\}_{n \geq 0}$ be the sequence of orthonormal polynomials with respect to (1). Then,

$$
\lim _{n \rightarrow \infty} \frac{q_{n}(\varphi(n) z)}{p_{n}(\varphi(n) z)}=1,
$$

uniformly on compact subsets of $\mathbb{C} \backslash[A, B]$ where $[A, B]$ is the smallest interval containing $\{0\}$ and $[\eta-2 \lambda, \eta+2 \lambda]$.

Proof: Scaling the relation (5) we have

$$
\lim _{n \rightarrow \infty} \frac{q_{n}(\varphi(n) z)}{p_{n}(\varphi(n) z)}=\lim _{n \rightarrow \infty} \frac{\tilde{\gamma}_{n}}{\gamma_{n}}\left(1-\frac{\frac{M_{n} p_{n}^{(j)}(c)}{1+M_{n} K_{n-1}^{(j, j)}(c, c)} K_{n-1}^{(0, j)}(\varphi(n) z, c)}{p_{n}(\varphi(n) z)}\right) .
$$

As in Theorem 3, it is enough to prove that the expression in brackets tends to 1 when $n \rightarrow \infty$. Thus, for a fixed $z \in \Omega$ being $\Omega$ an arbitrary compact subset of $\mathbb{C} \backslash[A, B]$ we deduce

$$
\begin{aligned}
& \lim _{n \rightarrow \infty} \frac{\frac{M_{n} p_{n}^{(j)}(c)}{1+M_{n} K_{n-1}^{(j, j)}(c, c)} K_{n-1}^{(0, j)}(\varphi(n) z, c)}{p_{n}(\varphi(n) z)} \\
= & \lim _{n \rightarrow \infty} \frac{M_{n} K_{n-1}^{(j, j)}(c, c) \frac{n^{2 f(j)+1}}{K_{n-1}^{(j, j}(c, c)} n^{-(2 f(j)+1)} \frac{p_{n}^{(j)}(c)}{n^{f(j)}} n^{f(j)}}{1+M_{n} K_{n-1}^{(j, j)}(c, c)} \\
\times & \lambda_{n} n^{f(j)+1}\left(\left(\sum_{i=0}^{j} \frac{p_{n-1}^{(i)}(c) j!}{n^{f(j)+1}(\varphi(n) z-c)^{j-i+1} i!}\right)\right. \\
- & \left.\frac{p_{n-1}(\varphi(n) z)}{p_{n}(\varphi(n) z)}\left(\sum_{i=0}^{j} \frac{p_{n}^{(i)}(c) j!}{n^{f(j)+1}(\varphi(n) z-c)^{j-i+1} i!}\right)\right)=0,
\end{aligned}
$$

uniformly on $\Omega$, where we have applied (4), (6), (32), (33), and $\lim _{n \rightarrow \infty} \frac{\lambda_{n}}{(\varphi(n) z-c)^{j-i+1}}=M(z)$ with $|M(z)|<+\infty$.

Corollary 1 Assuming that (31) and (32) hold, then

$$
\lim _{n \rightarrow \infty} \frac{q_{n-1}(\varphi(n) z)}{q_{n}(\varphi(n) z)}=\frac{2 \lambda}{z-\eta+\sqrt{(z-\eta)^{2}-4 \lambda^{2}}},
$$

uniformly on compact subsets of $\mathbb{C} \backslash[A, B]$, where $[A, B]$ is the smallest interval containing $\{0\}$ and $[\eta-2 \lambda, \eta+2 \lambda]$.

Proof: From Theorem 4 and (33), we have

$$
\begin{aligned}
\lim _{n \rightarrow \infty} \frac{q_{n-1}(\varphi(n) z)}{q_{n}(\varphi(n) z)} & =\lim _{n \rightarrow \infty} \frac{q_{n-1}(\varphi(n) z)}{p_{n-1}(\varphi(n) z)} \frac{p_{n-1}(\varphi(n) z)}{p_{n}(\varphi(n) z)} \frac{p_{n}(\varphi(n) z)}{q_{n}(\varphi(n) z)} \\
& =\frac{2 \lambda}{z-\eta+\sqrt{(z-\eta)^{2}-4 \lambda^{2}}} .
\end{aligned}
$$

