Extension of isometries and the Mazur-Ulam property

María de Nazaret Cueto Avellaneda
Supervised by Antonio M. Peralta Pereira

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Extensión de isometrías y la propiedad de Mazur–Ulam

María de Nazaret Cueto Avellaneda
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Suena: *Nuvole Bianche* - Ludovico Einaudi.

Solo Dios basta
Santa Teresa de Ávila

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Prefacio

El trabajo que refleja la presente memoria ha sido realizado en el marco del Programa interuniversitario de Doctorado en Matemáticas[1] - 8910 ofrecido por la Universidad de Almería[2] y que imparte de manera conjunta con las Universidades de Granada, Málaga, Jaén y Cádiz. El citado programa, implantado en el curso 2013/14, fue evaluado favorablemente el 1 de Julio de 2011 obteniendo la Mención hacia la Excelencia con referencia 2011-00209. El convenio de colaboración[3] suscrito entre dichas universidades permitió que en el curso académico 2017/18 la doctoranda iniciase su formación investigadora de tercer ciclo bajo la dirección y supervisión del Profesor Antonio M. Peralta Pereira, miembro del grupo de investigación FQM-375 *Análisis Funcional: C^*-álgebras y Teoría de Operadores* de la Universidad de Granada.

La línea de investigación donde se sitúa este proyecto es la denominada *Análisis Funcional. Espacios y Álgebras de Banach. Aplicaciones.* Dentro de la misma, el Proyecto de Investigación presentado al inicio del doctorado al órgano responsable, a saber, la Escuela Internacional de Doctorado de la Universidad de Almería[4], y validado positivamente durante los consecutivos cursos académicos por la comisión académica del programa de doctorado, ha supuesto una guía de actuación metodológica y temporal que culmina con la presente tesis doctoral.

La Normativa de Estudios Oficiales de Doctorado en la Universidad de Almería[5] regula los estudios de doctorado que conducen a la obtención del título de Doctor por la Universidad de Almería, de acuerdo con la ordenación de los estudios universitarios oficiales establecida por el Real Decreto 99/2011 de 28 de enero. En su Artículo 24 se contempla la posibilidad de presentar la tesis doctoral en la modalidad de *Tesis por compendio de publicaciones.* Los requisitos mínimos para esta modalidad incluyen la presentación de al menos 3 contribuciones, dos de las cuales pertenezcan a la categoría A de la escala de valoración de los resultados de

[1] Página web del Doctorado en Matemáticas de la UAL
[2] Página web de la Universidad de Almería
[3] Convenio de colaboración
[4] Página web de la EIDUAL
[5] Normativa de Estudios Oficiales de Doctorado en la UAL

investigación contenida en el Plan Propio de Investigación y Transferencia de la Universidad de Almería aprobado en el corespondiente año, y una tercera, a la categoría B del mismo documento.

De acuerdo con la normativa expuesta, y acogiéndonos a la modalidad comentada, la presente tesis supone la colección de 5 artículos de investigación:

1. *The Mazur–Ulam property for commutative von Neumann algebras*,
   M. Cueto-Avellaneda, A.M. Peralta,
   Linear and Multilinear Algebra **68**, no.2, 337-362 (2020).

   Este artículo fue aceptado por la citada revista el 21 de Julio de 2018 y posteriormente publicado electrónicamente el 3 de Septiembre de 2018. La revista se encuentra en el cuartil **Q2** del *Journal Citation Reports*[6] (JCR) correspondiente al año 2018 en la categoría *Mathematics* y, por tanto, el artículo pertenece a la **categoría B** del Plan Propio de Investigación y Transferencia de la Universidad de Almería del año 2020[7]. Una prepublicación o preimpresión del trabajo se encuentra disponible en la base de datos arXiv.

2. *On the extension of isometries between the unit spheres of a JBW*-triple and a Banach space*,
   J. Becerra Guerrero, M. Cueto-Avellaneda, F.J. Fernández-Polo, A.M. Peralta,
   to appear in *Journal of the Institute of Mathematics of Jussieu.*

   Este artículo fue aceptado por la citada revista el 2 de Marzo de 2019 y posteriormente publicado electrónicamente el 15 de Abril de 2019. La revista se encuentra en el primer cuartil **Q1** del *Journal Citation Reports* (JCR) correspondiente al año 2018 en la categoría *Mathematics* y, por tanto, el artículo pertenece a la **categoría A** del Plan Propio de Investigación y Transferencia de la Universidad de Almería del año 2020. Una prepublicación o preimpresión del trabajo se encuentra disponible en la base de datos arXiv.

3. *On the Mazur–Ulam property for the space of Hilbert-space-valued continuous functions*,
   M. Cueto-Avellaneda, A.M. Peralta,
   Journal of Mathematical Analysis and Appllications **479** 875-902 (2019).

   Este artículo fue publicado electrónicamente el 25 de Junio de 2019. La revista citada se encuentra en **Q1** del *Journal Citation Reports* (JCR) correspondiente al año 2018 en la categoría *Mathematics* y, por tanto, el artículo pertenece a la **categoría A** del Plan Propio de Investigación

[6] JCR 2018
[7] PPIT UAL 2020

y Transferencia de la Universidad de Almería del año 2020. Una prepublicación o preimpresión del trabajo se encuentra disponible en la base de datos arXiv.

4. *Metric characterisation of unitaries in JB*-algebras*,
M. Cueto-Avellaneda, A.M. Peralta,
to appear in *Mediterranean Journal of Mathematics*.

   Este artículo ha sido aceptado para su publicación en la citada revista con fecha 3 de Junio de 2020, y se encuentra en proceso de publicación electrónica, con DOI: 10.1007/s00009-020-01556-w. Mediterranean Journal of Mathematics se encuentra en el cuartil **Q1** del *Journal Citation Reports* (JCR) correspondiente al año 2018 en la categoría *Mathematics*. Por tanto, el artículo pertenece a la **categoría A** del Plan Propio de Investigación y Transferencia de la Universidad de Almería del año 2020. Una prepublicación o preimpresión del trabajo se encuentra disponible en la base de datos arXiv.

5. *Can one identify two unital JB*-algebras by the metric spaces determined by their sets of unitaries?*
M. Cueto-Avellaneda, A.M. Peralta,
preprint 2020. Sometido para su publicación.

   Una prepublicación o preimpresión del trabajo se encuentra disponible en la base de datos arXiv.

La estructura de esta memoria responde a la naturaleza de los resultados que en ella se presentan. No obstante, estos resultados se encuentran organizados esencialmente en orden cronológico, de acuerdo al desarrollo de los artículos que la avalan. Tres grandes capítulos articulan este documento, precedidos por una introducción, y cuya conclusión da lugar a un cuarto apartado de problemas abiertos derivados de la tesis. Además, se ha incluído un apéndice que pone en valor ciertos resultados con una importancia rotunda, pero cuya intervención pasa desapercibida. Un segundo apéndice muestra algunas de las principales actividades en las que la doctoranda ha participado como parte de su formación durante el programa de doctorado. La bibliografía en la que se apoya nuestro trabajo se encuentra incluída al final de la memoria, y da paso a la exposición de los artículos citados anteriormente. Así, a lo largo de este documento se reseñarán aquellos enunciados más relevantes. Tanto las demostraciones como el resto de detalles, podrán consultarse en los trabajos adjuntos.

A continuación, analizaremos la estructura de la tesis mediante un repaso de los resultados que recoge. Dado que el objetivo de este primer apartado es el de motivar y justificar nuestro trabajo, así como resumir el contenido de la tesis, este recorrido se hará de manera ágil y breve, procurando no entrar en tecnicismos que distraigan al lector del espiritu común que se desprende del compendio de artículos presentados. Con la intención de optar a la Mención Internacional, el documento está redactado en inglés en su totalidad, a excepción del presente prefacio o resumen.

La Introducción que da inicio a este proyecto ofrece un contexto histórico en el que situar los resultados obtenidos. Así mismo, justifica y motiva las cuestiones tratadas. Como el título de esta tesis indica, los problemas de extensión de isometrías constituyen el núcleo e hilo conductor de esta memoria. El Capítulo 1 recoge las nociones y resultados básicos de aquellas estructuras que conformarán nuestro ambiente de trabajo, esto es, C*-álgebras, JB*-álgebras y JB*-triples, así como sus análogos sobre el cuerpo de los números reales. La intención de este capítulo no es la de profundizar plenamente en cada estructura sino la de introducir cada uno de estos objetos matemáticos de una manera constructiva. Toda C*-álgebra puede verse como una JB*-álgebra cuando consideramos la involución y la norma originales, pero definimos un nuevo producto llamado producto de Jordan. Así mismo, toda JB*-álgebra, y por tanto toda C*-álgebra, tiene una estructura más general de JB*-triple. La perspectiva que dan los JB*-triples resulta ser extremadamente ventajosa para tratar cuestiones en álgebras de Banach o álgebras de Jordan. Este es el espíritu con el que los artículos [33, 12, 34, 35] y [36] han sido concebidos. Por tanto, tendremos especial detalle en resaltar aquellos argumentos que reflejan esta filosofía, y de los que se desprende la posibilidad de poder trabajar sobre un mismo objeto desde el punto de vista de diferentes estructuras, beneficiándonos, lo máximo posible, de cada teoría involucrada.

La estrategia empleada en nuestros artículos para abordar los problemas de extensión isométrica dieron lugar al estudio de cuestiones paralelas. Todas ellas se recogen en el Capítulo 2. Por una parte, los artículos [12] y [34] culminan el estudio de la estructura facial en JB*-triples, iniciado en 1988. Los resultados obtenidos a este respecto se encuentran recogidos en la sección 2.1. El carácter geométrico inherente a los problemas de extensión de isometrías se ve igualmente reflejado en el artículo [35], cuyo objetivo es establecer una caracterización de aquellos puntos extremos de la bola cerrada unidad de una JB*-álgebra unital que son elementos unitarios. La sección 2.2 agrupa las novedades obtenidas en este sentido.

El Capítulo 3, compuesto por tres secciones, trata de manera directa el objetivo principal de esta tesis. La sección 3.1 está dedicada a la propiedad fuerte de Mankiewicz, utilizada como método para abordar la extensión de isometrías. En la sección 3.2, centraremos nuestra atención en determinar cuándo un espacio de Banach satisface la propiedad de Mazur–Ulam. Se trata de una generalización del problema de Tingley, planteado en 1987 y que permanece abierto desde entonces. Los artículos [33, 12] y [34] dan una respuesta positiva a estos problemas para espacios concretos, a saber, las álgebras de von Neumann conmutativas (subsección 3.2.1), los JBW*-triples (subsección 3.2.2) y los espacios de funciones continuas que toman valores en un espacio de Hilbert (subsección 3.2.3), respectivamente. En esta misma línea, establecemos un teorema de tipo Hatori-Molnár para estructuras de Jordan en la sección 3.3, con los resultados obtenidos en el artículo [36]. En particular, damos una respuesta positiva a una variante del problema de Tingley en la que se extienden $\mathbb{R}$-linealmente isometrías sobreyectivas entre los conjuntos de elementos unitarios de dos JBW*-álgebras.

Sean (X, d_X) e (Y, d_Y) dos espacios métricos, reales o complejos, donde d_X y d_Y denotan las distancias en X e Y, respectivamente. La noción de *isometría* adoptada en este proyecto es la de una aplicación $T : X \to Y$ que preserva distancias, es decir, T es una isometría si verifica

$$d_Y(T(x), T(y)) = d_X(x, y), \quad \forall x, y \in X.$$

Supongamos ahora que X e Y son espacios normados (reales o complejos), y consideremos las distancias inducidas por las normas. Diremos entonces que una aplicación $T : X \to Y$ es una isometría siempre que verifique $\|T(x) - T(y)\|_Y = \|x - y\|_X$, para todo x, y en X.

El origen de los problemas de extensión de isometrías se encuentra en el Teorema de Mazur–Ulam, que afirma que toda isometría sobreyectiva entre dos espacios normados reales es afín, es decir, una traslación de una aplicación lineal. Este teorema fue enunciado por S. Mazur y S. Ulam en 1932 como respuesta a un problema planteado por S. Banach, quizás en el famoso *Café Escocés* ([119]). Una profunda conclusión se desprende del teorema de Mazur-Ulam, a saber, la estructura lineal de los espacios normados está determinada por su estructura métrica.

A la luz de esta interpretación del enunciado expuesto, surgen diferentes generalizaciones entre las que destacamos la establecida por P. Mankiewicz en 1972 ([118]), en la que prueba que toda isometría sobreyectiva entre las bolas cerradas unidad de dos espacios normados reales X e Y admite una (única) extensión a una isometría lineal y sobreyectiva entre X e Y. Informalmente hablando, P. Mankiewicz prueba que toda la información *genética* de un espacio normado se aloja en su bola cerrada unidad.

En esta línea de pensamiento, es lógico que numerosos matemáticos se hayan sentido atraídos por la idea de intentar optimizar el teorema de Mankiewicz. Intentar precisar qué parte de la estructura algebraica de un espacio normado está determinada por la estructura de un cierto subespacio métrico subyacente resulta un desafío considerablemente intrigante.

En 1987, D. Tingley escoge intuitivamente la esfera unidad para reemplazar a la bola cerrada unidad en el teorema de Mankiewicz ([161]). Centra sus esfuerzos en los espacios normados finito-dimensionales y prueba que toda isometría sobreyectiva entre las esferas unidad de dos de estos espacios aplica puntos antípoda en puntos antípoda. La estrategia de D.

Tingley a la hora de abordar el problema pasa por considerar la extensión homogénea. Esto es, dados dos espacios de Banach X e Y, supongamos que $\Delta : S(X) \to S(Y)$ es una isometría sobreyectiva, donde $S(Z)$ denota la esfera unidad de cualquier espacio normado Z. Bajo estas hipótesis, siempre podemos considerar la biyección $T : X \to Y$ dada por

$$T(x) = \begin{cases} \|x\| \Delta\left(\frac{x}{\|x\|}\right) & x \neq 0 \\ \\ 0 & x = 0. \end{cases}$$

Se trata de una biyección positivamente homogénea, es decir, que actúa de forma homogénea sobre los escalares no negativos. Sin embargo, probar que T es de hecho una isometría no resulta una cuestión trivial. A cambio, el teorema de Mazur–Ulam garantiza la linealidad una vez que salvemos ese obstáculo. Nos referiremos a T como la *extensión homogénea* de Δ.

El problema planteado en [161] atrae el interés de un gran número de matemáticos y, en consecuencia, adquiere nombre propio.

Problema 1. [161, Problema de Tingley (1987)] *Sean X e Y dos espacios de Banach. Supongamos que $\Delta : S(X) \to S(Y)$ es una isometría sobreyectiva. ¿Existe una isometría sobreyectiva $\mathbb{R}$-lineal $T : X \to Y$ tal que $T(x) = \Delta(x)$ para todo x en $S(X)$?*

¿Es toda isometría sobreyectiva entre las esferas unidad de dos espacios de Banach la restricción a la esfera unidad de una isometría sobreyectiva $\mathbb{R}$-lineal entre los espacios? El problema de Tingley pretende concentrar todo aquello que hace a un espacio normado *ser* un espacio normado, dentro de su esfera unidad. Es interesante notar que $S(X)$ es un conjunto con muy pocas propiedades topológicas (no es, en general, compacto y tiene interior vacío).

La simplicidad del enunciado del problema de Tingley esconde una ardua cuestión que, tal y como ya observa el propio D. Tingley en [161], permanece abierta incluso para dos espacios de Banach de dimensión dos. A pesar de todo, numerosas respuestas parciales han sido proporcionadas en espacios concretos. Destacan, mayoritariamente en la primera etapa, las contribuciones de G.G. Ding y sus estudiantes. Existen respuestas positivas al problema de Tingley para espacios de Hilbert y espacios de sucesiones (ver [42, 46, 45, 43, 47]), espacios de funciones medibles $L^p((\Omega, \mu), \mathbb{R})$, (ver [154, 155, 156]), espacios de funciones continuas (ver [164, 44, 60] y [153]) y espacios polihedrales de dimensión finita ([28, 96]). Los surveys [48] y [169] contienen información detallada sobre estas primeras aportaciones al problema de Tingley.

De entre las estrategias empleadas para abordar el problema, destaca la tendencia a fijar un espacio de Banach y estudiar las isometrías sobreyectivas entre su esfera unidad y la esfera unidad de un segundo espacio de Banach arbitrario. Como respuesta a esta línea de investigación, y con el respaldo de las numerosas respuestas positivas en espacios concretos, L. Cheng y Y. Dong introducen en 2011 la propiedad de Mazur–Ulam.

Definición 2. [28, Propiedad de Mazur–Ulam] *Un espacio de Banach X satisface la* propiedad de Mazur–Ulam *si para cualquier espacio de Banach Y, toda isometría sobreyectiva $\Delta : S(X) \to S(Y)$, es la restricción a la esfera unidad de una isometría $\mathbb{R}$-lineal y sobreyectiva entre X e Y.*

Es claro que dicha propiedad está íntimamente relacionada con el problema de Tingley. De hecho, puede reformularse como sigue: un espacio de Banach X posee la propiedad de Mazur–Ulam si el problema de Tingley asociado al mismo admite una respuesta positiva para toda isometría sobreyectiva entre $S(X)$ y la esfera unidad de cualquier espacio de Banach Y. Tal y como comentamos anteriormente, muchos de los resultados obtenidos para el problema de Tingley, determinan en realidad espacios de Banach con la propiedad de Mazur–Ulam. Ejemplos de esto son el espacio $c_0(\mathbb{R})$, formado por todas las sucesiones de números reales que convergen a cero ([47, Corollary 2]), el espacio $\ell_\infty(\Gamma, \mathbb{R})$, de las funciones acotadas y $\mathbb{R}$-valuadas sobre un espacio discreto Γ ([112, Main Theorem]), el espacio $C(K, \mathbb{R})$, de todas las funciones continuas $\mathbb{R}$-valuadas definidas sobre un espacio compacto y Hausdorff K [112, Corollary 6], y los espacios $L^p((\Omega, \mu), \mathbb{R})$, de las funciones medibles $\mathbb{R}$-valuadas en un espacio de medida σ-finito (Ω, μ) para todo $1 \leq p \leq \infty$ [155, 154, 156].

Tanto el problema de Tingley, como su generalización en la propiedad de Mazur–Ulam, suponen áreas de investigación actualmente activas. La lista de artículos abordando exitosamente estos problemas en espacios concretos no deja de crecer. El survey [136] es una excelente referencia para ver la situación actual del problema de Tingley, donde se incluyen numerosas aportaciones como [70, 71, 72, 134, 136, 139, 158, 159, 160, 153, 61, 69, 62, 125, 24, 165].

En la tarea de investigación en Matemáticas, el éxito a la hora de abordar un problema no está plenamente garantizado. Algunas conjeturas pueden quedar bloqueadas durante muchos años y otras incluso descartarse. Así mismo, la culminación de cuestiones que han permanecido abiertas durante largos períodos de tiempo, no solo supone una satisfacción en sí misma, sino que normalmente implica el desarrollo previo de métodos y teorías novedosos para abordar dichas cuestiones. En este sentido, la dificultad inherente a los problemas de extensión de isometrías, lejos de ser un obstáculo, ha dado lugar a una inmensa cantidad de nuevas herramientas, dedicadas plenamente a resolver el problema, y que adicionalmente cobran interés por derecho propio. Es el caso de los resultados expuestos en el Capítulo 2, donde se recogen esencialmente algunas de las novedades obtenidas en nuestros artículos [12], [35] y [34].

En primer lugar, centramos nuestros esfuerzos en el estudio de las caras de la bola cerrada unidad de estructuras triples. En efecto, la estabilidad facial que presentan las isometrías sobreyectivas entre las esferas unidad de dos espacios de Banach justifica el estudio de la teorıa facial.

El estudio de la estructura facial en JB*-triples fue iniciado en 1988 por C.M. Edwards y G.T. Rüttimann en el artículo [55], donde proporcionan una descripción de las caras débil*-cerradas de la bola unidad cerrada de un JBW*-triple, es decir, aquellos JB*-triples que son duales como espacios de Banach. También exploran las caras norma-cerradas en el predual de un JBW*-triple. C.M. Edwards y G.T. Rüttimann establecen una correspondencia bi-unívoca entre el conjunto de los tripotentes de un JBW*-triple M y las caras débil*-cerradas de su bola unidad cerrada y las caras norma-cerradas de la bola unidad cerrada de su predual M_*.

Una correspondencia análoga es establecida por los mismos autores en 2001, cuando centran sus estudios en la estructura facial de la bola unidad cerrada de un JBW*-triple real ([57]). No es hasta el año 2010 cuando se concluye el estudio de la estructura facial de la bola unidad cerrada de un JB*-triple arbitrario. Una primera aportación por parte de C.M. Edwards, C.S. Hoskin, F.J. Fernández-Polo y A.M. Peralta en [52] determina las caras norma-cerradas de la bola unidad cerrada de un JB*-triple. En una segunda aproximación al problema en [67], F.J. Fernández-Polo y A.M. Peralta determinan las caras débil*-cerradas de la bola unidad cerrada del dual de cualquier JB*-triple. En estos dos últimos casos, se prueba que existe una correspondencia bi-unívoca entre tripotentes compactos del bidual y las caras. Así, los resultados de C.M. Edwards y G.T. Rüttimann en [56] sobre compacidad en el ambiente triple resultan fundamentales.

Llegados a este punto, se presentan claramente dos problemas abiertos cuya solución enunciamos en la sección 2.1. El primero se trata de distinguir entre aquellas caras débil*-cerradas en la bola unidad cerrada del bidual de un JB*-triple, que siempre puede verse con estructura de JBW*-triple, que están asociadas a un tripotente compacto en virtud de los resultados en [67], de aquellas que están asociadas a un tripotente no compacto, según la correspondencia establecida en [57]. Resolvemos este interrogante mediante el concepto topológico de cara *abierta relativa* en el artículo [12]. La segunda cuestión sugiere abordar el estudio de la estructura facial en JB*-triples reales. Así, el artículo [34], donde describimos las caras norma-cerradas de la bola unidad cerrada de un JB*-triple real y las débil*-cerradas de la bola unidad cerrada de su dual, supone la culminación de una tarea iniciada hace treinta y dos años. Además, para este estudio se requiere la introducción del concepto de compacidad en ambientes reales. Análogamente al caso complejo, llegamos mediante la noción de cara abierta relativa a la distinción entre aquellas caras débil*-cerradas en el bidual de un JB*-triple real asociadas a tripotentes compactos de aquellas asociadas a tripotentes no compactos. A lo largo del desarrollo de estos enunciados, los argumentos se han basado de una manera rotunda en la posibilidad de trasladar los resultados del ambiente real al ambiente complejo y viceversa, aprovechando la definición de un JB*-triple real como forma real de su complexificación.

Al trabajar en una C*-álgebra A, los conceptos de invertibilidad, de punto extremo o de elemento unitario, son sorprendentemente independientes de si la estructura algebraica considerada en A es la associativa, la Jordan, o la de JB*-triple. Aprovechamos que todas estas nociones se ven preservadas al paso a JB*-triples para establecer en la sección 2.2 una caracterización de los unitarios en JB*-álgebras. Dicha caracterización extiende la dada para C*-álgebras por M. Mori en [125] y presenta como novedad frente a las ya existentes que evita trabajar con duales o preduales. Se trata de una respuesta meramente métrica a la tarea de determinar cuáles son aquellos puntos extremos de la bola unidad de una JB*-álgebra unital que son de hecho unitarios.

El Capítulo 3 es el núcleo del documento y recoge todos los resultados propiamente concernientes a los problemas de extensión de isometrías. Al inicio se exponen una serie de estrategias que con frecuencia guían los intentos de resolver este tipo de problemas. La sección 3.1 está dedicada a la *propiedad fuerte de Mankiewicz*, o *strong Mankiewicz property*, introducida por M. Mori y N. Ozawa en [126]. Diremos que un subconjunto convexo $\mathcal{K}$ de un espacio normado X satisface la propiedad fuerte de Mankiewicz si para toda isometría sobreyectiva Δ de $\mathcal{K}$ a otro subconjunto convexo L de un espacio normado Y es afín. Los citados autores, establecen una condición suficiente para que un espacio de Banach satisfaga dicha propiedad. Entran en juego así los teoremas de tipo Krein–Milman o de tipo Russo–Dye, que expresan la bola unidad cerrada como la envolvente convexo-cerrada de los puntos extremos o de los unitarios cuando procede, respectivamente. En [12] y [34] contribuímos a esta línea de investigación proporcionando nuevos ejemplos de espacios de Banach cuyas bolas unidad cerradas tienen la propiedad fuerte de Mankiewicz, a saber, todo JBW*-triple, el espacio de funciones continuas H-valuadas en un espacio compacto de Hausdorff, donde H es un espacio de Hilbert real con dimensión mayor estricta que uno, así como algunos subtriples de este último espacio.

La propiedad recién comentada proporciona un método novedoso de abordar la propiedad de Mazur–Ulam, protagonista de la sección 3.2. M. Mori y N. Ozawa la emplean para probar que toda C*-álgebra unital y toda álgebra de von Neumann real satisface la propiedad de Mazur–Ulam. Siguiendo esta estrategia, utilizamos los resultados anteriormente expuestos, junto con toda una maquinaria desarrollada especificamente en la que se incluyen los resultados sobre estructuras faciales, para concluir en las secciones 3.2.2 y 3.2.3 que todo JBW*-triple con rango distinto de dos, todo espacio de Hilbert y el espacio $C(K, H)$ de las funciones continuas

H-valuadas en un espacio compacto de Hausdorff K, donde H es un espacio de Hilbert real o complejo, satisfacen la propiedad de Mazur–Ulam.

Por otra parte, la sección 3.2.1 refleja los resultados obtenidos en [33]. En ellos se concluye que toda álgebra de von Neumann conmutativa tiene la propiedad de Mazur–Ulam. En este caso, los argumentos empleados se basan en los desarrollados anteriormente por A. Jiménez-Vargas, A. Morales-Campoy, A.M. Peralta y M.I. Ramírez en [94] o [135]. Estos dos últimos artículos destacan por abordar los problemas de extensión isométrica en ambientes complejos. La relevancia radica en que en el problema de Tingley no cabe esperar más que $\mathbb{R}$-linealidad. La conjugación en la esfera de $\mathbb{C}$ es un contraejemplo sencillo para ver que no toda isometría sobreyectiva sobre la esfera de dos espacios de Banach complejos puede extenderse complejo-linealmente, ni siquiera conjugado-linealmente. Siguiendo esta línea, trabajamos sobre funciones continuas complejo-valuadas y definidas en un espacio *estoniano*. Este concepto topológico nos permite trabajar con abundancia de proyecciones para más tarde inferir, efectivamente, nuestros resultados a espacios de funciones medibles y a álgebras de von Neumann conmutativas, generalizando por el camino los resultados de D. Tan [154].

El broche del Capítulo 3, y esencialmente de la totalidad del proyecto, lo constituye la versión del teorema de Hatori-Molnár para JB*- y JBW*-álgebras. Los resultados del artículo [36] avalan la sección 3.3, donde se plantea la sustitución de las esferas en el problema de Tingley por subconjuntos estrictamente contenidos en ellas. Tras coleccionar resultados tanto positivos como negativos, revisamos el teorema de Hatori-Molnár, enunciado en [86, Teorema 1] por O. Hatori y L. Molnár. En el mismo, se prueba que dos C*-álgebras unitales cuyos grupos unitarios son isométricos, son necesariamente Jordan *-isomorfas, y se especifica una descripción completa de dicha isometría en los unitarios de la forma e^{ih} para cualquier h hermítico. En el caso de las álgebras de von Neumann, dado que todo unitario es de la forma anterior, dicho teorema supone una respuesta positiva al problema de Tingley modificado. Puesto que las técnicas utilizadas por O. Hatori y L. Molnár no son compatibles con estructuras de Jordan, a lo largo de la sección 3.3 exponemos versiones Jordan de las mismas. Lejos de ser una simple traducción al lenguaje Jordan, requieren herramientas totalmente novedosas en este ambiente, como el teorema de Stone para grupos uniparamétricos en JB*-álgebras. El teorema de Shirshov-Cohn afirma que toda JB*-subálgebra de una JB*-álgebra generada por dos elementos hermíticos es una JC*-álgebra, esto es, una JB*-subálgebra norma-cerrada de una C*-álgebra. Dicho resultado será extraordinariamente fructífero en nuestro argumentos. De hecho, una versión de este teorema será enunciada para dos tripotentes ortogonales en el estudio de elementos unitarios realizado en la sección 2.2.

Tras este breve repaso por la estructura y contenido de la presente memoria, cabe destacar que todos los resultados obtenidos en el periodo de formación de la doctoranda contribuyen considerablemente a la teoría de las JB*-álgebras y a la teoría de los JB*-triples. En particular, proporcionan novedades en dos problemas abiertos desde hace más de treinta años, a saber, la estructura facial de los JB*-triples, y la propiedad de Mazur–Ulam. Lejos de que esto suponga un punto y final, y desde la perspectiva del inicio de una nueva etapa, se plantean en el Capítulo 4 una serie de cuestiones intrigantes íntimamente relacionadas con las tratadas en los artículos que componen esta tesis.

A lo largo del desarrollo de los artículos que componen esta tesis, fue necesario aplicar una serie de resultados propios del folclore del Análisis Funcional cuya presencia pasa desapercibida y que, sin embargo, tienen un impacto rotundo en nuestros argumentos. Es el caso de los llamados *resultados de separación* que constituyen el Apéndice A y que incluyen el Lema de Urysohn y el Teorema de separación de Eidelheit. Este último es acertadamente usado por R. Tanaka en [158] y [159] de manera directa para probar que un subconjunto convexo de la esfera unidad de un espacio normado es maximal como subconjunto de la esfera si y solo si es una cara propia maximal de la bola unidad cerrada. Esta afirmación proporciona conclusiones básicas pero importantes a la hora de abordar los problemas de extensión de isometrías mediante la estructura facial. Por su parte, el Lema de Urysohn es aplicado numerosas veces tanto en su versión más básica, como en su generalización al ambiente de JB*-triples enunciada en [68] por F.J. Fernández-Polo y A.M. Peralta.

Finalmente, el Apéndice B refleja parte de las actividades realizadas por la doctoranda durante el programa de doctorado. Todas ellas han contribuído indudablemente a su formación académica.

Introduction

According to the rules set by the University of Almería concerning the official PhD studies, and in agreement with the *regrouping modality*, we present this final project as a collection of all the novelties provided in the papers [33, 12, 34, 35] and [36]. The authorship of the quoted works belongs to Antonio M. Peralta and the doctoral candidate, with the exception of the paper [12], which is a joint work of the mentioned authors in collaboration with F.J. Fernández-Polo and J. Becerra Guerrero. Along the rest of the document, the name of the quoted authors will be omitted for briefness, and assumed when the corresponding papers are mentioned.

The isometric extension problems are the core of the study developed in the framework of this thesis. More concretely, the task of determining those Banach spaces satisfying the Mazur–Ulam property signifies the centre of our research. The main purpose of this introduction is to provide the reader with a general overview, not just of the existing results related to this topic, but of the chronological development of themselves. The questions treated in the papers supporting this thesis find their origin more than thirty years ago, and the lacking, until now, of an answer to the isometric extension problem in the general case manifests how hard the problems are, even though the list of attempts successfully addressing these questions in concrete spaces has been actively widened during all these years, and still growing.

Let us briefly review how our achievements are organised in the present document. The starting point of this memoir is Chapter 1, which provides a background on the structures on which our work relies, that is, C*-algebras, JB*-algebras and JB*-triples, as well as their analogues in the real setting. This chapter is not intended to deepen in the quoted mathematical objects but to introduce them in a constructive way. We shall show that any C*-algebra can be regarded as a JB*-algebra when equipped with its natural Jordan product, and the original norm and involution. In the same manner, it will be exposed that any JB*-algebra, and hence any C*-algebra, has a more general algebraic structure of JB*-triple. The perspective given by the JB*-triple theory is extremely advantageous in order to treat with Banach or Jordan algebras. This is precisely the spirit of our papers [33, 12, 34, 35] and [36], and thus we shall point out along Chapter 1 those results concerned with this philosophy.

The strategy followed in our papers to address the isometric extension problems derived in the study of important parallel questions. All of them are exposed in Chapter 2. On the one hand, the papers [12] and [34] culminate the study of the facial structure in JB*-triples, initiated in 1988. The obtained results in this line can be found in section 2.1. On the other hand, the geometric nature of the isometric extension problems is also explored in [35], whose main goal is to characterise those extreme points of the closed unit ball of a unital JB*-algebra which are unitary. Section 2.2 collects the achieved novelties in that sense.

Chapter 3, divided in three sections, is directly devoted to the principal objective of this thesis, that is, the extension of isometries. Section 3.1 is concerned with the strong Mankiewicz property, employed as a method to tackle the isometric extension problems in our papers [12] and [34]. In section 3.2, we shall centre our attention on the task of determining whether a specific Banach space satisfies the Mazur–Ulam property. This is the final purpose of the papers [33, 12], and [34], whose novelties are exposed. In [33] we are concerned with commutative von Neumann algebras, and pursue our goal dealing exclusively with continuous functions (section 3.2.1). On the other hand, the papers [12] and [34], respond to a similar structure. That is, both of them are aimed to obtain a primary statement involving the Mazur–Ulam property in certain Banach spaces, and additionally, each one of these two works goes necessarily through the facial structure in the development of its arguments. While [12] is focused on general JBW*-triples (section 3.2.2), [34] is intended to vindicate the usefulness of techniques in JB*-triple theory to solve natural problems in Functional Analysis. In the last case, the more general perspective given by triple structures allowed us to tackle successfully the Mazur–Ulam property in spaces of Hilbert-space-valued continuous functions on a compact Hausdorff space (section 3.2.3). In the same spirit, we establish in section 3.3 a Hatori-Molnár type theorem for Jordan structures. The quoted statement is achieved in our paper [36], where it is considered the extension of surjective isometries between the unitary sets of two unital JB*-algebras to a real linear surjective isometry between the spaces.

Two appendices are included in this document. The first one, Appendix A, is intended to highlight some statements known as *separation results*. In particular, Urysohn's lemma and Eidelheit's separation theorem have turned out to be useful tools. The presence of these statements, which were required during the developments of the papers supporting this thesis, could be unperceived, but it had an invaluable impact over our arguments. Finally, and with a completely different soul, Appendix B is a collection of the main activities in which the doctoral candidate has participated, and which have complemented her academic formation.

As commented before, since this thesis responds to a regrouping modality of our papers [33, 12, 34, 35] and [36], the reader is referred to the quoted documents (included at the end of this memoir) for the proofs and details of the results we shall state along the different chapters.

The notion of JB*-triple and its characterisation as those complex Banach spaces whose open unit ball is a bounded symmetric domain is due to W. Kaup (see [106]). These structures will conform our natural environment of work during the whole memoir. We recall that a JB*-triple is a complex Banach space X admitting a continuous triple product $\{\cdot,\cdot,\cdot\}_X : X \times X \times X \to X$, which is symmetric and linear in the outer variables, conjugate-linear in the middle one, and satisfies the following axioms:

1. $L(a,b)L(x,y) = L(x,y)L(a,b) + L(L(a,b)x,y) - L(x,L(b,a)y)$, for every $a,b,x,y \in X$, where $L(a,b)$ is the operator on X given by $L(a,b)x = \{a,b,x\}_X$;
2. For all $a \in X$, $L(a,a)$ is a hermitian operator with non-negative spectrum;
3. $\| \{a,a,a\}_X \|_X = \|a\|_X^3$, for all $a \in X$.

A JBW*-triple is a JB*-triple which is a dual Banach space. According to the anticipating comments, let us observe that any C*-algebra A can be regarded as a JB*-triple when equipped with the triple product defined by

$$\{x,y,z\}_A = \frac{1}{2}(xy^*z + zy^*x), \quad x,y,z \in A. \tag{1}$$

The Jordan structures also enlarge the class of JB*-triples if we consider, for instance, a JB*-algebra M in the sense employed in [82, §3.8], under the triple product

$$\{x,y,z\}_M = (x \circ y^*) \circ z + (z \circ y^*) \circ x - (x \circ z) \circ y^*, \quad x,y,z \in M. \tag{2}$$

Along this introduction, many other structures and concepts will be mentioned. Since we would like to emphasize here the significance of our achievements contrasting them with the forerunners, that will be made by avoiding deep technical notions. According to the chosen style, for additional details the reader is referred to Chapter 1 for the basis on Banach algebras, Jordan algebras, and triple structures. When a specific notion is commented, and as long as its understanding seems to be relevant, an appropriate reference will be provided.

Let (X, d_X) and (Y, d_Y) be two (real or complex) metric spaces, where d_X and d_Y denote the distances in X and Y, respectively. The notion of *isometry* adopted in this project is that of a mapping $T : X \to Y$ which preserves distances, that is,

$$d_Y(T(x), T(y)) = d_X(x, y), \quad \forall x, y \in X.$$

If we assume that X and Y are (real or complex) normed spaces, and considering the distances induced by the norms, we shall say that a mapping $T : X \to Y$ is an isometry whenever $\|T(x) - T(y)\|_Y = \|x - y\|_X$ holds for every $x, y \in X$.

The Mazur–Ulam theorem, stated in 1932 (see [119]), means a source of inspiration for those problems concerning extension of isometries. The celebrated result, due to S. Mazur and S. Ulam, affirms that every surjective isometry between two real normed spaces is an affine mapping, that is, linear up to a translation. The interest provoked by the previous assertion relies on the conclusion which is hidden behind its statement, namely, the algebraic structure of a (real) normed space is determined by its underlying metric space.

Among the subsequent generalisations derived from this theorem, we highlight the results due to P. Mankiewicz in 1972, who devoted his paper [118] to determining whether an isometry $T : U \to Y$ from a subset $U \subseteq X$ of a real normed space X into a real normed space Y admits an extension to an isometry from X onto Y. It should be noted that if the isometric extension were found, the affinity would follow immediately from the Mazur–Ulam theorem. As the own author indicated in the paper, there exist counterexamples for the general case [118, Remark 4 and Example 8], but under certain assumptions over U and its image, it turns out to be true. P. Mankiewicz gave a positive answer for the case in which U is non-empty, open and connected, and $T(U)$ is an open subset of Y. Consequently, he proved that the same conclusion holds if U and $T(U)$ are both *convex bodies*, that is, convex sets with non-empty interiors ([118, Theorem 5 and Remark 7]).

Theorem A. [118, Mankiewicz's theorem (1972)] *Every surjective isometry between convex bodies in real normed spaces can be uniquely extended to an affine isometry between the whole spaces.*

The theorem above applies particularly to any surjective isometry between the closed unit balls of two real normed spaces. Consequently, a reinterpretation of Mankiewicz's theorem in such a setting affirms, roughly speaking, that the entire information of a real normed space is clustered inside its closed unit ball.

Following the same research line, in 1987 D. Tingley focused, intuitively, his attention on the unit spheres, and posed what is now known as *Tingley's problem* (see [161]).

Problem B. [161, Tingley's problem (1987)] *Is every surjective isometry between the unit spheres of two Banach spaces the restriction to the unit sphere of a surjective linear isometry between the whole spaces?*

Henceforth, let the symbol $S(X)$ stand for the unit sphere of any real or complex normed space X, that is, the set of all norm-one elements in X. Let us consider X and Y two (real or complex) Banach spaces. Problem B inquires if given a surjective isometry $\Delta : S(X) \to S(Y)$, it is possible to find a real linear extension of Δ between the whole spaces, that is, a surjective real linear isometry $T : X \to Y$ such that $T(x) = \Delta(x)$ for every $x \in S(X)$. Of course, it is straightforward to see that, in case that such a extension exists, T is unique. D. Tingley proved that surjective isometries between unit spheres of finite-dimensional normed spaces map antipodal points to antipodal points (cf. [161, Theorem]).

The first observation to be done is the importance of the scalar field of the involved normed spaces. Indeed, let us consider the complex Banach space $X = Y = \mathbb{C} \oplus^{\infty} \mathbb{C}$, equipped with the maximum norm, namely, $\|(z_1, z_2)\|_{\infty} = \max\{|z_j| : j = 1, 2\}$, $(z_1, z_2) \in X$. It is easy to see that the surjective isometry $\Delta : S(X) \to S(X)$ given by $\Delta(z_1, z_2) = (z_1, \bar{z}_2)$, for each (z_1, z_2) in $S(X)$, cannot be extended complex-linearly (nor even conjugate-linearly) to a surjective isometry from X to itself. This is just an example which manifests that it is hopeless to expect complex-linearity or conjugate-linearity in Tingley's problem, since a surjective isometry between the unit spheres of two complex Banach spaces need not admit an extension to a surjective complex linear nor conjugate-linear isometry between the whole spaces.

Despite the simplicity of its statement, Tingley's problem is a hard question which remains unsolved even for surjective isometries between the unit spheres of an arbitrary couple of two-dimensional normed spaces. Dealing with unit spheres, where there is no linear or convex structure, forces to develop new techniques to tackle the problem. However, the intrinsic difficulty to the quoted isometric extension problem, far from being an obstacle, seems to have worked during the last thirty three years more as a trigger in the seeking of positive answers to Tingley's problem in concrete spaces. Actually, a vast collection of partial solutions has been provided in the three decades elapsed after Tingley's paper. We shall make a review of the most relevant results obtained in the topic. For a more detailed overview in that sense, the surveys [48, 169], and [136] will allow the reader to get updated about the state-of-the-art of this problem.

From now on, $\mathbb{K}$ will denote the fields $\mathbb{R}$ or $\mathbb{C}$ indistinctly. Let L be a locally compact Hausdorff space. In 1994, R.S. Wang headed the first approach to the problem by extending surjective isometries between unit spheres of two $C_0(L, \mathbb{K})$-type spaces, where $C_0(L, \mathbb{K})$ denotes the Banach space of all $\mathbb{K}$-valued continuous functions defined on L vanishing at infinity, equipped with the supremum norm ([164]). The proofs, based on Urysohn's lemma (Appendix A) and some facts of geometric nature, contemplate both cases, the real $\mathbb{K} = \mathbb{R}$, and the complex one $\mathbb{K} = \mathbb{C}$, but the obtained extension is, in general, just real linear according to the comments made above.

Theorem C. [164, Theorem B and C] *Let L_1 and L_2 be two locally compact Hausdorff spaces.*

(a) Suppose $\Delta : S(C_0(L_1,\mathbb{R})) \to S(C_0(L_2,\mathbb{R}))$ is a surjective isometry. Then there exists a surjective real linear isometry T from $C_0(L_1,\mathbb{R})$ onto $C_0(L_2,\mathbb{R})$ such that $T|_{S(C_0(L_1,\mathbb{R}))} \equiv \Delta$.

(b) Suppose $\Delta : S(C_0(L_1,\mathbb{C})) \to S(C_0(L_2,\mathbb{C}))$ is a surjective isometry. Then there exists a surjective real linear isometry T from $C_0(L_1,\mathbb{C})$ onto $C_0(L_2,\mathbb{C})$ such that $T|_{S(C_0(L_1,\mathbb{C}))} \equiv \Delta$.

Furthermore, there exist two disjoint clopen subsets A and B of L_1 such that $A \cup B = L_1$, $T|_{C_0(A,\mathbb{C})}$ is complex linear, and $T|_{C_0(B,\mathbb{C})}$ is conjugate-linear, where

$$C_0(X,\mathbb{C}) := \{f \in C_0(L_1,\mathbb{C}) : f|_{L_1\setminus X} \equiv \mathbf{0}\},$$

for $X = A, B$.

By virtue of the commutative Gelfand-Naimark theorem (see [144, Theorem 1.16.6] or Theorem 1.1.10), this pioneering theorem solves Tingley's problem for any surjective isometry between the unit spheres of two commutative C*-algebras.

Let X and Y be two Banach spaces. Suppose $\Delta : S(X) \to S(Y)$ is a surjective isometry. We can always consider the positive homogeneous extension, that is, the bijection $T : X \to Y$ given by

$$T(x) = \begin{cases} \|x\|\,\Delta\left(\frac{x}{\|x\|}\right) & x \neq 0 \\ 0 & x = 0. \end{cases}$$

The mapping T is a positively homogeneous bijection by obvious reasons. Nevertheless, the task of deciding whether T is an isometry is a really hard question. In return, by the Mazur–Ulam theorem, T will be real linear as soon as it is an isometry.

The first decade of the XXIst century witnessed the real attraction caused by Tingley's problem, being gathered during this period of time an avalanche of partial solutions. G.G. Ding and his students, included R.S. Wang, totally engaged with the isometric extension problem, provided a huge amount of contributions in this line. To be more detailed, we firstly stand out the papers [42, 43, 46] and [45] published between 2002 and 2004. In Theorem 2.2 of [42], G.G. Ding establishes an isometric extension result for 1-Lipschitz mappings between the unit spheres of two Hilbert spaces by assuming particular conditions inspired by the results obtained in [161]. As a consequence, he gives a positive answer to Tingley's problem for surjective isometries between the unit spheres of two Hilbert spaces. On the other hand, the arguments of [43, 46] and [45] consist of developing appropriate Banach-Stone representation type theorems for surjective isometries between the unit spheres of certain sequence spaces, and applying them to solve Tingley's problem. That strategy goes through expressing the homogenous extension in terms of the description given by such representation theorems. That makes easier to prove that it is actually a surjective real linear isometry between the whole spaces involved. The author obtains a positive solution to Tingley's problem for X and Y being $\mathcal{L}^p(\Gamma,\mathbb{R})$-type spaces, with $p>1$ ([42] for $p=2$, and [43]), $\mathcal{L}^1(\Gamma,\mathbb{R})$-type spaces ([46]), and $\mathcal{L}^\infty(\Gamma,\mathbb{R})$-type spaces ([45]), where Γ is an index set. Here, a $\mathcal{L}^\infty(\Gamma,\mathbb{R})$-type space is a normed space of real functions on an index set Γ endowed with the supremum norm. Therefore, the last class of normed spaces includes $\ell^\infty(\Gamma,\mathbb{R})$, $c(\Gamma,\mathbb{R})$ and $c_0(\Gamma,\mathbb{R})$, that is, the spaces of all real bounded, convergent, and null functions on Γ, respectively. Related to these sequence spaces, it is interesting to observe that an affirmative answer to Tingley's problem for X and Y being $\mathcal{L}^\infty(\Gamma,\mathbb{C})$-type spaces follows from Theorem C. Actually, by the same theorem, every surjective isometry from $S(L^\infty((\Omega,\Sigma,\mu),\mathbb{C}))$ onto itself admits a real linear extension between the whole spaces, where (Ω,Σ,μ) is a σ-finite measure space and $L^\infty((\Omega,\Sigma,\mu),\mathbb{C})$ denotes the space of all measurable essentially bounded complex functions endowed with the essential supremum norm, that is, $\|f\|_{\text{ess}} = \operatorname{ess\,sup}\{|f(t)| : t\in\Omega\}$ for each $f\in L^\infty((\Omega,\Sigma,\mu),\mathbb{C})$.

The paper [44] represents the first attempt of addressing Tingley's problem in the case of two Banach spaces of different nature. In the just quoted work, G.G. Ding considers a real normed space Y satisfying some density-conditions, and extends real-linearly any surjective isometry from the unit sphere of Y onto the unit sphere of the commutative C*-algebra $C(K,\mathbb{K})$ of all $\mathbb{K}$-valued continuous functions on K, K being a compact Hausdorff space. The conclusions hold, for instance, for $Y=L^1((\Omega,\mu),\mathbb{R})$, $Y=C(K,\mathbb{R})$, $Y=\ell^\infty(\mathbb{N},\mathbb{R})$ or for any Y separable or reflexive Banach space.

X.N. Fang and J.H. Wang generalised the main result in [44] by removing all the conditions on the real normed space Y but considering $C(K,\mathbb{R})$ with K assumed to be a compact metric space. They prove that, under the mentioned hypothesis, for any real normed space Y, every surjective isometry from $S(Y)$ onto $S(C(K,\mathbb{R}))$ can be extended to be a real linear surjective isometry on the whole Y, and hence the corresponding Tingley's problem is solved positively.

G.G. Ding gives in [47] a positive answer to Tingley's problem for surjective isometries between $S(c_0(\mathbb{N},\mathbb{R}))$ and the unit sphere of an arbitrary real Banach space.

It deserves to be mentioned at this point the achievements due to R. Liu in 2007. In the Main Theorem of the paper [112], R. Liu solves Tingley's problem for a surjective isometry from the unit sphere of a $\mathcal{L}^\infty(\Gamma,\mathbb{R})$-type space onto the unit sphere of any real Banach space. Additionally, in [112] R. Liu assures that the same statement is true for any surjective isometry from the unit sphere of $C(K,\mathbb{R})$ with K a compact Hausdorff space onto the unit sphere of any real Banach space.

Theorem D. [112, Corollary 6] *Let Y be a real Banach space, and let K be a compact Hausdorff space. Suppose $\Delta : S(C(K,\mathbb{R})) \to S(Y)$ is a surjective isometry. Then Δ can be extended to a real linear surjective isometry from $C(K,\mathbb{R})$ onto Y.*

Let (Ω,Σ,μ) be a σ-finite measure space, consider the space $L^\infty((\Omega,\Sigma,\mu),\mathbb{R})$ of all measurable essentially bounded real functions equipped with the essential supremum norm. D. Tan shows in [154, Theorem 2.5] that, given any real Banach space Y, every surjective isometry from $S(L^\infty((\Omega,\Sigma,\mu),\mathbb{R}))$ onto $S(Y)$ can be extended to be a linear isometry on the whole space $L^\infty((\Omega,\Sigma,\mu),\mathbb{R})$. $L^p((\Omega,\Sigma,\mu),\mathbb{R})$-spaces with $1 \leq p < \infty$ were also studied in [155, 156] yielding that an arbitrary real Banach space Y is real-linearly isometric to $L^p((\Omega,\Sigma,\mu),\mathbb{R})$ if and only if their unit spheres are isometric.

We have just exposed that most of the classical Banach spaces responds positively to the isometric extension problem. It is significant to observe that, as the list of positive solutions to Tingley's problem in concrete spaces was enlarging, it was appearing a clear tendency in these early approaches consisting of fixing a Banach space and considering a surjective isometry from its unit sphere onto the unit sphere of any other Banach space. In 2011, L. Cheng and Y. Dong, encouraged by the abundance of positive partial answers, and possibly inspired by the previous comments about the procedure of tackling the problem, introduced in [28] the *Mazur–Ulam property.*

Definition E. [28, Mazur–Ulam property] *A Banach space X satisfies the* Mazur–Ulam property *if for any Banach space Y, every surjective isometry $\Delta : S(X) \to S(Y)$ admits an extension to a surjective real linear isometry from X onto Y.*

Of course, an equivalent reformulation tells that X satisfies the Mazur–Ulam property if Tingley's problem admits a positive solution for every surjective isometry from $S(X)$ onto the unit sphere of any Banach space Y. Therefore, the first examples of Banach spaces satisfying the Mazur–Ulam property already came from [60, 112, 47, 154, 155], and [156], and include the space $c_0(\Gamma, \mathbb{R})$ of real null sequences ([47, Corollary 2]), and the space $\ell_\infty(\Gamma, \mathbb{R})$ of all real-valued bounded functions ([112, Main Theorem]) on a discrete set Γ, the space $C(K, \mathbb{R})$ of all real-valued continuous functions on a compact Hausdorff space K [112, Corollary 6], and the spaces $L^p((\Omega, \mu), \mathbb{R})$ of real-valued measurable functions on an arbitrary σ-finite measure space (Ω, μ) for all $1 \leq p \leq \infty$ [155, 154, 156]. L. Cheng and Y. Dong also contributed by proving in [28] that any somewhere-flat space, and any CL-space admitting a smooth point satisfies the Mazur–Ulam property.

The study of those Banach spaces satisfying the Mazur–Ulam property continues being an active area, as well as the task of providing positive solutions to Tingley's problem. Frequently, handling one of these isometric extension questions becomes the first step in order to achieve the other. Undoubtedly, any successful approach to any of the two problems enriches both of them. In that sense, the last five years have accumulated a numerous list of results in a wide range of spaces by the contributions of mathematicians like R. Tanaka, A.M. Peralta, F.J. Fernández-Polo, M. Mori, N. Ozawa, A. Jiménez-Vargas, A. Morales-Campoy, M.I. Ramírez, V. Kadets, M. Martín, J.J. Garcés, E. Jordá, J. Cabello-Sánchez or I. Villanueva, among others.

Tingley's problems admits a positive solution for any surjective isometry $\Delta : S(X) \to S(Y)$ whenever (X, Y) is a couple of Banach spaces in the following list: X and Y are infinite-dimensional polyhedral Banach spaces [96], $n \times n$ complex matrix algebras and finite von Neumann algebras [158, 160], spaces of compact operators on complex Hilbert spaces and compact C*-algebras [139], weakly compact JB*-triples [69], spaces of bounded linear operators on complex Hilbert spaces [70], atomic von Neumann algebras and, more generally, atomic JBW*-triples [71], von Neumann algebras [72], and spaces of p-Schatten von Neumann operators on a complex Hilbert space with $1 \leq p \leq \infty$ [62]. Concerning preduals of von Neumann algebras, we highlight the positive answer given for spaces of trace class operators on complex Hilbert spaces (see [61]). It is worth noting that the space of trace class operators can be regarded as the dual of the space of compact operators and as the predual of the space of bounded linear operators. Actually, Tingley's problem for surjective isometries between the unit spheres of general von Neumann algebra preduals finds its corresponding solution in [125], as well as for surjective isometries between the spheres of self-adjoint parts of two von Neumann algebras. Additionally, the reader is invited to take a look to the recent papers [165] and [24], where the particular case of two-dimensional Banach spaces is treated.

Our knowledge on the class of Banach spaces satisfying the Mazur–Ulam property is a bit more reduced. The reason of this fact is probably a simple matter of time. A special mention should be made to those results which explore this property in the complex settings. We have already commented the existence of real linear surjective isometries which are not complex linear nor conjugate-linear. The consequent reticence respect to the complex framework resulted in a restriction of the study to real Banach spaces. The task was fortunately iniciated in [94] and [134], where it was proved that the space of complex null sequences $c_0(\Gamma, \mathbb{C})$ and the space $\ell_\infty(\Gamma, \mathbb{C})$ of all complex-valued bounded functions on Γ satisfy the Mazur–Ulam property, where Γ denotes an infinite set equipped with the discrete topology.

Motivated by the encouraging affirmative results already obtained, and comparing with the forerunners in Tingley's problem, we devoted the paper [33] to filling the existing gap in the complex setting by studying the Mazur–Ulam property in the space $L^\infty((\Omega, \mu), \mathbb{C})$ of all complex-valued measurable essentially bounded functions on an arbitrary σ-finite measure space (Ω, μ), endowed with the (essential) supremum norm. The following theorem is an extension to complex-valued functions of the real version due to D. Tan [154].

Theorem F. [33, Theorem 3.14, Linear and Multilinear Algebra]
Let (Ω, μ) be a σ-finite measure space, and let Y be a complex Banach space. Suppose $\Delta : S(L^\infty((\Omega, \mu), \mathbb{C})) \to S(Y)$ is a surjective isometry. Then there exists a surjective real linear isometry $T : L^\infty((\Omega, \mu), \mathbb{C}) \to Y$ whose restriction to $S(L^\infty((\Omega, \mu), \mathbb{C}))$ is Δ. □

An appropriate topological argument allowed us to state the above theorem by working, only and exclusively, with continuous functions. In fact, let K be a compact Hausdorff space. We recall that K is called *Stonean* if the closure of every open set in K is open. A Stonean space K is said to be *hyper-Stonean* if it admits a faithful family of positive normal measures (cf. [152, Definition 1.14]). It is known that for any σ-finite measure space (Ω, μ), the complex space $L^\infty((\Omega, \mu), \mathbb{C})$ is a commutative von Neumann algebra. Furtheremore, the commutative C*-algebra $C(K) := C(K, \mathbb{C})$, of all complex-valued continuous functions on a compact Hausdorff space K, is a dual Banach space (equivalently, a von Neumann algebra) if and only if K is hyper-Stonean (cf. [50]). Thus, from the metric point of view of Functional Analysis, $L^\infty((\Omega, \mu), \mathbb{C})$ is isometrically C*-isomorphic equivalent to some $C(K)$, where K is a hyper-Stonean space. Therefore, Theorem F follows as a corollary of the main result in [33] which, in the spirit of R. Wang, reads as follows:

Theorem G. [33, Theorem 3.11, Linear and Multilinear Algebra]
Let $\Delta : S(C(K)) \to S(Y)$ be a surjective isometry, where K is a Stonean space and Y is a complex Banach space. Then there exist two disjoint clopen subsets K_1 and K_2 of K with $K = K_1 \cup K_2$ satisfying that if K_1 (respectively,

K_2) is non-empty, then there exist a closed subspace Y_1 (respectively, Y_2) of Y and a complex linear (respectively, conjugate-linear) surjective isometry $T_1 : C(K_1) \to Y_1$ (respectively, $T_2 : C(K_2) \to Y_2$) such that $Y = Y_1 \oplus^\infty Y_2$, and $\Delta(a) = T_1(\pi_1(a)) + T_2(\pi_2(a))$, for every $a \in S(C(K))$, where π_j is the natural projection of $C(K)$ onto $C(K_j)$ given by $\pi_j(a) = a|_{K_j}$. In particular, Δ admits an extension to a surjective real linear isometry from $C(K)$ onto Y. □

Following standard terminology, a *localizable measure space* (Ω, ν) is a measure space which can be obtained as a direct sum of finite measure spaces $\{(\Omega_i, \mu_i) : i \in \mathcal{I}\}$. The Banach space $L^\infty((\Omega, \nu), \mathbb{C})$ of all locally ν-measurable essentially bounded functions on Ω is a dual Banach space and a commutative von Neumann algebra. Actually, every commutative von Neumann algebra is C*-isomorphic and isometric to some $L^\infty((\Omega, \nu), \mathbb{C})$ for some localizable measure space (Ω, ν) (see [144, Proposition 1.18.1]). Thus, it derives from the successful efforts in [33] that any commutative von Neumann algebra joins the Mazur–Ulam property.

An outstanding result due to M. Mori and N. Ozawa recently added more attractiveness to the study of isometric extension of isometries.

Theorem H. [126, Theorem 1] *Every unital complex C*-algebra, as a real Banach space, and every real von Neumann algebra has the Mazur–Ulam property.*

The result, which generalises Theorem G, is very conclusive on its own. Moreover, beyond the statement, the proofs follow an innovative argument. Indeed, the observations made by R. Tanaka in [157, Lemma 3.5], and in subsequent papers as [158, Lemma 3.3] or [159, Lemma 4.1], expose in the most general setting how fruitful is a meticulous knowledge of the facial structure of the Banach space involved, since any surjective isometry between unit spheres maps maximal proper faces to maximal proper faces. Accordingly, the main arguments have strongly relied, until now, on the facial structure. M. Mori and N. Ozawa, not unaware of this assertion, but on the contrary, went a step further in [126] by realising that in the setting of unital C*-algebras, the Russo–Dye theorem ([143]) and Mankiewicz's theorem ([118]) can be effectively combined to answer positively the isometric extension problem. These ideas were generalised to introduce the *strong Mankiewicz property*, and materialised in a sufficient condition to get such a property in any convex body.

Definition I. [126, Strong Mankiewicz property] *A convex subset K of a normed space X satisfies the* strong Mankiewicz property *if every surjective isometry Δ from K onto an arbitrary convex subset L in a normed space Y is affine.*

Let $\mathcal{B}_X$ stand for the closed unit ball of a Banach space X. The set of all extreme points of a convex set C will be denoted by $\partial_e(C)$.

Theorem J. [126, Theorem 2] *Let X be a Banach space such that the closed convex hull of the extreme points, $\partial_e(\mathcal{B}_X)$, of the closed unit ball, $\mathcal{B}_X$, of X has non-empty interior in X. Then every convex body $K \subset X$ satisfies the strong Mankiewicz property.*

The celebrated Russo–Dye theorem (see [143, Theorem I.8.4]) precisely assures that every (complex) unital C*-algebra satisfies the hypotheses in the previous theorem. And the same conclusion can be deduced from a result due to B. Li (see [111, Theorem 7.2.4]) in the class of real von Neumann algebras (see also [129, Corollary 6]).

We have followed this new path iniciated by M. Mori and N. Ozawa in two papers, [12] and [34]. The final purpose of both of them was to explore the Mazur–Ulam property, in any JBW*-triple in the paper [12], and in the space $C(K, H)$ of all continuous functions on a compact Hausdorff space K valued in a Hilbert space H, in [34].

On the one hand, in [12] we took advantage of a result due to A.A. Siddiqui, which says that every element in the unit ball of a JBW*-triple is the average of two extreme points (see [148, Theorem 5]). By Theorem J, the closed unit ball, and hence every convex body, of any JBW*-triple M satisfies the strong Mankiewicz property ([12, Corollary 2.2]).

After that, it resulted to be necessary, and extremely interesting, to deepen in the facial structure of JB*-triples. A brief overview of the theory known until that time points to the paper [55], written by C.M. Edwards and G.T. Rüttimann in 1988, as the precursor in the study of faces of the closed unit ball of triple structures. These authors gave a description of the weak*-closed faces of the closed unit ball of any JBW*-triple M and the norm-closed faces of the closed unit ball of its predual via a one-to-one correspondence with the set of tripotents in M ([55]). Analogous results were established in 2001 for real JBW*-triples by the same authors([57]). We had to wait until 2010 to have a successful approach to the norm-closed faces of the closed unit ball of a general JB*-triple. The conclusions were obtained in two steps. The first one, devoted to determining the norm-closed faces of the closed unit ball of any JB*-triple E, was obtained by C.M. Edwards, C.S. Hoskin, F.J. Fernández-Polo, and A.M. Peralta in [52]; and the second one, in [67], where the last two mentioned authors gave a full description of the weak*-closed faces of the closed unit ball of the dual space of any JB*-triple. In both papers, a one-to-one correspondence is also found, but in this case with the set of compact tripotents in the bidual of E (see Section 2.1 or [56] for additional details).

In view of these statements, it emerged the natural question whether we can topologically distinguish between weak*-closed faces in the closed unit ball of the bidual, X^{**}, of a JB*-triple, X, associated with compact tripotents in X^{**}, from weak*-closed faces in $\mathcal{B}_{X^{**}}$ associated with non-compact tripotents in X^{**}. We addressed the problem in [12], and answered the question by considering those faces in the bidual of a JB*-triple X which are *open relative* to X in the sense of [65, 56].

Theorem K. [12, Theorem 3.6, J. Inst. Math. Jussieu] *Let X be a JB*-triple. Suppose F is a proper weak*-closed face of the closed unit ball of X^{**}. Then the following statements are equivalent:*

(*a*) *F is open relative to X, that is, $F \cap X$ is weak*-dense in F;*

(*b*) *F is a weak*-closed face associated with a non-zero compact tripotent in X^{**}, that is, there exists a unique non-zero compact tripotent u in X^{**} satisfying $F = u + \mathcal{B}_{X_0^{**}(u)}$.* □

The previous theorem allows us to state as a consequence that, given a decreasing net of compact tripotents in the second dual of a JB*-triple, the norm-closed face associated with its infimum coincides with the norm-closure of the union of all the norm-closed faces associated with the compact tripotents in the net. By keeping an eye on our goal of solving the surjective extension problem, we apply these conclusions to show that the restriction of a surjective isometry from the unit sphere of a JBW*-triple M onto the unit sphere of any real Banach space, to each norm-closed proper face of $\mathcal{B}_M$ is an affine function. After some hard technical results, we are able to partially state the main theorem. We recall that the rank of a JB*-triple X is the minimal cardinal number r satisfying $\text{card}(S) \leq r$ whenever S is an orthogonal subset of X, that is, $0 \notin S$ and $x \perp y$ for every $x \neq y$ in S.

Theorem L. [12, Theorem 4.14, J. Inst. Math. Jussieu] *Let M be a JBW*-triple with rank bigger than or equal to three. Then, every surjective isometry from the unit sphere of M onto the unit sphere of a real Banach space Y admits a unique extension to a surjective real linear isometry from M onto Y.* □

The following result is a combination of [39, Theorem 2.1], together with Tingley's problem for Hilbert spaces ([42]), and it covers the case of JBW*-triples of rank 1 (cf. [107, Table 1 in page 210]).

Proposition M. [12, Proposition 4.15, J. Inst. Math. Jussieu] *Every Hilbert space satisfies the Mazur–Ulam property. Every rank one JBW*-triple satisfies the Mazur–Ulam property.* □

Recently, it was proved by O.F. K. Kalenda and A.M. Peralta, that all JBW*-triples of rank 2 also satisfy the Mazur–Ulam property ([100, Theorem 1.1]). Therefore, by combining this result with our conclusions in Theorem L and Proposition M ([12, Theorem 4.14 and Proposition 4.15]), it can be concluded that any JBW*-triple satisfies the Mazur–Ulam property ([100, Corollary 1.2]).

On the other hand, the final objective of the paper [34] was to prove that, given any Banach space Y, every surjective isometry from $S(C(K,H))$ onto $S(Y)$ admits an extension to a surjective real linear isometry from $C(K,H)$ to Y, where $C(K,H)$ is the space of all H-valued continuous functions on K, where K is a compact Hausdorff space, and H is a real or complex Hilbert space. As reviewed, the case in which H is a real Hilbert space with dimension equal to 1 or 2 was already covered in [112] and [126], respectively. Therefore, we restricted our first efforts to dimensions bigger than 2 in the real case, and bigger than 1 in the complex one. We justified the novelties that we intended to expose in [34, Theorems 2.1 and 2.2], by illustrating the fact that the unit sphere of $C(K,H)$ is metrically distinguishable from the unit sphere of a unital C*-algebra when H assumed to be complex and of dimension bigger than or equal to 2, and from the unit sphere of a real von Neumann algebra when H is a real Hilbert space of dimension 3 or bigger than or equal to 5.

We also contributed to enlarge the list of convex sets in normed spaces satisfying the strong Mankiewiz property in the paper [34]. We revisit some results in [141, 27, 132] to establish that for any compact Hausdorff space K, and every real Hilbert space $\mathcal{H}$ with dimension bigger than or equal to 2, the closed unit ball of the space $C(K,\mathcal{H})$, of all $\mathcal{H}$-valued continuous functions on K, coincides with the closed convex hull of its extreme points. As an immediate consequence, every convex body in $C(K,\mathcal{H})$ satisfies the strong Mankiewicz property. In the same paper, we further prove that certain real JB*-subtriples of $C(K,\mathcal{H})$ satisfy the hypothesis of Theorem J, and hence any convex body of these subtriples also has the strong Mankiewicz property.

Our strategy along the paper [34] relies on the natural JB*-triple structure associated with the space $C(K,H)$, since it can be regarded as a Hilbert $C(K)$-module in the sense introduced by I. Kaplansky in [101] (see also [89]). This triple structure provided the key tools and results to pursue our primary goal. In order to cover also the real case, it became necessary to develop facial arguments for real JB*-triples, in the sense introduced in [90].

We culminated in [34] the facial study of the closed unit ball of real JB*-triples. We firstly made an extension of the notion of compactness in real JB*-triples, and showed that $\text{Trip}_c(E^{**})^{\sim}$, the set of all compact tripotents in the bidual of a real JB*-triple E with a largest element adjoined, is a complete lattice. After some technical lemmata, a first statement concerning norm-closed faces in the closed unit ball of a real JB*-triple was established and it reads as follows:

Theorem N. [34, Theorem 3.5, J. Math. Anal. Appl.] *Let τ be a conjugation on a JB*-triple X, and let $E = X^\tau$. Then for each norm-closed proper face F of $\mathcal{B}_E$ there exists a unique compact tripotent $u \in E^{**}$*

satisfying $F = (u + \mathcal{B}_{E_0^{**}(u)}) \cap E$. *Furthermore, the mapping*

$$u \mapsto (\{u\}_{\prime,E^*})_{\prime,E} = (u + \mathcal{B}_{E_0^{**}(u)}) \cap E$$

is an anti-order isomorphism from the complete lattice $\mathrm{Trip}_c(E^{**})^{\sim}$ *onto the complete lattice* $\mathcal{F}_n(\mathcal{B}_E)$ *of all norm-closed faces in the closed unit ball of* E. □

It was additionally proved that the weak*-closed faces in the closed unit ball of the dual of a real JB*-triple are also in one-to-one correspondence with the compact tripotents in the bidual.

Theorem O. [34, Theorem 3.7, J. Math. Anal. Appl.] *Let* τ *be a conjugation on a JB*-triple* X*, and let* $E = X^{\tau}$. *Then for each weak*-closed proper face* F *of* $\mathcal{B}_{E^*}$ *there exists a unique compact tripotent* $u \in E^{**}$ *satisfying* $F = \{u\}_{\prime,E^*}$. *Furthermore, the mapping*

$$u \mapsto \{u\}_{\prime,E^*}$$

is an order isomorphism from the complete lattice $\mathrm{Trip}_c(E^{**})^{\sim}$ *onto the complete lattice* $\mathcal{F}_{w^*}(\mathcal{B}_{E^*})$ *of all weak*-closed faces in the closed unit ball of* E^*. □

Analogously to the complex setting, Theorem 3.6 in [34] assures that those proper weak*-closed faces of the closed unit ball of the bidual of a real JB*-triple E associated with a compact tripotent are precisely those which are open relative to E^{**}. That was crucial to obtain the principal theorem in [34].

Theorem P. [34, Theorem 5.6, Corollaries 5.7 and 5.8, J. Math. Anal. Appl.] *Let* K *be a compact Hausdorff space and let* H *be a real or complex Hilbert space. Then the Banach space* $C(K, H)$ *satisfies the Mazur–Ulam property.* □

In the spirit of P. Mankiewicz and with the deep insight of D. Tingley, the question at this stage is whether in Tingley's problem, and hence more generally in the Mazur–Ulam property, the unit spheres can be reduced to strictly smaller subsets. Actually, in some operator algebras the unit spheres have been successfully replaced by the spheres of positive operators (see [123, 122, 127, 128] and [135]). In the final remark of the paper [33], we consider the tempting possibility of extending real-linearly surjective isometries between the sets of extreme points of the closed unit balls of two Banach spaces. Even in the most favourable case of a finite dimensional normed space X, we cannot always conclude that every surjective isometry on the set of extreme points of the closed unit ball of X can be extended to a surjective real linear isometry on X (see [33, Remark 3.15]). In other words, the set of extreme points is not enough to determine a surjective real linear isometry. On the other hand, the existence of an additional structure on X provides new candidates to establish a variant of Tingley's problem by replacing the unit spheres with a proper subset. For instance, in a unital C*-algebra A, the set $\mathcal{U}(A) := \{u \in A : uu^* = u^*u = \mathbf{1}_A\}$ of all unitary elements in A is, in general, strictly contained in the set of all extreme points of the closed unit ball of A. The next theorem has been recently obtained by M. Mori in [125], and it characterises unitaries among extreme points of the closed unit ball of a unital C*-algebra A in terms of the elements in $\partial_e(\mathcal{B}_A)$ at distance $\sqrt{2}$ from the element under study.

Theorem Q. [125, Lemma 3.1] *Let A be a unital C*-algebra, and let $u \in \partial_e(\mathcal{B}_A)$. Then the following statements are equivalent:*

(*a*) *u is a unitary (i.e., $uu^* = u^*u = 1$);*

(*b*) *The set $\mathcal{A}_u = \{e \in \partial_e(\mathcal{B}_A) : \|u \pm e\| = \sqrt{2}\}$ contains an isolated point.*

Other previous approaches to the problem of characterising unitaries in geometric terms can be found in the literature, for example, we highlight the result of C.A. Akemann and N. Weaver for unital C*-algebras (cf. [5, Theorem 3]), and its appropriate version in the settings of JB*-algebras and JB*-triples, established by A. Rodríguez Palacios in [142, Theorem 3.1] and [25, Theorem 4.2.24].

The description provided by C.A. Akemann and N. Weaver goes necessarily through the Banach space structure of the dual space A^*, or through the predual A_*, when A is a von Neumann algebra, while in contrast, in Mori's characterisation it is just required to deal with extreme points of $\mathcal{B}_A$ at certain distance.

Analogously, the result due to A. Rodríguez Palacios forces to deal with the elements in the dual of a JB*-algebra. In the paper [35], we aimed to explore the validity of the characterisation exposed in Theorem Q in the setting of unital JB*-algebras. As it can be found in Chaper 1 of this memoir, any C*-algbera A can be regarded as a JB*-algebra when equipped with the natural Jordan product given by $a \circ b := 1/2(ab + ba)$ for each $a, b \in A$, and the original norm and involution. Assuming that A is unital, the set $\mathcal{U}(A) := \{u \in A : uu^* = u^*u = \mathbf{1}_A\}$ of all unitary elements in A coincides with the set of all Jordan unitary elements in A (i.e. those $u \in A$ such that $u \circ u^* = \mathbf{1}_A$, and $u^2 \circ u^* = u$), when the latter is regarded as a JB*-algebra. However, since there is no associativity in the Jordan setting, many obstacles are found in the task of extending Theorem Q.

According to the philosophy of this thesis, the arguments employed are undoubtedly benefited from results in JB*-triple theory. Any JB*-algebra is included in the more general class of JB*-triples with the triple product in (2). When the triple structure is considered in those algebras, the concepts of unitary element and unitary tripotent are equivalent. The equivalence of these notions yields a wider perspective when all these different structures are involved, and it allows us to apply results obtained in different categories.

In one of the first results in [35], we prove that for each tripotent u in a JB*-triple X, the set of those tripotents e in the 2-Peirce space $X_2(u) := \{x \in X : \{u,u,x\}_X = x\}$ such that $\|u \pm e\|_X \leq \sqrt{2}$ coincides with

$$\{i(p-q) : p, q \in \mathcal{P}(X_2(u)) \text{ with } p \perp q\},$$

where the symbol $\mathcal{P}(X_2(u))$ stands for the projections in the unital JB*-algebra $X_2(u)$ [35, Corollary 3.3].

One of the most successful tools in the theory of Jordan algebras is the Shirshov-Cohn theorem, which affirms that the JB*-subalgebra of a JB*-algebra generated by two symmetric elements (and possibly the unit element) is a JC*-algebra, that is, a JB*-subalgebra of some $B(H)$ (cf. [82, Theorem 7.2.5] and [166, Corollary 2.2]). In [35], we establish an appropriate version of the Shirshov-Cohn theorem, in which two orthogonal tripotents play the role of the symmetric elements.

Lemma R. [35, Lemma 3.6, Mediterr. J Math.] *Let u_1 and u_2 be two orthogonal tripotents in a unital JB*-algebra M. Then the JB*-subalgebra N of M generated by u_1, u_2 and the unit element is a JC*-algebra, that is, there exists a complex Hilbert space H satisfying that N is a JB*-subalgebra of $B(H)$, we can further assume that the unit of N coincides with the identity on H.* □

After some technical results inspired from prominent recent achievements by J. Hamhalter, O. F. K. Kalenda, H. Pfitzner, and A.M. Peralta in [81], we arrive to our main result in [35, Theorem 3.8].

Theorem S. [35, Theorem 3.8, Mediterr. J Math.] *Let u be an extreme point of the closed unit ball of a unital JB*-algebra M. Then the following statements are equivalent:*

(a) *u is a unitary tripotent;*

(b) *The set $\mathcal{M}_u = \{e \in \partial_e(\mathcal{B}_M) : \|u \pm e\| \leq \sqrt{2}\}$ contains an isolated point.*

□

Theorem Q becomes now a corollary for unital C*-algebras.

Unitaries in unital C*-algebras and JB*-algebras have been intensively studied. They constitute the central notion in the already mentioned Russo–Dye theorem [143] and its JB*-algebra-analogue in the Wright–Youngson–Russo–Dye theorem [167], which are outstanding results in the field of functional analysis.

Going back to the different versions of the isometric extension problem, and still concerned with unitaries, it is extremely remarkable in that sense a result due to O. Hatori and L. Molnár, in which it is proved that every surjective isometry $\Delta : \mathcal{U}(A) \to \mathcal{U}(B)$, where A and B are von Neumann algebras, admits an extension to a surjective real linear isometry between these algebras (see [86, Corollary 3]). That result is consequence of a more general statement (cf. [86, Theorem 1]).

Theorem T. [86, Theorem 1] *Let $\Delta : \mathcal{U}(A) \to \mathcal{U}(B)$ be a surjective isometry, where A and B are two unital C*-algebras. Then the identity*

$$\Delta(e^{iA_{sa}}) = e^{iB_{sa}}$$

*holds, and there is a central projection $p \in B$ and a Jordan *-isomorphism $J : A \to B$ satisfying*

$$\Delta(e^{ix}) = \Delta(1)\left(pJ(e^{ix}) + (1-p)J(e^{ix})^*\right), \quad (x \in A_{sa}).$$

Let us simply observe that the arguments in the proofs of O. Hatori and L. Molnár cannot be applied in the Jordan setting, even under the more favourable hypothesis of working with a JC*-algebra, that is, a norm-closed Jordan *-subalgebra of a C*-algebra.

One of the key ingredients in [86] is the use of one-parameter unitary groups, motivated by previous results on uniformly continuous group isomorphisms of unitary groups in AW*-factors due to Sakai (see [145]). Another fundamental tool is the well-known Stone's one-parameter theorem, which affirms that for each strongly continuous one-parameter unitary group $\{E(t) : t \in \mathbb{R}\}$ on a complex Hilbert space H there exists a self-adjoint operator $h \in B(H)$ such that $E(t) = e^{ith}$, for every $t \in \mathbb{R}$ (see [32, 5.6, Chapter X]). However, the set $\mathcal{U}(M)$ of all unitaries in a unital JB*-algeba M is not, in general, a group nor a subgroup of the unitary set of some $B(H)$. The set $\mathcal{U}(M)$ is not even stable under Jordan products. A full new machinery was needed to establish in [36] a Hatori-Molnár type theorem for Jordan algebras.

The foundation of our arguments in [36] in the setting of JB*-algebras mainly relies on two ideas. The first one is the opportunity, provided by the JB*-triple theory, of changing appropriately the Jordan product of a JB*-algebra with a new Jordan product given by each unitary element. Arguing with the new product, we can infer the conclusions through the immutable triple product to the original JB*-algebra structure. The second one is the excellent tool provided by the Shirshov-Cohn theorem.

The U-operators naturally arise in the theory of JB*-algebras. Let M be a JB*-algebra. Given $a, b \in M$, we shall write $U_{a,b} : M \to M$ for the bounded linear operator defined by

$$U_{a,b}(x) = (a \circ x) \circ b + (b \circ x) \circ a - (a \circ b) \circ x,$$

for all $x \in M$. We shall write U_a for $U_{a,a}$. It is known that given a unitary element u in M, the Banach space of M becomes a unital JB*-algebra with unit u for the (Jordan) product defined by $x \circ_u y := U_{x,y}(u^*) = \{x, u, y\}_X$ and the involution $*_u$ defined by $x^{*_u} := U_u(x^*) = \{u, x, u\}_M$. This JB*-algebra $M(u) = (M, \circ_u, *_u)$ is called the *u-isotope* of M. The theory of isotopes is strongly applied in the whole paper, and one more time, the results for JB*- and JBW*-triples simply our arguments.

One of the advantage of this approach relies on the *uniqueness* of the triple product. That is, a JB*-algebra may admit two different Jordan products compatible with the same norm, this is the case of all isotopes. However, when JB*-algebras are regarded as JB*-triples, any surjective linear isometry between them is a triple isomorphism (see [106, Proposition 5.5]). The following result, which is a Jordan version of the Stone's one-parameter theorem, is an example of the benefits of opening the perspective to triple techniques when it is possible. In that case, the results in [78] involving triple derivations and uniformly continuous unitary one-parameter groups on JB*-algebras, are fundamental in the proof.

Theorem U. [36, Theorem 3.1, Preprint 2020] *Let M be a unital JB*-algebra. Suppose $\{u(t) : t \in \mathbb{R}\}$ is a family in $\mathcal{U}(M)$ satisfying $u(0) = \mathbf{1}$, and $U_{u(t)}(u(s)) = u(2t+s)$, for all $t, s \in \mathbb{R}$. We also assume that the mapping $t \mapsto u(t)$ is continuous. Then there exists $h \in M_{sa}$ such that $u(t) = e^{ith}$ for all $t \in \mathbb{R}$.*

With all these tools, a first statement is established. We impose two sufficient conditions in order to obtain the desired result for JB*-algebras.

Theorem V. [36, Theorem 3.4, Preprint 2020] *Let $\Delta : \mathcal{U}(M) \to \mathcal{U}(N)$ be a surjective isometry, where M and N are two unital JB*-algebras. Suppose that one of the following holds:*

(1) $\|\mathbf{1}_N - \Delta(\mathbf{1}_M)\| < 2$;

(2) *There exists a unitary ω_0 in N such that $U_{\omega_0}(\Delta(\mathbf{1}_M)) = \mathbf{1}_N$.*

Then there exists a unitary ω in N satisfying

$$\Delta(e^{iM_{sa}}) = U_{\omega^*}(e^{iN_{sa}}).$$

*Furthermore, there exists a central projection $p \in N$ and a Jordan *-isomorphism $\Phi : M \to N$ such that*

$$\begin{aligned}\Delta(e^{ih}) &= U_{\omega^*}\left(p \circ \Phi(e^{ih})\right) + U_{\omega^*}\left((\mathbf{1}_N - p) \circ \Phi(e^{ih})^*\right) \\ &= P_2(U_{\omega^*}(p))U_{\omega^*}(\Phi(e^{ih})) + P_2(U_{\omega^*}(\mathbf{1}_N - p))U_{\omega^*}(\Phi((e^{ih})^*)),\end{aligned}$$

for all $h \in M_{sa}$. Consequently, the restriction $\Delta|_{e^{iM_{sa}}}$ admits a (unique) extension to a surjective real linear isometry from M onto N.

Let us observe that in the previous theorem condition (1) implies condition (2) (see [36, Remark 3.2]).

A nice consequence follows from the result above, and says that the Banach spaces underlying two unital JB*-algebras are isometrically isomorphic if and only if the metric spaces determined by the unitary sets of these algebras are isometric.

Corollary W. [36, Corollary 3.8, Preprint 2020] *The following statements are equivalent for any two unital JB*-algebras M and N:*

(*a*) *M and N are isometrically isomorphic as (complex) Banach spaces;*

(*b*) *M and N are isometrically isomorphic as real Banach spaces;*

(*c*) *There exists a surjective isometry $\Delta : \mathcal{U}(M) \to \mathcal{U}(N)$.*

Finally, we relax some of the hypotheses in Theorem V at the cost of considering surjective isometries between the unitary sets of two JBW*-algebras.

Theorem X. [36, Theorem 3.9, Preprint 2020] *Let $\Delta : \mathcal{U}(M) \to \mathcal{U}(N)$ be a surjective isometry, where M and N are two JBW*-algebras. Then there exist a unitary ω in N, a central projection $p \in N$, and a Jordan *-isomorphism $\Phi : M \to N$ such that*

$$\begin{aligned}\Delta(u) &= U_{\omega^*}\left(p \circ \Phi(u)\right) + U_{\omega^*}\left((\mathbf{1}_N - p) \circ \Phi(u)^*\right)\\ &= P_2(U_{\omega^*}(p))U_{\omega^*}(\Phi(u)) + P_2(U_{\omega^*}(\mathbf{1}_N - p))U_{\omega^*}(\Phi(u^*)),\end{aligned}$$

for all $u \in \mathcal{U}(M)$. Consequently, Δ admits a (unique) extension to a surjective real linear isometry from M onto N.

The Hatori-Molnár type theorem for JB*- and JBW*-algebras closes successfully a thesis based on isometric extension problems, and opens also the door to address new challenging questions which are waiting for an answer (Chapter 4).

We conclude by borrowing some words from a recommendation letter written by E. Oddel on support a nomination of Professor G.G. Ding to an award, which is quoted in paper by the latter in 2012, and refer to Tingley's problem: "This is a very difficult problem that remains unsolved after 25 years". Almost ten years latter, the only possible answer is the same,

"Wir müssen wissen, wir werden wissen" - D. Hilbert.

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Chapter 1

Preliminaries

The aim of this chapter is to provide a comprehensive background on the structures on which our work relies. As commented in the introduction, this memoir means a collection of all those new achievements obtained in the papers [33, 12, 34, 35] and [36]. Therefore, we shall primarily introduce the fundamental concepts and background related to associative algebras, Jordan structures and triple systems. It is our purpose to equip the reader through this initial step with the tools required to make a fruitful first incursion in the quoted papers. The subsequent chapters will contain additional specific notions involving the results each one of them is concerned with. We shall provide general references for each topic, as well as the sources of those statements which are significant in our goals. Certain proofs will be extremely occasionally included with the only intention of highlighting the arguments used in them. We shall be particularly interested in the proofs which show the equivalence of certain notions in the different structures involved during the whole work.

The use of the term *algebra* has been traditionally restricted to those vector spaces enjoying a product which is associative. This work is intended as an attempt to bring together different structures, not always associative (C*-algebras, JB*-algebras and JB*-triples), towards a main shared focus: addressing the general problem of extension of isometries. It will be shown that the mentioned settings actually coexist in several scenes, offering us an extremely wide variety of tools to work with as a consequence. The procedure followed in [33, 12, 34, 35] and [36], and hence during this entire memoir, has been precisely thought to take advantage of this fact. Since the associativity condition is not assumed in all the structures we shall manage, a common framework is introduced for this goal by considering an algebra as a vector space together with a product satisfying no more conditions beyond bilinearity. Therefore, the whole machinery exposed along these lines will evolve from a joint foundation.

Throughout these notes the symbol $\mathbb{K}$ will be used to denote indistinctly a field that is either the real field $\mathbb{R}$ or the complex field $\mathbb{C}$. We fix now some notation which will remain valid for the rest of this document. Let $(X, \|\cdot\|)$ be a normed space. The norm of X will be denoted by $\|\cdot\|_X$ when it is necessary to emphasise X. The closed unit ball of X will be denoted by $\mathcal{B}_X$. The symbol $S(X)$ will stand for the unit sphere of X. The rest of the notation will be introduced progressively, when necessary.

An *algebra* A over $\mathbb{K}$ is a vector space, over $\mathbb{K}$, together with a bilinear mapping

$$\begin{aligned} A \times A &\to A \\ (a, b) &\mapsto ab \end{aligned}$$

called product. The product in an algebra will be denoted by juxtaposition, unless any other particular symbol is specified.

We shall say that A is a *real algebra* whenever $\mathbb{K} = \mathbb{R}$, and if $\mathbb{K} = \mathbb{C}$, we shall refer to A as a *complex algebra.* We will occasionally use the word *algebra* as including both settings if no distinction between the real and complex cases is required, and so $\mathbb{K}$ could be omitted whenever this does not lead to error.

The field $\mathbb{K}$ over which the structure of an algebra is constructed is not a minor matter. In spite of the fact that sometimes the real case and the complex one are analogous, the nature of the scalars in the addressed vector space will lead our arguments through different paths. It is worthwhile to remind that any vector space X over $\mathbb{C}$ can be regarded as a real vector space when we consider the same set of vectors X but $\mathbb{R}$ as the field of scalars. The symbol $X_{\mathbb{R}}$ will stand for the quoted underlying real vector space. According to the last observations, throughout this project we shall conveniently identify any complex algebra A with an algebra over $\mathbb{R}$ via $A_{\mathbb{R}}$.

Let A be a real or complex algebra. A is said to be *associative* (respectively, *commutative*) if the equality $(ab)c = a(bc)$ (respectively, $ab = ba$) is satisfied for every $a, b, c \in A$. Additionally, A will be called *unital* if there exists an element $\mathbf{1}_A \in A$, known as the *unit* of A, such that $\mathbf{1}_A a = a\mathbf{1}_A = a$, for every $a \in A$. In such a case, the unit is unique. Note that if $A = \{0\}$, A is trivially unital. We shall frequently find situations in which it is advantageous to find a smaller algebra structure inside the main algebra. A *subalgebra* B of A is a vector subspace of A which is closed under the product in A, that is, B is such that $b_1 b_2 \in B$ for every $b_1, b_2 \in B$. Therefore, B can be regarded as an algebra itself when endowed with the restriction of the product of A.

Let A be an algebra over $\mathbb{K}$. Take $z \in A$. The powers of the element z in the algebra A will be written as follows:

$$\begin{aligned} x^1 &= x; \\ x^{n+1} &= xx^n, \quad n \geq 1. \end{aligned}$$

If A is unital, we set $x^0 = \mathbf{1}_A$.

The natural morphisms between algebras, which make possible the interaction of these objects, are those linear mappings preserving the considered products. Namely, let A and B be two algebras, and let $\varphi : A \to B$ be a linear mapping. We shall say that φ is a *homomorphism* from A to B whenever $\varphi(ab) = \varphi(a)\varphi(b)$, for every $a, b \in A$. It is said that φ is a *monomorphism* (respectively, an *epimorphism*) if φ is injective (respectively, surjective). If φ is bijective, it will be called *isomorphism*. Suppose now that A and B are unital. A homomorphism $\varphi : A \to B$ is called *unital* if φ maps the unit in A to the unit in B, that is, $\varphi(\mathbf{1}_A) = \mathbf{1}_B$.

1.1 C*-algebras

The exposed results can be found in any basic reference about Banach and C*-algebras as [144, 99, 152], or [130]. The theory of C*-algebras find its most remote origin probably in the contributions made by P. Jordan, J. von Neumann and E. Heissenberg in the framework of quantum mechanics. But it was at the hands of I.M. Gelfand and M.A. Naimark that a proper abstract characterisation of a C*-algebra was provided in 1943 (cf. [80]). Since then, the theory of C*-algebras has become the core of a vast number of research lines, and the foundation of another huge amount of theories.

Let A be a complex (respectively, real) associative algebra. We shall say that A is a *complex* (respectively, *real*) *normed algebra* if A is a normed space with a norm $\|\cdot\|_A$ satisfying $\|ab\|_A \leq \|a\|_A \|b\|_A$, for every $a, b \in A$. A *complex* (respectively, *real*) *Banach algebra* is a complex (respectively, real) normed algebra whose norm is complete. Roughly speaking, there are two structures, one of algebraic nature and another of analytic essence, co-living inside a real or complex normed algebra. The Banach condition makes compatible these two structures by supporting the harmony between them.

The following examples show that some of the classical spaces we are used to work with are actually Banach algebras.

Example 1.1.1. (1) *$\mathbb{R}$ and $\mathbb{C}$, equipped with the natural product, are commutative unital associative algebras over $\mathbb{R}$ and over $\mathbb{C}$, respectively. It is clear that $\mathbf{1}_{\mathbb{R}} = \mathbf{1}_{\mathbb{C}} = 1$. $\mathbb{R}$ is a real Banach algebra with the absolute value as a norm. Additionally, $\mathbb{C}$ is a complex Banach algebra when the modulus is considered. Of course, $\mathbb{C}$ can be regarded as a real Banach algebra via $\mathbb{C}_{\mathbb{R}}$.*

(2) *Let K be a compact Hausdorff space. The space $C(K, \mathbb{K})$, of all continuous functions defined on K which take values in $\mathbb{K}$, is a commutative associative algebra over $\mathbb{K}$ with the point-wise product, that is, given $f, g \in C(K, \mathbb{K})$, $(fg)(t) = f(t)g(t)$, for any $t \in K$. Furthermore, the constant one*

function $\mathbf{1} : K \to \mathbb{K}$ *given by* $\mathbf{1}(t) = 1$ $(t \in K)$, *acts as a unit, and hence* $C(K, \mathbb{K})$ *is unital.*

Let us now extend slightly the previous example. Let L *be a topological space. We recall that a function* $f : L \to \mathbb{K}$ *is said to vanish at infinity if for every* $\epsilon > 0$, *the set* $\{\omega \in L : |f(\omega)| \geq \epsilon\}$ *is compact. We write* $C_0(L, \mathbb{K})$ *for the set of all continuous functions from* L *to* $\mathbb{K}$ *vanishing at infinity. If* L *is a locally compact Hausdorff space,* $C_0(L, \mathbb{K})$ *becomes a commutative associative algebra over* $\mathbb{K}$ *with respect to the point-wise product. It is remarkable that* $C_0(L, \mathbb{K})$ *is a unital algebra if and only if* L *is compact, in which case it coincides with* $C(L, \mathbb{K})$.

Let K *be a compact Hausdorff space, and let* L *be a locally compact Hausdorff space.* $C(K, \mathbb{K})$ *and* $C_0(L, \mathbb{K})$ *are both Banach algebras over* $\mathbb{K}$ *sharing the same norm, namely, the supremum norm defined in the more general case by* $\|a\|_\infty = \sup\{|a(t)| : t \in L\}$, *for any* $a \in C_0(L, \mathbb{K})$.

From now on, we shall denote $C_0(L) := C_0(L, \mathbb{C})$, *and* $C(K) := C(K, \mathbb{C})$.