## 6 A numerical experiment

Theorem 1 has already been illustrated for concrete choices of the measure $\mu$ in previous papers. For example, in [6] where $\mu$ is the Jacobi weight and in [7] for Laguerre case. In those papers
other methods were used to do the computations. Thus, we show a numerical experiment for the symmetric case considering the Hermite weight $e^{-x^{2}}$ on the real axis and $c=0$. Then, (1) becomes

$$
\begin{equation*}
(f, g)_{S}=\int_{-\infty}^{+\infty} f(x) g(x) e^{-x^{2}} d x+M_{n} f^{(j)}(0) g^{(j)}(0), \quad j \geq 0 \tag{34}
\end{equation*}
$$

where $\left\{M_{n}\right\}_{n \geq 0}$ is a sequence satisfying (2). Applying the symmetrization process described in Section 3 we get the following inner products:

If $j=2 r$

$$
\begin{gather*}
(p, q)_{S_{1}}=\int_{0}^{+\infty} p(x) q(x) x^{-1 / 2} e^{-x} d x+M_{2 n}\left((r+1)_{r}\right)^{2} p^{(r)}(0) q^{(r)}(0)  \tag{35}\\
(p, q)_{S_{2}}=\int_{0}^{+\infty} p(x) q(x) x^{1 / 2} e^{-x} d x \tag{36}
\end{gather*}
$$

and, if $j=2 r+1$

$$
\begin{gathered}
(p, q)_{S_{3}}=\int_{0}^{+\infty} p(x) q(x) x^{-1 / 2} e^{-x} d x \\
(p, q)_{S_{4}}=\int_{0}^{+\infty} p(x) q(x) x^{1 / 2} e^{-x}+M_{2 n+1}\left((r+1)_{r+1}\right)^{2} p^{(r)}(0) q^{(r)}(0)
\end{gathered}
$$

We are going to compare numerically the zeros of the orthonormal polynomials $q_{n}$ with respect to (34) with the zeros of the corresponding limit functions given by Mehler-Heine formulae in Theorem 2.

Observe that, for even $j$ and using the symmetrization process, the zeros of the orthogonal polynomials $q_{2 n}$ and $q_{2 n+1}$ are the square roots of the zeros of those polynomials orthogonal with respect to (35) and (36), respectively. Since (36) is a classical inner product, we only pay attention to the zeros of $q_{2 n}$ in the numerical experiment. We have the analogue setting when $j$ is odd, then we focus our attention on the zeros of the polynomials $q_{2 n+1}$.

Next, we describe the steps to construct the numerical experiment.
Step 1. Determining $a_{n}$ and $b_{n}$ in (20-21). In this case, Mehler-Heine asymptotics for Hermite orthonormal polynomials, $h_{n}$, is well known. Indeed,

$$
\begin{aligned}
\lim _{n \rightarrow \infty}(-1)^{n} n^{1 / 4} h_{2 n}(z / \sqrt{n}) & =z^{1 / 2} J_{-1 / 2}(2 z)=\frac{\cos (2 z)}{\sqrt{\pi}} \\
\lim _{n \rightarrow \infty}(-1)^{n} n^{1 / 4} h_{2 n+1}(z / \sqrt{n}) & =z^{1 / 2} J_{1 / 2}(2 z)=\frac{\sin (2 z)}{\sqrt{\pi}}
\end{aligned}
$$

uniformly on compact subsets of the complex plane. Therefore, $a_{n}=n^{1 / 2}$ and $b_{n}=\sqrt{n}$. Then, all the hypothesis in [15, Lemma 3] hold. Thus, Mehler-Heine formulae (20-21) also
hold for the polynomials $p_{n}^{[2 i]}$, which in this case are the generalized Hermite orthonormal polynomials, where the values of the constants in (22) are

$$
A=1, \quad a=-1 / 4, \quad B=1, \quad b=1 / 2, \quad \alpha=-1 / 2 .
$$

In general, for Freud weights the quantities $a_{n}$ are related to Maskhar-Rakhmanov-Saff numbers and $b_{n}=c \frac{n}{a_{n}}$, where $c$ is a known constant (for example, see [4, Ch. 10]). One should be careful with the parameters in this formulae because texts and papers work with slightly different standardizations but they are relevant in the numerical experiments. For example, it usual to consider $b_{n}=2 \sqrt{n}$ in the above formulae for Hermite polynomials.

Step 2. Computing the zeros of $q_{n}\left(x / b_{n}\right)$. We calculate the scaled zeros of these polynomials using (5).