(3) *Consider the following sequence spaces:*

$$\ell_\infty := \left\{ \{x_n\}_n \in \mathbb{K}^{\mathbb{N}} : \{x_n\}_n \text{ is bounded } \right\};$$
$$c := \left\{ \{x_n\}_n \in \mathbb{K}^{\mathbb{N}} : \lim_{n \to +\infty} \{x_n\}_n \in \mathbb{K} \right\};$$
$$c_0 := \left\{ \{x_n\}_n \in \mathbb{K}^{\mathbb{N}} : \lim_{n \to +\infty} \{x_n\}_n = 0 \right\};$$
$$c_{00} := \left\{ \{x_n\}_n \in \mathbb{K}^{\mathbb{N}} : \{n \in \mathbb{N} : x_n \neq 0\} \text{ is finite } \right\}.$$

The spaces above are all commutative associative algebras over $\mathbb{K}$ *with respect to the same coordinate-wise product* $\{x_n\}_n\{y_n\}_n = \{x_n y_n\}_n$. *Moreover, as the inclusions* $c_{00} \subseteq c_0 \subseteq c \subseteq \ell_\infty$ *hold, we can consider each sequence space as a subalgebra of the subsequent algebras in which it is included. Let us focus our attention on* $\{\mathbf{1}_n\}_n$, *where* $\mathbf{1}_n = 1$ *for any* $n \in \mathbb{N}$. *The constant one sequence so defined makes* ℓ_∞ *and* c *unital algebras. On the contrary, if* c_0 *or* c_{00} *were unital, the unit element would be necessarily* $\{\mathbf{1}_n\}_n$. *However, it is evident that* $\{\mathbf{1}_n\}_n \notin c_0$.

The associative algebras ℓ_∞, c *and* c_0, *together with the norm* $\|\{a_n\}_n\|_\infty = \sup\{|a_n| : n \in \mathbb{N}\}$, *are Banach algebras over* $\mathbb{K}$. *We cannot affirm the same of* c_{00} *though. The space* c_{00} *is not even closed in* c_0 *nor complete (actually* $\overline{c_{00}} = c_0$*).*

The operator algebra is probably the standard example of a non-commutative algebra.

Example 1.1.2. *Let* X *and* Y *be two vector spaces over the same field, and consider the space* $\mathcal{L}(X, Y)$, *of all linear maps from* X *to* Y. $\mathcal{L}(X, X)$

is usually denoted by $\mathcal{L}(X)$*. Given* $T, S \in \mathcal{L}(X)$*, it is evident that we can always compose* T *and* S *to yield another linear mapping* $T \circ S$ *from* X *to itself. It is easily seen that* $\mathcal{L}(X)$ *has a structure of associative algebra with the composition as product. This algebra is not, in general, commutative.*

Suppose now that X *and* Y *are two normed spaces. It is customary to write* $B(X, Y)$ *for the space of all bounded linear operators from* X *into* Y *(the notation* $L(X, Y)$ *can be also found in the literature). We shall simply write* $B(X)$ *for* $B(X, X)$*. The usual algebraic operators, addition and scalar multiplication, are well defined in* $B(X)$*. Furthermore, the composition of operators defines an associative product in* $B(X)$*.Therefore,* $B(X)$ *can be regarded as an associative algebra over* $\mathbb{K}$*, and it is unital since the identity mapping* $I_X : X \to X$*, with* $I_X(x) = x$ *(*$x \in X$*), lies in* $B(X)$*, acting as a unit.*

Furthermore, $B(X)$ *is a Banach algebra whenever* X *is a Banach space. The norm involved is the operator norm:*

$$\|T\|_{B(X)} = \sup\{\|T(x)\| : x \in S(X)\}, \quad T \in B(X).$$

The subspace $K(X)$*, of all compact operators in* $B(X)$ *is a Banach subalgebra of* $B(X)$*. As we have already commented,* $B(X)$ *is always unital, while in contrast* $K(X)$ *has a unit just in the case in which* X *is finite dimensional.*

Definition 1.1.3. *Let* A *be a unital associative algebra. An element* $a \in A$ *is* invertible *if there exists* $b \in A$ *such that* $ab = ba = \mathbf{1}_A$*. In such a case, the element* b *is unique (it is called the* inverse *of* a *in* A*) and will be denoted by* a^{-1}*.*

We shall denote by $\mathrm{Inv}(A)$ the set of all invertible elements in a unital associative algebra A. The set $\mathrm{Inv}(A)$ is a multiplicative subgroup of A. We present some useful properties related to invertible elements.

Proposition 1.1.4. [99, Propositions 3.1.5 and 3.1.6],[152, Proposition 1.6, Corollary 1.8] *Let* A *be a unital Banach algebra, and let* $a, b \in A$*. Then:*

(i) *If* $\|a\| < 1$*, the element* $\mathbf{1}_A - a$ *is invertible in* A *with*

$$(\mathbf{1}_A - a)^{-1} = \sum_{n=0}^{\infty} a^n;$$

(ii) *If* $\|\mathbf{1}_A - a\| < 1$*, the element* a *lies in* $\mathrm{Inv}(A)$*;*

(iii) $\mathrm{Inv}(A)$ *is open in* A*, and the mapping* $a \mapsto a^{-1}$ *on* $\mathrm{Inv}(A)$ *is continuous.*

In the setting of associative algebras we can consider new products based on the original one. This is the case of the *Lie bracket* or *commutator*, a bilinear mapping $[\cdot,\cdot] : A \times A \to A$ defined over an associative algebra A as

$$[x,y] = xy - yx, \quad x,y \in A. \tag{1.1}$$

As a remarkable property, $[x,y] = -[y,x]$, for any $x,y \in A$. A *Lie algebra* is an algebra $\mathcal{L}$ whose product $[\cdot,\cdot] : \mathcal{L} \times \mathcal{L} \to \mathcal{L}$ satisfies the following two conditions:

$$\begin{aligned}&[x,x] = 0, \quad \forall x \in \mathcal{L},\\ &[x,[y,z]] + [z,[x,y]] + [y,[z,x]] = 0, \quad \forall x,y,z \in \mathcal{L},\end{aligned} \tag{1.2}$$

where (1.2) is known as *Jacobi identity*. It is clear that any associative algebra can be regarded as a Lie algebra when we consider the natural Lie product. As a consequence of the Poincaré-Birkhoff-Witt theorem, a converse statement can be affirmed, indeed, for every Lie algebra $\mathcal{L}$ there exists an associative algebra A such that $\mathcal{L}$ is isomorphic to a subalgebra of A when the latest is endowed with the Lie bracket.

Another fruitful example of a product in an associative algebra A derived from the associative one is the Jordan product. Let us consider the bilinear mapping $\circ : A \times A \to A$ given by

$$a \circ b := \frac{1}{2}(ab + ba), \quad a,b \in A. \tag{1.3}$$

When equipped with the above product, A becomes a commutative algebra, at the cost of losing, in general, its original associativity. We shall repeatedly refer to the product in (1.3) as the *Jordan product* in A.

It is not a waste of time noticing that any unital associative algebra keeps its unit when endowed with the product in (1.3). A routine exercise shows the following fact: let $\mathbf{1}_A$ denote the unit element in a unital associative algebra A, and take any $x \in A$. Then $\mathbf{1}_A \circ x = \frac{1}{2}(\mathbf{1}_A x + x\mathbf{1}_A) = \frac{1}{2}(x+x) = x$. Fortunately, the reciprocal is also true. That is, the existence of an element $\mathbf{1}_A$ in an associative algebra A acting as a unit for the abelian product implies the existence of a unit element in the associative sense, and moreover, both of them coincide. Indeed, for any $x \in A$ we have

$$\mathbf{1}_A x \mathbf{1}_A = 2(\mathbf{1}_A \circ x) \circ \mathbf{1}_A - (\mathbf{1}_A \circ \mathbf{1}_A) \circ x = x.$$

Therefore

$$\mathbf{1}_A x = \mathbf{1}_A(\mathbf{1}_A x \mathbf{1}_A) = \mathbf{1}_A^2 x \mathbf{1}_A = (\mathbf{1}_A \circ \mathbf{1}_A) x \mathbf{1}_A = \mathbf{1}_A x \mathbf{1}_A = x.$$

Analogously, $x\mathbf{1}_A = x$, and hence $\mathbf{1}_A x = x\mathbf{1}_A = x$, for every $x \in A$.

Along with these lines, whenever we consider an associative algebras the previous Jordan product, the definition of the Jordan morphisms experiences substantial changes. We shall say that a linear mapping $\varphi : A \to B$ is a *Jordan homomorphism* between two associative algebras A and B if it verifies

$$\varphi\left(\frac{1}{2}ab + \frac{1}{2}ba\right) = \frac{1}{2}\varphi(a)\varphi(b) + \frac{1}{2}\varphi(b)\varphi(b),$$

for every $a, b \in A$, equivalently, $\varphi(a \circ b) = \varphi(a) \circ \varphi(b)$, for every $a, b \in A$. It follows from the definition that any homomorphism between associative algebras is a Jordan homomorphism. The reciprocal is not in general true. By a *Jordan monomorphism* (respectively, *Jordan epimorphism*) we mean a Jordan homomorphism which is injective (respectively, surjective). Therefore, a *Jordan isomorphism* is a bijective Jordan homomorphism. If A and B are unital associative algebras, a Jordan homomorphism $\varphi : A \to B$ will be called *unital* whenever φ maps $\mathbf{1}_A$ to $\mathbf{1}_B$. Given an associative algebra A, any subalgebra of A with respect to the Jordan product (1.3) will be called a *Jordan subalgebra* of A.

We are ready to enrich the structures we are working with in order to arrive to our main goal, the notion of C*-algebra. Let A be a complex algebra. A conjugate-linear mapping $* : A \to A$ is called an *(algebra) involution* if the following conditions are satisfied:

$$(a^*)^* = a, \quad (ab)^* = b^*a^*, \quad \forall a, b \in A.$$

A complex Banach algebra endowed with an involution is called *Banach *-algebra*. For each element a in a Banach *-algebra A, we shall refer to the element a^* in A as the *adjoint* of a. An element $a \in A$ is called self-adjoint (or hermitian) if $a = a^*$. We write $A_{sa} := \{a \in A : a = a^*\}$, the set of all self-adjoint elements in A. The morphisms preserving the structure of Banach *-algebra are called **-homomorphisms*. Analogously, when the Jordan product (1.3) is considered in two Banach *-algebras A and B, we shall say that a Jordan homomorphism $\varphi : A \to B$ is a *Jordan *-homomorphism* if φ satisfies $\varphi(a^*) = \varphi(a)^*$, for every $a \in A$.

Definition 1.1.5. *A C*-algebra is a Banach *-algebra A satisfying*

$$\|aa^*\|_A = \|a\|_A^2, \quad \textit{for every } a \in A. \tag{1.4}$$

The C*-condition expressed in (1.4) is known as the *Gelfand-Naimark axiom*. It is worth noting that the Gelfand-Naimark axiom combines the three key ingredients of a C*-algebra, namely, the product, the norm, and the involution. In [80], I.L. Gelfand and M.A. Naimark assumed an extra condition in Definition 1.1.5 when they introduced the abstract concept of C*-algebra, namely, $\mathbf{1}_A + a^*a$ is invertible for every a in A. They conjectured that this condition could be removed with no effects, and in fact M. Fukamiya ([77]), J.L. Kelley and R.L. Vaught ([108]), and I. Kaplansky proved that this assumption is superfluous.

Most of the examples reviewed in Example 1.1.1 and Example 1.1.2 are actually C*-algebras.

Example 1.1.6. (1) *The complex field is a C*-algebra with the conjugation as involution, since* $|zz^*| = |z\bar{z}| = |z^2|$*, for every* $z \in \mathbb{C}$.

(2) *Let* L *be a locally compact Hausdorff space. The complex Banach algebra* $C_0(L)$ *is a C*-algebra with involution* $^* : C_0(L) \to C_0(L)$ *given by the assignment* $f \mapsto f^*$*, where* $f^*(t) = \overline{f(t)}$*. We shall see that this is actually the prototype of commutative C*-algebra.*

(3) *Let* H *be a complex Hilbert space, and consider the complex Banach algebra* $B(H)$*. It is well known that for each* $T \in B(H)$ *there exists a unique* $T^* \in B(H)$ *such that* $\langle T(x)|y\rangle_H = \langle x|T^*(y)\rangle_H$*, for every* $x, y \in H$*. The mapping* $^* : B(H) \to B(H)$*,* $T \mapsto T^*$ *is an involution on* $B(H)$*. Since the Gelfand-Naimark axiom is satisfied by this involution, the product given by the composition, and the operator norm, we can conclude that* $B(H)$ *is a C*-algebra.*

It can be derived from the Gelfand-Naimark axiom (1.4) that the involution in a C*-algebra A is an isometry, and hence $\|a^*\|_A = \|a\|_A$, for all $a \in A$. Moreover, if A is a unital C*-algebra, we have that $\|\mathbf{1}_A\|_A = 1$, and $\mathbf{1}_A^* = \mathbf{1}_A$.

Let A be a Banach *-algebra. The space $\mathbb{C}1 \oplus A$ can be equipped with the structure of *-algebra with the product given by

$$(\lambda 1 + a)(\mu 1 + b) = \lambda\mu 1 + \lambda b + \mu a + ab, \tag{1.5}$$

and the involution

$$(\lambda 1 + a)^* = \bar{\lambda} 1 + a^*, \tag{1.6}$$

for each $\lambda, \mu \in \mathbb{C}$, and each $a, b \in A$. The following proposition guarantees the existence of a C*-algebra norm in $\mathbb{C}1 \oplus A$.

Proposition 1.1.7. [144, Yood theorem, Proposition 1.1.7] *Let* A *be a C*-algebra without unit, and let* $\tilde{A} := \mathbb{C}1 \oplus A$ *with the product in* (1.5)*, and the involution in* (1.6)*. Then* $\tilde{A}$ *is a unital C*-algebra with the norm defined by*

$$\|\lambda 1 + a\| = \sup\{\frac{\|\lambda y + ay\|}{\|y\|} : \|y\| \neq 0\},$$

for each $a \in A$*, and each* $\lambda \in \mathbb{C}$.

Therefore, whenever we consider a C*-algebra A which is not unital, $\tilde{A}$ will denote the unital C*-algebra obtained by the above procedure, and be called *unitization* of A.

Let A be a unital complex Banach algebra. For each $a \in A$, the set

$$\sigma_A(a) := \{\lambda \in \mathbb{C} : a - \lambda \mathbf{1}_A \notin \mathrm{Inv}(A)\}$$

is called the *spectrum* of a in A. Let us suppose that A is a C*-algebra without unit, then the spectrum of an element a in A will be the spectrum of a in the unitization $\tilde{A}$. That is, for each $a \in A$, $\sigma_A(a) := \sigma_{\tilde{A}}(a)$.

Theorem 1.1.8. [152, Gelfand theorem, Proposition I.2.3 and Theorem I.2.5] *Let A be a unital complex Banach algebra. Then the spectrum of any element of A is a non-empty compact set.*

Let A be a C*-algebra, and consider an element a in A. We shall say that a is:

(i) *normal* if $a^*a = aa^*$. Clearly, any self-adjoint element is normal;

(ii) a *projection* if a is a self-adjoint idempotent, that is, $a^2 = a = a^*$. We shall write $\mathcal{P}(A)$ for the set of all projections in A.

(iii) a *partial isometry* if aa^* is a projection (equivalently, a^*a is a projection). Actually, a is a partial isometry if and only if $aa^*a = a$;

(iv) *positive* if $a \in A_{sa}$ and $\sigma_A(a) \subset \mathbb{R}_0^+$. A result due to I. Kaplansky assures that $a \in A$ is positive if and only if $a = x^*x$, for some $x \in A$ ([144, Theorem 1.4.4]).

(v) a *symmetry* if $a \in A_{sa}$ and $a^2 = \mathbf{1}_A$, provided that A is unital. The set of all symmetries in a unital C*-algebra A will be denoted by $\mathrm{Symm}(A)$;

(vi) *central* if a commutes with any other element in A.

Let A be a commutative C*-algebra. The *spectrum* of A is the set of all non-zero homomorphisms from A onto the complex field. It will be denoted by $\Omega(A)$. It is known that $\Omega(A)$ is a subset of $\mathcal{B}_{A^*}$ ([152, Proposition I.3.9]). The set $\Omega(A)$ is locally weak*-compact, and if A is assumed to be unital, $\Omega(A)$ is weak*-compact ([152, Proposition I. 3.10]).

Theorem 1.1.9. [144, Commutative Gelfand-Naimark theorem, Theorem 1.2.1 and Corollary 1.2.2] *Let A be a commutative C*-algebra. Then A is isometrically *-isomorphic to the C*-algebra $C_0(\Omega(A))$, where $\Omega(A)$ is the spectrum of A. If A is unital, A is isometrically *-isomorphic to $C(\Omega(A))$.*

Let H be a complex Hilbert space. A significant observation comes from realising that any self-adjoint (i.e. $T^* \in A$ for every $T \in A$) norm-closed subalgebra A of $B(H)$ is a C*-algebra. The reciprocal statement has its place in history.

Theorem 1.1.10. [144, Gelfand-Naimark theorem, Theorem 1.16.6] *Every C*-algebra is isometrically *-isomorphic to a self-adjoint norm-closed subalgebra of $B(H)$, for some complex Hilbert space H.*

The following theorem shows the local theory of C*-algebras.

Theorem 1.1.11. [144, Corollary 1.2.3] *Let a be a normal element in a C*-algebra A. Then the C*-subalgebra of A generated by a, that is, the smallest C*-subalgebra of A containing a, is isometrically *-isomorphic to $C_0(\sigma_A(a) \cup \{0\})$. If A is unital, the C*-subalgebra of A generated by a and the unit, that is, the smallest C*-subalgebra of A containing a and $\mathbf{1}_A$, is isometrically *-isomorphic to $C(\sigma_A(a) \cup \{0\})$.*

A *von Neumann algebra* is a C*-algebra which is also a dual Banach space. It is known, by a celebrated result due to S. Sakai, that every von Neumann algebra has a unique (isometric) predual, its involution is weak*-continuous, and its product is separately weak*-continuous (cf. [144, §1.7]).

We shall be interested in the facial structure of certain Banach spaces in Chapter 2. Let us introduce the notion of face and extreme point. Let X be a vector space, and consider a convex subset C of X. A non-empty convex subset F of C is said to be a *face* of C if $\alpha x + (1-\alpha)y \in F$ with $x, y \in C$ and $0 < \alpha < 1$, implies $x, y \in F$. A face of C is said to be *proper* if it differs from C and $\emptyset$. An element x in C such that $\{x\}$ is a face of C is called an *extreme point* of C. The symbol $\partial_e(C)$ stands for the set of all extreme points of a convex set C. We will be mainly interested in the set of extreme points of the closed unit ball of a Banach space.

The following proposition, due to R.V. Kadison, is one of the most celebrated results in the theory of C*-algebras.

Theorem 1.1.12. ([97, Theorem 1], [144, Proposition 1.6.1 and Theorem 1.6.4]) *Let A be a C*-algebra. Then the closed unit ball of A has an extreme point if and only if A is unital. Then the extreme points of the closed unit ball of A are precisely the maximal partial isometries in A, that is, those elements u of A satisfying*

$$(\mathbf{1}_A - uu^*)A(\mathbf{1}_A - u^*u) = \{0\}.$$

On the other hand an element u in a unital C*-algebra A is called *unitary* if u is invertible in A with inverse u^*, that is, if $uu^* = u^*u = \mathbf{1}_A$. Every unitary in A is an extreme point of its closed unit ball, but the reciprocal implication is not always true, consider, for example, the right shift operator in $B(H)$, where H is an infinite dimensional complex Hilbert space. We shall explore these inclusions in section 2.2.

R.R. Phelps showed in [141] that the closed unit ball of the commutative unital C*-algebra $C(K)$, where K is a compact Hausdorff space, coincides with the closed convex hull of its extreme points. Since the extreme points of the closed unit ball of $C(K)$ are precisely the unitary elements in $C(K)$, R.R. Phelps provided in fact a particular case of the celebrated Russo–Dye theorem ([143, Theorem I.8.4]), which states that the closed unit ball of any unital C*-algebra agrees with the closed convex hull of its unitary elements.

Concerning the real setting, a *real C*-algebra* is defined as a real norm-closed self-adjoint subalgebra of a C*-algebra (cf. [111]). Real C*-algebras can be also obtained as *real forms* of C*-algebras, that is, given a real C*-algebra A there exists a unique (complex) C*-algebra structure on its algebraic complexification $B = A \oplus iA$, and a conjugation (i.e. a conjugate-linear isometry of period 2) τ on B such that $A = B^\tau = \{b \in B : \tau(b) = b\}$, ([111, Proposition 5.1.3]).

A real C*-algebra which is a dual Banach space will be called *real von Neumann algebra*. A real version of the Russo–Dye theorem in the setting of real von Neumann algebras follows from a result due to B. Li (see [111, Theorem 7.2.4]). An explicit statement is provided by J.C. Navarro and M.A. Navarro in [129, Corollary 6], in which it is asserted that the open unit ball of a real von Neumann algebra A is contained in the sequentially convex hull of the set of unitary elements in A. We shall focus on these real structures in the more general setting of real JB*-triples.

We finish this section by stating a proposition, which can be found in our paper [35], involving the relationship between the notions of central projection and isolated symmetry. The implication $(a) \Rightarrow (b)$ is proved in [125, Proof of Lemma 3.1], and in case of von Neumann algebras, the equivalence of (a) and (b) was proved by Y. Kato in [102].

Proposition 1.1.13. [35, Proposition 2.1, Mediterr. J. Math.]*Let p be a projection in a unital C*-algebra A. Then the following statements are equivalent:*

(a) *p is (norm) isolated in $\mathcal{P}(A)$;*

(b) *p is a central projection in A;*

(c) *$1 - 2p$ is (norm) isolated in Symm(A).* □

The space of complex-valued continuous functions

Let K be a compact Hausdorff space. We briefly draw the attention of the reader to the special example of C*-algebra given by the space $C(K)$ of all complex-valued continuous functions on K, equipped with the supremum norm. We will be interested in some of its geometric properties when we try to solve the isometric extension problem for commutative C*-algebras in section 3.2.1.

By the commutative Gelfand-Naimark theorem (see Theorem 1.1.9), the space $C(K)$ is the prototype of unital commutative C*-algebra. By assuming additional conditions over K, we enrich the structure of $C(K)$, and build a link with the space of complex-valued measurable essentially bounded functions. In order to show that link, we recall that K is called *Stonean* if the closure of every open set in K is open, and a Stonean space K is said to be *hyper-Stonean* if it admits a faithful family of positive normal measures (cf. [152, Definition 1.14]).

It is known that if K is a Stonean space, then every element a in the C*-algebra $C(K)$ can be uniformly approximated by finite linear combinations of projections (see [144, Proposition 1.3.1]). On the other hand, the C*-algebra $C(K)$ is a dual Banach space (equivalently, a von Neumann algebra) if and only if K is hyper-Stonean (cf. [50]).

Following standard terminology, a *localizable measure space* (Ω, ν) is a measure space which can be obtained as a direct sum of finite measure spaces $\{(\Omega_i, \mu_i) : i \in \mathcal{I}\}$. The Banach space $L^\infty(\Omega, \nu)$ of all locally ν-measurable essentially bounded functions on Ω is a dual Banach space and a commutative von Neumann algebra. Actually, every commutative von Neumann algebra is C*-isomorphic and isometric to some $L^\infty(\Omega, \nu)$ for some localizable measure space (Ω, ν) (see [144, Proposition 1.18.1]). From the point of view of Functional Analysis, the commutative von Neumann algebras $L^\infty(\Omega, \nu)$ and $C(K)$ with K hyper-Stonean are isometrically equivalent. Therefore, all those results stated for $C(K)$ with K compact Hausdorff space hold true for any commutative von Neumann algebra.

On the other hand, we shall see the importance of having control on the faces of the closed unit ball of the Banach space we are working with. For each $t_0 \in K$ and each $\lambda \in \mathbb{T}$ we set

$$A(t_0, \lambda) := \{f \in S(C(K)) : f(t_0) = \lambda\}, \tag{1.7}$$

where $\mathbb{T}$ denotes the unit sphere of $\mathbb{C}$. Then, it is well known, and easy to see, that $A(t_0, \lambda)$ is a maximal norm-closed proper face of $\mathcal{B}_{C(K)}$ and a maximal convex subset of $S(C(K))$.

1.2 JB*-algebras

In [116], K. McCrimmon makes a review of the historical development of Jordan algebras. The author distinguishes between a first stage, where the Jordan algebras are born from a physical motivation, and a second stage in which these structures gain mathematical interest. To be precise, the origins of the Jordan algebras can be found in the paper [95], dated in 1934, and written by P. Jordan, J. von Neumann and E. Wigner. The appearance of the work [93] in 1966 means a new approach to the theory through the quadratic U-operators, which, according to K. McCrimmon, *reveal the essential algebraic properties of Jordan algebras much more clearly than the linear approach via the multiplication operators.* The references guiding our results will be [82, 25, 147].

A *Jordan algebra* M over $\mathbb{K}$ is an algebra (over $\mathbb{K}$) whose product $\circ : M \times M \to M$ is abelian, and satisfies the following axiom

$$(a \circ b) \circ a^2 = a \circ (b \circ a^2), \tag{1.8}$$

for every $a, b \in M$. The identity (1.8) is known as the *Jordan identity*, and far from being an associativity condition, it expresses somehow a weak form of it. If $\mathbb{K} = \mathbb{C}$, we shall say that M is a *complex Jordan algebra*. On the other hand, whenever $\mathbb{K} = \mathbb{R}$, we shall refer to M as a *real Jordan algebra*. Following the terminology employed in Section 1.1, the term *Jordan algebra* will be used when there is no need to distinguish between the real and the complex settings. The omission of $\mathbb{K}$ in these cases will be a matter of briefless. The product $\circ$ in a Jordan algebra is usually called the *Jordan product*. Observe that in Jordan algebras the associativity is not necessarily assumed. Actually, this handicap will be crucial in the development of our arguments when treating with Jordan structures. By a *Jordan subalgebra* of a Jordan algebra we mean merely a subalgebra respect to the Jordan product. Any Jordan subalgebra of a Jordan algebra is a Jordan algebra via the induced Jordan product.

The abelian product defined in (1.3) for associative algebras provides us with the first example of Jordan algebra.

Example 1.2.1. *Every real or complex associative algebra A is a real Jordan algebra when endowed with the abelian product*

$$a \circ b := \frac{1}{2}(ab + ba), \quad a, b \in A.$$

Looking at the previous example, it may be instantly understandable that the natural morphisms for Jordan algebras are the Jordan homomorphisms. Actually, the terminology used for this kind of mappings in the setting of associative algebras makes sense now for these more general structures. Let us recall that a linear mapping $\varphi : M \to N$ between two Jordan algebras M and N is a Jordan homomorphism if $\varphi(a \circ b) = \varphi(a) \circ \varphi(b)$, for every $a, b \in A$. The mapping φ is a Jordan isomorphism (respectively, a Jordan monomorphism or a Jordan epimorphism) if it is a bijective Jordan homomorphism (respectively, an injective or surjective Jordan homomorphism).

A *complex* (respectively, *real*) *normed Jordan algebra* is a complex (respectively, real) Jordan algebra M equipped with a norm, $\|\cdot\|_M$, satisfying $\|a \circ b\|_M \leq \|a\|_M \, \|b\|_M$, for every $a, b \in M$. If the norm is complete, we shall say that M is a *complex* (respectively, *real*) *Jordan Banach algebra*.

Example 1.2.2. *Every associative Banach algebra over $\mathbb{K}$ is a Jordan Banach algebra over $\mathbb{K}$ with respect to the product defined in* (1.3) *and the original norm. In particular, every complex associative Banach algebra is a real Jordan Banach algebra with the quoted product and norm.*

Let M be a Jordan Banach algebra. The *Jordan multiplication operator* by an element $a \in M$ will be denoted by M_a, that is, $M_a(b) = b \circ a$ ($b \in M$). Given two elements a and b in M, and following the longstanding notation (see for instance [92, 82, 116] or [25]), we shall frequently consider the bounded linear operator $U_{a,b} : M \to M$ defined as follows:

$$U_{a,b}(x) = (a \circ x) \circ b - (a \circ b) \circ x + (x \circ b) \circ a, \quad x \in M. \tag{1.9}$$

The mapping $U_{a,a}$ will be simply denoted by U_a. With the aim of simplifying some computations, it is worth noting that, given $a, b \in M$, $U_{a,b}$ and M_a can be connected through the expression

$$U_{a,b} = M_a M_b + M_b M_a - M_{a \circ b},$$

and in particular

$$U_a = 2\,(M_a)^2 - M_{a^2},$$

([82, 2.4.16]).

The Jordan identity (1.8) in a Jordan algebra M can be expressed in terms of the commutator of Jordan multiplication operators, we observe that the latter belong to the algebra $\mathcal{L}(M)$. Let us take an arbitrary element a in M, then the Jordan identity is equivalent to

$$[M_a, M_{a^2}] = 0. \tag{1.10}$$

Indeed, take any $b \in M$. In this case we have

$$\begin{aligned} [M_a, M_{a^2}](b) &= (M_a M_{a^2} - M_{a^2} M_a)(b) \\ &= \left(b \circ a^2\right) \circ a - (b \circ a) \circ a^2 \\ &= a \circ \left(b \circ a^2\right) - (a \circ b) \circ a^2 = 0. \end{aligned}$$

A first and easy application of (1.10) is to prove that, given an element a in a Jordan algebra M, $[M_a, U_a] = 0$. Namely, it follows from the linearity of the commutator that

$$[M_a, U_a] = [M_a, 2\,(M_a)^2 - M_{a^2}] = 2[M_a, (M_a)^2] - [M_a, M_{a^2}] = 0.$$

The behaviour of U_a, for a given $a \in M$, over products of the type $a \circ b$, for any $b \in M$, can be deduced from this fact. Namely,

$$0 = [M_a, U_a](b) = M_a(U_a(b)) - U_a(M_a(b)) = U_a(b) \circ a - U_a(b \circ a),$$

and hence $U_a(b \circ a) = U_a(b) \circ a$, for every b in M.

The equivalence expressed in (1.10) can be developed and, through linearization, it is proved that the following equalities hold in any Jordan algebra M:

$$[M_{x^2}, M_y] = [M_{y^2}, M_x] = 0, \quad \forall x, y \in M \tag{1.11}$$

(cf. [147]). In terms of the Jordan product, we have the following generalised Jordan identity:

$$(y \circ z) \circ x^2 = y \circ (z \circ x^2), \quad \forall x, y, z \in M. \tag{1.12}$$

An important step forward an analogy with associativity in Jordan algebras is the *power associativity*. An algebra is called *power associative* if the subalgebras generated by single elements are associative. If M is a Jordan algebra, the power associativity is equivalent to say that the following identities hold for all $a \in M$,

$$a^m \circ a^n = a^{m+n}, \quad m, n \in \mathbb{N}.$$

Any Jordan algebra is power associative ([82, Lemma 2.4.5]).

Seeking associativity inside Jordan structures leads us naturally to the special Jordan algebras. A Jordan algebra M is said to be *special* if there exists an associative algebra A containing M as a Jordan subalgebra. The definition of special Jordan algebras responds somehow to a partial reciprocal statement for the Example 1.2.2, considering that any associative algebra can be regarded as a special Jordan algebra of itself. That fact agrees with the name used in [82] to call the Jordan product in (1.3)) as the *special Jordan product*. However, there are Jordan algebras which are not special and are called *exceptional* (see [82, Corollary 2.8.5], [25, Example 3.1.56]). Macdonald's theorem ([113]), one of the deepest results in Jordan theory, overcomes this fact and takes advantage of the special Jordan algebras. It essentially states that any polynomial identity in three variables which is linear in one of them and holds in any special Jordan algebra, actually holds in any Jordan algebra ([147, §3.3, Corollary 2] or [82, 2.4.13]).

In the light of the quoted result, trying to find special Jordan algebras inside a Jordan structure becomes an attractive objective. Some years before I.G. Macdonald stated the theorem named after him, in 1956, A.I. Shirshov and P.M. Cohn opportunely proved that any Jordan algebra generated by two elements, and the unit, is special (see [146], and [31]). The potential of this result, known as the *Shirshov-Cohn theorem*, is undeniable. Among its applications, for instance, we point out that it makes possible to prove the validity of any identity involving two elements in a Jordan algebra, by just verifying it in any special Jordan algebra (cf. [25, Theorem 3.1.55], [82, 2.4.14] or [147, §3.3]).

MacDonald's theorem provides one of the cardinal identities in Jordan algebras, which is known in the literature as *MacDonald's identity*, and it reads as follows: let M be a Jordan algebra, then the following equality holds

$$U_{U_y(x)} = U_y U_x U_y, \tag{1.13}$$

for every $x, y \in M$ (cf. [82, 2.4.18], [147, §3.3, (48)]).

We are aimed to work with the concept of invertibility in Jordan structures. Therefore, it is necessary to distinguish those Jordan algebras which are provided with a unit. As in the associative case, a Jordan homomorphism $\varphi : M \to N$ between unital Jordan algebras M and N will be called unital whenever $\varphi(\mathbf{1}_M) = \mathbf{1}_N$ holds.

Definition 1.2.3. *Let M be a unital Jordan algebra. An element $a \in M$ is* Jordan invertible *if there exists $b \in M$ such that*

$$a \circ b = \mathbf{1}_M, \quad \text{and} \quad a^2 \circ b = a.$$

*In such a case, the element b is unique (called the (*Jordan*)* inverse *of a in M), and it will be denoted by a^{-1} (cf.* [82, 3.2.9] and [25, Definition 4.1.2]*).*

The following lemma is borrowed from [147]. The proof is included to highlight the key role of the U-operators in Jordan algebras, as well as the efficiency of Macdonald's identity.

Lemma 1.2.4. [147, §14.2, Lemma 4] *Let M be a unital Jordan algebra. The following statements are equivalent for any $a, b \in M$:*

(i) a is Jordan invertible with (Jordan) inverse b;

(ii) $U_a(b) = a$, and $U_a(b^2) = \mathbf{1}_M$;

(iii) b is Jordan invertible with (Jordan) inverse a.

Proof. $(i) \Rightarrow (ii)$ Let us suppose that a is Jordan invertible and b is the Jordan inverse of a, that is, $a \circ b = \mathbf{1}_M$ and $a^2 \circ b = a$. We can easily deduce that

$$\begin{aligned} U_a(b) &= 2\,(M_a)^2\,(b) - M_{a^2}(b) \\ &= 2(b \circ a) \circ a - b \circ a^2 \\ &= 2a - a = a. \end{aligned}$$

Making use of (1.12), we have

$$\begin{aligned} U_a(b^2) &= 2\,(M_a)^2\,(b^2) - M_{a^2}(b^2) \\ &= 2(b^2 \circ a) \circ a - b^2 \circ a^2 \\ &= 2(a \circ a) \circ b^2 - b^2 \circ a^2 \\ &= a^2 \circ b^2 = a^2 \circ (b \circ b) \\ &= (a^2 \circ b) \circ b = a \circ b = \mathbf{1}_M. \end{aligned}$$

$(ii) \Rightarrow (iii)$ Applying Macdonald's identity (1.13) to b^2 and a, we have $U_{U_a(b^2)} = U_a U_{b^2} U_a = U_{\mathbf{1}_M}$, that is, $U_{U_a(b^2)} : M \to M$ is the identity operator. As a consequence, U_a is invertible (as well as U_{b^2}), and hence, bijective. On the other hand, it can be deduced that

$$U_a(\mathbf{1}_M - b \circ a) = U_a(\mathbf{1}_M) - U_a(b \circ a) = a^2 - U_a(b) \circ a = a^2 - a \circ a = 0,$$

$$U_a(b - b^2 \circ a) = U_a(b) - U_a(b^2 \circ a) = a - U_a(b^2) \circ a = a - \mathbf{1}_M \circ a = 0.$$

By injectivity of U_a, the equalities $\mathbf{1}_M - b \circ a = 0$ and $b - b^2 \circ a = 0$ hold, that is, b is Jordan invertible and its Jordan inverse coincides with a. This concludes the proof since the arguments for the implications $(iii) \Rightarrow (ii) \Rightarrow (i)$ are exactly the same but replacing the roles of a and b each other. □

The previous lemma guarantees, beyond the definition, the uniqueness of the Jordan inverse, in case it exists.

Let A be an associative algebra and let a be a fixed element in A. The operator $U_a : A \to A$ defined in (1.9) has an elegant expression for each $x \in A$ from which we shall profit, indeed, $U_a(x) = axa$.

Proposition 1.2.5. [147, §14.2, Proposition 2] *Let A be a unital associative algebra, and let $a \in A$. The following statements are equivalent:*

i) a is invertible in A;

ii) a is Jordan invertible in A when the latter is regarded as a Jordan algebra.

If any of the previous statements is satisfied, the inverse of a coincides in both products, the associative and the Jordan one.

Proof. To prove $i) \Rightarrow ii)$ let us suppose that a is invertible in the associative-algebra sense, that is, there exists $a^{-1} \in A$ such that $aa^{-1} = a^{-1}a = \mathbf{1}_A$. By considering the product defined in (1.3), A can be regarded as a unital Jordan algebra with unit $\mathbf{1}_A$. Thus, we have $a \circ a^{-1} = \frac{1}{2}(aa^{-1} + a^{-1}a) = \frac{1}{2}(\mathbf{1}_A + \mathbf{1}_A) = \mathbf{1}_A$. The associativity of A makes possible to argue as follows:

$$\begin{aligned} a^2 \circ a^{-1} = (aa) \circ a^{-1} &= \frac{1}{2}\left((aa)a^{-1} + a^{-1}(aa)\right) = \frac{1}{2}\left(a(aa^{-1}) + (a^{-1}a)a\right) \\ &= \frac{1}{2}\left(a\mathbf{1}_A + \mathbf{1}_A a\right) = \frac{1}{2}(a + a) = a. \end{aligned}$$

Therefore, a is Jordan invertible in A with a^{-1} as Jordan inverse.

$ii) \Rightarrow i)$ If a is Jordan invertible in A (when equipped with the product in (1.3)) with Jordan inverse a^{-1}, we have $a \circ a^{-1} = \mathbf{1}_A$, and $a^2 \circ a^{-1} = a$ by definition. In terms of the associative product that means

$$\frac{1}{2}aa^{-1} + \frac{1}{2}a^{-1}a = \mathbf{1}_A, \quad \text{and} \quad \frac{1}{2}aaa^{-1} + \frac{1}{2}a^{-1}aa = a.$$

On the other hand, by Lemma 1.2.4 (ii), the identities $U_a(a^{-1}) = a$ and $U_a\left((a^{-1})^2\right) = \mathbf{1}_A$ hold.

Applying associativity and the facts exposed above, it can be derived

$$\begin{aligned}
\mathbf{1}_A &= U_a\left((a^{-1})^2\right) = a(a^{-1})^2 a = (aa^{-1})(a^{-1}a) \\
&= (aa^{-1})\left(aa^{-1} - aa^{-1} + a^{-1}a\right) = (aa^{-1})\left(-aa^{-1} + 2a \circ a^{-1}\right) \\
&= (aa^{-1})\left(-aa^{-1} + 2\mathbf{1}_A\right) = -(aa^{-1})(aa^{-1}) + 2aa^{-1} \\
&= -U_a(a^{-1})a^{-1} + 2aa^{-1} = -aa^{-1} + 2aa^{-1} = aa^{-1}.
\end{aligned}$$

An identical argument gives $\mathbf{1}_A = a^{-1}a$, witnessing that a is invertible in the associative sense and a^{-1} is its inverse in A. □

Proposition 1.2.5 allows us to denote by $\mathrm{Inv}(M)$ the set of all Jordan invertible elements in a unital Jordan algebra without any abuse of notation. We collect now a full list of interesting properties involving Jordan invertible elements and U-operators.

Proposition 1.2.6. [25, Theorem 4.1.3] *Let M be a unital Jordan algebra, and let $a, b \in M$.*

(i) *a is Jordan invertible in M if and only if U_a is invertible (and hence bijective) in $\mathcal{L}(M)$;*

(ii) *If a is Jordan invertible in M, $U_{a^{-1}} = U_a^{-1}$, and $M_{a^{-1}} = U_a^{-1} M_a$;*

(iii) *The elements a and b are Jordan invertibles in M if and only if $U_a(b)$ is Jordan invertible in M;*

(iv) *If a is Jordan invertible in M, the equality $(a \circ x) \circ a^{-1} = a \circ (x \circ a^{-1})$ holds for every x in M.*

Additional properties can be deduced when a norm is considered in the Jordan structure.

Proposition 1.2.7. [25, Theorem 4.1.6, 4.1.7 and 4.1.15] *Let M be a unital Jordan Banach algebra, and let $a, b \in M$.*

(i) *$a \in \mathrm{Inv}(M)$ if and only if $U_a \in \mathrm{Inv}(B(M))$;*

(ii) *If $\|a\|_M < 1$, the element $\mathbf{1}_M - a$ is invertible in M with*

$$(\mathbf{1}_M - a)^{-1} = \sum_{n=0}^{\infty} a^n.$$

Moreover, $(\mathbf{1}_M - a)^{-1}$ lies in the Jordan subalgebra generated by $\mathbf{1}_M$ and a;

(iii) Inv(M) *is open in* M*;*

(iv) *The mapping* $a \mapsto a^{-1}$ *is continuous on* Inv(M)*.*

In [115], K. McCrimmon extends Macdonald's theorem and obtains a variant of it which involves two invertible generators and their inverses. As a consequence of this result, a Shirshov-Cohn theorem with inverses, which says that any Jordan algebra generated by two Jordan invertible elements and its inverses is special, is stated.

A *JB-algebra* is a real Jordan Banach algebra J in which the norm satisfies the following two axioms for all $a, b \in J$:

(i) $\|a^2\| = \|a\|^2$;

(ii) $\|a^2\| \leq \|a^2 + b^2\|$.

These structures were studied by E.M. Alfsen, F.W. Shultz and E. Størmer in [6], the monograph [82] contains all basic results and references on JB-algebras.

A Jordan Banach *-algebra M is a complex Jordan Banach algebra endowed with a continuous involution $^* : M \to M$. Analogously to the notation in the setting of C*-algebras, we shall say that a Jordan homomorphism $J : M \to N$ between two Jordan Banach *-algebras M and N is a Jordan *-homomorphism if J preserves the involution, that is, if $J(a^*) = J(a)^*$, for every $a \in M$. We shall write $M_{sa} := \{a \in M : a^* = a\}$ for the set of all self-adjoint elements in a Jordan Banach *-algebra M.

Definition 1.2.8. *A* JB*-algebra M *is a Jordan Banach* **-algebra satisfying* $\|U_a(a^*)\| = \|a\|^3$*, for every* $a \in M$*.*

JB*-algebras were first considered by I. Kaplansky, who presented them at a lecture to the St. Andrews Colloquium for the Edinburgh Mathematical Society in 1976. We know from a result by M.A. Youngson that the involution of every JB*-algebra is an isometry (cf. [170, Lemma 4]). The hermitian part, M_{sa}, of a JB*-algebra, M, is always a JB-algebra, a fact noted by I. Kaplansky during his famous lecture. A celebrated theorem due to J.D.M. Wright asserts that, conversely, the complexification of every JB-algebra J is a JB*-algebra under a certain norm extending the norm of J (see [166]).

A JBW-algebra (respectively, a JBW*-algebra) is a JB-algebra (respectively, a JB*-algebra) which is a dual Banach space. Every JBW-algebra (respectively, every JBW*-algebra) M has a unique isometric predual which will be denoted by M_* ([82, Theorem 4.4.16]). It is also known that the Jordan product of each JBW-algebra (respectively, each JBW*-algebra) is separately weak*-continuous ([82, Corollary 4.1.6]).

Of course, any C^*-alegbra A is a JB*-algebra when equipped with the original norm and involution, and the natural Jordan product given in (1.3), that is, $a \circ b = \frac{1}{2}(ab + ba)$ $(a, b \in A)$. Norm-closed Jordan *-subalgebras of C*-algebras are called *JC*-algebras*. JC*-algebras which are also dual Banach spaces are called *JW*-algebras*. Any JW*-algebra is a weak*-closed Jordan *-subalgebra of a von Neumann algebra. The setting of JC*-algebras constitutes a favourable framework in which we shall employ some associative tools.

Let a be a hermitian element in a JB*-algebra M, the theorem [82, Theorem 3.2.4] assures that the JB*-subalgebra of M generated by a, M_a, is isometrically JB*-isomorphic to a commutative C*-algebra. We shall say that a self-adjoint element a in a JB*-algebra M is *positive* in M if a is positive in the commutative C*-algebra M_a.

From the perspective of JB*-algebras, it deserves to be mentioned a re-reading of the celebrated Shirshov-Cohn theorem, namely, it affirms that the JB*-subalgebra of a JB*-algebra generated by two self-adjoint elements (and possibly the unit element) is a JC*-algebra, that is, a JB*-subalgebra of some $B(H)$ (cf. [82, Theorems 2.4.14 and 7.2.5] and [166, Corollary 2.2 and subsequent comments]).

Involving the norm in a JB*-algebra M, it is known that

$$\|U_{x,y}(z)\|_M \leq \|x\| \, \|y\| \, \|z\|, \quad \forall x, y, z \in M,$$

(cf. [25, Proposition 3.4.17]).

Definition 1.2.9. *Let M be a unital JB*-algebra. An element $u \in M$ is called* Jordan unitary *if u is Jordan invertible in M with u^* as Jordan inverse, that is, $u \circ u^* = \mathbf{1}_M$, and $u^2 \circ u^* = u$. The symbol $\mathcal{U}(M)$ will stand for the set of all Jordan unitary elements in M.*

Remark 1.2.10. *It is worth observing that in a unital C*-algebra, we can assure via Proposition 1.2.5 that unitary elements in the C*-algebra sense and Jordan unitaries coincide. Therefore, we find no obstacle in adopting the notation $\mathcal{U}(M)$ for the set of all Jordan unitary elements in a unital JB*-algebra.*

When the notion of Jordan unitary shows up, it is inevitable to draw our attention to the Wright–Youngson extension of the Russo–Dye theorem for JB*-algebras. This milestone result assures that the closed unit ball of any unital JB*-algebra M coincides with the closed convex hull of $\mathcal{U}(M)$ ([167], see also [25, Corollary 3.4.7 and Fact 4.2.39], or [150]). Different applications can be found on the study of surjective isometries between JB- and JB*-algebras (see [168, 91] and [25, Proposition 4.2.44]).

Two elements a, b in a Jordan algebra M are said to *operator commute* if

$$(a \circ c) \circ b = a \circ (c \circ b),$$

for all $c \in M$ (cf. [82, 4.2.4]). By the *center* of M we mean the set of all elements of M which operator commute with any other element in M. Any element in the center is called *central*. Two elements a, b in a JB*-algebra M are *orthogonal* whenever $a \circ b = 0$.

A self-adjoint idempotent in a JB*-algebra is called *projection*. The set of all projections in a JB*-algebra M will be denoted by $\mathcal{P}(M)$. Let us observe, that when a C*-algebra A is regarded as a JB*-algebra with the Jordan product given in (1.3), an element $p \in A$ is a projection in the C*-algebra A if and only if it is a projection in the JB*-algebra A. That shows that the notation is consistent.

An element s in a unital JB-algebra J is called a *symmetry* if $s^2 = \mathbf{1}_J$. If M is a unital JB*-algebra, the symmetries in M are defined as the symmetries in its self-adjoint part M_{sa}. Let a be a hermitian element in a JB*-algebra M, using the spectral theorem [82, Theorem 3.2.4], it can be seen that we can write a as the difference of two orthogonal positive elements in M_{sa}. When M is unital we obtain

$$\partial_e(\mathcal{B}_{M_{sa}}) = \mathrm{Symm}(M) = \{s \in M_{sa} : s^2 = 1\}$$

(cf. [168] or [25, Proposition 3.1.9]).

1.3 JB*-triples

This section is intended to familiarise the reader with triple structures by presenting a basic background in the theory of JB*-triples. Some of the concepts exhibited will represent a generalisation of those introduced for C*- and JB*-algebras. Most of the results will be accompanied by the appropriate references.

Let H and K be two complex Hilbert spaces. A J*-algebra, in the sense introduced by L.A. Harris in [83], is a closed complex subspace X of $B(H, K)$ such that $aa^*a \in X$ whenever $a \in X$. L.A. Harris proved in [83, Corollary 2] that the open unit ball, $\overset{\circ}{\mathcal{B}}_X$, of every J*-algebra X is a bounded symmetric domain (i.e. for each $x \in \overset{\circ}{\mathcal{B}}_X$ there exists a biholomorphic mapping in Fréchet's sense $h : \overset{\circ}{\mathcal{B}}_X \to \overset{\circ}{\mathcal{B}}_X$ such that h has x as its only fixed point and h^2 is the identity map on $\overset{\circ}{\mathcal{B}}_X$). However, J*-algebras are not the unique complex Banach spaces whose open unit ball is a bounded symmetric domain. W. Kaup established in [106] that the open unit ball of a complex Banach space X is a bounded symmetric domain if and only if X is a JB*-triple.

According to the definition introduced by W. Kaup in [106], a *JB*-triple* is a complex Banach space X admitting a continuous triple product $\{\cdot,\cdot,\cdot\} : X \times X \times X \to X$, which is symmetric and linear in the outer variables, conjugate-linear in the middle one, and satisfies the following axioms:

1. Jordan identity:

$$L(a,b)L(x,y) = L(x,y)L(a,b) + L(L(a,b)x,y) - L(x,L(b,a)y),$$

   for all a,b,x,y in X, where $L(a,b) : X \to X$ is the operator on X given by $L(a,b)x = \{a,b,x\}$ $(x \in X)$;

2. For all $a \in X$, $L(a,a)$ is a hermitian operator with non-negative spectrum;

3. $\| \{a,a,a\} \|_X = \|a\|_X^3$, for all $a \in X$.

We shall write $\{\cdot,\cdot,\cdot\}_X$ when it is necessary to specify (or emphasise) the Banach space X. It should be recalled that a bounded linear operator T on a complex Banach space is said to be *hermitian* if $\| \exp(i\alpha T)\| = 1$, for all real α ([17, §10 Corollary 13 and page 205]). The symmetry of the triple product makes possible to express any element in a JB*-triple X as a linear combination of products of the form $\{a,b,a\}$ $(a,b \in X)$.

Every J*-algebra is a JB*-triple with respect to the triple product given by

$$\{x,y,z\} = \frac{1}{2}(xy^*z + zy^*x). \tag{1.14}$$

Consequently, C*-algebras and complex Hilbert spaces are JB*-triples with respect to the above triple product. Other interesting examples are given by Jordan structures; for example, every JB*-algebra in the sense considered in [166, 167] and [148, 150] is a JB*-triple under the triple product

$$\{x,y,z\} = (x \circ y^*) \circ z + (z \circ y^*) \circ x - (x \circ z) \circ y^* \tag{1.15}$$

([18, Theorem 3.3]).

A subspace E of a JB*-triple X is said to be a *subtriple* of X if it is closed for the triple product in X, that is, if $\{E,E,E\} \subseteq E$. Equivalently, the subspace E of X is a subtriple if and only if, for each element x in E, $\{x,x,x\}$ lies in E. Every norm-closed subtriple of a JB*-triple becomes a JB*-triple when equipped with the inherited triple product [106]. We shall refer to the norm-closed subtriples of a JB*-triple as JB*-subtriples. A subtriple I of a JB*-triple X is an *ideal* of X if $\{X,X,I\} + \{X,I,X\} \subseteq I$. We shall say that I is an *inner ideal* whenever $\{I,X,I\} \subseteq I$.

The building blocks in the Gelfand-Naimark theorem for JB*-triples are given by the so-called *Cartan factors*. There are six types of Cartan factors defined as follows:

Cartan factor of type 1: the complex Banach space $B(H,K)$, of all bounded linear operators between two complex Hilbert spaces, H and K, whose triple product is given by (1.14).

Given a conjugation $j : H \to H$ (i.e. a conjugate-linear isometry of period 2) on a complex Hilbert space H, we can define a linear involution on $B(H)$ defined by $x \mapsto x^t := jx^*j$.

Cartan factor of type 2: the subtriple of $B(H)$ formed by the skew-symmetric operators for the involution t.

Cartan factor of type 3: the subtriple of $B(H)$ formed by the t-symmetric operators.

Cartan factor of type 4 or *spin:* a complex Banach space X admitting a complete inner product $(.|.)$ and a conjugation $x \mapsto \overline{x}$, for which the triple product and norm are given by

$$\{x, y, z\} = (x|y)z + (z|y)x - (x|\overline{z})\overline{y}, \quad \text{and}$$

$$\|x\|^2 = (x|x) + \sqrt{(x|x)^2 - |(x|\overline{x})|^2}, \text{ respectively.}$$

Cartan factors of types 5 and 6 (also called *exceptional* Cartan factors) consist of matrices over the eight-dimensional complex algebra of Cayley numbers; the type 6 consists of all 3 by 3 self-adjoint matrices and has a natural Jordan algebra structure, and the type 5 is the subtriple consisting of all 1 by 2 matrices. These Cartan factors are all finite-dimensional.

The *Gelfand–Naimark theorem for JB*-triples* affirms that each JB*-triple can be embedded into an ℓ_∞-sum of Cartan factors (cf. [76]).

The natural morphisms acting between JB*-triples (called *triple homomorphisms*) are those linear mappings preserving triple products. Concretely, let X and Y be two JB*-triples. A linear mapping $\varphi : X \to Y$ is called a *triple homomorphism* if $\varphi(\{a, b, c\}_X) = \{\varphi(a), \varphi(b), \varphi(c)\}_Y$, for every $a, b, c \in X$. We shall say that a triple homomorphism φ is a *triple isomorphism* (respectively, *monomorphism* or *epimorphism*) if it is bijective (respectively, injective or surjective).

A milestone result concerning triple isomorphisms is the celebrated *Kaup-Banach-Stone theorem*, which states that a linear bijection between JB*-triples is an isometry if and only if it is a triple isomorphism (cf. [106, Proposition 5.5]). Any unital *-triple homomorphism between JB*-algebras is a Jordan homomorphism. An important conclusion can be derived now, namely, the norm of the underlying complex Banach space of a JB*-triple and the algebraic structure given by the triple product are uniquely determined each other. This fact will be essential in the development of the arguments in the paper [36].

The triple product of every JB*-triple is a non-expansive mapping, that is,

$$\|\{a, b, c\}\| \leq \|a\|\|b\|\|c\| \quad \text{for all } a, b, c \text{ (see [76, Corollary 3]).} \tag{1.16}$$

Elements a, b in a JB*-triple X are called *orthogonal* (written $a \perp b$) if $L(a, b) = 0$. It is known that $a \perp b \Leftrightarrow \{a, a, b\} = 0 \Leftrightarrow \{b, b, a\} = 0 \Leftrightarrow b \perp a$; (see, for example, [23, Lemma 1]).

The rank of a JB*-triple X is the minimal cardinal number r satisfying $\mathrm{card}(S) \leq r$ whenever S is an orthogonal subset of X, that is, $0 \notin S$ and $x \perp y$ for every $x \neq y$ in S. A real linear version of the Kaup-Banach-Stone theorem was stated by T. Dang in 1992, by proving that if $T : X \to Y$ is a real linear surjective isometry between two JB*-triples X and Y, then T is a real linear triple isomorphism provided that X^{**} does not contain non-trivial rank-1 Cartan factors ([37, Theorem 3.1].

A JBW*-triple is a JB*-triple which is also a dual Banach space (with a unique isometric predual, see [9]). It is known that the second dual of a JB*-triple is a JBW*-triple (compare [41]). An extension of Sakai's theorem assures that the triple product of every JBW*-triple is separately weak* continuous (cf. [9] or [88]).

Let A be a C*-algebra regarded as a JB*-triple with the product given in (1.14). It is easy to see that partial isometries in A are precisely those elements e in A such that $\{e, e, e\} = e$. An element e in a JB*-triple X is said to be a *tripotent* if $\{e, e, e\} = e$. Any non-zero tripotent is an element of the unit sphere by the axioms of JB*-triple. We shall denote by $\mathrm{Trip}(X)$ the set of all tripotents in a JB*-triple X.

The extreme points of the closed unit ball of a JB*-triple X can only be understood in terms of those tripotents satisfying an additional property. For each tripotent $e \in X$, there exists an algebraic decomposition of X, known as the *Peirce decomposition* associated with e, which involves the eigenspaces of the operator $L(e, e)$. Namely,

$$X = X_2(e) \oplus X_1(e) \oplus X_0(e),$$

where $X_i(e) = \{x \in X : \{e, e, x\} = \frac{i}{2}x\}$ for each $i = 0, 1, 2$. It is easy to see that every *Peirce subspace* $X_i(e)$ is a JB*-subtriple of X.

The so-called Peirce arithmetic assures that

$$\{X_i(e), X_j(e), X_k(e)\} \subseteq X_{i-j+k}(e), \qquad \text{if } i - j + k \in \{0, 1, 2\},$$

$$\{X_i(e), X_j(e), X_k(e)\} = \{0\}, \text{ otherwise},$$

and

$$\{X_2(e), X_0(e), X\} = \{X_0(e), X_2(e), X\} = 0.$$

The projection $P_k(e)$ of X onto $X_k(e)$ is called the Peirce k-projection. It is known that Peirce projections are contractive (cf. [75, Corollary 1.2]) and satisfy that $P_2(e) = Q(e)^2$, $P_1(e) = 2(L(e, e) - Q(e)^2)$, and $P_0(e) = \mathrm{Id}_X - 2L(e, e) + Q(e)^2$, where $Q(e) : X \to X$ is the conjugate-linear mapping defined by $Q(e)(x) = \{e, x, e\}$. A tripotent e in X is called *unitary* (respectively, *complete* or *maximal*) if $X_2(e) = X$ (respectively, $X_0(e) = \{0\}$). Finally, a tripotent e in X is said to be *minimal* if $X_2(e) = \mathbb{C}e \neq \{0\}$. It is also known that $X_2(e) = X^1(e) \oplus X^{-1}(e)$, where

$X^j(e) = \{x \in X : Q(e)(x) = jx\}$ $(j = \pm 1)$. The space $X^0(e) = \ker(Q(e))$ coincides with $X_1(e) \oplus X_0(e)$, and

$$X = X^0(e) \oplus X^1(e) \oplus X^{-1}(e).$$

The natural projection of X onto $X^j(e)$ will be denoted by $P^j(e)$.

It is worth remarking that the Peirce space $X_2(e)$ is a unital JB*-algebra with unit e, product $x \circ_e y := \{x, e, y\}$ and involution $x^{*_e} := \{e, x, e\}$, respectively. Actually, the Kaup-Banach-Stone theorem [106, Proposition 5.5] implies that the triple product in $X_2(e)$ is uniquely determined by the identity

$$\{a, b, c\} = (a \circ_e b^{*_e}) \circ_e c + (c \circ_e b^{*_e}) \circ_e a - (a \circ_e c) \circ_e b^{*_e}, \quad (\forall a, b, c \in X_2(e)).$$

Furthermore, for each $x \in X$ the element

$$P_2(e)\{x, x, e\} = \{P_2(e)(x), P_2(e)(x), e\} + \{P_1(e)(x), P_1(e)(x), e\} \tag{1.17}$$

is positive in $X_2(e)$, and $P_2(e)\{x, x, e\} = 0$ if and only if $P_j(e)(x) = 0$ for every $j = 1, 2$ (see [75, Lemma 1.5 and preceding comments] or [133]).

It follows from the Peirce arithmetic that $a \perp b$ for every $a \in X_2(e)$ and every $b \in X_0(e)$. Let e and u be tripotents in a JB*-triple X, then

$$u \perp e \Leftrightarrow \text{the elements } u \pm e \text{ are tripotents.} \tag{1.18}$$

(see [90, Lemma 3.6]). It is known that $a \perp b$ in X implies that $\|\lambda a + \mu b\| = \max\{\|\lambda a\|, \|\mu b\|\}$ (compare [75, Lemma 1.3(a)]).

We shall consider the following natural partial order on the set $\mathrm{Trip}(X)$, of all tripotents in a JB*-triple X, defined by $u \leq e$ if $e - u$ is a tripotent in X with $e - u \perp u$. It is known that $u \leq e$ if and only if u is a projection in the JB*-algebra $X_2(e)$ (see [11]).

Similarly as there exist C*-algebras containing no non-zero projections, we can find JB*-triples containing no non-trivial tripotents. Another geometric property of JB*-triples provides an algebraic characterisation of the extreme points of their closed unit balls.

Proposition 1.3.1. [18, Lemma 4.1], [103, Proposition 3.5], [90, Lemma 3.3] *Let X be a real or complex JB*-triple. The extreme points of the closed unit ball of X coincide with the complete tripotents in X.*

For each element a in a JB*-triple X, we shall define inductively the *triple powers* of a in X as follows:

$$a^{[1]} = a, \quad a^{[3]} = \{a, a, a\}, \text{ and } \quad a^{[2n+1]} = \left\{a, a, a^{[2n-1]}\right\} \quad (n \geq 2).$$

Any JB*-triple X is *triple power associative*, that is, for all x in X, the triple product verifies the following identity

$$\left\{a^{[2n-1]}, a^{[2m-1]}, a^{[2k-1]}\right\} = a^{[2(n+m+k)-3]} \quad \forall n, m, k \in \mathbb{N},$$

(cf. [30, Lemma 1.2.10])

A JB*-triple X is said to be *commutative* or *abelian* if the identities

$$\left\{\{x, y, z\}, a, b\right\} = \left\{x, \{y, z, a\}, b\right\} = \left\{x, y, \{z, a, b\}\right\}$$

hold for all $x, y, z, a, b \in X$.

Given a subset M of a JB*-triple, the symbol X_M will stand for the (norm-closed) JB*-subtriple of X generated by M, which is the smallest norm-closed subtriple of X containing M. When $M = \{a\}$, we shall simply write X_a instead of X_M. It follows from the triple power associativity that the linear span of all odd powers of a coincides with the subtriple of X generated by a. Moreover, any JB*-subtriple of a JB*-triple generated by a single element is abelian.

The rich local theory of JB*-triples allows to identify any JB*-subtriple (i.e. any norm-closed subtriple) generated by any single element with a $C_0(L)$-space, for some L locally compact Hausdorff space. This fact shows the manifest advantage of considering the triple structure in some strictly smaller classes of Banach spaces as C*-algebras, where the same conclusions hold but at the cost of assuming normality on the generating element.

Let a be an element of a JB*-triple X, and let us consider X_a, the JB*-subtriple of X generated by a, that is, the closed subspace generated by all odd powers $a^{[2n+1]}$. It is known that there exists an isometric triple isomorphism $\Psi : X_a \to C_0(L)$ satisfying $\Psi(a)(s) = s$, for all s in L (see [106, (1.15) Corollary], [104, 4.8 Corollary]), where $C_0(L)$ is the abelian C*-algebra of all complex-valued continuous functions on L vanishing at 0, L being a locally compact subset of $(0, \|a\|]$ satisfying that $\|a\| \in L \cup \{0\}$ is compact.

If $f : L \cup \{0\} \to \mathbb{C}$ is a continuous function vanishing at 0, the *triple functional calculus* of f at the element a is the unique element $f_t(a) \in X_a$, given by $f_t(a) = \Psi^{-1}(f)$. We can define this way $a^{\left[\frac{1}{2n+1}\right]} := (r_n)_t(a)$, where $r_n(s) = s^{\frac{1}{2n+1}}$ $(s \in L)$ and $n \in \mathbb{N}$.

When X is regarded as a JB*-subtriple of X^{**}, the triple functional calculus $f \mapsto f_t(a)$ admits an extension, denoted by the same symbol, from $C_0(L)$ to the commutative W*-algebra W generated by $C_0(L)$, onto the JBW*-subtriple X_a^{**} of X^{**} generated by a. If we assume that a is a norm-one element, the sequences $(a^{\left[\frac{1}{2n-1}\right]})_n$ and $(a^{[2n-1]})_n$ in $C_0(L)$ converge in the weak*-topology of $C_0(L)^{**}$ to the characteristic functions χ_L and $\chi_{\{1\}}$ of the sets L and $\{1\}$, respectively. The corresponding limits define two tripotents in X_a^{**} which are denoted by $r_{X^{**}}(a)$ and $u_{X^{**}}(a)$, and called the *range tripotent* and the *support tripotent* of a, respectively. The tripotent $r_{X^{**}}(a)$ is the smallest tripotent $e \in \mathrm{Trip}(X^{**})$ in the bidual, X^{**}, satisfying that a is positive in the JBW*-algebra $X_2^{**}(e)$. In addition, the support tripotent $u_{X^{**}}(a)$ is the biggest projection in $X_2^{**}(r(a))$ such that $u_{X^{**}}(a) \leq a \leq r_{X^{**}}(a)$ in $X_2^{**}(r(a))$ (compare [56, Lemma 3.3] or [52, Lemma 3.2 (ii)]).

We have seen before that an element in a unital C*-algebra is a unitary if and only if it is a Jordan unitary, when endowed with the Jordan product in (1.3). Since any C*-algebra can be regarded as a JB*-triple with respect to the triple product in (1.15), it would be desirable that the concepts of unitary element and unitary tripotent give the same elements. The following proposition shows that, in fact, this is true in a more general setting (cf. [18, Proposition 4.3] or [25, Theorem 4.2.24, Definition 4.2.25 and Fact 4.2.26]).

Proposition 1.3.2. (cf. [18, Proposition 4.3]) *Let M be a unital JB*-algebra, and let $u \in M$. Then u is a Jordan unitary in M if and only if u is a unitary tripotent when M is regarded as a JB*-triple.*

Proof. Suppose u is Jordan unitary in M, that is, $u \circ u^* = \mathbf{1}_M$ and $u^2 \circ u^* = u$. Let us now regard M as a JB*-triple together with the triple product in (1.15). It is clear that u is a tripotent in M. Indeed,

$$\begin{aligned}\{u,u,u\} &= (u \circ u^*) \circ u - (u \circ u) \circ u^* + (u^* \circ u) \circ u \\ &= 2(\mathbf{1}_M \circ u) - u^2 \circ u^* = 2u - u = u.\end{aligned}$$

Making use of (1.12), we have

$$\begin{aligned}\{u,u,x\} &= (u \circ u^*) \circ x - (u \circ x) \circ u^* + (u^* \circ x) \circ u \\ &= \mathbf{1}_M \circ x - (u \circ x) \circ u^* + (u^* \circ x) \circ u \\ &= x - (u \circ x) \circ u^* + (u^* \circ x) \circ (u^2 \circ u^*) \\ &= x - (u \circ x) \circ u^* + \big((u^* \circ x) \circ u^2\big) \circ u^* \\ &= x - (u \circ x) \circ u^* + \big((u^* \circ u^2) \circ x\big) \circ u^* \\ &= x - (u \circ x) \circ u^* + (u \circ x) \circ u^* = x\end{aligned}$$

Therefore, $M_2(u) = M$ and hence u is a unitary tripotent in M.

To prove the other implication, let us suppose that u is a unitary tripotent in M. By definition, $\{u,u,x\} = x$ for every x in M. Thus, we have

$$\begin{aligned}\mathbf{1}_M = \{u,u,\mathbf{1}_M\} &= (u \circ u^*) \circ \mathbf{1}_M - (u \circ \mathbf{1}_M) \circ u^* + (u^* \circ \mathbf{1}_M) \circ u \\ &= u \circ u^* - u \circ u^* + u^* \circ u = u \circ u^*,\end{aligned}$$

and hence

$$\begin{aligned} u = \{u,u,u\} &= (u\circ u^*)\circ u - (u\circ u)\circ u^* + (u^*\circ u)\circ u \\ &= \mathbf{1}_M \circ u - u^2\circ u^* + \mathbf{1}_M\circ u = 2u - u^2\circ u^*. \end{aligned}$$

We have just shown that $u^2\circ u^* = u$, which concludes the proof. □

Proposition 1.3.3. *Let M be a unital JB*-algebra, and let $u\in \operatorname{Trip}(M)$. Suppose u is Jordan invertible, then u is a Jordan unitary.*

Proof. Let u^{-1} denote the Jordan inverse of u in M. Lemma 1.2.4 assures that $U_u(u^{-1}) = u$. Since u is a tripotent, we have $U_u(u^*) = \{u,u,u\} = u$. By Proposition 1.2.6, U_u is an invertible operator, and hence bijective. Therefore, $U_u(u^{-1}) = u = U_u(u^*)$, which implies $u^{-1} = u^*$. □

We collect next a series of properties of unitary elements in unital JB*-algebras. These properties are the starting point of the paper [36], in which the surjective isometries between the sets of unitary elements of two unital JB*-algebras are studied. The conclusions of the following lemma show the strong links between JB*-algebras and JB*-triples.

Lemma 1.3.4. [25, Lemma 4.2.41, Theorem 4.2.28, Corollary 3.4.32], [168], [91] *Let M be a unital JB*-algebra, and let u be a unitary element in M. Then the following statements hold:*

(*a*) *The Banach space of M becomes a unital JB*-algebra with unit u for the (Jordan) product defined by $x\circ_u y := U_{x,y}(u^*) = \{x,u,y\}$ and the involution $*_u$ defined by $x^{*_u} := U_u(x^*) = \{u,x,u\}$. (This JB*-algebra $M(u) = (M,\circ_u,*_u)$ is called the* u-isotope *of M.)*

(*b*) *The unitary elements of the JB*-algebras M and $(M,\circ_u,*_u)$ are the same, and they also coincide with the unitary tripotents of M when the latter is regarded as a JB*-triple.*

(*c*) *The triple product of M satisfies*

$$\begin{aligned} \{x,y,z\} &= (x\circ y^*)\circ z + (z\circ y^*)\circ x - (x\circ z)\circ y^* \\ &= (x\circ_u y^{*_u})\circ_u z + (z\circ_u y^{*_u})\circ_u x - (x\circ_u z)\circ_u y^{*_u}, \end{aligned}$$

for all $x,y,z\in M$. Actually, the previous identities hold when $\circ$ is replaced with any Jordan product on M making the latter a JB-algebra with the same norm.*

(*d*) *The mapping $U_u : M\to M$ is a surjective isometry and hence a triple isomorphism. Consequently, $U_u(\mathcal{U}(M)) = \mathcal{U}(M)$. Furthermore, the operator $U_u : (M,\circ_{u^*},*_{u^*}) \to (M,\circ_u,*_u)$ is a Jordan *-isomorphism.*

Concerning the predual of a JBW*-triple, a couple of results due to Y. Friedman and B. Russo should be mentioned here. The first one is a consequence of [75, Proposition 1(a)] and reads as follows:

$$\begin{gathered}\text{Let } e \text{ be a tripotent in a JB}^*\text{-triple } X \text{ and let } \varphi \text{ be a functional in } X^* \\ \text{satisfying } \varphi(e) = \|\varphi\|, \text{ then } \varphi = \varphi P_2(e).\end{gathered} \tag{1.19}$$

The second result tells that the extreme points of the closed unit ball of the predual, M_*, of a JBW*-triple M are in one-to-one correspondence with the minimal tripotents in M via the following correspondence:

$$\begin{gathered}\text{For each } \varphi \in \partial_e(\mathcal{B}_{M_*}) \text{ there exists a unique minimal tripotent } e \in M \\ \text{satisfying } \varphi(x)e = P_2(e)(x) \text{ for all } x \in M,\end{gathered} \tag{1.20}$$

(see [75, Proposition 4]). By analogy with the notation employed in the setting of C*-algebras, the elements in $\partial_e(\mathcal{B}_{M_*})$ are usually called *pure atoms*. For each minimal tripotent e in M, we shall write φ_e for the unique pure atom associated with e.

Different real analogues of JB*-triples have been also studied in recent years. Actually, four notions of JB*-triples over the real field have been introduced by H. Upmeier [162], T. Dang and B. Russo [38], J.M. Isidro, W. Kaup, and A. Rodríguez [90], and A.M. Peralta [138], respectively. All those definitions yield strong algebraic, topological and geometrical axioms which model certain structures strictly bigger than the class of complex JB*-triples. This memoir has adopted the notion of real JB*-triple given in [90] by J.M. Isidro, W. Kaup, and A. Rodríguez, but it is worth to observe that the class of J*B-triples introduced by T. Dang and B. Russo includes (complex) JB*-triples as well as real JB*-triples in the sense of [90], and moreover, it is shown in [38, Proposition 1.4] that (complex) JB*-triples are precisely those complex Jordan Banach triples whose underlying real Banach space is a J*B-triple. In the same line, every real JB*-triple in the sense of [90], is a numerically positive J*B-triple as defined in [138].

A *real JB*-triple* is by definition a real closed subtriple of a JB*-triple (see [90]). Every JB*-triple is a real JB*-triple when it is regarded as a real Banach space. Real JB*-triples can be obtained as *real forms* of JB*-triples. More concretely, given a real JB*-triple E, there exists a unique (complex) JB*-triple structure on its algebraic complexification $X = E \oplus iE$, and a conjugation (i.e. a conjugate-linear isometry of period 2) τ on X such that

$$E = X^\tau = \{z \in X : \tau(z) = z\},$$

(see [90]). Consequently, every real C*-algebra is a real JB*-triple with respect to the product given in (1.14), and the Banach space $B(H_1, H_2)$ of all bounded real linear operators between two real, complex, or quaternionic Hilbert spaces also is a real JB*-triple with the same triple product.

As in the complex case, an element e in a real JB*-triple E is said to be a *tripotent* if $\{e, e, e\} = e$. We shall also write $\text{Trip}(E)$ for the set of all tripotents in E. It is known that an element $e \in E$ is a tripotent in E if and only if it is a tripotent in the complexification of E. Each tripotent e in E induces a Peirce decomposition of E in similar terms to those we commented in page 24 with the exception that $E_2(e)$ is not, in general, a JB*-algebra but a real JB*-algebra (i.e. a closed *-invariant real subalgebra of a (complex) JB*-algebra). Unitary and complete tripotents are defined analogously to the complex setting. Furthermore, the extreme points of $\mathcal{B}_E$ coincide with the complete tripotents in the real JB*-triple E (cf. [90, Lemma 3.3]).

There are numerous results which do not belong to the folklore since they are quite recent. We collect next some of them which have been useful in the development or even have been stated in the papers produced for this thesis.It can be observed that we are mostly interested in results in whose statement the triple structure is not probably the protagonist. However, following the spirit of this project, the proofs of such conclusions (which can be consulted in the corresponding references) are based on triple techniques, and thus, those results manifest the usefulness of considering the JB*-triple structure in the strictly smaller subclasses of C*- and JB*-algebras.

We start with a Jordan version of Proposition 1.1.13, which was originally established in [91, Proposition 1.3], and a new proof can be consulted in [25, Proposition 3.1.24 and Remark 3.1.25]. An alternative proof, based on the structure of real JB*-triples, is included in [35, Proposition 2.2].

Proposition 1.3.5. [91, Proposition 1.3], [25, Proposition 3.1.24], [35, Proposition 2.2, Mediterr. J. Math.] *Let p be a projection in a unital JB*-algebra M. Then the following statements are equivalent:*

(a) p is (norm) isolated in $\mathcal{P}(M)$;

(b) p is a central projection;

(c) $1 - 2p$ is (norm) isolated in Symm(M). □

We highlight next an outstanding result achieved by J. Hamhalter, O.F. K. Kalenda, A.M. Peralta, and H. Pfitzner in [81], a paper which contains an abundant collections of new results in the theory of JB*-triples.

Lemma 1.3.6. [81, Lemma 6.1] *Let A be a unital C*-algebra, and let $u \in A$ be a complete tripotent. Then there exist a complex Hilbert space H and an isometric unital Jordan $*$-monomorphism $\psi : A \to B(H)$ such that $\psi(u)^*\psi(u) = \mathbf{1}_{B(H)}$.*

As we commented in previous sections, one of the most successful tools in the theory of Jordan algebras is the Shirshov-Cohn theorem. The next lemma is an appropriate version, in which two orthogonal tripotents play the role of the symmetric elements.

Lemma 1.3.7. [35, Lemma 3.6, Mediterr. J. Math.] *Let u_1 and u_2 be two orthogonal tripotents in a unital JB*-algebra M. Then the JB*-subalgebra N of M generated by u_1, u_2 and the unit element is a JC*-algebra, that is, there exists a complex Hilbert space H satisfying that N is a JB*-subalgebra of $B(H)$, we can further assume that the unit of N coincides with the identity on H.* □

Lemma 6.3 in [81] is the Jordan version of Lemma 1.3.6, and says that given a complete tripotent u in a unital JB*-algebra M there exists a unital Jordan *-monomorphism $\psi : N \to B(H)$, for some H complex Hilbert space, such that $\psi(u)^*\psi(u) = \mathbf{1}_{B(H)}$, where N denotes the closed unital Jordan *-subalgebra of M generated u. The result is a consequence of the just commented Lemma 1.3.6 together with Lemma 6.2 in the same paper, which assures that given a complete tripotent u in a unital JB*-algebra M, the JB*-subalgebra N of M generated by u and the unit is a JC*-algebra of some C*-algebra B, and we can further assume that u is a complete tripotent in the C*-subalgebra of B generated by N.

Inspired by these results, we produce our version of the Shirshov-Cohn theorem in Lemma 1.3.7 to state an analogous proposition.

Proposition 1.3.8. [35, Proposition 3.7, Mediterr. J. Math.] *Let u_1 and u_2 be two orthogonal tripotents in a unital JB*-algebra M satisfying the following properties:*

(a) $u = u_1 + u_2$ a complete tripotent in M;

(b) u_1, u_2 are central projections in the JB-algebra $M_2(u)$.*

Let N denote the JB-subalgebra of M generated by u_1, u_2 and the unit element. Then N is a JC*-subalgebra of some C*-algebra B, and u is a complete tripotent in the C*-subalgebra A of B generated by N. Moreover, the elements u_1, u_2 are central projections in the JB*-algebra $A_2(u)$.* □

The space of Hilbert-space-valued continuous functions

One of the papers endorsing this project, [34], is focused on the space $C(K, H)$ of all continuous functions defined on a compact Hausdorff space K with values in a Hilbert space H. It is known that $C(K, H)$ is a Banach space when equipped with the supremum norm, that is,

$$\|a\|_\infty = \sup\{\|a(t)\|_H : t \in K\}, \quad a \in C(K, H).$$

As commented in previous sections, the particular case in which $H = \mathbb{C}$, means the prototype of a unital commutative C*-algebra. However, the algebraic structure of $C(K, H)$ is not so simple when we consider an arbitrary complex Hilbert space H. This space cannot be treated, in general, as a C*-algebra. The strategy followed in [34] in order to work with $C(K, H)$ goes through realising that it can be regarded as a Hilbert $C(K)$-module in the sense introduced by I. Kaplansky in [101], and consequently there exists a motivating triple structure in $C(K, H)$ by virtue of Theorem 1.4 in [89].

The *Hilbert C*-modules* appear for the first time in [101], where I. Kaplansky generalises Hilbert spaces by allowing the inner product to take values in a (commutative) C*-algebra rather than in $\mathbb{C}$, as the own author explains in the quoted paper. These structures were originally conceived to be able of tackling automorphisms and derivations in some special cases of C*-algebras, avoiding the lack of machinery in the topic at that time. However, far from being an isolated trick, a whole theory has been born around them in the course of time, with W. Paschke and M. Rieffel as pioneers. The monographs [114] and [109] could be excellent texts for the basic notions in this topic.

We recall that a *right R-module* (on an associative ring R) is an additive abelian group $\mathcal{M}$ together with a mapping $\mathcal{M} \times R \to \mathcal{M}$, $(x, a) \mapsto xa$, such that for all $x, y \in \mathcal{M}$ and $a, b \in R$,

i) $(x + y)a = xa + ya$;

ii) $x(a + b) = xa + xb$;

iii) $(xa)b = x(ab)$.

Left R-modules can be defined by simply letting R act from the left hand side of $\mathcal{M}$. We shall refer to $\mathcal{M}$ as an R-bimodule whenever $\mathcal{M}$ is both a right and a left R-module. Let us observe that the validity of the concepts of right and left A-module is clearly guaranteed for any associative algebra A.

Let A be a C*-algebra. We shall now invoke the concept of Hilbert C^*-module introduced in [101]. A *pre-Hilbert A-module E* is a complex vector space equipped with two binary mappings

$$\begin{array}{ll} E \times A \to E & \langle \cdot | \cdot \rangle_E : \ E \times E \to A \\ (x, a) \mapsto xa & \qquad (x, y) \mapsto \langle x | y \rangle_E \end{array}$$

satisfying the following properties:

(1) E together with the first binary operation is a right A-module with compatible scalar product, that is, $\lambda(xa) = (\lambda x)a = x(\lambda a)$, for every $\lambda \in \mathbb{C}$, every $x \in E$ and every $a \in A$;

(2) $\langle\cdot|\cdot\rangle_E$ is a sesquilinear form, that is, linear in the first variable and conjugate-linear in the second one;

(3) $\langle x|y\rangle_E = \langle y|x\rangle_E^*$, for every $x, y \in E$;

(4) $\langle x|ya\rangle_E = \langle x|y\rangle_E \, a$, for every $x, y \in E$, and every $a \in A$;

(5) $\langle x|x\rangle_E \geq 0$, for any $x \in E$;

(6) $\langle x|x\rangle_E = 0$ implies $x = 0$, for every $x \in E$.

We shall refer to the mapping $\langle\cdot|\cdot\rangle_E$ as the *A-valued inner product* in E. Let us consider on E the norm

$$\|x\|_E := \|\langle x|x\rangle_E\|_A^{1/2} \qquad (x \in E). \tag{1.21}$$

It is not hard to see that $\|\cdot\|_E$ is actually a norm on E, and the inequalities

(i) $\|xa\|_E \leq \|x\|_E \|a\|_A$,

(ii) $\langle x|y\rangle_E \langle y|x\rangle_E \leq \|y\|_E \langle x|x\rangle_E$,

(iii) $\|\langle x|y\rangle_E\|_E \leq \|x\|_E \|y\|_E$,

hold for every $x, y \in E$, and every $a \in A$. A pre-Hilbert A-module E is called a *Hilbert C*-module* over the C*-algebra A (or simply a *Hilbert A-module*) if it is complete respect to the norm $\|\cdot\|_E$ defined in (1.21).

Let K stand for a compact Hausdorff space, and suppose H is a complex Hilbert space. The Banach space $C(K, H)$ can be treated as a $C(K)$-bimodule by considering the mappings

$$\begin{aligned} C(K,H) \times C(K) &\longrightarrow C(K,H) \\ (a, f) &\longmapsto af, \end{aligned} \qquad \begin{aligned} C(K) \times C(K,H) &\longrightarrow C(K,H) \\ (f, a) &\longmapsto fa, \end{aligned}$$

where for every $a \in C(K, H)$, and every $f \in C(K)$,

$$(af)(t) = a(t)f(t) = f(t)a(t) = (fa)(t) \quad (t \in K).$$

A $C(K)$-valued mapping on $C(K, H)$ can be also defined by the assignment

$$\begin{aligned} \langle\cdot|\cdot\rangle_{C(K,H)} : C(K,H) \times C(K,H) &\longrightarrow C(K) \\ (a, b) &\longmapsto \langle a|b\rangle_{C(K,H)}, \end{aligned}$$

where for every $a, b \in C(K, H)$, $\langle a|b\rangle_{C(K,H)}(t) := \langle a(t)|b(t)\rangle_H \quad (t \in K)$. It is not hard to check that $\langle\cdot|\cdot\rangle_{C(K,H)}$ is a sesquilinear mapping. Moreover, the equalities

$$\langle b|a\rangle_{C(K,H)}^*(t) = \overline{\langle b(t)|a(t)\rangle_H} = \langle a(t)|b(t)\rangle_H = \langle a|b\rangle_{C(K,H)}(t), \quad t \in K$$

hold for every $a, b \in C(K, H)$. On the other hand, the sesquilinearity of the inner product in H makes possible to argue as follows for every $a, b \in C(K, H)$, and every $f \in C(K)$,

$$\begin{aligned}\langle fa|b\rangle_{C(K,H)}(t) &= \langle (fa)(t)|b(t)\rangle_H = \langle f(t)a(t)|b(t)\rangle_H \\ &= f(t)\langle a(t)|b(t)\rangle_H = f\langle a|b\rangle_{C(K,H)}(t), \quad t \in K.\end{aligned}$$

Finally, it is clear that $\langle a|a\rangle_{C(K,H)} \geq 0$, with $\langle a|a\rangle_{C(K,H)} = 0$ if and only if $a = 0$, because of the properties of $\langle\cdot|\cdot\rangle_H$. We have just shown that $C(K, H)$, endowed with the $C(K)$-valued inner product $\langle\cdot|\cdot\rangle_{C(K,H)}$ satisfies the properties required to be a Hilbert C*-module over the C*-algebra $C(K)$ in the sense introduced by I. Kaplansky in [101].

Since any complex Hilbert space can be regarded as a JB*-triple, finding a triple structure in any Hilbert C*-module seems to be a more than a suggestive task. That reasonable presumption was ratified by J.M. Isidro in [89]. From the quoted paper, we borrow the main theorem which signifies a new example of JB*-triple.

Theorem 1.3.9. [89, Theorem 1.4] *Every Hilbert C*-module E is a JB*-triple respect to the triple product given by*

$$\{x, y, z\}_E = \frac{1}{2}\langle x|y\rangle_E\ z + \frac{1}{2}\langle z|y\rangle_E\ x, \quad (x, y, z \in E). \tag{1.22}$$

Consequently, we can conclude that $C(K, H)$ has a JB*-triple-structure when endowed with the triple product in (1.22). It is worth observing that the norm in a JB*-triple is uniquely determined by the triple product considered (cf. [106, Proposition 5.5]). Therefore, a priori the results involving the norm from the $C(K)$-valued inner product may not imply or be inferred from those involving the supremum norm. Fortunately for our purposes, both norms, $\|\cdot\|_{C(K,H)}$ in (1.21) and $\|\cdot\|_\infty$, coincide. Therefore, the Banach space of $C(K, H)$ equipped with the supremum norm becomes a JB*-triple when the triple product in (1.22) is considered.

The case in which H is a real Hilbert space instead of a complex one is also addressed in [34]. Actually, several distinctions over the nature of H are necessarily made in order to deal with $C(K, H)$ properly. Let us fix a complex Hilbert space H, whose inner product is denoted by $\langle\cdot|\cdot\rangle_H$. Its underlying real space $H_{\mathbb{R}}$, becomes a real Hilbert space with the inner product $(a|b)_H = \Re\mathrm{e}\langle a|b\rangle_H$ $(a, b \in H)$. The questions treated in the paper [34] are deeply related with the norm of the space $C(K, H)$. Changing the nature of the Hilbert space H could make differ such a norm. However, the norms of $C(K, H)$ and $C(K, H_{\mathbb{R}})$ are exactly the same. Therefore, we can concentrate our efforts in the case in which H is a real Hilbert space. That decision does not imply a loss of the triple structure obtained above. On the contrary, let $\mathcal{H}$ be a real Hilbert space whose inner product is denoted by

$(\cdot|\cdot)_{\mathcal{H}}$. The real Banach space $C(K,\mathcal{H})$, equipped with the supremum norm, is a real JB*-triple respect to the triple product given by

$$\{a,b,c\}_{C(K,\mathcal{H})} := \frac{1}{2}(a|b)_{\mathcal{H}}c + \frac{1}{2}(c|b)_{\mathcal{H}}a. \tag{1.23}$$

The following technical lemma, proved in [34], is required for later purposes in section 3.2.3. Some conditions on the dimension of the Hilbert space must be assumed. The proof follows the arguments of R.C. Sine and N.T. Peck (see [132, proof of Theorem 1]).

Lemma 1.3.10. [34, Lemma 4.6, J. Math. Anal. Appl.] *Let K be a compact Hausdorff space and let $\mathcal{H}$ be a real Hilbert space with dim$(\mathcal{H}) = n \geq 2$. Suppose $t_0 \in K$ and $x_0 \in S(\mathcal{H})$. If $a \in \mathcal{B}_{C(K,\mathcal{H})}$ is such that $a(t_0) \in \mathbb{R}x_0$ and $\varepsilon > 0$ is small enough. Then the following statements hold:*

(a) *If $\mathcal{H}$ is infinite dimensional, then there exists a non-vanishing function b in $\mathcal{B}_{C(K,\mathcal{H})}$ such that $b(t_0) \in \mathbb{R}x_0$ and $\|a-b\| < \varepsilon$. If $a(t_0) \neq 0$, we can also assume that $b(t_0) = a(t_0)$;*

(b) *If $\mathcal{H}$ is finite dimensional, then there exist non-vanishing continuous functions $b_1,\ldots,b_k$ in $\mathcal{B}_{C(K,\mathcal{H})}$ such that $b_j(t_0) \in \mathbb{R}x_0$, for every $j \in \{1,\ldots,k\}$, and $\left\|a - \frac{1}{k}\sum_{j=1}^{k} b_j\right\| \leq \varepsilon$. If $a(t_0) \neq 0$, we can also assume that $b_j(t_0) = a(t_0)$ for all $j \in \{1,\ldots,k\}$.*

Furthermore, for each j in $\{0,\ldots,k\}$ there exists $v_j \in C(K,\mathcal{H})$ satisfying $\|v_j(t)\| = 1$, and $(b_j(t)|v_j(t)) = 0$, for all $t \in K$, and thus $u_j = b_j + (1-\|b_j(\cdot)\|^2)^{\frac{1}{2}}v_j$, and $w_j = b_j - (1-\|b_j(\cdot)\|^2)^{\frac{1}{2}}v_j$ both lie in $\partial_e(\mathcal{B}_{C(K,\mathcal{H})})$ and $b_j = \frac{1}{2}(u_j + w_j)$.

□

Let K be a compact Hausdorff space, and let H be a real or complex Hilbert space. We are interested in describing the maximal norm-closed proper faces in $C(K,H)$. A generalisation of (1.7) provides us with the geometric information required when we consider the Mazur–Ulam property in $C(K,H)$. Namely, for each $t_0 \in K$ and each $x_0 \in S(H)$ we set

$$A(t_0,x_0) := \{a \in S(C(K,H)) : a(t_0) = x_0\}.$$

It is not hard to check that $A(t_0,x_0)$ is a maximal norm-closed proper face of $\mathcal{B}_{C(K,H)}$ and a maximal convex subset of $S(C(K,H))$. Actually, every maximal convex subset of the unit sphere of $C(K,H)$ is of this form.

We shall take advantage of our knowledge on the maximal norm-closed proper faces of $C(K,H)$ to state one of the principal technical tools in [34]. In the spirit of the whole paper, the proof of this result profits from the triple structure of $C(K,H)$ exposed above, and it is combined with the theory of JB*-algebras through the Peirce 2-subspace associated with certain tripotent. The spectral theorem ([82, Theorem 3.2.4]) reduces the arguments to the setting of commutative unital C*-algebras, and hence, [126, Lemma 18] can be applied, as well as the fact that the extreme points of the closed unit ball of any commutative unital C*-algebra are precisely its unitary elements. Finally, we stand out Lemma 4 of A.A. Siddiqui in [148], which illustrates the return path to the triple structure, and allows us to state the following corollary of Lemma 1.3.10.

Corollary 1.3.11. [34, Corollary 4.7 J. Math. Anal. Appl.] *Let K be a compact Hausdorff space and let H be a complex Hilbert space. Suppose $t_0 \in K$ and $x_0 \in S(H)$. Then every element in $A(t_0, x_0)$ can be approximated in norm by a finite convex combination of elements in $A(t_0, x_0) \cap \partial_e(\mathcal{B}_{C(K,H)})$.*

The case of real Hilbert spaces is treated in the next result.

Corollary 1.3.12. [34, Corollary 4.8, J. Math. Anal. Appl.] *Let K be a compact Hausdorff space and let $\mathcal{H}$ be a finite-dimensional real Hilbert space with $dim(\mathcal{H}) = n \geq 2$. Suppose $t_0 \in K$ and $x_0 \in S(\mathcal{H})$. Then every element in $A(t_0, x_0)$ can be approximated in norm by a finite convex combination of elements in $A(t_0, x_0) \cap \partial_e(\mathcal{B}_{C(K,\mathcal{H})})$.*

Chapter 2

New geometric results

As the main title indicates, the core of this project consists of new achievements related to extension of isometries. More precisely, the papers [33, 12], and [34] are concerned with the Mazur–Ulam property, which asks about the chances of extending linearly surjective isometries defined between the unit spheres certain Banach spaces ([28]). An accurate knowledge on the geometric structure of the addressed Banach space seems to be a useful tool to attack the problem. According to the exposed comment, the first target in this chapter is to explore the facial structure of the JB*-triples. We shall review the results due to the forerunners on the topic, and present our own contributions, which are part of the paper [34].

The extreme points of the closed unit ball of a Banach space are minimal faces. These elements play a fundamental role in some Krein–Milmann type theorems containing attempts of recovering the closed unit ball of a Banach space by convex combinations of its extreme points. The origins of these statements can be found in the so-called Russo–Dye theorem ([143]), whose framework allows to restrict the assertions to the class of unitary elements. Several analogous versions came after the Russo–Dye theorem through algebraic structures where the notion of unitary is still applicable (see, for instance, [167, 150]). In view of the motivations presented, we shall dedicate a section to go further in the study of unitary elements in unital JB*-algebras. Concretely, a metric characterisation in which only the Banach structure of the Jordan algebra is employed will be exhibited. In contrast with the historical precedents no information on the dual space is required. Such a metric characterisation is the main achievement of the study developed in [35]. Some related results can be also derived from it in the setting of JB*-triples.

2.1 Facial structure in JB*-triples

The following lemma, essentially due to L. Cheng and Y. Ding [28], and later rediscovered by R. Tanaka [157] (see also [158]), shows the importance of possessing an accurate knowledge of the facial structure of the closed unit ball of a Banach space when studying Tingley's problem and the Mazur–Ulam property.

Lemma 2.1.1. [28, Lemma 5.1], [157, Lemma 3.5] and [158, Lemma 3.3] *Let X, Y be Banach spaces, and let $\Delta : S(X) \to S(Y)$ be a surjective isometry. Then C is a maximal convex subset of $S(X)$ if and only if $\Delta(C)$ is that of $S(Y)$. Furthermore, C is a maximal proper (norm-closed) face of $\mathcal{B}_X$ if and only if $\Delta(C)$ is a maximal proper (norm-closed) face of $\mathcal{B}_Y$.* □

The stability of maximal proper faces under surjective isometries between unit spheres justifies and increases our interest in the study of the facial structure of JB*-triples, a question which is actually interesting by itself.

The task of describing the faces of the closed unit ball of triple structures was initiated in 1988 by C.M. Edwards and G.T. Rüttimann in the setting of JBW*-triples (see [55]). These authors had previously study the facial structure of the closed unit ball of JB- and JBW-algebras, via techniques which are not valid for complex JB*-triples. The *facear* and *prefacear* operations are our starting point.

Let X be a real or complex Banach space with dual space X^*. Suppose F and G are two subsets of $\mathcal{B}_X$ and $\mathcal{B}_{X^*}$, respectively. Then we set

$$F' = F^{\prime,X^*} = \{a \in \mathcal{B}_{X^*} : a(x) = 1 \ \forall x \in F\}, \tag{2.1}$$

$$G_\prime = G_{\prime,X} = \{x \in \mathcal{B}_X : a(x) = 1 \ \forall a \in G\}.$$

The notation $F^{\prime,X^*}$ and $G_{\prime,X}$ (instead of F' and $G_\prime$) will be mainly used to avoid confusion when different Banach spaces are involved. Let us observe that F' is a weak*-closed face of $\mathcal{B}_{X^*}$, and $G_\prime$ is a norm-closed face of $\mathcal{B}_X$.

Some notation and terminology are needed in order to review the results in this topic. The set of all norm-closed faces of the closed unit ball $\mathcal{B}_X$ of X will be denoted by $\mathcal{F}_n(\mathcal{B}_X)$. The symbol $\mathcal{F}_{w^*}(\mathcal{B}_{X^*})$ stands for the set of all weak*-closed faces of $\mathcal{B}_{X^*}$ in X^*. The intersection of an arbitrary family of elements of $\mathcal{F}_n(\mathcal{B}_X)$ (respectively, of $\mathcal{F}_{w^*}(\mathcal{B}_{X^*})$) lies again in $\mathcal{F}_n(\mathcal{B}_X)$ (respectively, in $\mathcal{F}_{w^*}(\mathcal{B}_{X^*})$).

A subset F of $\mathcal{B}_X$ is said to be a *norm-semi-exposed face* of $\mathcal{B}_X$ if $F = (F')_\prime$, and a subset G of $\mathcal{B}_{X^*}$ is said to be a *weak*-semi-exposed face* of $\mathcal{B}_{X^*}$ if $G = (G_\prime)'$. The symbols $\mathcal{S}_n(\mathcal{B}_X)$ and $\mathcal{S}_{w^*}(\mathcal{B}_{X^*})$ will stand for the set of all norm-semi-exposed faces of $\mathcal{B}_X$, and the set of all weak*-semi-exposed faces of $\mathcal{B}_{X^*}$, respectively. The intersection of an arbitrary family of norm-semi-exposed faces is a norm-semi-exposed face. Analogously, the intersection of any family of elements in $\mathcal{S}_{w^*}(\mathcal{B}_{X^*})$ belongs to $\mathcal{S}_{w^*}(\mathcal{B}_{X^*})$.

We recall that a partially ordered set $\mathcal{P}$ is said to be a *lattice* if for any pair (a, b) of elements of $\mathcal{P}$, the supremum and the infimum of $\{a, b\}$ exist. The partially ordered set $\mathcal{P}$ is called a *complete lattice* if, for any subsets $\mathcal{S} \subseteq \mathcal{P}$, the supremum and the infimum of $\mathcal{S}$ exist in $\mathcal{P}$. Therefore, by considering the ordering of set inclusion, both $\mathcal{F}_n(\mathcal{B}_X)$ and $\mathcal{S}_n(\mathcal{B}_X)$ form a complete lattice. The same conclusion is true for weak*-semi-exposed faces, that is, $\mathcal{F}_{w^*}(\mathcal{B}_{X^*})$ and $\mathcal{S}_{w^*}(\mathcal{B}_{X^*})$ can be regarded as complete lattices respect to the ordering of set inclusion. It is worth noting that the *facear* and *prefacear* mappings $F \mapsto F'$ and $G \mapsto G_{\prime}$ given by (2.1) are anti-order isomorphisms between the complete lattices $\mathcal{S}_n(\mathcal{B}_X)$ and $\mathcal{S}_{w^*}(\mathcal{B}_{X^*})$. Moreover, these mappings are inverses of each other ([55, Lemma 2.1]).

Let us consider now the set $\mathrm{Trip}(W)$ of all tripotents in a JBW*-triple W. Following [55], let $\mathrm{Trip}(W)^{\sim}$ denote the disjoint union of $\mathrm{Trip}(W)$ and a one-point set $\{\omega\}$. Extending the partial order on $\mathrm{Trip}(W)$ given in (1.3), a partial order on $\mathrm{Trip}(W)^{\sim}$ can be defined by setting $u \leq e$ whenever both u and e lie in $\mathrm{Trip}(W)$, and $u \leq e$ respect to the natural partial order in $\mathrm{Trip}(W)$, or if u is an arbitrary element in $\mathrm{Trip}(W)^{\sim}$ and $e = \omega$. The set $\{\omega\}_{\prime}$ is conveniently defined as the closed unit ball of the predual of W, and thus $(\{\omega\}_{\prime})'$ is the empty set. Lemma 4.3 in [55] assures that $\mathrm{Trip}(W)^{\sim}$ form a complete lattice respect to the partial order described above.

In one of the first results on faces proved by C.M. Edwards and G.T. Rüttimann they proved that any norm-closed face of the closed unit ball of the predual of a JBW*-triple is actually norm-semi-exposed ([55, Corollary 4.5], see also [55, Proof of Theorem 4.4]). The main characterization of norm closed faces of the closed unit ball of the predual reads as follows:

Theorem 2.1.2. [55, Theorem 4.4] *Let W be a JBW*-triple with predual W_*. Then the mapping*

$$\begin{aligned} \mathrm{Trip}(W)^{\sim} &\longrightarrow \mathcal{F}_n(\mathcal{B}_{W_*}) \\ e &\longmapsto \{e\}_{\prime} \end{aligned}$$

is an order isomorphism from the complete lattice $\mathrm{Trip}(W)^{\sim}$ *onto the complete lattice* $\mathcal{F}_n(\mathcal{B}_{W_*})$.

In the same paper, Edwards and Rüttimann also determine all weak*-closed faces of the closed unit ball of a JB*-triple. The arguments are based on an anti-order isomorphism from the complete lattice of tripotents in W with a largest element adjoined onto the complete lattice $\mathcal{S}_n(\mathcal{B}_{W_*})$, which can be derived from Theorem 2.1.2 and Corollary 3.3 in [55]. The concrete description is given in the following terms:

Theorem 2.1.3. [55, Theorem 4.6] *Let W be a JBW*-triple, and let F be a non-empty weak*-closed face of the unit ball, $\mathcal{B}_W$, of W. Then, there exists a tripotent e in W such that*

$$F = e + \mathcal{B}_{W_0(e)} = (\{e\}_\prime)',$$

where $\mathcal{B}_{W_0(e)}$ denotes the unit ball of the Peirce zero space $W_0(e)$ in W. Furthermore, the mapping

$$\begin{aligned} \mathrm{Trip}(W)^{\sim} &\longrightarrow \mathcal{F}_{w^*}(\mathcal{B}_W) \\ e &\longmapsto (\{e\}_\prime)' \end{aligned}$$

is an anti-order isomorphism from the complete lattice $\mathrm{Trip}(W)^{\sim}$ onto the complete lattice $\mathcal{F}_{w^*}(\mathcal{B}_W)$. □

The set of tripotents in a C*-algebra coincides with the set of partial isometries when the latter is regarded as a JB*-triple. The particular case of von Neumann algebras is also treated in [55].

In 1992, the description of the facial structure of the closed unit ball of any von Neumann algebra is re-discovered and extended by C.A. Akemann and G.K. Pedersen by considering a general C*-algebra (see [4]). A different approach allowed them to prove that every norm-closed face of the closed unit ball of a C*-algebra A is norm-semi-exposed [4, Theorem 4.10], and every weak*-closed face of $\mathcal{B}_{A^*}$ is weak*-semi-exposed [4, Theorem 4.11]. So, in the strictly smaller class of C*-algebras the norm closed faces of their closed unit ball were described by Akemann and Pedersen. The weak*-closed faces of the closed unit ball of the dual space of a C*-algebra are also described by these authors.

Let A be a C*-algebra. A projection p in A** is said to be *open* if $A \cap (pA^{**}p)$ is weak*-dense in $pA^{**}p$, equivalently, there exists an increasing net of positive elements in A, all of them bounded by p, converging to p in the strong*topology of A** (see [130, §3.11], [152, §III.6 and Corollary III.6.20]). A projection $p \in A^{**}$ is called *closed* if $1 - p$ is open. A closed projection p in A^{**} is called *compact* if $p \leq x$ for some norm-one positive element $x \in A$.

Making use of the concepts above, C.A. Akemann and G.K. Pedersen introduced in [4] a new notion involving partial isometries which turned out to be a key concept for the description of the facial structure of the ball of C*-algebras. A partial isometry $v \in A^{**}$ *belongs locally* to A if v^*v is a compact projection and there exists a norm-one element x in A satisfying $v = xv^*v$ (compare [4, Remark 4.7]). A partial isometry v in A^{**} belongs locally to A if and only if v^* belongs locally to A (see [4, Lemma 4.8]). Actually, the authors proved that norm closed faces of $\mathcal{B}_A$ are in one-to-one correspondence with the compact partial isometries in A**.

The addressed task experimented a significant progress in 2001, thanks again to the studies of C.M. Edwards and G.T. Rüttimann, who dedicated the paper [57] to explore the facial structure of real JBW*-triples (in the sense of [90]). Analogously to the complex setting, it is shown that the algebraic structure of a real JBW*-triple is closely related to the geometry of its closed unit ball and the closed unit ball of its predual. The strategy followed in their approach is based on the results for complex JBW*-triples previously stated by the same authors.

Let τ be a conjugation on a JBW*-triple W, and let $M = W^\tau := \{x \in W : \tau(x) = x\}$. Lemma 3.4 in [57] assures, among other things, that the set $\mathrm{Trip}(M)^\sim$, of all tripotents in $\mathrm{Trip}(W)^\sim$ which are τ-invariant, is a sub-complete lattice of the complete lattice $\mathrm{Trip}(W)^\sim$ consisting of all the tripotents in the real JBW*-triple M with the same largest element adjoined. We gather next the main conclusions.

Theorem 2.1.4. [57, Theorems 3.7 and 3.9] *Let τ be a conjugation on a JBW*-triple W, and let $M = W^\tau$, with predual M_*.*

(i) *Then the mapping*

$$\begin{aligned} \mathrm{Trip}(M)^\sim &\longrightarrow \mathcal{F}_n(\mathcal{B}_{M_*}) \\ e &\longmapsto \{e\}_\prime \end{aligned}$$

is an order isomorphism from the sub-complete lattice $\mathrm{Trip}(M)^\sim$ of the complete lattice $\mathrm{Trip}(W)^\sim$ onto the complete lattice $\mathcal{F}_n(\mathcal{B}_{M_*})$;

(ii) *Let F be a weak*-closed face of the unit ball $\mathcal{B}_M$ in M. Then, there exists a tripotent e in M such that*

$$F = e + \mathcal{B}_{M_0(e)} = (\{e\}_\prime)' \cap M,$$

where $\mathcal{B}_{M_0(e)}$ denotes the unit ball of the 0-Peirce space $M_0(e)$ in M. Furthermore, the mapping

$$\begin{aligned} \mathrm{Trip}(M)^\sim &\longrightarrow \mathcal{F}_{w^*}(\mathcal{B}_M) \\ e &\longmapsto (\{e\}_\prime)' \cap M \end{aligned}$$

is an anti-order isomorphism from $\mathrm{Trip}(M)^\sim$ onto the complete lattice $\mathcal{F}_{w^*}(\mathcal{B}_M)$.

□

It is consequently stated that any norm-closed face of the closed unit ball of the predual, M_*, of a real JBW*-triple M is norm-semi-exposed, and that any weak*-closed face, F, of $\mathcal{B}_M$ is weak*-semi-exposed, and if non-empty, of the form $F = e + \mathcal{B}_{M_0(e)}$ for some tripotent e in M ([57, Corollary 3.11]).

One of the advantages of working with dual spaces in the triple theory is the abundance of (complete) tripotents (cf. the Kreim-Milman theorem and [18, Lemma 4.1] and [103, Proposition 3.5], and [90, Lemma 3.3]). However, the situation is not as prosperous for a general JB*-triple, since we can find JB*-triples containing no non-trivial tripotents. For instance, the space $C_0[0,1]$ of complex-valued continuous functions on $[0,1]$ vanishing at 0. On the other hand, it is known that the bidual X^{**} of a JB*-triple X is a JBW*-triple (compare [41]). In contrast with the possible scarcity of tripotents in X, the set $\mathrm{Trip}(X^{**})$ is too big to be in one-to-one correspondence with the set of norm-closed faces of $\mathcal{B}_X$. A new class of tripotents is required in order to have a description of the norm-closed faces of the closed unit ball of a general JB*-triple.

The notion of *compactness* in triple theory appears originally in [56]. The terminology used in [56] is motivated by the behaviour of self-adjoint tripotents in the bidual of any commutative C*-algebra (cf. [4]).

Let us fix a JBW*-triple M. According to what is commented in Section 1.3, it is known that, for each $a \in S(M)$, the sequence $(a^{[2n-1]})$ converges in the weak*-topology of M to the (possibly zero) support tripotent $u_{_M}(a)$ in M (compare [55, Lemma 3.3] or [52, page 130]). The following equality $a = u_{_M}(a) + P_0(u_{_M}(a))(a)$ holds for every a in the above conditions. For a norm-one element a in a JB*-triple X, $u_{_{X^{**}}}(a)$, the support tripotent of a in X^{**}, is always non-zero. Given a in M the support tripotents $u_{_M}(a)$ and $u_{_{M^{**}}}(a)$ need not coincide. To avoid confusion, given a norm-one element a in a JBW*-triple M, unless otherwise stated, we shall write $u(a)$ for the support tripotent of a in M^{**}.

A tripotent u in the JBW*-triple X^{**} is said to be *compact-G_δ* if u coincides with the support tripotent of a norm-one element in X. The tripotent u is said to be *compact* if $u = 0$ or there exists a decreasing net of compact-G_δ tripotents in X^{**} whose infimum is u (compare [56, §4]). We shall write $\mathrm{Trip}_c(X^{**})$ for the set of all compact tripotents in the bidual X^{**} of a JB*-triple X, and $\mathrm{Trip}_c(X^{**})^\sim$ for $\mathrm{Trip}_c(X^{**})$ with a largest element adjoined.

C.M. Edwards and G.T. Rüttimann explored in [56, §5] the consequences of the new notion of compactness for the bidual of a JB*-triple in the particular case of C*-algebras. It was proved in [56, Theorem 5.1] that a partial isometry v in A^{**} belongs locally to A in the sense adopted by C.A. Akemann and G.K. Pedersen if and only if it is compact in the sense introduced in [56]. This generalisation to the triple setting set the course for subsequent results.

The description of the facial structure of the closed unit ball of a general JB*-triple remained as an open question until 2010, when C.M. Edwards, F.J. Fernández-Polo, C.S. Hoskin and A.M. Peralta made clear that compact tripotents should be the key notion for solving this problem. Corollary 3.11 in [52] affirms that every norm-closed face of $\mathcal{B}_X$ in a JB*-triple X is norm-semi-exposed.

Theorem 2.1.5. [52, Theorem 3.10 and Corollary 3.12] *Let X be a JB*-triple, and let F be a norm closed face of the unit ball, $\mathcal{B}_X$, in X. Then, there exists a (unique) compact tripotent e in X^{**} such that*

$$F = F_e^X = (e + \mathcal{B}_{X_0^{**}(e)}) \cap X = (\{e\}_\prime)_\prime ,$$

where $\mathcal{B}_{X_0^{**}(u)}$ denotes the unit ball of the 0-Peirce $X_0^{**}(e)$ in X^{**}. Furthermore, the mapping

$$\begin{aligned} \mathrm{Trip}_c(X^{**})^\sim &\longrightarrow \mathcal{F}_n(\mathcal{B}_X) \\ e &\longmapsto F_e^X = (\{e\}_\prime)_\prime = (e + \mathcal{B}_{X_0^{**}(e)}) \cap X \end{aligned}$$

is an anti-order isomorphism from the complete lattice $\mathrm{Trip}_c(X^{**})^\sim$ *onto the complete lattice $\mathcal{F}_n(\mathcal{B}_X)$.* □

Concerning the dual space of a JB*-triple, the study is succesfully completed by F.J. Fernández-Polo and A.M. Peralta in the same year in the paper [67].

Theorem 2.1.6. *[67, Theorem 2] Let X be a JB*-triple, and let F be a proper weak*-closed face of the closed unit ball of X^*. Then, there exists a (unique) compact tripotent e in X^{**} such that $F = \{e\}_\prime$. Further, the mapping*

$$\begin{aligned} \mathrm{Trip}_c(X^{**})^\sim &\longrightarrow \mathcal{F}_{w^*}(\mathcal{B}_{X^*}) \\ e &\longmapsto \{e\}_{\prime,X^*} \end{aligned}$$

is an order isomorphism from $\mathrm{Trip}_c(X^{**})^\sim$ *onto the complete lattice $\mathcal{F}_{w^*}(\mathcal{B}_{X^*})$.*

It should be remarked that Theorems 2.1.5 and 2.1.6 throw new light on the facial structure of JB*-algebras, a setting in which the solution of this problem was unknown until that time.

The following couple of questions had remained open.

(Q1) Let X be a JB*-triple, and consider the JBW*-triple X^{**}. Comparing Theorems 2.1.3 and 2.1.5, is it possible to distinguish between weak*-closed faces in $\mathcal{B}_{X^{**}}$ associated with compact tripotents in X^{**} and weak*-closed faces in $\mathcal{B}_{X^{**}}$ associated with non-compact tripotents in X^{**}?

(Q2) Let τ be a conjugation on a JB*-triple X, and let $E = X^\tau$. Does there exist a one-to-one correspondence between the norm-closed faces of the closed unit ball of the real JB*-triple E and the set of compact tripotents in E^{**}? Is it possible to establish a correspondence between weak*-closed faces of the closed unit ball of the dual space E^* and compact tripotents in E^{**}?

The problem **(Q1)** posed above found a positive answer in our paper [12]. In order to understand some of the results achieved in the quoted paper, let us introduce a topological notion required to carry out the task. This notion was introduced in [65].

Let X be a Banach space, E a weak*-dense subset of X^* and S a non-zero subset of X^*. Following the notation in [65, §2], we shall say that S is *open relative to* E if $S \cap E$ is $\sigma(X^*, X)$ dense in $\overline{S}^{\sigma(X^*,X)}$. Let X be a JB*-triple. A tripotent e in X^{**} is called *closed relative to* X if $X_0^{**}(e)$ is an open subset of X^{**} relative to X. We shall say that e is *bounded relative to* X if there exists x in the unit sphere of X satisfying that $\{e, e, x\} = e$ (or equivalently, $x = e + P_0(e)(x)$ in X^{**}). One of the main achievements in [65] shows that a tripotent u in X^{**} is compact if and only if it is closed and bounded (cf. [65, Theorem 2.6]).

Theorem 3.6 in [12] shows that any weak*-closed face F in the closed unit ball of the bidual, X^{**}, of a JB*-triple X is associated with a compact tripotent in X^{**} if and only if F is precisely open relative to X.

Theorem 2.1.7. [12, Theorem 3.6, J. Inst. Math. Jussieu] *Let X be a JB*-triple. Suppose F is a proper weak*-closed face of the closed unit ball of X^{**}. Then the following statements are equivalent:*

(a) *F is open relative to X;*

(b) *F is a weak*-closed face associated with a non-zero compact tripotent in X^{**}, that is, there exists a unique non-zero compact tripotent u in X^{**} satisfying that $F = F_u^{X^{**}} = u + \mathcal{B}_{X_0^{**}(u)}$.* □

The arguments employed to prove the characterisation above demand a skilful understanding of the strong*-topology, originally introduced in [10], and subsequently developed in [140] (see also [26, §5.10.2]). Suppose φ is a norm-one normal functional in the predual M_* of a JBW*-triple M. If z is a norm-one element in M satisfying $\varphi(z) = 1$, then the assignment

$$(x, y) \mapsto \varphi\{x, y, z\}$$

defines a positive sesquilinear form on M, which does not depend on the choice of z. We therefore have a prehilbert seminorm on M defined by $\|x\|_\varphi^2 := \varphi\{x, x, z\}$. The *strong*-topology* of M is the topology on M induced by the seminorms $\|x\|_\varphi$ when φ ranges in the unit sphere of M_*. Among the properties of this topology we note that the strong*-topology of M is compatible with the duality (M, M_*) (see [10, Theorem 3.2]). By combining this property with the bipolar theorem, we deduce that the identity

$$\overline{C}^{\sigma(M,M_*)} = \overline{C}^{\text{strong}^*} \tag{2.2}$$

holds for every convex subset $C \subseteq M$. According to this conclusions, it is worth remarking that the strong*-topology is conceived in [10] precisely as a topology for which the Kaplansky density theorem holds ([10, Corollary 3.3]). Another interesting property asserts that the triple product of M is jointly strong*-continuous on bounded sets of M (see [140] or [26, Theorem 5.10.133]).

Theorem 2.1.7 makes also use of some technical lemmata involving geometric inequalities in different settings: C*-algebras ([12, Lemma 3.3]), JB*-algebras ([12, Lemma 3.4]), and finally, JB*-triples ([12, Lemma 3.5]).

The following property is derived from the proof of Theorem 2.1.7, and it will be effectively applied in the subsequent problems of extension of isometries.

Proposition 2.1.8. [12, Proposition 3.7, J. Inst. Math. Jussieu] *Let $(u_\lambda)_{\lambda\in\Lambda}$ be a decreasing net of compact tripotents in the second dual of a JB*-triple X. Suppose $u \neq 0$ is the infimum of the net $(u_\lambda)_{\lambda\in\Lambda}$ in X^{**}. For each λ in the index set, let $F^X_{u_\lambda} = (u_\lambda + \mathcal{B}_{X_0^{**}(u_\lambda)}) \cap X$ and $F^X_u = (u + \mathcal{B}_{X_0^{**}(u)}) \cap X$ denote the corresponding norm closed faces of $\mathcal{B}_X$ associated with u_λ and u, respectively. Then the identity*

$$F^X_u = \overline{\bigcup_{\lambda\in\Lambda} F^X_{u_\lambda}}^{\|\cdot\|}$$

holds. □

On the other hand, it is worth realising that trying to satisfy the interrogations posed in **(Q2)** supposes more than just establishing certain correspondence between tripotents and faces. It means actually to culminate the study of the facial structure in the setting of real JB*-triples. The paper [34] provides a full description of the norm-closed faces of the closed unit ball of a real JB*-triple and the weak*-closed faces of the ball of its dual space.

With the purpose of exhibiting the results in [34], we start by invoking the definition of support tripotent of a functional. Proposition 2 in [75] assures that for each functional ϕ in the predual, M_*, of a JBW*-triple M there exists a unique tripotent $s(\phi)$ in M such that $\phi = \phi P_2(s(\phi))$, and the restriction of ϕ to the JBW*-algebra $M_2(s(\phi))$ is a faithful normal positive functional. We call such an element the *support tripotent* of ϕ in M.

The behaviour of the support tripotents of a functional in the dual space of a JB*-triple gives an interesting identity. Indeed, suppose now that a is a norm-one element in a JB*-triple X. Since $a = u(a) + (a - u(a))$ with $u(a) \perp (a - u(a))$ in X^{**}, it follows from [75, Proposition 1] that $\{u(a)\}_{\prime, X^*} \subseteq \{a\}^{\prime, X^*}$. However, if $\phi \in X^*$ satisfies $\|\phi\| = 1 = \phi(a)$, we deduce from the definition of the support tripotent of ϕ in the JBW*-triple X^{**} that $P_2(s(\phi))(a) = s(\phi)$, and hence $a = s(\phi) + P_0(s(\phi))(a)$ in X^{**} (cf.

[75, Lemma 1.6]). We therefore conclude that $u(a) \geq s(\phi)$ in X^{**}, and thus $\phi(u(a)) = 1$, witnessing that $\{u(a)\}_{\prime,X^*} = \{a\}^{\prime,X^*}$ and consequently,

$$\left(\{a\}^{\prime,X^*}\right)^{\prime,X^{**}} = \left(\{u(a)\}_{\prime,X^*}\right)^{\prime,X^{**}}. \tag{2.3}$$

Henceforth we assume that X is a JB*-triple equipped with a conjugation $\tau : X \to X$, and set $E = X^\tau = \{x \in X : \tau(x) = x\}$. Thus E is a real JB*-triple. Proposition 5.5 in [106] assures that τ is a conjugate linear triple automorphism. The mapping $\tau^\sharp : X^* \to X^*$, given by

$$\tau^\sharp(\varphi)(x) := \overline{\varphi(\tau(x))} \quad x \in X, \varphi \in X^*$$

is a conjugation on X^*, and the correspondence $\varphi \mapsto \varphi|_E$ defines a surjective real linear isometry from $(X^*)^{\tau^\sharp}$ onto E^*. We can similarly define a conjugation $\tau^{\sharp\sharp}$ on X^{**} satisfying that $(X^{**})^{\tau^{\sharp\sharp}}$ is isometrically isomorphic to E^{**}. In particular, the weak*-topology of E^{**} coincides with the restriction to E^{**} of the weak*-topology of X^{**}. Clearly, if a functional φ in X^* is a $\tau^\sharp$-symmetric (equivalently, $\varphi \in E^*$), its support tripotent in X^{**} is $\tau^{\sharp\sharp}$-symmetric and hence lies in E^{**}.

As one deepens in the theory of real JB*-triples (in the sense adopted in [90]), it emerges the intimate relation existing between them and its complexifications. If the attention is turned to the tripotents, it is not hard to see that

$$\mathrm{Trip}(E) = \mathrm{Trip}(X)^\tau = \{e \in \mathrm{Trip}(X) : \tau(e) = e\},$$

and what is even more interesting

$$\mathrm{Trip}(E^{**}) = \mathrm{Trip}(X^{**})^{\tau^{\sharp\sharp}} = \{e \in \mathrm{Trip}(X^{**}) : \tau^{\sharp\sharp}(e) = e\}.$$

Therefore, the result obtained in [57, Lemma 3.4(ii)] for real JBW*-triples can be translated to the bidual of a real JB*-triple and combined with the previous comments to assert that the set $\mathrm{Trip}(E^{**})^\sim$, of all tripotents in E^{**} with a largest element adjoined, is a sub-complete lattice of $\mathrm{Trip}(X^{**})^\sim$.

Let F be a subset of $\mathcal{B}_E$. We set $\mathfrak{F} := P^{-1}(F) \cap \mathcal{B}_X$, where the mapping $P : X \to X$ defined by $P(x) = \frac{1}{2}(x + \tau(x))$ $(x \in X)$ is a contractive real linear projection whose image is E. It can be shown that

$$F \in \mathcal{F}_n(\mathcal{B}_E) \Leftrightarrow \mathfrak{F} \in \mathcal{F}_n(\mathcal{B}_X).$$

That equivalence is an standard example to emphasize the unquestionable connection between a JB*-triple and its real form.

The results accomplished for general JB*-triples in [52] suggest the set of all compact tripotents as the candidate for establishing a link between the triple and the facial structures in the real framework. Therefore, the first step should be to certify that the concept of compactness makes also sense for a real JB*-triple.

If a is a norm-one element in E, $\tau(a) = a$ by definition. Since

$$\tau(a^{[\frac{1}{2n-1}]}) = \tau(a)^{[\frac{1}{2n-1}]} = a^{[\frac{1}{2n-1}]}, \quad \text{and} \quad \tau(a^{[2n-1]}) = \tau(a)^{[2n-1]} = a^{[2n-1]},$$

for all natural n, E^{**} is weak*-closed in X^{**}, and $\tau^{\sharp\sharp}$ is weak*-continuous, it can be deduced that $\tau^{\sharp\sharp}(r(a)) = r(a)$ and $\tau^{\sharp\sharp}(u(a)) = u(a)$, that is, the range and support tripotents of a in X^{**} are $\tau^{\sharp\sharp}$-symmetric elements in X^{**}, and thus they both are tripotents in E^{**}, called *range* and *support* tripotents of a in E^{**}. Combining (2.3) with the previous conclusions we get

$$\{a\}^{\prime,E^{*}} = \{u(a)\}_{\prime,E^{*}}, \text{ and } \left(\{a\}^{\prime,E^{*}}\right)^{\prime,E^{**}} = (\{u(a)\}_{\prime,E^{*}})^{\prime,E^{**}}. \tag{2.4}$$

The above facts allow us to consider the notion of compactness in the setting of real JB*-triples. A tripotent u in E^{**} will be called *compact-G_δ* if u coincides with the support tripotent of a norm-one element in E. The tripotent u is called *compact* if $u = 0$ or there exists a decreasing net of compact-G_δ tripotents in E^{**} whose infimum is u. Following the notation used in the complex setting, we shall write $\mathrm{Trip}_c(E^{**})$ for the set of all tripotents in E^{**} which are compact, and $\mathrm{Trip}_c(E^{**})^{\sim}$ for the set $\mathrm{Trip}_c(E^{**})$ with a largest element adjoined.

The following lemma is borrowed from [57]. We state it in terms of the bidual of a JB*-triple.

Lemma 2.1.9. [57, Lemma 3.6] *Let τ be a conjugation on a JB*-triple X, and let $E = X^{\tau}$. Then for each tripotent $u \in E^{**} = (X^{**})^{\tau^{\sharp\sharp}}$ we have*

$$\{u\}_{\prime,E^{*}} = (\{u\}_{\prime,X^{*}})^{\tau^{\sharp}} = \{u\}_{\prime,X^{*}} \cap E^{*} = \{u\}_{\prime,X^{*}}.$$

Given a tripotent u in the bidual, X^{**}, of a JB*-triple X, C.M. Edwards and G.T. Rüttimann proved that u is compact in X^{**} if and only if the associated weak*-closed face $\{u\}_{\prime,X^{*}}$ of $\mathcal{B}_{X^{*}}$ is actually weak*-semi-exposed (cf. [56, Theorem 4.2]). We proved in Proposition 3.2 of [34] that the same characterisation holds for those tripotents in the bidual of a real JB*-triple which are compact. The extension statement reads as follows:

Proposition 2.1.10. [34, Proposition 3.2, J. Math. Anal. Appl.] *Let τ be a conjugation on a JB*-triple X, and let $E = X^{\tau}$. A tripotent u in the real JBW*-triple E^{**} is compact if and only if $\{u\}_{\prime,E^{*}}$ is weak*-semi-exposed in $\mathcal{B}_{E^{*}}$.*

The arguments followed in the proof of the above proposition take advantage of a plain fact: according to the definition, every compact(-G_δ) tripotent in E^{**} is a $\tau^{\sharp\sharp}$-symmetric compact(-G_δ) tripotent in X^{**}. The next result builds a solid bridge between JB*-triples and real JB*-triples in terms of the compactness of tripotents by showing that the reciprocal is also true.

Corollary 2.1.11. [34, Corollary 3.3, J. Math. Anal. Appl.] *Let τ be a conjugation on a JB*-triple X, and let $E = X^\tau$. Suppose u is a tripotent in E^{**}. Then the following assertions are equivalent:*

(a) *u is compact in E^{**};*

(b) *u is compact in X^{**}.*

Now that the path to travel between the real and the complex settings is properly constructed, we can state Theorem 3.5 from [34], which collects some of the center conclusions in the paper. In particular, a complete description of the norm-closed faces of the closed unit ball of a real JB*-triple is provided.

Theorem 2.1.12. [34, Theorem 3.5, J. Math. Anal. Appl.] *Let τ be a conjugation on a JB*-triple X, and let $E = X^\tau$. Then for each norm-closed proper face F of $\mathcal{B}_E$ there exists a unique compact tripotent $u \in E^{**}$ satisfying $F = (u + \mathcal{B}_{E_0^{**}(u)}) \cap E$. Furthermore, the mapping*

$$u \mapsto (\{u\}_{\prime, E^*})_{\prime, E} = (u + \mathcal{B}_{E_0^{**}(u)}) \cap E$$

is an anti-order isomorphism from $\mathrm{Trip}_c(E^{**})^\sim$ onto $\mathcal{F}_n(\mathcal{B}_E)$.

Since the bidual of any real JB*-triple is a real JBW*-triple, Theorem 2.1.12 combined with Theorem 2.1.4 suggests a sub-problem analogous to the question stated in **(Q1)**. Namely, the necessity of a topological notion which enables us to distinguish between those proper weak*-closed faces of the closed unit ball of the bidual of a real JB*-triple associated to a compact tripotent, and those associated to a non-compact tripotent. Theorem 3.6 in [34] is an extension of Theorem 2.1.7 to the real setting.

Theorem 2.1.13. [34, Theorem 3.6, J. Math. Anal. Appl.] *Let τ be a conjugation on a JB*-triple X, and let $E = X^\tau$. Suppose F is a proper weak*-closed face of the closed unit ball of E^{**}. Then the following statements are equivalent:*

(a) *F is open relative to E, that is, $F \cap E$ is weak*-dense in F;*

(b) *F is a weak*-closed face associated with a non-zero compact tripotent in E^{**}, that is, there exists a unique non-zero compact tripotent u in E^{**} satisfying $F = F_u^{E^{**}} = u + \mathcal{B}_{E_0^{**}(u)}$.*

A suitable application of Corollary 2.1.11 together with some constructions of contractive real linear projections, makes possible to prove the last theorem of this section. Theorem 2.1.6 is crucial to guarantee the existence of a compact tripotent, associated with a given weak*-closed proper face in the dual of a real JB*-triple.

Theorem 2.1.14. [34, Theorem 3.7, J. Math. Anal. Appl.] *Let τ be a conjugation on a JB*-triple X, and let $E = X^\tau$. Then for each weak*-closed proper face F of $\mathcal{B}_{E^*}$ there exists a unique compact tripotent $u \in E^{**}$ satisfying $F = \{u\}_{\prime,E^*}$. Furthermore, the mapping*

$$u \mapsto \{u\}_{\prime,E^*}$$

is an order isomorphism from $\mathrm{Trip}_c(E^{**})^\sim$ *onto* $\mathcal{F}_{w^*}(\mathcal{B}_{E^*})$.

The theorem above culminates the study of the facial structure in the theory of JB*-triples, a task initiated more than thirty years ago. Although the results obtained are interesting by their own right, they will be crucial in the study of the Mazur–Ulam property for certain Banach spaces.

2.2 Unitary elements in Jordan structures

As manifested in previous sections, the natural framework of this thesis includes the structures of C*- and JB*-algebras, as well as their real analogous. Chapter 1 has illustrated that they can be regarded from a more general perspective through the theory of JB*-triples. This fact, far from being a simple chain of inclusions between these three classes of Banach spaces, entails the possibility of going through the three theories and benefiting as much as possible from each other. Indeed, let A be a C*-algebra. Any partial isometry in A can be considered as a tripotent when A is endowed with the triple product in (1.14). In the same manner, the set of maximal partial isometries in A coincides with the set of maximal (or complete) tripotents (see Proposition 1.3.1), and any unitary element in A will be a unitary tripotent. This section is concerned precisely with the unitary elements in Jordan structures. The reader is referred to Chapter 1 for the corresponding definitions. The achievements of our paper [35] will be reviewed, and the triple theory will conform our fundamental tool.

The set of all extreme points, $\partial_e(\mathcal{B}_A)$, of the closed unit ball, $\mathcal{B}_A$, of a unital C^*-algebra A was identified by R.V. Kadison in [97, Theorem 1] (Theorem 1.1.12) as the set of all maximal partial isometries in A. An explicit expression can be provided, namely, an element u in A is an extreme point of the closed unit ball if and only if the following identity holds,

$$(1 - uu^*)A(1 - u^*u) = \{0\}.$$

The equivalence above makes easy to see that, by definition, every unitary in A is an extreme point of its closed unit ball. However, the reciprocal implication is not always true, and thus, the task of determining whether an extreme point in A is a unitary element seems to be exciting.

In 2002, C.A. Akemann and N. Weaver searched for a characterisation of partial isometries, unitaries, and invertible elements in a unital C*-algebra

A in terms of the Banach space structure of certain subsets of A, the dual space, A^*, or the predual, A_*, when A is a von Neumann algebra (cf. [5]). The resulting characterisations are called geometric because only the Banach space structure of A is employed. It should be noted that the geometric characterisation of partial isometries in a C*-algebra was subsequently extended to a geometric characterisation of tripotents in a general JB*-triple (see, [64, 73]).

The geometric characterisation of unitaries actually relies on a good knowledge on the *set of states* of a Banach space X relative to an element x in its unit sphere, $S(X)$, defined by

$$S_x := \{\varphi \in X^* : \varphi(x) = \|\varphi\| = 1\}.$$

The element x is called a *vertex* of $\mathcal{B}_X$ (respectively, a *geometric unitary* of X) if S_x separates the points of X (respectively, spans X^*).

In the terminology of C.A. Akemann and N. Weaver, a unitary element x in a unital C*-algebra A in the sense adopted in this document (i.e. $xx^* = x^*x = \mathbf{1}_A$) is called *algebraic unitary.* They proved that a norm-one element x in a C*-algebra A is an algebraic unitary if and only if S_x spans A^*. In a von Neumann algebra W an analogous characterisation holds when one uses the predual, W_*, in lieu of the dual space and the *set of normal states relative to* x, $S^x = \{\varphi \in W_* : \varphi(x) = \|\varphi\| = 1\}$, in place of S_x (cf. [5, Theorem 3]).

Let A be a unital C*-algebra, and consider its natural Jordan product in (1.3). We exhibited in Proposition 1.2.5 and subsequent comments, that an element u in A is unitary if and only if u is Jordan unitary. Therefore, the characterisations in [5] lead naturally to explore the same conclusions in the Jordan setting. An appropriate version was explored by A. Rodríguez Palacios in [142], where the result is proved in the more general setting of JB*-triples.

Theorem 2.2.1. [142, Theorem 3.1] and [25, Theorem 4.2.24] *Let u be norm-one element in a JB*-algebra M. The following conditions are equivalent:*

(1) *u is Jordan unitary in M;*

(2) *u is a geometric unitary in M;*

(3) *u is a vertex of the closed unit ball of M.*

Surprisingly, as shown by C.-W. Leung, C.-K. Ng, N.-C. Wong in [110], the case of JB-algebras diverges from the result stated for JB*-algebras. Suppose x is a norm-one element in a JB-algebra J, then the following statements are equivalent:

(a) x is a geometric unitary in J;

(b) x is a vertex of the closed unit ball of J;

(c) x is an isolated point of $\mathrm{Symm}(J)$ (endowed with the norm topology);

(d) x is a central unitary in J;

(e) The multiplication operator $M_x : z \mapsto x \circ z$ satisfies $M_x^2 = \mathrm{id}_J$,

(see [110, Theorem 2.6] or [25, Proposition 3.1.15]).

Except perhaps statement (c) above, the previous characterisations rely on the set of states S_x of the underlying Banach space at an element x in the unit sphere, that is, they are geometric characterisations in which the structure of the whole dual space plays an important role.

Unitaries in unital C*- and JB*-algebras have been intensively studied. They constitute the central notion in the Russo–Dye theorem [143] and its JB*-algebra-analogue in the Wright–Youngson–Russo–Dye theorem [167, 148], which are milestone results in the field of Functional Analysis. Actually, the Russo–Dye theorem is effectively applied by M. Mori in [125] to address the problem of extending a surjective isometry between the unit spheres of two unital C*-algebras to a surjective real linear isometry between the whole spaces. This new approach took him to state a new characterisation of unitaries in a unital C*-algebra among the extreme points of its closed unit ball.

Theorem 2.2.2. *[125, Lemma 3.1] Let A be a unital C*-algebra, and let $u \in \partial_e(\mathcal{B}_A)$. Then the following statements are equivalent:*

(a) *u is a unitary (i.e., $uu^* = u^*u = 1$);*

(b) *The set $\mathcal{A}_u = \{e \in \partial_e(\mathcal{B}_A) : \|u \pm e\| = \sqrt{2}\}$ contains an isolated point.*

Our paper [35] emerged with the purpose of exploring the validity of this characterisation in the setting of JB*-algebras. Theorem 2.2.2 presents an advantage when compared with other characterisations of unitaries, namely, it avoids dealing with the dual space, but just with the extreme points of the closed unit ball which are at certain distance. The problem is solved merely as a metric question. This novelty makes even more interesting the task of extending Theorem 2.2.2 to Jordan structures.

In [97] it is affirmed that the only invertible extreme points of the closed unit ball of a unital C*-algebra are precisely the unitary elements. In this sense, the following result presented by G.K. Pedersen in [131] emerges as a meaningful statement. We present an alternative proof based on the different natures coexisting in a unital C*-algebra. It is a nice example of how to combine all the knowledge existing in the setting of associative and Jordan algebras with the more general triple structure, jumping conveniently from a theory to another to get the maximum benefit from each argument.

Proposition 2.2.3. [131, Proposition 3.3] *Let A be a unital C^*-algebra, and let $u \in \partial_e(\mathcal{B}_A)$. Then the following statements are equivalent:*

(i) *u is unitary in A;*

(ii) *$d_A(u, \mathrm{Inv}(A)) < 1$.*

Proof. The implication $(i) \Rightarrow (ii)$ holds since u is invertible with $u^{-1} = u^*$. To see $(ii) \Rightarrow (i)$ let us suppose that

$$1 > \mathrm{d}_A(u, \mathrm{Inv}(A)) := \inf\{\mathrm{d}_A(u, a) : a \in \mathrm{Inv}(A)\},$$

that is, there exists an element $a \in \mathrm{Inv}(A)$ such that $\mathrm{d}_A(u, a) < 1$, or equivalently, $\|u - a\|_A < 1$. By Proposition 3.5 in [103], u is a complete tripotent in A, when A is regarded as a JB*-triple. And hence we can apply Lemma 1.3.6 (see [81, Lemma 6.1]) and guarantee the existence of a complex Hilbert space H, and an isometric unital Jordan *-monomorphism $\psi : A \to B(H)$ such that $\psi(u)^*\psi(u) = \mathbf{1}_{B(H)}$. Since any unital Jordan homomorphism between unital associative algebras preserves invertible elements, we have $\psi(a)$ is an invertible element in $B(H)$.

Making use of the properties of ψ, and applying the hypothesis, we have

$$\begin{aligned}\|\psi(u)^*\psi(a) - \mathbf{1}_{B(H)}\|_{B(H)} &= \|\psi(u)^*\psi(a) - \psi(u)^*\psi(u)\|_{B(H)} \\ &= \|\psi(u)^*\left(\psi(a) - \psi(u)\right)\|_{B(H)} \\ &\leq \|\psi(u)^*\|_{B(H)}\|\psi(a) - \psi(u)\|_{B(H)} \\ &= \|\psi(u)\|_{B(H)}\|\psi(a) - \psi(u)\|_{B(H)} \\ &= \|u\|_A\|a - u\|_A = \|a - u\|_A < 1.\end{aligned}$$

Therefore, $\psi(u)^*\psi(a)$ is an invertible element in $B(H)$. The expression

$$\psi(u)^* = \psi(u)^*\left(\psi(a)(\psi(a))^{-1}\right) = \left(\psi(u)^*\psi(a)\right)(\psi(a))^{-1}$$

allows us to conclude that $\psi(u)^*$ is invertible in the C*-algebra sense (and hence a Jordan invertible) belonging to $B(H)$.

Let $(\psi(u)^*)^{-1}$ denote the inverse of $\psi(u)^*$ in $B(H)$, then by definition $(\psi(u)^*)^{-1}\psi(u)^* = \psi(u)^*(\psi(u)^*)^{-1} = \mathbf{1}_A$. And since $\psi(u)^*\psi(u) = \mathbf{1}_{B(H)}$, we have

$$\begin{aligned}\psi(u) &= \mathbf{1}_{B(H)}\psi(u) \\ &= \left((\psi(u)^*)^{-1}\psi(u)^*\right)\psi(u) \\ &= (\psi(u)^*)^{-1}\left(\psi(u)^*\psi(u)\right) \\ &= (\psi(u)^*)^{-1}\mathbf{1}_{B(H)} = (\psi(u)^*)^{-1}.\end{aligned}$$

We have just shown that the inverse of $\psi(u)^*$ in $B(H)$ is precisely $\psi(u)$. Thus, the equalities $\psi(u)^*\psi(u) = \psi(u)\psi(u)^* = \mathbf{1}_{B(H)}$ hold, and it follows that $\psi(u)$ is unitary in $B(H)$ in the C*-algebra sense. Equivalently, $\psi(u)$ is Jordan unitary in $B(H)$. It is clear that $\psi(u)^* = \psi(u^*)$ lies in $\psi(A)$. As $\psi(A)$ is a unital JB*-subalgebra of $B(H)$ with $\mathbf{1}_{\psi(A)} = \psi(\mathbf{1}_A) = \mathbf{1}_{B(H)}$, we can deduce that $\psi(u)$ is a Jordan unitary in $\psi(A)$.

From a triple point of view, this means that $\psi(u)$ is a unitary tripotent in the JB*-triple $\psi(A)$, and hence $\{\psi(u), \psi(u), \psi(x)\}_{\psi(A)} = \psi(x)$ holds for every $x \in A$. Since the linear bijection $\psi : A \to \psi(A)$ is an isometry between JB*-triples, the Kaup-Banach-Stone theorem (see [106, Proposition 5.5]) assures that ψ is in fact a triple isomorphism. Therefore, $\psi(\{u, u, x\}_A) = \psi(x)$ for every $x \in A$. Finally, it follows by injectivity that $\{u, u, x\}_A = x$ for every $x \in A$, and hence $A_2(u) = A$. We have just proved that u is a unitary tripotent in A, and equivalently, a unitary in A in the C*-algebra sense. □

The previous characterisation was extended to the Jordan setting by A.A. Siddiqui in [149, Lemma 6.6 and Theorem 6.7]. In a first step, the statement is proved for JC*-algebras. We stress here the main argument, which can be expressed, slightly generalised, in triple language as follows:

Proposition 2.2.4. [149, Proof of Lemma 6.6] *Let M be a unital JC*-algebra. Let $u \in \mathrm{Trip}(M)$ and $a \in \mathrm{Inv}(M)$. Suppose $P_0(u^*)(a^{-1}) = 0$, and $\|u - a\| < 1$. Then u is unitary in M.*

These examples above, and the effectiveness of triple techniques when travelling from C*-algebras to JB*-algebras, provide sufficient motivation for addressing an extension of Theorem 2.2.2 to unital JB*-algebras.

Some of the arguments in the proof of [125, Lemma 3.1] can be revisited under the point of view of Jordan algebras.

Proposition 2.2.5. [35, Proposition 3.1, Mediterr. J Math.], [125, Proof of Lemma 3.1] *Let e be a maximal partial isometry in a unital C*-algebra A, and let $l = ee^*$ and $r = e^*e$ denote the left and right projections of e. Suppose we can find two orthogonal projections $p, q \in A$ such that $l = p + q$. Then the element $y = i(p - q)e$ lies in $\mathcal{A}_e = \{y \in \partial_e(\mathcal{B}_A) : \|e \pm y\| = \sqrt{2}\}$, and for each $\theta \in \mathbb{R}$ the element*

$$y_\theta := P_2(e^*)(y) + \cos(\theta) P_1(e^*)(y) + \sin(\theta) P_1(e^*)(1)$$

is a maximal partial isometry in A.

If we further assume that p and q are central projections in lAl, the following statements hold:

(a) The elements $p' = epe^*$ and $q' = eqe^*$ are two orthogonal central projections in rAr, with $r = p' + q'$;

(*b*) *Suppose that e is not unitary in A, and take $y = i(p-q)e$. Then y lies in $\mathcal{A}_e$, and for each $\theta \in \mathbb{R}$ the element*

$$y_\theta := P_2(e^*)(y) + \cos(\theta)P_1(e^*)(y) + \sin(\theta)P_1(e^*)(1)$$

is a maximal partial isometry in A with $\|e \pm y_\theta\| = \sqrt{2}$, i.e. y_θ lies in $\mathcal{A}_e$ (actually, $\frac{e \pm y_\theta}{\sqrt{2}}$ is a maximal partial isometry in A), and $y_\theta \neq y$ for all θ in $\mathbb{R}\backslash \left(2\pi\mathbb{Z} \cup \pi\frac{1+2\mathbb{Z}}{2}\right)$. Furthermore, $\|y - P_2(y)(y_\theta)\| \leq 1 - \cos(\theta)$, and hence $P_2(y)(y_\theta)$ is invertible in $A_2(y)$ for θ close to zero.

□

Let us consider the C*-algebra given by the complex field $A = \mathbb{C}$. It is known that $\mathbb{T} = \partial_e(\mathcal{B}_A)$. Following the notation of Mori in [125], given an extreme point u of $\mathcal{B}_A$, we have

$$\mathcal{A}_u = \{e \in \partial_e(\mathcal{B}_A) : \|u \pm e\| = \sqrt{2}\} = \{iu, -iu\}.$$

But we can also add that $\mathcal{A}_u = \{e \in \partial_e(\mathcal{B}_{\mathbb{C}}) : \|u \pm e\| \leq \sqrt{2}\}$.

The following first approach makes use of the Shirshov-Cohn theorem ([82, Theorems 2.4.14 and 7.2.5]), combined with Wright's theorem ([166, Corollary 2.2 and subsequent comments]), and some tricks involving the U-operators.

Lemma 2.2.6. [35, Lemma 3.2, Mediterr. J Math.] *Let M be a unital JB*-algebra. Let e be a tripotent in M satisfying $\|1 \pm e\| \leq \sqrt{2}$. Then there exist two orthogonal projections p, q in M such that $e = i(p-q)$. Consequently,*

$$\left\{e \in \mathrm{Trip}(M) : \|1 \pm e\| \leq \sqrt{2}\right\} = \{i(p-q) : p, q \in \mathcal{P}(M) \text{ with } p \perp q\}.$$

□

Given a tripotent u in a JB*-triple E, the Peirce 2-subspace $E_2(u)$ is a unital JB*-algebra with unit u, and hence, the first statement in the next corollary is a straight consequence of our previous lemma.

Corollary 2.2.7. [35, Corollary 3.3, Mediterr. J Math.] *Let u be a tripotent in a JB*-triple E. Then*

$$\{e \in \mathrm{Trip}(E_2(u)) : \|u \pm e\| \leq \sqrt{2}\} = \{i(p-q) : p, q \in \mathcal{P}(E_2(u)) \text{ with } p \perp q\}.$$

Furthermore, if u is unitary in E, then

$$\mathcal{E}_u = \left\{e \in \partial_e(\mathcal{B}_E) : \|u \pm e\| \leq \sqrt{2}\right\} = iSymm(E_2(u)) \tag{2.5}$$

$$= \{i(p-q) : p, q \in Trip(E), p, q \leq u, p \perp q, p + q = u\}$$

and the elements $\pm iu$ are isolated in $\mathcal{E}_u$. □

The Jordan version of Theorem 2.2.2 $(a) \Rightarrow (b)$ has been established in Corollary 2.2.7 even in the setting of JB*-triples. A technical result is required to state the reciprocal implication.

The proof of the following proposition makes skilfully use of [75, Lemma 1.1], result in which Y. Friedman and B. Russo assure that, given an arbitrary tripotent u in a JB*-triple E, and for each $\lambda \in \mathbb{C}$ with $|\lambda| = 1$, the mapping

$$S_{\overline{\lambda}}(u) = \overline{\lambda}^2 P_2(u) + \overline{\lambda} P_1(u) + P_0(u)$$

is an isometric triple isomorphism on E.

Proposition 2.2.8. [35, Proposition 3.4, Mediterr. J Math.] *Let u be a tripotent in a JB*-triple E, and let*

$$\mathcal{E}_u = \{e \in \partial_e(\mathcal{B}_E) : \|u \pm e\| \leq \sqrt{2}\}.$$

Then every element $y \in \mathcal{E}_u$ with $P_1(u)(y) \neq 0$ or $P_0(u)(y) \neq 0$ is non-isolated in $\mathcal{E}_u$. Consequently, every isolated element $y \in \mathcal{E}_u$ belongs to $iSymm(E_2(u))$. □

A new statement with a slightly modification in the hypotheses derives from same arguments followed in the proposition above.

Proposition 2.2.9. [35, Remark 3.5, Mediterr. J Math.] *Let u be a tripotent in a JB*-triple E, and let*

$$\tilde{\mathcal{E}}_u = \{e \in Trip(E) : \|u \pm e\| \leq \sqrt{2}\}.$$

Then every element $y \in \tilde{\mathcal{E}}_u$ with $P_1(u)(y) \neq 0$ or $P_0(u)(y) \neq 0$ is non-isolated in $\tilde{\mathcal{E}}_u$. □

The Shirshov-Cohn type theorem established in Lemma 1.3.7, for two orthogonal tripotents and its adjoints, together with its consequence exposed in Proposition 1.3.8, provides the appropriate environment to apply Proposition 2.2.5. The collection of all the partial results established before, and the deep conclusions of the quoted proposition make possible to establish a generalisation of Theorem 2.2.2. This achievement is the main theorem of our paper [35], and reaches successfully its goal.

Theorem 2.2.10. [35, Theorem 3.8, Mediterr. J Math.] *Let u be an extreme point of the closed unit ball of a unital JB*-algebra M. Then the following statements are equivalent:*

(a) u is a unitary tripotent;

(b) The set $\mathcal{M}_u = \{e \in \partial_e(\mathcal{B}_M) : \|u \pm e\| \leq \sqrt{2}\}$ contains an isolated point.

□

Every JB*-triple E admitting a unitary element u is a unital JB*-algebra $E = E_2(u)$ with Jordan product and involution given by $x \circ_u y = \{x, u, y\}$, and $x^{*_u} = \{u, x, u\}$. Actually, there is a one-to-one (geometric) correspondence between the class of unital JB*-algebras and the class of JB*-triples admitting a unitary element. The next corollary is thus a rewording of our Theorem 2.2.10.

Corollary 2.2.11. [35, Corollary 3.9, Mediterr. J Math.] *Let E be a JB*-triple admitting a unitary element. Suppose u is an extreme point of the closed unit ball of E. Then the following statements are equivalent:*

(a) *u is a unitary tripotent;*

(b) *The set $\mathcal{E}_u = \{e \in \partial_e(\mathcal{B}_E) : \|u \pm e\| \leq \sqrt{2}\}$ contains an isolated point.*

□

Chapter 3

Extension of isometries

This chapter comprises the core of the present thesis, which is focused on the extension of isometries. Therefore, the main results obtained in the papers [33, 12, 34], and [36] will be exposed along these lines. We shall first deal with the Mazur–Ulam property as a natural generalisation of Tingley's problem, and continue with a brief incursion into the strategies frequently used to determine whether a Banach space has the Mazur–Ulam property. Three axes will conform the structure of the rest of the chapter, that is, the strong Mankiewicz property and the novelties stated in that sense, new examples of Banach spaces satisfying the Mazur–Ulam property, and a Hatori-Molnár type theorem intended to extend surjective isometries between unitary sets in Jordan structures.

Let (X, d_X) and (Y, d_Y) be two (real or complex) metric spaces, where d_X and d_Y denote the distances in X and Y, respectively. The notion of *isometry* adopted in this memoir is that of a mapping $T : X \to Y$ which preserves distances, that is,

$$d_Y(T(x), T(y)) = d_X(x, y), \quad \forall x, y \in X.$$

If we assume that X and Y are (real or complex) normed spaces, and considering the distances induced by the norms, we shall say that a mapping $T : X \to Y$ is an isometry whenever $\|T(x) - T(y)\|_Y = \|x - y\|_X$ holds for every $x, y \in X$.

We recall that a mapping $T : X \to Y$ between two vector spaces X and Y is *affine* if the identity $T(\alpha x + (1-\alpha)y) = \alpha T(x) + (1-\alpha)T(y)$ holds for every $\alpha \in \mathbb{R}$, and every $x, y \in X$. Every linear mapping is clearly affine, and it can be easily observed that any affine mapping $T : X \to Y$ such that $T(0) = 0$ is linear.

Inspired by the Mazur–Ulam property ([119]), P. Mankiewicz devoted his paper [118] to determining whether an isometry $T : U \to Y$ from a subset U of a real normed space X into a real normed space Y admits an extension to an isometry from X onto Y. This author proved that every surjective isometry between convex bodies in real normed spaces can be uniquely extended to an affine isometry between the whole spaces. This conclusion particularly holds for the closed unit balls. Therefore, Mankiewicz's theorem assures that the algebraic structure of a (real) normed space is determined by the metric space given by its closed unit ball.

In the spirit of P. Mankiewicz, D. Tingley focused, intuitively, his attention on the unit spheres, and proposed, in [161], an isometric extension problem. The so-called *Tingley's problem* asks if every surjective isometry between the unit spheres of two Banach spaces X and Y admits an extension to a surjective real linear isometry between the spaces. Tingley's problem has attracted the attention of a wide audience all over the world since it was posed in 1987. Its simple statement hides a hard question, and a deep conclusion. Indeed, two Banach spaces X and Y solving affirmatively Tingley's problem are real-linearly isometrically isomorphic if and only if their unit spheres are isometrically isomorphic as metric spaces. Let us observe that a surjective isometry between the unit spheres of two complex Banach spaces need not admit an extension to a surjective complex linear nor conjugate-linear isometry between the whole spaces (it can be considered, for instance, the conjugation in $\mathbb{C}$).

More than thirty years since Tingley's problem was formulated, the question remains unsolved, even in the case of two Banach spaces of dimension two. Nevertheless, the wide list of positive solutions to Tingley's problem (see for example, [164, 43, 44, 45, 46, 47, 60, 112, 155, 154, 156, 70, 71, 72, 134, 136, 139, 158, 159, 160, 153, 61, 69, 62, 125, 24, 165]) shows the lively interest provoked. The introduction of this project (page xii) includes a brief journey into the development of Tingley's problem in some classical Banach spaces, and indicates an ongoing collection of pairs (X, Y) of Banach spaces for which Tingley's problem admits a positive solution. The reader is also referred to the papers [48, 169], where the history of the first stages of this problem can be found, and to the survey [136] where a detailed overview of the more recent state-of-the-art is available.

L. Cheng and Y. Dong introduced the Mazur–Ulam property in 2011 (see [28]). A Banach space X satisfies the *Mazur–Ulam property* if for any Banach space Y, every surjective isometry $\Delta : S(X) \to S(Y)$ admits an extension to a surjective real linear isometry from X onto Y, where $S(X)$ and $S(Y)$ denote the unit spheres of X and Y, respectively. An equivalent reformulation tells that X satisfies the Mazur–Ulam property if Tingley's problem admits a positive solution for every surjective isometry from $S(X)$ onto the unit sphere of any Banach space Y.

Our knowledge on the class of Banach spaces satisfying the Mazur–Ulam property is a bit more reduced than on those solving Tingley's problem. The reason of this fact is probably a simple matter of time. This class includes classical Banach spaces like $c_0(\Gamma, \mathbb{R})$ of real null sequences ([47, Corollary 2]), the space $\ell_\infty(\Gamma, \mathbb{R})$ of all real-valued bounded functions on a discrete set Γ ([112, Main Theorem]), the space $C(K, \mathbb{R})$ of all real-valued continuous functions on a compact Hausdorff space K [112, Corollary 6], and the spaces $L^p((\Omega, \mu), \mathbb{R})$ of real-valued measurable functions on an arbitrary σ-finite measure space (Ω, μ) for all $1 \leq p \leq \infty$ [155, 154, 156]. More recent achievements will be reviewed along the next sections.

Since Tingley's problem was posed, and later the Mazur–Ulam property introduced, different approaches have taken place in order to handle these isometric extension problems. Let X and Y be two Banach spaces. Suppose $\Delta : S(X) \to S(Y)$ is a surjective isometry. We can always consider the positive homogeneous extension, that is, the bijection $T : X \to Y$ given by

$$T(x) = \begin{cases} \|x\| \, \Delta\left(\frac{x}{\|x\|}\right) & x \neq 0 \\ \\ 0 & x = 0. \end{cases}$$

The mapping T is clearly a positively homogeneous bijection. Nevertheless, the task of deciding whether T is an isometry is a really hard question. In return, T will be real linear as soon as it is an isometry by the Mazur–Ulam theorem.

Results of geometric nature are frequently used as helpful tools. We highlight next one of this kind established by X.N. Fang and J.H. Wang in [60]. In the field $\mathbb{K}$, it is known that two elements x and y in the closed unit ball of $\mathbb{K}$ are such that $|x + y| = 2$ if and only if $x = y$ and they both lie in $S(\mathbb{K})$.

Theorem 3.0.1. [60, Corollary 2.2] *Let X and Y be normed spaces, and let $\Delta : S(X) \to S(Y)$ be a surjective isometry. Then, for any x, y in $S(X)$, we have $\|x + y\|_X = 2$ if and only if $\|\Delta(x) + \Delta(y)\|_Y = 2$.*

This result plays a role, for example, in some of the proofs in [61] and [135]. It is also usefully applied in some arguments of our paper [33], whose main achievements related to the Mazur–Ulam property in the space of complex-valued continuous functions will be exposed in subsequent sections.

In the spirit of G.G. Ding and R. Liu, Lemma 2.1 in [60] constitutes another effective extension result which deserves to be stated. It was also proved by X.N. Fang and J.H. Wang, and we make use of it to get some of the main conclusions in [33] and [34].

Lemma 3.0.2. [60, Lemma 2.1] *Let X and Y be real normed spaces. Suppose $\Delta : S(X) \to S(Y)$ is an onto isometry. If for any $x, y \in S(X)$, we have*

$$\|\Delta(y) - \lambda\Delta(x)\| \leq \|y - \lambda x\|,$$

for all $\lambda > 0$, then Δ can be extended to a surjective real linear isometry from X onto Y. □

M. Mori and N. Ozawa state the following extension lemma inspired by the previous one. It should be noted that it is a kind of reformulation of the previous lemma.

Lemma 3.0.3. [126, Lemma 6] *Let X and Y be real normed spaces. Suppose $\Delta : S(X) \to S(Y)$ is an onto isometry. Assume that there are families $\{\varphi_j\}_j \subset \mathcal{B}_{X^*}$ and $\{\psi_j\}_j \subset \mathcal{B}_{Y^*}$ such that $\varphi_j = \psi_j \circ \Delta$ and that the family $\{\varphi_j\}_j$ is norming for X. Then Δ extends to a linear isometry.*

The strategy mainly followed in the isometric extension problems, and in the Mazur–Ulam property concretely, relies on a good knowledge of the facial structure of the Banach space involved. This is probably one of the reasons of having no solution for these questions in the general case. As commented in section 2.1, Lemma 2.1.1 allows to have the most general perspective possible, and hence becomes a key result in that sense. By adding some conditions, the result was improved in Proposition 2.4 of [61]. We now indicate a corollary of the quoted statement. It should be observed that the hypothesis assumed are satisfied by X and Y being C*-algebras ([4, Theorems 4.10 and 4.11]), hermitian parts of C*-algebras ([54, Corollary 5.1] and [4, Theorem 3.11]), preduals of von Neumann algebra ([55, Theorems 5.3 and 5.4]), preduals of the hermitian part of a von Neumann algebra ([53, Theorem 4.4] and [55, Theorem 4.1]) or more generally, JB*-triples (cf. [52, Corollary 3.11] and [67, Corollary 1]) or preduals of JBW*-triple ([55, Corollaries 4.5 and 4.7]).

Corollary 3.0.4. [61, Corollary 2.5] *Let X and Y be Banach spaces satisfying the following two properties:*

(i) *Every norm-closed face of $\mathcal{B}_X$ is norm-semi-exposed, and every norm-closed face of $\mathcal{B}_Y$ is norm-semi-exposed;*

(ii) *Every weak*-closed proper face of $\mathcal{B}_{X^*}$ is weak*-semi-exposed, and every weak*-closed proper face of $\mathcal{B}_{Y^*}$ is weak*-semi-exposed.*

Let $\Delta : S(X) \to S(Y)$ be a surjective isometry. The following statements hold:

1. *Let F be a convex set in $S(X)$. Then F is a norm-closed face of $\mathcal{B}_X$ if and only if $\Delta(F)$ is a norm-closed face of $\mathcal{B}_Y$;*

2. *Given $e \in S(X)$, we have $s \in \partial_e(\mathcal{B}_X)$ if and only if $\Delta(e) \in \partial_e(\mathcal{B}_Y)$.*

The facial stability under surjective isometries between unit spheres of Banach spaces is also shown in the following proposition, which has been borrowed from [136]. The result is deduced from similar arguments to those used by M. Mori in [125, Proposition 2.3].

Proposition 3.0.5. [136, Proposition 2.4] *Let $\Delta : S(X) \to S(Y)$ be a surjective isometry between the unit spheres of two Banach spaces. Then the following statements hold:*

(a) If $\mathcal{M}$ is a maximal proper face of $\mathcal{B}_X$, then $\Delta(-\mathcal{M}) = -\Delta(\mathcal{M})$.

(b) If X and Y satisfy the hypotheses of Corollary 3.0.4, then the identity $\Delta(-F) = -\Delta(F)$ holds for every proper norm-closed face of $\mathcal{B}_X$.

Related to the behaviour of surjective isometries between unit spheres of Banach spaces, support functionals associated with maximal proper norm-closed faces become useful ingredients. The following lemmata can be derived from similar ideas to those in [44, 112, 94, 134], and prove the existence of supporting functionals in $C(K, H)$-type spaces, as well as in any JB*-triple. The support functionals are associated with the $A(t_0, \lambda)$-type maximal norm-closed faces in the case of $C(K)$, which are defined in (1.7).

Lemma 3.0.6. [33, Lemma 2.1, Linear and Multilinear Algebra]
Let $\Delta : S(C(K)) \to S(X)$ be a surjective isometry, where K is a compact Hausdorff space and X is a complex Banach space. Then for each $t_0 \in K$ and each $\lambda \in \mathbb{T}$ the set

$$supp_{\Delta}(t_0, \lambda) := \{\varphi \in X^* : \|\varphi\| = 1, \ and \ \varphi^{-1}(\{1\}) \cap \mathcal{B}_X = \Delta(A(t_0, \lambda))\}$$

is a non-empty weak-closed face of $\mathcal{B}_{X^*}$.* □

More generally, let K be a compact Hausdorff space and let H be a complex Hilbert space with $\dim(H) \geq 2$. Considering the $A(t_0, x_0)$-faces in (1.7), we can always find support functionals for each (t_0, x_0) in $K \times S(H)$.

Lemma 3.0.7. [34, Lemma 5.1, J. Math. Anal. Appl.]
Let K be compact Hausdorff space, and let H be a complex Hilbert space with $\dim(H) \geq 2$. Let $\Delta : S(C(K, H)) \to S(Y)$ be a surjective isometry, where Y is a real Banach space. Then for each $t_0 \in K$ and each $x_0 \in S(H)$ the set

$$supp_{\Delta}(t_0, x_0) := \{\psi \in Y^* : \|\psi\| = 1, \ and \ \psi^{-1}(\{1\}) \cap \mathcal{B}_Y = \Delta(A(t_0, x_0))\}$$

is a non-empty weak-closed face of $\mathcal{B}_{Y^*}$, where, in this case, $A(t_0, x_0) := \{f \in S(C(K, H)) : f(t_0) = x_0\}$.* □

Concerning JB*-triples, it is worth observing that Kadison's transitivity theorem ([152, Theorem II.4.18]) was extended by L.J. Bunce, F.J. Fernández-Polo, J. Martínez Moreno, and A.M. Peralta to the setting of JB*-triples (cf. [22, Theorem 3.3]). Suppose X is a JB*-triple. Then every maximal norm-closed proper face of $\mathcal{B}_X$ is of the form

$$F_e^X = (e + \mathcal{B}_{X_0^{**}(e)}) \cap X, \tag{3.1}$$

where e is a minimal tripotent in X^{**} (see [22, Corollary 3.5]).

Lemma 3.0.8. [12, Lemma 4.7, J. Inst. Math. Jussieu] *Let X be a JB*-triple and let Y be a real Banach space. Suppose $\Delta : S(X) \to S(Y)$ is a surjective isometry. Then for each maximal proper norm-closed face F of the closed unit ball of X the set*

$$supp_{\Delta}(F) := \{\psi \in Y^* : \|\psi\| = 1, \text{ and } \psi^{-1}(\{1\}) \cap \mathcal{B}_Y = \Delta(F)\}$$

is a non-empty weak-closed face of $\mathcal{B}_{Y^*}$; in other words, for each minimal tripotent e in X^{**} the set*

$$supp_{\Delta}(F_e^X) := \{\psi \in Y^* : \|\psi\| = 1, \text{ and } \psi^{-1}(\{1\}) \cap \mathcal{B}_Y = \Delta(F_e^X)\}$$

is a non-empty weak-closed face of $\mathcal{B}_{Y^*}$.* □

3.1 Strong Mankiewicz property

Among other outstanding achievements, M. Mori and N. Ozawa introduce in the paper [126] a new point of view in order to address isometric extension problems like the Mazur–Ulam property. Following the just quoted authors, we shall say that a convex subset $\mathcal{K}$ of a normed space X satisfies the *strong Mankiewicz property* if every surjective isometry Δ from $\mathcal{K}$ onto an arbitrary convex subset L in a normed space Y is affine. Every convex subset of a strictly convex normed space satisfies the strong Mankiewicz property because it is uniquely geodesic (see [8, Lemma 6.1]), and there exist examples of convex subsets of $L^1[0,1]$ which do not satisfy this property (see [126, Example 5]). In [126, Theorem 2] a sufficient condition to get that property is provided by showing that some of the hypotheses in Mankiewicz's theorem can be somehow relaxed. The conclusions exposed in the following result have been borrowed from the statement of [126, Theorem 2] and its proof.

Theorem 3.1.1. [126, Theorem 2] *Let X be a Banach space such that the closed convex hull of the extreme points, $\partial_e(\mathcal{B}_X)$, of the closed unit ball, $\mathcal{B}_X$, of X has non-empty interior in X. Then every convex body $K \subset X$ satisfies the strong Mankiewicz property. Furthermore, suppose L is a convex subset of a normed space Y, and $\Delta : \mathcal{B}_X \to L$ is a surjective isometry. Then Δ can be uniquely extended to an affine isometry from X onto a norm-closed subspace of Y.*

The Russo–Dye theorem is effectively applied by M. Mori and N. Ozawa in [126, Corollary 3] to deduce that every convex body of a unital C*-algebra satisfies the strong Mankiewicz property. As the authors also showed, the same conclusion holds for real von Neumann algebras, by the appropriate real version of the Russo–Dye theorem due to B. Li (see [111, Theorem 7.2.4]), or directly from the one established by J.C. Navarro and M.A. Navarro in [129, Corollary 6].

J.D.M. Wright and M.A. Youngson extended the Russo–Dye theorem to the setting of unital JB*-algebras in 1977 (see [167]) and, from a different point of view, A.A. Siddiqui arrived to the same conclusions in 2010 (cf. [150]). Concerning triple structures, we cannot always assume the existence of unitary tripotents. In the case of JBW*-triples, the abundance of extreme points of the closed unit ball will make up for that absence. By extending the result [29, Theorem 3], due to H. Choda, which shows that every element in the unit ball of a von Neumann algebra is the average of two extreme points of the unit ball, A.A. Siddiqui proved in [148, Theorem 5] that every element in the unit ball of a JBW*-triple is the average of two extreme points of the closed unit ball. In the setting in which the notion of unitary is no longer applicable, these results will be called Krein–Milman type theorems.

The application of the strong Mankiewicz property is decisive in the study of the Mazur–Ulam property for unital C*-algebras and real von Neumann algebras in [126] developed by M. Mori and N. Ozawa. Therefore, Theorem 3.1.1 seems to be a powerful tool for our purposes of extending isometries. Looking at the hypotheses of the just quoted result, we search for those Banach spaces whose closed unit ball coincides with the closed convex hull of its extreme points, that is, Banach spaces satisfying a Krein–Milman type theorem.

Our first result in the topic belongs to the paper [12] and it is a straight consequence of the Krein–Milman type theorem for JBW*-triples [148, Theorem 5] and Theorem 3.1.1.

Corollary 3.1.2. [12, Corollary 2.2, J. Inst. Math. Jussieu] *The closed unit ball of every JBW*-triple M satisfies the strong Mankiewicz property. Consequently, every convex body in a JBW*-triple satisfies the same property. Furthermore, if L is a convex subset of a normed space Y, then every surjective isometry $\Delta : \mathcal{B}_M \to L$ can be uniquely extended to an affine isometry from M onto a norm closed subspace of Y.* □

In the same line, a version using the facial structure of JB*-triples reads as follows:

Lemma 3.1.3. [12, Lemma 4.10, J. Inst. Math. Jussieu] *Let e be a non-zero compact tripotent in the second dual of a JBW*-triple M. Let a be an element in the norm-closed face F_e^M asociated with e. Then a can be written as the average of two extreme points of $\mathcal{B}_M$ belonging to the face F_e^M.* □

Let K be a compact Hausdorff space and let $\mathcal{H}$ be a real Hilbert space. We gather some existing results to establish a Krein–Milman type theorem.

W.G. Bade proved that $\mathrm{co}(\partial_e \mathcal{B}_{C(K,\mathbb{R})})$ is dense in the closed unit ball of the space $C(K,\mathbb{R})$, of all real-valued continuous functions on a compact Hausdorff space K, if and only if K is totally disconnected (see [7]). The complex case was considered by R.R. Phelps in [141], where he showed that the closed unit ball of the commutative unital C*-algebra $C(K)$ coincides with the closed convex hull of its extreme points. Since the extreme points of the closed unit ball of $C(K)$ are precisely the unitary elements in $C(K)$, Phelps provided in fact a particular case of the celebrated Russo–Dye theorem (cf. [143]). When the complex field is replaced with a general Banach space X with $\dim(X) \geq 3$, the problem of determining whether its closed unit ball coincides with the closed convex hull of its extreme points was explored by authors like J. Cantwell [27], N.T. Peck [132], J.F. Mena-Jurado, J.C. Navarro-Pascual and V.I. Bogachev [121, 16].

Proposition 3.1.4. [141, 27, 132], [34, Proposition 4.5, J. Math. Anal. Appl.] *Let K be a compact Hausdorff space and let $\mathcal{H}$ be a real Hilbert space with $\dim(\mathcal{H}) \geq 2$. Then the closed unit ball of $C(K, \mathcal{H})$ coincides with the closed convex hull of its extreme points. Consequently, every convex body in $C(K, \mathcal{H})$ satisfies the strong Mankiewicz property.*

The previous proposition can be slightly generalised with subtle variations of the arguments from the same sources. We shall denote by $\dim(K)$ the *covering dimension* of K (see [59, page 385]). We simply observe that if K is Stonean or extremely disconnected then it is strongly zero dimensional, and hence $\dim(K) = 0$ (see [59, Theorem 6.2.25 and page 385]).

Proposition 3.1.5. *Let X be a strictly convex real Banach space and let K be a compact Hausdorff space satisfying one of the following hypotheses:*

$(h.1)$ *X is infinite dimensional;*

$(h.2)$ *If $\dim(X) = n$ $(n \in \mathbb{N})$, then $\dim(K) \leq n - 1$.*

Then every convex body in $C(K, X)$ satisfies the strong Mankiewicz property.

Let A be a unital C*-algebra, let $\varphi : A \to \mathbb{C}$ be a (continuous) multiplicative functional, and let $A_{\mathbb{R}}^{\varphi} := \varphi^{-1}(\mathbb{R}) = \{a \in A : \varphi(a) \in \mathbb{R}\}$. Clearly $A_{\mathbb{R}}^{\varphi}$ is a real C*-subalgebra of A. M. Mori and N. Ozawa prove in [126, Lemma 19] that $\mathcal{B}_{A_{\mathbb{R}}^{\varphi}}$ coincides with the closed convex hull of the unitary elements in $A_{\mathbb{R}}^{\varphi}$.

New examples of Banach spaces satisfying a Krein–Milman type theorem are provided in our paper [34] following the previous method in the JB*-triple setting.

Proposition 3.1.6. [34, Proposition 4.4, J. Math. Anal. Appl.] *Let A be a unital C*-algebra and let $\psi : A \to \mathbb{C}$ be a (continuous) non-zero triple homomorphism. Then the closed unit ball of the real JB*-triple $A_{\mathbb{R}}^{\psi} := \psi^{-1}(\mathbb{R})$ coincides with the closed convex hull of unitary tripotents in $A_{\mathbb{R}}^{\psi}$. Consequently, $\mathcal{B}_{A_{\mathbb{R}}^{\psi}}$ and every convex body $\mathcal{K} \subset A_{\mathbb{R}}^{\psi}$ satisfy the strong Mankiewiecz property.*

In [34, Theorem 4.9] we established a version of Lemma 19 in [126], which makes use of norm-closed JB*-subtriples of $C(K, H)$, and required technical conclusions from [34, Lemma 4.6], and the application of Proposition 3.1.6.

Proposition 3.1.7. [34, Proposition 4.9, J. Math. Anal. Appl.]
Let K be a compact Hausdorff space and let H be a complex Hilbert space. Suppose $x_0 \in S(H)$ and $\mathcal{O} \neq \emptyset$ is an open subset of K. Let us denote $p = \chi_{\mathcal{O}}$, $N = \{a \in C(K, H) : ap = \mu x_0 \otimes p, \text{ for some } \mu \in \mathbb{C}\}$, and $\varphi : N \to \mathbb{C}$ the triple homomorphism defined by $\varphi(a) = \langle a(t_0)|x_0\rangle$ $(a \in N)$, where t_0 is any element in $\mathcal{O}$. Then the closed unit ball of $N_{\mathbb{R}}^{\varphi} := \varphi^{-1}(\mathbb{R})$ coincides with the closed convex hull of its extreme points. Consequently, $\mathcal{B}_{N_{\mathbb{R}}^{\varphi}}$ satisfies the strong Mankiewicz property.

A similar statement in the finite dimensional real setting reads as follows:

Proposition 3.1.8. [34, Proposition 4.10, J. Math. Anal. Appl.]
Let K be a compact Hausdorff space and let $\mathcal{H}$ be a finite-dimensional real Hilbert space with dim$(\mathcal{H}) \geq 2$. Suppose $x_0 \in S(\mathcal{H})$ and $\mathcal{O} \neq \emptyset$ is an open subset of K. Let us denote $p = \chi_{\mathcal{O}}$, and

$$N = \{a \in C(K, \mathcal{H}) : ap = \mu x_0 \otimes p, \text{ for some } \mu \in \mathbb{R}\}.$$

Then the closed unit ball of the real JB-triple N coincides with the closed convex hull of its extreme points. Consequently, $\mathcal{B}_N$ satisfies the strong Mankiewicz property.*

3.2 The Mazur–Ulam property

This section is devoted to presenting new examples of Banach spaces satisfying the Mazur–Ulam property. We shall be concerned with commutative von Neumann algebras in section 3.2.1, where the novelties of the paper [33] will be exposed. Section 3.2.2 will explore the isometric extension problem in the setting of JBW*-triples, a task almost successfully addressed in the paper [12]. Finally, the space $C(K, H)$, of all continuous functions on a compact Hausdorff space K which take values in a Hilbert space H, will conform the centre of section 3.2.3, which exhibits the achievements obtained in [34].

3.2.1 Commutative von Neumann algebras

Our main goal is to explore the Mazur–Ulam property in any commutative von Neumann algebra. As commented in section 1.1, we shall pursue our objective by restricting the study to the unital commutative C*-algebra $C(K)$ of all continuous functions on a compact Hausdorff space K, assuming some conditions over K to avoid difficulties.

Henceforth K will be a compact Hausdorff space, X will be an arbitrary complex Banach space, and $\Delta : S(C(K)) \to S(X)$ a surjective isometry. Let $t_0 \in K$ and $\lambda \in \mathbb{T}$. We recall that $A(t_0, \lambda) = \{f \in S(C(K)) : f(t_0) = \lambda\}$ defined in (1.7) is a maximal norm-closed face of $\mathcal{B}_{C(K)}$, and Lemma 3.0.6 guarantees the existence of elements in the set $\text{supp}_\Delta(t_0, \lambda)$ of associated support functionals.

The development of the arguments in the paper [33] is based on a series of technical results. We begin precisely exploring the interaction of support functionals associated to any maximal face $A(t_0, \lambda)$ and Δ. The proof of all these statements uses, among other ingredients, the conclusions of Theorem 3.0.1, several applications of Urysohn's lemma (Appendix A) and a technical consequence of the parallelogram law.

Lemma 3.2.1. [33, Lemma 2.4, Linear and Multilinear Algebra] *Suppose K is a compact Hausdorff space. Let $\Delta : S(C(K)) \to S(X)$ be a surjective isometry, where X is a complex Banach space. Then for each t_0 in K and each $\lambda \in \mathbb{T}$ we have*

$$\varphi\Delta(f) = -1, \quad \forall f \in A(t_0, -\lambda), \ \forall \varphi \in supp_\Delta(t_0, \lambda).$$

Consequently,

$$supp_\Delta(t_0, -\lambda) = -supp_\Delta(t_0, \lambda), \quad and \quad \Delta(-A(t_0, \lambda)) = -\Delta(A(t_0, \lambda)).$$

□

We highlight the following two results which contain a generalised version of [94, Lemma 2.3 and Proposition 2.4]. Theorem 3.0.1 and Urysohn's lemma are again useful tools in the proof, as well as Lemma 3.2.1.

Lemma 3.2.2. [33, Lemma 2.5, Linear and Multilinear Algebra] *Suppose K is a compact Hausdorff space. Let $\Delta : S(C(K)) \to S(X)$ be a surjective isometry, where X is a complex Banach space. Then the following statements hold:*

(a) *For every $t_0 \neq t_1$ in K and every $\lambda, \mu \in \mathbb{T}$ we have*

$$supp_\Delta(t_0, \lambda) \cap supp_\Delta(t_1, \mu) = \emptyset;$$

(b) *Given $\mu, \nu \in \mathbb{T}$ with $\mu \neq \nu$, and t_0 in K, we have*

$$supp_\Delta(t_0, \nu) \cap supp_\Delta(t_0, \mu) = \emptyset.$$

□

Proposition 3.2.3. [33, Proposition 2.6, Linear and Multilinear Algebra] *Suppose K is a compact Hausdorff space, X is a complex Banach space, and $\lambda \in \mathbb{T}$. Let $\Delta : S(C(K)) \to S(X)$ be a surjective isometry. Let t_0 be an element in K and let φ be an element in $supp_\Delta(t_0, \lambda)$. Then $\varphi\Delta(f) = 0$, for every $f \in S(C(K))$ with $f(t_0) = 0$. Furthermore, $|\varphi\Delta(f)| < 1$, for every $f \in S(C(K))$ with $|f(t_0)| < 1$, and every $\varphi \in supp_\Delta(t_0, \lambda)$.* □

For a general compact Hausdorff space K, the C*-algebra $C(K)$ rarely contains an abundant collection of projections. For example, $C[0, 1]$ only contains trivial projections. If we assume that K is Stonean, then the characteristic function χ_A of every non-empty clopen set $A \subset K$ is a continuous function and a projection in $C(K)$, and thus $C(K)$ contains an abundant family of non-trivial projections.

Throughout the rest of this section we shall assume that K is a Stonean space. That restriction does not interfere with our main goal, but gives us a wider range of tools to work with. In fact, it is known that if K is a Stonean space, then every element a in the C*-algebra $C(K)$ can be uniformly approximated by finite linear combinations of projections (see [144, Proposition 1.3.1]).

Arguing by contradiction, it can be proved the following proposition, which means somehow a reciprocal statement of Proposition 3.2.3. Its conclusions are repeatedly applied in our arguments.

Proposition 3.2.4. [33, Proposition 3.1, Linear and Multilinear Algebra] *Suppose K is a Stonean space and X is a complex Banach space. Let $\Delta : S(C(K)) \to S(X)$ be a surjective isometry. Let t_0 be an element in K. If b is an element in $S(C(K))$ satisfying $\varphi\Delta(b) = 0$, for every $\varphi \in supp_\Delta(t_0, \mu)$ and for every $\mu \in \mathbb{T}$, then $b(t_0) = 0$.* □

The desired real linear extension arrives as a consequence of numerous results exploring the behaviour of the surjective isometry Δ from $S(C(K))$ onto $S(X)$, for any complex Banach space X and with K Stonean space, on different kind of elements. In a chained study, each conclusion improves the previous statement and it is used in the next one.

Firstly, we consider finite linear combinations of mutually orthogonal projections. In [33, Proposition 3.2] it is proved that if A is a non-empty clopen subset of K, $\lambda, \gamma \in \mathbb{T}$, and $b \in S(C(K))$ such that $\Delta(b) = \lambda\Delta(\gamma\chi_A)$, then $b = b\chi_A$ and $|b(t)| = 1$, for every $t \in A$. The information provided is completed in [33, Proposition 3.4], which affirms that the spectrum of the element b is contained in the set $\{\lambda\gamma, \overline{\lambda}\gamma, 0\}$. Consequently, there exist two disjoint clopen sets A_1 and A_2 (one of which could be empty) such that $A = A_1 \cup A_2$ and $b = \lambda\gamma\chi_{A_1} + \overline{\lambda}\gamma\chi_{A_2}$. Consequently, $\Delta(-\gamma\chi_A) = -\Delta(\gamma\chi_A)$. The existence of certain clopen subsets and the application of Lemma 3.2.1 is essential in the proof.

The notion of completely M-orthogonality enables us to go one step further in the search of real-linearity. We recall that a set $\{x_1, \ldots, x_k\}$ in a complex Banach space X is called *completely M-orthogonal* if

$$\Big\| \sum_{j=1}^{k} \alpha_j x_j \Big\| = \max\{\|\alpha_j x_j\| : 1 \leq j \leq k\},$$

for every $\alpha_1, \ldots, \alpha_k$ in $\mathbb{C}$. If $\{x_1, \ldots, x_k\} \subset S(X)$, then it is completely M-orthogonal if and only if the equality

$$\left\| \sum_{j=1}^{k} \lambda_j x_j \right\| = 1$$

holds for every $\lambda_1, \ldots, \lambda_k$ in $\mathbb{T}$ and $\lambda_{j_0} = 1$ for some $j_0 \in \{1, \ldots, k\}$ (see [94, Lemma 3.4] and [134, Lemma 2.3]).

Proposition 3.2.5. [33, Proposition 3.5, Linear and Multilinear Algebra] *Suppose K is a Stonean space. Let A and B be two non-empty disjoint clopen subsets of K. Let $\Delta : S(C(K)) \to S(X)$ be a surjective isometry, where X is a complex Banach space. Then the following statements hold:*

(a) *For every $\gamma, \mu \in \mathbb{T}$, the set $\{\Delta(\gamma\chi_{A}), \Delta(\mu\chi_{B})\}$ is completely M-orthogonal;*

(b) $\Delta(\sigma_1\gamma\chi_{A} + \sigma_2\mu\chi_{B}) = \sigma_1\Delta(\gamma\chi_{A}) + \sigma_2\Delta(\mu\chi_{B})$, *for every $\sigma_1, \sigma_2 \in \{\pm 1\}$ and every $\gamma, \mu \in \mathbb{T}$;*

(c) *For each $\lambda \in \mathbb{T}$, there exist two disjoint clopen sets A_1 and A_2 (one of which could be empty) such that $A = A_1 \cup A_2$,*

$$\lambda\Delta(\chi_{A_1}) + \lambda\Delta(\chi_{A_2}) = \Delta(\lambda\chi_{A_1}) + \Delta(\overline{\lambda}\chi_{A_2}) = \Delta(\lambda\chi_{A_1} + \overline{\lambda}\chi_{A_2}) = \lambda\Delta(\chi_{A}),$$

$$\Delta(\lambda\chi_{A_1}) = \lambda\Delta(\chi_{A_1}),\ \Delta(\lambda\chi_{A_2}) = \overline{\lambda}\Delta(\chi_{A_2}),$$

$$\overline{\lambda}\Delta(\chi_{A_1}) + \overline{\lambda}\Delta(\chi_{A_2}) = \Delta(\overline{\lambda}\chi_{A_1}) + \Delta(\lambda\chi_{A_2}) = \Delta(\overline{\lambda}\chi_{A_1} + \lambda\chi_{A_2}) = \overline{\lambda}\Delta(\chi_{A}),$$

$$\Delta(\overline{\lambda}\chi_{A_1}) = \overline{\lambda}\Delta(\chi_{A_1}),\ \text{and}\ \Delta(\overline{\lambda}\chi_{A_2}) = \lambda\Delta(\chi_{A_2}).$$

□

Proposition 3.6 in [33] conforms an appropriate generalisation of [94, Proposition 3.3], namely, let A be a non-empty clopen subset of K, and $\lambda \in \mathbb{T}\backslash\mathbb{R}$. If we additionally assume that $\Delta(\lambda\chi_{A}) = \lambda\Delta(\chi_{A})$ (respectively, $\Delta(\lambda\chi_{A}) = \overline{\lambda}\Delta(\chi_{A})$), the quoted result proves that $\Delta(\mu\chi_{A}) = \mu\Delta(\chi_{A})$ (respectively, $\Delta(\mu\chi_{A}) = \overline{\mu}\Delta(\chi_{A})$), for every $\mu \in \mathbb{T}$. Furthermore, under these hypotheses, the same conclusions hold for every non-empty clopen set in K contained in A. [33, Proposition 3.6] combines with Proposition 3.2.5 to yield the following corollary.

Corollary 3.2.6. [33, Corollary 3.7, Linear and Multilinear Algebra] *Suppose K is a Stonean space and X is a complex Banach space. Let $\Delta : S(C(K)) \to S(X)$ be a surjective isometry. Then there exists a clopen subset $K_1 \subseteq K$ such that $\Delta(\lambda\chi_{K_1}) = \lambda\Delta(\chi_{K_1})$ and $\Delta(\lambda\chi_{K\setminus K_1}) = \overline{\lambda}\Delta(\chi_{K\setminus K_1})$, for every $\lambda \in \mathbb{T}$. Consequently, if B_1 is a clopen subset of K contained in K_1 and B_2 is a clopen subset of K contained in $K_2 = K\setminus K_1$, then $\Delta(\mu\chi_{B_1}) = \mu\Delta(\chi_{B_1})$ and $\Delta(\mu\chi_{B_2}) = \overline{\mu}\Delta(\chi_{B_2})$, for every $\mu \in \mathbb{T}$.* □

From now on, the symbols K_1 and K_2 will denote the clopen subsets in Corollary 3.2.6. Under these hypotheses we define a new product

$$\begin{aligned} \mathbb{C} \times C(K) &\to C(K) \\ (\alpha, a) &\mapsto \alpha \odot a, \end{aligned}$$

where $(\alpha \odot a)(t) := \alpha\ a(t)$ for each $t \in K_1$, and $(\alpha \odot a)(t) := \overline{\alpha}\ a(t)$, otherwise.

Let $\gamma_1, \ldots, \gamma_n \in \mathbb{T}$, and let $B_1, \ldots, B_n$ be non-empty disjoint clopen subsets of K. We consider $v = \sum_{k=1}^{m} \lambda_k \chi_{A_k}$ an algebraic partial isometry in $C(K)$, where $\lambda_1, \ldots, \lambda_m \in \mathbb{T}$, $A_1, \ldots, A_m$ are non-empty disjoint clopen sets in K such that $A_k \cap B_j = \emptyset$, for every $k \in \{1, \ldots, m\}$ and every $j \in \{1, \ldots, n\}$. [33, Proposition 3.8] deals with these algebraic elements, and extends the previous results concerning finite linear combinations of mutually orthogonal projections by proving that the set $\{\Delta(v), \Delta(\gamma_1\chi_{B_1}), \ldots, \Delta(\gamma_n\chi_{B_n})\}$ is completely M-orthogonal, and the following equality holds,

$$\Delta(v) + \sum_{j=1}^{n} \Delta(\gamma_j\chi_{B_j}) = \Delta\left(v + \sum_{j=1}^{n} \gamma_j\chi_{B_j}\right).$$

A little bit more can be said. Indeed, the case in which $B_1, \ldots, B_{j_0}$ are contained in K_1 and $B_{j_0+1}, \ldots, B_n \subseteq K\setminus K_1$ with $j_0 \in \{0, 1, \ldots, n+1\}$ is carefully covered by [33, Proposition 3.9]. From the quoted results follows a complex linear behaviour when working in K_1 and a conjugate-linear behaviour of Δ in K_2. The proof makes use of the M-orthogonality assured by [33, Proposition 3.8] together with several applications of Propositions 3.2.4 and 3.2.3. As a consequence, a culminating corollary can be stated.