Step 3. Computing the zeros of limit functions $\Phi_{\alpha, r, L}$ and $\Phi_{\alpha, r, L}^{*}$ in (25) and (26), respectively. To do this we have to calculate in a recursive way the coefficients $d_{i, 1}(L)$ and $d_{i, 2}(L)$ given by (27) and (28), respectively. Obviously, this computation depends on the value of $L$. Then, we have six cases. Furthermore, the computation of (27) and (28) involves the computation of $\tau_{k, r, L}$ and $\varrho_{k, r, L}$, respectively, and of the constants $C_{s, t}$ and $\widetilde{C}_{s, t}$ which in turn depend on the explicit values of the functions $g$ and $g^{*}$. Summarizing,
3. a) Computing $g, g^{*}$ and the constants $C_{s, t}$ and $\widetilde{C}_{s, t}$. In [3, Corol. 2] the asymptotic behavior of the values $\left(\hat{p}_{2 n}^{[m]}\right)^{(2 k)}(0)$ and $\left(\hat{p}_{2 n+1}^{[m]}\right)^{(2 k+1)}(0)$ was obtained, where $\hat{p}_{n}^{[m]}$ are the orthonormal polynomials with respect to the weight function $x^{2 m} \exp \left(-2|x|^{\beta}\right), \beta>1$, and $m$ is a nonnegative integer. In our case, $\beta=2$ and the weight function is $x^{2 m} \exp \left(-x^{2}\right)$ whose corresponding orthonormal polynomials are the generalized Hermite. Thus, applying [3, Corol. 2] we have,

$$
\begin{equation*}
\left(p_{2 n}^{[4 i]}\right)^{(2 k)}(0)=\frac{1}{2^{i / 2+k+1 / 4}}\left(\hat{p}_{2 n}^{[2 i]}\right)^{(2 k)}(0) \simeq \frac{(-1)^{k}(2 k)!}{k!\Gamma(k+2 i+1 / 2)}(-1)^{n} n^{i+k-1 / 4} \tag{37}
\end{equation*}
$$

In particular, for $i=0$ we get

$$
\begin{equation*}
p_{2 n}^{(2 k)}(0) \simeq \frac{(-1)^{k}(2 k)!}{k!\Gamma(k+1 / 2)}(-1)^{n} n^{k-1 / 4} . \tag{38}
\end{equation*}
$$

Notice that in that paper they use other standardizations, concretely $b_{n}=2 \sqrt{n}$.
One can observe that in this case the polynomials $p_{2 n}$ are the classical Hermite polynomials and we know their explicit value, i.e., $p_{2 n}(0)=(-1)^{n} \Gamma(2 n+1) /\left(\pi^{1 / 4} 2^{n} \Gamma(n+\right.$ 1) $\Gamma^{1 / 2}(2 n+1)$ ), so after some easy computations we obtain (38). However, (37) is necessary to calculate the constants $C_{s, t}$ appearing in the computation of $d_{i, 1}(L)$ via (27). Furthermore, if we work with another Freud weight we do not know the explicit values
of the corresponding orthonormal polynomials at the origin. From (18) and (38), we deduce that $g(k)=f(2 k)=k-1 / 4$. From (37), we get

$$
C_{k, i}=\frac{(-1)^{k}(2 k)!}{k!\Gamma(k+2 i+1 / 2)} .
$$

In the same way, we obtain the nondecreasing function $g^{*}$ and the constants $\widetilde{C}_{k, i}$. More precisely,

$$
g^{*}(k)=k+1 / 4, \quad \widetilde{C}_{k, i}=\frac{(-1)^{k}(2 k+1)!}{k!\Gamma(k+2 i+3 / 2)} .
$$

3. b) Computing $\tau_{k, r, L}$ and $\varrho_{k, r, L}$ via formulae (23) and (24), respectively. We need $g$ and $g^{*}$ calculated in Step 3 a).
4. c) Computing recursively $d_{i, 1}(L)$ and $d_{i, 2}(L)$ via formulae (27) and (28), respectively. We need $\tau_{k, r, L}$ and $\varrho_{k, r, L}$ calculated in Step 3 b ) and the constants $C_{s, t}$ obtained in Step 3 a).
5. d) Finally, we construct the limit functions $\Phi_{\alpha, r, L}$ and $\Phi_{\alpha, r, L}^{*}$ via formulae (25) and (26). Then, we compute their zeros.

To illustrate this algorithm we choose: $r=4$ and the sequence $\left\{M_{n}\right\}_{n \geq 0}$ is taken as a potential sequence, i.e., $M_{n}=M n^{\delta}$ with $M>0$ and $\delta \in \mathbb{R}$.