Corollary 3.2.7. [33, Corollary 3.10, Linear and Multilinear Algebra] *Let $\Delta : S(C(K)) \to S(X)$ be a surjective isometry, where K is a Stonean space and X is a complex Banach space. Let $v_1, \ldots, v_n$ be mutually orthogonal algebraic partial isometries in $C(K)$. Then, given $\alpha_1, \ldots, \alpha_n \in \mathbb{C}\setminus\{0\}$ with $\max\{|\alpha_j| : j \in \{1, \ldots, n\}\} = 1$, we have*

$$\sum_{j=1}^{n} \alpha_j \Delta(v_j) = \Delta\left(\sum_{j=1}^{n} \alpha_j \odot v_j\right).$$

□

The chained study concludes with the main theorem, which reads as follows:

Theorem 3.2.8. [33, Theorem 3.11, Linear and Multilinear Algebra]
Let $\Delta : S(C(K)) \to S(X)$ be a surjective isometry, where K is a Stonean space and X is a complex Banach space. Then there exist two disjoint clopen subsets K_1 and K_2 of K such that $K = K_1 \cup K_2$ satisfying that if K_1 (respectively, K_2) is non-empty, then there exist a closed subspace X_1 (respectively, X_2) of X and a complex linear (respectively, conjugate linear) surjective isometry $T_1 : C(K_1) \to X_1$ (respectively, $T_2 : C(K_2) \to X_2$) such that $X = X_1 \oplus^\infty X_2$, and $\Delta(a) = T_1(\pi_1(a)) + T_2(\pi_2(a))$, for every $a \in S(C(K))$, where π_j is the natural projection of $C(K)$ onto $C(K_j)$ given by $\pi_j(a) = a|_{K_j}$. In particular, Δ admits an extension to a surjective real linear isometry from $C(K)$ onto X. □

In [33] we present two different proofs of the above theorem. The first approach is completely constructive in the line followed in [134, Theorem 1.1]. We consider the clopen subsets of K_1 and K_2 given in Corollary 3.2.6, and express $C(K)$ as $C(K_1) \oplus^\infty C(K_2)$. The corresponding homogeneous extensions $T_j : C(K_j) \to X$ (see page 59) with $j = 1, 2$, are Lipschitz mappings (cf. [134, Proof of Theorem 1.1]). Moreover, by Corollary 3.2.7, the identity $T_j(\widehat{a} + \widehat{b}) = T_j(\widehat{a}) + T_j(\widehat{b})$ holds for every $j = 1, 2$, and for any two algebraic elements in $C(K_1)$ (respectively, $C(K_2)$) of the form

$$\widehat{a} = \sum_{k=1}^{n} \alpha_k \odot v_k, \text{ and } \widehat{b} = \sum_{k=1}^{n} \beta_k \odot v_k,$$

where $v_1, \ldots, v_n$ are mutually orthogonal non-zero algebraic partial isometries in K_1 (respectively, K_2), $\alpha_1, \ldots, \alpha_n$, $\beta_1, \ldots, \beta_n \in \mathbb{C}\backslash\{0\}$ with $\max\{|\alpha_k| : k \in \{1, \ldots, n\}\} = \|\widehat{a}\|$, and $\max\{|\beta_k| : k \in \{1, \ldots, n\}\} = \|\widehat{b}\|$.

Since for each $j = 1, 2$, any $a \in C(K_j)$ can be approximated in norm by an algebraic element of the form of $\widehat{a}$, and T_j is continuous, it is shown that T_1 is complex linear and T_2 is conjugate-linear. Furthermore, $T_1(a_1) = \Delta(a_1)$ and $T_2(a_2) = \Delta(a_2)$ for every $a_j \in S(C(K_j))$, $j = 1, 2$. In particular, T_1 and T_2 are isometries, and $X_j = T_j(C(K_j))$ is a closed subspace of X for every $j = 1, 2$. Essentially the same arguments work together to obtain the joint statement in $C(K_1) \oplus^\infty C(K_2)$ through the corresponding projections onto each $C(K_j)$.

The alternative proof included in [33] for Theorem 3.2.8 is closer to the arguments employed by G.G. Ding ([47]), R. Liu ([112]), and X.N. Fang and J.H. Wang ([60]). Actually, it consists of an appropriate application of the support functionals to show that

$$\|\Delta^{-1}(y) - \lambda\Delta^{-1}(x)\| \leq \|y - \lambda x\|,$$

for every $x, y \in S(X)$, and every $\lambda > 0$. The extension Lemma 3.0.2 gives the desired conclusion for Δ^{-1}, and hence for Δ.

The next result is a corollary of our previous Theorem 3.2.8. It constitutes an extension of the corresponding theorem due to D. Tan [154] to complex-valued functions, and proves the Mazur–Ulam property for any commutative von Neumann algebra.

Theorem 3.2.9. [33, Theorem 3.14, Linear and Multilinear Algebra] *Let (Ω, μ) be a σ-finite measure space, and let X be a complex Banach space. Suppose $\Delta : S(L^\infty(\Omega, \mu)) \to S(X)$ is a surjective isometry. Then there exists a surjective real linear isometry $T : L^\infty(\Omega, \mu) \to X$ whose restriction to $S(L^\infty(\Omega, \mu))$ is Δ.* □

3.2.2 JBW*-triples

The paper [12] is intended to prove that any JBW*-triple which is not a Cartan factor of rank two satisfies the Mazur–Ulam property. The approach adopted is totally different from that shown in the case of commutative von Neumann algebras. We shall define two families of support functionals associated with minimal tripotents satisfying the hypotheses of Lemma 3.0.3, and thus the real linear extension will be guaranteed.

Given an element x_0 in a Banach space X, let $\mathcal{T}_{x_0} : X \to X$ denote the translation mapping with respect to the vector x_0 (i.e. $\mathcal{T}_{x_0}(x) = x + x_0$, for all $x \in X$). Let M be a JBW*-triple. We begin the study by showing that any surjective isometry from $S(M)$ onto the unit sphere of any Banach space Y is affine in the norm-closed faces associated with any non-zero tripotent in M.

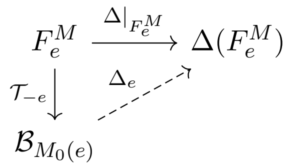
Corollary 3.2.10. [12, Corollary 4.1, J. Inst. Math. Jussieu] *Let M be a JBW*-triple, let Y be a Banach space, and let $\Delta : S(M) \to S(Y)$ be a surjective isometry. Suppose e is a non-zero tripotent in M, and let*

$$F_e^M = e + \mathcal{B}_{M_0(e)} = \left(e + \mathcal{B}_{M_0^{**}(e)}\right) \cap M$$

denote the proper norm-closed face of $\mathcal{B}_M$ associated with e. Then the restriction of Δ to F_e^M is an affine function. Furthermore, there exists an affine isometry T_e from $M_0(e)$ onto a norm closed subspace of Y satisfying $\Delta(\mathcal{T}_e(x)) = T_e(x)$ for all $x \in \mathcal{B}_{M_0(e)}$. □

The proof of the previous corollary relies fundamentally on the strong Mankiewicz property proved in Corollary 3.1.2 when combined with the facial structure of JB*-triples, reviewed in section 2.1. Just in order to emphasise the potential of these two tools, let us outline the arguments exposed using the notation of the statement above. The ideas in [61, Proof of Proposition 2.4 and comments after and before Corollary 2.5] assure that

F_e^M coincides with the intersection of all maximal proper norm-closed faces containing it, that is, F_e^M is an intersection face in the sense of [126]. Therefore, by Lemma 8 in [126], $\Delta(F_e^M)$ is also an intersection face, and in particular a convex set. Here it is when the strong Mankiewicz property plays a fundamental role, applied in this case to the JBW*-triple $M_0(e)$. The dashed arrow in the diagram



defines a surjective isometry Δ_e, which extends (uniquely) by Corollary 3.1.2 to be an affine isometry from $M_0(e)$ onto a norm-closed subspace of Y. The desired conclusion follows from the commutativity of the above diagram and the fact that $\mathcal{T}_{-e}$ is an affine mapping.

Our next result is a generalisation of [126, Lemma 15] to the context of JB*-triples. It demands a full knowledge in triple theory, with application of the triple continuous functional calculus. The advantages of the locally triple theory are exposed through the arguments.

Lemma 3.2.11. [12, Lemma 4.4, J. Inst. Math. Jussieu] *Let M be a JBW*-triple, and let u be a compact-G_δ tripotent in M^{**} associated with a norm-one element $a \in M$. Then there exists a decreasing sequence of non-zero tripotents $(e_n)_n$ in M (actually in the JBW*-subtriple of M generated by a) satisfying that for each $x \in F_u^M$ the sequence $\Theta_n(x) := e_n + P_0(e_n)(x)$ converges to x in the norm topology of M.* □

We prove in [12, Proposition 4.5] that Δ is affine in those norm-closed proper faces of $\mathcal{B}_M$ associated with compact-G_δ tripotents in the bidual of a JBW*-triple.

Proposition 3.2.12. [12, Proposition 4.5, J. Inst. Math. Jussieu] *Let M be a JBW*-triple, let Y be a real Banach space, and let $\Delta : S(M) \to S(Y)$ be a surjective isometry. Then the restriction of Δ to each norm-closed (proper) face of $\mathcal{B}_M$ associated with a compact-G_δ tripotent u in M^{**} is an affine function. Furthermore, for each $\psi \in Y^*$, there exist $\phi \in M^*$ and $\gamma \in \mathbb{R}$ such that $\|\phi\|, |\gamma| \leq \|\psi\|$, and*

$$\psi\Delta(x) = \Re e\, \phi(x) + \gamma, \text{ for all } x \in F_u^M.$$

□

That assertion allows us to deal with general proper norm-closed faces in the closed unit ball of a JBW*-triple. Indeed, Proposition 4.6 of [12] assumes the same hypotheses of Proposition 3.2.12 to affirm that the restriction of Δ to each norm-closed proper face F of $\mathcal{B}_M$ is an affine function, and moreover, for each $\psi \in Y^*$, there exist $\phi \in M^*$ and $\gamma \in \mathbb{R}$ such that $\|\phi\|, |\gamma| \leq \|\psi\|$, and

$$\psi\Delta(x) = \Re\mathrm{e}\, \phi(x) + \gamma, \text{ for all } x \in F.$$

The proof collects several conclusions as the facial structure of JB*-triples in Theorem 2.1.5, Proposition 2.1.8, and Proposition 3.2.12 when the situation is reduced to faces of $\mathcal{B}_M$ associated with compact-G_δ tripotents.

At this point, the approach is centred in proving that the most of JBW*-triples satisfies the following property.

Definition 3.2.13. [12, Property $(\mathcal{P})$, Definition 4.8, J. Inst. Math. Jussieu] *Let E be a JB*-triple. We shall say that E satisfies property $(\mathcal{P})$ if for each minimal tripotent e in E^{**} and each complete tripotent u in E (that is $u \in \partial_e(\mathcal{B}_E)$), there exists another minimal tripotent w in E^{**} satisfying $w \perp e$ and $u = w + P_0(w)(u)$.*

In [12, Proposition 4.9] we use, among other results, a non-commutative generalisation of Urysohn's lemma established in [68, Proposition 3.7] (cf. Appendix A). The result explores the behaviour of the support functionals associated with norm-closed proper faces of the form F_e^M for a minimal tripotent e in the bidual M^{**} of a JBW*-triple M. We state a corollary concerning such a behaviour in the elements of the unit sphere of M. We shall assume the one-to-one correspondence exposed in (1.20) between minimal tripotents and pure atoms.

Corollary 3.2.14. [12, Corollary 4.11, J. Inst. Math. Jussieu] *Let M be a JBW*-triple satisfying property $(\mathcal{P})$. Let $\varphi_e \in \partial_e(\mathcal{B}_{M^*})$ denote the unique pure atom associated with a minimal tripotent e in M^{**}. Suppose $\Delta : S(M) \to S(Y)$ is a surjective isometry from the unit sphere of M onto the unit sphere of a real Banach space Y. Then for each ψ in $supp_\Delta(F_e^M)$ we have $\psi\Delta(x) = \Re e\varphi_e(x)$ for every $x \in S(M)$.* □

We explore the connection between rank and property $(\mathcal{P})$ in the case of Cartan factors.

Proposition 3.2.15. [12, Proposition 4.13, J. Inst. Math. Jussieu] *Every Cartan factor of rank bigger than or equal to three satisfies property $(\mathcal{P})$.* □

The final statement provides a positive answer to the Mazur–Ulam property for any JBW*-triple M whose rank is bigger than two. All the machinery developed until now is used in the proof of [12, Theorem 4.14] to show that M satisfies the property $(\mathcal{P})$ in Definition 3.2.13. That is fruitful for our purpose since it enables us to consider, by virtue of Corollary 3.2.14, the families $\{\varphi_e\}_e$ and $\{\psi_e\}_e$ with e running in $\mathrm{Trip}_{\min}(M^{**})$, where for each

$e \in \mathrm{Trip}_{\min}(M^{**})$, ψ_e is a support functional associated with the face F_e^M, and φ_e is its unique the pure atom. Those families are under the hypotheses of Lemma 3.0.3, from which the desired conclusions hold.

Theorem 3.2.16. [12, Theorem 4.14, J. Inst. Math. Jussieu] *Let M be a JBW*-triple with rank bigger than or equal to three. Then, every surjective isometry from the unit sphere of M onto the unit sphere of a real Banach space Y admits a unique extension to a surjective real linear isometry from M onto Y.* □

The arguments employed above to arrive to property $(\mathcal{P})$ rely on the atomic decomposition on M^{**}. The situation is reduced to Cartan factors, and hence the Proposition 3.2.15 is fundamental.

The rank-one case is also covered in the paper [12]. Suppose M is a JBW*-triple of rank one. It is known that M must be reflexive (see, for example, [21, Proposition 4.5]). In particular M must coincide with a rank one Cartan factor, and hence it must be isometrically isomorphic to a complex Hilbert space (cf. [107, Table 1 in page 210]). Tingley's problem for Hilbert spaces was solved by G.G. Ding. On the other hand, M.M. Day stated in [39] a version of the Jordan-Von Neumann theorem which determines those normed spaces with an inner-product just by the elements in its unit sphere. Combining these two results with a facial argument (cf. [126, Lemma 8]), it can be proved the following proposition.

Proposition 3.2.17. [12, Proposition 4.15, J. Inst. Math. Jussieu] *Every Hilbert space satisfies the Mazur–Ulam property. Every rank one JBW*-triple satisfies the Mazur–Ulam property.* □

In conclusion, we have proved that any JBW*-triple with rank different from 2 satisfies the Mazur–Ulam property. The recent achievements of O.F.K. Kalenda and A.M. Peralta in [100] complete the study by showing that any JBW*-triple of rank 2 also satisfies the Mazur–Ulam property ([100, Theorem 1.1]). Finally, combining Theorem 3.2.16, Proposition 3.2.17, and Theorem 1.1 in [100], we are in position to state the final result.

Theorem 3.2.18. [100, Corollary 1.2] *Every JBW*-triple M satisfies the Mazur–Ulam property, that is, every surjective isometry from its unit sphere onto the unit sphere of an arbitrary real Banach space Y admits an extension to a surjective real linear isometry from M onto Y .*

3.2.3 C(K,H)-spaces

As commented in the introduction of this project, one of the aims of the paper [34] is to exhibit the usefulness of a good knowledge on real linear isometries between JB*-triples to study the Mazur–Ulam property on new classes of Banach spaces of continuous functions. In [126], M. Mori and N.

Ozawa proved that any unital C*-algebra and any real von Neumann algebra satisfies the Mazur–Ulam property. These novelties together with our own results in [12], exposed in section 3.2.2, naturally led us to consider in [34] the Mazur–Ulam property on the space $C(K, H)$ of all continuous functions from a compact Hausdorff space K into a real or complex Hilbert space H. This space is not, in general, a C*-algebra nor a JBW*-triple because it is neither a dual Banach space. However, it possesses a motivating structure of Hilbert $C(K)$-module in the sense introduced by I. Kaplansky ([106]), and consequently, a structure of JB*-triple ([89]), where $C(K)$ stands for the space $C(K, \mathbb{C})$ (see section 1.3 for more details).

The following theorem should convince the reader that the achievements obtained in [126] for unital C*-algebras are not enough to conclude that $C(K, H)$-spaces satisfy the Mazur–Ulam property. The proof argues by contradiction, and makes use of the real-linear version of the Kaup-Banach-Stone theorem for a real linear surjective isometry between a C*-algebra and a JB*-triple ([37, Theorem 3.1] or [63, Theorem 3.2 and Corollary 3.4], see also [72, Theorem 3.1]).

Theorem 3.2.19. [34, Theorem 2.1, J. Math. Anal. Appl.] *Let K be a compact Hausdorff space, and let H be a complex Hilbert space with dimension bigger than or equal to 2. Then there exists no surjective isometry from the unit sphere of $C(K, H)$ onto the unit sphere of a C*-algebra.* □

Let K be a compact Hausdorff space, and let $\mathcal{H}$ be a real Hilbert space. The Banach space $C(K, \mathcal{H})$ can be regarded as a real JB*-triple, in the sense of [90], when the triple product defined in (1.23) is considered. The set of all extreme points of the dual space of $C(K, \mathcal{H})$ is known by [151, Lemma 1.7 in page 197]. It can be deduced that every real or complex Cartan factor in the atomic part of $C(K, \mathcal{H})^{**}$ coincides with the real Hilbert space $\mathcal{H}$ equipped with the triple product $\{a, b, c\} := \frac{1}{2}(a|b)c + \frac{1}{2}(c|b)a$. Applying this, and arguing by contradiction, the result in [126] for real von Neumann algebras can be also discarded by virtue of the following statement.

Theorem 3.2.20. [34, Theorem 2.2, J. Math. Anal. Appl.] *Let K be a compact Hausdorff space, and let $\mathcal{H}$ be a real Hilbert space with $\dim(\mathcal{H}) = 3$ or $\dim(\mathcal{H}) \geq 5$. Then there exists no surjective isometry from the unit sphere of $C(K, \mathcal{H})$ onto the unit sphere of a real von Neumann algebra.* □

The sketch of the ideas developed in the paper [34] is similar to the strategy followed in [12] and outlined in section 3.2.2. We need to distinguish between the real and complex cases, even though the ideas will follow an analogous path. We begin by partially stating the main conclusions of the paper [34] in the complex setting.

Theorem 3.2.21. [34, Theorem 5.6, J. Math. Anal. Appl.] *Let K be a compact Hausdorff space and let H be a complex Hilbert space. Then the*

Banach space $C(K,H)$ satisfies the Mazur–Ulam property (as a real Banach space), that is, for each surjective isometry $\Delta : S(C(K,H)) \to S(Y)$, where Y is a real Banach space, there exists a surjective real linear isometry from $C(K,H)$ onto Y whose restriction to $S(C(K,H))$ is Δ. □

The case $H = \mathbb{C}$ is a consequence of [126, Theorem 1]. Throughout the rest of the section K and H will denote a compact Hausdorff space and a complex Hilbert space with $\dim(H) \geq 2$, respectively. Some notation is needed to draw the proof of the theorem above. Given $\eta \in H$ and a mapping $f : K \to \mathbb{K}$, the symbol $\eta \otimes f$ will denote the mapping from K to H defined by $\eta \otimes f(t) = f(t)\eta$, $(t \in K)$. We note that $\eta \otimes f$ is continuous whenever f lies in $C(K)$. We will use the juxtaposition for the point-wise product between maps whenever such a product makes sense. For each x_0 in H, we shall write x_0^* for the unique functional in H^* defined by $x_0^*(x) = \langle x | x_0 \rangle$ $(x \in H)$. Given $t_0 \in K$, $\delta_{t_0} : C(K,H) \to H$ will stand for the bounded linear operator defined by $\delta_{t_0}(a) = a(t_0)$ for each a in $C(K,H))$. Finally, let $x_0^* \otimes \delta_{t_0}$ denote the functional on $C(K,H)$ given by $(x_0^* \otimes \delta_{t_0})(a) := x_0^*(a(t_0))$, for each a in $C(K,H)$.

The final objective in the proof of Theorem 3.2.21 is the application of the extension Lemma 3.0.3 for the families

$$\{\Re\mathrm{e} x_0^* \otimes \delta_{t_0} : t_0 \in K, x_0 \in S(H)\} \subseteq S(C(K,H)^*_{\mathbb{R}}),$$

$$\{\psi : \psi \in \mathrm{supp}_{\Delta}(t_0, x_0), t_0 \in K, x_0 \in S(H)\} \subseteq S(Y^*),$$

where Y is a real Banach space. A hard technical work is previously necessary in order to be able of consider the specified families under the appropriate conditions. Let us observe, before starting to review the results developed to pursue our goal in Theorem 3.2.21, that for a fixed $t_0 \in K$, if $K = \{t_0\}$, then $C(K,H)$ is isometrically isomorphic to H, and thus the desired conclusion follows, for example, from [12, Proposition 4.15]. On the other hand, the existence of support functionals is guaranteed by Lemma 3.0.7. Actually, in the hypotheses of the just quoted lemma, it is known that each $A(t_0, x_0)$ is an intersection face in the sense employed in [126]. Therefore, Lemma 8 in [126] assures that $\Delta(-A(t_0, x_0)) = -\Delta(A(t_0, x_0))$, and consequently, $\psi\Delta(a) = -1$, for every $a \in -A(t_0, x_0)$, and for every $\psi \in \mathrm{supp}_{\Delta}(t_0, x_0)$.

The first claim in the proof of Theorem 3.2.21 is the validity of the equality

$$\psi\Delta(u) = \Re\mathrm{e}\langle u(t_0) | x_0 \rangle \tag{3.2}$$

for every $u \in \partial_e(\mathcal{B}_{C(K,H)})$.

Proposition 3.2.22. [34, Proposition 5.4, J. Math. Anal. Appl.]
Let $\Delta : S(C(K,H)) \to S(Y)$ be a surjective isometry, where Y is a real Banach space. Suppose $t_0 \in K$ and $x_0 \in S(H)$. Then there exist a net $(\mathcal{R}_\lambda)_\lambda$

of convex subsets of $A(t_0, x_0)$ and a net $(\theta_\lambda)_\lambda$ of affine contractions from $A(t_0, x_0)$ into $\mathcal{R}_\lambda$ such that $\theta_\lambda \to Id$ in the point-norm topology. Moreover, for each λ, $\mathcal{R}_\lambda$ satisfies the strong Mankiewicz property and $\Delta(\mathcal{R}_\lambda)$ is convex. Consequently $\Delta|_{A(t_0,x_0)}$ is affine. □

The previous proposition shows that any surjective isometry between the unit spheres of $C(K, H)$ and any Banach space Y are affine on the maximal proper faces of $\mathcal{B}_{C(K,H)}$ by using an adaptation of the arguments in [126, Proposition 20]. The strong Mankiewicz property obtained in Proposition 3.1.4, and Proposition 3.1.7 plays a fundamental role in the technical proof of Proposition 3.2.22. A more elaborated discussion is required for the claim stated in (3.2).

Proposition 3.2.23. [34, Proposition 5.5, J. Math. Anal. Appl.]
Let $\Delta : S(C(K, H)) \to S(Y)$ be a surjective isometry, where Y is a real Banach space. Suppose $t_0 \in K$ and $x_0 \in S(H)$. Then for each $\psi \in Y^*$ there exist ϕ_0 in $C(K, H)^*_{\mathbb{R}}$ and $\gamma_0 \in \mathbb{R}$ satisfying $\|\phi_0\| \leq \|\psi\|$ and

$$\psi\Delta(a) = \phi_0(a) + \gamma_0, \text{ for all } a \in A(t_0, x_0).$$

□

It can be shown that the claim (3.2) is true by an appropriate combination of the proposition above together with [34, Lemma 5.2]. Moreover, the same conclusion holds for every a in a maximal face of the form $A(t_2, x_2)$ with $t_2 \in K, x_2 \in S(H)$. That is thanks to Corollary 1.3.11 combined with (3.2) and the final conclusion in Proposition 3.2.22.

Since every $a \in S(C(K, H))$ belongs to a maximal face of the form $A(t_2, x_2)$ with $t_2 \in K, x_2 \in S(H)$, we conclude that $\psi\Delta(a) = \Re\text{e}\langle a(t_0)|x_0\rangle$, for all $a \in S(C(K, H))$. And thus, Lemma 3.0.3 assures the existence of a real linear extension of Δ as exposed in Theorem 3.2.21.

It is worth noting at this point that the case in which H is a real Hilbert space is not fully covered by our theorem. R. Liu proved in [112, Corollary 6] that $C(K, \mathbb{R})$ satisfies the Mazur–Ulam property whenever K is a compact Hausdorff space. Let $\mathcal{H} = \ell_2(\Gamma, \mathbb{R})$ be a real Hilbert space with inner product $(\cdot|\cdot)$. Suppose $\dim(\mathcal{H})$ is even or infinite. We can write Γ as the disjoint union of two subsets Γ_1, Γ_2 for which there exists a bijection $\sigma : \Gamma_1 \to \Gamma_2$. Let $H = \ell_2(\Gamma_1)$ denote the usual complex Hilbert space with inner product $\langle\cdot|\cdot\rangle$, and $(H_{\mathbb{R}}, \Re\text{e}\langle\cdot|\cdot\rangle)$ the underlying real Hilbert space. The mapping $(\lambda_j)_{j\in\Gamma_1} + \left(\lambda_{\sigma(j)}\right)_{j\in\Gamma_1} \mapsto \left(\lambda_j + i\lambda_{\sigma(j)}\right)_{j\in\Gamma_1}$ is a surjective real linear isometry from $\mathcal{H}$ onto $H_{\mathbb{R}}$. The next result is a straightforward consequence of our previous Theorem 3.2.21.

Corollary 3.2.24. [34, Corollary 5.7, J. Math. Anal. Appl.] *Let K be a compact Hausdorff space and let $\mathcal{H}$ be a real Hilbert space with $\dim(\mathcal{H})$ even* *or infinite. Then the real Banach space $C(K,\mathcal{H})$ satisfies the Mazur–Ulam property.* □

There are certain obstacles that prevent to apply the tools developed in Proposition 3.1.7, and some other technical lemmata in the case of $C(K,\mathcal{H})$ when $\mathcal{H}$ is a finite-dimensional real Hilbert space with odd dimension. The difficulties in Proposition 3.1.7 can be solved with Proposition 3.1.8. On the other hand Lemma 5.3 in [34] makes use of Theorem 2.1.7. If that result is replaced with its real version, Theorem 2.1.13, then the same conclusion holds for real Hilbert spaces. It is a bit more laborious, but no more than a routine exercise, to check that the arguments in Propositions 3.2.22 and 3.2.23, and in Theorem 3.2.21 are literally valid to get the following result.

Corollary 3.2.25. [34, Corollary 5.8, J. Math. Anal. Appl.] *Let K be a compact Hausdorff space and let $\mathcal{H}$ be a finite-dimensional real Hilbert space with odd dimension. Then the real Banach space $C(K,\mathcal{H})$ satisfies the Mazur–Ulam property.* □

The final statement gathering all the achievements reads as follows:

Theorem 3.2.26. [34, Theorem 5.6, Corollaries 5.7 and 5.8, J. Math. Anal. Appl.] *Let K be a compact Hausdorff space and let H be a real or complex Hilbert space. Then $C(K,H)$ satisfies the Mazur–Ulam property, that is, for each surjective isometry $\Delta : S(C(K,H)) \to S(Y)$, where Y is a real Banach space, there exists a surjective real linear isometry from $C(K,H)$ onto Y whose restriction to $S(C(K,H))$ is Δ.* □

The pioneer achievements of M. Jerison provide generalised versions of the Banach–Stone theorem for spaces of vector-valued continuous functions. Combining Theorem 3.2.21 and Corollary 3.2.24 with the Banach–Stone theorem in [74, Theorem 7.2.16] (see also [74, Definition 7.1.2]) we obtain next a description of the surjective isometries between the unit spheres of two $C(K,H)$-spaces.

Corollary 3.2.27. [34, Corollary 5.9, J. Math. Anal. Appl.] *Let K_1, K_2 be two compact Hausdorff spaces, let H be a real or complex Hilbert space, and let Y be a strictly convex real Banach space. Suppose $\Delta : S(C(K_1,H)) \to S(C(K_2,Y))$ is a surjective isometry. Then there exist a homeomorphism $h : K_2 \to K_1$ and a mapping which maps each $t \in K_2$ to a surjective linear isometry $V(t) : H \to Y$, which is continuous from K_2 into the space $B(H,Y)$ of bounded linear operators from H to Y with the strong operator topology, such that*

$$\Delta(a)(t) = V(t)(a(h(t))),$$

for all $a \in S(C(K_1,H)), t \in K_2$. □

3.3 Surjective isometries between unitary sets

The celebrated Mazur-Ulam theorem assures that every surjective isometry between two real normed spaces is an affine function (cf. [119]). P. Mankiewicz established an amazing generalisation of the Mazur-Ulam theorem by showing that every bijective isometry between the closed unit balls of two real normed spaces admits a unique extension to a bijective affine isometry between the corresponding spaces (see [118, Theorem 5 and Remark 7]). Tingley's problem asks if every surjective isometry between the unit spheres of two normed spaces admits an extension to a surjective real linear isometry between the spaces (cf. [161]). In view of this quick summary, the reader could feel tempted to ask if the unit spheres can be replaced in the isometric extension problem with some strictly smaller sets.

Problem 3.3.1. *Let X and Y be two Banach spaces, and consider two subsets $\mathcal{S}_1 \subseteq S(X)$ and $\mathcal{S}_2 \subseteq S(Y)$. Suppose $\Delta : \mathcal{S}_1 \to \mathcal{S}_2$ is a surjective isometry. Is Δ necessarily the restriction to $\mathcal{S}_1$ of a real linear surjective isometry from X onto Y? In other words, does there exist a surjective real linear isometry $T : X \to Y$ such that $T(x) = \Delta(x)$ for every $x \in \mathcal{S}_1$?*

In some operator algebras, it has been established a *positive Tingley's problem* by considering the spheres of positive operators instead of the whole unit spheres. The replacement has been successfully addressed in several papers (see, for example, [123, 122, 127, 128] and [135]).

From a purely geometric perspective, the set of all extreme points of the closed unit ball of a Banach space seems to be a natural candidate to replace the unit sphere in Tingley's problem. We made a brief incursion in this possibility in [33, Remark 3.15], and conclude that even in the case of a finite-dimensional normed space X, we cannot always assure that every surjective isometry on the set of extreme points of the closed unit ball of X can be extended to a surjective real linear isometry on the whole X. A counterexample can be found, for instance, with the real Banach space $X = \mathbb{R} \oplus^{\infty} \mathbb{R}$, where the extreme points of its closed unit ball are given by $\partial_e(\mathcal{B}_X) = \{p_1 = (1,1), p_2 = (1,-1), p_3 = (-1,1), p_4 = (-1,-1)\}$.

The surjective isometry $\Delta : \partial_e(\mathcal{B}_X) \to \partial_e(\mathcal{B}_X)$ defined by

$$\Delta(p_j) = p_{j+1},\ 1 \leq j \leq 3 \ \text{ and } \ \Delta(p_4) = p_1,$$

cannot be extended to a surjective real-linear isometry on X. It has been manifested that the set of extreme points of the closed unit ball is not enough to determine a surjective real linear isometry.

Let us address the question by enriching the structure of the Banach space treated. Let H_1 and H_2 be two Hilbert spaces. It is known that the set of all extreme points of the closed unit ball of any Hilbert space coincides, by strictly convexity, with its unit sphere. Therefore, when for each $j = 1, 2$, $\partial_e(\mathcal{B}_{H_j})$ plays the role of $\mathcal{S}_j$, Problem 3.3.1 turns out to be the original Tingley's problem, solved by G.G. Ding in [42] and extended in [12], where we prove that any Hilbert space satisfies the Mazur–Ulam property (see Proposition 3.2.17).

A different point of view is provided by the setting of unital C*-algebras, where the concept of unitary element shows up significantly. The space $C(K)$, of all complex-valued continuous functions on a compact Hausdorff space K, is such that $\mathcal{U}(C(K)) = \partial_e(\mathcal{B}_{C(K)})$. The same conclusion holds for the operator algebra $B(H)$ whenever the Hilbert space H is finite-dimensional, while in contrast, $\mathcal{U}(B(H)) \subsetneq \partial_e(\mathcal{B}_{B(H)})$ if H has infinite dimension. In section 2.2 we have explored the reciprocal inclusion, characterising those extreme points which are unitary elements in the setting of unital C*-algebras (see Theorem 2.2.2, due to M. Mori), and more generally, in the setting of unital JB*-algebras (cf. Theorem 2.2.10).

O. Hatori and L. Monar contributed definitely to this research line in the paper [86], where they prove that unital C*-algebras whose unitary groups are isometric, are necessarily Jordan *-isomorphic (see [86, Corollary 2]). The last conclusion is a consequence of the main theorem in [86].

We recall that given an element x in a unital Banach algebra A, e^x is defined as the element in A given by $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ (cf. [25, § 1.1.29]).

Theorem 3.3.2. [86, Theorem 1] *Let $\Delta : \mathcal{U}(A) \to \mathcal{U}(B)$ be a surjective isometry, where A and B are two unital C*-algebras. Then the identity*

$$\Delta(e^{iA_{sa}}) = e^{iB_{sa}}$$

*holds, and there is a central projection $p \in B$ and a Jordan *-isomorphism $J : A \to B$ satisfying*

$$\Delta(e^{ix}) = \Delta(1) \left(pJ(e^{ix}) + (1-p)J(e^{ix})^*\right), \quad (x \in A_{sa}).$$

It is worth observing that not every unitary element in a unital C*-algebra is of the form e^{ix} for some $x \in A_{sa}$ (cf. discussion preceding Proposition 4.4.10 in [99]). However, if we restrict our study to von Neumann algebras, we have

$$\mathcal{U}(W) = \{e^{ih} : h \in W_{sa}\} \tag{3.3}$$

for any von Neumann algebra W. This fact allowed O. Hatori and L. Molnár to provide a complete description of surjective isometries between unitary groups of von Neumann algebras by applying Theorem 3.3.2 in this setting. Consequently, they provided a positive solution to Problem 3.3.1 in the case $\mathcal{S}_j = \mathcal{U}(W_j)$, where W_j is a von Neumann algebra for each $j = 1, 2$. That is, every surjective isometry between the unitary groups of two von Neumann algebras admits an extension to a surjective real linear isometry between these algebras (see [86, Corollary 3]). Moreover, these influencing results have played an important role in some of the recent advances on the original Tingley's problem (see, for example, [125]).

By exploring the tools employed in the Hatori–Molnár theorem, we find that S. Sakai already employed uniformly continuous one-parameter groups in 1955 (see [145]). S. Sakai proved that if M and N are AW*-factors, $\mathcal{U}(M), \mathcal{U}(N)$ their respective unitary groups, and ρ a uniformly continuous group-isomorphism from $\mathcal{U}(M)$ into $\mathcal{U}(N)$, then there is a unique map f from M onto N which is either a linear or a conjugate-linear *-isomorphism and agrees with ρ on $\mathcal{U}(M)$. In the case of W*-factors not of type I_{2n} the continuity assumption was shown to be superfluous by H.A. Dye in [51, Theorem 2]. In the results by O. Hatori and L. Molnár, the mapping Δ is merely a distance preserving bijection between the unitary groups of two unital C*-algebras or two von Neumann algebras.

The proofs of the Hatori–Molnár theorems are based, among other things, on a study on isometries and mappings compatible with inverted Jordan triple products on groups by O. Hatori, G. Hirasawa, T. Miura, L. Molnár [84]. Despite of the attractive terminology, the study of the surjective isometries between the sets of unitaries of two unital JB*-algebras has not been considered. There are proper difficulties which are inherent to the Jordan setting. Our paper [36] was born precisely with the aim of filling this gap, and providing an appropriate Jordan version of the Hatori-Molnár theorem.

The first obstacle that one finds when trying to translate the previous results in the setting of C*-algebra to the Jordan setting, is that unitary elements are not stable under Jordan product. Indeed, let A be a unital C*-algebra. It is well known that the set $\mathcal{U}(A)$ is contained in the unit sphere of A and it is a subgroup of A which is also self-adjoint (i.e., u^* and uv lie in $\mathcal{U}(A)$ for all $u, v \in \mathcal{U}(A)$). We recall the results from Chapter 1 assuring that A can be regarded as a unital JB*-algebra when equipped with its natural Jordan product defined in (1.3) and the original norm and involution. In this context, an element u in A is unitary in the C*-algebra sense if and only if u is Jordan unitary (see Remark 1.2.10). However, given $u, v \in \mathcal{U}(A)$, the element $u \circ v$ is unitary in A if and only if u and v commute respect to the associative product. In [36], this early barrier will be resolved by making use of the U-operators, since expressions of the form uvu lie in $\mathcal{U}(A)$ for all $u, v \in \mathcal{U}(A)$.

In the setting of JB*-algebras, the foundation of our arguments in [36] mainly relies on two ideas. The first one is the opportunity, provided by the JB*-triple theory, of changing appropriately the Jordan product of a JB*-algebra with a new Jordan product given by each unitary element. Arguing with the new product, we can infer the conclusions through the immutable triple product to the original JB*-algebra structure. In the second one we apply the excellent tool provided by the Shirshov-Cohn theorem to establish a Jordan version of the outstanding Stone's one-parameter theorem. Let us see in detail these ideas.

A JB*-algebra may admit two different Jordan products compatible with the same norm. Actually, the isotopes associated to unitary elements in a unital JB*algebra, whose notion was introduced in Lemma 1.3.4, support this assertion. We recall from the quoted lemma that for each unitary u in a unital JB*-algebra M, we can always consider the u-isotope denoted by $M(u)$ and consisting of the Banach space M endowed with the Jordan product defined by

$$x \circ_u y := U_{x,y}(u^*) = \{x, u, y\}_M, \tag{3.4}$$

and the involution $*_u$ defined by

$$x^{*_u} := U_u(x^*) = \{u, x, u\}_M. \tag{3.5}$$

Therefore, the u-isotope $M(u) = (M, \circ_u, *_u)$ becomes a unital JB*-algebra with unit u, and the original norm in M. The theory of isotopes is a frequent method employed as a tool for convenient computations in unital JB*-algebras.

Lemma 1.3.4 also contains a series of properties related to the u-isotopes. We note the item (c) in this lemma, which says that, given a unitary element u in a unital JB*-algebra M, the triple product of M satisfies

$$\begin{aligned} \{x, y, z\}_M &= (x \circ y^*) \circ z + (z \circ y^*) \circ x - (x \circ z) \circ y^* \\ &= (x \circ_u y^{*_u}) \circ_u z + (z \circ_u y^{*_u}) \circ_u x - (x \circ_u z) \circ_u y^{*_u}, \end{aligned} \tag{3.6}$$

for all $x, y, z \in M$, where $\circ_u$ and $*_u$ are the product and involution of the u-isotope $M(u)$, defined in (3.4) and (3.5), respectively. Actually, the previous identities hold when $\circ$ is replaced with any Jordan product on M making the latter a JB*-algebra with the same norm (see [106, Proposition 5.5]). The identity (3.6) reveals the immutability of the triple structure, in particular under isotopes associated with unitary elements. From the same lemma, we also recall item (b), which affirms that $\mathcal{U}(M)$ coincides with $\mathcal{U}(M(u))$, and of course they also coincide with the unitary tripotents of M when the latter is regarded as a JB*-triple. That will be a key ingredient in our arguments.

On the other hand, by analogy with the associative case, if M is a unital Jordan Banach algebra, the closed subalgebra generated by an element $x \in M$ and the unit is associative, and hence we can always consider the elements of the form e^x in M, defined by $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ (cf. [25, § 1.1.29]).

As in the case of unital C*-algebras, there exits unitary elements in unital JB*-algebras which cannot be written in the form e^{ih} for some $h = h^*$.

Our first goal is to find a sufficient condition to guarantee that a unitary in a unital JB*-algebra writes as the exponential of some skew symmetric element. Let u be a unitary element in a unital C*-algebra A. It is known that $\|\mathbf{1}_A - u\|_A < 2$ implies that $u = e^{ih}$ for some $h \in A_{sa}$ (see [99, Exercise 4.6.6]). Changing the unit by considering isotopes, and using appropriately the celebrated Shirshov-Cohn theorem, we can apply the previous fact and state a similar Jordan version.

Lemma 3.3.3. [36, Lemma 2.2, Preprint 2020] *Let u, v be two unitaries in a unital JB*-algebra M. Let us suppose that $\|u - v\|_M = \eta < 2$. Then the following statements hold:*

(a) *There exists a self-adjoint element h in the u-isotope JB*-algebra $M(u) = (M, \circ_u, *_u)$ such that $v = e^{ih}$, where the exponential is computed in the JB*-algebra $M(u)$.*

(b) *There exists a unitary w in M satisfying $U_w(u^*) = v$.*

Moreover, if $\|u - v\|_M = \eta = \left|1 - e^{it_0}\right| = \sqrt{2}\sqrt{1 - \cos(t_0)}$, for some $t_0 \in (-\pi, \pi)$, we can further assume that

$$\|w - u\|_M, \|w - v\|_M \leq \sqrt{2}\sqrt{1 - \cos(\frac{t_0}{2})}.$$

□

Suppose now that u is a unitary element in a unital JB*-algebra M such that $\|\mathbf{1} - u\| < 2$. By Lemma 3.3.3(a) above we can find a self-adjoint element $h \in M_{sa}$ satisfying $u = e^{ih}$. Let us consider the unitary $\omega = e^{-i\frac{h}{2}} \in \mathcal{U}(M)$, and the mapping $U_\omega : M \to M$. Let us observe that $U_\omega(u) = \mathbf{1}_M$. It can be proved that $U_\omega : M(u) = (M, \circ_u, *_u) \to M$ is a unital surjective isometry, and by virtue of Theorem 6 in [168], a Jordan *-isomorphism. The role played by this Jordan *-isomorphism will be crucial in the main theorem.

Henceforth, let $\mathcal{G}$ be a group and let (X, d) be a non-trivial metric space such that X is a subset of $\mathcal{G}$ and

$$yx^{-1}y \in X \text{ for all } x, y \in X$$

(note that we are not assuming that X is a subgroup of $\mathcal{G}$).

Definition 3.3.4. *Let us fix a, b in X. We shall say that condition $B(a, b)$ holds for (X, d) if the following properties hold:*

(B.1) *For all $x, y \in X$ we have $d(bx^{-1}b, by^{-1}b) = d(x, y)$.*

(B.2) *There exists a constant $K > 1$ satisfying*

$$d(bx^{-1}b, x) \geq K d(x, b),$$

for all $x \in L_{a,b} = \{x \in X : d(a, x) = d(ba^{-1}b, x) = d(a, b)\}$.

Definition 3.3.5. *Let us fix $a, b \in X$. We shall say that condition $C_1(a,b)$ holds for (X,d) if the following properties hold:*

$(C.1)$ *For every $x \in X$ we have $ax^{-1}b, bx^{-1}a \in X$;*

$(C.2)$ $d(ax^{-1}b, ay^{-1}b) = d(x,y)$*, for all $x, y \in X$.*

We shall say that condition $C_2(a,b)$ holds for (X,d) if there exists $c \in X$ such that $ca^{-1}c = b$ and $d(cx^{-1}c, cy^{-1}c) = d(x,y)$ for all $x, y \in X$.

An element $x \in X$ is called *2-divisible* if there exists $y \in X$ such that $y^2 = x$. X is called *2-divisible* if every element in X is 2-divisible. Furthermore, X is a called *2-torsion free* if it contains the unit of $\mathcal{G}$ and the condition $x^2 = 1$ with $x \in X$ implies $x = 1$.

These conditions, introduced by O. Hatori, G. Hirasawa, T. Miura and L. Molnár in [84] for the study of isometries between metric groups, are the primary tools used in [86] to state Theorem 3.3.2. However, its applicability in the Jordan setting is extremely limited. Namely, let $\mathcal{A}$ be a unital JC*-algebra which will be regarded as a JB*-subalgebra of some $B(H)$. Let us observe that the unit of $\mathcal{A}$ must be a projection $\mathbf{1}_{\mathcal{A}}$ in $B(H)$, and thus by replacing H with $\mathbf{1}_{\mathcal{A}}(H)$, we can always assume that $\mathcal{A}$ and $B(H)$ share the same unit. We shall denote the product of $B(H)$ by mere juxtaposition. The set $\mathcal{U}(\mathcal{A})$ of all unitaries in $\mathcal{A}$ is not in general a subgroup of $\mathcal{U}(B(H))$ –the latter is not even stable under Jordan products–, however $U_u(v) = uvu$, and u^* lie in $\mathcal{U}(\mathcal{A})$ for all $u, v \in \mathcal{U}(\mathcal{A})$ (cf. Lemma 1.3.4). The set $\mathcal{U}(B(H))$ is a group for its usual product and will be equipped with the distance provided by the operator norm. Conditions of the type $C_1(a,b)$ do not hold for $(\mathcal{U}(\mathcal{A}), \|.\|)$ because products of the form $ax^{-1}b$ do not necessarily lie in $\mathcal{U}(\mathcal{A})$ for all $a, b, x \in \mathcal{U}(\mathcal{A})$. The set $\mathcal{U}(\mathcal{A})$ is not 2-torsion free since $-1 \in \mathcal{U}(\mathcal{A})$. We have therefore justified that [84, Corollaries 3.9, 3.10 and 3.11] cannot be applied in the Jordan setting, even under the more favourable hypothesis of working with a JC*-algebra.

Hidden within the proof of [85, Theorem 6], it is shown that for a complex Banach space Z, condition $B(a,b)$ is satisfied for elements a, b in the group of all surjective linear isometries on Z which are at distance strictly smaller than 1/2 (cf. [36, Lemma 2.7]). The arguments employed are valid for a JC*-algebra M by just regarding M as a unital Jordan *-subalgebra of some $B(H)$ with the same unit. However, this is far to be true for a general JB*-algebra. The existence of exceptional JB*-algebras which cannot be embedded as Jordan *-subalgebras of $B(H)$ (see [82, Corollary 2.8.5], [25, Example 3.1.56]), forces us to develop a new argument based on Lemma 3.3.3, the Shirshov-Cohn theorem, and the uniqueness of the triple product commented before in (3.6) (cf. [106, Proposition 5.5]). We create a framework in which [85, Proof of Theorem 6] can be applied, and obtain conclusions in the setting of JB*-algebras through the U-operators and the triple product.

Lemma 3.3.6. [36, Lemma 2.8, Preprint 2020] *Let u, v be two elements in $\mathcal{U}(M)$, where M is a unital JB*-algebra. Suppose $\|u - v\| < 1/2$. Then the Jordan version of condition $B(u, v)$ holds for $\mathcal{U}(M)$, that is,*

$(B.1)$ *For all $x, y \in \mathcal{U}(M)$ we have $\|U_v(x^*) - U_v(y^*)\| = \|x - y\|$.*

$(B.2)$ *The constant $K = 2 - 2\|u - v\| > 1$ satisfies that*

$$\|U_v(w^*) - w\| \geq K\|w - v\|,$$

for all w in the set

$$L_{u,v} = \{w \in \mathcal{U}(M) : \|u - w\| = \|U_v(u^*) - w\| = \|u - v\|\}.$$

□

The just stated lemma guarantees that the condition $B(u, v)$ holds for $\mathcal{U}(M)$. We go a step further with the following key theorem.

Theorem 3.3.7. [36, Theorem 2.9, Preprint 2020] *Let $\Delta : \mathcal{U}(M) \to \mathcal{U}(N)$ be a surjective isometry, where M and N are unital JB*-algebras. Suppose $u, v \in \mathcal{U}(M)$ with $\|u - v\| < \frac{1}{2}$. Then the following statements are true:*

(1) *The Jordan version of condition $B(u, v)$ holds for $\mathcal{U}(M)$;*

(2) *The Jordan version of condition $C_2(\Delta(u), \Delta(U_v(u^*)))$ holds for $\mathcal{U}(N)$;*

(3) *The identity $\Delta(U_v(u^*)) = U_{\Delta(v)}(\Delta(u)^*) =$ holds.*

□

We recall that a *one-parameter group* of bounded linear operators on a Banach space Z is a mapping $\mathbb{R} \to B(Z)$, $t \mapsto E(t)$ satisfying $E(0) = I$ and $E(t + s) = E(s)E(t)$, for all $s, t \in \mathbb{R}$. A one-parameter group $\{E(t) : t \in \mathbb{R}\}$ is uniformly continuous (at the origin) if $\lim\limits_{t\to 0} \|E(t) - I\| = 0$. It is known that for each uniformly continuous one-parameter group there exists a bounded linear operator $R \in B(Z)$ such that $E(t) = e^{tR}$ for all $t \in \mathbb{R}$, where the exponential is computed in the Banach algebra $B(Z)$ (see, for example, [19, Proposition 3.1.1]). A one-parameter group $\{E(t) : t \in \mathbb{R}\}$ on a complex Hilbert space H is called *strongly continuous* if for each ξ in H the mapping $t \mapsto E(t)(\xi)$ is continuous ([32, Definition 5.3, Chapter X]). A *one-parameter unitary group* on H is a one-parameter group on H such that $E(t)$ is a unitary element for each $t \in \mathbb{R}$.

The celebrated Stone's one-parameter theorem affirms that for each strongly continuous one-parameter unitary group $\{E(t) : t \in \mathbb{R}\}$ on a complex Hilbert space H there exists a self-adjoint operator $h \in B(H)$ such that $E(t) = e^{ith}$, for every $t \in \mathbb{R}$ ([32, 5.6, Chapter X]).

The study of uniformly continuous one-parameter groups of surjective isometries (i.e. triple isomorphisms), Jordan *-isomorphisms, and orthogonality preserving operators on JB*-algebras has been recently initiated in [78]. We contribute in [36] to this study by stating a Jordan version of Stone's one-parameter theorem for uniformly continuous one-parameter unitary groups. Triple derivations are essential in the proof of the theorem bellow. Therefore, we recall that a *triple derivation* on a JB*-triple X is a linear mapping $\delta : X \to X$ satisfying a ternary version of Leibniz' rule, that is,

$$\delta \{a, b, c\}_X = \{\delta(a), b, c\}_X + \{a, \delta(b), c\}_X + \{a, b, \delta(c)\}_X, \quad (a, b, c \in X).$$

We shall apply that every triple derivation is automatically continuous (see [10, Corollary 2.2]). If $\delta : M \to M$ is a triple derivation on a unital JB*-algebra, it is known that $\delta(\mathbf{1}_M)^* = -\delta(\mathbf{1}_M)$, that is, $i\delta(\mathbf{1}_M) \in M_{sa}$ (cf. [87, Proof of Lemma 1]). The arguments also go through several applications of MacDonald's identity (1.13), and induction computations. The Jordan version of the Stone's one-parameter theorem reads as follows:

Theorem 3.3.8. [36, Theorem 3.1, Preprint 2020] *Let M be a unital JB*-algebra. Suppose $\{u(t) : t \in \mathbb{R}\}$ is a family in $\mathcal{U}(M)$ satisfying $u(0) = \mathbf{1}$, and $U_{u(t)}(u(s)) = u(2t+s)$, for all $t, s \in \mathbb{R}$. We also assume that the mapping $t \mapsto u(t)$ is continuous. Then there exists $h \in M_{sa}$ such that $u(t) = e^{ith}$ for all $t \in \mathbb{R}$.* □

We have gathered all the ingredients employed to address the purpose of our paper [36], namely, to establish a Jordan version of the Hatori-Molnár theorem. The first main result in [36] asserts that, under some mild conditions, for each surjective isometry Δ between the unitary sets of two unital JB*-algebras M and N we can find a surjective real linear isometry $\Psi : M \to N$ which coincides with Δ on the subset $e^{iM_{sa}}$.

Theorem 3.3.9. [36, Theorem 3.4, Preprint 2020] *Let $\Delta : \mathcal{U}(M) \to \mathcal{U}(N)$ be a surjective isometry, where M and N are two unital JB*-algebras. Suppose that one of the following holds:*

(1) $\|\mathbf{1}_N - \Delta(\mathbf{1}_M)\| < 2$;

(2) *There exists a unitary ω_0 in N such that $U_{\omega_0}(\Delta(\mathbf{1}_M)) = \mathbf{1}_N$.*

Then there exists a unitary ω in N satisfying

$$\Delta(e^{iM_{sa}}) = U_{\omega^*}(e^{iN_{sa}}).$$

Furthermore, there exists a central projection $p \in N$ and a Jordan *-isomorphism $\Phi : M \to N$ such that

$$\begin{aligned}\Delta(e^{ih}) &= U_{\omega^*}\left(p \circ \Phi(e^{ih})\right) + U_{\omega^*}\left((\mathbf{1}_N - p) \circ \Phi(e^{ih})^*\right) \\ &= P_2(U_{\omega^*}(p))U_{\omega^*}(\Phi(e^{ih})) + P_2(U_{\omega^*}(\mathbf{1}_N - p))U_{\omega^*}(\Phi((e^{ih})^*)),\end{aligned}$$

for all $h \in M_{sa}$. Consequently, the restriction $\Delta|_{e^{iM_{sa}}}$ admits a (unique) extension to a surjective real linear isometry from M onto N. □

We re-write the conclusion of the previous Theorem 3.3.9 from the point of view of JB*-triples.

Theorem 3.3.10. [36, Remark 3.7, Preprint 2020] *Under the same hypotheses of Theorem 3.3.9, there exist two orthogonal tripotents u_1 and u_2 in M and two orthogonal tripotents $\widetilde{u}_1$ and $\widetilde{u}_2$ in N, a linear surjective isometry (i.e. triple isomorphism) $\Psi_1 : M_2(u_1) \to N_2(\widetilde{u}_1)$ and a conjugate linear surjective isometry (i.e. triple isomorphism) $\Psi_2 : M_2(u_2) \to N_2(\widetilde{u}_2)$ such that $M = M_2(u_1) \oplus^\infty M_2(u_2)$, $N = N_2(\widetilde{u}_1) \oplus^\infty N_2(\widetilde{u}_2)$, and the surjective real linear isometry $\Psi = \Psi_1 + \Psi_2 : M_2(u_1) \oplus^\infty M_2(u_2) \to N_2(\widetilde{u}_1) \oplus^\infty N_2(\widetilde{u}_2)$ restricted to $e^{iM_{sa}}$ coincides with Δ.* □

A consequence of Theorem 3.3.9 asserts that the Banach spaces underlying two unital JB*-algebras are isometrically isomorphic if and only if the metric spaces determined by the unitary sets of these algebras are isometric.

Corollary 3.3.11. [36, Corollary 3.8, Preprint 2020] *Two unital JB*-algebras M and N are Jordan *-isomorphic if and only if there exists a surjective isometry $\Delta : \mathcal{U}(M) \to \mathcal{U}(N)$ satisfying one of the following*

(1) $\|\mathbf{1}_N - \Delta(\mathbf{1}_M)\| < 2$;

(2) *There exists a unitary ω in N such that $U_\omega(\Delta(\mathbf{1}_M)) = \mathbf{1}_N$.*

Furthermore, the following statements are equivalent for any two unital JB-algebras M and N:*

(*a*) *M and N are isometrically isomorphic as (complex) Banach spaces;*

(*b*) *M and N are isometrically isomorphic as real Banach spaces;*

(*c*) *There exists a surjective isometry $\Delta : \mathcal{U}(M) \to \mathcal{U}(N)$.* □

Analogously to the associative setting, if we assume that M and N are JBW*-algebras then Theorem 3.3.9 yields a full description of any surjective isometry between the unitary sets of M and N. The reason being that

$$\mathcal{U}(M) = \{e^{ih} : h \in M_{sa}\}$$

for all JBW*-algebra M. Consequently, Problem 3.3.1 finds a positive answer whenever $\mathcal{S}_j = \mathcal{U}(M_j)$ with M_j being a JBW*-algebra for $j = 1, 2$.

Theorem 3.3.12. [36, Theorem 3.9, Preprint 2020] *Let* $\Delta : \mathcal{U}(M) \to \mathcal{U}(N)$ *be a surjective isometry, where* M *and* N *are two JBW***-algebras. Then there exist a unitary* ω *in* N, *a central projection* $p \in N$, *and a Jordan* **-isomorphism* $\Phi : M \to N$ *such that*

$$\begin{aligned}\Delta(u) &= U_{\omega^*}\left(p \circ \Phi(u)\right) + U_{\omega^*}\left((\mathbf{1}_N - p) \circ \Phi(u)^*\right)\\ &= P_2(U_{\omega^*}(p))U_{\omega^*}(\Phi(u)) + P_2(U_{\omega^*}(\mathbf{1}_N - p))U_{\omega^*}(\Phi(u^*)),\end{aligned}$$

for all $u \in \mathcal{U}(M)$. *Consequently,* Δ *admits a (unique) extension to a surjective real linear isometry from* M *onto* N. □

Chapter 4

Open problems

The final chapter of this project is devoted to stating some open problems. We begin with the most natural question.

Problem 4.0.1. *Let L be a locally compact Hausdorff space. Does the space $C_0(L,\mathbb{K})$, of all $\mathbb{K}$-valued continuous functions on L vanishing at zero, satisfy the Mazur–Ulam property?*

In 1994, R.S. Wang proved that any surjective isometry between the unit spheres of two $C_0(L,\mathbb{K})$-spaces admits an extension to a surjective real linear isometry between the whole spaces (see [164]). The above Problem 4.0.1 is intended to generalise this positive partial solution to Tingley's problem by showing that $C_0(L,\mathbb{K})$ actually satisfies the stronger Mazur–Ulam property. It is known that the C*-algebra $C_0(L,\mathbb{K})$ is unital if and only if L is compact. In such a case, $C_0(L,\mathbb{K})$ coincides with $C(L,\mathbb{K})$. The stunning results obtained by M. Mori and N. Ozawa in [126] prove that any unital C*-algebra satisfies the Mazur–Ulam property. The lacking of a unit element in $C_0(L,\mathbb{K})$ for a general locally compact Hausdorff space L makes the problem even more challenging. The new achievements due to A.M. Peralta in the preprint [137] contain the first approach to the Mazur–Ulam property in the setting of non-commutative and non-unital C*-algebras, and provide a new hope to afford the problem. In the just quoted paper, the author explores the Mazur–Ulam property for the space $K(H)$, of all compact operators on an infinite dimensional complex Hilbert space H, from a more general point of view, namely, the frame of weakly compact JB*-triples. The lacking of extreme points of the closed unit ball of $K(H)$ makes impossible to apply the arguments of Proposition 3.1.1 to prove that $K(H)$ has the strong Mankiewicz property.

Problem 4.0.2. *Let Δ be a surjective isometry between the unit spheres of two JBW*-algebra preduals. Does Δ admit an extension to a surjective real linear isometry between the whole spaces? The same question applies to a surjective isometry between the unit spheres of two JBW*-triple preduals.*

The characterisation of unitary elements given by M. Mori in [125, Lemma 3.1], and the Hatori-Molnár theorem [86, Corollary 3] for von Neumann algebras are the fundamental ingredients in the proof of the positive answer to Tingley's problem for surjective isometries between the unit spheres of two von Neumann algebra preduals recently achieved by M. Mori in [125]. The Jordan-analogue results obtained in [35], and [36] lead us to conjecture that Tingley's problem admits a positive answer for surjective isometries between the unit spheres of two JBW*-algebra preduals.

Problem 4.0.3. *Do von Neumann algebra preduals, JBW*-algebra preduals and JBW*-triple preduals satisfy the Mazur–Ulam property?*

Problem 4.0.4. *Do real JBW*-triples satisfy the Mazur–Ulam property?*

We claim that the problem above could find a satisfactory answer by following the same strategy used in the paper [12], where the same question is addressed for (complex) JBW*-triples. A real version of the Russo–Dye theorem is required in the setting of JBW*-algebras. The approximation of the elements in the closed unit ball given in [129] for real von Neumann algebras could provide a hint.

Problem 4.0.5. *Does every Banach space satisfying the conclusion of the Krein–Milmann theorem satisfy the Mazur–Ulam property? Is the strong Mankiewicz property a sufficient condition for having the Mazur–Ulam property?*

Problem 4.0.6. *Let X be a JB*-triple satisfying property $(\mathcal{P})$. Does X satisfy the Mazur–Ulam property?*

We have shown in the first part of the proof of Theorem 3.2.16 that if M is a JBW*-triple satisfying property $(\mathcal{P})$, then, every surjective isometry from the unit sphere of M onto the unit sphere of a Banach space Y admits a unique extension to a surjective real linear isometry from M onto Y. The proof of Theorem 3.2.16 actually shows that every JBW*-triple with rank bigger than or equal to three satisfies property $(\mathcal{P})$. There are other examples of JBW*-triples satisfying property $(\mathcal{P})$. In [12, Remark 4.16] we show that any JBW*-triple M which is not a factor satisfies property $(\mathcal{P})$, or more generally, any JBW*-triple M such that the atomic part of M^{**} is not a Cartan factor of rank one or two has this property too.

Problem 4.0.7. *Let E be a JB*-triple admitting no unitary elements. Suppose u is a complete tripotent in E (which is obviously non-unitary). Does the set*

$$\mathcal{E}_u = \{e \in \partial_e(\mathcal{B}_E) : \|u \pm e\| \leq \sqrt{2}\}$$

contain no isolated points?

The question above is posed at the end of [35], where we additionally explored some examples of JB*-triples admitting no unitary tripotents. More concretely, we studied this result in the case of a rectangular type 1 Cartan factor of the form $C = B(H, K)$, of all bounded linear operators between two complex Hilbert spaces H and K, with $\dim(H) > \dim(K)$.

In the simplest case $K = \mathbb{C}$ is one dimensional, and hence $C = H$ is a Hilbert space which can be regarded as a JB*-triple with triple product associated to a type 1 Cartan factor. Every norm-one element in C is an extreme point of its closed unit ball, that is, $\partial_e(\mathcal{B}_C) = S(C)$. Let us fix $u \in S(C)$. By assuming $\dim(C) \geq 2$ it is not hard to see that

$$\begin{aligned}\mathcal{C}_u &= \{e \in \partial_e(\mathcal{B}_C) : \|u \pm e\| \leq \sqrt{2}\} \\ &= \{itu + x : t \in \mathbb{R}, x \in C, \langle e, x\rangle = 0, t^2 + \|x\|^2 = 1\},\end{aligned}$$

is pathwise-connected.

In the case in which $\dim(K) \geq 2$, every complete tripotent in C must be a partial isometry u satisfying $uu^* = \mathrm{id}_K$ (and clearly, $u^*u \neq \mathrm{id}_H$). Let us take $y \in \mathcal{C}_u = \{e \in \partial_e(\mathcal{B}_C) : \|u \pm e\| \leq \sqrt{2}\}$. It can be checked that y is non-isolated in $\mathcal{C}_u$.

Problem 4.0.8. *Does Tingley's problem admit a positive solution for a surjective isometry between the units spheres of two Lipschitz spaces? Do Lipschitz spaces satisfy the Mazur–Ulam property?*

The first attempts to address Tingley's problem, for instance, the approaches of G.G. Ding and his students in [47, 46, 45, 43], relies on representation theorems for surjective linear isometries between the corresponding spaces. In the case of Lipschitz spaces, the lacking of a complete knowledge on the facial structure is an important obstacle. However, the study of the Banach-Stone type theorems for Lipschitz spaces is an active topic which suggests a possible path to explore the isometric extension problems.

Problem 4.0.9. *Let E be a Hilbert C*-module in the sense introduced by I. Kaplansky [101]. Suppose $\Delta : S(E) \to S(X)$ is a surjective isometry, where X is any Banach space. Does there exists a surjective real linear isometry $T : E \to X$ such that $T(x) = \Delta(x)$, for every $x \in S(E)$?*

Hilbert C-modules* appear for first time in [101], where I. Kaplansky generalises Hilbert spaces by allowing the inner product to take values in a (commutative) C*-algebra rather than in $\mathbb{C}$. In the paper [34], we address the Mazur–Ulam property by considering the Hilbert $C(K)$-module structure of the space of all Hilbert-valued continuous functions on a compact Hausdorff space K. This procedure allows us to apply techniques of JB*-triple theory. On the other hand, it is known that any unital C*-algebra and any Hilbert space satisfies the Mazur–Ulam property ([126], [12]).

Appendix A

Separation results

During the developments of the papers supporting this thesis, it was required to apply a huge amount of results whose presence could be unperceived, but it had an invaluable impact over our arguments. This is the case of some results known as *separation results*, which have turned out to be useful tools.

Urysohn's lemma

The well-known Urysohn's lemma, due to P. Urysohn [163], was stated in 1925, and it seems to be a really useful separation result with applications in the most assorted situations. In contrast with the classical proofs, R. Blair proposes in [14] one based on Zorn's lemma.

Lemma 1. [163, §25, Urysohn's lemma] *Let A and B be two disjoint closed subsets of a normal topological space X. Then there exists a continuous function $f : X \to [0, 1]$ such that $f(a) = 0$ for every a in A, and $f(b) = 1$ for every b in B.*

A version for locally compact Hausdorff spaces reads as follows:

Lemma 2. *Let L be a locally compact Hausdorff space, suppose K is a compact set in L, and U is an open subset of L such that $K \subseteq U$. Then there exists a continuous function $f : L \to [0, 1]$ such that $f(x) = 1$ for every $x \in K$, and $f(x) = 0$ for every $x \in L\backslash U$.*

Let K be a topological compact Hausdorff space, and suppose C and O are subsets of K such that C is closed, O is open, and $C \subseteq O$. From a re-reading of the original statement it can be derived the existence of a positive element a in the C*-algebra $C(K)$ of all complex-valued continuous functions on K, satisfying

$$0 \leq a \leq 1, \qquad a\chi_C = \chi_C \quad \text{and} \quad a\chi_{K\backslash O} = 0,$$

where χ_A denotes the characteristic function of any subset $A \subseteq K$. An abundant collection of results generalising Urysohn's lemma to non-necessarily commutative settings can be found in the literature. Concerning (non-commutative) C*-algebras, the study was initiated by C.A. Akemann [2, Theorem I.1]. The contributions of this author in [1] and [3], as well as, those due to L.G. Brown [20, §3], C.A. Akemann and G.K. Pedersen in [4, Lemma 2.7], supposed subsequent improvements on the topic.

Lemma 3. (Cf. [4, Lemma 2.7] or [20, Corollary 3.16]) *Let A be a C*-algebra. Suppose p and q are two projections in A^{**} with p compact and q open relative to A, such that $pq = p$. Then there exists a positive element x in A with $p \leq x \leq q$.*

The setting of JB*-triples is the natural environment of the present thesis, and as we have seen each C*-algebra is a JB*-triple. When a C*-algebra A is regarded as a JB*-triple, the set of tripotents Trip(A) coincides with the set of partial isometries in A. Several versions of Urysohn's lemma have been established in the framework of JB*-triples and TRO's. For instance, the reader is referred to [22, Theorem 3.3], [66, Theorems 1.4 and 1.10] or [15, Theorems 3.19 and 3.20]. As reviewed in section 2.1, the concepts of *open* and *compact* tripotents in the bidual of a JB*-triple were introduced by C.M. Edwards and G.T. Rüttimann in [56], while the notions of *closed* and *bounded* tripotents in the bidual of a JB*-triples are due to F.J. Fernández-Polo and A.M. Peralta (see [65]). We highlight the following generalisation of Urysohn's lemma in the setting of JB*-triples.

Proposition 4. [68, Proposition 3.7] *Let X be a weak*-dense JB*-subtriple of a JBW*-triple W. Let u, v be two non-zero orthogonal compact tripotents in W relative to X. Then there exist two orthogonal norm-one elements a, b in X such that $a = v + P_0(v)(a)$, and $b = P_0(u)(b)$. In particular, $u + v$ is compact relative to E.*

Eidelheit's separion theorem

It has been exposed along the different sections the intimate relation existing between the isometric extension problems and the facial structure of the Banach space involved. In that sense, Eidelheit's separation theorem plays a silent protagonist role. We borrow the statement from [120, Theorem 2.2.26], but the original source from 1936 can be consulted in [58] (see also [40] from 1941).

Theorem 5. [120, Theorem 2.2.26] *Let X be a topological vector space, and let C_1 and C_2 be non-empty convex subsets of X such that C_2 has non-empty interior. If $C_1 \cap \mathring{C}_2 = \emptyset$, then there is a member φ in X^*, and a real number s such that*

1. *$\Re e\, \varphi(x) \geq s$, for each x in C_1;*

2. *$\Re e\, \varphi(x) \leq s$, for each x in C_2;*

3. *$\Re e\, \varphi(x) < s$, for each x in $\mathring{C}_2$;*

The following result, due to R. Tanaka in [158], is based on a straightforward application of Eidelheit's separation theorem.

Lemma 6. [158, Lemma 3.3] *Let X be a Banach space, and let C be a maximal convex subset of $S(X)$. Then $C = f^{-1}(1) \cap \mathcal{B}_X$, for some $f \in S(X^*)$.*

It derives from this fact that a convex subset of $S(X)$ is maximal as a convex subset of $S(X)$ if and only if it is a maximal proper face of $\mathcal{B}_X$ ([159, Lemma 3.2]).

In the line of the comments above, Eidelheit's separation theorem is essential to guarantee in Lemmas 3.0.6, 3.0.7 and 3.0.8 the existence of support functionals, whose role in the study of the Mazur–Ulam property in [33, 12] and [34] (or previously in [94]) is indisputably useful. The separation theorem fits perfectly with Lemma 2.1.1, and hence provides basic but important conclusions in the facial structure approach to the isometric extension results.

Appendix B

Qualifications

The academic formation of the doctoral candidate has been complemented during the PhD program with courses, seminars, workshops and congresses. The following list includes the main activities where she has participated.

Courses

1. *VIII Escuela-Taller de Análisis Funcional.*
   5th-8th March 2018, BCAM Bilbao, Spain

2. *Doc-course: Modelos Matemáticos de la Ciencia.*
   Modelos Matemáticos en Medicina - V. Pérez.
   24th April 2018, IEMath, University of Granada

3. *Doc-course: Modelos Matemáticos de la Ciencia.*
   Dinámica poblacional - M. Piñar, T.E. Pérez.
   17th May 2018, IEMath, University of Granada

4. *Primera escuela de divulgación de las matemáticas: aprende a divulgar.*
   25th-28th June 2019, CIEM, Castro Urdiales, Cantabria, Spain

5. *XX Cursos de verano UAL: Divulgación y comunicación científica. Hacia la sociedad del conocimiento.*
   1st-3rd July 2019, University of Almería

Attendance at seminars and congresses

1. *Seminario de Jóvenes Investigadores en Matemáticas.*
   L. García Lirola - *Estructura extremal de espacios Lipschitz-libres.*
   14th February 2018, IEMath, University of Granada

2. *Jornada Mujer en la Ciencia.*
   15-16 February 2018, IEMath, University of Granada

3. *XIV Encuentro Red de Análisis Funcional y Aplicaciones.*
   8th-10th March 2018, BCAM Bilbao, Spain

4. T.S.S.R.K. Rao - *Operators on separable L1-predual spaces.*
   24th April 2018, Departamento de Análisis Matemático,
   Facultad de Ciencias, University of Granada

5. L. Molnár - *Bures isometries between density spaces of C^*-algebras and some related maps.*
   26th April 2018, Departamento de Análisis Matemático,
   Facultad de Ciencias, University of Granada

6. L. Molnár - *Strength functions: a strange function space associated to the positive semidefinite cone of Hilbert space operators.*
   27th April 2018, Departamento de Matemáticas, University of Almería

7. A.B.A. Essaleh - *On certain preservers of λ-Aluthge transforms.*
   11th May 2018, Departamento de Análisis Matemático,
   Facultad de Ciencias, University of Granada

8. O. Hatori - *A geometric inequality and its application.*
   6th September 2018, IEMath, University of Granada

9. T. Miura - *Isometries on a Lipschitz space of analytic functions.*
   7th September 2018, IEMath, University of Granada

10. *Workshop on the frontiers between Functional Analysis and Algebra (WFFAA) in tribute to Professor Amin Kaidi.*
    13th-14th September 2018, University of Almería

11. *Seminario de Jóvenes Investigadores en Matemáticas.*
    R. Chiclana - *Funciones Lipschitz que alcanzan fuertemente su norma.*
    8th November 2018, IEMath, University of Granada

12. *Seminario de Jóvenes Investigadores en Matemáticas.*
    A. Quero - *Índice numérico respecto a un operador.*
    29th November 2018, IEMath, University of Granada

13. *Seminario de Jóvenes Investigadores en Matemáticas.*
    J. Langemets - *Bidual octahedral renormings and strong regularity in Banach spaces.*
    30th April 2019, IEMath, University of Granada

14. M. Mbekhta - *Approximation of the polar factor of an operator acting on Hilbert spaces.*
    5th September 2019, University of Almería

15. M. Villegas - *Bilinear isometries on spaces of Lipschitz functions.*
    5th September 2019, University of Almería

16. A.M. Peralta - *Grothendieck's inequalities for JB*-triples.*
    5th September 2019, University of Almería

17. *Seminario de Jóvenes Investigadores en Matemáticas.*
    A. Rueda Roca - *Propiedad de Dauganet en tensor proyectivo.*
    30th October 2019, IEMath, University of Granada

18. H. Queffelec - *Groupe de Travail.*
    6th March 2020, Université de Lille, France

19. V. Müller - *Groupe de Travail 1.*
    6th March 2020, Université de Lille, France

20. V. Müller - *Groupe de Travail 2.*
    13th March 2020, Université de Lille, France

21. *Zagreb Workshop on Operator Theory 2020.*
    29th-30th June 2020

Talks delivered

1. *Seminario de Jóvenes Investigadores en Matemáticas.*
   Extension of isometries. On the Mazur-Ulam property for $C(K)$.
   17th January 2019, IEMath University of Granada

2. *Preserver Weekend in Szeged.*
   Extension of isometries on the unit sphere of some spaces of continuous functions.
   12th-14th April 2019, Szeged, Hungary

3. *Women in Operator Theory and its Applications WOT 19.*
   Jordan structures in certain real operator spaces.
   17th-19th June 2019, Lisbon, Portugal

4. *XVI Function Theory on Infinite Dimensional Spaces.*
   On the Mazur-Ulam property for continuous function spaces.
   18th-21st November 2019, Universidad Complutense de Madrid

Posters presented

1. *VIII Simposio de Investigación en Ciencias Experimentales.*
   Extension of isometries on the unit sphere of $C(K, H)$.
   14th-15th November 2019, University of Almería

Several events related to the education of the doctoral candidate has been cancelled because of the world pandemic caused by the Covid-19. Nevertheless, numerous initiatives have emerged via online. The following list includes the webinars where the doctoral candidate has taken part.

Webinars

1. *Groups, Operators, and Banach Algebras Webinar*, 2020
2. *Preserver Webinar*, 2020
3. *UK Virtual Operator Algebras Seminar*, 2020
4. *Triple Product Physics Lecture*, 2020

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The Mazur–Ulam property for commutative von Neumann algebras

María Cueto-Avellaneda and Antonio M. Peralta

Departamento de Análisis Matemático, Facultad de Ciencias, Universidad de Granada, Granada, Spain

ABSTRACT
Let (Ω, μ) be a σ-finite measure space. Given a Banach space X, let the symbol $S(X)$ stand for the unit sphere of X. We prove that the space $L^\infty(\Omega, \mu)$ of all complex-valued measurable essentially bounded functions equipped with the essential supremum norm satisfies the Mazur–Ulam property, that is, if X is any complex Banach space, every surjective isometry $\Delta : S(L^\infty(\Omega, \mu)) \to S(X)$ admits an extension to a surjective real linear isometry $T : L^\infty(\Omega, \mu) \to X$. This conclusion is derived from a more general statement which assures that every surjective isometry $\Delta : S(C(K)) \to S(X)$, where K is a Stonean space, admits an extension to a surjective real linear isometry from $C(K)$ onto X.

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1. Introduction

A Banach space X satisfies the *Mazur–Ulam property* if for any Banach space Y, every surjective isometry $\Delta : S(X) \to S(Y)$ admits an extension to a surjective real linear isometry from X onto Y, where $S(X)$ and $S(Y)$ denote the unit spheres of X and Y, respectively. An equivalent reformulation tells that X satisfies the Mazur–Ulam property if the so-called Tingley's problem admits a positive solution for every surjective isometry from $S(X)$ onto the unit sphere of any Banach space Y . Positive solutions to Tingley's problem have been established when X and Y are sequence spaces [1–4], $L^p(\Omega, \Sigma, \mu)$ spaces with $1 \leq p \leq \infty$ [5–7], $C(K)$ spaces [8], spaces of compact operators on complex Hilbert spaces and compact C*-algebras [9,10], spaces of bounded linear operators on complex Hilbert spaces, atomic von Neumann algebras and JBW*-triples [11,12], a von Neumann algebras [13], spaces of trace class operators [14], preduals of von Neumann algebras [15], and spaces of p-Schatten von Neumann operators on a complex Hilbert space [16]. We refer to the surveys [17–19] for a detailed overview on Tingley's problem.

CONTACT Antonio M. Peralta aperalta@ugr.es Departamento de Análisis Matemático, Facultad de Ciencias, Universidad de Granada, 18071 Granada, Spain

Our knowledge on the class of Banach spaces satisfying the Mazur–Ulam property is a bit more reduced. This class includes the space $c_0(\Gamma, \mathbb{R})$ of real null sequences, and the space $\ell_\infty(\Gamma, \mathbb{R})$ of all real-valued bounded functions on a discrete set Γ (see [[20, Corollary 2],[21, Main Theorem]]), the space $C(K, \mathbb{R})$ of all real-valued continuous functions on a compact Hausdorff space K [21, Corollary 6], and the spaces $L^p((\Omega, \mu), \mathbb{R})$ of real-valued measurable functions on an arbitrary σ-finite measure space (Ω, μ) for all $1 \leq p \leq \infty$ [5–7]. For some time the study of those Banach spaces satisfying the Mazur–Ulam property was restricted to real Banach spaces. The existence of real linear surjective isometries which are not complex linear nor conjugate linear was a serious obstacle. Two recent contributions initiate the study of the Mazur–Ulam property in the setting of complex Banach spaces. Let Γ be an infinite set, then the space of complex null sequences $c_0(\Gamma)$ satisfies the Mazur–Ulam property (see [22]). The space $\ell_\infty(\Gamma)$ of all complex-valued bounded functions on Γ also satisfies the Mazur–Ulam property [23].

In [24], Tan et al. introduce the notions of *generalized lush* (GL) spaces and local-GL-spaces in the study of the Mazur–Ulam property by showing that every local-GL-space satisfies this property. Among the consequences of this, it is established that if E is a local-GL-space and K is a compact Hausdorff space, then $C(K, E)$ has the Mazur–Ulam property (see [24, Proposition 3.11]). It should be observed that every CL-space in the sense of Fullerton [25], and every almost-CL-space in the sense employed by Lima in [26] is a GL-space. Let us briefly recall that a Banach space X is a GL space if for every $x \in S(X)$ and every $0 < \varepsilon < 1$ there exists a slice $S = S(\varphi, \varepsilon) = \{z \in X : \|z\| \leq 1, \Re e\varphi(z) > 1 - \varepsilon\}$ (with $\varphi \in S(X^*)$) such that $x \in S$ and

$$\operatorname{dist}(y, S) + \operatorname{dist}(y, -S) < 2 + \varepsilon,$$

for all $y \in S(X)$. It is not hard to check that $\mathbb{C}$ is not a (local-)GL-space. Therefore, the result established by Tan et al. in [24, Proposition 3.11] does not throw any new light for the space $C(K)$ of all complex-valued functions on a compact Hausdorff space K.

The space $L^\infty(\Omega, \mu)$ of complex-valued measurable essentially bounded functions on an arbitrary σ-finite measure space (Ω, μ) is beyond from our current knowledge on the class of complex Banach spaces satisfying the Mazur–Ulam property. This paper is devoted to fill this gap and clear our doubts.

The natural path is to explore the interesting proof provided by Tan in the case of $L^\infty(\Omega, \mu, \mathbb{R})$ in [5]. A detailed checkup of the arguments in [5] should convince the reader that those arguments are optimized for the real setting and it is hopeless to deal with complex scalars with the tools in [5]. To avoid difficulties we extend our study to a wider setting of complex Banach spaces, the space of all complex-valued continuous functions on a Stonean space.

Let K be a compact Hausdorff space. We recall that K is called *Stonean* or *extremally disconnected* if the closure of every open set in K is open. It is known that if K is a Stonean space, then every element a in the C*-algebra $C(K)$, of all continuous complex-valued functions on K, can be uniformly approximated by finite linear combinations of projections (see [27, Proposition 1.3.1]). This topological notion has a straight connection with the property of being *monotone complete*. More concretely, let K be a compact Hausdorff space, then every bounded increasing directed set of real-valued non-negative functions (f_α) in $C(K)$ has a least upper bound in $C(K)$ if and only if K is Stonean (cf. [28,29] or [[27, Proposition 1.3.2],[30, Proposition III.1.7]]). Let us mention, by the way, that a reader interested on a systematic comprehensive insight into the bewildering variety of monotone complete C*-algebras beyond von Neumann algebras and commutative AW*-algebras can consult the recent monograph [31] by Saitô and Wright.

The C*-algebra $C(K)$ is a dual Banach space (equivalently, a von Neumann algebra) if and only if K is hyper-Stonean (cf. [29]). We recall that a Stonean space K is said to be hyper-Stonean if it admits a faithful family of positive normal measures (cf. [30, Definition 1.14]).

Following standard terminology, a *localizable measure space* (Ω, ν) is a measure space which can be obtained as a direct sum of finite measure spaces $\{(\Omega_i, \mu_i) : i \in \mathcal{I}\}$. The Banach space $L^\infty(\Omega, \nu)$ of all locally ν-measurable essentially bounded functions on Ω is a dual Banach space and a commutative von Neumann algebra. Actually, every commutative von Neumann algebra is C*-isomorphic and isometric to some $L^\infty(\Omega, \nu)$ for some localizable measure space (Ω, ν) (see [27, Proposition 1.18.1]). From the point of view of Functional Analysis, the commutative von Neumann algebras $L^\infty(\Omega, \nu)$ and $C(K)$ with K hyper-Stonean are isometrically equivalent.

In this paper we establish that if K is a Stonean space, X is an arbitrary complex Banach space, and $\Delta : S(C(K)) \to S(X)$ is a surjective isometry, then there exist two disjoint clopen subsets K_1 and K_2 of K such that $K = K_1 \cup K_2$ satisfying that if K_1 (respectively, K_2) is non-empty then there exist a closed subspace X_1 (respectively, X_2) of X and a complex linear (respectively, conjugate linear) surjective isometry $T_1 : C(K_1) \to X_1$ (respectively, $T_2 : C(K_2) \to X_2$) such that $X = X_1 \oplus^\infty X_2$, and $\Delta(a) = T_1(\pi_1(a)) + T_2(\pi_2(a))$ for every $a \in S(C(K))$, where π_j is the natural projection of $C(K)$ onto $C(K_j)$ given by $\pi_j(a) = a|_{K_j}$. In particular, Δ admits an extension to a surjective real linear isometry from $C(K)$ onto X (see Theorem 3.11).

Let (Ω, μ) be a σ-finite measure space, and let X be a complex Banach space. A consequence of our main result shows that for every surjective isometry $\Delta : S(L^\infty(\Omega, \mu)) \to S(X)$, there exists a surjective real linear isometry $T : L^\infty(\Omega, \mu) \to X$ whose restriction to $S(L^\infty(\Omega, \mu))$ is Δ (see Theorem 3.14).

We finish this note with a discussion on the chances of extending a surjective isometry between the sets of extreme points of two Banach spaces.

2. Geometric properties for general compact Hausdorff spaces

In this section we shall gather a collection of results which are motivated by previous contributions in [5,8,20–23,32].

Henceforth, given a Banach space X, the symbol $\mathcal{B}_X$ will denote the closed unit ball of X.

Let us consider a compact Hausdorff space K and the C*-algebra $C(K)$. For each $t_0 \in K$ and each $\lambda \in \mathbb{T}$ we set

$$A(t_0, \lambda) := \{f \in S(C(K)) : f(t_0) = \lambda\},$$

where $\mathbb{T}$ denotes the unit sphere of $\mathbb{C}$. Then $A(t_0, \lambda)$ is a maximal norm-closed proper face of $\mathcal{B}_{C(K)}$ and a maximal convex subset of $S(C(K))$. As in previous papers, we consider a special subset of $A(t_0, \lambda)$ defined by

$$\text{Pick}(t_0, \lambda) := \{f \in S(C(K)) : f(t_0) = \lambda \text{ and } |f(t)| < 1,\ \forall t \neq t_0\}.$$

It is known that in a compact metric space the set $\mathrm{Pick}(t_0, \lambda)$ is non-empty for every $t_0 \in K$. The same statement is actually true whenever K is a first countable compact Hausdorff space (see [33, proof of Theorem 2.2]).

Similar arguments to those given in [22, Lemma 2.1] can be applied to establish our first result.

Lemma 2.1: *Let $\Delta : S(C(K)) \to S(X)$ be a surjective isometry, where K is a compact Hausdorff space and X is a complex Banach space. Then for each $t_0 \in K$ and each $\lambda \in \mathbb{T}$ the set*

$$supp_{\Delta}(t_0, \lambda) := \{\varphi \in X^* : \|\varphi\| = 1 \text{ and } \varphi^{-1}(\{1\}) \cap \mathcal{B}_X = \Delta(A(t_0, \lambda))\}$$

is a non-empty weak-closed face of $\mathcal{B}_{X^*}$.*

Proof: Since $A(t_0, \lambda)$ is a maximal convex subset of $S(C(K))$, we deduce from [34, Lemma 5.1(ii)] (see also [35, Lemma 3.5]) that $\Delta(A(t_0, \lambda))$ is a maximal convex subset of X. Thus, by Eidelheit's separation Theorem [36, Theorem 2.2.26] there is a norm-one functional $\varphi \in X^*$ such that $\varphi^{-1}(\{1\}) \cap \mathcal{B}_X = \Delta(A(t_0, \lambda))$ (compare the proof of [37, Lemma 3.3]). The rest can be straightforwardly checked by the reader. ■

Our next lemma was essentially shown in [[5, Lemma 2.4],[22, Lemma 2.2],[32, Lemmas 3.1 and 3.5]]. We include a sketch of the proof for completeness. We recall first that given a norm-one element x in a Banach space X, the *star-like subset of* $S(X)$ *around* x, $\mathrm{St}(x)$, is the set given by

$$\mathrm{St}(x) := \{y \in S(X) : \|x + y\| = 2\}.$$

It is known that $\mathrm{St}(x)$ is precisely the union of all maximal convex subsets of $S(X)$ containing x, moreover,

$$\mathrm{St}(x) = \{y \in X : [x, y] = \{tx + (1 - t)y : t \in [0, 1]\} \subseteq S(X)\}.$$

Lemma 2.2 ([22, Lemma 2.2],[32, Lemmas 3.1 and 3.5]): *Suppose K is a first countable compact Hausdorff space, where X is a complex Banach space. Let $\Delta : S(C(K)) \to S(X)$ be a surjective isometry. Then for each t_0 in K and each $\lambda \in \mathbb{T}$ we have $\varphi\Delta(f) = -1$, for every f in $A(t_0, -\lambda)$ and every $\varphi \in supp_{\Delta}(t_0, \lambda)$.*

Proof: Let us take $f \in A(t_0, -\lambda)$ and $\varphi \in \mathrm{supp}_{\Delta}(t_0, \lambda)$. We can always pick g_0 in $\mathrm{Pick}(t_0, \lambda)$ (here we need the hypotheses assuring that K is a first countable compact Hausdorff space). Clearly

$$\|\Delta(f) - \Delta(g_0)\| = \|f - g_0\| = 2,$$

and hence $-\Delta(f) \in \mathrm{St}(\Delta(g_0))$.

By mimicking the proof in [32, Lemma 3.1] we can show that $\mathrm{St}(\Delta(g_0)) = \Delta(A(t_0, \lambda))$. Explicitly speaking, $z \in \mathrm{St}(\Delta(g_0))$ if and only if $\|z + \Delta(g_0)\| = 2$. Applying [32, Corollary 2.2] we have $\|z + \Delta(g_0)\| = 2 \Leftrightarrow \|\Delta^{-1}(z) + g_0\| = 2 \Leftrightarrow \Delta^{-1}(z) \in \mathrm{St}(g_0) = A(t_0, \lambda)$. This shows that $-\Delta(f) \in \mathrm{St}(\Delta(g_0)) = \Delta(A(t_0, \lambda))$, and hence

$$-\varphi(\Delta(f)) = \varphi(-\Delta(f)) = 1.$$

■

We shall need an appropriate version of the above result in which K is replaced with a compact Hausdorff space. We begin with a technical consequence of the parallelogram law.

Lemma 2.3: *Let λ_1, λ_2 be two different numbers in $\mathbb{T}$. Then for every $0 < \rho < dist(\lambda_1, [0,1]\lambda_2)$ we have $|\alpha + \beta| < \sqrt{4 - (dist(\lambda_1, [0,1]\lambda_2) - \rho)^2} < 2$, for every $\alpha \in \mathcal{B}_{\mathbb{C}}$ with $|\alpha - \lambda_1| < \rho$ and every $\beta \in [0,1]\lambda_2$.*

Proof: Let us denote $\theta = \text{dist}(\lambda_1, [0,1]\lambda_2) > 0$, and take any $0 < \rho < \theta$. It is standard to check that $|\alpha - \beta| > \theta - \rho > 0$. By the parallelogram law we have

$$|\alpha + \beta|^2 + |\alpha - \beta|^2 = 2(|\alpha|^2 + |\beta|^2) \leq 4,$$

and thus

$$|\alpha + \beta| \leq \sqrt{4 - |\alpha - \beta|^2} < \sqrt{4 - (\theta - \rho)^2} < 2.$$

■

The extension of Lemma 2.2 for general compact Hausdorff spaces can be stated now.

Lemma 2.4: *Suppose K is a compact Hausdorff space. Let $\Delta : S(C(K)) \to S(X)$ be a surjective isometry, where X is a complex Banach space. Then for each t_0 in K and each $\lambda \in \mathbb{T}$ we have*

$$\varphi\Delta(f) = -1 \quad \textit{for every } f \textit{ in } A(t_0, -\lambda) \textit{ and every } \varphi \in supp_\Delta(t_0, \lambda).$$

Consequently, $supp_\Delta(t_0, -\lambda) = -supp_\Delta(t_0, \lambda)$, and $\Delta(-A(t_0, \lambda)) = -\Delta(A(t_0, \lambda))$.

Proof: Let us take $f \in A(t_0, -\lambda)$ and $\varphi \in \text{supp}_\Delta(t_0, \lambda)$. The element $-\Delta(f) \in S(X)$, and thus there exists $h \in S(C(K))$ satisfying $\Delta(h) = -\Delta(f)$. We consider any $g \in A(t_0, \lambda)$. Since $\|f - g\| = 2 = \|\Delta(f) - \Delta(g)\| = 2$, we deduce that $\Delta(h) = -\Delta(f) \in \text{St}(\Delta(g))$. We have shown that $\|\Delta(h) + \Delta(g)\| = 2$, for all $g \in A(t_0, \lambda)$. Corollary 2.2 in [32] implies

$$\|h + g\| = 2 \quad \text{for all } g \in A(t_0, \lambda). \tag{1}$$

Consequently, for each $g \in A(t_0, \lambda)$ there exists $t_g \in K$ such that

$$2 \leq |h(t_g) + g(t_g)| \leq |h(t_g)| + |g(t_g)| \leq 2.$$

That is, $|h(t_g)| = 1$.

For each open set $\mathcal{O} \subseteq K$ with $t_0 \in \mathcal{O}$, we find, via Urysohn's lemma, $g_{\mathcal{O}} \in A(t_0, \lambda)$ with $g_{\mathcal{O}}|_{K\setminus\mathcal{O}} = 0$. The above arguments show the existence of $t_{\mathcal{O}} \in \mathcal{O}$ satisfying $|h(t_{\mathcal{O}})| = 1$ for every $\mathcal{O}$. When the family of open subsets of K containing t_0 are ordered by inclusion, the net $(t_{\mathcal{O}})_{\mathcal{O}}$ converges to t_0. The continuity of h gives $(1)_{\mathcal{O}} = (|h(t_{\mathcal{O}})|)_{\mathcal{O}} \to |h(t_0)|$. Therefore, $|h(t_0)| = 1$.

If $h(t_0) \neq \lambda$, we find, via Lemma 2.3, $0 < \rho < \text{dist}(h(t_0), [0,1]\lambda) = \theta$ such that $|\alpha + \beta| \leq \sqrt{4 - (\theta - \rho)^2} < 2$, for every $\alpha \in \mathcal{B}_{\mathbb{C}}$ with $|\alpha - h(t_0)| < \rho$ and $\beta \in [0,1]\lambda$. The set $U := \{s \in K : |h(s) - h(t_0)| < \rho\}$ is an open neighbourhood of t_0. Applying Urysohn's lemma we find $k \in C(K)$ with $0 \leq k \leq 1$, $k(t_0) = 1$, and $k|_{K\setminus U} = 0$. The function $\lambda k \in$

$A(t_0,\lambda)$, and then (1) implies that $\|h+\lambda k\| = 2$. Since $\lambda k(K) \subseteq [0,1]\lambda$, $|h(s)| \leq 1$ and $|h(s) - h(t_0)| < \rho$ for every $s \in U$, and $k|_{K\setminus U} = 0$, we apply the above property of ρ to prove that $2 = \|h+\lambda k\| \leq \sqrt{4-(\theta-\rho)^2} < 2$, which is impossible. Therefore, $h(t_0) = \lambda$, and hence $h \in A(t_0,\lambda)$ and $1 = \varphi\Delta(h) = \varphi(-\Delta(f)) = -\varphi\Delta(f)$.

We have seen that $\varphi\Delta(f) = -1$, for every f in $A(t_0,-\lambda)$ and every $\varphi \in \text{supp}_\Delta(t_0,\lambda)$. Therefore, $\Delta(A(t_0,-\lambda)) = \varphi^{-1}(\{-1\}) \cap \mathcal{B}_X = (-\varphi)^{-1}(\{1\}) \cap \mathcal{B}_X = -(\varphi)^{-1}(\{1\}) \cap \mathcal{B}_X = -\Delta(A(t_0,\lambda))$, for every $\varphi \in \text{supp}_\Delta(t_0,\lambda)$. This shows that

$$\text{supp}_\Delta(t_0,-\lambda) = -\text{supp}_\Delta(t_0,\lambda). \qquad \blacksquare$$

The next two results contain a generalized version of [22, Lemma 2.3 and Proposition 2.4], the arguments here need an application of Urysohn's lemma.

Lemma 2.5: *Suppose K is a compact Hausdorff space. Let $\Delta : S(C(K)) \to S(X)$ be a surjective isometry, where X is a complex Banach space. Then the following statements hold*:

(a) *For every $t_0 \neq t_1$ in K and every $\lambda, \mu \in \mathbb{T}$ we have $supp_\Delta(t_0,\lambda) \cap supp_\Delta(t_1,\mu) = \emptyset$.*
(b) *Given $\mu, \nu \in \mathbb{T}$ with $\mu \neq \nu$, and t_0 in K, we have $supp_\Delta(t_0,\nu) \cap supp_\Delta(t_0,\mu) = \emptyset$.*

Proof: (a) Arguing by contradiction we assume the existence of $\varphi \in \text{supp}_\Delta(t_0,\lambda) \cap \text{supp}_\Delta(t_1,\mu)$. Let us find, via Urysohn's lemma, two functions $0 \leq f_0, f_1 \leq 1$ such that $f_0 f_1 = 0$ and $f_j(t_j) = 1$ for $j = 0, 1$. Under these conditions we have $\lambda f_0 \in A(t_0,\lambda)$ and $\mu f_1 \in A(t_1,\mu)$.

Since $-\mu f_1 \in A(t_1,-\mu)$, Lemma 2.4 implies that $\varphi\Delta(-\mu f_1) = -1$. By definition $\varphi\Delta(\lambda f_0) = 1$, and then

$$\begin{aligned} 2 &= \varphi\Delta(\lambda f_0) - \varphi\Delta(-\mu f_1) = |\varphi\Delta(\lambda f_0) - \varphi\Delta(-\mu f_1)| \\ &\leq \|\Delta(\lambda f_0) - \Delta(-\mu f_1)\| = \|\lambda f_0 + \mu f_1\| = 1, \end{aligned}$$

which is impossible.

(b) Arguing as in the previous case, let us take $\varphi \in \text{supp}_\Delta(t_0,\nu) \cap \text{supp}_\Delta(t_0,\mu)$, with $\mu \neq \nu$, and $f_0 \in A(t_0,1)$. Since $\mu f_0 \in A(t_0,\mu)$ and $\nu f_0 \in A(t_0,\nu)$, we get

$$2 = \varphi\Delta(\nu f_0) + \varphi\Delta(\mu f_0) \leq \|\Delta(\nu f_0) + \Delta(\mu f_0)\| \leq 2,$$

and by [32, Corollary 2.2] we have $2 = \|\nu f_0 + \mu f_0\| = |\mu + \nu|$, which holds if and only if $\mu = \nu$. $\blacksquare$

Proposition 2.6: *Suppose K is a compact Hausdorff space, X is a complex Banach space, and $\lambda \in \mathbb{T}$. Let $\Delta : S(C(K)) \to S(X)$ be a surjective isometry. Let t_0 be an element in K and let φ be an element in $supp_\Delta(t_0,\lambda)$. Then $\varphi\Delta(f) = 0$, for every $f \in S(C(K))$ with $f(t_0) = 0$. Furthermore, $|\varphi\Delta(f)| < 1$, for every $f \in S(C(K))$ with $|f(t_0)| < 1$, and every $\varphi \in supp_\Delta(t_0,\lambda)$.*

Proof: Let us take $g \in S(C(K))$ such that $g(t) = 0$ for every t in an open neighbourhood U of t_0. Take, via Urysohn's lemma, a function $f_0 \in S(C(K))$ with $0 \leq f_0 \leq 1$, $f_0(t_0) = 1$ and $f_0|_{K\setminus U} \equiv 0$. The functions $g \pm \lambda f_0 \in S(C(K))$ with $\lambda f_0 \in A(t_0,\lambda)$ and

$-\lambda f_0 \in A(t_0, -\lambda)$. Let us fix $\varphi \in \text{supp}_\Delta(t_0, \lambda)$. Lemma 2.4 implies that $\varphi\Delta(-\lambda f_0) = -1$, and clearly $\varphi\Delta(\lambda f_0) = 1$. Thus

$$|\varphi\Delta(g) \pm 1| = |\varphi\Delta(g) \pm \varphi\Delta(\lambda f_0)| = |\varphi\Delta(g) - \varphi\Delta(\mp\lambda f_0)|$$
$$\leq \|\varphi\| \, \|\Delta(g) - \Delta(\mp\lambda f_0)\| = \|g \pm \lambda f_0\| = 1,$$

which assures that $\varphi\Delta(g) = 0$.

Since every function $f \in S(C(K))$ with $f(t_0) = 0$ can be approximated in norm by functions in $S(C(K))$ vanishing in an open neighbourhood of t_0, we deduce from the continuity of $\varphi\Delta$ and the property proved in the previous paragraph that $\varphi\Delta(f) = 0$, for every such f.

For the last statement, let us take $f \in S(C(K))$ with $|f(t_0)| < 1$, and $\varphi \in \text{supp}_\Delta(t_0, \lambda)$. Let us find $1 > \varepsilon > 0$ such that $|f(t_0)| < 1 - \varepsilon$. We consider the non-empty closed set $C_\varepsilon := \{t \in K : |f(t)| \geq 1 - \varepsilon\}$ and the open complement $\mathcal{O}_\varepsilon = K \backslash C_\varepsilon \ni t_0$. We can find, via Urysohn's lemma, a function $h \in S(C(K))$ with $0 \leq h \leq 1$, $h|_{C_\varepsilon} \equiv 1$, and $h(t_0) = 0$. It is easy to check that $fh \in S(C(K))$, $(fh)(t_0) = 0$, and $\|f - fh\| \leq 1 - \varepsilon < 1$.

Since $(fh)(t_0) = 0$, the first statement of this proposition proves that $\varphi\Delta(fh) = 0$, and thus

$$|\varphi\Delta(f)| = |\varphi\Delta(f) - \varphi\Delta(fh)| \leq \|\Delta(f) - \Delta(fh)\| = \|f - fh\| < 1 - \varepsilon < 1. \qquad \blacksquare$$

Next, we derive a first consequence of the previous proposition.

Corollary 2.7: *Suppose K is a compact Hausdorff space, X is a complex Banach space, and $\lambda \in \mathbb{T}$. Let $\Delta : S(C(K)) \to S(X)$ be a surjective isometry. If we take $b, c \in S(C(K))$ such that $\Delta(b) = \lambda\Delta(c)$, then $|b(t)| < 1$, for every $t \in K$ satisfying $|c(t)| < 1$.*

Proof: Let us take $t \in K$ satisfying $|c(t)| < 1$. By the final statement in Proposition 2.6 we have $|\varphi\Delta(c)| < 1$, for every $\mu \in \mathbb{T}$ and every $\varphi \in \text{supp}_\Delta(t, \mu)$. If $|b(t)| = 1$, we can find $\phi \in \text{supp}_\Delta(t, b(t))$ (see Lemma 2.1). Since $b \in A(t, b(t))$, we have $1 = \phi\Delta(b) = \phi(\lambda\Delta(c)) = \lambda\phi\Delta(c)$, and thus, $1 = |\lambda| \, |\phi\Delta(c)| < 1$, which leads to a contradiction. $\blacksquare$

3. Geometric properties for Stonean spaces

For a general compact Hausdorff space K, the C*-algebra $C(K)$ rarely contains an abundant collection of projections. For example, $C[0, 1]$ only contains trivial projections. If we assume that K is Stonean, then the characteristic function, χ_A, of every non-empty clopen set $A \subset K$ is a continuous function and a projection in $C(K)$, and thus $C(K)$ contains an abundant family of non-trivial projections. Throughout this section we shall work with continuous functions on a Stonean space.

Our first result is a reciprocal of Proposition 2.6 and will be repeatedly applied in our arguments.

Proposition 3.1: *Suppose K is a Stonean space and X is a complex Banach space. Let $\Delta : S(C(K)) \to S(X)$ be a surjective isometry. Let t_0 be an element in K. If b is an element in $S(C(K))$ satisfying $\varphi\Delta(b) = 0$, for every $\varphi \in \text{supp}_\Delta(t_0, \mu)$ and for every $\mu \in \mathbb{T}$, then $b(t_0) = 0$.*

Proof: Arguing by contradiction, we suppose that $b(t_0) \neq 0$. If $|b(t_0)| = 1$, we can pick $\varphi \in \text{supp}_\Delta(t_0, b(t_0))$ (compare Lemma 2.1). It is clear that $b \in A(t_0, b(t_0))$, and hence $\varphi\Delta(b) = 1$, which contradicts the hypothesis in the proposition.

We deal now with the case $0 < |b(t_0)| < 1$. Since K is Stonean, we can always find a clopen subset W satisfying

$$t_0 \in W \subseteq \left\{ s \in K : |b(s) - b(t_0)| < \frac{|b(t_0)|}{2} \right\}.$$

Let us observe that $0 < |b(t_0)|/2 < |b(s)|$, for every $s \in W$. Having in mind the last observation, we consider the function $c = b(1 - \chi_W) + b|b|^{-1}\chi_W \in C(K)$. Clearly $\|c\| \leq 1$ and $c(t_0) = b(t_0)/|b(t_0)| \in \mathbb{T}$, therefore $c \in S(C(K))$. It is not hard to check that

$$\begin{aligned} \|c - b\| &= \|(b|b|^{-1} - b)\chi_W\| \leq \sup_{s\in W} |b(s)(|b(s)|^{-1} - 1)| = \sup_{s\in W} |1 - |b(s)|| \\ &\leq 1 - \inf_{s\in W} |b(s)| \leq 1 - (|b(t_0)|/2). \end{aligned}$$

The element c lies in $A(t_0, b(t_0)/|b(t_0)|)$, and so we can conclude, by taking $\mu \in \mathbb{T}$, $\varphi \in \text{supp}_\Delta(t_0, \mu)$ and applying the hypothesis, that

$$1 = \varphi\Delta(c) - \varphi\Delta(b) \leq \|\Delta(c) - \Delta(b)\| = \|c - b\| \leq 1 - \frac{|b(t_0)|}{2},$$

leading to $|b(t_0)|/2 \leq 0$, which is impossible. ■

Our next results are devoted to determine the behaviour of a surjective isometry $\Delta : S(C(K)) \to S(X)$ on elements which are finite linear combinations of mutually orthogonal projections. We begin with a single characteristic function of a clopen set.

Proposition 3.2: *Suppose K is a Stonean space, A is a non-empty clopen subset of K, X is a complex Banach space, and $\lambda, \gamma \in \mathbb{T}$. Let $\Delta : S(C(K)) \to S(X)$ be a surjective isometry. If we take $b \in S(C(K))$ such that $\Delta(b) = \lambda\Delta(\gamma\chi_A)$, then $b = b\chi_A$ and $|b(t)| = 1$, for every $t \in A$.*

Proof: We shall first prove that $b = b\chi_A$. Let us fix $t_0 \in K\backslash A$. If we pick an arbitrary $\mu \in \mathbb{T}$ and $\varphi \in \text{supp}_\Delta(t_0, \mu)$, combining the hypothesis with Proposition 2.6 we get $\varphi\Delta(b) = \lambda\varphi\Delta(\gamma\chi_A) = 0$ which implies, via Proposition 3.1, that $b(t_0) = 0$. The arbitrariness of t_0 guarantees that $b = b\chi_A$.

Take now $t_0 \in A$. If $|b(t_0)| < 1$, the second statement in Proposition 2.6 assures that $|\varphi\Delta(b)| < 1$, for every $\varphi \in \text{supp}(t_0, \gamma)$. However, in this case, $1 > |\varphi\Delta(b)| = |\varphi(\lambda\Delta(\gamma\chi_A))| = |\lambda||\varphi\Delta(\gamma\chi_A)| = 1$, which leads to a contradiction. Therefore, $|b(t_0)| = 1$, for every $t_0 \in A$. ■

The next lemma is an elementary technical observation with a curious geometric interpretation.

Lemma 3.3: *Let δ be a real number with $0 < \delta < 2$. Then the set*

$$\{\zeta \in \mathbb{T} : |\zeta - 1|^2 \geq \delta^2,\ |\zeta + 1|^2 \geq 4 - \delta^2\}$$

coincides with $\{\lambda, \overline{\lambda}\}$ for a unique $\lambda \in \mathbb{T}$ with $|\zeta - 1|^2 = \delta^2$, and $|\zeta + 1|^2 = 4 - \delta^2$. Moreover, for each $\gamma \in \mathbb{T}$ we have

$$\{\zeta \in \mathbb{T} : |\zeta - \gamma|^2 \geq |\lambda - 1|^2,\ |\zeta + \gamma|^2 \geq |\lambda + 1|^2\} = \{\lambda\gamma, \overline{\lambda}\gamma\}.$$

Proof: Let us take $0 < \delta < 2$. It is standard to prove that the set $Z = \{\zeta \in \mathbb{T} : |\zeta - 1|^2 \geq \delta^2,\ |\zeta + 1|^2 \geq 4 - \delta^2\}$ is composed of just one complex number and its conjugate, both of them depending only on δ. Actually, if we solve the corresponding system of inequalities associated to the conditions required to be in Z, we find that the only two analytic solutions are $\lambda = \frac{1}{2}(2 - \delta^2 + i\delta\sqrt{4 - \delta^2})$ and $\overline{\lambda} = \frac{1}{2}(2 - \delta^2 - i\delta\sqrt{4 - \delta^2})$. It is worth to observe that Z is precisely the set of those elements in the complex unit sphere which are outside the open disc of centre $(1, 0)$ and with radius δ and outside the open disc of centre $(-1, 0)$ and radius $\sqrt{4 - \delta^2}$. Figure 1 illustrates this geometric interpretation.

According to the above observations, for each $\gamma \in \mathbb{T}$, the set $\{\zeta \in \mathbb{T} : |\zeta - \gamma|^2 \geq |\lambda - 1|^2,\ |\zeta + \gamma|^2 \geq |\lambda + 1|^2\}$ can be identified with an appropriate turn of Z. The parameter δ is exactly the distance from λ to 1 and $|\lambda + 1|^2 = 4 - \delta^2$. In this new setting, we work with the complex sphere and the circumferences centred at γ and $-\gamma$ with radii δ and $\sqrt{4 - \delta^2}$, respectively. Thus, the only two elements in this turned set are $\lambda\gamma$ and $\overline{\lambda}\gamma$. ■

We can now complete the information in Proposition 3.2. Henceforth, for each element a in a complex Banach algebra A, the symbol $\sigma(a)$ will stand for the spectrum of a.

Proposition 3.4: *Suppose K is a Stonean space, A is a non-empty clopen subset of K, λ, $\gamma \in \mathbb{T}$, and X is a complex Banach space. Let $\Delta : S(C(K)) \to S(X)$ be a surjective isometry. If we take $b \in S(C(K))$ such that $\Delta(b) = \lambda\Delta(\gamma\chi_A)$, then $b = b\chi_A$ and $\sigma(b) \subseteq \{\lambda\gamma, \overline{\lambda}\gamma, 0\}$. Consequently, there exist two disjoint clopen sets A_1 and A_2 (one of which could be empty) such that $A = A_1 \cup A_2$ and $b = \lambda\gamma\chi_{A_1} + \overline{\lambda}\gamma\chi_{A_2}$. Consequently, $\Delta(-\gamma\chi_A) = -\Delta(\gamma\chi_A)$.*

Proof: Proposition 3.2 implies that $b = b\chi_A$ and $|b(t)| = 1$, for every $t \in A$.

We assume first that $\lambda \neq \pm 1$ (i.e. $|\lambda - 1|, |\lambda + 1| \in (0, 2)$ and $|\lambda - 1|^2 + |\lambda + 1|^2 = 4$).

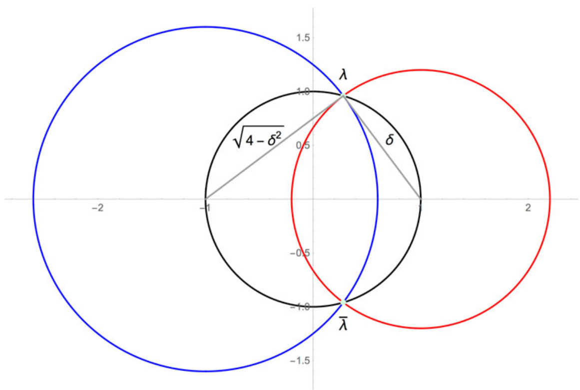

Figure 1. Particular case of Lemma 3.3 with $\delta = 1.2$.

We fix an arbitrary $t_0 \in A$. Let us observe the following property: for each $\varphi \in \text{supp}_\Delta(t_0, \gamma)$, we have $\varphi\Delta(b) = \varphi(\lambda\Delta(\gamma\chi_A)) = \lambda\varphi\Delta(\gamma\chi_A) = \lambda$. Therefore, by Lemma 2.4, for each $g \in A(t_0, \gamma)$ and each $k \in A(t_0, -\gamma)$, we have

$$|\lambda - 1| = |\varphi\Delta(b) - \varphi\Delta(g)| \leq \|\Delta(b) - \Delta(g)\| = \|b - g\|$$

and

$$|\lambda + 1| = |\varphi\Delta(b) - \varphi\Delta(k)| \leq \|\Delta(b) - \Delta(k)\| = \|b - k\|.$$

Since $A(t_0, \gamma) = -A(t_0, -\gamma)$, it follows that

$$|\lambda - 1| \leq \|b - g\| \quad \text{and} \quad |\lambda + 1| \leq \|b + g\| \quad \text{for all } g \in A(t_0, \gamma). \tag{2}$$

For each $0 < \varepsilon < 1$, let us find a clopen set W satisfying

$$t_0 \in W \subset \{s \in K : |b(s) - b(t_0)| < \varepsilon\}.$$

We consider the functions $g_\varepsilon^\pm = \pm b(1 - \chi_W) + \gamma\chi_W \in S(C(K))$, which clearly lie in $A(t_0, \gamma)$. By (2) we have

$$|\lambda - 1| \leq \|b - g_\varepsilon^+\| = \sup_{s\in W} |b(s) - \gamma| \leq |b(t_0) - \gamma| + \varepsilon$$

and

$$|\lambda + 1| \leq \|b + g_\varepsilon^-\| = \sup_{s\in W} |b(s) + \gamma| \leq |b(t_0) + \gamma| + \varepsilon,$$

which implies that $|b(t_0) \pm \gamma| \geq |\lambda \pm 1| - \varepsilon$. The arbitrariness of $0 < \varepsilon < 1$ gives $|b(t_0) \pm \gamma| \geq |\lambda \pm 1|$. Since $|b(t_0)| = 1$, we conclude that $b(t_0) \in \{\lambda\gamma, \overline{\lambda}\gamma\}$, for every $t_0 \in A$ (cf. Lemma 3.3). We have therefore shown that $\sigma(b) = b(K) \subseteq \{\lambda\gamma, \overline{\lambda}\gamma, 0\}$. The rest is clear.

We deal now with $\lambda = \pm 1$. The statement is clear for $\lambda = 1$ with $b = \gamma\chi_A$. Finally, let us assume that $\lambda = -1$. We fix an arbitrary $t_0 \in A$. By repeating the previous arguments, or by Lemma 2.4, we deduce that, for each $\varphi \in \text{supp}_\Delta(t_0, \gamma)$, we have $\varphi\Delta(b) = -1$, and thus

$$2 = |-1 - 1| = |\varphi\Delta(b) - \varphi\Delta(g)| \leq \|\Delta(b) - \Delta(g)\| = \|b - g\| \leq 2,$$

for every $g \in A(t_0, \gamma)$. As before, given $0 < \varepsilon < 1$, we consider a clopen set W such that $t_0 \in W \subset \{s \in K : |b(s) - b(t_0)| < \varepsilon\}$, and the function $g_\varepsilon^+ = b(1 - \chi_W) + \gamma\chi_W \in A(t_0, \gamma)$. Since

$$2 = \|b - g_\varepsilon^+\| = \sup_{s\in W} |b(s) - \gamma| \leq |b(t_0) - \gamma| + \varepsilon,$$

we deduce from the arbitrariness of $\varepsilon > 0$ that $2 \leq |b(t_0) - \gamma| \leq 2$, and thus $b(t_0) = -\gamma$. We have shown that $b(t_0) = -\gamma$ for every $t_0 \in A$. ■

We recall that a set $\{x_1, \ldots, x_k\}$ in a complex Banach space X is called *completely M-orthogonal* if

$$\left\| \sum_{j=1}^{k} \alpha_j x_j \right\| = \max\{\|\alpha_j x_j\| : 1 \leq j \leq k\},$$

for every $\alpha_1, \ldots, \alpha_k$ in $\mathbb{C}$. If $\{x_1, \ldots, x_k\} \subset S(X)$, then it is completely M-orthogonal if and only if the equality

$$\left\| \sum_{j=1}^{k} \lambda_j x_j \right\| = 1$$

holds for every $\lambda_1, \ldots, \lambda_k$ in $\mathbb{T}$ and $\lambda_{j_0} = 1$ for some $j_0 \in \{1, \ldots, k\}$ (see [[22, Lemma 3.4],[23, Lemma 2.3]]).

We can now complete the information given in Proposition 3.4.

Proposition 3.5: *Suppose K is a Stonean space. Let A and B be two non-empty disjoint clopen subsets of K. Let $\Delta : S(C(K)) \to S(X)$ be a surjective isometry, where X is a complex Banach space. Then the following statements hold*:

(a) *For every $\gamma, \mu \in \mathbb{T}$, the set $\{\Delta(\gamma \chi_A), \Delta(\mu \chi_B)\}$ is completely M-orthogonal.*
(b) $\Delta(\sigma_1 \gamma \chi_A + \sigma_2 \mu \chi_B) = \sigma_1 \Delta(\gamma \chi_A) + \sigma_2 \Delta(\mu \chi_B)$, *for every $\sigma_1, \sigma_2 \in \{\pm 1\}$ and every $\gamma, \mu \in \mathbb{T}$.*
(c) *For each $\lambda \in \mathbb{T}$, there exist two disjoint clopen sets A_1 and A_2 (one of which could be empty) such that $A = A_1 \cup A_2$,*

$$\lambda\Delta(\chi_{A_1}) + \lambda\Delta(\chi_{A_2}) = \Delta(\lambda\chi_{A_1}) + \Delta(\overline{\lambda}\chi_{A_2}) = \Delta(\lambda\chi_{A_1} + \overline{\lambda}\chi_{A_2}) = \lambda\Delta(\chi_A),$$
$$\Delta(\lambda\chi_{A_1}) = \lambda\Delta(\chi_{A_1}), \Delta(\lambda\chi_{A_2}) = \overline{\lambda}\Delta(\chi_{A_2}),$$
$$\overline{\lambda}\Delta(\chi_{A_1}) + \overline{\lambda}\Delta(\chi_{A_2}) = \Delta(\overline{\lambda}\chi_{A_1}) + \Delta(\lambda\chi_{A_2}) = \Delta(\overline{\lambda}\chi_{A_1} + \lambda\chi_{A_2}) = \overline{\lambda}\Delta(\chi_A),$$
$$\Delta(\overline{\lambda}\chi_{A_1})7 = \overline{\lambda}\Delta(\chi_{A_1}), \quad \text{and} \quad \Delta(\overline{\lambda}\chi_{A_2}) = \lambda\Delta(\chi_{A_2}).$$

Proof: (a) Let us take $\lambda, \mu, \gamma \in \mathbb{T}$. By Proposition 3.4 there exist two disjoint clopen sets A_1 and A_2 such that $A = A_1 \cup A_2$ and $\Delta(\lambda\gamma\chi_{A_1} + \overline{\lambda}\gamma\chi_{A_2}) = \lambda\Delta(\gamma\chi_A)$. Therefore, by the hypothesis, we have

$$\begin{aligned} \left\|\Delta(\mu\chi_B) \pm \lambda\Delta(\gamma\chi_A)\right\| &= \left\|\Delta(\mu\chi_B) - \Delta(\mp\lambda\gamma\chi_{A_1} \mp \overline{\lambda}\gamma\chi_{A_2})\right\| \\ &= \left\|\mu\chi_B \pm \lambda\gamma\chi_{A_1} \pm \overline{\lambda}\gamma\chi_{A_2}\right\| = 1, \end{aligned}$$

which proves the statement.

(b) Let us fix $\sigma_1, \sigma_2 \in \{\pm 1\}$. Since, by (a), $\{\Delta(\gamma\chi_A), \Delta(\mu\chi_B)\}$ is completely M-orthogonal, it follows that $\sigma_1\Delta(\gamma\chi_A) + \sigma_2\Delta(\mu\chi_B) \in S(X)$, and thus there exists $b \in S(C(K))$ satisfying $\Delta(b) = \sigma_1\Delta(\gamma\chi_A) + \sigma_2\Delta(\mu\chi_B)$. If we take $t_0 \in K\backslash(A \cup B)$, an arbitrary element α of $\mathbb{T}$ and $\varphi \in \text{supp}_\Delta(t_0, \alpha)$, then we have, via Proposition 2.6, that $\varphi\Delta(b) = \sigma_1\varphi\Delta(\gamma\chi_A) + \sigma_2\varphi\Delta(\mu\chi_B) = 0$ and Proposition 3.1 concludes that $b = b\chi_{A\cup B}$ because of the arbitrariness of t_0. By repeating the arguments in the proof of Proposition 3.2 we get $|b(t)| = 1$, for all $t \in A \cup B$.

Pick $t_0 \in A$ and $\varphi \in \text{supp}_\Delta(t_0, \sigma_1\gamma)$. By Proposition 2.6 we have $\varphi\Delta(b) = \varphi(\sigma_1\Delta(\gamma\chi_A) + \sigma_2\Delta(\mu\chi_B)) = 1$, and hence $\Delta(b) \in \varphi^{-1}(\{1\}) \cap \mathcal{B}_X = \Delta(A(t_0, \sigma_1\gamma))$ (cf. Lemma 2.1). This shows that $b(t_0) = \sigma_1\gamma$ for all $t_0 \in A$. Similarly, $b(t_0) = \sigma_2\mu$ for all $t_0 \in B$. We have therefore shown that $b = \sigma_1\gamma\chi_A + \sigma_2\mu\chi_B$ and

$$\Delta(\sigma_1\gamma\chi_A + \sigma_2\mu\chi_B) = \sigma_1\Delta(\gamma\chi_A) + \sigma_2\Delta(\mu\chi_B).$$

(c) We may assume that $\lambda \neq \pm 1$. Proposition 3.4 proves the existence of two disjoint clopen sets A_1 and A_2 such that $A = A_1 \cup A_2$ and

$$\Delta(\lambda\chi_{A_1}) + \Delta(\overline{\lambda}\chi_{A_2}) = \Delta(\lambda\chi_{A_1} + \overline{\lambda}\chi_{A_2}) = \lambda\Delta(\chi_A) = \lambda\Delta(\chi_{A_1}) + \lambda\Delta(\chi_{A_2}),$$

where in the first and last equalities we have applied (b). Therefore,

$$\Delta(\lambda\chi_{A_1}) - \lambda\Delta(\chi_{A_1}) = \lambda\Delta(\chi_{A_2}) - \Delta(\overline{\lambda}\chi_{A_2}).$$

We deduce from this identity and Proposition 2.6 that

$$\varphi(\Delta(\lambda\chi_{A_1}) - \lambda\Delta(\chi_{A_1})) = \varphi(\lambda\Delta(\chi_{A_2}) - \Delta(\overline{\lambda}\chi_{A_2})) = 0, \quad (3)$$

for every $t_0 \in A_1$, $\mu \in \mathbb{T}$ and $\varphi \in \text{supp}(t_0, \mu)$. However, a new application of Proposition 3.4 assures the existence of disjoint clopen sets A_{11} and A_{12} such that $A_1 = A_{11} \cup A_{12}$, and $\Delta(\lambda\chi_{A_{11}} + \overline{\lambda}\chi_{A_{12}}) = \lambda\Delta(\chi_{A_1})$, and by (b), we get

$$\Delta(\lambda\chi_{A_{11}}) + \Delta(\overline{\lambda}\chi_{A_{12}}) = \lambda\Delta(\chi_{A_{11}}) + \lambda\Delta(\chi_{A_{12}}).$$

If we can find $t_0 \in A_{12}$, then by (3), Proposition 2.6, and (b) we have $\varphi\Delta(\lambda\chi_{A_{12}}) = \varphi\Delta(\overline{\lambda}\chi_{A_{12}}) = 1$, for every $\varphi \in \text{supp}_\Delta(t_0, \overline{\lambda})$. Consequently,

$$2 = \varphi\Delta(\lambda\chi_{A_{12}}) + \varphi\Delta(g) \leq \|\Delta(\lambda\chi_{A_{12}}) + \Delta(g)\| \leq 2,$$

for all $g \in A(t_0, \overline{\lambda})$. Corollary 2.2 in [32] implies that $\|\lambda\chi_{A_{12}} + g\| = 2$, for all $g \in A(t_0, \overline{\lambda})$. In particular, for every clopen $W \subset A_{12}$ with $t_0 \in W$ (taking $g = \overline{\lambda}\chi_W$) we deduce the existence of $s_W \in W$ such that $|\lambda + \overline{\lambda}| = 2$, which is impossible. Therefore $A_{12} = \emptyset$, and thus $\Delta(\lambda\chi_{A_1}) = \lambda\Delta(\chi_{A_1})$.

Similar arguments lead to $\Delta(\lambda\chi_{A_2}) = \overline{\lambda}\Delta(\chi_{A_2})$.

We shall finally prove the last identities. By the above arguments there exist disjoint clopen sets A_3 and A_4 (one of which could be empty) such that $A = A_3 \cup A_4$,

$$\overline{\lambda}\Delta(\chi_{A_3}) + \overline{\lambda}\Delta(\chi_{A_4}) = \Delta(\overline{\lambda}\chi_{A_3}) + \Delta(\lambda\chi_{A_4}) = \Delta(\overline{\lambda}\chi_{A_3} + \lambda\chi_{A_4}) = \overline{\lambda}\Delta(\chi_A),$$

$\Delta(\overline{\lambda}\chi_{A_3}) = \overline{\lambda}\Delta(\chi_{A_3})$, and $\Delta(\overline{\lambda}\chi_{A_4}) = \lambda\Delta(\chi_{A_4})$. We shall finish by proving that $A_1 = A_3$ and $A_2 = A_4$. If there exists $t_0 \in A_1 \cap A_4$, we pick $\varphi \in \text{supp}_\Delta(t_0, \lambda)$ and, by Proposition 2.6, we compute

$$\varphi\Delta(\chi_A) = \varphi\left(\lambda\Delta(\overline{\lambda}\chi_{A_3} + \lambda\chi_{A_4})\right) = \varphi\left(\lambda\Delta(\overline{\lambda}\chi_{A_3}) + \lambda\Delta(\lambda\chi_{A_4})\right) = \lambda$$

and

$$\varphi\Delta(\chi_A) = \varphi\left(\overline{\lambda}\Delta(\lambda\chi_{A_1} + \overline{\lambda}\chi_{A_2})\right) = \varphi\left(\overline{\lambda}\Delta(\lambda\chi_{A_1}) + \overline{\lambda}\Delta(\overline{\lambda}\chi_{A_2})\right) = \overline{\lambda},$$

which is impossible because $\lambda \neq \pm 1$. This shows that $A_1 = A_3$ and $A_2 = A_4$. ∎

An appropriate generalization of [22, Proposition 3.3] is established next.

Proposition 3.6: *Suppose K is a Stonean space, A is a non-empty clopen subset of K, $\lambda \in \mathbb{T}\backslash\mathbb{R}$, and X is a complex Banach space. Let $\Delta : S(C(K)) \to S(X)$ be a surjective isometry. We additionally assume that $\Delta(\lambda\chi_A) = \lambda\Delta(\chi_A)$ (respectively, $\Delta(\lambda\chi_A) = \overline{\lambda}\Delta(\chi_A)$). Then $\Delta(\mu\chi_A) = \mu\Delta(\chi_A)$ (respectively, $\Delta(\mu\chi_A) = \overline{\mu}\Delta(\chi_A)$), for every $\mu \in \mathbb{T}$. Furthermore, if B is another non-empty clopen set in K contained in A, then $\Delta(\mu\chi_B) = \mu\Delta(\chi_B)$ (respectively, $\Delta(\mu\chi_B) = \overline{\mu}\Delta(\chi_B)$), for every $\mu \in \mathbb{T}$.*

Proof: We shall only prove the case in which $\Delta(\lambda\chi_A) = \lambda\Delta(\chi_A)$, the other statement is very similar. Let us take $\mu \in \mathbb{T}$. If $\mu = \pm 1$, then it is clear that the statement holds by Proposition 3.4. We can therefore assume that $\mu \in \mathbb{T}\backslash\mathbb{R}$. Proposition 3.5(*c*) proves the existence of two disjoint clopen sets A_1 and A_2 (one of which could be empty) such that $A = A_1 \cup A_2$,

$$\mu\Delta(\chi_{A_1}) + \mu\Delta(\chi_{A_2}) = \Delta(\mu\chi_{A_1}) + \Delta(\overline{\mu}\chi_{A_2}) = \Delta(\mu\chi_{A_1} + \overline{\mu}\chi_{A_2}) = \mu\Delta(\chi_A),$$

$$\Delta(\mu\chi_{A_1}) = \mu\Delta(\chi_{A_1}) \quad \text{and} \quad \Delta(\mu\chi_{A_2}) = \overline{\mu}\Delta(\chi_{A_2}).$$

We claim that $A_2 = \emptyset$. Otherwise, by Proposition 3.5 we have

$$\begin{aligned} |\lambda + \mu| &= \|\lambda\Delta(\chi_A) + \mu\Delta(\chi_A)\| = \|\Delta(\lambda\chi_{A_1}) + \Delta(\lambda\chi_{A_2}) + \Delta(\mu\chi_{A_1}) + \Delta(\overline{\mu}\chi_{A_2})\| \\ &= \max\{\|\Delta(\lambda\chi_{A_1}) + \Delta(\mu\chi_{A_1})\|, \|\Delta(\lambda\chi_{A_2}) + \Delta(\overline{\mu}\chi_{A_2})\|\} = (\text{by Proposition 3.4}) \\ &= \max\{\|\lambda\chi_{A_1} + \mu\chi_{A_1}\|, \|\lambda\chi_{A_2} + \overline{\mu}\chi_{A_2}\|\} = \max\{|\lambda + \mu|, |\lambda + \overline{\mu}|\}, \end{aligned}$$

and hence $|\lambda + \overline{\mu}| \leq |\lambda + \mu|$. By replacing μ with $-\mu$ in the above arguments we get $|\lambda - \overline{\mu}| \leq |\lambda - \mu|$. Combining the last two inequalities we have $\Re e(\lambda\overline{\mu}) = \Re e(\lambda\mu)$, or equivalently, $\lambda\overline{\mu} + \overline{\lambda}\mu = \lambda\mu + \overline{\lambda}\overline{\mu}$, which holds if and only if $\mu(\overline{\lambda} - \lambda) = \overline{\mu}(\overline{\lambda} - \lambda)$ and $\lambda(\overline{\mu} - \mu) = \overline{\lambda}(\overline{\mu} - \mu)$. The last equalities hold if and only if $\lambda, \mu \in \mathbb{R}$, which is impossible.

For the second statement, let us take a non-empty clopen set B in K contained in A. We assume $\Delta(\lambda\chi_A) = \lambda\Delta(\chi_A)$ (respectively, $\Delta(\lambda\chi_A) = \overline{\lambda}\Delta(\chi_A)$). The desired equality is clear if $\mu = \pm 1$, we thus assume that $\mu \in \mathbb{T}\backslash\mathbb{R}$. Proposition 3.5(*c*) guarantees the existence of two disjoint clopen sets B_1, B_2 in K such that $B = B_1 \cup B_2$ and $\mu\Delta(\chi_B) = \Delta(\mu\chi_{B_1}) + \Delta(\overline{\mu}\chi_{B_2})$. Observe that $A = B_1 \cup B_2 \cup (A\backslash B)$ and that $A\backslash B = A \cap (K\backslash B)$ is a clopen set in K. We therefore have

$$\mu\Delta(\chi_A) = \mu\Delta(\chi_{B_1}) + \mu\Delta(\chi_{B_2}) + \mu\Delta(\chi_{A\backslash B}) = \Delta(\mu\chi_{B_1}) + \Delta(\overline{\mu}\chi_{B_2}) + \mu\Delta(\chi_{A\backslash B})$$

and, by applying the first conclusion in this proposition and Proposition 3.5 we deduce that

$$\mu\Delta(\chi_A) = \Delta(\mu\chi_A) = \Delta(\mu\chi_{B_1}) + \Delta(\mu\chi_{B_2}) + \Delta(\mu\chi_{A\backslash B})$$

(respectively, $\mu\Delta(\chi_A) = \Delta(\overline{\mu}\chi_A) = \Delta(\overline{\mu}\chi_{B_1}) + \Delta(\overline{\mu}\chi_{B_2}) + \Delta(\overline{\mu}\chi_{A\backslash B})$). Then the identity

$$\Delta(\overline{\mu}\chi_{B_2}) + \mu\Delta(\chi_{A\backslash B}) = \Delta(\mu\chi_{B_2}) + \Delta(\mu\chi_{A\backslash B})$$

(respectively,

$$\Delta(\mu\chi_{B_1}) + \mu\Delta(\chi_{A\backslash B}) = \Delta(\overline{\mu}\chi_{B_1}) + \Delta(\overline{\mu}\chi_{A\backslash B}))$$

holds. Therefore, $\varphi(\Delta(\overline{\mu}\chi_{B_2}) - \Delta(\mu\chi_{B_2})) = \varphi(\Delta(\mu\chi_{A\backslash B}) - \mu\Delta(\chi_{A\backslash B})) = 0$, for every $t_0 \in B_2$, $\gamma \in \mathbb{T}$ and $\varphi \in \text{supp}_\Delta(t_0, \gamma)$. If we can find $t_0 \in B_2$, then for $\gamma = \mu$ we have

$\varphi(\Delta(\overline{\mu}\chi_{B_2})) = \varphi(\Delta(\mu\chi_{B_2})) = 1$, and hence $\Delta(\overline{\mu}\chi_{B_2}) \in \varphi^{-1}(\{1\}) \cap \mathcal{B}_X = \Delta(A(t_0, \mu))$ (cf. Lemma 2.1). Thus $\overline{\mu}\chi_{B_2} \in A(t_0, \mu)$, which is impossible. We have shown that $B_2 = \emptyset$, and hence $B = B_1$ and $\Delta(\mu\chi_B) = \mu\Delta(\chi_B)$.

In the case $\Delta(\lambda\chi_A) = \overline{\lambda}\Delta(\chi_A)$, similar arguments prove that $\Delta(\mu\chi_B) = \overline{\mu}\Delta(\chi_B)$. ■

A first corollary of the above proposition plays a fundamental role in our argument.

Corollary 3.7: *Suppose K is a Stonean space and X is a complex Banach space. Let* $\Delta : S(C(K)) \to S(X)$ *be a surjective isometry. Then there exists a clopen subset* $K_1 \subseteq K$ *such that* $\Delta(\lambda\chi_{K_1}) = \lambda\Delta(\chi_{K_1})$ *and* $\Delta(\lambda\chi_{K\setminus K_1}) = \overline{\lambda}\Delta(\chi_{K\setminus K_1})$, *for every* $\lambda \in \mathbb{T}$. *Consequently, if* B_1 *is a clopen subset of K contained in* K_1 *and* B_2 *is a clopen subset of K contained in* $K_2 = K\setminus K_1$, *then* $\Delta(\mu\chi_{B_1}) = \mu\Delta(\chi_{B_1})$ *and* $\Delta(\mu\chi_{B_2}) = \overline{\mu}\Delta(\chi_{B_2})$, *for every* $\mu \in \mathbb{T}$.

Proof: The proof follows straightforwardly from Propositions 3.5 and 3.6. ■

From now on, given a surjective isometry $\Delta : S(C(K)) \to S(X)$ where K is a Stonean space and X is a complex Banach space, the symbols K_1 and K_2 will denote the clopen subsets given by Corollary 3.7. Under these hypotheses we define a new product $\odot : \mathbb{C} \times C(K) \to C(K)$ given by

$$(\alpha \odot a)(t) := \alpha\, a(t) \text{ if } t \in K_1 \quad \text{and} \quad (\alpha \odot a)(t) := \overline{\alpha}\, a(t) \text{ otherwise.} \tag{4}$$

We observe that $\alpha \odot a = \alpha\, a$ whenever $\alpha \in \mathbb{R}$.

Our next results are devoted to determine the behaviour of a surjective isometry $\Delta : S(C(K)) \to S(X)$ on algebraic elements.

Proposition 3.8: *Suppose K is a Stonean space,* $\gamma_1, \ldots, \gamma_n \in \mathbb{T}$, $B_1, \ldots, B_n$ *are non-empty disjoint clopen subsets of K, and X is a complex Banach space. Let* $\Delta : S(C(K)) \to S(X)$ *be a surjective isometry and let* $v = \sum_{k=1}^{m} \lambda_k\chi_{A_k}$ *be an algebraic partial isometry in* $C(K)$, *where* $\lambda_1, \ldots, \lambda_m \in \mathbb{T}$, $A_1, \ldots, A_m$ *are non-empty disjoint clopen sets in K such that* $A_k \cap B_j = \emptyset$, *for every* $k \in \{1, \ldots, m\}$ *and every* $j \in \{1, \ldots, n\}$. *Then the set* $\{\Delta(v), \Delta(\gamma_1\chi_{B_1}), \ldots, \Delta(\gamma_n\chi_{B_n})\}$ *is completely M-orthogonal, and the equality*

$$\Delta(v) + \sum_{j=1}^{n} \Delta(\gamma_j\chi_{B_j}) = \Delta\left(v + \sum_{j=1}^{n} \gamma_j\chi_{B_j}\right)$$

holds.

Proof: We shall prove the statement arguing by induction on n. In the case $n = 1$, let us take $\mu_1 \in \mathbb{T}$. Since B_1 is a non-empty clopen set, by Proposition 3.4 there exist two disjoint clopen sets B_{11} and B_{12} such that $B_1 = B_{11} \cup B_{12}$ and $\mu_1\Delta(\gamma_1\chi_{B_1}) = \Delta(\mu_1\gamma_1\chi_{B_{11}} + \overline{\mu_1}\gamma_1\chi_{B_{12}})$. Since χ_{B_1} is orthogonal to v, it follows from Proposition 3.5 and the hypotheses that

$$\begin{aligned}\|\Delta(v)+\mu_1\Delta(\gamma_1\chi_{B_1})\| &= \|\Delta(v)+\Delta(\mu_1\gamma_1\chi_{B_{11}}+\overline{\mu_1}\gamma_1\chi_{B_{12}})\|\\ &= \|\Delta(v)-\Delta(-\mu_1\gamma_1\chi_{B_{11}}-\overline{\mu_1}\gamma_1\chi_{B_{12}})\|\\ &= \|v+\mu_1\gamma_1\chi_{B_{11}}+\overline{\mu_1}\gamma_1\chi_{B_{12}}\|=1.\end{aligned}$$

This proves that the set $\{\Delta(v),\Delta(\gamma_1\chi_{B_1})\}$ is completely M-orthogonal, and consequently $\Delta(v)+\Delta(\gamma_1\chi_{B_1})\in S(X)$. Then there exists $b\in S(C(K))$ satisfying $\Delta(b)=\Delta(v)+\Delta(\gamma_1\chi_{B_1})$.

We shall next show that $b=b\chi_{A_1\cup\cdots\cup A_m\cup B_1}$. To this end, take an arbitrary $t_0\in K\backslash(A_1\cup\cdots\cup A_m\cup B_1)$, $\alpha\in\mathbb{T}$ and $\varphi\in \operatorname{supp}_\Delta(t_0,\alpha)$. By Proposition 2.6 we have $\varphi\Delta(b)=\varphi\Delta(v)+\varphi\Delta(\gamma_1\chi_{B_1})=0$. Proposition 3.1 gives $b=b\chi_{A_1\cup\cdots\cup A_m\cup B_1}$.

Now, let us pick $t_0\in A_{k_0}$ for some $k_0\in\{1,\dots,m\}$, and $\varphi\in\operatorname{supp}_\Delta(t_0,\lambda_{k_0})$. Proposition 2.6 implies that $\varphi\Delta(b)=\varphi(\Delta(v)+\Delta(\gamma_1\chi_{B_1}))=1$, and hence $\Delta(b)\in\varphi^{-1}(\{1\})\cap\mathcal{B}_X=\Delta(A(t_0,\lambda_{k_0}))$ (cf. Lemma 2.1). Thus $b\in A(t_0,\lambda_{k_0})$, and it follows that $b(t_0)=\lambda_{k_0}$, for every $t_0\in A_{k_0}$. We conclude from the arbitrariness of k_0 that $b=\lambda_1\chi_{A_1}+\cdots+\lambda_m\chi_{A_m}+\gamma_1\chi_{B_1}=v+\gamma_1\chi_{B_1}$, which concludes the proof of the case $n=1$ in our induction argument.

Suppose by the induction hypothesis that the statement is true for any $1\le k\le n$. By the induction hypothesis for $k=1$ and $k=n$ with the algebraic partial isometry $w=v+\gamma_1\chi_{B_1}$, we get

$$\Delta(v)+\sum_{j=1}^{n+1}\Delta(\gamma_j\chi_{B_j})=\Delta(v)+\Delta(\gamma_1\chi_{B_1})+\sum_{j=2}^{n+1}\Delta(\gamma_j\chi_{B_j})$$

$$=\Delta(v+\gamma_1\chi_{B_1})+\sum_{j=2}^{n+1}\Delta(\gamma_j\chi_{B_j})=\Delta\left(w+\sum_{j=2}^{n+1}\gamma_j\chi_{B_j}\right)=\Delta\left(v+\sum_{j=1}^{n+1}\gamma_j\chi_{B_j}\right). \quad (5)$$

Let us take $\mu_1,\dots,\mu_{n+1}\in\mathbb{T}$. For each $j\in\{1,\dots,n+1\}$, Proposition 3.4 assures the existence of two disjoint clopen sets B_{j_1} and B_{j_2} such that $B_j=B_{j_1}\cup B_{j_2}$ and $\mu_j\Delta(\gamma_j\chi_{B_j})=\Delta(\mu_j\gamma_j\chi_{B_{j_1}}+\overline{\mu_j}\gamma_j\chi_{B_{j_2}})$. Therefore, we can conclude by the identity proved in (??), applied twice to v and $\{\mu_j\gamma_j\chi_{B_{j_1}}\}_j$ and to $w=v+\sum_{j=1}^{n+1}\mu_j\gamma_j\chi_{B_{j_1}}$ and $\{\overline{\mu_j}\gamma_j\chi_{B_{j_2}}\}_j$, and Proposition 3.5 that

$$\left\|\Delta(v)+\sum_{j=1}^{n+1}\mu_j\Delta(\gamma_j\chi_{B_j})\right\|=\left\|\Delta(v)+\sum_{j=1}^{n+1}\Delta\left(\mu_j\gamma_j\chi_{B_{j_1}}+\overline{\mu_j}\gamma_j\chi_{B_{j_2}}\right)\right\|$$

$$=\left\|\Delta(v)+\sum_{j=1}^{n+1}\Delta(\mu_j\gamma_j\chi_{B_{j_1}})+\sum_{j=1}^{n+1}\Delta(\overline{\mu_j}\gamma_j\chi_{B_{j_2}})\right\|$$

$$= \left\| \Delta\left(v + \sum_{j=1}^{n+1} \mu_j \gamma_j \chi_{B_{j_1}}\right) + \sum_{j=1}^{n+1} \Delta(\overline{\mu_j} \gamma_j \chi_{B_{j_2}}) \right\|$$

$$= \left\| \Delta\left(v + \sum_{j=1}^{n+1} \mu_j \gamma_j \chi_{B_{j_1}} + \sum_{j=1}^{n+1} \overline{\mu_j} \gamma_j \chi_{B_{j_2}}\right) \right\| = 1,$$

which finishes the induction argument and the proof. ■

Our next result is the technical core of the paper. In the statement we keep the notation given by Corollary 3.7 and (4).

Proposition 3.9: *Suppose K is a Stonean space, $\gamma_1, \ldots, \gamma_n \in \mathbb{T}$, $B_1, \ldots, B_n$ are non-empty disjoint clopen subsets of K such that $B_1, \ldots, B_{j_0}$ are contained in K_1 and $B_{j_0+1}, \ldots, B_n$ are contained in $K \backslash K_1$ with $j_0 \in \{0, 1, \ldots, n+1\}$. If $j_0 = 0$ (respectively, $j_0 = n+1$) we assume that $B_j \subseteq K \backslash K_1$ (respectively, $B_j \subseteq K_1$) for all $j = 1, \ldots, n$. Suppose X is a complex Banach space. Let $\Delta : S(C(K)) \to S(X)$ be a surjective isometry and let $v = \sum_{k=1}^m \lambda_k \chi_{A_k}$ be an algebraic partial isometry in $C(K)$, where $\lambda_1, \ldots, \lambda_m \in \mathbb{T}$, $A_1, \ldots, A_m$ are non-empty disjoint clopen sets in K such that $A_k \cap B_j = \emptyset$, for every $k \in \{1, \ldots, m\}$ and every $j \in \{1, \ldots, n\}$. Then, given $\alpha_1, \ldots, \alpha_n \in \mathbb{C} \backslash \{0\}$ with $\max\{|\alpha_j| : j \in \{1, \ldots, n\}\} < 1$, we have*

$$\Delta(v) + \sum_{j=1}^{n} \alpha_j \Delta(\gamma_j \chi_{B_j}) = \Delta\left(v + \sum_{j=1}^{j_0} \alpha_j \gamma_j \chi_{B_j} + \sum_{j=j_0+1}^{n} \overline{\alpha_j} \gamma_j \chi_{B_j}\right)$$

$$= \Delta\left(v + \sum_{j=1}^{n} \alpha_j \odot (\gamma_j \chi_{B_j})\right).$$

Proof: Since the set $\{\Delta(v), \Delta(\gamma_1 \chi_{B_1}), \ldots, \Delta(\gamma_n \chi_{B_n})\}$ is completely M-orthogonal (cf. Proposition 3.8), we can deduce that $\Delta(v) + \sum_{j=1}^n \alpha_j \Delta(\gamma_j \chi_{B_j}) \in S(X)$. Thus there exists $y \in S(C(K))$ such that $\Delta(y) = \Delta(v) + \sum_{j=1}^n \alpha_j \Delta(\gamma_j \chi_{B_j})$.

Let us fix $t_0 \in K \backslash (\bigcup_{k,j} A_k \cup B_j)$, an arbitrary element μ of $\mathbb{T}$, and $\varphi \in \operatorname{supp}_\Delta(t_0, \mu)$. Proposition 2.6 implies that $\varphi\Delta(y) = \varphi\Delta(v) + \sum_{j=1}^n \alpha_j \varphi\Delta(\gamma_j \chi_{B_j}) = 0$. The arbitrariness of μ allows us to apply Proposition 3.1 to deduce that $y(t_0) = 0$, which gives $y = y\chi_{(\cup_{k,j} A_k \cup B_j)}$ thanks to the arbitrariness of t_0.

Take now $t_0 \in A_{k_0}$ and $\varphi \in \operatorname{supp}_\Delta(t_0, \lambda_{k_0})$ for some $k_0 \in \{1, \ldots, m\}$. A new application of Proposition 2.6 implies that $\varphi\Delta(y) = \varphi\Delta(v) + \sum_{j=1}^n \alpha_j \varphi\Delta(\gamma_j \chi_{B_j}) = 1$, and hence $\Delta(y) \in \varphi^{-1}(\{1\}) \cap B_X = \Delta(A(t_0, \lambda_{k_0}))$, which assures that $y(t_0) = \lambda_{k_0}$, for every $t_0 \in A_{k_0}$ and for every $k_0 \in \{1, \ldots, m\}$. Therefore,

$$y = v + y(1 - \chi_{\cup_k A_k}) = v + \sum_{j=1}^{n} y\chi_{B_j}.$$

We shall prove the desired identity by induction on n. If $n = 1$, it follows from the above that there exists $y \in S(C(K))$ such that $\Delta(y) = \Delta(v) + \alpha_1 \Delta(\gamma_1 \chi_{B_1})$ and

$y = v + y\chi_{B_1}$, with $|\alpha_1| < 1$. We shall prove that $y\chi_{B_1} = \alpha_1 \odot (\gamma_1\chi_{B_1})$. The completely M-orthogonality of $\{\Delta(v), \Delta(\gamma_1\chi_{B_1})\}$ guarantees the existence of $z \in S(C(K))$ such that $\Delta(z) = \Delta(v) + (\alpha_1/|\alpha_1|)\Delta(\gamma_1\chi_{B_1})$ and since $B_1 \subseteq K_1$ or $B_1 \subseteq K_2$, the identity $z = v + (\alpha_1/|\alpha_1|) \odot (\gamma_1\chi_{B_1})$ holds by Proposition 3.8, Corollary 3.7 and (4). We also know that

$$1 - |\alpha_1| = \left|\frac{\alpha_1}{|\alpha_1|} - \alpha_1\right| = \|\Delta(y) - \Delta(z)\| = \|y - z\| = \left\|y\chi_{B_1} - \frac{\alpha_1}{|\alpha_1|} \odot (\gamma_1\chi_{B_1})\right\|$$

and

$$\begin{aligned} 1 < 1 + |\alpha_1| &= \left|\frac{\alpha_1}{|\alpha_1|} + \alpha_1\right| = \left\|\Delta(y) + \frac{\alpha_1}{|\alpha_1|}\Delta(\gamma_1\chi_{B_1})\right\| = \left\|y + \frac{\alpha_1}{|\alpha_1|} \odot (\gamma_1\chi_{B_1})\right\| \\ &= \left\|y\chi_{B_1} + \frac{\alpha_1}{|\alpha_1|} \odot (\gamma_1\chi_{B_1})\right\| \vee \|v\| = \left\|y\chi_{B_1} + \frac{\alpha_1}{|\alpha_1|} \odot (\gamma_1\chi_{B_1})\right\|. \end{aligned} \tag{6}$$

It follows from the previous two identities that

$$\left|y(t)\chi_{B_1}(t) - \frac{\alpha_1\gamma_1}{|\alpha_1\gamma_1|}\chi_{B_1}(t)\right| \leq 1 - |\alpha_1\gamma_1|$$

and

$$\left|y(t)\chi_{B_1}(t) + \frac{\alpha_1\gamma_1}{|\alpha_1\gamma_1|}\chi_{B_1}(t)\right| \leq 1 + |\alpha_1\gamma_1|,$$

for every element t in K_1, and

$$\left|y(t)\chi_{B_1}(t) - \frac{\overline{\alpha_1}\gamma_1}{|\overline{\alpha_1}\gamma_1|}\chi_{B_1}(t)\right| \leq 1 - |\overline{\alpha_1}\gamma_1|$$

and

$$\left|y(t)\chi_{B_1}(t) + \frac{\overline{\alpha_1}\gamma_1}{|\overline{\alpha_1}\gamma_1|}\chi_{B_1}(t)\right| \leq 1 + |\overline{\alpha_1}\gamma_1|,$$

for every element t in K_2. When particularized to an element $t \in B_1 \subseteq K_1$ and $t \in B_1 \subseteq K\backslash K_1$ the previous inequalities result in

$$\left|y(t) - \frac{\alpha_1\gamma_1}{|\alpha_1\gamma_1|}\right| \leq 1 - |\alpha_1\gamma_1| \quad \text{and} \quad \left|y(t) + \frac{\alpha_1\gamma_1}{|\alpha_1\gamma_1|}\right| \leq 1 + |\alpha_1\gamma_1|$$

and

$$\left|y(t) - \frac{\overline{\alpha_1}\gamma_1}{|\overline{\alpha_1}\gamma_1|}\right| \leq 1 - |\overline{\alpha_1}\gamma_1| \quad \text{and} \quad \left|y(t) + \frac{\overline{\alpha_1}\gamma_1}{|\overline{\alpha_1}\gamma_1|}\right| \leq 1 + |\overline{\alpha_1}\gamma_1|,$$

respectively, which gives $y(t) = \alpha_1\gamma_1$ and $y(t) = \overline{\alpha_1}\gamma_1$, respectively. Therefore $y\chi_{B_1} = \alpha_1 \odot (\gamma_1\chi_{B_1})$, which concludes the induction argument in the case $n = 1$.

Suppose now, by the induction hypothesis, that given $\alpha_1, \ldots, \alpha_n \in \mathbb{C}\backslash\{0\}$ with $\max\{|\alpha_j| : j \in \{1, \ldots, n\}\} < 1$, we have

$$\Delta(v) + \sum_{j=1}^{n} \alpha_j \Delta(\gamma_j \chi_{B_j}) = \Delta\left(v + \sum_{j=1}^{j_0} \alpha_j \gamma_j \chi_{B_j} + \sum_{j=j_0+1}^{n} \overline{\alpha_j} \gamma_j \chi_{B_j}\right)$$
$$= \Delta\left(v + \sum_{j=1}^{n} \alpha_j \odot (\gamma_j \chi_{B_j})\right),$$

whenever v is an algebraic partial isometry and $B_1, \ldots, B_n$ are as in the statement of the proposition.

Let $v = \sum_{k=1}^{m} \lambda_k \chi_{A_k}$ and $B_1, \ldots, B_{n+1}$ be as in the statement of the proposition. By the arguments exhibited at the beginning of the proof, we may assume the existence of $y \in S(C(K))$ such that $\Delta(y) = \Delta(v) + \sum_{j=1}^{n+1} \alpha_j \Delta(\gamma_j \chi_{B_j})$ and $y = v + \sum_{j=1}^{n+1} y\chi_{B_j}$. To prove that $y = v + \sum_{j=1}^{n+1} \alpha_j \odot (\gamma_j \chi_{B_j})$, it will suffice to show that $y\chi_{B_j} = \alpha_j \odot (\gamma_j \chi_{B_j})$ for every $j = 1, \ldots, n+1$.

Let us fix $j_1 \in \{1, \ldots, n+1\}$. Proposition 3.8 assures the existence of $z \in S(C(K))$ such that

$$\Delta(z) = \Delta(v) + \frac{\alpha_{j_1}}{|\alpha_{j_1}|} \Delta(\gamma_{j_1} \chi_{B_{j_1}}) + \sum_{j \neq j_1} \alpha_j \Delta(\gamma_j \chi_{B_j}).$$

Thus, by induction hypothesis, Corollary 3.7 and Proposition 3.8, we conclude that $z = v + (\alpha_{j_1}/|\alpha_{j_1}|) \odot (\gamma_{j_1} \chi_{B_{j_1}}) + \sum_{j \neq j_1} \alpha_j \odot (\gamma_j \chi_{B_j})$. Applying this identity we get

$$1 - |\alpha_{j_1}| = \left|\alpha_{j_1} - \frac{\alpha_{j_1}}{|\alpha_{j_1}|}\right| = \|\Delta(y) - \Delta(z)\| = \|y - z\|$$
$$= \left\|y\chi_{B_{j_1}} - \frac{\alpha_{j_1}}{|\alpha_{j_1}|} \odot (\gamma_{j_1} \chi_{B_{j_1}})\right\| \vee \max\left\{\left\|y\chi_{B_j} - \alpha_j \odot (\gamma_j \chi_{B_j})\right\| : j \neq j_1\right\}. \quad (7)$$

Consequently

$$\left\|y\chi_{B_{j_1}} - \frac{\alpha_{j_1}}{|\alpha_{j_1}|} \odot (\gamma_{j_1} \chi_{B_{j_1}})\right\| \leq 1 - |\alpha_{j_1}|. \quad (8)$$

Arguing as in (6) we also get

$$1 + |\alpha_{j_1}| = \left|\frac{\alpha_{j_1}}{|\alpha_{j_1}|} + \alpha_{j_1}\right| = \left\|\Delta(y) + \frac{\alpha_{j_1}}{|\alpha_{j_1}|} \Delta(\gamma_{j_1} \chi_{B_{j_1}})\right\| = \left\|y + \frac{\alpha_{j_1}}{|\alpha_{j_1}|} \odot (\gamma_{j_1} \chi_{B_{j_1}})\right\|$$
$$= \left\|y\chi_{B_{j_1}} + \frac{\alpha_{j_1}}{|\alpha_{j_1}|} \odot (\gamma_{j_1} \chi_{B_{j_1}})\right\| \vee \|v\| \vee \max\{\|y\chi_{B_j}\| : j \neq j_1\}$$
$$= \left\|y\chi_{B_{j_1}} + \frac{\alpha_{j_1}}{|\alpha_{j_1}|} \odot (\gamma_{j_1} \chi_{B_{j_1}})\right\|. \quad (9)$$

Evaluating at an element $t_0 \in B_{j_1}$ we deduce from (8) and (9) that

$$\left|y(t_0) - \frac{\alpha_{j_1}\gamma_{j_1}}{|\alpha_{j_1}\gamma_{j_1}|}\right| \leq 1 - |\alpha_{j_1}\gamma_{j_1}| \quad \text{and} \quad \left|y(t_0) + \frac{\alpha_{j_1}\gamma_{j_1}}{|\alpha_{j_1}\gamma_{j_1}|}\right| \leq 1 + |\alpha_{j_1}\gamma_{j_1}|,$$

if $t_0 \in K_1$, and

$$\left| y(t_0) - \frac{\overline{\alpha_{j_1}}\gamma_{j_1}}{|\overline{\alpha_{j_1}}\gamma_{j_1}|} \right| \leq 1 - |\overline{\alpha_{j_1}}\gamma_{j_1}| \quad \text{and} \quad \left| y(t_0) + \frac{\overline{\alpha_{j_1}}\gamma_{j_1}}{|\overline{\alpha_{j_1}}\gamma_{j_1}|} \right| \leq 1 + |\overline{\alpha_{j_1}}\gamma_{j_1}|,$$

if $t_0 \in K_2$, inequalities which give $y(t_0) = \alpha_{j_1}\gamma_{j_1}$ if $t_0 \in K_1$ and $y(t_0) = \overline{\alpha_{j_1}}\gamma_{j_1}$ if $t_0 \in K_2$, respectively. We have shown that $y\chi_{B_j} = \alpha_j \odot (\gamma_j \chi_{B_j})$, which finishes the proof. ■

The next corollary is a straightforward consequence of the previous proposition.

Corollary 3.10: *Let $\Delta : S(C(K)) \to S(X)$ be a surjective isometry, where K is a Stonean space and X is a complex Banach space. Let $v_1, \ldots, v_n$ be mutually orthogonal algebraic partial isometries in $C(K)$. Then, given $\alpha_1, \ldots, \alpha_n \in \mathbb{C}\backslash\{0\}$ with $\max\{|\alpha_j| : j \in \{1, \ldots, n\}\} = 1$, we have*

$$\sum_{j=1}^{n} \alpha_j \Delta(v_j) = \Delta\left(\sum_{j=1}^{n} \alpha_j \odot v_j\right).$$

□

Proposition 3.9 and its revision in Corollary 3.10 are the tools we need to get a first approach to our main result. In this first approach we follow the ideas in the proof of [23, Theorem 1.1] or in the line followed in [5].

Theorem 3.11: *Let $\Delta : S(C(K)) \to S(X)$ be a surjective isometry, where K is a Stonean space and X is a complex Banach space. Then there exist two disjoint clopen subsets K_1 and K_2 of K such that $K = K_1 \cup K_2$ satisfying that if K_1 (respectively, K_2) is non-empty, then there exist a closed subspace X_1 (respectively, X_2) of X and a complex linear (respectively, conjugate linear) surjective isometry $T_1 : C(K_1) \to X_1$ (respectively, $T_2 : C(K_2) \to X_2$) such that $X = X_1 \oplus^\infty X_2$, and $\Delta(a) = T_1(\pi_1(a)) + T_2(\pi_2(a))$, for every $a \in S(C(K))$, where π_j is the natural projection of $C(K)$ onto $C(K_j)$ given by $\pi_j(a) = a|_{K_j}$. In particular, Δ admits an extension to a surjective real linear isometry from $C(K)$ onto X.*

Proof: Let K_1 and K_2 be the clopen subsets given by Corollary 3.7. We can assume that $K_j \neq \emptyset$, for every $j = 1, 2$. Otherwise, the arguments are even easier. Clearly, $C(K) = C(K_1) \oplus^\infty C(K_2)$.

We consider the homogeneous extensions $T_j : C(K_j) \to X$, defined by $T_j(0) = 0$ and $T_j(a) = \|a\|\Delta(1/\|a\| a)$ for all $a \in C(K_j)\backslash\{0\}$.

Let us fix two algebraic elements in $C(K_1)$ (respectively, $C(K_2)$) of the form

$$\hat{a} = \sum_{k=1}^{n} \alpha_k \odot v_k \quad \text{and} \quad \hat{b} = \sum_{k=1}^{n} \beta_k \odot v_k,$$

where $v_1, \ldots, v_n$ are mutually orthogonal non-zero algebraic partial isometries in K_1 (respectively, K_2), $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n \in \mathbb{C}\backslash\{0\}$ with $\max\{|\alpha_k| : k \in \{1, \ldots, n\}\} = \|\hat{a}\|$, and $\max\{|\beta_k| : k \in \{1, \ldots, n\}\} = \|\hat{b}\|$.

If $\hat{a}+\hat{b}=0$, with $\hat{a}\neq 0$, then Corollary 3.10 assures that

$$T_j(\hat{a}) = \|\hat{a}\|\Delta\left(\frac{\hat{a}}{\|\hat{a}\|}\right) = \|\hat{a}\|\left(-\Delta\left(-\frac{\hat{a}}{\|\hat{a}\|}\right)\right) = -\|\hat{b}\|\Delta\left(\frac{\hat{b}}{\|\hat{b}\|}\right) = -T_j(\hat{b}),$$

and hence $T_j(\hat{a}+\hat{b}) = 0 = T_j(\hat{a}) + T_j(\hat{b})$, for every $j=1,2$.

If $\hat{a}+\hat{b}\neq 0$, a new application of Corollary 3.10 implies that

$$T_j(\hat{a}) = \|\hat{a}\|\Delta\left(\frac{1}{\|\hat{a}\|}\hat{a}\right) = \|\hat{a}\|\Delta\left(\sum_{k=1}^{n}\frac{\alpha_k}{\|\hat{a}\|}\odot v_k\right) = \|\hat{a}\|\left(\sum_{k=1}^{n}\frac{\alpha_k}{\|\hat{a}\|}\Delta(v_k)\right),$$

$$T_k(\hat{b}) = \|\hat{b}\|\Delta\left(\frac{1}{\|\hat{b}\|}\hat{b}\right) = \|\hat{b}\|\Delta\left(\sum_{k=1}^{n}\frac{\beta_k}{\|\hat{b}\|}\odot v_k\right) = \|\hat{b}\|\left(\sum_{k=1}^{n}\frac{\beta_k}{\|\hat{b}\|}\Delta(v_k)\right),$$

$$T_j(\hat{a}+\hat{b}) = \|\hat{a}+\hat{b}\|\Delta\left(\frac{1}{\|\hat{a}+\hat{b}\|}(\hat{a}+\hat{b})\right) = \|\hat{a}+\hat{b}\|\Delta\left(\sum_{k=1}^{n}\frac{\alpha_k+\beta_k}{\|\hat{a}+\hat{b}\|}\odot v_k\right)$$
$$= \sum_{k=1}^{n}(\alpha_k+\beta_k)\Delta(v_k).$$

Therefore, $T_j(\hat{a}+\hat{b}) = T_j(\hat{a}) + T_j(\hat{b})$, for every $j=1,2$.

It is known that T_j is a Lipschitz mapping for every $j=1,2$ (compare for example, the final part in the proof of [23, Theorem 1.1]).

Now we observe that for every $a,b\in C(K_j)$ and $\varepsilon>0$ we can find a set $\{v_1,\ldots,v_n\}$ of mutually orthogonal non-zero algebraic partial isometries in $C(K_j)$ and $\alpha_1,\beta_1,\ldots,\alpha_n,\beta_n\in\mathbb{C}\backslash\{0\}$ such that $\|a-\hat{a}\|<\varepsilon$ and $\|b-\hat{b}\|<\varepsilon$, where $\hat{a}=\sum_{k=1}^{n}\alpha_k\odot v_k$, and $\hat{b}=\sum_{k=1}^{n}\beta_k\odot v_k$. Since, by the arguments in the first part of this proof, we know that $T_j(\hat{a}+\hat{b}) = T_j(\hat{a}) + T_j(\hat{b})$, and T_j is a Lipschitz mapping, we deduce, from the arbitrariness of $\varepsilon>0$, that $T_j(a+b) = T_j(a)+T_j(b)$, for all $a,b\in C(K_j)$.

For $\alpha\in\mathbb{C}$ and a non-zero algebraic partial isometry $v\in C(K_j)$ we have

$$T_1(\alpha v) = |\alpha|\Delta\left(\frac{\alpha}{|\alpha|}v\right) = \alpha\Delta(v) = \alpha T_1(v),$$

if $v\in C(K_1)$, and

$$T_2(\alpha v) = |\alpha|\Delta\left(\frac{\alpha}{|\alpha|}v\right) = \overline{\alpha}\Delta(v) = \overline{\alpha}T_2(v),$$

if $v\in C(K_2)$ (compare Corollary 3.7). We can therefore conclude from the arguments in the previous paragraph that T_1 is complex linear and T_2 is conjugate linear. It is obvious from definitions that $T_1(a_1)=\Delta(a_1)$ and $T_2(a_2)=\Delta(a_2)$ for every $a_j\in S(C(K_j))$, $j=1$, 2. In particular, T_1 and T_2 are isometries, and $X_j = T_j(C(K_j))$ is a closed subspace of X for every $j=1,2$.

Furthermore, every $a \in S(C(K))$ can be approximated in norm by an algebraic element of the form

$$\hat{a} = \sum_{l=1}^{n} \alpha_l \odot v_l + \sum_{k=1}^{m} \beta_k \odot w_k = \sum_{l=1}^{n} \alpha_l v_l + \sum_{k=1}^{m} \beta_k \odot w_k,$$

where $v_1, \ldots, v_n$ and $w_1, \ldots, w_m$ are mutually orthogonal non-zero algebraic partial isometries in $C(K_1)$ and $C(K_2)$, respectively, $\alpha_1, \ldots, \alpha_n$, $\beta_1, \ldots, \beta_m \in \mathbb{C}\backslash\{0\}$ with $\max\{|\alpha_l| : l \in \{1, \ldots, n\}\} \vee \max\{|\beta_k| : k \in \{1, \ldots, m\}\} = 1$. It follows from previous arguments (essentially from Corollary 3.10) that

$$\Delta(\hat{a}) = \sum_{l=1}^{n} \alpha_l \Delta(v_l) + \sum_{k=1}^{m} \beta_k \Delta(w_k) = T_1(\pi_1(\hat{a})) + T_2(\pi_2(\hat{a})),$$

and by continuity

$$\Delta(a) = T_1(\pi_1(a)) + T_2(\pi_2(a)),$$

for every $a \in S(C(K))$. Suppose $x \in X_1 \cap X_2$ with $\|x\| = 1$. By construction, there exist $a_1 \in S(C(K_1))$ and $a_2 \in S(C(K_2))$ satisfying $\Delta(a_1) = x = \Delta(a_2)$, and hence $a_1 = a_2$, which is impossible because $C(K_1) \cap C(K_2) = \{0\}$. Therefore, $X_1 \cap X_2 = \{0\}$.

We shall finally show that $X = X_1 \oplus X_2$. Given $x \in X$, there exists $a = a_1 + a_2$ in $C(K)$, with $a_j \in C(K_j)$, satisfying

$$x = \Delta(a) = T_1(\pi_1(a)) + T_2(\pi_2(a)) = T_1(a_1) + T_2(a_2) \in X_1 \oplus X_2.$$

The rest is clear. ■

After presenting our first approach to obtain the final conclusion in the previous Theorem 3.11, we insert next a second approach which is closer to the arguments in [[20],[21, Corollaries 5 to 7],[32]]. This second approach conducts to a less conclusive result, we include it here for completeness and as a tribute to the pioneering works of Ding, Liu and Fang, Wang.

We recall next a lemma taken from [32].

Lemma 3.12 ([32, Lemma 2.1]): *Let X and Y be real normed spaces. Suppose $\Delta : S(X) \to S(Y)$ is an onto isometry. If for any $x, y \in S(X)$, we have*

$$\|\Delta(y) - \lambda\Delta(x)\| \leq \|y - \lambda x\|,$$

for all $\lambda > 0$, then Δ can be extended to a surjective real linear isometry from X onto Y. □

Let $\Delta : S(C(K)) \to S(X)$ be a surjective isometry, where K is a Stonean space and X is a complex Banach space. Let K_1 and K_2 be the clopen subsets given by Corolary 3.7. We define a new mapping $\sigma : K \times C(K) \to \mathbb{C}$, given by $\sigma(t, a) = a(t)$, if $t \in K_1$, and $\sigma(t, a) = \overline{a(t)}$, if $t \in K_2$. By a little abuse of notation, we write $\sigma(a(t)) := \sigma(t, a)$ $((t, a) \in K \times C(K))$.

Our next proposition is a generalization of [32, Theorem 3.1] for complex-valued functions.

Proposition 3.13: *Let $\Delta : S(C(K)) \to S(X)$ be a surjective isometry, where K is a Stonean space and X is a complex Banach space. Then for each $t_0 \in K$ and each $\varphi \in supp(t_0, 1)$ the identity*

$$\varphi\Delta(a) = \sigma(t_0, a) = \sigma(a(t_0))$$

holds for every $a \in S(C(K))$.

Proof: As in the proof of Theorem 3.11, every $a \in S(C(K))$ can be approximated in norm by an algebraic element of the form

$$\hat{a} = \sum_{j=1}^{n} \alpha_j \odot v_j + \sum_{k=1}^{m} \beta_k \odot w_k = \sum_{j=1}^{n} \alpha_j v_j + \sum_{k=1}^{m} \beta_k \odot w_k,$$

where $v_1, \ldots, v_n$ and $w_1, \ldots, w_m$ are mutually orthogonal non-zero algebraic partial isometries in $C(K_1)$ and $C(K_2)$, respectively, $\alpha_1, \ldots, \alpha_n$, $\beta_1, \ldots, \beta_m \in \mathbb{C}\backslash\{0\}$ with $\max\{|\alpha_j| : j \in \{1, \ldots, n\}\} \vee \max\{|\beta_k| : k \in \{1, \ldots, m\}\} = 1$. Corollary 3.10 implies that

$$\Delta(\hat{a}) = \sum_{j=1}^{n} \alpha_j \Delta(v_j) + \sum_{k=1}^{m} \beta_k \Delta(w_k).$$

It is easy to check that for $t_0 \in K$ and $\varphi \in \mathrm{supp}(t_0, 1)$ we have

$$\varphi\Delta(\hat{a}) = \sum_{j=1}^{n} \alpha_j \varphi\Delta(v_j) + \sum_{k=1}^{m} \beta_k \varphi\Delta(w_k) = \sigma(t_0, \hat{a}) = \sigma(\hat{a}(t_0)).$$

We can easily deduce from the continuity of Δ and σ, and the norm density commented above, that $\varphi\Delta(a) = \sigma(t_0, a) = \sigma(a(t_0))$. ■

Alternative proof to the final conclusion in Theorem 3.11.: In the hypotheses of this theorem, let $\Delta : S(C(K)) \to S(X)$ be a surjective isometry. By Proposition 3.13, for each $t_0 \in K$ and each $\varphi \in \mathrm{supp}(t_0, 1)$ the identity

$$\varphi\Delta(a) = \sigma(t_0, a) = \sigma(a(t_0))$$

holds for every $a \in S(C(K))$, equivalently,

$$\varphi(x) = \sigma(t_0, \Delta^{-1}(x)) = \sigma(\Delta^{-1}(x)(t_0)),$$

for every $x \in S(X)$. Let us pick $x, y \in S(X)$, $\lambda > 0$ and $\varphi_t \in \mathrm{supp}(t, 1)$ $(t \in K)$. Since

$$\begin{aligned}\|\Delta^{-1}(y) - \lambda\Delta^{-1}(x)\| &= \max_{t\in K} |\Delta^{-1}(y)(t) - \lambda\Delta^{-1}(x)(t)| \\ &= \max_{t\in K_1} |\Delta^{-1}(y)(t) - \lambda\Delta^{-1}(x)(t)| \vee \max_{t\in K_2} |\Delta^{-1}(y)(t) \\ &\quad - \lambda\Delta^{-1}(x)(t)| \\ &= \max_{t\in K_1} |\sigma(\Delta^{-1}(y)(t)) - \lambda\sigma(\Delta^{-1}(x)(t))| \vee \max_{t\in K_2} |\sigma(\Delta^{-1}(y)(t)) \\ &\quad - \lambda\sigma(\Delta^{-1}(x)(t))| \\ &= \max_{t\in K_1} |\varphi_t(y) - \lambda\varphi_t(x)| \vee \max_{t\in K_2} |\varphi_t(y) - \lambda\varphi_t(x)| \leq \|y - \lambda x\|,\end{aligned}$$

we conclude from 3.12 (see [32, Lemma 2.1]) that $\Delta^{-1}: S(X) \to S(C(K))$ admits a unique extension to a surjective real isometry from X to $C(K)$. The rest is clear. ■

We have commented at the introduction that for any σ-finite measure space (Ω, μ), the complex space, $L^\infty(\Omega, \mu)$, of all complex-valued measurable essentially bounded functions equipped with the essential supremum norm, is a commutative von Neumann algebra, and thus from the metric point of view of Functional Analysis, the commutative von Neumann algebra $L^\infty(\Omega, \mu)$ is (C*-isomorphic) isometrically equivalent to some $C(K)$, where K is a hyper-Stonean space. Consequently, the next result, which is an extension of a theorem due to Tan [5] to complex-valued functions, is a corollary of our previous Theorem 3.11.

Theorem 3.14: *Let (Ω, μ) be a σ-finite measure space, and let X be a complex Banach space. Suppose $\Delta : S(L^\infty(\Omega, \mu)) \to S(X)$ is a surjective isometry. Then there exists a surjective real linear isometry $T : L^\infty(\Omega, \mu) \to X$ whose restriction to $S(L^\infty(\Omega, \mu))$ is Δ.* □

Remark 3.15: The celebrated Mazur–Ulam theorem assures that every surjective isometry F between two real normed spaces X and Y is an affine function. Mankiewicz established an amazing generalization of the Mazur–Ulam theorem by showing that every bijective isometry between convex sets in normed linear spaces with non-empty interiors admits a unique extension to a bijective affine isometry between the corresponding spaces (see [38, Theorem 5 and Remark 7]). Tingley's problem asks if every surjective isometry between the unit spheres of two normed spaces admits an extension to a surjective real linear isometry between the spaces. Tingley's problem remains open for general Banach spaces. We have surveyed some positive solutions to Tingley's problem in the introduction. The reader could feel tempted to ask if the unit spheres can be replaced by a strictly smaller set. In some operator algebras the unit spheres have been successfully replaced by the spheres of positive operators (see [39–43]).

Let $\partial_e(\mathcal{B}_X)$ denote the set of all extreme points of the closed unit ball, $\mathcal{B}_X$, of a Banach space X. The set $\partial_e(\mathcal{B}_X)$ seems to be an appropriate candidate to replace the unit sphere of X. However, the answer under these weak conditions is not always positive. Consider, for example, the real Banach space $X = \mathbb{R} \oplus^\infty \mathbb{R}$. It is easy to check that $\partial_e(\mathcal{B}_X) = \{p_1 = (1,1), p_2 = (1,-1), p_3 = (-1,1), p_4 = (-1,-1)\}$, with $d(p_i, p_j) = \|p_i - p_j\| = 2(1 - \delta_{i,j})$, for every $i, j \in \{1, \dots, 4\}$. We can establish a surjective isometry $\Delta : \partial_e(\mathcal{B}_X) \to \partial_e(\mathcal{B}_X)$ defined by

$$\Delta(p_1) = p_2,\ \Delta(p_2) = p_3,\ \Delta(p_3) = p_4 \quad \text{and} \quad \Delta(p_4) = p_1.$$

If we could find an extension of Δ to a surjective real linear isometry $T : X \to X$, then there would exist a real matrix satisfying $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. However, by assumptions $T(p_1) = p_2 \Rightarrow a + b = 1$ and $T(p_4) = p_1 \Rightarrow -a - b = 1$, which is impossible.

After exhibiting the previous counterexample, we provide a list of examples where the previous Tingley's problem for extreme points admits a positive answer. If H and K are Hilbert spaces, we know well that $\partial_e(\mathcal{B}_H) = S(H)$ and $\partial_e(\mathcal{B}_K) = S(K)$. So, in this setting the set of extreme points coincides with the whole unit sphere. Ding proves in [1, Theorem

2.2] that every surjective isometry

$$\Delta : \partial_e(\mathcal{B}_H) = S(H) \to \partial_e(\mathcal{B}_K) = S(K)$$

admits an extension to a surjective real linear isometry from H onto K.

A similar example can be given in another context. Let $C_p(H)$ be the space of p-Schatten von Neumann operators on a complex Hilbert space H equipped with its natural norm $\|a\|_p^p := \text{tr}(|a|^p)$. It is known that $C_p(H)$ is uniformly convex (and hence strictly convex) for every $1 < p < \infty$ (compare the Clarkson–McCarthy inequalities [44]). In particular, $\partial_e(\mathcal{B}_{C_p(H)}) = S(C_p(H))$. A very recent theorem assures that for $1 \leq p \leq \infty$, every surjective isometry

$$\Delta : \partial_e(\mathcal{B}_{C_p(H)}) = S(C_p(H)) \to \partial_e(\mathcal{B}_{C_p(H)}) = S(C_p(H))$$

can be uniquely extended to a surjective real linear isometry on $C_p(H)$ (see [11,14,16, Theorem 2.15]).

We can also present an example of different nature. It is well known that in a finite von Neumann algebra M, the set of all extreme points of its closed unit ball is precisely the set $\mathcal{U}_M$ of all unitary operators in M (see [45–47]). An outstanding theorem due to Hatori and Molnár establishes that every surjective isometry between the unitary groups of two von Neumann algebras can be extended to a surjective real linear isometry between the corresponding von Neumann algebras (compare [48, Corollary 3]). Consequently, if N_1 and N_2 are finite von Neumann algebras (we could consider $N_1 = N_2 = \mathbb{C} \oplus^\infty \mathbb{C}$ or $N_1 = N_2 = M_n(\mathbb{C})$, and many other examples), every surjective isometry

$$\Delta : \partial_e(\mathcal{B}_{N_1}) = \mathcal{U}_{N_1} \to \partial_e(\mathcal{B}_{N_2}) = \mathcal{U}_{N_2}$$

can be uniquely extended to a surjective real linear isometry $T : N_1 \to N_2$.

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ORCID

María Cueto-Avellaneda http://orcid.org/0000-0002-1208-5684
Antonio M. Peralta http://orcid.org/0000-0003-2528-8357

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ON THE EXTENSION OF ISOMETRIES BETWEEN THE UNIT SPHERES OF A JBW*-TRIPLE AND A BANACH SPACE

JULIO BECERRA-GUERRERO, MARÍA CUETO-AVELLANEDA, FRANCISCO J. FERNÁNDEZ-POLO AND ANTONIO M. PERALTA

Departamento de Análisis Matemático, Facultad de Ciencias, Universidad de Granada, 18071 Granada, Spain (juliobg@ugr.es; mcueto@ugr.es; pacopolo@ugr.es; aperalta@ugr.es)

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Abstract We prove that if M is a JBW*-triple and not a Cartan factor of rank two, then M satisfies the Mazur–Ulam property, that is, every surjective isometry from the unit sphere of M onto the unit sphere of another real Banach space Y extends to a surjective real linear isometry from M onto Y.

Keywords: Tingley's problem; Mazur–Ulam property; extension of isometries; JBW*-triples

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1. Introduction

Inspired by the Mazur–Ulam theorem and the positive answers obtained to Tingley's problem in a wide range of concrete spaces, Cheng and Dong introduced in [10] the Mazur–Ulam property. A Banach space X satisfies the *Mazur–Ulam property* if for any Banach space Y, every surjective isometry $\Delta : S(X) \to S(Y)$ admits an extension to a surjective real linear isometry from X onto Y, where $S(X)$ and $S(Y)$ denote the unit spheres of X and Y, respectively.

The so-called *Tingley's problem* asks if every surjective isometry between the unit spheres of two Banach spaces X and Y admits an extension to a surjective real linear isometry between the spaces. This problem was first considered by Tingley in [61]. Recent positive solutions to Tingley's problem in concrete settings include surjective isometries $\Delta : S(X) \to S(Y)$ when X and Y are von Neumann algebras [28], compact C*-algebras and JB*-triples [50] and [29], atomic JBW*-triples [27], spaces of trace class operators [23], spaces of p-Schatten von Neumann operators with $1 \leqslant p \leqslant \infty$ [24], preduals of von Neumann algebras and the self-adjoint parts of two von Neumann algebras [44]. The reader is referred to the surveys [19, 48, 64] for a more thorough overview on Tingley's problem.

The available literature shows that some of the spaces for which Tingley's problem admits a positive solution actually satisfy the stronger Mazur–Ulam property. That is the case of $c_0(\Gamma, \mathbb{R})$, $\ell_\infty(\Gamma, \mathbb{R})$ (see [18, Corollary 2], [41, Main Theorem]), $C(K, \mathbb{R})$ where K is a compact Hausdorff space [41, Corollary 6], $L^p((\Omega, \mu), \mathbb{R})$ where (Ω, μ) is a σ-finite measure space and $1 \leqslant p \leqslant \infty$ [55–57], almost-CL-spaces admitting a smooth point [54, 58], $c_0(\Gamma, \mathbb{C})$ [35], $\ell_\infty(\Gamma, \mathbb{C})$ [47], and commutative von Neumann algebras [12]. The list has been widen in a very recent result by Mori and Ozawa in [45] where they prove that every unital complex C*-algebra and every real von Neumann algebra satisfies the Mazur–Ulam property.

Our main goal in this note is to establish a version of the results by Mori and Ozawa in the setting of JBW*-triples (see Section 2 for concrete definitions and examples). In our principal results (see Theorem 4.14, Proposition 4.15 and Remark 4.16) we prove that if M is a JBW*-triple and not a Cartan factor of rank two, then M satisfies the Mazur–Ulam property.

The starting point in our arguments is Corollary 2.2 where we check, by applying a result due to Mori and Ozawa [45], that the closed unit ball of a JBW*-triple satisfies the strong Mankiewicz property.

In Section 3 we deepen our knowledge on a class of faces of the closed unit ball of the bidual of a JB*-triple which remained unexplored until now. The main result in [20] shows that the proper norm closed faces of the closed unit ball, $\mathcal{B}_E$, of a JB*-triple, E, are in one-to-one correspondence with those tripotents in E^{**} which are compact. A preceding result due to Edwards and Rüttimann assures that weak* closed proper faces of the closed unit ball of E^{**} are in one-to-one correspondence with the set of tripotents in E^{**} (see [21]). On the other hand, following the notation in [25, § 2], we shall say that a set $S \subseteq E^{**}$ is *open relative to* E if $S \cap E$ is $\sigma(E^{**}, E^*)$ dense in $\overline{S}^{\sigma(E^{**},E^*)}$. It seems natural to ask whether relatively open faces in the closed unit ball of E^{**} can be characterized in terms of a set of tripotents in E^{**}. We shall show in Theorem 3.6 that a proper weak* closed face of the closed unit ball of E^{**} is open relative to E if and only if it is a weak* closed face associated with a compact tripotent in E^{**}.

Let E be a JB*-triple. The characterization of those proper weak* closed faces of the closed unit ball of E^{**} which are open relative to E in terms of the compact tripotents in E^{**} is applied to establish that if $\Delta : S(M) \to S(Y)$ is a surjective isometry, where M is a JBW*-triple and Y is a Banach space, the restriction of Δ to each norm closed proper face of $\mathcal{B}_M$ is an affine mapping (see Proposition 4.6).

2. Background on JB*-triples and the strong Mankiewicz property

Along this paper, given a complex Banach space X, its underlying real Banach space will be denoted by the same symbol X or by $X_\mathbb{R}$ in case of ambiguity. It is well known that $\varphi \mapsto \Re e\varphi$ is an isometric bijection from $(X^*)_\mathbb{R}$ onto $(X_\mathbb{R})^*$. If X is a real or complex Banach space, the symbol $\mathcal{B}_X$ will stand for the closed unit ball of X, while $S(X)$ will denote the unit sphere of X. We shall frequently regard X as being contained in X^{**} and we identify the weak*-closure in X^{**} of a closed subspace Y of X with Y^{**}.

A convex subset K of a normed space X is called a *convex body* if it has non-empty interior in X. The Mazur–Ulam theorem was extended by Mankiewicz in [42] by showing that every surjective isometry between convex bodies in two arbitrary normed spaces can be uniquely extended to an affine function between the spaces. This result is one of the main tools applied in those papers devoted to explore new progress to Tingley's problem and to determine new Banach spaces satisfying the Mazur–Ulam property.

In a very recent paper by Mori and Ozawa (see [45]), a new technical achievement has burst into the scene of the current research on those Banach spaces satisfying the Mazur–Ulam property. Following these authors, we shall say that a convex subset K of a normed space X satisfies the *strong Mankiewicz property* if every surjective isometry Δ from K onto an arbitrary convex subset L in a normed space Y is affine. As observed by Mori and Ozawa, every convex subset of a strictly convex normed space satisfies the strong Mankiewicz property because it is uniquely geodesic (see [2, Lemma 6.1]), and there exist examples of convex subsets of $L^1[0,1]$ which do not satisfy this property (see [45, Example 5]). In [45, Theorem 2] Mori and Ozawa show that some of the hypotheses in Mankiewicz's theorem can be somehow relaxed. The following result has been borrowed from [45, Theorem 2 and its proof].

Theorem 2.1 [45, Theorem 2]. *Let X be a Banach space such that the closed convex hull of the extreme points, $\partial_e(\mathcal{B}_X)$, of the closed unit ball, $\mathcal{B}_X$, of X has non-empty interior in X. Then every convex body $K \subset X$ satisfies the strong Mankiewicz property. Furthermore, suppose L is a convex subset of a normed space Y, and $\Delta : \mathcal{B}_X \to L$ is a surjective isometry. Then Δ can be uniquely extended to an affine isometry from X onto a norm closed subspace of Y.* □

The celebrated Russo–Dye theorem (see [51]) assures that every (complex) unital C*-algebra satisfies the hypotheses in the previous theorem. Actually, Mori and Ozawa show that any Banach space in the class of real von Neumann algebras also satisfies the desired hypotheses (see [45, Corollary 3]). This can be also deduced from the real version of the Russo–Dye theorem, established by Navarro and Navarro in [46, Corollary 6], which asserts that the open unit ball of a real von Neumann algebra A is contained in the sequentially convex hull of the set of unitary elements in A. As pointed out by Mori and Ozawa in [45, Proof of Corollary 3], the latter conclusion can be deduced from a result due to Li (see [40, Theorem 7.2.4]).

Let us continue this section by adding some new examples of Banach spaces fulfilling the hypotheses in the Mori–Ozawa Theorem 2.1. Henceforth, let $B(H, K)$ denote the Banach space of all bounded linear operators between two complex Hilbert spaces H and K. A J*-algebra in the sense introduced by Harris in [32] is a closed complex subspace E of $B(H, K)$ such that $aa^*a \in E$ whenever $a \in E$. Harris proved in [32, Corollary 2] that the open unit ball, $\overset{\circ}{\mathcal{B}}_E$, of every J*-algebra E is a bounded symmetric domain (i.e. for each $x \in \overset{\circ}{\mathcal{B}}_E$ there exists a biholomorphic mapping in Fréchet's sense $h : \overset{\circ}{\mathcal{B}}_E \to \overset{\circ}{\mathcal{B}}_E$ such that h has x as its only fixed point and h^2 is the identity map on $\overset{\circ}{\mathcal{B}}_E$). However, J*-algebras are not the unique complex Banach spaces whose open unit ball is a bounded symmetric domain. Kaup established in [37] that the open unit ball of a complex Banach space E is a bounded symmetric domain if and only if E is a JB*-triple, that is, there exists a continuous triple product $\{.,.,.\} : E \times E \times E \to E$, which is symmetric and linear in the first and third variables, conjugate linear in the second variable, and satisfies the following axioms:

(a) (Jordan identity) $L(a,b)L(x,y) = L(x,y)L(a,b) + L(L(a,b)x,y) - L(x,L(b,a)y)$, for every a,b,x,y in E, where $L(a,b)$ is the operator on E given by $L(a,b)x = \{a,b,x\}$;

(b) $L(a,a)$ is a hermitian operator with non-negative spectrum for all $a \in E$;

(c) $\|\{a,a,a\}\| = \|a\|^3$ for each $a \in E$.

Every J*-algebra is a JB*-triple with respect to the triple product given by

$$\{x,y,z\} = \tfrac{1}{2}(xy^*z + zy^*x). \tag{1}$$

Consequently, C*-algebras and complex Hilbert spaces are JB*-triples with respect to the above triple product. Other interesting examples are given by Jordan structures; for example every JB*-algebra in the sense considered in [62, 63] and [52, 53] is a JB*-triple under the triple product

$$\{x,y,z\} = (x \circ y^*) \circ z + (z \circ y^*) \circ x - (x \circ z) \circ y^*. \tag{2}$$

Another milestone result in the theory of JB*-triples is the so-called *Kaup–Banach–Stone theorem*, established by Kaup in [37, Proposition 5.5], which proves that a linear bijection between JB*-triples is an isometry if and only if it is a triple isomorphism.