Even case. Since $j=2 r$, then $j=8$. Taking into account the symmetrization process, after some computations, we get

$$
\begin{equation*}
L=\lim _{n \rightarrow \infty} M_{2 n} K_{2 n-1}^{(8,8)}(0,0)=\lim _{n \rightarrow \infty}(5)_{4} M_{2 n} \tilde{K}_{n-1}^{(4,4)}(0,0), \tag{39}
\end{equation*}
$$

where $\tilde{K}_{n-1}(x, y)$ are the kernel polynomials related to the symmetrized measure $\nu$ given in (35) as a particular case of (17). Then, applying adequately Lemma 2 we have

$$
\begin{equation*}
M_{2 n} K_{2 n-1}^{(8,8)}(0,0) \simeq M \frac{2^{\delta} C_{4,0}^{2}}{2 g(4)+1} n^{2 g(4)+1+\delta}=M \frac{2^{\delta} C_{4,0}^{2}}{2 g(4)+1} n^{\delta+17 / 2} \tag{40}
\end{equation*}
$$

Thus, we have three possible choices for $\delta$.

- If $\delta<-17 / 2$, then $\lim _{n \rightarrow \infty} M_{2 n} K_{2 n-1}^{(8,8)}(0,0)=0$. Thus, $L=0$ and for this example we have taken $\delta=-10$.
- If $\delta>-17 / 2$, then $\lim _{n \rightarrow \infty} M_{2 n} K_{2 n-1}^{(8,8)}(0,0)=+\infty$. Thus, $L=+\infty$ and for this example we have taken $\delta=-2$. Notice that by Proposition 3 the orthonormal polynomials with respect to (35) have always a zero in $(-\infty, 0)$. Therefore, the polynomial $q_{2 n}$ has two complex conjugate zeros whose real parts are zero.
- If $\delta=-17 / 2$, then $L=\lim _{n \rightarrow \infty} M_{2 n} K_{2 n-1}^{(8,8)}(0,0)=M \frac{2^{\delta} C_{4,0}^{2}}{2 g(4)+1}$. Thus,

$$
L=M \frac{2^{\delta} C_{4,0}^{2}}{2 g(4)+1}=M \frac{256 \sqrt{2}}{17 \pi} .
$$

This is the only case where $M$ plays a relevant role. Taking into account (39) and (40) and applying Proposition 3 there are two complex zeros whose real parts are zero if and only if

$$
M>\frac{(2 g(4)+1)(g(0)+g(4)+1)}{2^{\delta} C_{4,0}^{2}(g(4)-g(0))}=\frac{153 \sqrt{2} \pi}{4096} \approx 0.165957 .
$$

Thus, we show two examples in Table 1 with $M=1$ and $M=1 / 10$ to illustrate both cases.

In Table 1 we denote $y_{2 n, 1}$ the first positive zero of $q_{2 n}$ and so on. Notice than there are at least $2 n-2$ real zeros.

Table 1: Even case. $\mathbf{j = 8}$

|  | $n$ | $b_{n} y_{2 n, 1}$ | $b_{n} y_{2 n, 2}$ | $b_{n} y_{2 n, 3}$ | Complex zero |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $L=0$ | $n=100$ | 0.78412 | 2.35253 | 3.92133 | - |
|  | $n=200$ | 0.78479 | 2.35444 | 3.92422 | - |
|  | $n=300$ | 0.78501 | 2.35505 | 3.92517 | - |
|  | $\Phi_{\frac{-1}{2}, 4,0}(z)$ | 0.78540 | 2.35619 | 3.92699 | - |
| $\begin{aligned} L & =\frac{128 \sqrt{2}}{85 \pi} \\ M & =\frac{1}{10} \end{aligned}$ | $n=100$ | 0.64709 | 2.06989 | 3.63403 | - |
|  | $n=200$ | 0.62893 | 2.04445 | 3.61736 | - |
|  | $n=300$ | 0.62222 | 2.03553 | 3.61174 | - |
|  | $\Phi_{\frac{-1}{2}, 4, \frac{128 \sqrt{2}}{58 \pi}}(z)$ | 0.60764 | 2.01703 | 3.60044 | - |
| $\begin{aligned} L & =\frac{256 \sqrt{2}}{17 \pi} \\ M & =1 \end{aligned}$ | $n=100$ | 1.04652 | 3.00536 | 4.81142 | $3.01401 i$ |
|  | $n=200$ | 1.04553 | 3.00577 | 4.81360 | $3.06416 i$ |
|  | $n=300$ | 1.04531 | 3.00607 | 4.81451 | $3.07959 i$ |
|  | $\Phi_{\frac{-1}{2}, 4, \frac{256 \sqrt{2}}{1 / T}}(z)$ | 1.04501 | 3.00689 | 4.81658 | $3.10864 i$ |
| $L=+\infty$ | $n=100$ | 0.98736 | 2.90366 | 4.71791 | $4.31744 i$ |
|  | $n=200$ | 0.99062 | 2.91112 | 4.72651 | $4.27901 i$ |
|  | $n=300$ | 0.99172 | 2.91362 | 4.72944 | $4.26646 i$ |
|  | $\Phi_{\frac{-1}{2}, 4,+\infty}(z)$ | 0.99391 | 2.91865 | 4.73538 | $4.24173 i$ |