A JBW*-triple is a JB*-triple which is also a dual Banach space (with a unique isometric predual [3]). It is known that the second dual of a JB*-triple is a JBW*-triple (compare [15]). An extension of Sakai's theorem assures that the triple product of every JBW*-triple is separately weak* continuous (cf. [3] or [33]).

We shall only recall some basic facts and results in the theory of JB*-triples. Let A be a C*-algebra regarded as a JB*-triple with the product given in (1). It is easy to see that partial isometries in A are precisely those elements e in A such that $\{e,e,e\} = e$. An element e in a JB*-triple E is said to be a *tripotent* if $\{e,e,e\} = e$. The extreme points of the closed unit ball of a JB*-triple can only be understood in terms of those tripotents satisfying an additional property. For each tripotent e in E the eigenvalues of the operator $L(e,e)$ are contained in the set $\{0, 1/2, 1\}$, and E can be decomposed in the form

$$E = E_2(e) \oplus E_1(e) \oplus E_0(e),$$

where for $i = 0,1,2$, $E_i(e)$ is the $\frac{i}{2}$ eigenspace of $L(e,e)$. This decomposition is known as the *Peirce decomposition* associated with e. The so-called *Peirce arithmetic* affirms that for every $i,j,k \in \{0,1,2\}$ we have

- $\{E_i(e), E_j(e), E_k(e)\} \subseteq E_{i-j+k}(e)$ if $i-j+k$ belongs to the set $\{0,1,2\}$, and $\{E_i(e), E_j(e), E_k(e)\} = \{0\}$ otherwise;
- $\{E_2(e), E_0(e), E\} = \{E_0(e), E_2(e), E\} = \{0\}$.

For $k \in \{0,1,2\}$, the projection $P_k(e)$ of E onto $E_k(e)$ is called the Peirce-k projection. It is known that Peirce projections are contractive (cf. [30]) and satisfy that $P_2(e) = Q(e)^2$, $P_1(e) = 2(L(e,e) - Q(e)^2)$, and $P_0(e) = Id_E - 2L(e,e) + Q(e)^2$, where for each $a \in E$, $Q(a) : E \to E$ is the conjugate linear map given by $Q(a)(x) = \{a,x,a\}$. A tripotent e

in E is called *unitary* (respectively, *complete* or *maximal*) if $E_2(e) = E$ (respectively, $E_0(e) = \{0\}$). Finally, a tripotent e in E is said to be *minimal* if $E_2(e) = \mathbb{C}e \neq \{0\}$.

Additional properties of the Peirce decomposition assure that the Peirce space $E_2(e)$ is a unital JB*-algebra with unit e, product $x \circ_e y := \{x, e, y\}$ and involution $x^{*_e} := \{e, x, e\}$. It follows from Kaup–Banach–Stone theorem that the triple product in $E_2(e)$ is uniquely determined by the identity

$$\{a, b, c\} = (a \circ_e b^{*_e}) \circ_e c + (c \circ_e b^{*_e}) \circ_e a - (a \circ_e c) \circ_e b^{*_e}, \qquad (\forall a, b, c \in E_2(e)).$$

Furthermore, for each $x \in E$ the element

$$P_2(e)\{x, x, e\} = \{P_2(e)(x), P_2(e)(x), e\} + \{P_1(e)(x), P_1(e)(x), e\} \tag{3}$$

is positive in $E_2(e)$, and $P_2(e)\{x, x, e\} = 0$ if and only if $P_j(e)(x) = 0$ for every $j = 1, 2$ (see [30, Lemma 1.5 and preceding comments]).

Elements a, b in a JB*-triple E are called *orthogonal* (written $a \perp b$) if $L(a, b) = 0$. It is known that $a \perp b \Leftrightarrow \{a, a, b\} = 0 \Leftrightarrow \{b, b, a\} = 0 \Leftrightarrow b \perp a$; (see, for example, [8, Lemma 1]). Let e be a tripotent in E. It follows from Peirce arithmetic that $a \perp b$ for every $a \in E_2(e)$ and every $b \in E_0(e)$.

The rank of a JB*-triple E is the minimal cardinal number r satisfying $\operatorname{card}(S) \leqslant r$ whenever S is an orthogonal subset of E, that is, $0 \notin S$ and $x \perp y$ for every $x \neq y$ in S.

We shall consider the following partial order on the set of tripotents of a JB*-triple E defined by $u \leqslant e$ if $e - u$ is a tripotent in E and $e - u \perp u$. It is known that $u \leqslant e$ if and only if u is a projection in the JB*-algebra $E_2(e)$.

Similarly as there exist C*-algebras containing no non-zero projections, we can find JB*-triples containing no non-trivial tripotents. Another geometric property of JB*-triples provides an algebraic characterization of the extreme points of their closed unit balls. Concretely, the extreme points of the closed unit ball of a JB*-triple E are precisely the complete tripotents in E, that is

$$\partial_e(\mathcal{B}_E) = \{\text{complete tripotents in } E\}, \tag{4}$$

(cf. [5, Lemma 4.1] and [39, Proposition 3.5]).

An element u in a unital C*-algebra A is called *unitary* if $uu^* = u^*u = 1$. It is known that an element u in a JB*-algebra B is a unitary if and only if u is Jordan invertible in B and its unique Jordan inverse in B coincides with u^* (compare [63] and [52, 53]). If a unital C*-algebra A is regarded as a JB*-algebra with the natural Jordan product given by $a \circ b := \frac{1}{2}(ab + ba)$, then an element u in A is a unitary in the C*-algebra sense if, and only if, it is unitary in the JB*-algebra sense if, and only if, it is unitary (tripotent) in the JB*-triple sense. Clearly, every unitary element in a JB*-algebra is an extreme point of its closed unit ball.

After reviewing the basic facts on the extreme points of the closed unit ball of a JB*-triple, we can next consider the strong Mankiewicz property for convex bodies in a JBW*-triple. Let us recall that the Russo–Dye theorem is the tool employed by Mori and Ozawa to show, via Theorem 2.1 [45, Theorem 2], that every convex body of a unital C*-algebra satisfies the strong Mankiewicz property. The Russo–Dye theorem was extended to the setting of unital JB*-algebras by Wright and Youngson [63] and

Siddiqui [53]. In 2007, Siddiqui proved that every element in the unit ball of a JBW*-triple is the average of two extreme points (see [52, Theorem 5]). Our next result is a straight consequence of this result and [45, Theorem 2].

Corollary 2.2. *The closed unit ball of every JBW*-triple M satisfies the strong Mankiewicz property. Consequently, every convex body in a JBW*-triple satisfies the same property. Furthermore, if L is a convex subset of a normed space Y, then every surjective isometry $\Delta : \mathcal{B}_M \to L$ can be uniquely extended to an affine isometry from M onto a norm closed subspace of Y.*

3. Relatively open faces in the bidual of a JB*-triple

As in the study of the Mazur–Ulam property in the setting of unital C*- and von Neumann algebras (see [45]), the facial structure of JB*-triples plays a central role in our study of the Mazur–Ulam property in the class of JBW*-triples. For this purpose we shall require some basic notions.

In order to understand the nomenclature we refresh the usual 'facear' and 'pre-facear' operations. Let X be a complex Banach space with dual space X^*. For each subset $F \subseteq \mathcal{B}_X$ and each $G \subseteq \mathcal{B}_{X^*}$, we set

$$F' = \{a \in \mathcal{B}_{X^*} : a(x) = 1 \ \forall x \in F\}, \quad G_{\prime} = \{x \in \mathcal{B}_X : a(x) = 1 \ \forall a \in G\}. \tag{5}$$

Then, F' is a weak* closed face of $\mathcal{B}_{X^*}$ and $G_{\prime}$ is a norm closed face of $\mathcal{B}_X$. The subset F of $\mathcal{B}_X$ is said to be a *norm-semi-exposed face* of $\mathcal{B}_X$ if $F = (F')_{\prime}$ and the subset G of $\mathcal{B}_{X^*}$ is said to be a *weak*-semi-exposed face* of $\mathcal{B}_{X^*}$ if $G = (G_{\prime})'$. The mappings $F \mapsto F'$ and $G \mapsto G_{\prime}$ are anti-order isomorphisms between the partially ordered sets of norm-semi-exposed faces of $\mathcal{B}_X$ and of weak*-semi-exposed faces of $\mathcal{B}_{X^*}$ and are inverses of each other.

In a celebrated result published in [21], Edwards and Rüttimann proved that the weak* closed faces of the closed unit ball of a JBW*-triple M are in one-to-one correspondence with the tripotents in M. The concrete theorem reads as follows:

Theorem 3.1 [21]. *Let M be a JBW*-triple, and let F be a weak* closed face of the unit ball $\mathcal{B}_M$ in M. Then, there exists a tripotent e in M such that*

$$F = F_e^M = e + \mathcal{B}_{M_0(e)} = (\{e\}_{\prime})',$$

where $\mathcal{B}_{M_0(e)}$ denotes the unit ball of the Peirce zero space $M_0(e)$ in M. Furthermore, the mapping $e \mapsto F_e^M = (\{e\}_{\prime})'$ is an anti-order isomorphism from the partially ordered set $\mathcal{U}(M)$ of all tripotents in M onto the partially ordered set of weak closed faces of $\mathcal{B}_M$ excluding the empty set.* □

We continue by reviewing the notion of compact tripotent in the second dual of a JB*-triple. Given an element a in a JB*-triple, we set $a^{[1]} := a$, $a^{[3]} := \{a, a, a\}$, and $a^{[2n+1]} := \{a, a, a^{[2n-1]}\}$, $(n \in \mathbb{N})$. Let us fix a JBW*-triple M. It is known that, for each $a \in S(M)$, the sequence $(a^{[2n-1]})$ converges in the weak* topology of M to a (possibly zero) tripotent $u_M(a)$ or $u(a)$ in M (compare [21, Lemma 3.3] or [20, page 130]).

This tripotent $u_{_M}(a)$ is called the *support tripotent* of a. The equality $a = u(a) + P_0(u(a))(a)$ holds for every a in the above conditions. For a norm-one element a in a JB*-triple E, $u_{_{E^{**}}}(a)$ will denote the support tripotent of a in E^{**} which is always non-zero. Given a in M the support tripotents $u_{_M}(a)$ and $u_{_{M^{**}}}(a)$ need not coincide. To avoid confusion, given a norm-one element a in a JBW*-triple M, unless otherwise stated, we shall write $u(a)$ for the support tripotent of a in M^{**}.

According to the terminology introduced by Edwards and Rüttimann in [22], a tripotent e in the second dual, E^{**}, of a JB*-triple E is said to be *compact-G_δ* if there exists a norm-one element a in E satisfying $u(a) = u_{_{E^{**}}}(a) = e$. A tripotent e in E^{**} is *compact* if $e = 0$ or it is the infimum of a decreasing net of compact-G_δ tripotents in E^{**} converging to e in the weak* topology. Clearly, every tripotent in E is compact in E^{**}.

Akemann and Pedersen described in [1] the facial structure of a general C*-algebra, a task actually initiated and considered by Edwards and Rüttimann in [21]. The understanding of the facial structure of a general JB*-triple was completed by Edwards, Fernández-Polo, Hoskin and Peralta in [20]. The result required in this note is subsumed in the next theorem.

Theorem 3.2 [20, Theorem 3.10 and Corollary 3.12]. *Let E be a JB*-triple, and let F be a norm closed face of the unit ball $\mathcal{B}_E$ in E. Then, there exists a (unique) compact tripotent u in E^{**} such that*

$$F = F_u^E = (u + \mathcal{B}_{E_0^{**}(u)}) \cap E = (\{u\}_\prime)_\prime,$$

where $\mathcal{B}_{E_0^{**}(u)}$ denotes the unit ball of the Peirce zero space $E_0^{**}(u)$ in E^{**}. Furthermore, the mapping $u \mapsto F_u^E = (\{u\}_\prime)_\prime$ is an anti-order isomorphism from the partially ordered set of all compact tripotents in E^{**} onto the partially ordered set of norm closed faces of $\mathcal{B}_E$ excluding the empty set. □

The facial structure of the closed unit ball of a JB*-triple E assures that norm closed faces of $\mathcal{B}_E$ are in one-to-one correspondence with compact tripotents in E^{**}. Even in the case in which we are dealing with a JBW*-triple M, tripotents in M are not enough to determine all norm closed faces of $\mathcal{B}_M$.

The celebrated Kadison's transitivity theorem was extended by Bunce, Martínez–Moreno and the last two authors of this note to the setting of JB*-triples (cf. [7, Theorem 3.3]). Suppose E is a JB*-triple. A consequence of Kadison's transitivity theorem proves that every maximal norm closed proper face of $\mathcal{B}_E$ is of the form

$$F_e^E = (e + \mathcal{B}_{E_0^{**}(e)}) \cap E, \tag{6}$$

where e is a minimal tripotent in E^{**} (see [7, Corollary 3.5]).

When comparing Theorems 3.1 and 3.2 the natural question is whether we can topologically distinguish between weak* closed faces in $\mathcal{B}_{E^{**}}$ associated with compact tripotents in E^{**} from weak* closed faces in $\mathcal{B}_{E^{**}}$ associated with non-compact tripotents in E^{**}. We shall see in Theorem 3.6 that the required topological notion was already considered in [25].

Let X be a Banach space, E a weak* dense subset of X^* and S a non-zero subset of X^*. Following the notation in [25, § 2], we shall say that S is *open relative to E* if $S \cap E$

is $\sigma(X^*, X)$ dense in $\overline{S}^{\sigma(X^*,X)}$. Let E be a JB*-triple. A tripotent e in E^{**} is called *closed (relative to E)* if $E_0^{**}(e)$ is an open subset of E^{**} relative to E. We shall say that e is *bounded* (*relative to E*) if there exists x in the unit sphere of E satisfying that $\{e, e, x\} = e$ (or equivalently, $x = e + P_0(e)(x)$ in E^{**}). One of the main achievements in [25] shows that a tripotent u in E^{**} is compact if and only if it is closed and bounded (cf. [25, Theorem 2.6.]).

Other tools needed for our purposes are the triple functional calculus at an element in a JB*-triple E and the strong* topology. The symbol E_a will stand for the JB*-subtriple of E generated by the element a. It is known that E_a is JB*-triple isomorphic (and hence isometric) to $C_0(\Omega_a)$ for some locally compact Hausdorff space Ω_a contained in $[0, \|a\|]$, such that $\Omega_a \cup \{0\}$ is compact, where $C_0(\Omega_a)$ denotes the Banach space of all complex-valued continuous functions vanishing at 0. It is also known that the triple identification of E_a and $C_0(\Omega_a)$ can be assumed to satisfy that a corresponds to the function mapping each $\lambda \in \Omega_a$ to itself (cf. [37, Corollary 1.15] and [30]). The *triple functional calculus* at the element a is defined as follows. Given a function $f \in C_0(\Omega_a)$, $f_t(a)$ will stand for the (unique) element in E_a corresponding to the function f.

Let a be an element in a JB*-triple E. Let $g_t(a) =: a^{[\frac{1}{2}]} \in E_a$ where $g(\lambda) = \lambda^{\frac{1}{2}}$ $(\lambda \in \Omega_a)$. According to the notation in [25], along this paper, $P_0(a)$ will denote the bounded linear operator on E defined by

$$P_0(a)(y) = y - 2L(a^{[\frac{1}{2}]}, a^{[\frac{1}{2}]})(y) + Q(a^{[\frac{1}{2}]})^2(y). \tag{7}$$

In the literature this operator is called the Bergman operator associated with a. We should note that this notation is not ambiguous when $a = e$ is a tripotent, because $P_0(e)$ is precisely the Peirce projection of E onto $E_0(e)$. In the sequel we shall also write $a^{[2]}$ for the element $h_t(a) \in E_a$ where $h(\lambda) = \lambda^2$ $(\lambda \in \Omega_a)$. The elements $a^{[2]}$ and $a^{[\frac{1}{2}]}$ may be seen as artificial constructions in the triple setting; however, both of them lie in E_a.

Let a be a norm-one element in a JBW*-triple M. Lemma 3.3 in [22] implies the existence of a smallest tripotent $r(a)$ in M such that $a \in M_2(r(a))$ and a is positive in the latter JBW*-algebra. Furthermore, in the order of the JBW*-algebra $M_2^{**}(r(a))$, we have

$$0 \leqslant u_M(a) \leqslant u_{M^{**}}(a) \leqslant a^{[2n+1]} \leqslant a \leqslant r(a),$$

for every natural n. The tripotent $r(a) = r_M(a)$ is called the *range tripotent* of a. We have already commented that the support tripotent $u_M(a)$ might be zero; however, $u_{M^{**}}(a) \neq 0$.

It is time to recall the definition and basic properties of the strong* topology. Suppose φ is a norm-one normal functional in the predual M_* of a JBW*-triple M. If z is a norm-one element in M satisfying $\varphi(z) = 1$, then the assignment

$$(x, y) \mapsto \varphi\{x, y, z\}$$

defines a positive sesquilinear form on M, which does not depend on the choice of z. We therefore have a prehilbert seminorm on M defined by $\|x\|_\varphi^2 := \varphi\{x, x, z\}$. The *strong* topology* of M is the topology on M induced by the seminorms $\|x\|_\varphi$ when φ ranges in the unit sphere of M_*. The strong* topology was originally introduced in [4], and subsequently developed in [49] (see also [9, § 5.10.2]). Among the properties of this topology we note that the strong* topology of M is compatible with the duality (M, M_*) (see [4, Theorem 3.2]). By combining this property with the bipolar theorem, we deduce that the identity

$$\overline{C}^{\sigma(M,M_*)} = \overline{C}^{\text{strong}^*}, \tag{8}$$

holds for every convex subset $C \subseteq M$. Another interesting property asserts that the triple product of M is jointly strong* continuous on bounded sets of M (see [49] or [9, Theorem 5.10.133]).

We shall study next a series of geometric inequalities in different settings. The first case is probably part of the folklore in the theory of C*-algebras.

Lemma 3.3. *Let A be a unital C*-algebra. Suppose a and b are two elements in the closed unit ball of A with a positive. Then $\|1 - a(1+b)a\| \leqslant 1$.*

Proof. Let $z = 1 - a(1+b)a$. Since for each $y \in A$, the mapping $x \mapsto yxy^*$ is positive, we get

$$\begin{aligned} zz^* &= (1 - a(1+b)a)(1 - a(1+b)a)^* = 1 - 2a^2 - a(b+b^*)a + a(1+b)aa(1+b)^*a \\ &\leqslant 1 - 2a^2 - a(b+b^*)a + a(1+b)(1+b)^*a = 1 - a^2 + abb^*a \leqslant 1 - a^2 + a^2 = 1. \end{aligned}$$

It follows from the Gelfand–Naimark axiom that $\|1 - a(1+b)a\|^2 = \|z\|^2 = \|zz^*\| \leqslant 1$. □

The case of JB*-algebras is treated next. We first recall some notation. Given an element a in a JB*-algebra B, we shall write U_a for the linear mapping on B defined by $U_a(x) = 2(a \circ x) \circ a - a^2 \circ x$ $(x \in B)$. It is clear that if B is regarded as a JB*-triple with the product given in (2) then $U_a(x) = \{a, x^*, a\}$ for every $a, x \in B$.

Lemma 3.4. *Let B be a unital JB*-algebra. Suppose a and b are two elements in the closed unit ball of B with a positive. Then*

$$\|1 - U_a(1+b)\| = \|1 - \{a, 1+b^*, a\}\| \leqslant 1.$$

Proof. There is no loss of generality in assuming that B is a JBW*-algebra. Let us fix a unitary element u in B. By [62, page 294], the JBW*-subalgebra $\mathcal{C}$ of B generated by $1, u$ and u^* can be realized as a JW*-subalgebra of a von Neumann algebra A. In particular $\mathcal{C}$ is a commutative von Neumann algebra. Since u is a unitary element in $\mathcal{C}$, we can find a hermitian element $h \in \mathcal{C} \subset B$ such that $u = e^{ih}$ (cf. [36, Remark 10.2.2]). Let $\widetilde{\mathcal{C}}$ denote the JB*-subalgebra of B generated by 1, a and h. A new application of [62] implies that $\widetilde{\mathcal{C}}$ is isometrically JB*-isomorphic to a JB*-subalgebra of a unital C*-algebra $\widetilde{A}$. Since a and u are identified with elements in the unit ball of $\widetilde{A}$ with a positive, Lemma 3.3 implies that

$$1 \geqslant \|1 - a(1+u)a\|_{\widetilde{A}} = \|1 - U_a(1+u)\|_{\widetilde{\mathcal{C}}} = \|1 - U_a(1+u)\|_B = \|1 - \{a, 1+u^*, a\}\|_B.$$

Finally, by the Russo–Dye theorem for unital JB*-algebras (see [53]), we get the desired conclusion. □

We shall next establish a JB*-triple version of the previous two lemmata.

Lemma 3.5. *Suppose a and b are two elements in the closed unit ball of a JB*-triple E. Then*

$$\|2a - a^{[2]} + P_0(a)(b)\| = \left\|2a - a^{[2]} + b - 2L(a^{[\frac{1}{2}]}, a^{[\frac{1}{2}]})(b) + Q(a^{[\frac{1}{2}]})^2(b)\right\| \leqslant 1.$$

Proof. By [31, Corollary 1] we may suppose that E is a JB*-subtriple of a unital JB*-algebra B which is an ℓ_∞-sum of a type I von Neumann factor and an ℓ_∞-sum of finite dimensional simple JB*-algebras. Lemma 2.3 in [7] implies the existence of an isometric triple embedding, $\pi : B \to B$, such that $\pi(a)$ is a positive element in B. Since the elements $\pi(a)$, $1 - \pi(a)$, and $-\pi(b)$ lie in the closed unit ball of B, we deduce from Lemma 3.4 that

$$\|1 - U_{1-\pi(a)}(1 - \pi(b))\|_B = \|1 - \{1 - \pi(a), 1 - \pi(b)^*, 1 - \pi(a)\}\|_B \leqslant 1. \tag{9}$$

On the other hand, it is not hard to check that, since $\pi(a)$ is positive in B, we have

$$\begin{aligned}
&2\pi(a) - \pi(a)^{[2]} + \pi(b) - 2L(\pi(a)^{[\frac{1}{2}]}, \pi(a)^{[\frac{1}{2}]})(\pi(b)) + Q(\pi(a)^{[\frac{1}{2}]})^2(\pi(b)) \\
&\quad = 2\pi(a) - \pi(a)^{[2]} + \{1 - \pi(a), \pi(b)^*, 1 - \pi(a)\} = 1 + \{1 - \pi(a), \pi(b)^* - 1, 1 - \pi(a)\}.
\end{aligned}$$

Finally, since π is an isometric triple embedding we deduce that

$$\begin{aligned}
&\|2a - a^{[2]} + b - 2L(a^{[\frac{1}{2}]}, a^{[\frac{1}{2}]})(b) + Q(a^{[\frac{1}{2}]})^2(b)\|_E \\
&\quad = \|2\pi(a) - \pi(a)^{[2]} + \pi(b) - 2L(\pi(a)^{[\frac{1}{2}]}, \pi(a)^{[\frac{1}{2}]})(\pi(b)) + Q(\pi(a)^{[\frac{1}{2}]})^2(\pi(b))\|_B \\
&\quad = \|1 + \{1 - \pi(a), \pi(b)^* - 1, 1 - \pi(a)\}\|_B \\
&\quad = \|1 - \{1 - \pi(a), 1 - \pi(b)^*, 1 - \pi(a)\}\|_B \leqslant (\text{by } (9)) \leqslant 1.
\end{aligned}$$

□

The promised characterization of those weak* closed faces in the bidual of a JB*-triple E corresponding to compact tripotents in E^{**} can be now stated.

Theorem 3.6. *Let E be a JB*-triple. Suppose F is a proper weak* closed face of the closed unit ball of E^{**}. Then the following statements are equivalent:*

(a) *F is open relative to E;*

(b) *F is a weak* closed face associated with a non-zero compact tripotent in E^{**}, that is, there exists a unique non-zero compact tripotent u in E^{**} satisfying that $F = F_u^{E^{**}} = u + \mathcal{B}_{E_0^{**}(u)}$.*

Proof. $(b) \Rightarrow (a)$ Let us first assume that $F = F_u^{E^{**}} = u + \mathcal{B}_{E_0^{**}(u)}$, where u is a compact-G_δ tripotent in E^{**}, that is, $u = u(a)$ for some $a \in S(E)$. It is known that the sequence $(a^{[2n-1]})_n$ is decreasing in $E_2^{**}(r(a))$ and converges in the weak* topology (and hence in the strong* topology) of E^{**} to $u(a)$.

Pick an arbitrary $y \in F$ (i.e. $y = u + P_0(u)(y)$). Kaplansky's density theorem assures that $\mathcal{B}_E$ is strong* dense in $\mathcal{B}_{E^{**}}$ (cf. [4, Corollary 3.3] or just apply (8)); thus we can find a net $(y_\lambda)_{\lambda\in\Lambda} \subset \mathcal{B}_E$ converging to y in the strong* topology of E^{**}. We set $a_n := a^{[2n-1]}$ $(n \in \mathbb{N})$ and

$$x_{\lambda,n} := (2a_n - a_n^{[2]}) + P_0(a_n)(y_\lambda), \quad ((\lambda, n) \in \Lambda \times \mathbb{N}).$$

We have already commented that $(a_n)_n = (a^{[2n-1]})_n \to u$ in the strong* topology of E^{**}. Having in mind that the triple product of E^{**} is jointly strong* continuous, and identifying the JBW*-subtriple of E^{**} generated by a and its range tripotent, $r(a)$, with a commutative von Neumann algebra in which a is a positive generator, we can easily deduce that $(a_n^{[2]})_n = (\{a_n, r(a), a_n\})_n = (\{a_n, a_n, r(a)\})_n \to u$ in the strong* topology of E^{**}. Moreover, the support and the range tripotents of a coincide with the support and the range tripotent of $a^{[\frac{1}{4}]}$, respectively, and thus $(a_n^{[\frac{1}{2}]})_n = (\{(a^{[\frac{1}{4}]})^{[2n-1]}, r(a), (a^{[\frac{1}{4}]})^{[2n-1]}\})_n \to u$ in the strong* topology of E^{**}.

Clearly, the double indexed net $(x_{\lambda,n})_{\lambda,n}$ is contained in E, and by the joint strong* continuity of the triple product of E^{**} the net $(x_{\lambda,n})_{\lambda,n}$ tends to $2u - u + P_0(u)(y) = u + P_0(u)(y) = y$ in the strong* topology of E^{**}, and hence in the weak* topology of the latter space.

On the other hand, by considering the JBW*-subtriple of E^{**} generated by a, we can easily see that $2a_n - a_n^{[2]} = u + P_0(u)(2a_n - a_n^{[2]}) \in (u + E_0^{**}(u)) \cap E$. Since $a = u + P_0(u)(a)$, Lemma 2.5 in [25] assures that $P_0(a_n)(y_\lambda) \in E_0^{**}(u)$. Consequently,

$$x_{\lambda,n} = 2a_n - a_n^{[2]} + P_0(a_n)(y_\lambda) \in (u + E_0^{**}(u)) \cap E.$$

Lemma 3.5 proves that $x_{\lambda,n} \in \mathcal{B}_E$ for every $(\lambda, n) \in \Lambda \times \mathbb{N}$, and thus $x_{\lambda,n} \in F \cap E$, for every $(\lambda, n) \in \Lambda \times \mathbb{N}$. Since $(x_{\lambda,n})_{\lambda,n} \to y$ in the weak* topology of E^{**}, we get $y \in \overline{F \cap E}^{w^*}$. This concludes the proof in the case that u is compact-G_δ, that is

$$\overline{(u + \mathcal{B}_{E_0^{**}(u)}) \cap E}^{w^*} = u + \mathcal{B}_{E_0^{**}(u)},$$

for every compact-G_δ tripotent $u \in E^{**}$.

Suppose now that u is a non-zero compact tripotent in E^{**}. Then, by definition, we can find a decreasing net $(u_\mu)_\mu$ of compact-G_δ tripotents in E^{**} converging to u in the weak* topology of E^{**}, and hence $(u_\mu)_\mu \to u$ in the strong* topology. We have proved in the previous paragraphs that each $F_{u_\mu}^{E^{**}}$ is open relative to E, that is,

$$\overline{F_{u_\mu}^{E^{**}} \cap E}^{w^*} = \overline{(u_\mu + \mathcal{B}_{E_0^{**}(u_\mu)}) \cap E}^{w^*} = F_{u_\mu}^{E^{**}} = u_\mu + \mathcal{B}_{E_0^{**}(u_\mu)}, \tag{10}$$

for every μ. Given an arbitrary $y \in F = F_u^{E^{**}}$, the net $(u_\mu + P_0(u_\mu)(y)) \to u + P_0(u)(y) = y$ in the weak* topology. Since $F_{u_\mu}^{E^{**}} \subseteq F = F_u^{E^{**}}$ for every μ, the arbitrariness of y shows that

$$F = F_u^{E^{**}} = \overline{\bigcup_\mu F_{u_\mu}^{E^{**}}}^{w^*}. \tag{11}$$

Now, the relation

$$\overline{F \cap E}^{w^*} \supseteq \overline{\bigcup_\mu \overline{F_{u_\mu}^{E^{**}} \cap E}^{w^*}}^{w^*} = \overline{\left(\bigcup_\mu F_{u_\mu}^{E^{**}}\right) \cap E}^{w^*} = \overline{\bigcup_\mu F_{u_\mu}^{E^{**}}}^{w^*} = (\text{by } (11)) = F,$$

assures that F is open relative to E.

$(a) \Rightarrow (b)$ Since F is a weak* closed face of E^{**} we can find a tripotent $e \in E^{**}$ satisfying $F = F_e^{E^{**}} = e + \mathcal{B}_{E_0^{**}(e)}$ (cf. Theorem 3.1). Now, by applying that F is open relative to E, we deduce that $G = E \cap F = (e + \mathcal{B}_{E_0^{**}(e)}) \cap E$ is a non-empty norm closed face of $\mathcal{B}_E$ whose weak*-closure in E^{**} is F. Theorem 3.2 implies the existence of a non-zero compact tripotent $u \in E^{**}$ such that $G = (u + \mathcal{B}_{E_0^{**}(u)}) \cap E$. Finally, the implication $(b) \Rightarrow (a)$ gives $u + \mathcal{B}_{E_0^{**}(u)} = \overline{G}^{w^*} = F = F_e^{E^{**}} = e + \mathcal{B}_{E_0^{**}(e)}$, and hence, by Theorem 3.1, $e = u$ is a non-zero compact tripotent. □

A particularization of the implication $(b) \Rightarrow (a)$ in Theorem 3.6, in the case in which $E = A$ is a C*-algebra and F is a proper weak* closed face of the closed unit ball of A^{**} associated with a compact projection in A^{**}, is established by Mori and Ozawa in [45, Lemma 16].

We shall also need the next consequence of the above Theorem 3.6.

Proposition 3.7. *Let $(u_\lambda)_{\lambda\in\Lambda}$ be a decreasing net of compact tripotents in the second dual of a JB*-triple E. Suppose $u \neq 0$ is the infimum of the net $(u_\lambda)_{\lambda\in\Lambda}$ in E^{**}. For each λ in the index set, let $F_{u_\lambda}^E = (u_\lambda + \mathcal{B}_{E_0^{**}(u_\lambda)}) \cap E$ and $F_u^E = (u + \mathcal{B}_{E_0^{**}(u)}) \cap E$ denote the corresponding norm closed faces of $\mathcal{B}_E$ associated with u_λ and u, respectively. Then the identity*

$$F_u^E = \overline{\bigcup_{\lambda\in\Lambda} F_{u_\lambda}^E}^{\|\cdot\|}$$

holds.

Proof. Let us observe that u is compact by [22, Theorem 4.5], and $(u_\lambda)_\lambda$ converges in the weak* topology of E^{**} to u with $u \leqslant u_\lambda$ for every λ.

Since $u \leqslant u_\lambda$ for every λ, the inclusion $F_u^E \supset \overline{\bigcup_{\lambda\in\Lambda} F_{u_\lambda}^E}^{\|\cdot\|}$ always holds. Arguing by contradiction, we assume the existence of $z_0 \in F_u^E \backslash \overline{\bigcup_{\lambda\in\Lambda} F_{u_\lambda}^E}^{\|\cdot\|}$. Since Λ is a directed set and $(u_\lambda)_\lambda$ is a decreasing net, and hence $F_{u_{\lambda_1}}^E \subseteq F_{u_{\lambda_2}}^E$ for every $\lambda_1 \leqslant \lambda_2$, it is not hard to check that $\bigcup_{\lambda\in\Lambda} F_{u_\lambda}^E$ is a convex subset of $S(E)$. It follows that $\overline{\bigcup_{\lambda\in\Lambda} F_{u_\lambda}^E}^{\|\cdot\|}$ is a norm closed convex subset of $S(E)$. By applying the Hahn–Banach theorem we can find a functional $\phi \in E^*$ and a positive δ satisfying

$$\Re\mathrm{e}\phi(z_0) + \delta \leqslant \Re\mathrm{e}\phi(x), \text{ for all } x \in \overline{\bigcup_{\lambda\in\Lambda} F_{u_\lambda}^E}^{\|\cdot\|}. \tag{12}$$

Let $F_{u_\lambda}^{E^{**}}$ and $F_u^{E^{**}}$ be the corresponding weak* closed faces of $\mathcal{B}_{E^{**}}$ associated with u_λ and u, respectively. By repeating the same arguments we gave in the second part of the proof of $(b) \Rightarrow (a)$ in Theorem 3.6 it can be established that

$$F_u^{E^{**}} = \overline{F_u^{E^{**}} \cap E}^{w^*} = \overline{\left(\bigcup_\lambda F_{u_\lambda}^{E^{**}}\right) \cap E}^{w^*} = \overline{\bigcup_\lambda (F_{u_\lambda}^{E^{**}} \cap E)}^{w^*} = \overline{\bigcup_\lambda F_{u_\lambda}^{E^{**}}}^{w^*}. \tag{13}$$

Having in mind that $\phi \in E^*$, we deduce from (12) and from (13) that $\Re\mathrm{e}\phi(z_0) + \delta \leqslant \Re\mathrm{e}\phi(z)$ for all $z \in F_u^{E^{**}}$, which is impossible because $z_0 \in F_u^E \subseteq F_u^{E^{**}}$. □

4. JBW*-triples satisfying the Mazur–Ulam property

We begin this section with a straight consequence of Corollary 2.2 and the facial theory of JB*-triples. Given an element x_0 in a Banach space X, let $\mathcal{T}_{x_0} : X \to X$ denote the translation mapping with respect to the vector x_0 (i.e. $\mathcal{T}_{x_0}(x) = x + x_0$, for all $x \in X$).

Corollary 4.1. *Let M be a JBW*-triple, let Y be a Banach space, and let $\Delta : S(M) \to S(Y)$ be a surjective isometry. Suppose e is a non-zero tripotent in M, and let $F_e^M = e + \mathcal{B}_{M_0(e)} = (e + \mathcal{B}_{M_0^{**}(e)}) \cap M$ denote the proper norm closed face of $\mathcal{B}_M$ associated with e. Then the restriction of Δ to F_e^M is an affine function. Furthermore, there exists an affine isometry T_e from $M_0(e)$ onto a norm closed subspace of Y satisfying $\Delta(\mathcal{T}_e(x)) = T_e(x)$ for all $x \in \mathcal{B}_{M_0(e)}$.*

Proof. The arguments in [23, Proof of Proposition 2.4 and comments after and before Corollary 2.5] show that F_e^M coincide with the intersection of all maximal proper norm closed faces containing it, that is, F_e^M is an intersection face in the sense of [45]. Therefore, by [45, Lemma 8], $\Delta(F_e^M)$ also is an intersection face, and in particular a convex set.

Let us observe that $M_0(e)$ is a JBW*-triple and thus, by Corollary 2.2, $\mathcal{B}_{M_0(e)}$ satisfies the strong Mankiewicz property. The dashed arrow in the diagram

$$\begin{array}{ccc} F_e^M & \xrightarrow{\ \Delta|_{F_e^M}\ } & \Delta(F_e^M) \\ {\scriptstyle \mathcal{T}_{-e}}\downarrow & \overset{\Delta_e}{\nearrow} & \\ \mathcal{B}_{M_0(e)} & & \end{array}$$

defines a surjective isometry Δ_e, which must be affine by Corollary 2.2. Actually, the just quoted corollary proves the existence of a (unique) extension of Δ_e to an affine isometry T_e from $M_0(e)$ onto a norm closed subspace of Y. The desired conclusion follows from the commutativity of the above diagram and the fact that $\mathcal{T}_{-e}$ is an affine mapping. □

Let us refresh our knowledge on the predual of a JBW*-triple with a couple of results due to Friedman and Russo. The first one is a consequence of [30, Proposition 1(a)] and reads as follows:

$$\text{Let } e \text{ be a tripotent in a JB*-triple } E \text{ and let } \varphi \text{ be a functional in } E^* \qquad (14)$$
$$\text{satisfying } \varphi(e) = \|\varphi\|, \text{ then } \varphi = \varphi P_2(e).$$

The second result tells that the extreme points in the closed unit ball of the predual, M_*, of a JBW*-triple M are in one-to-one correspondence with the minimal tripotents in M via the following correspondence:

$$\text{For each } \varphi \in \partial_e(\mathcal{B}_{M_*}) \text{ there exists a unique minimal tripotent } e \in M \qquad (15)$$
$$\text{satisfying } \varphi(x)e = P_2(e)(x) \text{ for all } x \in M,$$

(see [30, Proposition 4]). By analogy with notation in the setting of C*-algebras, the elements in $\partial_e(\mathcal{B}_{M_*})$ are usually called *pure atoms*. For each minimal tripotent in M, we shall write φ_e for the unique pure atom associated with e.

The next lemma is a straight consequence of (14).

Lemma 4.2. *Let φ be a normal functional in the predual of a JBW*-triple M. Suppose $(x_\lambda)_\lambda$ is a net in M converging to a tripotent e in the weak* topology of M. If $(\varphi(x_\lambda))_\lambda \to \|\varphi\|$, then $\varphi = \varphi P_2(e)$. Consequently, if e is a minimal tripotent and $\|\varphi\| = 1$, then we have $\varphi = \varphi_e$.*

The following result is a quantitative version of a useful tool developed by Friedman and Russo in [30, Lemma 1.6]. The original argument in the just quoted paper is combined here with [7, Proposition 2.4].

Lemma 4.3. *Let e be a tripotent in a JB*-triple E, and let x be an element in the closed unit ball of E. Then $\|P_1(e)(x)\| \leqslant 4\sqrt{\|e - P_2(e)(x)\|}$.*

Proof. By [30, Lemma 1.1] the mapping $-S_i(e)(\cdot) = P_2(e) - iP_1(e) - P_0(e) : E \to E$ is an isometric triple isomorphism. Set $x_j = P_j(e)(x)$ for all $j \in \{0, 1, 2\}$, $y = -S_i(e)(x)$ and $z = \frac{1}{2}(x + y)$. Clearly, $\|y\| = \|x\| \leqslant 1$ and $\|z\| \leqslant 1$ as well. We also know that $z = x_2 + \lambda x_1$, with $\lambda = \frac{1-i}{2}$. By the axioms of JB*-triples, $\|\{z, z, z\}\| = \|z\|^3 \leqslant 1$, and by the contractivity of $P_2(e)$ and Peirce arithmetic we deduce that

$$\|\{x_2, x_2, x_2\} + \{x_1, x_1, x_2\}\| = \|P_2(e)\{z, z, z\}\| \leqslant \|\{z, z, z\}\| \leqslant 1.$$

Therefore

$$\begin{aligned}\|e + \{x_1, x_1, e\}\| &\leqslant \|\{e, e, e\} - \{x_2, x_2, x_2\}\| + \|\{x_1, x_1, e\} - \{x_1, x_1, x_2\}\| \\ &\quad + \|\{x_2, x_2, x_2\} + \{x_1, x_1, x_2\}\| \leqslant 4\|e - x_2\| + 1.\end{aligned}$$

Having in mind that $\{x_1, x_1, e\}$ is a positive element in the JB*-algebra $E_2(e)$ and e is its unit (cf. [30, Lemma 1.5]), we get

$$1 + \|\{x_1, x_1, e\}\| = \|e + \{x_1, x_1, e\}\| \leqslant 4\|e - x_2\| + 1.$$

Finally [7, Proposition 2.4] gives $\|x_1\| \leqslant 2\sqrt{\|\{x_1, x_1, e\}\|} \leqslant 4\sqrt{\|e - x_2\|}$. □

Our next result is a generalization of [45, Lemma 15] to the context of JB*-triples.

Lemma 4.4. *Let M be a JBW*-triple, and let u be a compact-G_δ tripotent in M^{**} associated with a norm-one element $a \in M$. Then there exists a decreasing sequence of non-zero tripotents $(e_n)_n$ in M (actually in the JBW*-subtriple of M generated by a) satisfying that for each $x \in F_u^M$ the sequence $\Theta_n(x) := e_n + P_0(e_n)(x)$ converges to x in the norm topology of M.*

Proof. By the assumptions $u = u(a)$ is the support tripotent of a in M^{**}. It is known that the JBW*-triple, W_a, of M generated by the element a is isometrically JBW*-triple isomorphic to a commutative von Neumann algebra W admitting a as a positive generator (cf. [33, Lemma 3.11] and [37]).

By the Borel functional calculus in $W \cong W_a$, we set $e_n = \chi_{(1-\frac{1}{n},1]}(a) \in W_a$ $(n \in \mathbb{N})$. Clearly, $(e_n)_n$ is a decreasing sequence of tripotents in $W_a \subset M$.

We fix $x \in F_u^M$. Let us insert some notation. The symbol $\mathcal{D}$ will stand for the set of all continuous functions $f : [0, 1] \to [0, 1]$ satisfying $f(0) = 0$ and $f(1) = 1$. Let $\mathcal{C}$ be the subset of M given by

$$\mathcal{C} = \{L(f_t(a)^{[\frac{1}{2}]}, f_t(a)^{[\frac{1}{2}]})(x) - f_t(a) : f \in \mathcal{D}\}.$$

We claim that $\mathcal{C}$ is a convex set. To prove the claim let $r = r_{M^{**}}(a)$ denote the range tripotent of a in M^{**}, and let π be a linear isometric triple homomorphism from M^{**} into a JBW*-algebra B such that $\pi(r)$ is a projection in B and $\pi|_{M_2^{**}(r)} : M_2^{**}(r) \to \pi(M^{**})_2(\pi(r))$ is a unital Jordan *-monomorphism (cf. [20, Lemma 3.9] or [7, Lemma 2.3]). It is not hard to see that

$$\begin{aligned}\pi(L(f_t(a)^{[\frac{1}{2}]}, f_t(a)^{[\frac{1}{2}]})(x) - f_t(a)) \\ &= L(f_t(\pi(a))^{[\frac{1}{2}]}, f_t(\pi(a))^{[\frac{1}{2}]})(\pi(x)) - f_t(\pi(a)) \\ &= L(f(\pi(a))^{\frac{1}{2}}, f(\pi(a))^{\frac{1}{2}})(\pi(x)) - f(\pi(a)) \\ &= f(\pi(a)) \circ \pi(x) - f(\pi(a))\end{aligned} \tag{16}$$

where $f(\pi(a))$ denotes the continuous functional calculus of the JBW*-algebra B at the element $\pi(a)$. The above observation implies that

$$\pi(\mathcal{C}) = \{f(\pi(a)) \circ \pi(x) - f(\pi(a)) : f \in \mathcal{D}\},$$

and thus $\pi(\mathcal{C})$ (and hence $\mathcal{C}$) is a convex set because $\mathcal{D}$ is.

It is well known that u lies in the strong*-closure in M^{**} of the set $\mathcal{D}(a) := \{f_t(a) : f \in \mathcal{D}\}$. Thus, we can find a net $(a_\lambda)_\lambda$ in $\mathcal{D}(a)$ such that $(a_\lambda^{[\frac{1}{2}]})_\lambda$ converges to u in the strong* topology of M^{**}. Having in mind that the triple product of M^{**} is jointly strong* continuous on bounded sets, it follows that

$$L(a_\lambda^{[\frac{1}{2}]}, a_\lambda^{[\frac{1}{2}]})(x) - a_\lambda \to L(u, u)(x) - u = 0,$$

in the strong* topology of M^{**}. Therefore $0 \in \overline{\mathcal{C}}^{\text{strong}^*}$. Since the strong* topology of M^{**} is compatible with the duality (M^{**}, M^*), and $\mathcal{C}$ is a convex subset of M, the closure of $\mathcal{C}$ in the strong* topology coincides with its weak*-closure in M^{**} (compare (8)). Furthermore, $0 \in \overline{\mathcal{C}}^{\text{strong}^*} = \overline{\mathcal{C}}^{w^*}$ assures that 0 belongs to the weak closure of $\mathcal{C}$ in M, and the latter with the norm closure of $\mathcal{C}$ in M. We can therefore conclude that

$$0 \text{ lies in the norm closure of the set } \mathcal{C} \text{ in } M. \tag{17}$$

Given an arbitrary $1 > \varepsilon > 0$, by (17), we can find an element $d = f_t(a)$ with $f \in \mathcal{D}$ such that $\|L(d^{[\frac{1}{2}]}, d^{[\frac{1}{2}]})(x) - d\| < \frac{\varepsilon^2}{128}$. Clearly, d and $d^{[\frac{1}{2}]}$ belong to the face F_u^M (cf. [20, Lemma 3.3]).

Let us be more precise. We consider the JB*-subtriple, M_a, of M generated by a, and its identification with a commutative C*-algebra of the form $C_0(\Omega_a)$, where $\Omega_a \subseteq [0, 1]$, with $\Omega_a \cup \{0\}$ compact and correspond to the function $t \mapsto t$ for every $t \in \Omega_a$ (cf. [37, Corollary 1.15] and [30]). Having in mind that $d = f_t(a)$ with $f \in \mathcal{D}$, we can find $t_0 \in [0, 1)$ such that $1 - \frac{\varepsilon^2}{256} \leqslant f(t) \leqslant 1$ for all $t \in [t_0, 1]$. There exits a natural n_0 satisfying $1 - \frac{1}{n_0} > t_0$. Let $g : [0, 1] \to [0, 1]$ be the continuous function defined by

$$g(t) = \begin{cases} f(t), & 0 \leqslant t \leqslant t_0, \\ \text{affine} & t_0 \leqslant t \leqslant 1 - \frac{1}{n_0}, \\ 1, & 1 - \frac{1}{n_0} \leqslant t \leqslant 1, \end{cases}$$

and define $c = g_t(a) \in M_a$. Clearly, $\|c-d\| \leqslant \frac{\varepsilon^2}{256}$ and $e_n \leqslant e_{n_0} \leqslant c \leqslant c^{[\frac{1}{2}]}$ in $W_a = \overline{M_a}^{\sigma(M,M_*)}$ for all $n \geqslant n_0$. Having in mind the isometric triple embedding π, it follows, as in (16), that

$$\begin{aligned}\|L(c^{[\frac{1}{2}]}, c^{[\frac{1}{2}]})(x) - c\| &= \|\pi L(c^{[\frac{1}{2}]}, c^{[\frac{1}{2}]})(x) - \pi(c)\| = \|\pi(c) \circ \pi(x) - \pi(c)\| \\ &\leqslant \|(\pi(c) - \pi(d)) \circ \pi(x) - (\pi(c) - \pi(d))\| + \|\pi(d) \circ \pi(x) - \pi(d)\| \\ &\leqslant 2\|\pi(c) - \pi(d)\| + \|\pi(d) \circ \pi(x) - \pi(d)\| = 2\|c-d\| + \|L(d^{[\frac{1}{2}]}, d^{[\frac{1}{2}]})(x) - d\| < \frac{\varepsilon^2}{64}.\end{aligned}$$

Since $e_n \leqslant c \leqslant c^{[\frac{1}{2}]}$ in W_a for all $n \geqslant n_0$, we deduce that $c = e_n + P_0(e_n)(c)$ and $c^{[\frac{1}{2}]} = e_n + P_0(e_n)(c^{[\frac{1}{2}]})$ for all $n \geqslant n_0$. Therefore, by applying Peirce arithmetic, we have $P_2(e_n)(L(c^{[\frac{1}{2}]}, c^{[\frac{1}{2}]})(x) - c) = P_2(e_n)(x) - e_n$, which combined with the contractiveness of $P_2(e_n)$ gives

$$\|P_2(e_n)(x) - e_n\| < \frac{\varepsilon^2}{64}, \quad \text{for all } n \geqslant n_0.$$

Lemma 4.3 assures that

$$\|P_1(e_n)(x)\| \leqslant 4\sqrt{\|P_2(e_n)(x) - e_n\|} < 4\sqrt{\frac{\varepsilon^2}{64}} < \frac{\varepsilon}{2}.$$

Finally we compute the distance between $\Theta_n(x) = e_n + P_0(e_n)(x)$ and x. In this case, for each $n \geqslant n_0$ we have

$$\begin{aligned}\|\Theta_n(x) - x\| &= \|e_n + P_0(e_n)(x) - x\| = \|e_n - P_2(e_n)(x) - P_1(e_n)(x)\| \\ &\leqslant \|e_n - P_2(e_n)(x)\| + \|P_1(e_n)(x)\| < \frac{\varepsilon^2}{64} + \frac{\varepsilon}{2} < \varepsilon,\end{aligned}$$

for every $n \geqslant n_0$. □

In the next results we shall establish a version of the conclusions in [45, Lemma 15] in the setting of JBW*-triples.

Proposition 4.5. *Let M be a JBW*-triple, let Y be a real Banach space, and let $\Delta : S(M) \to S(Y)$ be a surjective isometry. Then the restriction of Δ to each norm closed (proper) face of $\mathcal{B}_M$ associated with a compact-G_δ tripotent u in M^{**} is an affine function. Furthermore, for each $\psi \in Y^*$, there exist $\phi \in M^*$ and $\gamma \in \mathbb{R}$ such that $\|\phi\|, |\gamma| \leqslant \|\psi\|$, and*

$$\psi\Delta(x) = \Re e\, \phi(x) + \gamma, \quad \text{for all } x \in F_u^M.$$

Proof. Let a be a norm-one element in M, and let $u = u(a)$ be the support tripotent of a in M^{**}. By Lemma 4.4 there exists a sequence of non-zero tripotents $(e_n)_n$ in M satisfying that for each $x \in F_u^M$ the sequence $\Theta_n(x) := e_n + P_0(e_n)(x)$ converges to x in the norm topology of M. Clearly, $\Theta_n(y) \in F_{e_n}^M$ for all $n \in \mathbb{N}$, $y \in \mathcal{B}_M$.

Now, take $x, y \in F_u^M$ and $t \in]0,1[$. Since each Θ_n is an affine map and $\Delta|_{F_{e_n}^M}$ is also affine (see Corollary 4.1), we deduce that

$$\Delta(\Theta_n(tx + (1-t)y)) = \Delta(t\Theta_n(x) + (1-t)\Theta_n(y)) = t\Delta(\Theta_n(x)) + (1-t)\Delta(\Theta_n(y)),$$

for every $n \in \mathbb{N}$. Taking limits in $n \to \infty$, it follows from Lemma 4.4 and from the norm continuity of Δ that $\Delta(tx + (1-t)y) = t\Delta(x) + (1-t)\Delta(y)$, which proves that $\Delta|_{F_u^M}$ is affine.

For the last assertion, let us fix $\psi \in Y^*$. By Corollary 4.1, for each natural n, we can find a linear isometry $T_n : M_0(e_n) \to Y$ and a norm-one element $y_n = \Delta(e_n) \in S(Y)$ such that $\Delta(w) = T_n(w - e_n) + y_n$, for all $w \in F_{e_n}^M$. Let us define $\Re\mathrm{e}\, \phi_n = \psi T_n P_0(e_n) \in (M_{\mathbb{R}})^*$, with $\phi_n \in M^*$ and $\|\phi_n\| \leqslant \|\psi\|$. We can therefore write

$$\psi\Delta(w) = \Re\mathrm{e}\, \phi_n(w) + \gamma_n, \tag{18}$$

for all $w \in F_{e_n}^M$, where $\gamma_n := \psi(y_n)$ and $|\gamma_n| \leqslant \|\psi\|$. Find a subsequence $(\gamma_{\sigma(n)})_n$ converging to some $\gamma \in \mathbb{R}$ with $|\gamma| \leqslant \|\psi\|$. The sequence $(\phi_{\sigma(n)})_n$ is bounded in $\mathcal{B}_{M^*}$. Let $\phi \in M^*$ be a $\sigma(M^*, M)$-cluster point of $(\phi_{\sigma(n)})_n$ with $\|\phi\| \leqslant \|\psi\|$.

Take now an element $x \in F_u^M$. We deduce from Lemma 4.4 and the continuity of Δ that $\psi\Delta(\Theta_{\sigma(n)}(x)) \to \psi\Delta(x)$ in $\mathbb{R}$. It follows from (18) that

$$\psi\Delta\Theta_{\sigma(n)}(x) = \Re\mathrm{e}\, \phi_{\sigma(n)}\Theta_{\sigma(n)}(x) + \gamma_{\sigma(n)},$$

for all natural n. Since $\phi \in M^*$ is a $\sigma(M^*, M)$-cluster point of $(\phi_{\sigma(n)})_n$ and $\|\Theta_{\sigma(n)}(x) - x\| \to 0$, we conclude that $(\Re\mathrm{e}\, \phi_{\sigma(n)}\Theta_{\sigma(n)}(x))_n \to \Re\mathrm{e}\, \phi(x)$. By combining all these assertions we get $\psi\Delta(x) = \Re\mathrm{e}\, \phi(x) + \gamma$. □

By applying Proposition 4.5, we can now deal with general proper norm closed faces in the closed unit ball of a JBW*-triple.

Proposition 4.6. *Let M be a JBW*-triple, let Y be a real Banach space, and let $\Delta : S(M) \to S(Y)$ be a surjective isometry. Then the restriction of Δ to each norm closed proper face F of $\mathcal{B}_M$ is an affine function. Furthermore, for each $\psi \in Y^*$, there exist $\phi \in M^*$ and $\gamma \in \mathbb{R}$ such that $\|\phi\|, |\gamma| \leqslant \|\psi\|$, and*

$$\psi\Delta(x) = \Re e\, \phi(x) + \gamma, \quad \text{for all } x \in F.$$

Proof. Let F be a proper norm closed face of $\mathcal{B}_M$. We know from Theorem 3.2 that $F = F_u^M$, where u is a compact tripotent in M^{**}. Then there exists a net $(u_\lambda)_{\lambda\in\Lambda}$ of compact-G_δ tripotents in M^{**} decreasing in the weak* topology of M^{**} to u (cf. [22]). For each $\lambda \in \Lambda$ we write $F_{u_\lambda}^M = (u_\lambda + \mathcal{B}_{M_0^{**}(u_\lambda)}) \cap M$ for the proper norm closed face associated with u_λ.

Proposition 3.7 assures that $F = F_u^M = \overline{\bigcup_{\lambda\in\Lambda} F_{u_\lambda}^M}^{\|.\|}$. For each $\lambda \in \Lambda$, u_λ is a compact-G_δ tripotent in M, and thus Proposition 4.5 implies that the restriction of Δ to the face $F_{u_\lambda}^M = (u_\lambda + \mathcal{B}_{M_0^{**}(u_\lambda)}) \cap M$ is an affine function.

Now, taking $x, y \in \bigcup_{\lambda\in\Lambda} F_{u_\lambda}^M$ and $t \in]0,1[$, we can find $\lambda_0 \in \Lambda$ such that $x, y \in F_{u_{\lambda_0}}^M$. By applying that $\Delta|_{F_{u_{\lambda_0}}^M}$ is an affine mapping, we deduce that

$$\Delta(tx + (1-t)y) = t\Delta(x) + (1-t)\Delta(y).$$

This proves that $\Delta|_{\bigcup_{\lambda\in\Lambda} F_{u_\lambda}^M}$ is affine. The norm density of $\bigcup_{\lambda\in\Lambda} F_{u_\lambda}^M$ in F and the continuity of Δ can be now applied to deduce that $\Delta|_F$ is affine.

Let us prove the final statement. For this purpose we fix $\psi \in Y^*$. For each $\lambda \in \Lambda$, Proposition 4.5 implies the existence of a functional $\phi_\lambda \in M^*$ and $\gamma_\lambda \in \mathbb{R}$ such that $\|\phi_\lambda\|, |\gamma_\lambda| \leqslant \|\psi\|$, and

$$\psi\Delta(x) = \Re\mathrm{e}\,\phi_\lambda(x) + \gamma_\lambda, \text{ for all } x \in F^M_{u_\lambda}. \tag{19}$$

By the weak* compactness of $\mathcal{B}_{M^*}$ we can find common subnets (ϕ_μ) and (γ_μ) converging to $\phi \in M^*$ and $\gamma \in \mathbb{R}$, respectively. Clearly $\|\phi\| \leqslant \|\psi\|$ and $|\gamma| \leqslant \|\psi\|$. We claim that

$$\psi\Delta(x) = \Re\mathrm{e}\,\phi(x) + \gamma, \text{ for all } x \in F.$$

Namely, for each $\varepsilon > 0$, we can find μ_0 and $x_{\mu_0} \in F^M_{u_{\mu_0}}$ such that $\|x - x_{\mu_0}\| < \frac{\varepsilon}{6\|\psi\|}$, $|\gamma_{\mu_0} - \gamma| < \frac{\varepsilon}{3}$ and $|\phi(x) - \phi_{\mu_0}(x)| < \frac{\varepsilon}{3}$. We therefore conclude from (19) that

$$\begin{aligned}|\psi\Delta(x) - \Re\mathrm{e}\,\phi(x) - \gamma| &\leqslant |\psi\Delta(x) - \psi\Delta(x_{\mu_0})| + |\psi\Delta(x_{\mu_0}) - \Re\mathrm{e}\,\phi_{\mu_0}(x_{\mu_0}) - \gamma_{\mu_0}| \\ &\quad + |\Re\mathrm{e}\,\phi_{\mu_0}(x_{\mu_0}) - \Re\mathrm{e}\,\phi_{\mu_0}(x)| + |\Re\mathrm{e}\,\phi_{\mu_0}(x) - \Re\mathrm{e}\,\phi(x)| + |\gamma_{\mu_0} - \gamma| \\ &\leqslant (\|\psi\| + \|\phi_{\mu_0}\|)\,\|x_{\mu_0} - x\| + 2\frac{\varepsilon}{3} < \varepsilon.\end{aligned}$$

The desired statement follows from the arbitrariness of ε. □

We can now mimic the ideas in [17], [41], [35, Lemma 2.1], and [12, Lemma 2.1] to prove the existence of support functionals for faces.

Lemma 4.7. *Let E be a JB*-triple and let Y be a real Banach space. Suppose $\Delta : S(E) \to S(Y)$ is a surjective isometry. Then for each maximal proper norm closed face F of the closed unit ball of E the set*

$$supp_\Delta(F) := \{\psi \in Y^* : \|\psi\| = 1, \text{ and } \psi^{-1}(\{1\}) \cap \mathcal{B}_Y = \Delta(F)\}$$

is a non-empty weak closed face of $\mathcal{B}_{Y^*}$; in other words, for each minimal tripotent e in E^{**} the set*

$$supp_\Delta(F^E_e) := \{\psi \in Y^* : \|\psi\| = 1, \text{ and } \psi^{-1}(\{1\}) \cap \mathcal{B}_Y = \Delta(F^E_e)\}$$

is a non-empty weak closed face of $\mathcal{B}_{Y^*}$.*

Proof. By applying [10, Lemma 5.1(ii)] (see also [59, Lemma 3.5]) we deduce that the set $\Delta(F)$ is a maximal convex subset of $\mathcal{B}_Y$. It follows from Eidelheit's separation theorem [43, Theorem 2.2.26] that there exists a norm-one functional $\varphi \in Y^*$ such that $\varphi^{-1}(\{1\}) \cap \mathcal{B}_Y = \Delta(F)$ (compare the proof of [60, Lemma 3.3] or [12, Lemma 2.1]). □

For the sake of brevity and conciseness, we introduce the following notation.

Definition 4.8. Let E be a JB*-triple. We shall say that E satisfies property $(\mathcal{P})$ if for each minimal tripotent e in E^{**} and each complete tripotent u in E (that is $u \in \partial_e(\mathcal{B}_E)$), there exists another minimal tripotent w in E^{**} satisfying $w \perp e$ and $u = w + P_0(w)(u)$.

Another tool needed in the proof of our main result is established in the next result.

Proposition 4.9. *Let M be a JBW*-triple satisfying property $(\mathcal{P})$. Let $\varphi_e \in \partial_e(\mathcal{B}_{M^*})$ denote the unique pure atom associated with a minimal tripotent e in M^{**}. Suppose $\Delta : S(M) \to S(Y)$ is a surjective isometry from the unit sphere of M onto the unit sphere of a real Banach space Y. Then for each ψ in $\mathrm{supp}_\Delta(F_e^M)$ we have $\psi\Delta(u) = \Re e\varphi_e(u)$ for every non-zero tripotent u in $\partial_e(\mathcal{B}_M)$.*

Proof. Let us fix a minimal tripotent e in M^{**}, $u \in \partial_e(\mathcal{B}_M)$, and ψ in $\mathrm{supp}_\Delta(F_e^M)$. By the hypotheses on M we can find another minimal tripotent w in M^{**} satisfying $w \perp e$ and $u = w + P_0(w)(u)$. Proposition 4.6 implies the existence of $\lambda_w \in \mathbb{R}$ and $\varphi \in M^*$ such that $\|\varphi\| \leqslant 1$ and $\psi\Delta(x) = \lambda_w + \Re e\varphi(x)$ for every $x \in F_w^M$.

Since minimal tripotents in M^{**} are compact, we are in a position to apply the non-commutative generalization of Urysohn's lemma established in [26, Proposition 3.7]. By this result, we can find orthogonal norm-one elements $a_0, b_0 \in M$ such that $a_0 = e + P_0(e)(a_0)$ and $b_0 = w + P_0(w)(b_0)$, that is, $a_0 \in F_e^M$ and $b_0 \in F_w^M$. Since, by orthogonality, $\pm a_0 + b_0 \in (\pm F_e^M) \cap F_w^M$, we deduce from Lemma 4.7 and [45, Lemma 8] that

$$\pm 1 = \psi\Delta(\pm a_0 + b_0) = \lambda_w \pm \Re e\varphi(a_0) + \Re e\varphi(b_0) = \pm\Re e\varphi(a_0) + \psi\Delta(b_0),$$

which implies that $\psi\Delta(b_0) = 0$ and $\Re e\varphi(a_0) = 1$. In the above argument, a_0 can be arbitrarily replaced with any element c in the face F_e^M for which there exists $b_0 \in F_w^M$ with $c \perp b_0$. Arguing as in the proof of [1, Lemma 2.7] we can find a net (a_λ) in $M_2(r_M(a_0)) \subseteq M$ such that $a_\lambda = e + P_0(e)(a_\lambda)$ (equivalently $a_\lambda \in F_e^M$), and $(a_\lambda) \to e$ in the weak* topology of M^{**}. Since $a_\lambda \perp b_0$ for every λ, it follows from the above arguments that $\Re e\varphi(a_\lambda) = 1 = \|\varphi\|$ for all λ. Lemma 4.2 assures that $\varphi = \varphi_e$.

We have therefore shown that $\psi\Delta(x) = \lambda_w + \Re e\varphi_e(x)$ for every $x \in F_w^M$. Since $b_0 \in F_w^M$ and $b_0 \perp e$, we get $0 = \psi\Delta(b_0) = \lambda_w + \Re e\varphi_e(b_0)$, which implies that $\lambda_w = 0$, and $\psi\Delta(u) = \Re e\varphi_e(u)$ as desired. □

Let us recall another result proved by Mori and Ozawa in [45, Lemma 18]. Let A be a unital C*-algebra, and let p be a minimal projection in A^{**}. Then for each $a \in F_p^A = (p + A_0^{**}(p)) \cap \mathcal{B}_A$ and each $\varepsilon > 0$ there exist unitary elements $u_1, \ldots, u_m$ in F_p^A and $t_1, \ldots, t_m \in [0, 1]$ satisfying $\sum_{j=1}^m t_j = 1$ and $\|a - \sum_{j=1}^m t_j u_j\| < \varepsilon$. We shall establish a version of these results in the setting of JBW*-triples.

Lemma 4.10. *Let e be a non-zero compact tripotent in the second dual of a JBW*-triple M. Let a be an element in the norm closed face F_e^M. Then a can be written as the average of two extreme points of $\mathcal{B}_M$ belonging to the face F_e^M.*

Proof. Let $a \in F_e^M = (\{e\}_\prime)_\prime$. By [52, Theorem 5] a can be written in the form $a = \frac{u_1+u_2}{2}$, where $u_1, u_2 \in \partial_e(\mathcal{B}_M)$. Let us pick an arbitrary $\varphi \in \{e\}_\prime \subset S(M^*)$. Since $1 = \varphi(a) = \frac{\varphi(u_1)+\varphi(u_2)}{2}$, $\varphi(u_1) = \varphi(u_2) = 1$, it follows from the arbitrariness of $\varphi \in \{e\}_\prime \subset S(M^*)$ that $u_1, u_2 \in (\{e\}_\prime)_\prime = F_e^M$ as desired. □

Now, by combining Proposition 4.9 and Lemma 4.10 we get the following result.

Corollary 4.11. *Let M be a JBW*-triple satisfying property $(\mathcal{P})$. Let $\varphi_e \in \partial_e(\mathcal{B}_{M^*})$ denote the unique pure atom associated with a minimal tripotent e in M^{**}. Suppose $\Delta : S(M) \to S(Y)$ is a surjective isometry from the unit sphere of M onto the unit sphere of a real*

Banach space Y. Then for each ψ in $supp_{\Delta}(F_e^M)$ we have $\psi\Delta(x) = \Re e\varphi_e(x)$ for every $x \in S(M)$.

Proof. Let e be a minimal tripotent in M^{**}, $\varphi_e \in \partial_e(\mathcal{B}_{M^*})$ the unique pure atom associated with e, and let ψ be an element in $\mathrm{supp}_{\Delta}(F_e^E)$. Proposition 4.9 implies that $\psi\Delta(u) = \Re e\varphi_e(u)$ for every $u \in \partial_e(\mathcal{B}_M)$.

Let us fix $x \in S(M)$. By applying Zorn's lemma there exists a minimal tripotent $v \in M^{**}$ such that $x \in F_v^M = (v + M_0^{**}(v)) \cap \mathcal{B}_M$. Lemma 4.10 and Proposition 4.6 give the desired statement. □

Before approaching our main goal we shall establish a technical result. Let e and v be two tripotents in a JB*-triple E. According to the standard notation, we shall say that e and v are *collinear* if $e \in E_1(v)$ and $v \in E_1(e)$.

Friedman and Russo proved in [30, Proposition 6] that every JBW*-triple M satisfies a pre-variant of the so-called *'extreme ray property'*, that is, if u and e are tripotents in M and e is minimal, then $P_2(u)(e)$ is a scalar multiple of another minimal tripotent in M. Actually, the same conclusion holds when M is a JB*-triple E because minimal (respectively, complete) tripotents in E are minimal (respectively, complete) in E^{**}.

Lemma 4.12. *Let u be a complete tripotent in a JB*-triple E, that is, $u \in \partial_e(\mathcal{B}_E)$. Then every minimal tripotent e in E decomposes as a linear combination of the form $e = \lambda v + \mu w$, where v and w are two collinear tripotents in E which are minimal or zero, $v \in E_2(u)$, $w \in E_1(u)$, and $\lambda, \mu \in \mathbb{R}_0^+$ satisfy $\lambda^2 + \mu^2 = 1$.*

Proof. By applying that u is a complete tripotent (i.e. $E_0(u) = \{0\}$), we deduce that $e = e_2 + e_1$ where $e_k = P_k(u)(e) \in E_k(u)$ for $k = 1, 2$.

Since e is minimal, the pre-variant version of the extreme ray property (see [30, Proposition 6]) implies that $P_2(u)(e) = e_2 = \lambda v$, where v is a minimal tripotent in E and $\lambda \geqslant 0$. We observe that $e = e_1$ is a minimal tripotent whenever λ vanishes. We can thus assume that $\lambda > 0$. It is clear that e_1 belongs to $E_1(v)$. By the identity

$$\alpha e = \{e, e_2, e\} = \{e_2, e_2, e_2\} + 2\{e_2, e_2, e_1\} + \{e_1, e_2, e_1\} \text{ (with } \alpha \in \mathbb{C}),$$

combined with Peirce rules and the completeness of u, we obtain that $\{e_1, e_2, e_1\} = 0$, $\alpha\lambda v = \{e_2, e_2, e_2\} = |\lambda|^2\lambda v$ (and thus $\alpha = |\lambda|^2$), and $|\lambda|^2 e_1 = 2\{e_2, e_2, e_1\} = 2|\lambda|^2\{v, v, e_1\}$, which proves that $e_1 \in E_1(v)$.

Now, having in mind the identity

$$\gamma e = \{e, e_1, e\} = \{e_1, e_1, e_1\} + 2\{e_2, e_1, e_1\} + \{e_2, e_1, e_2\} \text{ (with } \gamma \in \mathbb{C}),$$

we get $\{e_1, e_1, e_1\} = \gamma e_1$, and hence, by the triple functional calculus, e_1 is a multiple of a tripotent in E. That is, $e_1 = \mu w$, where w is a tripotent in E, $\mu \geqslant 0$, and $|\mu|^2 = \gamma$. We may reduce to the case in which $\mu \neq 0$. We further know that $2\lambda\mu^2\{v, w, w\} = 2\{e_2, e_1, e_1\} = \gamma e_2 = \gamma\lambda v$, witnessing that $2\{v, w, w\} = v$. Therefore v and w are collinear tripotents in E. The *triple system analyzer* (see [13, Proposition 2.1 and Lemma in page 306]) gives the desired statement. □

The best known examples of JBW*-triples are given by the so-called Cartan factors. There are six types of Cartan factors defined as follows:

Cartan factor of type 1: the complex Banach space $B(H, K)$, of all bounded linear operators between two complex Hilbert spaces, H and K, whose triple product is given by (1).

Given a conjugation, j, on a complex Hilbert space, H, we can define a linear involution on $B(H)$ defined by $x \mapsto x^t := jx^*j$.

Cartan factor of type 2: the subtriple of $B(H)$ formed by the skew-symmetric operators for the involution t.

Cartan factor of type 3: the subtriple of $B(H)$ formed by the t-symmetric operators.

Cartan factor of type 4 or *spin:* a complex Banach space X admitting a complete inner product $(.|.)$ and a conjugation $x \mapsto \overline{x}$, for which the norm of X is given by

$$\|x\|^2 = (x|x) + \sqrt{(x|x)^2 - |(x|\overline{x})|^2}.$$

Cartan factors of types 5 and 6 (also called *exceptional* Cartan factors) consist of matrices over the eight dimensional complex algebra of Cayley numbers; the type 6 consists of all 3 by 3 self-adjoint matrices and has a natural Jordan algebra structure, and the type 5 is the subtriple consisting of all 1 by 2 matrices.

Our next goal is to show the connection between rank and property $(\mathcal{P})$ in the case of Cartan factors.

Proposition 4.13. *Every Cartan factor of rank bigger than or equal to three satisfies property* $(\mathcal{P})$.

Proof. Let M be a Cartan factor of rank $\geqslant 3$. Let e be a minimal tripotent in M^{**}, and let u be a complete tripotent in M (that is $u \in \partial_e(\mathcal{B}_M)$). By (3) the element $P_2(u)\{e, e, u\}$ is positive in $M_2^{**}(u)$. On the other hand, by Peirce arithmetic, $\{e, e, u\} = P_2(e)(u) + \frac{1}{2}P_1(e)(u)$, where $P_2(e)(u) \in M_2^{**}(e) = \mathbb{C}e$, and hence $P_2(e)(u) = \delta e$ for some $\delta \in \mathbb{C}$. We observe that u is also complete in M^{**}.

By [13, Corollary 2.2] the subtriple $M_1^{**}(e)$ has rank at most two. Therefore, $\frac{1}{2}P_1(e)(u) = \lambda_1 v_1 + \lambda_2 v_2$, where $\lambda_1, \lambda_2 \in \mathbb{R}_0^+$ and v_1 and v_2 are mutually orthogonal minimal tripotents in $M_1^{**}(e)$ or zero with $v_j \neq 0$ if $\lambda_j > 0$.

We shall distinguish several cases:

Case 1: $\lambda_1, \lambda_2 > 0$. By the triple system analyzer [13, Proposition 2.1(iii)], v_1, v_2 are minimal tripotents in M^{**} and the triplet (v_1, e, v_2) is a prequadrangle in the terminology of [13]. Therefore the three points e, v_1, v_2 are contained in a rank two JBW*-subtriple of M^{**}.

Case 2: $\lambda_1 > 0$, $\lambda_2 = 0$. By the triple system analyzer [13, Proposition 2.1(i) and (ii)], v_1 is a minimal tripotent in M^{**} and e, v_1 are collinear; or v_1 is a minimal tripotent in $M_1^{**}(e)$ but not minimal in M^{**} and there exists a minimal tripotent $\widetilde{e}$ in M^{**} such that $(e, v_1, \widetilde{e})$ is a trangle in the terminology of [13]. That is, the three points e, v_1, v_2 are contained in a rank one or two JBW*-subtriple of M^{**} (cf. [13, LEMMA in page 306]).

Case 3: $\lambda_1 = \lambda_2 = 0$, or equivalently, $P_1(e)(u) = 0$. In this case $\{e, e, u\} = \delta e$ is contained in a rank one JBW*-subtriple of M^{**}.

We shall first assume that $e \in M_2^{**}(u) \cup M_1^{**}(u)$.

It follows from the above cases that the set $\{P_2(e)(u), \frac{1}{2}P_1(e)(u)\}$ is contained in a JBW*-subtriple F of M^{**} of rank at most two. Since we have assumed that $e \in M_2^{**}(u) \cup M_1^{**}(u)$, the element $\{e, e, u\}$ coincides with $P_2(u)\{e, e, u\}$ and it is a positive element in the JBW*-algebra $M_2^{**}(u)$. We also know that $\{e, e, u\} = P_2(e)(u) + \frac{1}{2}P_1(e)(u) \in F$. The range tripotents of $\{e, e, u\}$ in F and in $M_2^{**}(u)$ give the same element which will be denoted by r. Clearly, r is a projection in $M_2^{**}(u)$ which must be minimal or the sum of two mutually orthogonal minimal projections in $M_2^{**}(u)$.

Now, having in mind that M, and hence $M_2(u)$ and $M_2^{**}(u)$, all have rank $\geqslant 3$ (cf. [38, Proposition 5.8]), we deduce the existence of a minimal projection w in $M_2^{**}(u)$ which is orthogonal to r (and hence to $\{e, e, u\}$). By applying that r is the range tripotent of $\{e, e, u\}$, and the fact that

$$M_2^{**}(u) = (M_2^{**}(u))_2(r) \oplus (M_2^{**}(u))_1(r) \oplus (M_2^{**}(u))_0(r),$$

where $(M_2^{**}(u))_0(r) = (M_2^{**}(u))_2(u-r)$ and $(M_2^{**}(u))_1(r) = (M_2^{**}(u))_1(u-r)$, we deduce from the Jordan identity that

$$\begin{aligned}\{e, e, u-r\} + \{e, e, r\} &= \{e, e, u\} = \{r, \{e, e, u\}, r\} \\ &= -\{e, e, \{r, u, r\}\} + 2\{\{e, e, r\}, u, r\} = -\{e, e, r\} + 2\{\{e, e, r\}, r, r\}.\end{aligned}$$

Therefore

$$\{e, e, u-r\} = -2\{e, e, r\} + 2\{\{e, e, r\}, r, r\} \in (M_2^{**}(u))_2(r) \oplus (M_2^{**}(u))_1(r),$$

which implies that $0 = P_2(u-r)\{e, e, u-r\} = \{P_2(u-r)(e), P_2(u-r)(e), u-r\} + \{P_1(u-r)(e), P_1(u-r)(e), u-r\}$. [30, Lemma 1.5], and the comments preceding it, now assure that $P_2(u-r)(e) = P_1(u-r)(e) = 0$, and thus $e = P_0(u-r)(e) \perp u-r$. Since w is a minimal projection in $M_2^{**}(u)$ with $w \leqslant u-r$, it follows that w is a minimal tripotent in M^{**} with $w \perp e$.

We consider now the general case in which $e \in M_2^{**}(u) \oplus M_1^{**}(u)$. By Lemma 4.12, e decomposes as a linear combination of the form $e = \lambda v_2 + \mu v_1$, where v_2 and v_1 are two collinear tripotents in M^{**} which are minimal or zero, $v_2 \in M_2^{**}(u)$, $v_1 \in M_1^{**}(u)$, and $\lambda, \mu \in \mathbb{R}_0^+$ satisfy $|\lambda|^2 + |\mu|^2 = 1$.

If $\lambda \neq 0$, we apply the first part of this proof to the element v_2 to find a minimal tripotent $w \in M^{**}$ such that $w \leqslant u$ and $w \perp v_2$. We shall next show that $w \perp v_1$. Indeed, the element $v_2 + w$ is a tripotent in $M_2^{**}(u)$, and the corresponding Peirce projections commute, that is, $P_j(v_2)P_k(w) = P_k(w)P_j(v_2)$ for all $j, k \in \{0, 1, 2\}$ (cf. [33, (1.10)]). The element $P_1(w)(v_1) = P_1(w)P_1(v_2)(v_1) = P_1(v_2)P_1(w)(v_1) \in M_1^{**}(w) \cap M_1^{**}(v_2) \subset M_2^{**}(w + v_2) \subset M_2^{**}(u)$. However, $P_1(w)(v_1) = P_1(w)P_1(u)(v_1) = P_1(u)P_1(w)(v_1) \in M_1^{**}(u)$, and thus $P_1(w)(v_1) = 0$. Moreover, $P_2(w)(v_1) \in M_2^{**}(w) \subset M_2^{**}(u)$, and $P_2(w)(v_1) = P_2(w)P_1(u)(v_1) = P_1(u)P_2(w)(v_1) \in M_1^{**}(u)$, which shows that $P_2(w)(v_1) = 0$, and consequently $v_1 = P_0(w)(v_1) \perp w$, as desired.

Finally, if $\lambda = 0$ we get $e = \mu v_1 \in M_1^{**}(u)$ and we finish by applying the first part of this proof. □

We can now prove that most of JBW*-triples satisfy the Mazur–Ulam property.

Theorem 4.14. *Let M be a JBW*-triple with rank bigger than or equal to three. Then, every surjective isometry from the unit sphere of M onto the unit sphere of a real Banach space Y admits a unique extension to a surjective real linear isometry from M onto Y.*

Proof. Let $\Delta : S(M) \to S(Y)$ be a surjective isometry from the unit sphere of M onto the unit sphere of a Banach space Y. If we show that M satisfies property $(\mathcal{P})$, then it follows from Corollary 4.11 that, for each minimal tripotent e in M^{**} and each ψ in $\text{supp}_\Delta(F_e^M)$ we have $\psi\Delta(x) = \Re e\varphi_e(x)$ for every $x \in S(M)$, where $\varphi_e \in \partial_e(\mathcal{B}_{M^*})$ is the unique pure atom associated with a minimal tripotent e in M^{**}. Let $\mathcal{U}_{\min}(M^{**})$ denote the set of all minimal tripotents in M^{**}. For each $e \in \mathcal{U}_{\min}(M^{**})$, we pick $\psi_e \in \text{supp}_\Delta(F_e^M)$ (cf. Lemma 4.7). We consider the families $\{\varphi_e\}_{e\in\mathcal{U}_{\min}(M^{**})}$ and $\{\psi_e\}_{e\in\mathcal{U}_{\min}(M^{**})}$. Since the set $\{\varphi_e\}_{e\in\mathcal{U}_{\min}(M^{**})}$ is norming on M, and $\psi_e\Delta(x) = \Re e\varphi_e(x)$ for every $x \in S(M)$, the conclusion of the theorem will follow from [45, Lemma 6].

We shall finally prove that M satisfies property $(\mathcal{P})$. Let e be a minimal tripotent in M^{**}, and let u be a complete tripotent in M (that is $u \in \partial_e(\mathcal{B}_M)$). By considering the atomic decomposition of M^{**}, we can write $M^{**} = \mathcal{A} \oplus \mathcal{N}$, as the ℓ_∞-direct (orthogonal) sum of its atomic and non-atomic parts. The atomic part of M^{**}, $\mathcal{A}$, is precisely the weak*-closure of the linear span of all minimal tripotents in M^{**} (see [30, Theorem 2]). It is also known that $\mathcal{A} = \bigoplus_{j\in\Lambda} C_j$, where $\{C_j : j \in \Lambda\}$ is a family of Cartan factors (cf. [34, Corollary 1.8] and [31, Proposition 2]). It is further known that if $\iota_M : M \to M^{**}$ and $\pi_{at} : M^{**} \to \mathcal{A}$ denote the canonical inclusion of M into its bidual and the projection of M^{**} onto $\mathcal{A}$, respectively, then the mapping $\Phi = \pi_{at} \circ \iota_M$ is an isometric triple isomorphism with weak* dense image.

The element $\Phi(u) = (u_j)_{j\in\Lambda}$ is a complete tripotent in $\mathcal{A}$, and e belongs to a unique C_{j_0}. If $\sharp\Lambda \geqslant 2$, we can find $j_1 \neq j_0$ in Λ, and in this case, any minimal tripotent $w \in C_{j_1}$ with $w \leqslant u_{j_1}$ satisfies $w \leqslant u_{j_1} \leqslant \Phi(u) \leqslant u$ and $w \perp e$. We can therefore reduce to the case in which Λ is a single element, and hence $\mathcal{A}$ is a Cartan factor. In the latter case the desired conclusion follows from Proposition 4.13 because the rank of M is smaller than or equal to the rank of $\mathcal{A}$. □

We shall finish this note by exploring the Mazur–Ulam property in the case of JBW*-triples of rank one. Suppose M is a JBW*-triple of rank one. It is known that M must be reflexive (see, for example, [6, Proposition 4.5]). In particular M must coincide with a rank one Cartan factor, and hence it must be isometrically isomorphic to a complex Hilbert space (cf. [38, Table 1 in page 210]). It is due to Ding that every surjective isometry between the unit spheres of two Hilbert spaces admits a unique extension to a surjective real linear isometry between the spaces (see [16]). Suppose now that $\Delta : S(H) \to S(Y)$ is a surjective isometry, where H is a Hilbert space and Y is a Banach space. Given $x, y \in S(Y)$ there exist $a, b \in S(H)$ satisfying $\Delta(a) = x$ and $\Delta(b) = y$. Having in mind that the set $\{b\}$ is a maximal norm closed face of $\mathcal{B}_H$, we deduce from [45, Lemma 8] that $\Delta(-b) = -y$. Therefore,

$$\begin{aligned}\|x+y\|^2 + \|x-y\|^2 &= \|\Delta(a)+\Delta(b)\|^2 + \|\Delta(a)-\Delta(b)\|^2 \\ &= \|a+b\|^2 + \|a-b\|^2 = 2\|a\|^2 + 2\|b\|^2 = 4.\end{aligned}$$

It follows from [14, Theorem 2.1] that Y is a Hilbert space. The previously quoted result of Ding in [16] gives the next proposition.

Proposition 4.15. *Every Hilbert space satisfies the Mazur–Ulam property. Every rank one JBW*-triple satisfies the Mazur–Ulam property.* □

Remark 4.16. We have shown in the first part of the proof of Theorem 4.14 that if M is a JBW*-triple satisfying property $(\mathcal{P})$, every surjective isometry from the unit sphere of M onto the unit sphere of a Banach space Y admits a unique extension to a surjective real linear isometry from M onto Y. The proof of Theorem 4.14 actually shows that every JBW*-triple with rank bigger than or equal to three satisfies property $(\mathcal{P})$. We shall see next that there are other examples of JBW*-triples satisfying property $(\mathcal{P})$.

Suppose M is a JBW*-triple such that the atomic part of M^{**} is not a Cartan factor of rank one or two (in particular when M is not a factor). We claim that M satisfies property $(\mathcal{P})$. Indeed, let $\mathcal{A}$ denote the atomic part of M^{**}. If $\mathcal{A}$ is a Cartan factor of rank bigger than or equal to three the proof of Theorem 4.14 shows that M satisfies property $(\mathcal{P})$. If $\mathcal{A}$ is an ℓ_∞-sum of at least two Cartan factors we have also seen in the proof of Theorem 4.14 that M satisfies property $(\mathcal{P})$.

Let E be a JB*-triple. If the atomic part, $\mathcal{A}$, of E^{**} is a Cartan factor of rank 2, or even more generally, a finite rank JBW*-triple, then $\mathcal{A}$ is a reflexive Banach space (cf. [6, Proposition 4.5] and [11, Theorem 6]). As we have already commented, E embeds isometrically into $\mathcal{A}$, and thus E is reflexive and $E = E^{**} = \mathcal{A}$.

In particular, if the atomic part of E^{**} reduces to a rank one Cartan factor, then E is a rank one JBW*-triple and satisfies the Mazur–Ulam property by Proposition 4.15.

Summarizing, if M is not a rank two Cartan factor, then M satisfies the Mazur–Ulam property.

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On the Mazur–Ulam property for the space of Hilbert-space-valued continuous functions

María Cueto-Avellaneda, Antonio M. Peralta *

Departamento de Análisis Matemático, Facultad de Ciencias, Universidad de Granada, 18071 Granada, Spain

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Let K be a compact Hausdorff space and let H be a real or complex Hilbert space with $\dim(H_{\mathbb{R}}) \geq 2$. We prove that the space $C(K, H)$, of all H-valued continuous functions on K, equipped with the supremum norm, satisfies the Mazur–Ulam property, that is, if Y is any real Banach space, every surjective isometry Δ from the unit sphere of $C(K, H)$ onto the unit sphere of Y admits a unique extension to a surjective real linear isometry from $C(K, H)$ onto Y. Our strategy relies on the structure of $C(K)$-module of $C(K, H)$ and several results in JB*-triple theory. For this purpose we determine the facial structure of the closed unit ball of a real JB*-triple and its dual space.

1. Introduction

A Banach space X satisfies the *Mazur–Ulam property* if for any Banach space Y, every surjective isometry $\Delta : S(X) \to S(Y)$ admits an extension to a surjective real linear isometry from X onto Y, where $S(X)$ and $S(Y)$ denote the unit spheres of X and Y, respectively. This property, which was first named by L. Cheng and Y. Dong in [13], is equivalent to say that Tingley's problem (see [54]) admits a positive solution for every surjective isometry from $S(X)$ onto the unit sphere of any other Banach space.

Behind their simple statements, Tingley's problem and the Mazur–Ulam property are hard problems which remain unsolved even for surjective isometries between the unit spheres of a couple of two dimensional normed spaces (the reader is invited to take a look to the recent papers [55] and [10], where this particular case is treated). Positive solutions to Tingley's problem have been found for surjective isometries $\Delta : S(X) \to S(Y)$ when X and Y are von Neumann algebras [29], compact C*-algebras [48], atomic JBW*-triples [28], spaces of trace class operators [24], spaces of p-Schatten von Neumann operators with $1 \leq p \leq \infty$ [25], preduals of von Neumann algebras and the self-adjoint parts of two von Neumann algebras [43]. The surveys [18,56], and [46] are appropriate references to the reader in order to check the state-of-the-art of this problem.

* Corresponding author.
E-mail addresses: mcueto@ugr.es (M. Cueto-Avellaneda), aperalta@ugr.es (A.M. Peralta).

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Apart from a wide list of classical Banach spaces satisfying the Mazur–Ulam property (cf. [56,46]), new achievements prove that this property is satisfied by commutative von Neumann algebras [15], unital complex C*-algebras and real von Neumann algebras [44], and more recently, JBW*-triples with rank one or rank bigger than or equal to three [6]. The latest two mentioned references naturally lead us to consider the Mazur–Ulam property on the space $C(K, H)$ of all continuous functions from a compact Hausdorff space K into a real or complex Hilbert space H. This space is not, in general, a C*-algebra nor a JBW*-triple because it is neither a dual Banach space. However, it possesses a motivating structure of Hilbert $C(K)$-module, and consequently, a structure of JB*-triple, where $C(K)$ stands for the space $C(K, \mathbb{C})$.

The main conclusions in this note prove that for every real Hilbert space $\mathcal{H}$ with $\dim(\mathcal{H}) \geq 2$, and every compact Hausdorff space K, the real Banach space $C(K, \mathcal{H})$ satisfies the Mazur–Ulam property (see Corollaries 5.7 and 5.8). We previously establish in Theorem 5.6 that the same property holds when we consider the complex Banach space of all continuous functions with values in a complex Hilbert space H, showing that each surjective isometry $\Delta : S(C(K, H)) \to S(Y)$, where Y is a real Banach space, extends to a surjective real linear isometry from $C(K, H)$ onto Y. Let us note that R. Liu proved in [40, Corollary 6] that the space $C(K, \mathbb{R})$, of all real continuous functions on K, satisfies the Mazur–Ulam property.

Our strategy relies on the natural JB*-triple structure associated with the space $C(K, H)$. This structure provides the key tools and results to pursue our goals. We would like to vindicate the usefulness of techniques in JB*-triple theory to solve natural problems in functional analysis. In subsection 1.1 we gather a basic background, definitions and results on JB*-triple theory required in this note.

The paper is structured in five sections, this first one serves as introduction and the fifth and last section contains the main conclusions. In section 2 we try to illustrate the fact that the unit sphere of $C(K, H)$ is metrically distinguishable from the unit sphere of a unital C*-algebra and from the unit sphere of a real von Neumann algebra. More precisely, we prove in Theorem 2.1 that for any complex Hilbert space H with dimension bigger than or equal to 2, there exists no surjective isometry from the unit sphere of $C(K, H)$ onto the unit sphere of a C*-algebra. Moreover, for a real Hilbert space $\mathcal{H}$ with $\dim(\mathcal{H}) = 3$ or $\dim(\mathcal{H}) \geq 5$, there exists no surjective isometry from the unit sphere of $C(K, \mathcal{H})$ onto the unit sphere of a real von Neumann algebra (cf. Theorem 2.2).

One of the most successful tools applied in recent studies on the Mazur–Ulam property is derived from an accurate knowledge of the facial structure of the closed unit ball of one of the involved Banach spaces. Weak*-closed faces of the closed unit ball of a JBW*-triple and norm-closed faces of the closed unit ball of its predual are well known thanks to the studies due to C.M. Edwards and G.T. Rüttimann [20]. Norm-closed faces of the closed unit ball of a general JB*-triple and weak*-closed faces of the closed unit ball of its dual space were completely determined in [19,27]. Edwards and Rüttimann enlarged our knowledge with the description of the weak*-closed faces of the closed unit ball of a real JBW*-triple, and of the norm-closed faces of the closed unit ball of its predual (cf. [22]). Until now the structure of norm-closed faces of the closed unit ball of a general real JB*-triple remains unexplored; we shall devote section 3 to culminate the study of the facial structure of the closed unit ball of a real JB*-triple and its dual space.

On the other hand, it is irrefutable that the extremal structure of Banach spaces has become a focus of attention, either as main topic or as a helpful tool for understanding the underlying geometry. In the setting of unital C*-algebras the Russo–Dye theorem asserts that the closure of the convex hull of the unitary elements is the closed unit ball. M. Mori and N. Ozawa prove in [44, Theorem 2] that every Banach space X such that the closed convex hull of the extreme points of its closed unit ball has non-empty interior satisfies that every convex body $\mathcal{K} \subset X$ has the strong Mankiewicz property, that is, every surjective isometry Δ from $\mathcal{K}$ onto an arbitrary convex subset L in a normed space Y is affine. This is a key ingredient to prove that unital C*-algebras, real von Neumann algebras and JBW*-triples of rank 1 or bigger than or equal to three satisfy the Mazur–Ulam property [44,6].

In Section 4 we revisit some results in [49,12,45] to establish a Krein–Milman type theorem showing that for any compact Hausdorff space K, and every real Hilbert space $\mathcal{H}$ with $\dim(\mathcal{H}) \geq 2$, the closed unit ball of $C(K, \mathcal{H})$ coincides with the closed convex hull of its extreme points (cf. Proposition 4.5). We also prove that, for each real Hilbert space $\mathcal{H}$ with dimension bigger than or equal to 2, every element in a maximal norm-closed proper face of the closed unit ball of $C(K, \mathcal{H})$ can be approximated in norm by a finite convex combination of elements in that face which are also extreme points of the closed unit ball of $C(K, \mathcal{H})$ (see Corollaries 4.7 and 4.8). We further prove that certain real JB*-subtriples of $C(K, H)$ satisfy the strong Mankiewicz property (cf. Propositions 4.9 and 4.10).

1.1. Basic background in JB-triple theory*

We recall that, according to [37], a JB*-triple is a complex Banach space X admitting a continuous triple product $\{.,.,.\} : X \times X \times X \to X$, which is symmetric and linear in the outer variables, conjugate linear in the middle one, and satisfies the following axioms:

(a) $L(a, b)L(x, y) = L(x, y)L(a, b) + L(L(a, b)x, y) - L(x, L(b, a)y)$, for all a, b, x, y in X, where $L(a, b)$ is the operator on X given by $L(a, b)x = \{a, b, x\}$;
(b) For all $a \in X$, $L(a, a)$ is a hermitian operator with non-negative spectrum;
(c) $\|\{a, a, a\}\| = \|a\|^3$, for all $a \in X$.

In order to provide some examples, let us consider two complex Hilbert spaces H_1 and H_2, and let $B(H_1, H_2)$ denote the Banach space of all bounded linear operators from H_1 into H_2. The space $B(H_1, H_2)$ is a JB*-triple with respect to the triple product defined by

$$\{x, y, z\} = \frac{1}{2}(xy^*z + zy^*x). \tag{1}$$

In particular, C*-algebras are JB*-triples when equipped with the above triple product. The Jordan structures enlarge the class of JB*-triples if we consider, for instance, the JB*-algebras in the sense employed in [32, §3.8] under the triple product

$$\{x, y, z\} = (x \circ y^*) \circ z + (z \circ y^*) \circ x - (x \circ z) \circ y^*.$$

Some basic facts and known results about JB*-triples will be needed in the development of this paper. Kaup's *Banach-Stone theorem* states that a linear bijection between JB*-triples is an isometry if and only if it is a triple isomorphism (cf. [37, Proposition 5.5]).

Let X be a JB*-triple. An element e in X is said to be a *tripotent* if $\{e, e, e\} = e$. In particular, the partial isometries of a C*-algebra A are precisely its tripotent elements if A is regarded as a JB*-triple respect to the triple product in (1). For each tripotent $e \in X$, there exists an algebraic decomposition of X, known as the *Peirce decomposition* associated with e, which involves the eigenspaces of the operator $L(e, e)$. Namely,

$$X = X_2(e) \oplus X_1(e) \oplus X_0(e),$$

where $X_i(e) = \{x \in X : \{e, e, x\} = \frac{i}{2}x\}$ for each $i = 0, 1, 2$. It is easy to see that every Peirce subspace $X_i(e)$ is a JB*-subtriple of X.

The so-called Peirce arithmetic assures that $\{X_i(e), X_j(e), X_k(e)\} \subseteq X_{i-j+k}(e)$ if $i - j + k \in \{0, 1, 2\}$, and $\{X_i(e), X_j(e), X_k(e)\} = \{0\}$ otherwise, and

$$\{X_2(e), X_0(e), X\} = \{X_0(e), X_2(e), X\} = \{0\}.$$

The projection $P_k(e)$ of X onto $X_k(e)$ is called the Peirce k-projection. It is known that Peirce projections are contractive (cf. [31, Corollary 1.2]) and satisfy that $P_2(e) = Q(e)^2$, $P_1(e) = 2(L(e, e) - Q(e)^2)$, and

$P_0(e) = \mathrm{Id}_X - 2L(e,e) + Q(e)^2$, where $Q(e) : X \to X$ is the conjugate linear map defined by $Q(e)(x) = \{e, x, e\}$. A tripotent e in X is called *unitary* (respectively, *complete* or *maximal*) if $X_2(e) = X$ (respectively, $X_0(e) = \{0\}$).

It is worth remarking that the Peirce-2 subspace $X_2(e)$ is a unital JB*-algebra with unit e, product $x \circ_e y := \{x, e, y\}$ and involution $x^{*_e} := \{e, x, e\}$. Actually, Kaup's Banach-Stone theorem [37, Proposition 5.5] implies that the triple product in $X_2(e)$ is uniquely determined by the identity

$$\{a,b,c\} = (a \circ_e b^{*_e}) \circ_e c + (c \circ_e b^{*_e}) \circ_e a - (a \circ_e c) \circ_e b^{*_e}, \qquad (\forall a,b,c \in X_2(e)).$$

Elements a, b in a JB*-triple X are said to be *orthogonal* (written $a \perp b$) if $L(a,b) = 0$. It is known that $a \perp b \Leftrightarrow \{a,a,b\} = 0 \Leftrightarrow \{b,b,a\} = 0 \Leftrightarrow b \perp a$. Let e be a tripotent in X. It follows from Peirce arithmetic that $a \perp b$ for every $a \in X_2(e)$ and every $b \in X_0(e)$. Let e and u be tripotents in X, then

$$u \perp e \Leftrightarrow u \pm e \text{ are tripotents}$$

(cf. [35, Lemma 3.6]).

We shall consider the following natural partial order on the set $\mathcal{U}(X)$, of all tripotents in a JB*-triple X, defined by $u \leq e$ if $e - u$ is a tripotent in X with $e - u \perp u$.

Complete tripotents play a fundamental role in the extremal structure of the closed unit ball of a JB*-triple X. Indeed, the extreme points of the closed unit ball of X coincide with the complete tripotents in X (cf. [9, Lemma 4.1] and [38, Proposition 3.5]).

A JBW*-triple is a JB*-triple which is also a dual Banach space (with a unique isometric predual [5]). It is known that the second dual of a JB*-triple is a JBW*-triple (compare [17]). An extension of Sakai's theorem assures that the triple product of every JBW*-triple is separately weak*-continuous (cf. [5] or [33]).

As we commented before, throughout this paper we shall exhibit some new spaces satisfying a Krein–Milman type theorem. The starting point is the celebrated Russo-Dye theorem (see [50]). This result naturally involves the concept of *unitary* element. Let A be a unital C*-algebra. An element $u \in A$ is a *unitary* if it is invertible with $u^{-1} = u^*$, i.e., $uu^* = u^*u = 1$. Similarly, an element u in a unital JB*-algebra B is called unitary if u is Jordan invertible in B and its (unique) Jordan inverse in B coincides with u^* (compare [32, §3.2]). Every unital C*-algebra A can be regarded as a unital JB*-algebra equipped with the Jordan product given by $a \circ b := \frac{1}{2}(ab + ba)$, and every JB*-algebra is included in the class of JB* triples. Fortunately, the three definitions of unitary elements given in previous paragraphs coincide for elements in A.

Finally, we shall make a brief incursion into the theory of real JB*-triples. A *real JB*-triple* is, by definition, a real closed subtriple of a JB*-triple (see [35]). Every JB*-triple is a real JB*-triple when it is regarded as a real Banach space. As in the case of real C*-algebras, real JB*-triples can be obtained as *real forms* of JB*-triples. More concretely, given a real JB*-triple E, there exists a unique (complex) JB*-triple structure on its algebraic complexification $X = E \oplus iE$, and a conjugation (i.e. a conjugate linear isometry of period 2) τ on X such that

$$E = X^\tau = \{z \in X : \tau(z) = z\},$$

(see [35]). Consequently, every real C*-algebra is a real JB*-triple with respect to the product given in (1), and the Banach space $B(H_1, H_2)$ of all bounded real linear operators between two real, complex, or quaternionic Hilbert spaces also is a real JB*-triple with the same triple product.

As in the complex case, an element e in a real JB*-triple E is said to be a *tripotent* if $\{e, e, e\} = e$. We shall also write $\mathcal{U}(E)$ for the set of all tripotents in E. It is known that an element $e \in E$ is a tripotent in E if and only if it is a tripotent in the complexification of E. Each tripotent e in E induces a Peirce decomposition of E in similar terms to those we commented in page 877 with the exception that $E_2(e)$ is not, in general, a JB*-algebra but a real JB*-algebra (i.e. a closed *-invariant real subalgebra of a (complex) JB*-algebra). Unitary and complete tripotents are defined analogously to the complex setting. Furthermore, the extreme points of $\mathcal{B}_E$ coincide with the complete tripotents in the real JB*-triple E (cf. [35, Lemma 3.3]).

Along this note, given a convex set L we denote by $\partial_e(L)$ the set of all extreme points in L. The symbol $\mathcal{B}_X$ will stand for the closed unit ball of a Banach space X.

2. Hilbert $C(K)$-modules whose unit spheres are not isometrically isomorphic to the unit sphere of a C*-algebra

One of the aims of this paper is to exhibit the usefulness of a good knowledge on real linear isometries between JB*-triples to study the Mazur–Ulam property on new classes of Banach spaces of continuous functions. We should convince the reader that the recent outstanding achievements obtained by Mori and Ozawa for unital C*-algebras in [44] are not enough to conclude that some natural spaces of vector-valued continuous functions satisfy the Mazur–Ulam property.

Suppose H is a complex Hilbert space whose inner product is denoted by $\langle .|. \rangle$, and let K be a compact Hausdorff space. It is clear that $C(K,H)$ is a $C(K)$-bimodule with respect to the module products defined by $(af)(t) = (fa)(t) = f(t)a(t)$ for all $t \in K$, $a \in C(K,H)$ and $f \in C(K)$. We consider a sesquilinear $C(K)$-valued mapping on $C(K,H)$ given by the following assignment

$$\langle .|. \rangle : C(K,H) \times C(K,H) \to C(K), \quad \langle a|b \rangle(t) := \langle a(t)|b(t) \rangle \quad (t \in K, a, b \in C(K,H)).$$

It is easy to check that this sesquilinear mapping satisfies the following properties:

(1) $\langle a|b \rangle = \langle b|a \rangle^*$;
(2) $\langle fa|b \rangle = f\langle a|b \rangle$;
(3) $\langle a|a \rangle \geq 0$ and $\langle a|a \rangle = 0$ if and only if $a = 0$,

for all $a, b \in C(K,H)$, $f \in C(K)$. We can therefore conclude that $C(K,H)$ is a Hilbert $C(K)$-module in the sense introduced by I. Kaplansky in [36], and consequently, $C(K,H)$ is a JB*-triple with respect to the triple product defined by

$$\{a, b, c\} = \frac{1}{2}\langle a|b \rangle c + \frac{1}{2}\langle c|b \rangle a, \quad (a, b, c \in C(K,H)), \tag{2}$$

(see [34, Theorem 1.4]). By a little abuse of notation, the symbol $\langle \cdot|\cdot \rangle$ will indistinctly stand for the inner product of H and the $C(K)$-valued inner product of $C(K,H)$.

Throughout this note $\mathbb{K}$ will stand for $\mathbb{R}$ or $\mathbb{C}$. Given $\eta \in H$ and a mapping $f : K \to \mathbb{K}$, the symbol $\eta \otimes f$ will denote the mapping from K to H defined by $\eta \otimes f(t) = f(t)\eta$ $(t \in K)$. We note that $\eta \otimes f$ is continuous whenever $f \in C(K)$. We will use the juxtaposition for the pointwise product between maps whenever such a product makes sense.

Let us consider vector-valued continuous functions on a compact Hausdorff space K with values in a Banach space X. It is known that if $e \in \partial_e(\mathcal{B}_{C(K,X)})$, then $\|e(t)\| = 1$ (that is, $e(t) \in S(X)$) for all $t \in K$ (cf. [2, Lemma 1.4]). The reciprocal implication is not true in general, however, if X is a strictly convex Banach space, then we have

$$e \in \partial_e(\mathcal{B}_{C(K,X)}) \text{ if and only if } \|e(t)\| = 1 \text{ for all } t \in K, \tag{3}$$

(cf. [2, Remark 1.5]).

Theorem 2.1. *Let K be a compact Hausdorff space, and let H be a complex Hilbert space with dimension bigger than or equal to 2. Then there exists no surjective isometry from the unit sphere of $C(K,H)$ onto the unit sphere of a C*-algebra.*

Proof. Arguing by contradiction we assume the existence of a C*-algebra A and a surjective isometry $\Delta : S(A) \to S(C(K,H))$. Since A and $C(K,H)$ are JB*-triples, it follows from [24, Corollary 2.5(b) and comments prior to it] that $\Delta(\partial_e(\mathcal{B}_A)) = \partial_e(\mathcal{B}_{C(K,H)})$. The non-emptiness of the set $\partial_e(\mathcal{B}_{C(K,H)})$ assures that $\partial_e(\mathcal{B}_A) \neq \emptyset$. It is well known that in such a case A must be unital (cf. [51, Proposition 1.6.1]). A recent result by Mori and Ozawa shows that every unital C*-algebra satisfies the Mazur–Ulam property (see [44, Theorem 1]). Therefore Δ extends to a surjective real linear isometry $T : A \to C(K,H)$.

Now, since $T : A \to C(K,H)$ is a surjective real linear isometry, A is a C*-algebra and $C(K,H)$ is a JB*-triple, we can apply [16, Theorem 3.1] or [26, Theorem 3.2 and Corollary 3.4] (see also [29, Theorem 3.1]) to deduce that T is a triple isomorphism when A and $C(K,H)$ are equipped with the triple products given in (1) and (2), respectively. Let $\mathbf{1}$ denote the unit element in A. Clearly, $A_2(\mathbf{1}) = A$. Since $\Delta(\mathbf{1})$ must be a unitary in $C(K,H)$, in particular $\Delta(\mathbf{1})(t) \in S(H)$ for every $t \in K$ (cf. (3)). Let us fix $t_0 \in K$. By applying that $\dim(H) \geq 2$, we can find $\eta \in S(H)$ satisfying $\langle \eta | \Delta(\mathbf{1})(t_0)\rangle = 0$. We consider the element $a = \eta \otimes 1$, where 1 is the unit element in $C(K)$. In this case $\{\Delta(\mathbf{1}), \Delta(\mathbf{1}), a\}(t_0) = \frac{1}{2}\eta \neq a(t_0)$, and thus $a \notin C(K,H)_2(\Delta(\mathbf{1}))$. $\square$

Let us observe another point of view to deal with $C(K,H)$ as a real JB*-triple. Indeed, suppose H is a complex Hilbert space with inner product $\langle .|.\rangle$. We can regard H as a real Hilbert space with its underlying real space and the inner product defined by $(a|b) = \Re\mathrm{e}\langle a|b\rangle$ $(a,b \in H)$, the latter real Hilbert space will be denoted by $H_{\mathbb{R}}$. In general, the inner product of a real Hilbert space will be denoted by $(a|b)$. Let K be a compact Hausdorff space. Let us observe that the norms of $C(K,H)$ and $C(K,H_{\mathbb{R}})$ both coincide. We can therefore reduce to the case in which H is a real Hilbert space. We shall always consider $C(K,H)$ as a real JB*-triple with respect to the triple product

$$\{a,b,c\} := \frac{1}{2}(a|b)c + \frac{1}{2}(c|b)a.$$

For each x_0 in H, we shall write x_0^* for the unique functional in H^* defined by $x_0^*(x) = \langle x|x_0\rangle$ $(x \in H)$. Given $t_0 \in K$, $\delta_{t_0} : C(K,H) \to H$ will stand for the bounded linear operator defined by $\delta_{t_0}(a) = a(t_0)$ $(a \in C(K,H))$. Finally, let $x_0^* \otimes \delta_{t_0}$ denote the functional on $C(K,H)$ given by $(x_0^* \otimes \delta_{t_0})(a) := x_0^*(a(t_0))$, for each $a \in C(K,H)$.

Theorem 2.2. *Let K be a compact Hausdorff space, and let $\mathcal{H}$ be a real Hilbert space with $\dim(\mathcal{H}) = 3$ or $\dim(\mathcal{H}) \geq 5$. Then there exists no surjective isometry from the unit sphere of $C(K,\mathcal{H})$ onto the unit sphere of a real von Neumann algebra.*

Proof. It is known that $\partial_e(\mathcal{B}_{C(K,\mathcal{H})^*}) = \{z^* \otimes \delta_t : t \in K,\ z \in S(\mathcal{H})\}$ (see [53, Lemma 1.7 in page 197]). It is easy to check that the norm-closed linear span of $\partial_e(\mathcal{B}_{C(K,\mathcal{H})^*})$ in $C(K,\mathcal{H})^*$ is precisely the space $\bigoplus_{t\in K}^{\ell_1}\mathcal{H}$. In particular, the atomic part of the real JBW*-triple $C(K,\mathcal{H})^{**}$, in the sense employed and studied in [47] and [26], coincides with the direct sum $\bigoplus_{t\in K}^{\ell_\infty}\mathcal{H}$. In other words, every real or complex Cartan factor in the atomic part of $C(K,\mathcal{H})^{**}$ coincides with the real Hilbert space $\mathcal{H}$ equipped with the triple product $\{a,b,c\} := \frac{1}{2}(a|b)c + \frac{1}{2}(c|b)a$.

We shall argue by contradiction. Suppose A is a real von Neumann algebra and $\Delta : S(A) \to S(C(K,\mathcal{H}))$ is a surjective isometry. Applying [44, Theorem 1(2)] we deduce the existence of a surjective real linear isometry $T : A \to C(K, \mathcal{H})$. The bitransposed mapping $T^{**} : A^{**} \to C(K, \mathcal{H})^{**}$ also is a surjective real linear isometry. It is known that the atomic part of A^{**} coincides with a direct sum of the form $\bigoplus_{\alpha\in\Lambda}^{\ell_\infty} B(\mathcal{H}_\alpha)$, where each $\mathcal{H}_\alpha$ is a Hilbert space over $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$ (see [14, Lemma 6.2] or [39, §5.3]). Arguing as in the proof of [26, Theorem 3.2] we deduce that T^{**} maps the atomic part of A^{**} onto the atomic part of $C(K, \mathcal{H})^{**}$, furthermore, each factor in the atomic part of A^{**} is isometrically mapped by T^{**} onto a factor in the atomic part of $C(K, \mathcal{H})^{**}$. That is, for each $\alpha \in \Lambda$ the restriction $T^{**}|_{B(\mathcal{H}_\alpha)} : B(\mathcal{H}_\alpha) \to \mathcal{H}$ is a surjective isometry. Since in $\mathcal{H}$, equipped with the product $\{a, b, c\} := \frac{1}{2}(a|b)c + \frac{1}{2}(c|b)a$, the rank is one (i.e. we cannot find two non-zero orthogonal tripotents), and $T^{**}|_{B(\mathcal{H}_\alpha)}$ preserves orthogonal tripotents (see [35, Theorem 4.8] or [26, Proposition 2.9]) it follows that $\mathcal{H}_\alpha$ must be a one dimensional Hilbert space over $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$, which is impossible because $T^{**}|_{B(\mathcal{H}_\alpha)} : B(\mathcal{H}_\alpha) \to \mathcal{H}$ is a surjective real linear isometry and $\dim(\mathcal{H}) \in \{3\} \cup \{4 + n : n \in \mathbb{N}\}$. □

3. Facial structure of real JB*-triples revisited

This section is devoted to explore the facial structure of the closed unit ball of a real JB*-triple and its dual space. It is an interesting question by its own right, and moreover, its application will be crucial later in the study of the Mazur–Ulam property.

The facial structure of the closed unit ball of a JBW*-triple and its predual was completely determined by C.M. Edwards and G.T. Rüttimann in [20]. In order to review the results, we shall recall some terminology. Let X be a real or complex Banach space with dual space X^*. Suppose F and G are two subsets of $\mathcal{B}_X$ and $\mathcal{B}_{X^*}$, respectively. Then we set

$$F' = F'^{,X^*} = \{a \in \mathcal{B}_{X^*} : a(x) = 1 \ \forall x \in F\},$$
$$G_\prime = G_{\prime,X} = \{x \in \mathcal{B}_X : a(x) = 1 \ \forall a \in G\}.$$

Clearly, F' is a weak*-closed face of $\mathcal{B}_{X^*}$ and $G_\prime$ is a norm-closed face of $\mathcal{B}_X$. We say that F is a *norm-semi-exposed face* of $\mathcal{B}_X$ (respectively, G is a *weak*-semi-exposed face* of $\mathcal{B}_{X^*}$) if $F = (F')_\prime$ (respectively, $G = (G_\prime)'$). It is known that the mappings $F \mapsto F'$ and $G \mapsto G_\prime$ are anti-order isomorphisms between the complete lattices $\mathcal{S}_n(\mathcal{B}_X)$, of norm-semi-exposed faces of $\mathcal{B}_X$, and $\mathcal{S}_{w^*}(\mathcal{B}_{X^*})$, of weak*-semi-exposed faces of $\mathcal{B}_{X^*}$, and are inverses of each other.

Recall that a partially ordered set $\mathcal{P}$ is called a *complete lattice* if, for any subset $\mathcal{S} \subseteq \mathcal{P}$, the supremum and the infimum of $\mathcal{S}$ exist in $\mathcal{P}$. It is shown in [20, Corollary 4.3] that, for each JBW*-triple M, the set $\tilde{\mathcal{U}}(M)$, of all tripotents in M with a largest element adjoined, is a complete lattice with respect to the ordering defined in page 878.

Let M be a JBW*-triple. The main achievements in [20] prove that every weak*-closed face of $\mathcal{B}_M$ is weak*-semi-exposed; furthermore, the mapping

$$u \mapsto (\{u\}_\prime)' = u + \mathcal{B}_{M_0(u)} \tag{4}$$

is an anti-order isomorphism from the complete lattice $\tilde{\mathcal{U}}(M)$ onto the complete lattice $\mathcal{F}_{w^*}(\mathcal{B}_M)$ of weak*-closed faces of $\mathcal{B}_M$ (cf. [20, Theorem 4.6]). Concerning the facial structure of M_*, the same authors proved in [20, Theorem 4.4] that every norm-closed face of $\mathcal{B}_{M_*}$ is norm-semi-exposed, and the mapping

$$u \mapsto \{u\}_\prime \tag{5}$$

is an order isomorphism from $\tilde{\mathcal{U}}(M)$ onto the complete lattice $\mathcal{F}_n(\mathcal{B}_{M_*})$ of norm-closed faces of $\mathcal{B}_{M_*}$.

In 1992, C.A. Akemann and G.K. Pedersen studied the norm-closed faces of the closed unit ball of a C*-algebra A and the weak*-closed faces of $\mathcal{B}_{A^*}$ (see [1]). We had to wait until 2010 to have a description of the norm-closed faces of the closed unit ball of a JB*-triple ([19]). A JB*-triple X might contain no non-trivial tripotents, while the set of all tripotents in X^{**} is too big to be in one-to-one correspondence with the set of norm-closed faces of $\mathcal{B}_X$. The appropriate set is the set of all compact tripotents in X^{**}. We continue refreshing the notion of compactness.

Let a be a norm-one element in a JB*-triple X, and let X_a denote the JB*-subtriple generated by a, that is, the closed subspace generated by all odd powers $a^{[2n+1]}$, where $a^{[1]} = a$, $a^{[3]} = \{a, a, a\}$, and $a^{[2n+1]} = \{a, a, a^{[2n-1]}\}$ $(n \geq 2)$. A well known result proves the existence of an isometric triple isomorphism $\Psi : X_a \to C_0(L)$ satisfying $\Psi(a)(s) = s$, for all s in L (compare [37, 1.15]), where $C_0(L)$ is the abelian C*-algebra of all complex-valued continuous functions on L vanishing at 0, L is a subset of $(0, \|a\|]$ satisfying that $\|a\| \in L$, and $L \cup \{0\}$ is compact. If $f : L \cup \{0\} \to \mathbb{C}$ is a continuous function vanishing at 0, the *triple functional calculus* of f at the element a is the unique element $f_t(a) \in X_a$, defined by $f_t(a) = \Psi^{-1}(f)$. We can define this way $a^{[\frac{1}{2n+1}]} := (r_n)_t(a)$, where $r_n(s) = s^{\frac{1}{2n+1}}$ $(s \in L)$ and $n \in \mathbb{N}$.

When X is regarded as a JB*-subtriple of X^{**}, the triple functional calculus $f \mapsto f_t(a)$ admits an extension, denoted by the same symbol, from $C_0(L)$ to the commutative W*-algebra W generated by $C_0(L)$, onto the JBW*-subtriple X_a^{**} of X^{**} generated by a. Observe that the sequences $(a^{[\frac{1}{2n-1}]})_n$ and $(a^{[2n-1]})_n$ in $C_0(L)$ converge in the weak*-topology of $C_0(L)^{**}$ to the characteristic functions χ_L and $\chi_{\{1\}}$ of the sets L and $\{1\}$, respectively. The corresponding limits define two tripotents in X_a^{**} which are called the *range tripotent* and the *support tripotent* of a, respectively. These tripotents will be denoted by $r(a)$ and $u(a)$, respectively.

For each functional φ in the predual, M_*, of a JBW*-triple M there exists a unique tripotent $s(\varphi)$ (called the *support tripotent* of φ) such that $\varphi = \varphi P_2(s(\varphi))$ and $\varphi|_{M_2(s(\varphi))}$ is a faithful normal positive functional on the JBW*-algebra $M_2(s(\varphi))$ (cf. [31, Proposition 2]).

We are interested in a special property satisfied by the support tripotent. Suppose a is a norm-one element in a JB*-triple X. Since $a = u(a) + (a - u(a))$ with $u(a) \perp (a - u(a))$ in X^{**}, it follows from [31, Proposition 1] that $\{u(a)\}_{\prime, X^*} \subseteq \{a\}^{\prime, X^*}$. However, if $\phi \in X^*$ satisfies $\|\phi\| = 1 = \phi(a)$, we deduce from the definition of the support tripotent of ϕ in X^{**} that $P_2(s(\phi))(a) = s(\phi)$, and hence $a = s(\phi) + P_0(s(\phi))(a)$ in X^{**} (cf. [31, Lemma 1.6]). We therefore conclude that $u(a) \geq s(\phi)$ in X^{**}, and thus $\phi(u(a)) = 1$, witnessing that $\{u(a)\}_{\prime, X^*} = \{a\}^{\prime, X^*}$ and consequently,

$$\left(\{a\}^{\prime, X^*}\right)^{\prime, X^{**}} = \left(\{u(a)\}_{\prime, X^*}\right)^{\prime, X^{**}}. \tag{6}$$

A tripotent u in the JBW*-triple X^{**} is said to be *compact-G_δ* if u coincides with the support tripotent of a norm-one element in X. The tripotent u is said to be *compact* if $u = 0$ or there exists a decreasing net of compact-G_δ tripotents in X^{**} whose infimum is u (compare [21, §4]). Henceforth we shall write $\tilde{\mathcal{U}}_c(X^{**})$ for the set of all compact tripotents in X^{**} with a largest element adjoined. Having these notions in mind we can understand the main result in [19]: Every norm-closed face of $\mathcal{B}_X$ is norm-semi-exposed and the mapping

$$u \mapsto (\{u\}_\prime)_\prime = (u + \mathcal{B}_{X_0^{**}(u)}) \cap X$$

is an anti-order isomorphism from $\tilde{\mathcal{U}}_c(X^{**})$ onto the complete lattice $\mathcal{F}_n(\mathcal{B}_X)$ of norm-closed faces of $\mathcal{B}_X$ (cf. [19, Corollaries 3.11 and 3.12]). The study is completed in [27], where it is shown that the mapping

$$u \mapsto \{u\}_\prime$$

is an order isomorphism from $\tilde{\mathcal{U}}_c(X^{**})$ onto the complete lattice $\mathcal{F}_{w^*}(\mathcal{B}_{X^*})$ of weak*-closed faces of $\mathcal{B}_{X^*}$.

In the setting of real JBW*-triples, C.M. Edwards and G.T. Rüttimann proved in [22] that the conclusions in (4) and (5) hold when M is a real JBW*-triple. However, as long as we know, the facial structure of the closed unit ball of a real JB*-triple remains unexplored. We shall try to fill this gap.

We begin with a very basic result. Let us consider a complex Banach space X equipped with a conjugation $\tau : X \to X$ (i.e., a conjugate linear isometry of period 2), and set $E = X^\tau = \{x \in X : \tau(x) = x\}$. The mapping $P : X \to X$ defined by $P(x) = \frac{1}{2}(x + \tau(x))$ $(x \in X)$, is a contractive real linear projection whose image is E. The mapping $\tau^\sharp : X^* \to X^*$, $\tau^\sharp(\varphi)(x) := \overline{\varphi(\tau(x))}$ $(x \in X, \varphi \in X^*)$ is a conjugation on X^*, and the correspondence $\varphi \mapsto \varphi|_E$ defines a surjective real linear isometry from $(X^*)^{\tau^\sharp}$ onto E^*. We can similarly define a conjugation $\tau^{\sharp\sharp}$ on X^{**} satisfying that $(X^{**})^{\tau^{\sharp\sharp}}$ is isometrically isomorphic to E^{**}. In particular, the weak*-topology of E^{**} coincides with the restriction to E^{**} of the weak*-topology of X^{**}. Clearly, if a functional φ in X^* is a $\tau^\sharp$-symmetric (equivalently, $\varphi \in E^*$), its support tripotent in X^{**} is $\tau^{\sharp\sharp}$-symmetric and hence lies in E^{**}.

Let F be a subset of $\mathcal{B}_E$. We set $\mathfrak{F} := P^{-1}(F) \cap \mathcal{B}_X$. It is standard to check that

$$F \in \mathcal{F}_n(\mathcal{B}_E) \Leftrightarrow \mathfrak{F} \in \mathcal{F}_n(\mathcal{B}_X). \tag{7}$$

Henceforth we assume that X is a complex JB*-triple, and thus E is a real JB*-triple. Proposition 5.5 in [37] assures that τ is a conjugate linear triple automorphism. It is not hard to see that $\mathcal{U}(E) = \mathcal{U}(X)^\tau = \{e \in \mathcal{U}(X) : \tau(e) = e\}$, and what is even more interesting $\mathcal{U}(E^{**}) = \mathcal{U}(X^{**})^{\tau^{\sharp\sharp}} = \{e \in \mathcal{U}(X^{**}) : \tau^{\sharp\sharp}(e) = e\}$. It follows from [22, Lemma 3.4(ii)] that the set $\tilde{\mathcal{U}}(E^{**})$ of all tripotents in E^{**} with a largest element adjoined is a sub-complete lattice of $\tilde{\mathcal{U}}(X^{**})$.

If a is a norm-one element in E (that is, an element in X with $\tau(a) = a$). Since $\tau(a^{[\frac{1}{2n-1}]}) = \tau(a)^{[\frac{1}{2n-1}]} = a^{[\frac{1}{2n-1}]}$ and $\tau(a^{[2n-1]}) = a^{[2n-1]}$, for all natural n, E^{**} is weak*-closed in X^{**}, and $\tau^{\sharp\sharp}$ is weak*-continuous, we deduce that $\tau^{\sharp\sharp}(r(a)) = r(a)$ and $\tau^{\sharp\sharp}(u(a)) = u(a)$, that is, the range and support tripotents of a in X^{**} are $\tau^{\sharp\sharp}$-symmetric elements in X^{**}, and thus they both are tripotents in E^{**}, called *range* and *support* tripotents of a in E^{**}, respectively. Combining (6) with the previous conclusions we get

$$\{a\}^{\prime,E^*} = \{u(a)\}_{\prime,E^*}, \text{ and } \left(\{a\}^{\prime,E^*}\right)^{\prime,E^{**}} = \left(\{u(a)\}_{\prime,E^*}\right)^{\prime,E^{**}}. \tag{8}$$

Thanks to the above facts, the notion of compact tripotent fits well in the setting of real JB*-triples. A tripotent u in E^{**} will be called *compact-G_δ* if u coincides with the support tripotent of a norm-one element in E. The tripotent u is called *compact* if $u = 0$ or there exists a decreasing net of compact-G_δ tripotents in E^{**} whose infimum is u. As in the complex setting, we shall write $\tilde{\mathcal{U}}_c(E^{**})$ for the set of all compact tripotents in E^{**} with a largest element adjoined.

It is absolutely clear that every compact(-G_δ) tripotent in E^{**} is a $\tau^{\sharp\sharp}$-symmetric compact(-G_δ) tripotent in X^{**}. The reciprocal is not obvious. To prove it we shall extend a result of Edwards and Rüttimann which affirms that a tripotent $u \in X^{**}$ is compact if and only if the face $\{u\}_{\prime,X^*}$ is weak*-semi-exposed in $\mathcal{B}_{X^*}$ (cf. [21, Theorem 4.2]). We recall first a lemma borrowed from [22].

Lemma 3.1. [22, Lemma 3.6] *Let τ be a conjugation on a JB*-triple X, and let $E = X^\tau$. Then for each tripotent $u \in E^{**} = (X^{**})^{\tau^{\sharp\sharp}}$ we have*

$$\{u\}_{\prime,E^*} = \left(\{u\}_{\prime,X^*}\right)^{\tau^\sharp} = \{u\}_{\prime,X^*} \cap E^* = \{u\}_{\prime,X^*}.$$

We establish next a real version of [21, Theorem 4.2].

Proposition 3.2. *Let τ be a conjugation on a JB*-triple X, and let $E = X^\tau$. A tripotent u in the real JBW*-triple E^{**} is compact if and only if $\{u\}_{\prime,E^*}$ is weak*-semi-exposed in $\mathcal{B}_{E^*}$.*

Proof. Suppose u is a non-trivial compact tripotent in E^{**}. According to what we commented before this proposition, u is a $\tau^{\sharp\sharp}$-symmetric compact tripotent in X^{**}. Theorem 4.2 in [21] implies that $\{u\}_{\prime,X^*}$ is weak*-semi-exposed in $\mathcal{B}_{X^*}$, that is

$$\left((\{u\}_{\prime,X^*})_{\prime,X}\right)^{\prime,X^*} = \{u\}_{\prime,X^*}.$$

It follows from Lemma 3.1 that

$$\{u\}_{\prime,E^*} = (\{u\}_{\prime,X^*})^{\tau^\sharp} = \{u\}_{\prime,X^*} \cap E^* = \{u\}_{\prime,X^*}.$$

We shall next show that the non-empty set $(\{u\}_{\prime,X^*})_{\prime,X} \subseteq S(X)$ is τ-symmetric. Take $x \in (\{u\}_{\prime,X^*})_{\prime,X}$ and $\varphi \in \{u\}_{\prime,X^*} = \{u\}_{\prime,E^*}$. Since $\tau^\sharp(\varphi) = \varphi \in \{u\}_{\prime,X^*}$ we have

$$1 = \tau^\sharp(\varphi)(x) = \overline{\varphi(\tau(x))} = \varphi(\tau(x)),$$

witnessing that $\tau(x) \in (\{u\}_{\prime,X^*})_{\prime,X}$, and thus $\tau\left((\{u\}_{\prime,X^*})_{\prime,X}\right) = (\{u\}_{\prime,X^*})_{\prime,X}$. We have also shown that for each $x \in (\{u\}_{\prime,X^*})_{\prime,X}$ and $\varphi \in \{u\}_{\prime,X^*}$ we have

$$1 = \varphi\left(\frac{x+\tau(x)}{2}\right) \leq \left\|\frac{x+\tau(x)}{2}\right\| \leq 1.$$

It follows from the above that $(\{u\}_{\prime,X^*})_{\prime,X} \cap E$ is a non-empty subset of $S(E)$ which coincides with $(\{u\}_{\prime,E^*})_{\prime,E}$ and

$$\left((\{u\}_{\prime,E^*})_{\prime,E}\right)^{\prime,E^*} = \{u\}_{\prime,E^*}, \tag{9}$$

which guarantees that $\{u\}_{\prime,E^*}$ is weak*-semi-exposed in $\mathcal{B}_{E^*}$.

Suppose now that $\{u\}_{\prime,E^*}$ is weak*-semi-exposed in $\mathcal{B}_{E^*}$, that is, the equality in (9) holds. We can literally follow the arguments contained in the proof of [21, Theorem 4.2]. The details are included here for completeness reasons. It follows from the equality in (9) that the convex set $(\{u\}_{\prime,E^*})_{\prime,E}$ is a non-empty norm-closed face of $\mathcal{B}_E$. For each $a \in (\{u\}_{\prime,E^*})_{\prime,E}$ let face(a) denote the smallest face of B_E containing $\{a\}$ and set $\Lambda = \{\text{face}(a) : a \in (\{u\}_{\prime,E^*})_{\prime,E}\}$. Since for each $a_1, a_2 \in (\{u\}_{\prime,E^*})_{\prime,E}$, both face$(a_1)$ and face(a_2) are contained in face$(\frac{1}{2}(a_1+a_2))$, we conclude that Λ is a partially ordered by set inclusion which is upward directed. We can further check that if

$$a_1 \in \text{face}(a_1) \subseteq \text{face}(a_2) \subseteq \left(\{a_2\}^{\prime,E^*}\right)^{\prime,E^{**}} = (\text{by } (8)) = (\{u(a_2)\}_{\prime,E^*})^{\prime,E^{**}},$$

then

$$(\{u(a_1)\}_{\prime,E^*})^{\prime,E^{**}} = \left(\{a_1\}^{\prime,E^*}\right)^{\prime,E^{**}} \subseteq \left(\left(\left(\{a_2\}^{\prime,E^*}\right)^{\prime,E^{**}}\right)_{\prime,E^*}\right)^{\prime,E^{**}}$$
$$= \left(\left((\{u(a_2)\}_{\prime,E^*})^{\prime,E^{**}}\right)_{\prime,E^*}\right)^{\prime,E^{**}} = (\{u(a_2)\}_{\prime,E^*})^{\prime,E^{**}}.$$

The description of the weak*-closed faces of $\mathcal{B}_{E^{**}}$ proved in [22, Theorem 3.9] gives $u(a_1) \geq u(a_2)$.

We define a net now. For each $\mu \in \Lambda$ we set $u_\mu = u(a)$, where $a \in (\{u\}_{\prime,E^*})_{\prime,E}$ satisfies $\mu = \text{face}(a)$. We have shown in the previous paragraphs that $\{u_\mu\}_{\mu\in\Lambda}$ is a decreasing net of compact-G$_\delta$ tripotents in E^{**}.

In particular, the net $\{u_\mu\}_{\mu\in\Lambda}$ converges in the weak*-topology of E^{**} to its infimum. Let v denote this infimum, which is, by definition, a compact tripotent in E^{**}.

For each $\mu \in \Lambda$, we have $u_\mu = u(a)$, with $a \in (\{u\}_{\prime,E^*})_{\prime,E}$. Therefore

$$(\{u_\mu\}_{\prime,E^*})^{\prime,E^{**}} = (\{u(a)\}_{\prime,E^*})^{\prime,E^{**}} = \left(\{a\}^{\prime,E^*}\right)^{\prime,E^{**}} \subseteq (\{u\}_{\prime,E^*})^{\prime,E^{**}},$$

which, by a new application of [22, Theorem 3.9], proves $u \le u_\mu$ for every $\mu \in \Lambda$, and consequently, $u \le v$.

Finally, for each $a \in (\{u\}_{\prime,E^*})_{\prime,E}$, we know that $v \le u(a) = u_\mu$ with $\mu = \text{face}(a)$, which implies that $\{v\}_{\prime,E^*} \subseteq \{u_\mu\}_{\prime,E^*} = \{a\}^{\prime,E^*}$. We deduce from the arbitrariness of $a \in (\{u\}_{\prime,E^*})_{\prime,E}$ that $\{v\}_{\prime,E^*} \subseteq \left((\{u\}_{\prime,E^*})_{\prime,E}\right)^{\prime,E^*} = \{u\}_{\prime,E^*}$, where the last equality follows from the hypothesis. Therefore $v \le u$ (cf. [22, Theorem 3.7 or 3.9]), witnessing that $u = v$ is a compact tripotent in E^{**}. $\square$

We can now prove that compact tripotents in the second dual of a real JB*-triple are compact in the second dual of its complexification.

Corollary 3.3. *Let τ be a conjugation on a JB*-triple X, and let $E = X^\tau$. Suppose u is a tripotent in E^{**}. Then the following assertions are equivalent:*

(a) u is compact in E^{**};
(b) u is compact in X^{**}.

Proof. The implication $(a) \Rightarrow (b)$ has been commented before Lemma 3.1.

$(b) \Rightarrow (a)$ Suppose that u is compact in X^{**}. Theorem 4.2 in [21] assures that $\{u\}_{\prime,X^*}$ is weak*-semi-exposed. Lemma 3.1 shows that $\{u\}_{\prime,X^*} = \{u\}_{\prime,E^*}$. The arguments in the proof of the "only if" implication in Proposition 3.2 assure that $\{u\}_{\prime,E^*}$ is weak*-semi-exposed in $\mathcal{B}_{E^*}$. The "if" implication of Proposition 3.2 proves that u is compact in E^{**}. $\square$

In the setting of (complex) JB*-triples a new characterization of compact tripotents in the second dual has been recently established in [6]. The concrete result reads as follows:

Theorem 3.4. [6, Theorem 3.6] *Let X be a JB*-triple. Suppose F is a proper weak*-closed face of the closed unit ball of X^{**}. Then the following statements are equivalent:*

(a) F is open relative to X, that is, $F \cap X$ is weak-dense in F;*
(b) F is a weak-closed face associated with a non-zero compact tripotent in X^{**}, that is, there exists a unique non-zero compact tripotent u in X^{**} satisfying $F = u + \mathcal{B}_{X_0^{**}(u)}$.* $\square$

We shall make use of the previous theorem to determine the norm-closed faces of the closed unit ball of a real JB*-triple.

Theorem 3.5. *Let τ be a conjugation on a JB*-triple X, and let $E = X^\tau$. Then for each norm-closed proper face F of $\mathcal{B}_E$ there exists a unique compact tripotent $u \in E^{**}$ satisfying $F = (u + \mathcal{B}_{E_0^{**}(u)}) \cap E$. Furthermore, the mapping*

$$u \mapsto (\{u\}_{\prime,E^*})_{\prime,E} = (u + \mathcal{B}_{E_0^{**}(u)}) \cap E$$

is an anti-order isomorphism from $\tilde{\mathcal{U}}_c(E^{**})$ onto $\mathcal{F}_n(\mathcal{B}_E)$.

Proof. Suppose F is a norm-closed proper face of $\mathcal{B}_E$. Let $P = \frac{1}{2}(Id_X + \tau)$. Then P is a contractive real linear projection on X whose image is E. By (7), the set $\mathfrak{F} := P^{-1}(F) \cap \mathcal{B}_X$ is a norm-closed proper face of $\mathcal{B}_X$. It is not hard to check that, since $P(\tau(x)) = P(x)$ for all $x \in X$, we have $\tau(\mathfrak{F}) = \mathfrak{F}$. By [19, Corollary 3.12] there exists a unique compact tripotent $u \in X^{**}$ satisfying $\mathfrak{F} = (\{u\}_{\prime, X^*})_{\prime, X} = (u + \mathcal{B}_{X_0^{**}(u)}) \cap X$. An application of Theorem 3.4 guarantees that

$$\tau^{\sharp\sharp}(u) + \mathcal{B}_{X_0^{**}(\tau^{\sharp\sharp}(u))} = \tau^{\sharp\sharp}(u + \mathcal{B}_{X_0^{**}(u)}) = \tau^{\sharp\sharp}(\overline{\mathfrak{F}}^{w^*}) = \overline{\tau(\mathfrak{F})}^{w^*} = \overline{\mathfrak{F}}^{w^*} = u + \mathcal{B}_{X_0^{**}(u)}.$$

A new application of [19, Corollary 3.12], implies that $\tau^{\sharp\sharp}(u) = u \in E^{**}$. Corollary 3.3 shows that u is compact in E^{**}, and it is not hard to check that $F = \mathfrak{F} \cap E = \mathfrak{F}^{\tau} = (u + \mathcal{B}_{E_0^{**}(u)}) \cap E$, as desired. The rest is clear. $\square$

We can now prove the main goal of this subsection which is a tool required for latter purposes.

Theorem 3.6. *Let τ be a conjugation on a JB*-triple X, and let $E = X^{\tau}$. Suppose F is a proper weak*-closed face of the closed unit ball of E^{**}. Then the following statements are equivalent:*

(a) *F is open relative to E, that is, $F \cap E$ is weak*-dense in F;*
(b) *F is a weak*-closed face associated with a non-zero compact tripotent in E^{**}, that is, there exists a unique non-zero compact tripotent u in E^{**} satisfying $F = F_u^{E^{**}} = u + \mathcal{B}_{E_0^{**}(u)}$.*

Proof. Let $P = \frac{1}{2}(Id_X + \tau^{\sharp\sharp})$. Then P is a contractive weak*-continuous real linear projection on X^{**} whose image is E^{**}. It is shown in [22, Theorem 3.9] that $F \subseteq \mathcal{B}_{E^{**}}$ is a proper weak*-closed face if and only if $\mathfrak{F} := P^{-1}(F) \cap \mathcal{B}_{X^{**}}$ is a proper weak*-closed face of $\mathcal{B}_{X^{**}}$. Since $\mathfrak{F}$ is $\tau^{\sharp\sharp}$-symmetric and $F = \mathfrak{F}^{\tau^{\sharp\sharp}} \cap E$, it is not hard to check that $\overline{\mathfrak{F} \cap X}^{w^*} = \mathfrak{F}$ if and only if $\overline{F \cap E}^{w^*} = F$. Therefore the desired equivalence is a consequence of Theorem 3.4 and Corollary 3.3. $\square$

It remains to determine the weak*-closed faces of the closed unit ball of the dual space of a real JB*-triple.

Theorem 3.7. *Let τ be a conjugation on a JB*-triple X, and let $E = X^{\tau}$. Then for each weak*-closed proper face F of $\mathcal{B}_{E^*}$ there exists a unique compact tripotent $u \in E^{**}$ satisfying $F = \{u\}_{\prime, E^*}$. Furthermore, the mapping*

$$u \mapsto \{u\}_{\prime, E^*}$$

is an order isomorphism from $\tilde{\mathcal{U}}_c(E^{**})$ onto $\mathcal{F}_{w^*}(\mathcal{B}_E)$.

Proof. As before, we set $P = \frac{1}{2}(Id_X + \tau)$ and $Q = \frac{1}{2}(Id_X + \tau^{\sharp})$. Then P and Q are contractive real linear projections on X and X^* whose images are E and E^*, respectively, and Q is weak*-continuous. The set F is a weak*-closed proper face of $\mathcal{B}_{E^*}$ if and only if the set $\mathfrak{F} := Q^{-1}(F) \cap \mathcal{B}_{X^*}$ is a weak*-closed proper face of $\mathcal{B}_{X^*}$. By [27, Theorem 2] there exists a (unique) compact tripotent $u \in X^{**}$ satisfying $\mathfrak{F} = \{u\}_{\prime, X^*}$. Clearly, $\mathfrak{F}$ is $\tau^{\sharp}$-symmetric and $F = \mathfrak{F}^{\tau^{\sharp}} = \mathfrak{F} \cap E^*$. We have commented in previous pages that τ and $\tau^{\sharp\sharp}$ are triple automorphisms on X and X^{**}, respectively. Then, we can easily check that

$$\{u\}_{\prime, X^*} = \mathfrak{F} = \tau^{\sharp}(\mathfrak{F}) = \tau^{\sharp}(\{u\}_{\prime, X^*}) = \{\tau^{\sharp\sharp}(u)\}_{\prime, X^*},$$

witnessing that $\tau^{\sharp\sharp}(u) = u$. Corollary 3.3 proves that u is a compact tripotent in E^{**}. Finally, $F = \mathfrak{F} \cap E^* = \{u\}_{\prime, E^*}$. $\square$

4. New spaces satisfying a Krein–Milman type theorem

A convex subset $\mathcal{K}$ of a normed space X is called a *convex body* if it has non-empty interior in X. The Mazur–Ulam theorem establishes that every surjective isometry between two normed real spaces is always affine. In [41] P. Mankiewicz extended this result by showing that any surjective isometry defined between convex bodies in two arbitrary normed spaces is the restriction of a unique affine isometry between the whole spaces. Mankiewicz's theorem has become a fundamental tool for researchers working on positive solutions to Tingley's problem or on new Banach spaces satisfying the Mazur–Ulam property.

In relation with these questions, M. Mori and N. Ozawa have recently contributed by introducing a new point of view (see [44]). Following the just quoted authors, we shall say that a convex subset $\mathcal{K}$ of a normed space X satisfies the *strong Mankiewicz property* if every surjective isometry Δ from $\mathcal{K}$ onto an arbitrary convex subset L in a normed space Y is affine. Every convex subset of a strictly convex normed space satisfies the strong Mankiewicz property because it is uniquely geodesic (see [4, Lemma 6.1]), and there exist examples of convex subsets of $L^1[0,1]$ which do not satisfy this property (see [44, Example 5]). In [44, Theorem 2] Mori and Ozawa establish the following variant of Mankiewicz's theorem.

Theorem 4.1. [44, Theorem 2] *Let X be a Banach space such that the closed convex hull of the extreme points, $\partial_e(\mathcal{B}_X)$, of the closed unit ball, $\mathcal{B}_X$, of X has non-empty interior in X. Then, every convex body $\mathcal{K} \subset X$ has the strong Mankiewicz property. Furthermore, suppose L is a convex subset of a normed space Y, and $\Delta : \mathcal{B}_X \to L$ is a surjective isometry. Then Δ can be uniquely extended to an affine isometry from X onto a norm-closed subspace of Y.* □

By combining the previous result with the Russo–Dye theorem, Mori and Ozawa proved that every convex body in a unital C*-algebra or in a real von Neumann algebra satisfies the strong Mankiewicz property (see [44, Corollary 3]). A deeper application of the facial structure of unital C*-algebras leads Mori and Ozawa to a significant achievement in the study of the Mazur–Ulam property.

Theorem 4.2. [44, Theorem 1] *Every unital complex C*-algebra (as a real Banach space) and every real von Neumann algebra has the Mazur–Ulam property.*

It is worth mentioning that concerning the strong Mankiewicz and the Mazur–Ulam properties, a version of the Mori-Ozawa theorem has been recently established in the wider setting of JBW*-triples.

Theorem 4.3. [6, Corollary 2.2, Theorem 4.14 and Proposition 4.15] *Every convex body in a JBW*-triple satisfies the strong Mankiewicz property. Every JBW*-triple which is not a Cartan factor of rank two satisfies the Mazur–Ulam property.*

The previous two theorems reveal the noticeable applicability of Theorem 4.1 in the study of those problems asking for extension of isometries between the spheres of two Banach spaces. This powerful tool is limited to those Banach spaces whose closed unit ball coincides with the closed convex hull of its extreme points. For this reason, we survey some forerunners where the latter property has been studied.

W.G. Bade proved that $\text{co}(\partial_e \mathcal{B}_{C(K,\mathbb{R})})$ (i.e., the convex hull of $\partial_e \mathcal{B}_{C(K,\mathbb{R})}$) is dense in the closed unit ball of $C(K,\mathbb{R})$ if and only if K is totally disconnected (see [3]). The complex case was considered by R.R. Phelps in [49], where he showed that the closed unit ball of the commutative unital C*-algebra $C(K)$ coincides with the closed convex hull of its extreme points. Since the extreme points of the closed unit ball of $C(K)$ are precisely the unitary elements in $C(K)$, Phelps provided in fact a particular case of the celebrated Russo–Dye theorem ([50]), which states that the closed unit ball of any unital C*-algebra agrees with the closed convex hull of its unitary elements.

When the complex field is replaced with a general Banach space X with $\dim(X) \geq 3$, the notion of unitary element does not make any sense in the space $C(K, X)$, of all X-valued continuous functions on K. In the setting of $C(K, X)$ spaces the problem of determining whether its closed unit ball coincides with the closed convex hull of its extreme points was explored by authors like J. Cantwell [12], N.T. Peck [45], J.F. Mena-Jurado, J.C. Navarro-Pascual and V.I. Bogachev [42,7]. Since the notion of unitary is no longer applicable, these results are called Krein–Milman type theorems.

All the comments above provide sufficient motivation for identifying new examples of Banach spaces satisfying a Krein–Milman type theorem. Some of them can be obtained by certain "hyperplanes" associated with multiplicative functionals on unital C*-algebras. Let A be a unital C*-algebra and suppose $\varphi : A \to \mathbb{C}$ is a homomorphism. We observe first that φ is automatically continuous (cf. [8, §16, Proposition 3]). We can therefore apply the Gleason-Kahane-Żelazko theorem [57, Theorem 2] to deduce that φ is in fact a *-homomorphism, that is, $\varphi(a^*) = \varphi(a)^*$, for every element a in A. Consequently, φ is a triple homomorphism when A and $\mathbb{C}$ both are equipped with the tripe product defined in (1). However, given $\lambda \in \mathbb{T} = S(\mathbb{C})$ with $\lambda \neq 1$, the non-zero functional $\psi = \lambda\varphi : A \to \mathbb{C}$ is a triple homomorphism which is not multiplicative.

It is worth noting that every triple homomorphism $\psi : A \to \mathbb{C}$ can be expressed as a product of an element $\lambda \in \mathbb{T}$ and a *-homomorphism $\varphi : A \to \mathbb{C}$. We observe that every triple homomorphism ψ from a JB*-triple E into $\mathbb{C}$ is automatically continuous (cf. [37, Lemma 1.6]). Suppose $\psi \neq 0$. Since for every $a \in A$ we have $\psi(a) = \psi\{a, 1, 1\} = \{\psi(a), \psi(1), \psi(1)\} = \psi(a)\overline{\psi(1)}\psi(1)$, it follows that $\psi(1) \in \mathbb{T}$ because $\psi \neq 0$. It is standard to check that the mapping $\varphi = \overline{\psi(1)}\psi$ is a Jordan *-homomorphism from A onto $\mathbb{C}$. We can therefore apply [57, proof of Theorem 1] to deduce that φ is a *-homomorphism, and $\psi = \psi(1)\varphi$.

Let A be a unital C*-algebra, let $\varphi : A \to \mathbb{C}$ be a (continuous) multiplicative functional, and let $A^{\varphi}_{\mathbb{R}} := \varphi^{-1}(\mathbb{R}) = \{a \in A : \varphi(a) \in \mathbb{R}\}$. Clearly $A^{\varphi}_{\mathbb{R}}$ is a real C*-subalgebra of A. M. Mori and N. Ozawa prove in [44, Lemma 19] that $\mathcal{B}_{A^{\varphi}_{\mathbb{R}}}$ coincides with the closed convex hull of the unitary elements in $A^{\varphi}_{\mathbb{R}}$. The next statement somehow extends this conclusion to the triple setting. The result also shows a new class of real JB*-triples satisfying a Krein–Milman type theorem.

Proposition 4.4. *Let A be a unital C*-algebra and let $\psi : A \to \mathbb{C}$ be a (continuous) non-zero triple homomorphism. Then the closed unit ball of the real JB*-triple $A^{\psi}_{\mathbb{R}} := \psi^{-1}(\mathbb{R})$ coincides with the closed convex hull of the unitary tripotents in $A^{\psi}_{\mathbb{R}}$. Consequently, $\mathcal{B}_{A^{\psi}_{\mathbb{R}}}$ and every convex body $\mathcal{K} \subset A^{\psi}_{\mathbb{R}}$ satisfy the strong Mankiewiecz property.*

Proof. The observations made above guarantee the existence of a non-zero and (continuous) multiplicative functional $\varphi : A \to \mathbb{C}$ and an element λ in $\mathbb{T}$ such that $\psi = \lambda\varphi$. If we write $A^{\varphi}_{\mathbb{R}} = \{b \in A : \varphi(b) \in \mathbb{R}\}$, it is clear that $A^{\psi}_{\mathbb{R}} = \psi^{-1}(\mathbb{R}) = (\lambda\varphi)^{-1}(\mathbb{R}) = \{a \in A : \varphi(a) \in \overline{\lambda}\mathbb{R}\} = \overline{\lambda}A^{\varphi}_{\mathbb{R}}$. Therefore, $\mathcal{B}_{A^{\psi}_{\mathbb{R}}} = \mathcal{B}_{\overline{\lambda}A^{\varphi}_{\mathbb{R}}} = \overline{\lambda}\mathcal{B}_{A^{\varphi}_{\mathbb{R}}}$.

Let us pick now $a \in \mathcal{B}_{A^{\psi}_{\mathbb{R}}}$ and $\varepsilon > 0$. We have shown that there exists $b \in \mathcal{B}_{A^{\varphi}_{\mathbb{R}}}$ such that $a = \overline{\lambda}b$. It is shown in the proof of [44, Lemma 19] that there exist unitary elements $u_1, \ldots, u_n$ in the real C*-algebra $A^{\varphi}_{\mathbb{R}}$ and $\alpha_1, \ldots, \alpha_n$ in $[0, 1]$ with $\sum\limits_{j=1}^{n} \alpha_j = 1$ satisfying $\left\| b - \sum\limits_{j=1}^{n} \alpha_j u_j \right\| < \varepsilon$. Therefore, $\left\| a - \sum\limits_{j=1}^{n} \alpha_j \overline{\lambda} u_j \right\| = \left\| \overline{\lambda} b - \sum\limits_{j=1}^{n} \alpha_j \overline{\lambda} u_j \right\| = \left\| b - \sum\limits_{j=1}^{n} \alpha_j u_j \right\| < \varepsilon$. Finally, we observe that $\overline{\lambda}u_1, \ldots, \overline{\lambda}u_n$ are unitary tripotents in $A^{\psi}_{\mathbb{R}}$. The final conclusion follows from Theorem 4.1 and [44, Lemma 4]. $\square$

A Krein-Milman type theorem for the space $C(K, \mathcal{H})$ is essentially known in the literature.

Proposition 4.5. [49,12,45] *Let K be a compact Hausdorff space and let $\mathcal{H}$ be a real Hilbert space with $\dim(\mathcal{H}) \geq 2$. Then the closed unit ball of $C(K, \mathcal{H})$ coincides with the closed convex hull of its extreme points. Consequently, every convex body in $C(K, \mathcal{H})$ satisfies the strong Mankiewicz property.*

Proof. If $\dim(\mathcal{H}) = n \in \mathbb{N}$, we can identify the Hilbert space $\mathcal{H}$ with $\ell_2^n(\mathbb{R})$. If $n = 2$, R. Phelps proved in [49, Theorem 1] that the convex hull of the extreme points of $B_{C(K)}$ is always dense in the closed unit ball. If $n > 2$, the same conclusion holds by [12, Theorem I and Remark]. On the other hand, if $\dim(\mathcal{H}) = \infty$, then the closed unit ball of $C(K, \mathcal{H})$ coincides with the convex hull of its extreme points by [45, Theorem 5]. Therefore, in both cases $C(K, \mathcal{H})$ satisfies a Krein–Milman type theorem and the thesis of our proposition derives from Theorem 4.1. $\square$

The following technical lemma is required for later purposes.

Lemma 4.6. *Let K be a compact Hausdorff space and let $\mathcal{H}$ be a real Hilbert space with $\dim(\mathcal{H}) = n \geq 2$. Suppose $t_0 \in K$ and $x_0 \in S(\mathcal{H})$. If $a \in \mathcal{B}_{C(K,\mathcal{H})}$ is such that $a(t_0) \in \mathbb{R}x_0$ and $\varepsilon > 0$ is small enough, then the following statements hold:*

(a) *If $\mathcal{H}$ is infinite dimensional, then there exists a non-vanishing function b in $\mathcal{B}_{C(K,\mathcal{H})}$ such that $b(t_0) \in \mathbb{R}x_0$ and $\|a - b\| < \varepsilon$. If $a(t_0) \neq 0$, we can also assume that $b(t_0) = a(t_0)$;*

(b) *If $\mathcal{H}$ is finite dimensional, then there exist non-vanishing continuous functions $b_1, \ldots, b_k$ in $\mathcal{B}_{C(K,\mathcal{H})}$ such that $b_j(t_0) \in \mathbb{R}x_0$, for every $j \in \{1, \ldots, k\}$, and $\left\| a - \frac{1}{k}\sum_{j=1}^k b_j \right\| \leq \varepsilon$. If $a(t_0) \neq 0$, we can also assume that $b_j(t_0) = a(t_0)$ for all $j \in \{1, \ldots, k\}$.*
Furthermore, for each j in $\{0, \ldots, k\}$ there exist $v_j \in C(K, \mathcal{H})$ satisfying $\|v_j(t)\| = 1$, and $(b_j(t)|v_j(t)) = 0$, for all $t \in K$, and thus $u_j = b_j + (1 - \|b_j(\cdot)\|^2)^{\frac{1}{2}} v_j$, $w_j = b_j - (1 - \|b_j(\cdot)\|^2)^{\frac{1}{2}} v_j$ both lie in $\partial_e(\mathcal{B}_{C(K,\mathcal{H})})$ and $b_j = \frac{1}{2}(u_j + w_j)$.

Proof. Take $a \in \mathcal{B}_{C(K,\mathcal{H})}$ such that $a(t_0) = \lambda x_0$, with $\lambda \in \mathbb{R}$, and $\varepsilon > 0$. We shall split the proof into two cases.

Case 1: Suppose $\mathcal{H}$ is infinite dimensional.

If $\lambda \in \mathbb{R} \setminus \{0\}$, then clearly $1 \geq \|a\| \geq |\lambda| > 0$. By [45, Corollary after Proposition 2] applied to $|\lambda|/2 > \varepsilon/2 > 0$, there exists $b \in \mathcal{B}_{C(K,\mathcal{H})}$ which is a non-vanishing function (i.e. $\|b(t)\| \geq m > 0$ for every $t \in K$, and some $m \in \mathbb{R}^+$) such that for each $t \in K$, $\|b(t)\| < \varepsilon/2$ if $\|a(t)\| < \varepsilon/2$, and $b(t) = a(t)$ if $\|a(t)\| \geq \varepsilon/2$. It is not hard to check that $\|a - b\| < \varepsilon$, and $b(t_0) = a(t_0) = \lambda x_0$ because $\|a(t_0)\| = |\lambda| > \varepsilon/2$.

On the other hand, if $\lambda = 0$, that is, if $a(t_0) = 0$, let us consider the open set $\mathcal{U}_\varepsilon = \{t \in K : \|a(t)\| < \varepsilon/2\}$. By Urysohn's lemma there exists a continuous function $f : K \to \mathbb{R}$ such that $0 \leq f \leq 1$, $f(t_0) = 1$ and $f|_{K\setminus\mathcal{U}_\varepsilon} \equiv 0$. Define $\widetilde{a} = a + (\varepsilon/2)x_0 \otimes f \in C(K, \mathcal{H})$, which lies in the closed unit ball for ε small enough ($\varepsilon \leq 1$). Note that $\|a - \widetilde{a}\| \leq \varepsilon/2$. Since $\widetilde{a}(t_0) = (\varepsilon/2)x_0$ and $\varepsilon/2 \neq 0$, we have shown before that there exists a non-vanishing function $b \in \mathcal{B}_{C(K,\mathcal{H})}$ such that for each $t \in K$, $\|b(t)\| < \varepsilon/4$ if $\|\widetilde{a}(t)\| < \varepsilon/4$, and $b(t) = \widetilde{a}(t)$ if $\|\widetilde{a}(t)\| \geq \varepsilon/4$ (cf. [45, Corollary after Proposition 2]). Therefore $b(t_0) = \widetilde{a}(t_0) = (\varepsilon/2)x_0$. It is also clear that $\|\widetilde{a} - b\| < \varepsilon/2$, and thus $\|a - b\| \leq \|a - \widetilde{a}\| + \|\widetilde{a} - b\| < \varepsilon/2 + \varepsilon/2 = \varepsilon$ as desired.

Case 2: Suppose $\infty > \dim(\mathcal{H}) = n \geq 2$.

As before, we shall distinguish the cases $\lambda = 0$ and $\lambda \neq 0$. Let us first assume that $|\lambda| \geq 2\varepsilon > 0$ with ε small enough. Following the arguments due to R.C. Sine and N.T. Peck (see [45, proof of Theorem 1]), for $\alpha, \beta > 0$ and $z_0 \in S(\mathcal{H})$, we shall consider $B(z_0, \alpha) = \{z \in S(\mathcal{H}) : \|z - z_0\| < \alpha\}$ and the wedge $W(z_0, \alpha, \beta) := \operatorname{co}(B(z_0, \alpha) \cup \{-\beta z_0\})$.

For every $\varepsilon > 0$, there exists $k \in \mathbb{N}$ such that $\frac{1}{k} < \frac{\varepsilon}{2}$. Find $z_1, \cdots, z_k \in S(\mathcal{H})$, $\alpha_1, \cdots, \alpha_k \in \mathbb{R}^+$ and $\beta_1, \cdots, \beta_k \in \mathbb{R}^+$, satisfying:

- The sets $\{W(z_j,\alpha_j,\beta_j) : j = 1,\ldots,k\}$ are pointwise disjoint outside the closed ball in $\mathcal{H}$ centered in zero with radius $\varepsilon/2$;
- $W(x_j,\alpha_j,\beta_j)\cap \mathbb{R}x_0 \subseteq [-\varepsilon x_0, \varepsilon x_0]$, for every $j = 1,\ldots,k$.

Let us now define $\varphi_j : \mathcal{B}_{\mathcal{H}} \to \mathcal{B}_{\mathcal{H}} \setminus \mathring{W}(z_j,\alpha_j,\beta_j)$ given by $\varphi_j(z) = z$ if $z \notin W(z_j,\alpha_j,\beta_j)$, and for $z \in W(z_j,\alpha_j,\beta_j)$, $\varphi_j(z)$ is obtained by projecting z parallel to $-z_j$ until it hits the boundary of $W(z_j,\alpha_j,\beta_j)$. The number β_j can be chosen such that $\|\varphi_j(z)\| \leq \varepsilon/2$, for every $\|z\| \leq \varepsilon/2$. We claim that

$$\left\| z - \frac{1}{k}\sum_{j=1}^{k}\varphi_j(z)\right\| \leq \varepsilon, \text{ for every } z \in \mathcal{B}_{\mathcal{H}}. \tag{10}$$

Namely, if we take $z \in \mathcal{B}_{\mathcal{H}}$ with $\|z\| \leq \varepsilon/2$, then $\left\| z - \frac{1}{k}\sum_{j=1}^{k}\varphi_j(z)\right\| \leq \|z\| + \frac{1}{k}\sum_{j=1}^{k}\|\varphi_j(z)\| \leq \varepsilon$. On the other hand, if we pick $z \in \mathcal{B}_{\mathcal{H}}$ with $\|z\| > \varepsilon/2$, then z lies in at most one $W(z_{j_0},\alpha_{j_0},\beta_{j_0})$, and that implies $\left\| z - \frac{1}{k}\sum_{j=1}^{k}\varphi_j(z)\right\| = \left\|\frac{z}{k} - \frac{\varphi_{j_0}(z)}{k}\right\| \leq \frac{2}{k} < \varepsilon$, as we were expecting.

Set $b_j := \varphi_j \circ a \in \mathcal{B}_{C(K,\mathcal{H})}$. Obviously, b_j is a non-vanishing function. It follows from (10) that $\left\| a - \frac{1}{k}\sum_{j=1}^{k} b_j\right\| = \left\| a - \frac{1}{k}\sum_{j=1}^{k}\varphi_j\circ a\right\| \leq \varepsilon$. Furthermore, since $\|a(t_0)\| = |\lambda| \geq 2\varepsilon > \varepsilon$ and $W(x_j,\alpha_j,\beta_j)\cap \mathbb{R}x_0 \subseteq [-\varepsilon x_0,\varepsilon x_0]$, it follows that $a(t_0) = \lambda x_0 \notin W(z_j,\alpha_j,\beta_j)$, for any $j \in \{1,\cdots,k\}$, and thus $b_j(t_0) = \varphi_j\circ a(t_0) = a(t_0) = \lambda x_0 \in \mathbb{R}x_0$, for every $j \in \{1,\cdots,k\}$.

If we assume $a(t_0) = 0$, we can argue as in the infinite dimensional case, and thus, for $0 < \varepsilon < 1$ we define $\widetilde{a} \in \mathcal{B}_{C(K,\mathcal{H})}$, with $\widetilde{a}(t_0) = (\varepsilon/2)x_0$ and such that $\|a - \widetilde{a}\| \leq \varepsilon/2$. Now we can apply the conclusions above which guarantee the existence of $k \in \mathbb{N}$, non-vanishing functions $b_1,\cdots,b_k \in \mathcal{B}_{C(K,\mathcal{H})}$ and such that $b_j(t_0) = (\varepsilon/2)x_0$, for every $j \in \{1,\cdots,k\}$. The desired conclusion follows from the inequality $\left\| a - \frac{1}{k}\sum_{j=1}^{k} b_j\right\| \leq \|a-\widetilde{a}\| + \left\|\widetilde{a} - \frac{1}{k}\sum_{j=1}^{k} b_j\right\| < \varepsilon$.

The rest of the argument is essentially in [45, proof of Theorem 1]. It is shown in the just quoted paper that, for each $j \in \{1,\cdots,k\}$ there exists a continuous field $\vartheta_j : S(\mathcal{H})\backslash B(z_j,\alpha_j) \to S(\mathcal{H})$ (i.e. a continuous mapping satisfying $(\vartheta_j(z)|z) = 0$ for all $z \in S(\mathcal{H})\backslash B(z_j,\alpha_j)$). Taking $v_j := \vartheta_j(\frac{b_j(\cdot)}{\|b_j(\cdot)\|})$ we get the desired statement. □

Let K be a compact Hausdorff space, and let H be a real or complex Hilbert space. For each $t_0 \in K$ and each $x_0 \in S(H)$ we set

$$A(t_0,x_0) := \{a \in S(C(K,H)) : a(t_0) = x_0\}.$$

It is not hard to check that $A(t_0,x_0)$ is a maximal norm-closed proper face of $\mathcal{B}_{C(K,H)}$ and a maximal convex subset of $S(C(K,H))$. Actually, every maximal convex subset of the unit sphere of $C(K,H)$ is of this form.

Our next corollary is one of the principal technical tools required for our main result.

Corollary 4.7. *Let K be a compact Hausdorff space and let H be a complex Hilbert space. Suppose $t_0 \in K$ and $x_0 \in S(H)$. Then every element in $A(t_0,x_0)$ can be approximated in norm by a finite convex combination of elements in $A(t_0,x_0)\cap\partial_e(\mathcal{B}_{C(K,H)})$.*

Proof. Take $a \in A(t_0, x_0)$. By Lemma 4.6 the element a can be approximated in norm by a finite convex combination of non-vanishing functions in $A(t_0, x_0) \cap \mathcal{B}_{C(K,H)}$. Let b be a non-vanishing functions in $A(t_0, x_0) \cap \mathcal{B}_{C(K,H)}$. The element $u(\cdot) = \frac{b(\cdot)}{\|b(\cdot)\|}$ lies in $A(t_0, x_0)$ and is a maximal tripotent in $C(K, H)$ (i.e., an element in $A(t_0, x_0) \cap \partial_e(\mathcal{B}_{C(K,H)})$).

To simplify the notation let us write E for $C(K, H)$. Clearly, b is a hermitian element in the JB*-algebra $E_2(u)$. Let A denote the JB*-subalgebra of $E_2(u)$ generated by b and u. It is known that A is isometrically isomorphic to a commutative unital C*-algebra (cf. [32, Theorem 3.2.4]). The intersection $F = A(t_0, x_0) \cap A \subseteq S(A)$ is a maximal norm-closed face of $\mathcal{B}_A$. Lemma 18 in [44] guarantees that $b \in F$ can be approximated in norm by a finite convex combination of elements in $F \cap \partial_e(\mathcal{B}_A)$. Every element in $\partial_e(\mathcal{B}_A)$ is a unitary element in A, and hence a unitary element in $E_2(u)$. We further know from Lemma 4 in [52] that every unitary element in $E_2(u)$ is an extreme point of $\mathcal{B}_E$. We can therefore conclude that $F \cap \partial_e(\mathcal{B}_A) \subseteq F \cap \partial_e(\mathcal{B}_E) \subseteq A(t_0, x_0) \cap \partial_e(\mathcal{B}_E)$, which finishes the proof. $\square$

The case of real Hilbert spaces is treated in the next result.

Corollary 4.8. *Let K be a compact Hausdorff space and let $\mathcal{H}$ be a finite-dimensional real Hilbert space with $\dim(\mathcal{H}) = n \geq 2$. Suppose $t_0 \in K$ and $x_0 \in S(\mathcal{H})$. Then every element in $A(t_0, x_0)$ can be approximated in norm by a finite convex combination of elements in $A(t_0, x_0) \cap \partial_e(\mathcal{B}_{C(K,\mathcal{H})})$.*

Proof. Let a be an element in $A(t_0, x_0)$. Since $a(t_0) = x_0 \in S(\mathcal{H})$, by Lemma 4.6($b$), for each $\varepsilon > 0$ small enough, there exist non-vanishing continuous functions $b_1, \ldots, b_k$ in $\mathcal{B}_{C(K,\mathcal{H})}$ such that $b_j(t_0) = a(t_0) = x_0$, for every $j \in \{1, \ldots, k\}$, and $\left\| a - \frac{1}{k} \sum_{j=1}^k b_j \right\| \leq \varepsilon$. Furthermore, for each j in $\{0, \ldots, k\}$ there exists $v_j \in C(K, \mathcal{H})$ satisfying $\|v_j(t)\| = 1$, and $(b_j(t)|v_j(t)) = 0$, for all $t \in K$, and thus $u_j = b_j + (1 - \|b_j(\cdot)\|^2)^{\frac{1}{2}} v_j$, and $w_j = b_j - (1 - \|b_j(\cdot)\|^2)^{\frac{1}{2}} v_j$ both lie in $\partial_e(\mathcal{B}_{C(K,\mathcal{H})})$ and $b_j = \frac{1}{2}(u_j + w_j)$. Having in mind that $\|a(t_0)\| = \|x_0\| = 1$, we can easily see that $u_j(t_0) = w_j(t_0) = b_j(t_0) = a(t_0) = x_0$, which guarantees that $u_j, w_j \in A(t_0, x_0) \cap \partial_e(\mathcal{B}_{C(K,\mathcal{H})})$. Finally,

$$\left\| a - \frac{1}{2k} \sum_{j=1}^k (u_j + w_j) \right\| = \left\| a - \frac{1}{k} \sum_{j=1}^k b_j \right\| \leq \varepsilon,$$

as desired. $\square$

Let H be a real or complex Hilbert space, and let K be a compact Hausdorff space. Let $\mathcal{O} \neq \emptyset$ be an open subset of K. We set $p := \chi_{\mathcal{O}}$ the characteristic function of $\mathcal{O}$. Note that we cannot, in general, assume that $p \in C(K)$.

Fix $x_0 \in S(H)$. Let us define some subsets of $C(K, H)$ whose elements are constant on $\mathcal{O}$:

$$\begin{aligned} F = F_{x_0 \otimes p} &:= \{a \in S(C(K, H)) : ap = x_0 \otimes p\}, \qquad (11) \\ B = B_p &:= \{a \in C(K, H) : ap = h \otimes p, \text{ for some } h \in H\}, \\ N = N_p^{x_0} &:= \{a \in C(K, H) : ap = \mu x_0 \otimes p, \text{ for some } \mu \in \mathbb{K}\}, \end{aligned}$$

where $\mathbb{K} = \mathbb{R}$ if H is a real Hilbert space and $\mathbb{K} = \mathbb{C}$ if H is a complex Hilbert space.

It is not difficult to check that N and B are norm-closed subtriples of $C(K, H)$ with $F \subseteq N \subseteq B \subseteq C(K, H)$.

Let us consider the mapping $T : B \to H$ defined by $T(a) = a(t_0)$ for each $a \in B$, where t_0 is any element in the open set $\mathcal{O}$. Clearly T is linear. We further know that T is a triple homomorphism. Indeed, if we take $a, b, c \in B$ and write $a(t_0) = x_a$, $b(t_0) = x_b$ and $c(t_0) = x_c$ the constant elements in the Hilbert space associated to each function, we have that

$$T\{a,b,c\} = \{a,b,c\}(t_0) = \frac{1}{2}\langle a(t_0)|b(t_0)\rangle c(t_0) + \frac{1}{2}\langle c(t_0)|b(t_0)\rangle a(t_0)$$
$$= \frac{1}{2}\langle x_a|x_b\rangle x_c + \frac{1}{2}\langle x_c|x_b\rangle x_a = \{x_a, x_b, x_c\} = \{T(a), T(b), T(c)\}.$$

The restriction $T|_N : N \to \mathbb{K}x_0 \subseteq H$ also is a triple homomorphism and $T|_N(a) = a(t_0) = \mu_a x_0$ for every $a \in N$, where $\mu_a \in \mathbb{K}$.

We are now in position to present an extension of [44, Lemma 19] and Proposition 4.4 to the setting of continuous functions valued in a Hilbert space.

Proposition 4.9. *Let K be a compact Hausdorff space and let H be a complex Hilbert space. Suppose $x_0 \in S(H)$ and $\mathcal{O} \neq \emptyset$ is an open subset of K. Let us denote $p = \chi_{\mathcal{O}}$, $N = \{a \in C(K,H) : ap = \mu x_0 \otimes p,$ for some $\mu \in \mathbb{C}\}$, and $\varphi : N \to \mathbb{C}$ the triple homomorphism defined by $\varphi(a) = \langle a(t_0)|x_0\rangle$ $(a \in N)$, where t_0 is any element in $\mathcal{O}$. Then the closed unit ball of $N^{\varphi}_{\mathbb{R}} := \varphi^{-1}(\mathbb{R})$ coincides with the closed convex hull of its extreme points. Consequently, $\mathcal{B}_{N^{\varphi}_{\mathbb{R}}}$ satisfies the strong Mankiewicz property.*

Proof. Let us pick $a \in \mathcal{B}_{N^{\varphi}_{\mathbb{R}}}$ and any $t_0 \in \mathcal{O}$. Since $\varphi(a) = \lambda \in \mathbb{R}$, we can assure that $a(t_0) = \lambda x_0 \in \mathbb{R}x_0$. Without loss of generality, we can assume, via Lemma 4.6, that a is a non-vanishing function. Define now $u \in C(K,H)$ given by $u(t) := \frac{a(t)}{\|a(t)\|}$, for every $t \in K$. Observe that $u \in \partial_e(\mathcal{B}_{C(K,H)})$. We further know that u lies in $N^{\varphi}_{\mathbb{R}}$ because $up = \frac{a(t_0)}{|a(t_0)|} x_0 \otimes p$ and $\varphi(u) = \frac{a(t_0)}{\|a(t_0)\|}$.

To simplify the notation we set $E = C(K,H)$. Since $a \in E^1(u)$ (actually $a \in N^1(u)$), the JB*-subtriple of E generated by a and u is JB*-triple isomorphic (and hence isometric) to a commutative unital C*-algebra. Let A denote the JB*-subtriple generated by a and u. Since $a, u \in N$, it follows that $A \subseteq N$, and hence the restriction $\varphi|_A : A \to \mathbb{C}$ is a non-zero triple homomorphism. By applying Proposition 4.4 to the real JB*-triple $A^{\varphi|_A}_{\mathbb{R}}$ we conclude that $a \in A^{\varphi|_A}_{\mathbb{R}}$ can be approximated in norm by convex combinations of unitary tripotents in $A^{\varphi|_A}_{\mathbb{R}}$.

Finally, every unitary element in $A^{\varphi|_A}_{\mathbb{R}}$ is a unitary element in the JB*-algebra $E_2(u)$. We observe now that, since u is an extreme point of $\mathcal{B}_E$, every unitary element in $E_2(u)$ is an extreme point of $\mathcal{B}_E$ (cf. [52, Lemma 4]), and thus every unitary element in $A^{\varphi|_A}_{\mathbb{R}}$ belongs to $\partial_e(\mathcal{B}_{N^{\varphi}_{\mathbb{R}}})$, because $A^{\varphi|_M}_{\mathbb{R}} \subseteq N^{\varphi}_{\mathbb{R}}$. Theorem 4.1 gives the final statement. □

We shall next establish a real version of Proposition 4.9. For reasons which will be better understood at the end of the next section, we shall restrict our interest to the finite dimensional case.

Proposition 4.10. *Let K be a compact Hausdorff space and let $\mathcal{H}$ be a finite-dimensional real Hilbert space with $dim(\mathcal{H}) \geq 2$. Suppose $x_0 \in S(\mathcal{H})$ and $\mathcal{O} \neq \emptyset$ is an open subset of K. Let us denote $p = \chi_{\mathcal{O}}$, and*

$$N = \{a \in C(K, \mathcal{H}) : ap = \mu x_0 \otimes p, \text{ for some } \mu \in \mathbb{R}\}.$$

Then the closed unit ball of the real JB-triple N coincides with the closed convex hull of its extreme points. Consequently, $\mathcal{B}_N$ satisfies the strong Mankiewicz property.*

Proof. Every function $a \in N$ is constant on the compact subset $\overline{\mathcal{O}}$. By replacing K with the compact quotient space $\tilde{K} = K/\overline{\mathcal{O}}$, we can assume without loss of generality, that $\overline{\mathcal{O}}$ is a single point t_0 in K and N is the real JB*-subtriple of $C(K, \mathcal{H})$ of all functions $a \in C(K, \mathcal{H})$ such that $a(t_0) \in \mathbb{R}x_0$.

Let $a \in \mathcal{B}_N$. If $a(t_0) = 0$, arguing as in the proof of Lemma 4.6(b), for each $0 < \varepsilon < 1$, we can find $\widetilde{a} \in \mathcal{B}_{C(K,\mathcal{H})}$, with $\widetilde{a}(t_0) = (\varepsilon/2)x_0$ and such that $\|a - \widetilde{a}\| \leq \varepsilon/2$. Clearly, $\tilde{a} \in \mathcal{B}_N$ and does not vanish on

t_0. By Lemma 4.6(b) applied to $\tilde{a}$, there exist non-vanishing continuous functions $b_1, \ldots, b_k$ in $\mathcal{B}_{C(K,\mathcal{H})}$ such that $b_j(t_0) = \tilde{a}(t_0) = (\varepsilon/2)x_0$, for every $j \in \{1, \ldots, k\}$, and $\left\| \tilde{a} - \frac{1}{k} \sum_{j=1}^{k} b_j \right\| \leq \varepsilon$. Furthermore, for each j in $\{0, \ldots, k\}$ there exist $v_j \in C(K, \mathcal{H})$ satisfying $\|v_j(t)\| = 1$, and $(b_j(t)|v_j(t)) = 0$, for all $t \in K$.

To simplify the notation we write $E = C(K, \mathcal{H})$. Let us fix a non-vanishing function $b \in \mathcal{B}_E$ with $b(t_0) \in \mathbb{R}x_0$ and $v \in E$ satisfying $\|v(t)\| = 1$, and $(b(t)|v(t)) = 0$, for all $t \in K$. We set $u(\cdot) := \frac{b(\cdot)}{\|b(\cdot)\|} \in E$. It is not hard to check that u and v are tripotents in E with $E_2(u) = E^1(u) = C(K, \mathbb{R})u$, $E_2(v) = E^1(v) = C(K, \mathbb{R})v$, $\{u, u, v\} = \frac{1}{2}v$, and $\{v, v, u\} = \frac{1}{2}u$.

Let us consider the real JB*-subtriple $F = E_2(u) \oplus E_2(v) = C(K, \mathbb{R})u \oplus C(K, \mathbb{R})v$. Clearly, $N \cap F$ is a non-trivial real JB*-subtriple of F containing b. The mapping $\Psi : F \to C(K)$, $\Psi(fu + gv) = f + ig$, is a surjective isometric triple isomorphism between real JB*-triples which maps $N \cap F$ to $C(K)^{t_0}_{\mathbb{R}} = \{h \in C(K) : h(t_0) \in \mathbb{R}\}$. It follows from [44, Lemma 19] that every element in the closed unit ball of $C(K)^{t_0}_{\mathbb{R}}$ (in particular $\Psi(b)$) can be approximated in norm by convex combinations of unitary tripotents in $C(K)^{t_0}_{\mathbb{R}}$. Finally, if v is a unitary element in $C(K)^{t_0}_{\mathbb{R}}$, then $w = \Re\mathrm{e}(v)u + \Im\mathrm{m}(v)v \in N \cap F$ with $w(t_0) = \Re\mathrm{e}(v)(t_0)u(t_0) = v(t_0)u(t_0) \in \mathbb{R}x_0$ and $\|w(t)\|^2_{\mathcal{H}} = |\Re\mathrm{e}(v)(t)|^2 + |\Im\mathrm{m}(v)(t)|^2 = 1$, for all $t \in K$, witnessing that $w \in \partial_e(\mathcal{B}_E)$. □

5. $C(K, H)$ satisfies the Mazur–Ulam property

Throughout this section K and H will denote a compact Hausdorff space and a complex Hilbert space with $\dim(H) \geq 2$, respectively.

Given an element y_0 in a Banach space Y, we write τ_{y_0} for the translation by the element y_0, that is, $\tau_{y_0}(y) = y + y_0$, for all $y \in Y$.

Our first lemma is essentially contained in [40] and [15, Lemma 2.1], and its proof can be easily deduced from the arguments in the just quoted references.

Lemma 5.1. *Let $\Delta : S(C(K, H)) \to S(Y)$ be a surjective isometry, where Y is a real Banach space. Then for each $t_0 \in K$ and each $x_0 \in S(H)$ the set*

$$supp_{\Delta}(t_0, x_0) := \{\psi \in Y^* : \|\psi\| = 1, \text{ and } \psi^{-1}(\{1\}) \cap \mathcal{B}_Y = \Delta(A(t_0, x_0))\}$$

is a non-empty weak-closed face of $\mathcal{B}_{Y^*}$.* □

In the hypothesis of the previous lemma, it is known that each $A(t_0, x_0)$ is an intersection face in the sense employed in [44]. Therefore, Lemma 8 in [44] assures that $\Delta(-A(t_0, x_0)) = -\Delta(A(t_0, x_0))$, and consequently,

$$\psi\Delta(a) = -1, \text{ for all } a \in -A(t_0, x_0), \text{ and all } \psi \in \mathrm{supp}_{\Delta}(t_0, x_0). \tag{12}$$

The following technical lemma might be known, although an explicit reference is out from our knowledge. We include here a proof, which seems to be new, and is based on techniques of real JB*-triples.

Lemma 5.2. *Let $(\mathcal{H}, (.|.))$ be a real Hilbert space, K a compact Hausdorff space, and φ a non-zero functional in $C(K, \mathcal{H})^*$. Suppose there exist $t_0 \in K$, $x_0 \in S(\mathcal{H})$, and an open neighborhood $\mathcal{O}$ of t_0 satisfying $\varphi(b) = \|\varphi\|$ for every $b \in A(t_0, x_0)$ whose cozero-set is contained in $\mathcal{O}$. Then $\varphi(a) = \|\varphi\|(a(t_0)|x_0) = \|\varphi\|(x_0^* \otimes \delta_{t_0})(a)$, for all $a \in C(K, \mathcal{H})$.*

Proof. Let us assume that $\|\varphi\| = 1$. Let 1 denote the unit element in $C(K)$. Since the element $e = x_0 \otimes 1$ is a non-zero tripotent in the real JB*-triple $C(K, \mathcal{H})$ with $e \in A(t_0, x_0)$, it follows from the hypothesis that

$\varphi(e) = \|\varphi\| = 1$. An application of [47, Lemma 2.7] shows that $\varphi(a) = \varphi P^1(e)(a)$, for every $a \in C(K, \mathcal{H})$. It is not hard to see that

$$\{e, a, e\}(t) = (e(t)|a(t))e(t) = (a(t)|x_0)x_0 = x_0 \otimes (a|x_0)(t),$$

and hence $P^1(e)(a) = (a|x_0)x_0 = x_0 \otimes (a|x_0)$, and

$$\varphi(a) = \varphi P^1(e)(a) = \varphi(x_0 \otimes (a|x_0))$$

for every $a \in C(K, \mathcal{H})$. This shows that $\varphi = \varphi|_{C(K,\mathbb{R}x_0)}$ can be identified with a norm-one functional in $C(K, \mathbb{R}x_0)^* \cong C(K, \mathbb{R})^*$. The norm-one functional $\psi = \varphi|_{C(K,\mathbb{R}x_0)} \in C(K, \mathbb{R})^*$ satisfies that $\psi(f) = 1$ for every $f \in C(K)$ with $\|f\| = 1 = f(t_0)$. It is not hard to see, via Urysohn's lemma, that $\ker(\psi)$ contains all $f \in \mathcal{B}_{C(K,\mathbb{R})}$ vanishing on a open neighborhood of t_0 contained in $\mathcal{O}$. Therefore, ψ vanishes on every function $f \in C(K)$ with $f(t_0) = 0$, and thus $\psi(g) = g(t_0)$ for all $g \in C(K, \mathbb{R})$, and consequently $\varphi(a) = (a(t_0)|x_0)$, for all $a \in C(K, \mathcal{H})$. $\square$

According to the notation in [44], given a face F contained in the unit sphere of a Banach space X and $\lambda \in [-1, 1]$ we set

$$\begin{aligned} F_\lambda &:= \{s \in S(X) : \operatorname{dist}(x, F) \le 1 - \lambda,\ \operatorname{dist}(x, -F) \le 1 + \lambda\} \\ &= \{s \in S(X) : \operatorname{dist}(x, F) = 1 - \lambda,\ \operatorname{dist}(x, -F) = 1 + \lambda\}. \end{aligned}$$

Let p be a projection in the bidual, A^{**}, of a C*-algebra A, whose cone of positive elements will be denoted by A^+. Following [1,21], we say that p is *compact* if p is closed relative to A (i.e. $A \cap (1-p)A^{**}(1-p)$ is weak*-dense in $(1-p)A^{**}(1-p)$) and there exists a norm-one element $x \in A^+$ such that $p \le x$ (compare [1, page 422]). In our setting, for each closed (i.e. compact) subset $C \subseteq K$, the projection χ_C is compact in $C(K)^{**}$ and rarely lies in $C(K)$.

As in [44], for $\lambda \in [-1, 1]$, we define

$$F^A(p, \lambda) := \{x \in S(A) : xp = px = \lambda p\} = S(A) \cap \{\lambda p + y : y \in \mathcal{B}_{(1-p)A^{**}(1-p)}\}.$$

We observe that $F^A(p, 1) = F^A(p) = A \cap (p \oplus \mathcal{B}_{(1-p)A^{**}(1-p)})$ is precisely the norm-closed face of $\mathcal{B}_A$ associated with the projection p (compare [1]). It is established in [44, Lemma 17] that, under these circumstances, we have $F^A(p, \lambda) = \left(F^A(p)\right)_\lambda$. Our next goal is to obtain a version of this fact in the setting of continuous functions valued in a Hilbert space.

Lemma 5.3. *Let $\mathcal{C}$ be a closed subset of K, and let x_0 be a norm-one element in H. Let $p = \chi_{\mathcal{C}}$, and let $F_{x_0 \otimes p}$ be the set defined in (11). For each $\lambda \in [0, 1]$ set*

$$F(x_0 \otimes p, \lambda) := \{a \in S(C(K, H)) : ap = \lambda x_0 \otimes p\}.$$

Then $F(x_0 \otimes p, \lambda) = (F_{x_0 \otimes p})_\lambda$.

Proof. ($\supseteq$) Let $a \in (F_{x_0 \otimes p})_\lambda$. We fix $t_0 \in \mathcal{C}$. For each $\varepsilon > 0$ there exist $b \in F_{x_0 \otimes p}$ and $c \in -F_{x_0 \otimes p}$ such that $\|a(t_0) - x_0\|_H \le \|a - b\| < 1 - \lambda + \varepsilon$ and $\|a(t_0) + x_0\|_H \le \|a - c\| < 1 + \lambda + \varepsilon$. The arbitrariness of $\varepsilon > 0$ implies that $\|a(t_0) - x_0\|_H \le 1 - \lambda$ and $\|a(t_0) + x_0\|_H \le 1 + \lambda$, which proves that $a(t_0) = \lambda x_0$.

($\subseteq$) Let us take $a \in F(x_0 \otimes p, \lambda)$. To simplify the notation, let us write $E = C(K, H)$. Since H is a (complex) Hilbert space, we can identify E^{**} with the Banach space $C(\tilde{K}, (H, w))$ of all continuous functions from $\tilde{K}$ to H when this latter space is provided with its weak topology, where $\tilde{K}$ is a compact Hausdorff space such that $C(K)^{**} \equiv C(\tilde{K})$ (see [11, Theorem 2]).

The set $F_{x_0\otimes p}$ is a proper norm-closed face of $\mathcal{B}_E$, it is actually the face associated with the compact tripotent $e = x_0 \otimes p \in E^{**} \equiv C(\tilde{K}, (H, w))$. It has been recently shown in [6, Theorem 3.6] that the weak*-closure of $F_{x_0\otimes p}$ in E^{**} is precisely the proper weak*-closed face of $\mathcal{B}_{C(K,H)^{**}}$ associated with the compact tripotent e, that is, $\overline{F_{x_0\otimes p}}^{w^*} = F_e^{E^{**}} = e + \mathcal{B}_{E_0^{**}(e)}$. Clearly, the element $e + a(1-p)$ belongs to $F_e^{E^{**}} = e+\mathcal{B}_{E_0^{**}(e)}$ and $\|a-(e+a(1-p))\| = \|\lambda x_0\otimes p - x_0\otimes p\| = 1-\lambda$. We deduce that $\operatorname{dist}(a, \overline{F_{x_0\otimes p}}^{w^*}) \leq 1-\lambda$. Now, an application of the Hahn-Banach separation theorem gives $\operatorname{dist}(a, F_{x_0\otimes p}) = \operatorname{dist}(a, \overline{F_{x_0\otimes p}}^{w^*}) \leq 1-\lambda$. If in the above argument we replace $e+a(1-p)$ by $-e+a(1-p)$, we derive $\operatorname{dist}(a, -F_{x_0\otimes p}) \leq 1+\lambda$. □

The following proposition is a first step to obtain a linear extension of a surjective isometry between the unit spheres of $C(K,H)$ and any Banach space Y. We shall show that such isometries are affine on the maximal proper faces of $\mathcal{B}_{C(K,H)}$ using an adaptation of the arguments in [44, Proposition 20].