Odd case. Since $j=2 r+1$, then $j=9$. We can proceed in the same way that as for the even case. To avoid repeating all the details, we only highlight the most relevant differences with the even case. Obviously, 0 is a zero of the polynomial $q_{2 n+1}$.
In this case the critical value is $\delta=-19 / 2$. In this setting,

$$
L=M \frac{2^{\delta} \widetilde{C}_{4,0}^{2}}{2 g^{*}(4)+1}=M \frac{519 \sqrt{2}}{19 \pi},
$$

and there are two complex zeros whose real parts are zero if and only if

$$
M>\frac{\left(2 g^{*}(4)+1\right)\left(g^{*}(0)+g^{*}(4)+1\right)}{2^{\delta} \widetilde{C}_{4,0}^{2}\left(g^{*}(4)-g^{*}(0)\right)}=\frac{209 \sqrt{2} \pi}{8192} \approx 0.11335 .
$$

Thus, we show two examples in Table 2 with $M=1$ and $M=1 / 20$ to illustrate both cases. The cases $L=0$ and $L=+\infty$ are also pointed out. As for the even case in Table 1 we denote $y_{2 n+1,1}$ the first positive zero of $q_{2 n+1}$ and so on.

Table 2: Odd case. $\mathrm{j}=9$

|  | $n$ | $b_{n} y_{2 n+1,1}$ | $b_{n} y_{2 n+1,2}$ | $b_{n} y_{2 n+1,3}$ | Complex zero |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $L=0$ | $n=100$ | 1.56407 | 3.12850 | 4.69348 | - |
|  | $n=200$ | 1.56751 | 3.13516 | 4.70300 | - |
|  | $n=300$ | 1.56864 | 3.13736 | 4.70617 | - |
|  | $\Phi_{\frac{-1}{2}, 4,0}^{*}(z)$ | 1.57080 | 3.14159 | 4.71239 | - |
| $\begin{aligned} L & =\frac{128 \sqrt{2}}{95 \pi} \\ M & =\frac{1}{20} \end{aligned}$ | $n=100$ | 1.38432 | 2.87997 | 4.45036 | - |
|  | $n=200$ | 1.36013 | 2.85813 | 4.43808 | - |
|  | $n=300$ | 1.35109 | 2.85035 | 4.43385 | - |
|  | $\Phi_{\frac{-1}{2}, 4, \frac{128 \sqrt{2}}{95 \pi}}^{*}(z)$ | 1.33136 | 2.83407 | 4.42525 | - |
| $\begin{aligned} L & =\frac{512 \sqrt{2}}{19 \pi} \\ M & =1 \end{aligned}$ | $n=100$ | 1.95555 | 3.80915 | 5.56992 | $3.93083 i$ |
|  | $n=200$ | 1.95994 | 3.81767 | 5.58152 | $3.96696 i$ |
|  | $n=300$ | 1.96151 | 3.82067 | 5.58560 | $3.97757 i$ |
|  | $\Phi_{\frac{-1}{2}, 4, \frac{512 \sqrt{2}}{19 \pi}}^{*}(z)$ | 1.96481 | 3.82690 | 5.59407 | 3.99681i |
| $L=+\infty$ | $n=100$ | 1.91046 | 3.75234 | 5.51669 | $4.82743 i$ |
|  | $n=200$ | 1.91861 | 3.76570 | 5.53290 | $4.78443 i$ |
|  | $n=300$ | 1.92134 | 3.77020 | 5.53840 | $4.77038 i$ |
|  | $\Phi^{-\frac{1}{2}, 4,+\infty}$ (z) | 1.92685 | 3.77929 | 5.54960 | $4.74269 i$ |

Finally, we include some plots of the limit functions appearing in Table 1 and Table 2. In Figure 3 we can see how the limit function $\Phi_{\frac{-1}{2}, 4, L}(z)$ changes depending on the value of $M$ and it is nice to observe when the complex zeros appear. All the computations have been done with the software Mathematica ${ }^{\circledR} 11$.


Figure 1: Graphics of $\Phi_{\frac{-1}{2}, 4, L}(z)$ for different values of $L$.


Figure 2: Graphics of $\Phi_{\frac{-1}{2}, 4, L}^{*}(z)$ for different values of $L$.


Figure 3: Graphics of $\Phi_{\frac{-1}{2}, 4, L}(z)$ for different values of $M$.

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