Proposition 5.4. *Let $\Delta : S(C(K,H)) \to S(Y)$ be a surjective isometry, where Y is a real Banach space. Suppose $t_0 \in K$ and $x_0 \in S(H)$. Then there exist a net $(\mathcal{R}_\lambda)_\lambda$ of convex subsets of $A(t_0,x_0)$ and a net $(\theta_\lambda)_\lambda$ of affine contractions from $A(t_0,x_0)$ into $\mathcal{R}_\lambda$ such that $\theta_\lambda \to Id$ in the point-norm topology. Moreover, for each λ, $\mathcal{R}_\lambda$ satisfies the strong Mankiewicz property and $\Delta(\mathcal{R}_\lambda)$ is convex. Consequently $\Delta|_{A(t_0,x_0)}$ is affine.*

Proof. Fix $x_0 \in S(H)$ and $t_0 \in K$. Let us write $\varphi_0 = x_0^* \otimes \delta_{t_0} \in S(C(K,H)^*)$, and consider the norm-closed inner ideal of $C(K,H)$

$$L = \{b \in C(K,H) : \|b\|_{\varphi_0} = 0\} = \{b \in C(K,H) : b(t_0) = 0\},$$

where $\|b\|_{\varphi_0}^2 = \varphi_0\{b, b, x_0 \otimes 1\}$ for each $b \in C(K,H)$. We can always find, via Urysohn's lemma, two nets $(f_\lambda)_\lambda$, $(e_\lambda)_\lambda$ in $C(K)$ satisfying the following properties: $0 \leq e_\lambda \leq f_\lambda \leq 1$, $e_\lambda f_\lambda = e_\lambda$ for every $\lambda \in \Lambda$, $e_\mu \geq e_\lambda$ and $f_\mu \geq f_\lambda$ for every $\mu \geq \lambda$, and

$$\|f_\lambda b - b\| \underset{\lambda}{\to} 0,\ \|bf_\lambda - b\| \underset{\lambda}{\to} 0,\ \|e_\lambda b - b\| \underset{\lambda}{\to} 0,\ \text{and } \|be_\lambda - b\| \underset{\lambda}{\to} 0, \quad \forall b \in L.$$

We shall say that $(f_\lambda)_\lambda$ and $(e_\lambda)_\lambda$ are *module-approximate units* for L. We can actually assume that each f_λ (and hence each e_λ) vanishes on an open neighborhood of t_0.

We define now $\theta_\lambda : C(K,H) \to C(K,H)$, $\theta_\lambda(c) := x_0 \otimes (1-e_\lambda) + ce_\lambda$. Since for each $a \in A(t_0,x_0)$ the element $a - x_0 \otimes 1$ belongs to L, we deduce that

$$\|\theta_\lambda(a) - a\| = \|(a - x_0 \otimes 1)e_\lambda - (a - x_0 \otimes 1)\| \underset{\lambda}{\to} 0.$$

Clearly θ_λ is an affine mapping for every λ, and $c \in \mathcal{B}_{C(K,H)}$, $\theta_\lambda(c)$ lies in $A(t_0,x_0)$. Finally it is worth noting that θ_λ is contractive.

From now on we fix a subindex λ, and thus we shall write e, f and θ for e_λ, f_λ and θ_λ, respectively. Let us consider the open subset $\mathcal{O} \subseteq K$ given by $\mathcal{O} := (1-f)^{-1}(\mathbb{R} \setminus \{0\})$. By construction $t_0 \in \mathcal{O}$ and $p := \chi_{\mathcal{O}} \in C(K)^{**}$ is the range projection of $(1-f)$ in $C(K)^{**}$.

We consider next the norm-closed subtriples $F \subseteq N \subseteq B \subseteq C(K,H)$ defined in (11), that is,

$$F = F_{x_0\otimes p} := \{a \in S(C(K,H)) : ap = x_0 \otimes p\},$$
$$B = B_p := \{a \in C(K,H) : ap = h \otimes p,\ \text{for some } h \in H\},$$
$$N = N_p^{x_0} := \{a \in C(K,H) : ap = \mu x_0 \otimes p,\ \text{for some } \mu \in \mathbb{C}\}.$$

Given $a \in A(t_0, x_0)$, we have $\theta(a)(1-f) = x_0 \otimes (1-f)$, which proves that $\theta(a) \in F$. We therefore conclude that $\theta|_{A(t_0,x_0)} : A(t_0, x_0) \to F \subseteq A(t_0, x_0)$.

As we previously commented in section 4, we cannot, in general, assume that $p \in C(K)$, so, we shall distinguish the different cases.

Case 1: We assume that $p \in C(K)$. In this case we consider the following norm-closed face of $\mathcal{B}_{C(K,H)}$

$$\mathcal{R} = \mathcal{R}_\lambda = (x_0 \otimes p) + \mathcal{B}_{p^\perp C(K,H)} = F \subseteq A(t_0, x_0),$$

where $\mathcal{B}_{p^\perp C(K,H)} \equiv \mathcal{B}_{C((1-p)K,H)}$. Proposition 4.5 implies that $\mathcal{R}$ satisfies the strong Mankiewicz property because the translation $\tau_{-x_0 \otimes p}$ is a surjective affine isometry.

It is not hard to see that

$$\theta(a) = x_0 \otimes (1-e) + ae = x_0 \otimes p + (x_0 \otimes (1-e-p)) + ae \in \mathcal{R},$$

and thus $\theta(A(t_0, x_0)) \subseteq \mathcal{R}$.

Having in mind that $\mathcal{R}$ is an intersection face in the sense employed in [44, Lemma 8], the just quoted result implies that $\Delta(F)$ also is an intersection face, and in particular a non-empty convex set. Since $\mathcal{R}$ satisfies the strong Mankiewicz property we deduce that $\Delta|_{\mathcal{R}}$ is affine.

Case 2: We assume that $p \notin C(K)$. We claim that, in this case, $\overline{\mathcal{O}} \cap (K \setminus \mathcal{O}) \neq \emptyset$. Otherwise, $\overline{\mathcal{O}} \cap (K \setminus \mathcal{O}) = \emptyset$, and hence $K = \mathcal{O} \mathring{\cup} (K \setminus \mathcal{O}) \subseteq \overline{\mathcal{O}} \mathring{\cup} (K \setminus \mathcal{O}) \subseteq K$, which proves that $\mathcal{O}$ is clopen. Therefore $p = \chi_{\mathcal{O}}$ is continuous, leading to a contradiction.

Following the construction in Section 4, we shall consider the linear mapping $T : B \to H$ given by $T(a) = a(t_0)$ for each $a \in B$. We have seen in Section 4 that T is a triple homomorphism. Let us now take $a \in B$ and write $a(t_0) = x_a$. By applying that $\overline{\mathcal{O}} \cap (K \setminus \mathcal{O}) \neq \emptyset$ we deduce that $\|a\| = \|a|_{(K \setminus \mathcal{O})}\| \geq \|a|_{\mathcal{O}}\| = \|x_a\|$. It follows that $\|T(a)\| = \|a(t_0)\| = \|x_a\| \leq \|a\|$. The arbitrariness of $a \in B$ proves that T is continuous and contractive.

Since N is a JB*-triple of B the restriction $T|_N : N \to H$ also is a triple homomorphism, and thus the linear functional $\varphi \equiv x_0^* \circ T|_N : N \to \mathbb{C}$ is a continuous triple homomorphism. Proposition 4.9 now assures that the closed unit ball of $N_{\mathbb{R}}^{\varphi} := \varphi^{-1}(\mathbb{R})$ satisfies the strong Mankiewicz property.

In this case we set

$$\mathcal{R}_\lambda = \mathcal{R} := (x_0 \otimes (1-f)) + f\mathcal{B}_{N_{\mathbb{R}}^{\varphi}} \subseteq A(t_0, x_0).$$

Clearly $\mathcal{R}$ satisfies the strong Mankiewicz property.

Let us take $a \in F$. Since $(1-f)a = x_0 \otimes (1-f)$, we deduce that $a = (1-f)a + fa = x_0 \otimes (1-f) + fa$, with $ap = x_0 \otimes p$, $\|a\| = 1$ and $\varphi(a) = 1$. We have therefore shown that $F \subseteq \mathcal{R}$.

Let us prove that $\theta(A(t_0, x_0)) \subseteq \mathcal{R}$. Namely, for each $a \in A(t_0, x_0)$ we write

$$\theta(a) = (x_0 \otimes (1-e)) + ae = (x_0 \otimes (1-f)) + x_0 \otimes (f-e) + ae$$
$$= (x_0 \otimes (1-e)) + f\,(x_0 \otimes (1-e) + ae)\,,$$

where $p\,(x_0 \otimes (1-e) + ae) = x_0 \otimes p$ and $\varphi\,(x_0 \otimes (1-e) + ae) = 1$, which implies that $\theta(a) \in \mathcal{R}$.

We shall next show that $\Delta(\mathcal{R})$ is convex. The rest of the proof is just an adaptation of the proof of [44, Lemma 20], the argument is included here for completeness.

Let us follow the notation in [44]. Given $\gamma \in [-1, 1]$ we define $h_\gamma : [0, 1] \to [-1, 1]$, $h_\gamma(t) := t + (1-t)\gamma$. For $i \in \{1, 2\}$ and $m \in \mathbb{N}$ we set

$$G_m^i := A(t_0,x_0)\bigcap\left(\bigcap_{\substack{k=1\\ \chi_{[\frac{2k-2+i}{2m},\frac{2k-1+i}{2m}]}(1-f)\neq 0}}^{2m} F\left(x_0\otimes\chi_{[\frac{2k-2+i}{2m},\frac{2k-1+i}{2m}]}(1-f), h_\gamma(\frac{k}{m})\right)\right),$$

and $H_m^i(\gamma) := A(t_0,x_0)\cap\mathcal{N}_{\frac{1}{m}}(G_m^i(\gamma))$, where $\mathcal{N}_\delta(G_m^i(\gamma))$ is the δ-neighborhood around $G_m^i(\gamma)$.

Given $\gamma_1,\gamma_2\in[-1,1]$ and $\alpha\in[0,1]$, Lemma 5.3 and [44, Lemma 10] assure that for $\gamma_3=\alpha\gamma_1+(1-\alpha)\gamma_2$ we have

$$\alpha G_m^i(\gamma_1)+(1-\alpha)G_m^i(\gamma_2)\subseteq G_m^i(\gamma_3),$$

and

$$\alpha H_m^i(\gamma_1)+(1-\alpha)H_m^i(\gamma_2)\subseteq H_m^i(\gamma_3).$$

Following the ideas in the proof of [44, Proposition 20] we shall next show that

$$\mathcal{R}(\gamma) := \{a\in A(t_0,x_0) : pa = x_0\otimes h_\gamma(1-f)\} = \bigcap_{m\in\mathbb{N}}\left(H_m^1(\gamma)\cap H_m^2(\gamma)\right). \tag{13}$$

($\subseteq$) Consider the function $g_m:[0,1]\to\mathbb{R}$ given by

$$g_m(t) := \begin{cases}\gamma, & \text{if } t=0\\ h_\gamma(\frac{k}{m}), & \text{if } t\in[\frac{2k-1}{2m},\frac{2k}{2m}] \text{ with } k\in\{1,\ldots,m\}\\ \text{affine}, & \text{in the rest.}\end{cases}$$

By definition $\|g_m-h_\gamma\|_{C(K)}\leq\frac{1}{m}$ and $(g_m-h_\gamma)(0)=0$, which assures that $(g_m-h_\gamma)(1-f)=p(g_m-h_\gamma)(1-f)\in pC(K)$. For each $a\in\mathcal{R}(\gamma)$ we have

$$\begin{aligned}p\Big(a+x_0\otimes(g_m-h_\gamma)(1-f)\Big) &= x_0\otimes h_\gamma(1-f)+x_0\otimes(g_m-h_\gamma)(1-f)\\ &=\sum_{k=1}^m x_0\otimes h_\gamma(\frac{k}{m})\chi_{[\frac{2k-1}{2m},\frac{2k}{2m}]}(1-f),\end{aligned}$$

therefore $b_m := a+x_0\otimes(g_m-h_\gamma)(1-f)\in G_m^1(\gamma)$ and $\|a-b_m\| = \|x_0\otimes(g_m-h_\gamma)(1-f)\|\leq\|g_m-h_\gamma\|\leq\frac{1}{m}$, witnessing that $a\in H_m^1(\gamma)$. We can similarly show that $a\in H_m^1(\gamma)$ for every natural m.

($\supseteq$) Take now $a\in\bigcap_{m\in\mathbb{N}}\left(H_m^1(\gamma)\cap H_m^2(\gamma)\right)$. For each natural m, we can find $b_m^i\in G_m^i(\gamma)$ satisfying $\|a-b_m^i\|\leq\frac{1}{m}$. Let us consider the projection $p_m^i=\sum_{k=1}^m\chi_{[\frac{2k-2+i}{2m},\frac{2k-1+i}{2m}]}(1-f)\in C(K)^{**}$, where $i\in\{1,2\}$. Since $b_m^i\in G_m^i(\gamma)$, we have $\|x_0\otimes h_\gamma(1-f)p_m^i-b_m^ip_m^i\|\leq\frac{1}{m}$, and hence

$$\|ap_m^i-x_0\otimes h_\gamma(1-f)p_m^i\|\leq\frac{2}{m},\ \text{and}\ \|a(p_m^1+p_m^2)-x_0\otimes h_\gamma(1-f)(p_m^1+p_m^2)\|\leq\frac{2}{m},$$

for every natural m. Having in mind that $p_m^1+p_m^2=\chi_{[\frac{1}{2m},1]}(1-f)$, we deduce that $ap=x_0\otimes h_\gamma(1-f)$, which finishes the proof of (13).

Finally, since clearly $\mathcal{R}=\bigcup_{\gamma\in[-1,1]}\mathcal{R}(\gamma)$ and by [44, Lemma 11]

$$\Delta^{-1}(\alpha\Delta(\mathcal{R}(\gamma_1))+(1-\alpha)\Delta(\mathcal{R}(\gamma_2)))\subseteq\bigcap_{m\in\mathbb{N}}\left(H_m^1(\gamma_3)\cap H_m^2(\gamma_3)\right),$$

for all $\alpha \in [0,1]$, $\gamma_1, \gamma_2 \in [-1,1]$ and $\gamma_3 = \alpha\gamma_1 + (1-\alpha)\gamma_2$, we prove that $\Delta(\mathcal{R})$ is convex.

Summarizing, we have proved that each $\mathcal{R}_\lambda$ satisfies the strong Mankiewicz property, $\Delta(\mathcal{R}_\lambda)$ is convex, $\theta_\lambda(A(t_0,x_0)) \subseteq \mathcal{R}_\lambda$ and $\|\theta_\lambda(a) - a\| \to 0$ for each $a \in A(t_0,x_0)$. Therefore $\Delta|_{\mathcal{R}_\lambda}$ is an affine mapping, and consequently, $\Delta|_{A(t_0,x_0)}$ is affine too. □

We shall need a more elaborated discussion on the conclusions of Proposition 5.4.

Proposition 5.5. *Let $\Delta : S(C(K,H)) \to S(Y)$ be a surjective isometry, where Y is a real Banach space. Suppose $t_0 \in K$ and $x_0 \in S(H)$. Then for each $\psi \in Y^*$ there exist ϕ_0 in $C(K,H)^*_{\mathbb{R}}$ and $\gamma_0 \in \mathbb{R}$ satisfying $\|\phi_0\| \le \|\psi\|$ and*

$$\psi\Delta(a) = \phi_0(a) + \gamma_0, \text{ for all } a \in A(t_0,x_0).$$

Proof. Let us fix $t_0 \in K$ and $x_0 \in S(H)$. Let us fix $\psi \in Y^*$. We can assume, without loss of generality, that $\|\psi\| = 1$. By Proposition 5.4 and its proof there exist a net $(\mathcal{R}_\lambda)_\lambda$ of convex subsets of $A(t_0,x_0)$ and a net $(\theta_\lambda)_\lambda$ of affine contractions from $A(t_0,x_0)$ into $\mathcal{R}_\lambda$ such that $\theta_\lambda \to \mathrm{Id}$ in the point-norm topology. Moreover, for each λ, $\mathcal{R}_\lambda$ satisfies the strong Mankiewicz property and $\Delta(\mathcal{R}_\lambda)$ is convex. We further know that one of the following statements holds for each $\mathcal{R}_\lambda$:

Case 1: $\mathcal{R}_\lambda = (x_0 \otimes p) + \mathcal{B}_{(1-p)C(K,H)}$, where p is a projection in $C(K)$. We consider in this case the surjective isometry $\Delta_\lambda : \mathcal{B}_{(1-p)C(K,H)} \to \Delta(\mathcal{R}_\lambda)$ defined by the following diagram:

$$\begin{array}{ccc} \mathcal{R}_\lambda & \xrightarrow{\ \Delta|_{\mathcal{R}_\lambda}\ } & \Delta(\mathcal{R}_\lambda) \\ {\scriptstyle \tau_{-x_0\otimes p}}\downarrow & \nearrow{\scriptstyle \Delta_\lambda} & \\ \mathcal{B}_{(1-p)C(K,H)} & & \end{array}$$

By Proposition 4.5 and Theorem 4.1 there exist $c_\lambda \in \Delta(\mathcal{R}_\lambda)$ and a linear isometry $T_\lambda : (1-p)C(K,H) \to Y$ such that $\Delta(b) = c_\lambda + T_\lambda(b - x_0 \otimes p)$ for all $b \in \mathcal{R}_\lambda$. By regarding T_λ as a linear contraction, $\tilde{T}_\lambda$, from $C(K,H)$ to Y defined by $\tilde{T}_\lambda(a) := T_\lambda(a(1-p))$, we deduce that

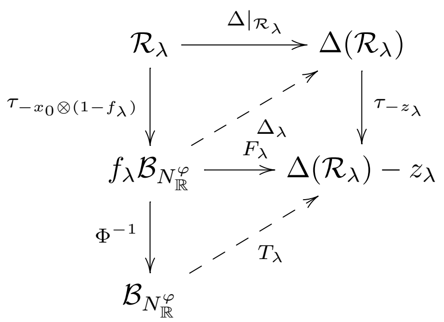
$$\psi\Delta(b) = \gamma_\lambda + \phi_\lambda(b), \text{ for all } b \in \mathcal{R}_\lambda,$$

where $\gamma_\lambda = \psi(c_\lambda)$ is a real number in $[-1,1]$ and $\phi_\lambda = \psi \circ \tilde{T}_\lambda \in \mathcal{B}_{C(K,H)^*_{\mathbb{R}}}$.

Case 2: $\mathcal{R}_\lambda = (x_0 \otimes (1-f_\lambda)) + f_\lambda \mathcal{B}_{N^\varphi_{\mathbb{R}}}$, where $f_\lambda \in S(C(K))$ with $0 \le f_\lambda \le 1$, $p_\lambda \in C(K)^{**} \backslash C(K)$ is the range projection of $1 - f_\lambda$ and N_λ is the JB*-subtriple of $C(K,H)$ defined by $N_\lambda = \{a \in C(K,H) : ap_\lambda = \mu x_0 \otimes p_\lambda, \text{ for some } \mu \in \mathbb{C}\}$. Furthermore, suppose p_λ is the characteristic function of the open set $\mathcal{O}_\lambda$, then we have $\overline{\mathcal{O}_\lambda} \cap (K \setminus \mathcal{O}_\lambda) \neq \emptyset$. We consider in this case the surjective isometry $\Delta_\lambda : f_\lambda \mathcal{B}_{N^\varphi_{\mathbb{R}}} \to \Delta(\mathcal{R}_\lambda)$ defined by the following diagram:

$$\begin{array}{ccc} \mathcal{R}_\lambda & \xrightarrow{\ \Delta|_{\mathcal{R}_\lambda}\ } & \Delta(\mathcal{R}_\lambda) \\ {\scriptstyle \tau_{-x_0\otimes(1-f_\lambda)}}\downarrow & \nearrow{\scriptstyle \Delta_\lambda} & \\ f_\lambda \mathcal{B}_{N^\varphi_{\mathbb{R}}} & & \end{array}$$

Proposition 4.9 assures that $f_\lambda \mathcal{B}_{N_{\mathbb{R}}^{\varphi}}$ satisfies the strong Mankiewicz property, and hence Δ_λ is affine. Since $f_\lambda(1-p_\lambda) = 1 - p_\lambda$, we can easily deduce that the mapping $\Phi_\lambda : a \mapsto f_\lambda a$ is a surjective affine isometry from $\mathcal{B}_{N_{\mathbb{R}}^{\varphi}}$ onto $f_\lambda \mathcal{B}_{N_{\mathbb{R}}^{\varphi}}$. Let $z_\lambda := \Delta(x_0 \otimes (1 - f_\lambda))$. We complete now the previous diagram



The mappings $F_\lambda = \Delta_\lambda - z_\lambda$ and T_λ are affine and map zero to zero. Let $\tilde{T}_\lambda : N_{\mathbb{R}}^{\varphi} \to Y$ be a bounded linear operator whose restriction to $\mathcal{B}_{N_{\mathbb{R}}^{\varphi}}$ is T_λ. Clearly, $\|\tilde{T}_\lambda\| \leq 1$. Let $\phi_\lambda \in \mathcal{B}_{C(K,H)_{\mathbb{R}}^*}$ be a Hahn-Banach extension of $\psi \circ \tilde{T}_\lambda \circ \Phi^{-1} \in \left(N_{\mathbb{R}}^{\varphi}\right)^*$. It follows from the previous diagram that

$$\psi\Delta(b) = \phi_\lambda(b) + \gamma_\lambda,$$

for every $b \in \mathcal{R}_\lambda$, where $\gamma_\lambda = -\phi_\lambda(x_0 \otimes (1 - f_\lambda)) + \psi(z_\lambda)$ is a real number in the interval $[-2, 2]$.

We have therefore shown that for each index λ there exist a functional ϕ_λ in $\mathcal{B}_{C(K,H)_{\mathbb{R}}^*}$ and a real $\gamma_\lambda \in [-2, 2]$ satisfying

$$\psi\Delta(b) = \phi_\lambda(b) + \gamma_\lambda, \text{ for every } b \in \mathcal{R}_\lambda. \tag{14}$$

Having in mind that $\mathcal{B}_{C(K,H)_{\mathbb{R}}^*}$ is weak*-compact (and the compactness of $\mathcal{B}_{\mathbb{R}}$), we can find $\phi_0 \in \mathcal{B}_{C(K,H)_{\mathbb{R}}^*}$, $\gamma_0 \in \mathbb{R}$, and common subnets $(\phi_\mu)_\mu$ and $(\gamma_\mu)_\mu$ converging to ϕ_0 and to γ_0 in the weak* and norm topologies of $C(K,H)_{\mathbb{R}}^*$ and $\mathbb{R}$, respectively. Since, for each $a \in A(t_0, x_0)$ the net $(\theta_\mu(a))_\mu \subseteq \mathcal{R}_\lambda$ converges in norm to a, we can easily deduce from (14) that $\psi\Delta(a) = \phi_0(a) + \gamma_0$, for every $a \in A(t_0, x_0)$. $\square$

We can now state the main result of this section.

Theorem 5.6. *Let K be a compact Hausdorff space and let H be a complex Hilbert space. Then the Banach space $C(K,H)$ satisfies the Mazur–Ulam property (as a real Banach space), that is, for each surjective isometry $\Delta : S(C(K,H)) \to S(Y)$, where Y is a real Banach space, there exists a surjective real linear isometry from $C(K,H)$ onto Y whose restriction to $S(C(K,H))$ is Δ.*

Proof. Let us fix $t_0 \in K$, $x_0 \in S(H)$, and $\psi \in \operatorname{supp}_\Delta(t_0, x_0)$ (cf. Lemma 5.1). We first observe that if $K = \{t_0\}$, then $C(K,H)$ is isometrically isomorphic to H, and thus the desired conclusion follows, for example, from [6, Proposition 4.15].

We claim that

$$\psi\Delta(u) = \Re\mathrm{e}\langle u(t_0)|x_0\rangle, \text{ for all } u \in \partial_e(\mathcal{B}_{C(K,H)}). \tag{15}$$

Let us take $t_1 \in K \backslash \{t_0\}$ and open neighborhoods $\mathcal{O}_1$, $\mathcal{O}_2$ and $\mathcal{O}_3$ such that $\overline{\mathcal{O}_1} \subset \mathcal{O}_2$, $t_0 \in \mathcal{O}_1$, $t_1 \in \mathcal{O}_3$, and $\mathcal{O}_2 \cap \mathcal{O}_3$. Let $f, g \in C(K)$ whose cozero-sets are contained in $\mathcal{O}_2$ and $\mathcal{O}_3$, respectively, $f(t_0) = 1$ and $g(t_1) = 1$. Given $u \in \partial_e(\mathcal{B}_{C(K,H)})$, Proposition 5.5, applied to the face $A(t_1, u(t_1))$ and ψ, implies the existence of a functional $\phi \in \mathcal{B}_{C(K,H)_{\mathbb{R}}^*}$ and a real γ satisfying

$$\psi\Delta(a) = \phi(a) + \gamma, \text{ for all } a \in A(t_1, u(t_1)). \tag{16}$$

For each $b \in A(t_0, x_0)$, the elements $gu \pm fb$ belong to $A(t_1, u(t_1))$ and to $\pm A(t_0, x_0)$. Combining (16), Lemma 5.1 and (12), we get

$$\pm 1 = \psi\Delta(gu \pm fb) = \pm\phi(fb) + \phi(gu) + \gamma = \pm\phi(fb) + \psi\Delta(gu).$$

We therefore deduce that $\psi\Delta(gu) = 0$ and $\phi(fb) = 1$ for every $b \in A(t_0, x_0)$ and every f as above. In particular, $\phi(fb) = 1$ for every $b \in A(t_0, x_0)$ whose cozero-set is contained in $\mathcal{O}_1$. Lemma 5.2 assures that $\phi(a) = \Re\text{e}\langle a(t_0)|x_0\rangle$, for all $a \in C(K, \mathcal{H})$. Since $u \in A(t_1, u(t_1))$, (16) implies that $\phi(u) = \Re\text{e}\langle u(t_0)|x_0\rangle$, which finishes the proof of the claim in (15).

Now, Corollary 4.7 combined with (15) and the final conclusion in Proposition 5.4 prove that $\psi\Delta(a) = \Re\text{e}\langle a(t_0)|x_0\rangle$, for all a in a maximal face of the form $A(t_2, x_2)$ with $t_2 \in K, x_2 \in S(H)$. Since every $a \in S(C(K, H))$ belongs to a maximal face of the form $A(t_2, x_2)$ with $t_2 \in K, x_2 \in S(H)$, we conclude that

$$\psi\Delta(a) = \Re\text{e}\langle a(t_0)|x_0\rangle, \text{ for all } a \in S(C(K, H)). \tag{17}$$

Finally, we consider the families $\{\Re\text{e}x_0^* \otimes \delta_{t_0} : t_0 \in K, x_0 \in S(H)\} \subseteq S(C(K,H)^*_{\mathbb{R}})$ and $\{\psi : \psi \in \text{supp}_\Delta(t_0, x_0), t_0 \in K, x_0 \in S(H)\} \subseteq S(Y^*)$. Since the first family is norming for $C(K, H)$, the desired conclusion follows from (17) and [44, Lemma 6] (alternatively, [23, Lemma 2.1]). $\square$

The conclusion of Theorem 5.6 in the case $H = \mathbb{C}$ is a consequence of [44, Theorem 1]. The case in which H is a real Hilbert spaces is not fully covered by our theorem. R. Liu proved in [40, Corollary 6] that $C(K, \mathbb{R})$ satisfies the Mazur–Ulam property whenever K is a compact Hausdorff space. Let $\mathcal{H} = \ell_2(\Gamma, \mathbb{R})$ be a real Hilbert space with inner product $(\cdot|\cdot)$. Suppose $\dim(\mathcal{H})$ is even or infinite. We can write Γ as the disjoint union of two subsets Γ_1, Γ_2 for which there exists a bijection $\sigma : \Gamma_1 \to \Gamma_2$. Let $H = \ell_2(\Gamma_1)$ denote the usual complex Hilbert space with inner product $\langle\cdot|\cdot\rangle$, and $(H_{\mathbb{R}}, \Re\text{e}\langle\cdot|\cdot\rangle)$ the underlying real Hilbert space. The mapping $(\lambda_j)_{j\in\Gamma_1} + \left(\lambda_{\sigma(j)}\right)_{j\in\Gamma_1} \mapsto \left(\lambda_j + i\lambda_{\sigma(j)}\right)_{j\in\Gamma_1}$ is a surjective real linear isometry from $\mathcal{H}$ onto $H_{\mathbb{R}}$. The next result is a straightforward consequence of our previous Theorem 5.6.

Corollary 5.7. *Let K be a compact Hausdorff space and let $\mathcal{H}$ be a real Hilbert space with $\dim(\mathcal{H})$ even or infinite. Then the real Banach space $C(K, \mathcal{H})$ satisfies the Mazur–Ulam property.*

There are certain obstacles that prevent to apply the tools developed in Proposition 4.9, and Lemma 5.3 in the case of $C(K, \mathcal{H})$ when $\mathcal{H}$ is a finite-dimensional real Hilbert space with odd dimension. The difficulties in Proposition 4.9 can be solved with Proposition 4.10. If in the proof of Lemma 5.3, Theorem 3.6 replaces [6, Theorem 3.6] then the same conclusion holds for real Hilbert spaces. It is a bit more laborious, but no more than a routine exercise, to check that the arguments in Propositions 5.4 and 5.5 and Theorem 5.6 are literally valid to get the following result.

Corollary 5.8. *Let K be a compact Hausdorff space and let $\mathcal{H}$ be a finite-dimen-sional real Hilbert space with odd dimension. Then the real Banach space $C(K, \mathcal{H})$ satisfies the Mazur–Ulam property.*

The pioneer achievements of M. Jerison provide generalized versions of the Banach–Stone theorem for spaces of vector-valued continuous functions. Combining Theorem 5.6 and Corollary 5.7 with the Banach–Stone theorem in [30, Theorem 7.2.16] (see also [30, Definition 7.1.2]) we obtain next a description of the surjective isometries between the unit spheres of two $C(K, H)$ spaces.

Corollary 5.9. *Let K_1, K_2 be two compact Hausdorff spaces, let H be a real or complex Hilbert space, and let Y be a strictly convex real Banach space. Suppose $\Delta : S(C(K_1, H)) \to S(C(K_2, Y))$ is a surjective isometry.*

Then there exist a homeomorphism $h : K_2 \to K_1$ and a mapping which maps each $t \in K_2$ to a surjective linear isometry $V(t) : H \to Y$, which is continuous from K_2 into the space $B(H, Y)$ of bounded linear operators from H to Y with the strong operator topology, such that

$$\Delta(a)(t) = V(t)(a(h(t))),$$

for all $a \in S(C(K_1, H)), t \in K_2$. □

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METRIC CHARACTERISATION OF UNITARIES IN JB*-ALGEBRAS

MARÍA CUETO-AVELLANEDA, ANTONIO M. PERALTA

ABSTRACT. Let M be a unital JB*-algebra whose closed unit ball is denoted by $\mathcal{B}_M$. Let $\partial_e(\mathcal{B}_M)$ denote the set of all extreme points of $\mathcal{B}_M$. We prove that an element $u \in \partial_e(\mathcal{B}_M)$ is a unitary if and only if the set

$$\mathcal{M}_u = \{e \in \partial_e(\mathcal{B}_M) : \|u \pm e\| \leq \sqrt{2}\}$$

contains an isolated point. This is a new geometric characterisation of unitaries in M in terms of the set of extreme points of $\mathcal{B}_M$.

1. Introduction

We know from a celebrated result of R.V. Kadison that the extreme points of the closed unit ball of a C*-algebra A are precisely the maximal partial isometries in A, that is, the elements u in A such that $(1 - uu^*)A(1 - u^*u) = \{0\}$ (see [14]). Every unitary in A is an extreme point of its closed unit ball, but the reciprocal implication is not always true. In 2002, C.A. Akemann and N. Weaver searched for a characterisation of partial isometries, unitaries, and invertible elements in a unital C*-algebra A in terms of the Banach space structure of certain subsets of A, the dual space, A^*, or the predual, A_*, when A is a von Neumann algebra (cf. [1]). The resulting characterisations are called geometric because only the Banach space structure of A is employed. It should be noted that the geometric characterisation of partial isometries in a C*-algebra was subsequently extended to a geometric characterisation of tripotents in a general JB*-triple (see, [6, 7]). The geometric characterisation of untaries actually relies on a good knowledge on the *set of states* of a Banach space X relative to an element x in its unit sphere, $S(X)$, defined by

$$S_x := \{\varphi \in X^* : \varphi(x) = \|\varphi\| = 1\}.$$

The element x is called a *vertex* of the closed unit ball of X (respectively, a *geometric unitary* of X) if S_x separates the points of X (respectively, spans X^*).

Akemann and Weaver proved that a norm-one element x in a C*-algebra A is (an *algebraic*) *unitary* (i.e. $xx^* = x^*x = 1$) if and only if S_x spans A^*. In a von Neumann algebra W an analogous characterisation holds when one uses the predual, W_*, in lieu of the dual space and the *set of normal states relative to* x, $S^x = \{\varphi \in W_* : \varphi(x) = \|\varphi\| = 1\}$, in place of S_x (cf. [1, Theorem 3]).

An appropriate versions of the just commented result in the setting of JB*-algebras and JB*-triples was established by A. Rodríguez Palacios in [22] (see section 2 for the missing notions). We recall that a complex (respectively, real) *Jordan*

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algebra M is a (not-necessarily associative) algebra over the complex (respectively, real) field whose product is abelian and satisfies $(a\circ b)\circ a^2 = a\circ (b\circ a^2)$ $(a,b\in M)$. A *normed Jordan algebra* is a Jordan algebra M equipped with a norm, $\|.\|$, satisfying $\|a\circ b\|\leq \|a\|\ \|b\|$ $(a,b\in M)$. A *Jordan Banach algebra* is a normed Jordan algebra whose norm is complete. Every real or complex associative Banach algebra is a real Jordan Banach algebra with respect to the product $a\circ b := \frac12 (ab+ba)$.

An element a in a unital Jordan Banach algebra J is called invertible whenever there exists $b\in J$ satisfying $a\circ b = 1$ and $a^2\circ b = a$. The element b is unique and it will be denoted by a^{-1} (cf. [10, 3.2.9]).

A *JB***-algebra* is a complex Jordan Banach algebra M equipped with an algebra involution * satisfying $\| \{a,a,a\} \| = \|a\|^3$, $a\in M$ (we recall that $\{a,a,a\} = 2(a\circ a^*)\circ a - a^2\circ a^*$). A *JB-algebra* is a real Jordan Banach algebra J in which the norm satisfies the following two axioms for all $a,b\in J$

(i) $\|a^2\| = \|a\|^2$;
(ii) $\|a^2\|\leq \|a^2+b^2\|$.

The hermitian part, M_{sa}, of a JB*-algebra, M, is always a JB-algebra. A celebrated theorem due to J.D.M. Wright asserts that, conversely, the complexification of every JB-algebra is a JB*-algebra (see [24]). We refer to the monographs [10] and [5] for the basic notions and results in the theory of JB- and JB*-algebras.

Every C*-algebra A is a JB*-algebra when equipped with its natural Jordan product $a\circ b = \frac12 (ab+ba)$ and the original norm and involution. Norm-closed Jordan *-subalgebras of C*-algebras are called *JC***-algebras*.

Two elements a,b in a Jordan algebra M are said to *operator commute* if

$$(a\circ c)\circ b = a\circ (c\circ b),$$

for all $c\in M$. By the *centre* of M we mean the set of all elements of M which operator commute with any other element in M.

We recall that an element u in a unital JB*-algebra M is a *unitary* if it is invertible and its inverse coincides with u^*. An element s in a unital JB-algebra J is called a *symmetry* if $s^2 = 1$. The set of all symmetries in J will be denoted by $\mathrm{Symm}(J)$. If M is a JB*-algebra, we shall write $\mathrm{Symm}(M)$ for $\mathrm{Symm}(M_{sa})$.

The geometric characterisation of unitaries in JB*-algebras reads as follows: For a norm-one element u in a JB*-algebra M, the following conditions are equivalent:

(1) u is a unitary in M;
(2) u is a geometric unitary in M;
(3) u is a vertex of the closed unit ball of M,

(see [22, Theorem 3.1] and [5, Theorem 4.2.24], where the result is proved in the more general setting of JB*-triples).

Surprisingly, as shown by C.-W. Leung, C.-K. Ng, N.-C. Wong in [17], the case of JB-algebras differs slightly from the result stated for JB*-algebras. Suppose x is a norm-one element in a JB-algebra J, then the following statements are equivalent:

(a) x is a geometric unitary in J;
(b) x is a vertex of the closed unit ball of J;
(c) x is an isolated point of $\mathrm{Symm}(J)$ (endowed with the norm topology);
(d) x is a central unitary in J;
(e) The multiplication operator $M_x : z\mapsto x\circ z$ satisfies $M_x^2 = \mathrm{id}_J$,

(see [17, Theorem 2.6] or [5, Proposition 3.1.15]).

Except perhaps statement (c) above, the previous characterisations rely on the set of states S_x of the underlying Banach space at an element x in the unit sphere, that is, they are geometric characterisations in which the structure of the whole dual space plays an important role.

From a completely independent setting, the different attempts to solve the problem of extending a surjective isometry between the unit spheres of two Banach spaces to a surjective real linear isometry between the spaces (known as Tingley's problem) have produced a substantial collection of new ideas and devices which are, in most of cases, interesting by themselves (cf., for example, [2, 4, 19, 20, 21]). Let us borrow some words from [4] *"...it is really impressive the development of machinery and technics that this problem (Tingley's problem) has led to."*. We shall place our focus on the next result, included by M. Mori in [19], which provides a new characterisation of unitaries in a unital C*-algebra.

From now on, the closed unit ball of a Banach space X will be denoted by $\mathcal{B}_X$. The set of all extreme points of a convex set C will be denoted by $\partial_e(C)$.

Theorem 1.1. [19, Lemma 3.1] *Let A be a unital C*-algebra, and let $u \in \partial_e(\mathcal{B}_A)$. Then the following statements are equivalent:*

(a) *u is a unitary (i.e., $uu^* = u^*u = 1$);*
(b) *The set $\mathcal{A}_u = \{e \in \partial_e(\mathcal{B}_A) : \|u \pm e\| = \sqrt{2}\}$ contains an isolated point.*

The advantage of the previous result is that it characterises unitaries among extreme points of the closed unit ball of a unital C*-algebra A in terms of the subset of all points in $\partial_e(\mathcal{B}_A)$ at distance $\sqrt{2}$ from the element under study. We do not need to deal with the dual of A.

The purpose of this note is to explore the validity of this characterisation in the setting of JB*-algebras. In a first result we prove that for each tripotent u in a JB*-triple E the equality

$$\{e \in \text{Trip}(E_2(u)) : \|u \pm e\| \leq \sqrt{2}\} = \{i(p - q) : p, q \in \mathcal{P}(E_2(u)) \text{ with } p \perp q\}$$

holds true, where given a JB*-triple E, the symbol Trip(E) stands for the set of all tripotents in E. Furthermore, if u is unitary in E, then

$$\mathcal{E}_u = \left\{e \in \partial_e(\mathcal{B}_E) : \|u \pm e\| \leq \sqrt{2}\right\} = i\text{Symm}(E_2(u))$$

$$= \{i(p - q) : p, q \in \text{Trip}(E), p, q \leq u, p \perp q, p + q = u\}$$

and the elements $\pm iu$ are isolated in $\mathcal{E}_u$ (Corollary 3.3).

After some technical results inspired from recent achievements by J. Hamhalter, O. F. K. Kalenda, H. Pfitzner, and the second author of this note in [9], we arrive to our main result in Theorem 3.8, where we prove the following: Let u be an extreme point of the closed unit ball of a unital JB*-algebra M. Then the following statements are equivalent:

(a) u is a unitary tripotent;
(b) The set $\mathcal{M}_u = \{e \in \partial_e(\mathcal{B}_M) : \|u \pm e\| \leq \sqrt{2}\}$ contains an isolated point.

2. Background on JB*-algebras and JB*-triples

Suppose A is a unital C*-algebra whose set of projections (i.e. symmetric idempotents) will be denoted by $\mathcal{P}(A)$. It is known that the distance from 1 to any projection in $\mathcal{P}(A)\backslash\{1\}$ is 1, that is, $\|1-q\| \in \{0,1\}$ for all $q \in \mathcal{P}(A)$. Suppose p is a central projection in A. In this case, A writes as the orthogonal sum of pAp and $(1-p)A(1-p)$, and every projection q in A is of the form $q = q_1 + q_2$, where $q_1 \leq p$ and $q_2 \leq 1-p$. Then it easily follows that $\|p-q\| = \max\{\|p-q_1\|, \|q_2\|\} \in \{0,1\}$ for each $q \in \mathcal{P}(A)$, which shows that p is isolated (in the norm topology) in $\mathcal{P}(A)$. An easy example of a non-isolated projection can be given with 2 by 2 matrices. It is known that every rank one projection in $M_2(\mathbb{C})$ can be written in the form $p = \begin{pmatrix} t & \gamma\sqrt{t(1-t)} \\ \overline{\gamma}\sqrt{t(1-t)} & 1-t \end{pmatrix}$, where $\gamma \in \mathbb{C}$ with $|\gamma| = 1$ and $t \in [0,1]$. The mapping $q : [0,1] \to \mathcal{P}(M_2(\mathbb{C}))$, $q(s) = \begin{pmatrix} s & \gamma\sqrt{s(1-s)} \\ \overline{\gamma}\sqrt{s(1-s)} & 1-s \end{pmatrix}$ is continuous and shows that p is non-isolated in $\mathcal{P}(M_2(\mathbb{C}))$. The natural question is whether p being isolated in $\mathcal{P}(A)$ implies that p is central in A. This question has been explicitly treated by M. Mori in [19, Proof of Lemma 3.1]. The argument is as follows, suppose p is isolated in $\mathcal{P}(A)$, for each $a = a^*$ in A, we consider the mapping $\omega : \mathbb{R} \to \mathcal{P}(A)$, $\omega(t) := e^{ita}pe^{-ita}$, which is differentiable with $\omega(0) = p$. We deduce from the assumption on p that ω must be constant, and thus taking derivative at $t = 0$ we get $iap - ipa = 0$, which implies that p commutes with every hermitian element in A. That is every isolated projection in $\mathcal{P}(A)$ is central in A. We gather this information in the next result.

Proposition 2.1. *Let p be a projection in a unital C*-algebra A. Then the following statements are equivalent:*

(a) p is (norm) isolated in $\mathcal{P}(A)$;
(b) p is a central projection in A;
(c) $1-2p$ is (norm) isolated in Symm(A).

Proof. The implication $(a) \Rightarrow (b)$ is proved in [19, Proof of Lemma 3.1], while $(b) \Rightarrow (a)$ has been commented before. Finally it is easy to see that a sequence $(q_n) \subseteq \mathcal{P}(A)\backslash\{p\}$ converges in norm to p if and only if the sequence $(1-2q_n) \subseteq \mathrm{Symm}(A)\backslash\{1-2p\}$ converges in norm to $1-2p$. □

A Jordan version of Proposition 2.1 was considered by J.D.M. Wright and M.A. Youngson in [25]. Before going into details, let us note that the lacking of associativity for the product of a JB*-algebra makes invalid the arguments presented above, and specially the use of products of the form $e^{ita}pe^{-ita}$ is not always possible in the Jordan analogue of $(a) \Rightarrow (b)$.

In our approach to the Jordan setting, JB*-algebras and JB-algebras will be regarded as JB*-triples and real JB*-triples, respectively. According to the original definition, introduced by W. Kaup in [15], a JB*-triple is a complex Banach space E equipped with a continuous triple product $\{.,.,.\} : E \times E \times E \to E$, $(a,b,c) \mapsto \{a,b,c\}$, which is bilinear and symmetric in (a,c) and conjugate linear in b, and satisfies the following axioms for all $a,b,x,y \in E$:

(a) $L(a,b)L(x,y) = L(x,y)L(a,b) + L(L(a,b)x,y) - L(x,L(b,a)y)$, where $L(a,b) : E \to E$ is the operator defined by $L(a,b)x = \{a,b,x\}$;
(b) $L(a,a)$ is a hermitian operator with non-negative spectrum;

(c) $\|\{a,a,a\}\| = \|a\|^3$.

Examples of JB*-triples include all C*-algebras and JB*-algebras with triple products of the form

$$\{x,y,z\} = \frac{1}{2}(xy^*z + zy^*x), \tag{1}$$

and

$$\{x,y,z\} = (x\circ y^*)\circ z + (z\circ y^*)\circ x - (x\circ z)\circ y^*, \tag{2}$$

respectively.

Given an element x in a JB*-triple E, we shall write $x^{[1]} := x$, $x^{[3]} := \{x,x,x\}$, and $x^{[2n+1]} := \left\{x,x,x^{[2n-1]}\right\}$, $(n\in\mathbb{N})$.

Analogously, as real C*-algebras are defined as real norm closed hermitian subalgebras of C*-algebras (cf. [18]), a real closed subtriple of a JB*-triple is called a *real JB*-triple* (see [11]). Every JB*-triple is a real JB*-triple when it is regarded as a real Banach space. In particular every JB-algebra is a real JB*-triple with the triple product defined in (2) (see [11]).

An element e in a real or complex JB*-triple E is said to be a *tripotent* if $\{e,e,e\} = e$. Each tripotent $e\in E$, determines a decomposition of X, known as the *Peirce decomposition* associated with e, in the form

$$E = E_2(e)\oplus E_1(e)\oplus E_0(e),$$

where $E_j(e) = \{x\in E : \{e,e,x\} = \frac{j}{2}x\}$ for each $j = 0,1,2$.

Triple products among elements in the Peirce subspaces satisfy the following *Peirce arithmetic*: $\{E_i(e),E_j(e),E_k(e)\}\subseteq E_{i-j+k}(e)$ if $i-j+k\in\{0,1,2\}$, and $\{E_i(e),E_j(e),E_k(e)\} = \{0\}$ otherwise, and

$$\{E_2(e),E_0(e),E\} = \{E_0(e),E_2(e),E\} = 0.$$

Consequently, each Peirce subspace $E_j(e)$ is a real or complex JB*-subtriple of E.

The projection $P_k(e)$ of E onto $E_k(e)$ is called the *Peirce k-projection*. It is known that Peirce projections are contractive (cf. [8, Corollary 1.2]) and determined by the following identities $P_2(e) = Q(e)^2$, $P_1(e) = 2(L(e,e) - Q(e)^2)$, and $P_0(e) = \mathrm{Id}_E - 2L(e,e) + Q(e)^2$, where $Q(e) : E\to E$ is the conjugate or real linear map defined by $Q(e)(x) = \{e,x,e\}$. A tripotent e in E is called *unitary* (respectively, *complete* or *maximal*) if $E_2(e) = E$ (respectively, $E_0(e) = \{0\}$). This definition produces no contradiction because unitary elements in a unital JB*-algebra are precisely the unitary tripotents in M when the latter is regarded as a JB*-triple (cf. [3, Proposition 4.3]). A tripotent e in X is called *minimal* if $E_2(e) = \mathbb{C}e \neq \{0\}$. The set of all tripotents (respectively, of all complete tripotents) in a JB*-triple E will be denoted by $\mathrm{Trip}(E)$ (respectively, $\mathrm{Trip}_{max}(E)$).

It is worth remarking that if E is a complex JB*-triple, the Peirce 2-subspace $E_2(e)$ is a unital JB*-algebra with unit e, product $x\circ_e y := \{x,e,y\}$ and involution $x^{*_e} := \{e,x,e\}$, respectively.

Let us recall that a couple of elements a,b in a real or complex JB*-triple E are called *orthogonal* (written $a\perp b$) if $L(a,b) = 0$. It is known that $a\perp b \Leftrightarrow \{a,a,b\} = 0 \Leftrightarrow \{b,b,a\} = 0 \Leftrightarrow b\perp a$. If e is a tripotent in E, it follows from Peirce rules that $a\perp b$ for every $a\in E_2(e)$ and every $b\in E_0(e)$. Two projections p,q in a JB*-algebra are orthogonal if and only if $p\circ q = 0$. An additional geometric property of orthogonal elements shows that $\|a \pm b\| = \max\{\|a\|, \|b\|\}$ whenever a and b are orthogonal elements in a real or complex JB*-triple (cf. [8, Lemma 1.3]).

Henceforth the set, $\mathrm{Trip}(E)$, of all tripotents in a JB*-triple E, will be equipped with the natural partial order defined by $u \leq e$ in $\mathrm{Trip}(E)$ if $e - u$ is a tripotent in E with $e - u \perp u$, equivalently, if u is a projection in the JB*-algebra $E_2(e)$.

One of the useful geometric properties of a real or complex JB*-triple, E, asserts that the extreme points of its closed unit ball, $\mathcal{B}_E$, are precisely the complete tripotents in E, that is,

$$\partial_e(\mathcal{B}_E) = \mathrm{Trip}_{max}(E) \tag{3}$$

(cf. [16, Proposition 3.5] and [11, Lemma 3.3]).

Let a be a hermitian element in a JB*-algebra M, the spectral theorem [10, Theorem 3.2.4] assures that the JB*-subalgebra of M generated by a is isometrically JB*-isomorphic to a commutative C*-algebra. In particular, we can write a as the difference of two orthogonal positive elements in M_{sa}. By applying this result it can be seen that every tripotent in M_{sa} is the difference of two orthogonal projections in M, and furthermore, when M is unital we obtain

$$\partial_e(\mathcal{B}_{M_{sa}}) = \mathrm{Symm}(M) = \{s \in M_{sa} : s^2 = 1\} \tag{4}$$

(cf. [25] or [5, Proposition 3.1.9]). As in the associative case, the symbol $\mathcal{P}(M)$ will stand for the set of all projections (i.e., self-adjoint idempotents) in a JB*-algebra M.

The next result, which is a Jordan version of Proposition 2.1, was originally established in [12, Proposition 1.3], and a new proof can be consulted in [5, Proposition 3.1.24 and Remark 3.1.25]. An alternative proof, based on the structure of real JB*-triples, is included here for the sake of completeness.

Proposition 2.2. [12, Proposition 1.3], [5, Proposition 3.1.24] *Let p be a projection in a unital JB*-algebra M. Then the following statements are equivalent:*

(*a*) *p is (norm) isolated in $\mathcal{P}(M)$;*

(*b*) *p is a central projection;*

(*c*) *$1 - 2p$ is (norm) isolated in Symm(M).*

Proof. The equivalence $(c) \Leftrightarrow (a)$ follows by the same arguments employed in the case of C*-algebras.

$(c) \Rightarrow (b)$ Suppose $1 - 2p$ is (norm) isolated in $\mathrm{Symm}(M)$. We consider M_{sa} as a real JB*-triple. Given $a, b \in M_{sa}$, by the axioms in the definition of JB*-triples, the mapping

$$\Phi_t^{a,b} = \exp(t(L(a,b) - L(b,a)) = \sum_{n=0}^{\infty} \frac{t^n}{n!}(L(a,b) - L(b,a))^n : M \to M$$

is a surjective linear isometry for all $t \in \mathbb{R}$, and clearly maps M_{sa} into itself. One of the starring results in the theory of JB*-triples asserts that every surjective linear isometry between JB*-triples is a triple isomorphism (cf. [15, Proposition 5.5]). Therefore $\Phi_t^{a,b}$ and $\Phi_t^{a,b}|_{M_{sa}} : M_{sa} \to M_{sa}$ are (isometric) triple isomorphisms. Since $1 - 2p$ is an extreme point of the closed unit ball of M_{sa}, we deduce that $\Phi_t^{a,b}(1 - 2p)$ must be an extreme point of the closed unit ball of M_{sa}, and hence a complete tripotent in M_{sa}, or equivalently, a symmetry in M. Therefore the mapping $\omega : \mathbb{R} \to \mathrm{Symm}(M)$, $t \mapsto \omega(t) = \Phi_t^{a,b}(1 - 2p)$ is differentiable with

$\omega(0) = 1-2p$. Since $1-2p$ is isolated in $\mathrm{Symm}(M)$, the mapping $\omega(t)$ must be constant, and thus, by taking derivative at $t=0$ we get

$$0 = (L(a,b) - L(b,a))(1-2p) = \{a,b,1-2p\} - \{b,a,1-2p\},$$

equivalently,

$$((1-2p)\circ a)\circ b = ((1-2p)\circ b)\circ a,$$

for all $a,b\in M_{sa}$ (and for all $a,b\in M$). This shows that $1-2p$ (and hence p) lies in the center of M as desired.

$(b)\Rightarrow(a)$ If p is a central projection in M, we know from [10, Lemma 2.5.5] that $M = U_p(M)\oplus U_{1-p}(M)$, where for each $z\in M$, $U_z(x) = \{z,x^*,z\} = 2(z\circ x)\circ z - z^2\circ x$ ($\forall x\in M$). We further know that every element in $U_p(M)$ is orthogonal to every element in $U_{1-p}(M)$. Arguing as in the associative case (see Proposition 2.1 above), we prove that for each projection q in M we have $\|p-q\|\in\{0,1\}$, which concludes the proof. □

3. Metric characterisation of unitaries

Let us revisit some of the arguments in the proof of [19, Lemma 3.1] under the point of view of Jordan algebras.

Proposition 3.1. *Let e be a maximal partial isometry in a unital C*-algebra A, and let $l = ee^*$ and $r = e^*e$ denote the left and right projections of e. Suppose we can find two orthogonal projections $p,q\in A$ such that $l = p+q$. Then the element $y = i(p-q)e$ lies in $\mathcal{A}_e = \left\{y\in\partial_e(\mathcal{B}_A) : \|e\pm y\| = \sqrt{2}\right\}$, and for each $\theta\in\mathbb{R}$ the element*

$$y_\theta := P_2(e^*)(y) + \cos(\theta)P_1(e^*)(y) + \sin(\theta)P_1(e^*)(1)$$

is a maximal partial isometry in A.

If we further assume that p and q are central projections in lAl, the following statements hold:

- (a) *The elements $p' = epe^*$ and $q' = eqe^*$ are two orthogonal central projections in rAr, with $r = p'+q'$;*
- (b) *Suppose that e is not unitary in A, and take $y = i(p-q)e$. Then y lies in $\mathcal{A}_e$, and for each $\theta\in\mathbb{R}$ the element $y_\theta := P_2(e^*)(y) + \cos(\theta)P_1(e^*)(y) + \sin(\theta)P_1(e^*)(1)$ is a maximal partial isometry in A with $\|e\pm y_\theta\| = \sqrt{2}$ (actually, $\frac{e\pm y_\theta}{\sqrt{2}}$ is a maximal partial isometry in A), and $y_\theta\neq y$ for all θ in $\mathbb{R}\backslash\left(2\pi\mathbb{Z}\cup\pi\frac{1+2\mathbb{Z}}{2}\right)$. Furthermore, $\|y - P_2(y)(y_\theta)\|\leq 1-\cos(\theta)$, and hence $P_2(y)(y_\theta)$ is invertible in $A_2(y)$ for θ close to zero.*

Proof. Let us prove the first statement. Clearly, $y = i(p-q)e$ lies in $\mathcal{A}_e$. By [9, Lemma 6.1] there exist a complex Hilbert space H and an isometric unital Jordan $*$-monomorphism $\Psi : A\to B(H)$ such that $\Psi(e)^*\Psi(e) = 1$. Let us denote $v = \Psi(e)$, $z = \Psi(y)$, and $z_\theta = \Psi(y_\theta)$. We observe that

$$z_\theta = P_2(v^*)(z) + \cos(\theta)P_1(v^*)(z) + \sin(\theta)P_1(v^*)(1),$$

because Ψ is a unital Jordan *-monomorphism, and hence it preserves triple products and involution. Clearly, $v = \Psi(e)$ is a maximal partial isometry (actually, an isometry $v^*v = 1$) in $B(H)$. We shall write B for $B(H)$. Having the above properties in mind we can rewrite z_θ in the form

$$z_\theta = v^*vzvv^* + \cos(\theta)\left((1-v^*v)zvv^* + v^*vz(1-vv^*)\right)$$

$$+\sin(\theta)\left((1-v^*v)1vv^*+v^*v1(1-vv^*)\right)$$

$$= zvv^* + \cos(\theta)z(1-vv^*) + \sin(\theta)(1-vv^*).$$

Let us observe that the latter expression already appears in the proof of [19, Lemma 3.1].

Let us examine the element z_θ more closely. It follows from the properties commented above that

$$\begin{aligned} z_\theta^* z_\theta &= vv^*z^*zvv^* + \cos(\theta)vv^*z^*z(1-vv^*) + \sin(\theta)vv^*z^*(1-vv^*) \\ &+ \cos(\theta)(1-vv^*)z^*zvv^* + \cos^2(\theta)(1-vv^*)z^*z(1-vv^*) \\ &+ \cos(\theta)\sin(\theta)(1-vv^*)z^*(1-vv^*) + \sin(\theta)(1-vv^*)zvv^* \\ &+ \sin(\theta)\cos(\theta)(1-vv^*)z(1-vv^*) + \sin^2(\theta)(1-vv^*) \\ &= vv^* + \cos^2(\theta)(1-vv^*) + \sin^2(\theta)(1-vv^*) = vv^* = 1, \end{aligned}$$

witnessing that z_θ is an isometry in B. It then follows from the properties of Ψ that $y_\theta = \Psi^{-1}\left(\Psi(y_\theta)\right) \in \partial_e(\mathcal{B}_A)$ is a complete tripotent in A.

Concerning the second statement, let us analyze the element $w = e \pm y_\theta$. As before, up to an application of [9, Lemma 6.1], we can suppose that $r = e^*e = 1$. We set $l = ee^*$. Assuming that e is not unitary the projection $1 - l = 1 - ee^*$ is not zero. We therefore have

$$w = e \pm y_\theta = (e \pm y)l + (e \pm \cos(\theta)y)(1-l) + \sin(\theta)(1-l),$$

and we shall compute w^*w.

(a) Let us make some observations. The mappings $\Phi_1 : lAl \to rAr$, $x \mapsto e^*xe$ and $\Phi_2 : rAr \to lAl$, $y \mapsto eye^*$ are well defined, linear, and contractive. It is easy to see that $x = lxl = e(e^*xe)e^* = \Phi_2\Phi_1(x)$ and $y = e^*(eye^*)e = \Phi_1\Phi_2(y)$, for all $x \in lAl$ and $y \in rAr$. Therefore Φ_2 and Φ_1 are linear bijections and inverses each other. Furthermore, for all $x, z \in lAl$, we have $\Phi_1(x)\Phi_1(z) = (e^*xe)(e^*ze^*) = e(xz)e^* = \Phi_1(xz)$, and $\Phi_1(x)^* = (e^*xe)^* = e^*x^*e = \Phi_1(x^*)$, for all $x \in lAl$, which shows that the first mapping is a unital C*-isomorphism. Then the elements $p' = \Phi_1(p)$ and $q' = \Phi_1(q)$ are two orthogonal central projections in $rAr = A$ with $1 = r = p' + q'$.

(b) We derive from the above that $pe = ep'$, and $qe = eq'$, essentially because $pe \perp eq'$ and $qe \perp ep'$. Consequently,

$$y = i(p-q)e = ie(p'-q'),\ \ e \pm y = e(\mu_\pm p' + \overline{\mu_\pm} q'),$$

$$\text{and } e \pm \cos(\theta)y = e\left(\lambda_\pm p' + \overline{\lambda_\pm} q'\right),$$

where $\mu_\pm = 1 \pm i$, and $\lambda_\pm = 1 \pm i\cos(\theta)$. We study next all summands involved in the product w^*w:

$$\begin{aligned} ((x \pm y)l)^*((x \pm y)l) &= l(x \pm y)^*(x \pm y)l = l(\overline{\mu_\pm}p' + \mu_\pm q')e^*e(\mu_\pm p' + \overline{\mu_\pm} q') \\ &= 2l(p' + q')l = 2l; \end{aligned}$$

$$\sin(\theta)((x \pm y)l)^*(1-l) = \sin(\theta)l(\overline{\mu_\pm}p' + \mu_\pm q')e^*(1-l) = 0;$$

$$((x\pm y)l)^*(e\pm\cos(\theta)y)(1-l) = l(\overline{\mu_\pm}p'+\mu_\pm q')e^*e\left(\lambda_\pm p'+\overline{\lambda_\pm}q'\right)(1-l)$$
$$= l(\lambda_\pm\overline{\mu_\pm}p'+\overline{\lambda_\pm}\mu_\pm q')(1-l);$$
$$(1-l)(e\pm\cos(\theta)y)^*(x\pm y)l = (1-l)\left(\overline{\lambda_\pm}p'+\lambda_\pm q'\right)e^*e(\mu_\pm p'+\overline{\mu_\pm}q')l$$
$$= (1-l)\left(\overline{\lambda_\pm}\mu_\pm p'+\lambda_\pm\overline{\mu_\pm}q'\right)l;$$
$$(1-l)(e\pm\cos(\theta)y)^*(e\pm\cos(\theta)y)(1-l) = (1-l)\left(\overline{\lambda_\pm}p'+\lambda_\pm q'\right)e^*e\left(\lambda_\pm p'+\overline{\lambda_\pm}q'\right)(1-l)$$
$$= (1-l)\left(\overline{\lambda_\pm}p'+\lambda_\pm q'\right)\left(\lambda_\pm p'+\overline{\lambda_\pm}q')\right)(1-l) = |\lambda_\pm|^2(1-l)\left(p'+q'\right)(1-l)$$
$$= (1+\cos^2(\theta))(1-l);$$
$$((e\pm\cos(\theta)y)(1-l))^*(\sin(\theta)(1-l)) = \sin(\theta)(1-l)\left(\overline{\lambda_\pm}p'+\lambda_\pm q'\right)e^*(1-l) = 0;$$
$$(\sin(\theta)(1-l))^*(x\pm y)l = \sin(\theta)(1-l)e(\mu_\pm p'+\overline{\mu_\pm}q')l = 0;$$
$$(\sin(\theta)(1-l))^*(e\pm\cos(\theta)y)(1-l) = \sin(\theta)(1-l)e\left(\lambda_\pm p'+\overline{\lambda_\pm}q'\right)(1-l) = 0;$$
$$(\sin(\theta)(1-l))^*(\sin(\theta)(1-l)) = \sin^2(\theta)(1-l).$$

By adding the previous nine identities, and having in mind that p' and q' are central projections, we get

$$\frac{w^*w}{2} = l+\frac{1}{2}(1+\cos^2(\theta))(1-l)+\frac{1}{2}\sin^2(\theta)(1-l)$$
$$+\frac{1}{2}l(\alpha p'+\overline{\alpha}q')(1-l)+\frac{1}{2}(1-l)\left(\overline{\alpha}p'+\alpha q'\right)l = 1,$$

which proves that $\frac{w}{\sqrt{2}} = \frac{e\pm y_\theta}{\sqrt{2}}$ is an isometry, and consequently, $\|e\pm y_\theta\| = \sqrt{2}$.

Let us now check that $y_\theta\neq y$ for all θ in $\mathbb{R}\backslash\left(2\pi\mathbb{Z}\cup\pi\frac{1+2\mathbb{Z}}{2}\right)$. Note that $l\neq 1$. Since

$$(y-y_\theta)^*(y-y_\theta) = (1-\cos(\theta))^2(1-l)y^*y(1-l)+\sin^2(\theta)(1-l)$$
$$-(1-\cos(\theta))\sin(\theta)(1-l)(y+y^*)(1-l)$$
$$= 2(1-\cos(\theta))(1-l)-2(1-\cos(\theta))\sin(\theta)a$$
$$= 2(1-\cos(\theta))((1-l)-\sin(\theta)a)\neq 0,$$

where $a = (1-l)\frac{y+y^*}{2}(1-l)$ is a hermitian element in the closed unit ball of $(1-l)A(1-l)$, and hence $\|\sin(\theta)a\|\leq|\sin(\theta)|<1$.

Finally, the identity

$$P_2(y)(y_\theta) = lylr+\cos(\theta)ly(1-l)r+\sin(\theta)l(1-l)r = lyl+\cos(\theta)ly(1-l)$$

allows us to conclude that $\|y-P_2(y)(y_\theta)\| = \|(1-\cos(\theta))ly(1-l)\|\leq 1-\cos(\theta)$, which finishes the proof. □

Our goal in this section is to establish a similar characterisation of unitaries to that given in Theorem 1.1 in the setting of JB*-algebras and JB*-triples. It should be noted that the characterisation of unitaries in the case of JB*-algebras is far from being a consequence of the result in the associative case. We begin by describing the set of partial isometries at distance smaller than or equal to $\sqrt{2}$ from the unit of a JB*-algebra. As observed by Mori in [19], in the easiest case $A=\mathbb{C}$, for $u\in\partial_e(\mathcal{B}_A) = \{z\in\mathbb{C}:|z|=1\}$, we have $\mathcal{A}_u = \{e\in\partial_e(\mathcal{B}_\mathbb{C}):\|u\pm e\|=\sqrt{2}\} = \{iu,-iu\}$. But we can also add that $\mathcal{A}_u = \{e\in\partial_e(\mathcal{B}_\mathbb{C}):\|u\pm e\|\leq\sqrt{2}\}$.

Lemma 3.2. *Let M be a unital JB*-algebra. Let e be a tripotent in M satisfying $\|1 \pm e\| \leq \sqrt{2}$. Then there exist two orthogonal projections p, q in M such that $e = i(p-q)$. Consequently,*

$$\left\{e \in \mathit{Trip}(M) : \|1 \pm e\| \leq \sqrt{2}\right\} = \{i(p-q) : p, q \in \mathcal{P}(M) \text{ with } p \perp q\}.$$

Proof. Let N denote the JB*-subalgebra of M generated by $1, e$ and e^*. It follows from the Shirshov-Cohn theorem [10, Theorems 2.4.14 and 7.2.5], combined with Wright's theorem [24, Corollary 2.2 and subsequent comments], that N is special, that is, there exists a unital C*-algebra A containing N as unital JB*-subalgebra. The C*-algebra A contains 1 and the partial isometry e and we have $\|1 \pm e\| \leq \sqrt{2}$. Let us write $l = ee^*$ and $r = e^*e$ for the left an right projections of e in N. Then, it follows that

$$0 \leq \frac{1}{2}(1+l\pm(e+e^*)) = \frac{1}{2}(1\pm e)(1\pm e)^* \leq \frac{1}{2}\|(1\pm e)(1\pm e)^*\|1 = \frac{\|1\pm e\|^2}{2}1 \leq 1, \tag{5}$$

which implies that $2l \pm U_l(e+e^*) \leq 2l$, where we have applied that the mapping U_l is positive. Therefore $U_l(e+e^*) = 0$, and it follows from the definition of l that $U_{1-l}(e) = U_{1-l}(e^*) = 0$. Back to (5) we get

$$2l + (1-l) \pm e(1-l) \pm (1-l)e^* = 1 + l \pm (e+e^*) \leq 2\,1 = 2l + 2(1-l),$$

inequality which implies that

$$\pm(e(1-l) + (1-l)e^*) \leq 1 - l$$

and hence $e(1-l) + (1-l)e^* = 0$, or equivalently, $e(1-l) = -(1-l)e^*$. We have shown that

$$e + e^* = U_l(e+e^*) + (1-l)(e+e^*)l + l(e+e^*)(1-l) + U_{1-l}(e+e^*) = 0,$$

that is $e = -e^*$ is a skew symmetric partial isometry in A, and thus there exist two orthogonal projections p, q in A such that $e = i(p-q)$. Since $e = i(p-q) \in M$, it follows that $e^2 = -p-q$ and $p-q$ both belong to M, and consequently, $p, q \in M$, which concludes the proof. □

Given a tripotent u in a JB*-triple E, the Peirce 2-subspace $E_2(u)$ is a unital JB*-algebra with unit u (see page 5). So, the first statement in the next corollary is a straight consequence of our previous lemma.

Corollary 3.3. *Let u be a tripotent in a JB*-triple E. Then*

$$\{e \in \mathit{Trip}(E_2(u)) : \|u \pm e\| \leq \sqrt{2}\} = \{i(p-q) : p, q \in \mathcal{P}(E_2(u)) \text{ with } p \perp q\}.$$

Furthermore, if u is unitary in E, then

$$\begin{aligned} \mathcal{E}_u &= \left\{e \in \partial_e(\mathcal{B}_E) : \|u \pm e\| \leq \sqrt{2}\right\} = i\mathit{Symm}(E_2(u)) \\ &= \{i(p-q) : p, q \in \mathit{Trip}(E), p, q \leq u, p \perp q, p+q = u\} \end{aligned} \tag{6}$$

and the elements $\pm iu$ are isolated in $\mathcal{E}_u$.

Proof. The first statement is a consequence of Lemma 3.2. If u is unitary the equality $E = E_2(u)$ holds. Having in mind that $\partial_e(\mathcal{B}_E) = \mathrm{Trip}_{max}(E)$, we deduce from the first statement that

$$\mathcal{E}_u \subseteq \{i(p-q) : p, q \in \mathrm{Trip}(E), p, q \leq u, p \perp q\}.$$

But every $e = i(p-q) \in \mathcal{E}_u$ must be also a complete tripotent in E, which forces $p + q = u$, otherwise $r = u - p - q$ would be a non-zero element in $E_0(e)$, which is impossible, so (6) is clear. It is obvious that $\pm iu \in \mathcal{E}_u$ and for any $i(p-q) \in \mathcal{E}_u \backslash \{\pm iu\}$ we have

$$\|iu \pm i(p-q)\| = \|i(1\pm 1)p + i(1\mp 1)q\| = \max\{\|(1\pm 1)p\|, \|(1\mp 1)q\|\} = 2.$$

This proves that $\pm iu$ are isolated in $\mathcal{E}_u$. □

The Jordan version of the Theorem 1.1$(a) \Rightarrow (b)$ has been established in Corollary 3.3 even in the setting of JB*-triples. For the reciprocal implication we shall first prove a technical result which also holds for JB*-triples.

Proposition 3.4. *Let u be a tripotent in a JB*-triple E, and let*

$$\mathcal{E}_u = \{e \in \partial_e(\mathcal{B}_E) : \|u \pm e\| \leq \sqrt{2}\}.$$

Then every element $y \in \mathcal{E}_u$ with $P_1(u)(y) \neq 0$ or $P_0(u)(y) \neq 0$ is non-isolated in $\mathcal{E}_u$. Consequently, every isolated element $y \in \mathcal{E}_u$ belongs to iSymm$(E_2(u))$.

Proof. Let us take $y \in \mathcal{E}_u$ with $P_1(u)(y) \neq 0$ or $P_0(u)(y) \neq 0$. By [8, Lemma 1.1] for each $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ the mapping $S_{\overline{\lambda}}(u) = \overline{\lambda}^2 P_2(u) + \overline{\lambda} P_1(u) + P_0(u) = \overline{\lambda}^2 P_2(u) + \overline{\lambda} P_1(u) + P_0(u)$ is an isometric triple isomorphism on E. Therefore the mapping $R_\lambda(u) = \lambda^2 S_{\overline{\lambda}}(u) = P_2(u) + \lambda P_1(u) + \lambda^2 P_0(u)$ is an isometric triple isomorphism on E for all λ in the unit sphere of $\mathbb{C}$. Since Peirce projections are contractive

$$\|y - R_\lambda(u)(y)\| \geq \max\left\{|\lambda - 1|\|P_1(u)(y)\|, |\lambda^2 - 1|\|P_0(u)(y)\|\right\} > 0,$$

for all $\lambda \in \mathbb{T}\backslash\{\pm 1\}$. Clearly, $R_\lambda(u)(y) \stackrel{\lambda \to 1}{\longrightarrow} y$ in norm.

On the other hand, $R_\lambda(u)(u) = u$ for all $|\lambda| = 1$. Since $R_\lambda(u)$ is an isometric triple automorphism on E and $y \in \partial_e(\mathcal{B}_E)$ we deduce that $R_\lambda(u)(y) \in \partial_e(\mathcal{B}_E)$, and

$$\|u \pm R_\lambda(u)(y)\| = \|R_\lambda(u)(u) \pm R_\lambda(u)(y)\| = \|R_\lambda(u)(u \pm y)\| = \|u \pm y\| \leq \sqrt{2},$$

for all $|\lambda| = 1$. Therefore y is non-isolated in $\mathcal{E}_u$, which concludes the proof of the first statement.

For the last statement, let us assume that $y \in \mathcal{E}_u$ is an isolated point. It follows from the first statement that $P_1(u)(y) = 0 = P_0(u)(y)$. That is, $y \in \partial_e(\mathcal{B}_E) \cap E_2(u)$ with $\|u \pm y\| \leq \sqrt{2}$. We conclude from Corollary 3.3 that $y \in i\text{Symm}(E_2(u))$. □

Remark 3.5. The arguments given the proof of Proposition 3.4 are valid to establish the following: Let u be a tripotent in a JB*-triple E, and let

$$\tilde{\mathcal{E}}_u = \{e \in \text{Trip}(E) : \|u \pm e\| \leq \sqrt{2}\}.$$

Then every element $y \in \tilde{\mathcal{E}}_u$ with $P_1(u)(y) \neq 0$ or $P_0(u)(y) \neq 0$ is non-isolated in $\tilde{\mathcal{E}}_u$.

We continue gathering the tools and results needed in our characterisation of unitaries in JB*-algebras. One of the most successful tools in the theory of Jordan algebras is the Shirshov-Cohn theorem, which affirms that the JB*-subalgebra of a JB*-algebra generated by two symmetric elements (and the unit element) is a JC*-algebra, that is, a JB*-subalgebra of some $B(H)$ (cf. [10, Theorem 7.2.5] and [24, Corollary 2.2]). The next lemma is an appropriate version of the Shirshov-Cohn theorem.

Lemma 3.6. *Let u_1 and u_2 be two orthogonal tripotents in a unital JB*-algebra M. Then the JB*-subalgebra N of M generated by u_1, u_1^*, u_2, u_2^* and the unit element is a JC*-algebra, that is, there exists a complex Hilbert space H satisfying that N is a JB*-subalgebra of $B(H)$, we can further assume that the unit of N coincides with the identity on H.*

Proof. Let us fix $t \in (0,1)$. We consider the element $e = u_1 + tu_2$. Let N_0 denote the JB*-subalgebra of M generated by e, e^* and the unit element. It follows from the Shirshov-Cohn theorem that N_0 is a JC*-algebra. We observe that N_0 is a JB*-subtriple of M, therefore the element $e^{[2n-1]}$ belongs to N_0 for all natural n. Now, applying that u_1 and u_2 are two orthogonal tripotents, we can deduce that

$$e^{[2n-1]} = u_1 + t^{(2n-1)} u_2.$$

The sequence $(e^{[2n-1]})_n = (u_1 + t^{(2n-1)} u_2)_n$ converges in norm to u_1, and thus u_1 lies in N_0. Consequently, u_1 and u_2 both belong to N_0.

Since N_0 and N are JB*-subalgebras of M, $u_1, u_2 \in N_0$ and clearly $e \in N$, it follows from their definition that $N = N_0$ is a JC*-algebra.

The final statement can be obtained as in the proof of [9, Lemma 6.2]. □

The next result is inspired by [9, Lemmata 6.2 and 6.3].

Proposition 3.7. *Let u_1 and u_2 be two orthogonal tripotents in a unital JB*-algebra M satisfying the following properties:*

(a) $u = u_1 + u_2$ a complete tripotent in M;
(b) u_1, u_2 are central projections in the JB-algebra $M_2(u)$.*

Let N denote the JB-subalgebra of M generated by u_1, u_2 and the unit element. Then N is a JC*-subalgebra of some C*-algebra B, and u is a complete tripotent in the C*-subalgebra A of B generated by N. Moreover, the elements u_1, u_2 are central projections in the JB*-algebra $A_2(u)$.*

Proof. Lemma 3.6 guarantees that N is a JB*-subalgebra of a unital C*-algebra B, and we can also assume that N contains the unit of B. Clearly, u, u_1 and u_2 are partial isometries in A. Let $l_i = u_i u_i^*$ and $r_i = u_i^* u_i$ denote the left and right projections of u_i in A $(i = 1, 2)$. We shall also write $l = uu^* = l_1 + l_2$ and $r = u^*u = r_1 + r - 2$, for the left and right projections of u in A, respectively. Let us note that $l_1 \perp l_2$ and $r_1 \perp r_2$.

By hypothesis, u_1, u_2 are central projections in the JB*-algebra $M_2(u)$, and hence in $N_2(u)$. It then follows that the identity

$$lNr = N_2(u) = N_2(u_1) \oplus^\infty N_2(u_2) = l_1 N r_1 \oplus^\infty l_2 N r_2$$

holds. Having in mind that $1 \in N$, we deduce that $lr = l_1 r_1 + l_2 r_2$, and so $l_1 r = l_1 r_1$, which proves that $l_1 r_2 = 0$. We can similarly prove that $l_2 r_1 = 0$.

Let A denote the C*-subalgebra of B generated by N. We shall next show that u is a complete tripotent in A. We know that u is a complete tripotent in M, and hence in N. Clearly u is a tripotent in A. The Peirce 0-projection on A is given by $P_0(u)(x) = (1-l)x(1-r)$ $(x \in A)$. We therefore know that $(1-l)x(1-r) = 0$, for all $x \in N$. We shall prove that $(1-l)x(1-r) = 0$ for all $x \in A$. For this purpose we shall adapt some technique from the proof of [9, Lemma 6.2].

Since N is a JB*-subalgebra of A, for each $n \in \mathbb{N} \cup \{0\}$, the elements $(u_1^*)^n$ and $(u_2^*)^n$ lie in N, and hence $(1-l)(u_1^*)^n(1-r) = (1-l)(u_2^*)^n(1-r) = 0$, or equivalently,

$$\begin{aligned} l_1(u_1^*)^n(1-r) &= l(u_1^*)^n(1-r) = (u_1^*)^n(1-r), \text{ and} \\ l_2(u_2^*)^n(1-r) &= l(u_2^*)^n(1-r) = (u_2^*)^n(1-r), \end{aligned} \tag{7}$$

where in the first two equalities we applied that $l_1 r_2 = l_2 r_1 = 0$.

Fix $t \in (0,1)$. We have shown in the proof of Lemma 3.6 that N coincides with the JB*-subalgebra of M generated by $e = u_1 + t u_2$ and 1. Let A_0 denote the set of all finite products of e and e^* and 1. Since A is the closed linear span of A_0 we only need to prove that $(1-l)x(1-r) = 0$, for all $x \in A_0$.

We say that an element $x \in A$ satisfies property $(\diamond)$ if

$$x(1-r) = 0, \text{ or } x(1-r) = (1-r), \text{ or } x(1-r) = (u_1^*)^n(1-r) + t^m (u_2^*)^n (1-r),$$

for some $n, m \in \mathbb{N}$.

Let us fix an element $x \in A$ satisfying property $(\diamond)$. If $x(1-r) = 0$, we have $e^* x(1-r) = 0$, and $ex(1-r) = 0$. If $x(1-r) = (1-r)$, it follows that

$$e^* x(1-r) = e^*(1-r) = u_1^*(1-r) + t e_2^*(1-r), \text{ and } ex(1-r) = e(1-r) = 0.$$

If $x(1-r) = (u_1^*)^n(1-r) + t^m (u_2^*)^n (1-r)$, for some $n, m \in \mathbb{N}$, it can be seen that

$$e^* x(1-r) = e^*(u_1^*)^n(1-r) + t^m e^*(u_2^*)^n(1-r) = (u_1^*)^{n+1}(1-r) + t^{m+1}(u_2^*)^{n+1}(1-r),$$

where we applied that $u_1 \perp u_2$, $l_1 r_2 = 0$, and $l_2 r_1 = 0$. This shows that $e^* x$ satisfies property $(\diamond)$.

In the latter case, by applying $u_1 \perp u_2$, $l_1 r_2 = 0$, and $l_2 r_1 = 0$, we also have

$$\begin{aligned} ex(1-r) &= e(u_1^*)^n(1-r) + t^m e(u_2^*)^n(1-r) \\ &= u_1(u_1^*)^n(1-r) + t^{m+1} u_2 (u_2^*)^n (1-r) \\ &= (u_1 u_1^*)(u_1^*)^{n-1}(1-r) + t^{m+1}(u_2 u_2^*)(u_2^*)^{n-1}(1-r) \\ &= l_1(u_1^*)^{n-1}(1-r) + t^{m+1} l_2 (u_2^*)^{n-1}(1-r) \\ &= (\text{by } (7)) = (u_1^*)^{n-1}(1-r) + t^{m+1}(u_2^*)^{n-1}(1-r), \end{aligned}$$

witnessing that ex satisfies property $(\diamond)$.

We have proved that if x satisfies property $(\diamond)$, then ex and $e^* x$ both satisfy property $(\diamond)$. It is not hard to check that $1, e$, and e^* satisfy property $(\diamond)$. We can thus conclude that every element in A_0 satisfies property $(\diamond)$. So, for each $x \in A_0$ we have $(1-l)x(1-r) = 0$ if $x(1-r) = 0$. If $x(1-r) = (1-r)$, it follows from the fact that $1 \in N$ and u is complete in N, that

$$(1-l)x(1-r) = (1-l)(1-r) = (1-l)1(1-r) = 0.$$

Finally, if $x(1-r) = (u_1^*)^n(1-r) + t^m (u_2^*)^n (1-r)$, for some $n, m \in \mathbb{N}$, we easily check that

$$(1-l)x(1-r) = (1-l)(u_1^*)^n(1-r) + t^m (1-l)(u_2^*)^n(1-r) = 0,$$

where in the last equality we applied that $(u_1^*)^n, (u_2^*)^n \in N$ and u is a complete tripotent in N. This proves that $(1-l)A_0(1-r) = \{0\}$, and hence u is complete in A.

It remains to prove that u_1 and u_2 are central projections in $A_2(u)$. We claim that

$$l_1 A r_2 = l_2 A r_1 = \{0\}. \tag{8}$$

Indeed, it is enough to prove that

$$l_1(x_1 \cdots x_m) r_2 = l_2(x_1 \cdots x_m) r_1 = 0, \tag{9}$$

for all natural m and $x_1, \ldots, x_m \in \{e, e^*\}$ because N is the JB*-subalgebra of M generated by e, e^* and the unit. We shall prove (9) by induction on m. We know from the hypotheses that $l_1 N r_2 = l_2 N r_1 = \{0\}$, so the case, $m = 1$ is clear.

The case $m = 2$ is worth to be treated independently. The products of three elements are the following: $e^2, (e^*)^2, ee^*$ and e^*e. The elements e^2 and $(e^*)^2$ belong to N, and thus $l_1 e^2 r_2 = l_2 e^2 r_1 = l_1 (e^*)^2 r_2 = l_2 (e^*)^2 r_1 = 0$. By the properties seen in the above paragraphs we have

$$l_1 ee^* r_2 = e r_1 e^* r_2 = ee^* l_1 r_2 = 0.$$

Since $e \circ e^* \in N$, it follows that $l_1(ee^* + e^*e) r_2 = 0$. The last two equalities together give

$$l_1 ee^* r_2 = l_1 e^* e r_2 = 0.$$

Similar arguments show that

$$l_2 ee^* r_1 = l_2 e^* e r_1 = 0.$$

Suppose by the induction hypothesis that (9) for all natural numbers $2 \leq m \leq m_0$. Let us make an observation, for any natural $k \leq m_0 - 1$ it follows from the induction hypothesis that

$$l_1(x_1 \cdots x_k) l_2 e = l_1 x_1 \cdots x_k e r_2 = 0,$$

therefore

$$\begin{aligned} 0 &= (l_1(x_1 \cdots x_k) l_2 e)(l_1(x_1 \cdots x_k) l_2 e)^* = l_1(x_1 \cdots x_k) l_2 ee^* l_2 (x_k^* \cdots x_1^*) l_1 \\ &= l_1(x_1 \cdots x_k) l_2 l_2 (x_k^* \cdots x_1^*) l_1 = \left(l_1(x_1 \cdots x_k) l_2\right) \left(l_1(x_1 \cdots x_k) l_2\right)^*, \end{aligned}$$

witnessing that

$$l_1(x_1 \cdots x_k) l_2 = 0, \text{ for all natural } k \leq m_0 - 1. \tag{10}$$

We deal next with the case $m_0 + 1$. We pick $x_1, \ldots x_{m_0}, x_{m_0+1} \in \{e, e^*\}$. Since $e^{m+1}, (e^*)^{m+1} \in N$, the desired conclusion is clear for $x_1 = \ldots = x_{m+1} = e$ and $x_1 = \ldots = x_{m+1} = e^*$. We can therefore assume the existence of $j \in \{1, \ldots, m_0\}$ such that $x_j x_{j+1} = e^* e = 1$ or $x_j x_{j+1} = ee^*$. In the first case

$$l_1 x_1 \cdots x_{m_0+1} r_2 = l_1 x_1 \cdots x_{j-1} x_{j+1} \cdots x_{m_0+1} r_2 = 0,$$

by the induction hypothesis. In the second case we have

$$\begin{aligned} l_1 x_1 \cdots x_{m_0+1} r_2 &= l_1 x_1 \cdots x_{j-1} l x_{j+1} \cdots x_{m_0+1} r_2 \\ &= l_1 x_1 \cdots x_{j-1} l_1 x_{j+1} \cdots x_{m_0+1} r_2 + l_1 x_1 \cdots x_{j-1} l_2 x_{j+1} \cdots x_{m_0+1} r_2 = 0, \end{aligned}$$

where in the last equality we applied (10) and the induction hypothesis.

Similar ideas to those we gave above are also valid to establish

$$l_2 x_1 \cdots x_m r_1 = 0, \text{ for all } m \in \mathbb{N}, x_1 \cdots x_m \in \{e, e^*\}.$$

This finishes the induction argument and the proof of the claim in (8). It follows from (8) that u_1 and u_2 are central projections in $A_2(u)$. $\square$

The desired characterisation of unitaries in a unital JB*-algebra is now established in our main result.

Theorem 3.8. *Let u be an extreme point of the closed unit ball of a unital JB*-algebra M. Then the following statements are equivalent:*

(a) *u is a unitary tripotent;*

(b) *The set $\mathcal{M}_u = \{e \in \partial_e(\mathcal{B}_M) : \|u \pm e\| \leq \sqrt{2}\}$ contains an isolated point.*

Proof. Corollary 3.3 gives $(a) \Rightarrow (b)$.

$(b) \Rightarrow (a)$ We shall show that if u is not a unitary tripotent then every point $y \in \mathcal{M}_u$ is non-isolated. We therefore assume that u is not a unitary tripotent. Let us fix $y \in \mathcal{M}_u$. If $P_1(u)(y) \neq 0$, Proposition 3.4 implies that y is non-isolated in $\mathcal{M}_u$. We can therefore assume that $P_1(u)(y) = 0$, and hence $y = P_2(u)(y)$. So, y and u lie in the JB*-algebra $M_2(u)$ (we observe that the latter need not be a JB*-subalgebra of M). Since y also is an extreme point of the closed unit ball of $M_2(u)$ and $\|u \pm y\| \leq \sqrt{2}$, Corollary 3.3 implies that y lies in $i\mathrm{Symm}(M_2(u))$, therefore, there exist orthogonal tripotents $u_1, u_2 \in M$ with $u_1, u_2 \leq u$, $u_1 + u_2 = u$ and $y = i(u_1 - u_2)$.

If u_2 is non-isolated in $\mathcal{P}(M_2(u))$, then there exists a sequence $(q_n)_n \subseteq \mathcal{P}(M_2(u))$ with $q_n \neq u_2$, for all n, converging to u_2 in norm. In this case the sequence $(i(u - 2q_n))_n$ is contained in $\mathcal{M}_u \backslash \{y = i(u_1 - u_2)\}$ (let us observe that $u - 2q_n$ is a symmetry in $M_2(u)$ and since $u \in \partial_e(\mathcal{B}_M)$, [23, Lemma 4] implies that $i(u - 2q_n) \in \partial_e(\mathcal{B}_M)$ for all $n \in \mathbb{N}$, and clearly $\|u \pm i(u - 2q_n)\| = \sqrt{2}$) and converges to y in norm. We have therefore shown that y is non-isolated in $\mathcal{M}_u$.

We finally assume that u_2 is isolated in $\mathcal{P}(M_2(u))$. In this case Proposition 2.2 proves that u_2 (and hence u_1) is a central projection in $M_2(u)$. We are in position to apply Proposition 3.7 to the tripotents u_1, u_2 and $u = u_1 + u_2$ in M. Let N denote the JB*-subalgebra of M generated by u_1, u_2 and the unit element. By the just quoted proposition, N is a JC*-subalgebra of some C*-algebra B, u is a complete tripotent in the C*-subalgebra A of B generated by N, and the elements u_1, u_2 are central projections in the JB*-algebra $A_2(u)$. Let us observe that u and y both belong to N (and to A). Proposition 3.1, applied to A, u, $p = u_1 u_1^*$, $q = u_2 u_2^*$, and y, implies that for each $\theta \in \mathbb{R}$ the element

$$y_\theta := P_2(u^*)(y) + \cos(\theta) P_1(u^*)(y) + \sin(\theta) P_1(u^*)(1)$$

is a maximal partial isometry in A with $\|u \pm y_\theta\| = \sqrt{2}$, and $y_\theta \neq y$ for all θ in $\mathbb{R} \backslash \left(2\pi\mathbb{Z} \cup \pi \frac{1+2\mathbb{Z}}{2}\right)$ because u is not unitary in N nor in A. We further know from the just quoted proposition that $\|y - P_2(y)(y_\theta)\| \leq 1 - \cos(\theta)$, and hence $P_2(y)(y_\theta)$ is invertible in $N_2(y)$ for θ close to zero. Since $y \in \partial_e(\mathcal{B}_M)$, it follows from [13, Lemma 2.2] that y_θ is Brown-Pedersen quasi-invertible in the terminology of [13], which combined with the fact that y_θ is a tripotent in N (and hence in M), trivially implies that $y_\theta \in \partial_e(\mathcal{B}_M)$. Therefore, for θ close to zero, $y_\theta \in \mathcal{M}_u \backslash \{y\}$ and $y_\theta \to y$ in norm when $\theta \to 0$, witnessing that y is non-isolated in $\mathcal{M}_u$. □

Let us conclude this note with some afterthoughts on JB*-triples. Let E be a JB*-triple with dimension at least 2. Suppose u is a complete tripotent in E which is not unitary. In view of Corollary 3.3 and Theorem 3.8, a natural topic remains to be studied: Does the set $\mathcal{E}_u = \{e \in \partial_e(\mathcal{B}_E) : \|u \pm e\| \leq \sqrt{2}\}$ contains no isolated points?

Every JB*-triple E admitting a unitary element is a unital JB*-algebra with Jordan product and involution given in (2). Actually, there is a one-to-one (geometric) correspondence between the class of unital JB*-algebras and the class of JB*-triples admitting a unitary element. The next corollary is thus a rewording of our Theorem 3.8.

Corollary 3.9. *Let E be a JB*-triple admitting a unitary element. Suppose u is an extreme point of the closed unit ball of E. Then the following statements are equivalent:*

(a) *u is a unitary tripotent;*

(b) *The set $\mathcal{E}_u = \{e \in \partial_e(\mathcal{B}_E) : \|u \pm e\| \leq \sqrt{2}\}$ contains an isolated point.*

A typical example of a JB*-triple admitting no unitary tripotents is a rectangular Cartan factor of type 1 of the form $C = B(H, K)$, of all bounded linear operators between two complex Hilbert spaces H and K, with $\dim(H) > \dim(K)$.

In the simplest case $K = \mathbb{C}$ is one dimensional, and hence $C = H$ is a Hilbert space with triple product $\{a, b, c\} = \frac{1}{2}(\langle a, b\rangle c + \langle c, b\rangle a)$ $(a, b, c \in H)$. Every norm-one element in C is an extreme point of its closed unit ball, that is, $\partial_e(\mathcal{B}_C) = S(C)$. Let us fix $u \in S(C)$. By assuming $\dim(C) \geq 2$ it is not hard to see that

$$\mathcal{C}_u = \{e \in \partial_e(\mathcal{B}_C) : \|u \pm e\| \leq \sqrt{2}\} = \{itu + x : t \in \mathbb{R}, x \in C, \langle e, x\rangle = 0, t^2 + \|x\|^2 = 1\},$$

is pathwise-connected.

In the case in which $\dim(K) \geq 2$, every complete tripotent in C must be a partial isometry u satisfying $uu^* = \mathrm{id}_K$ (and clearly, $u^*u \neq \mathrm{id}_H$). Let us take $y \in \mathcal{C}_u = \{e \in \partial_e(\mathcal{B}_C) : \|u \pm e\| \leq \sqrt{2}\}$. We shall see that y is non-isolated in $\mathcal{C}_u$. By Corollary 3.3 and Proposition 3.4 we can assume that $y \in i\mathrm{Symm}(C_2(u))$, that is, there exist two orthogonal tripotents u_1, u_2 with $u_1, u_2 \leq u$, $u_1 + u_2 = u$, and $y = i(u_1 - u_2)$. We may assume that $u_2 \neq 0$. Let us take a minimal tripotent e such that $e \leq u_2$, that is, $u_2 = (u_2 - e) + e$ with $(u_2 - e) \perp e$. In this case $e = \xi \otimes \eta : \zeta \mapsto \langle \zeta, \eta\rangle \xi$ with $\eta \in S(H)$, $\xi \in S(K)$. Since $u^*u \neq \mathrm{id}_H$, we can pick $\tilde{\eta} \in S(H)$ with $\langle \tilde{\eta}, u^*u(H)\rangle = \{0\}$. The element $\tilde{e} = \xi \otimes \tilde{\eta}$ is a minimal tripotent in C with $\tilde{e} \perp u_1, u_2 - e$. It is not hard to check that, for each real θ, the element $y_\theta := i(u_1 - (u_2 - e) - \cos(\theta) e + \sin(\theta)\tilde{e})$ is a complete tripotent in C, by orthogonality and from the fact that $\|\alpha e + \beta \tilde{e}\|^2 = |\alpha|^2 + |\beta|^2$ for all $\alpha, \beta \in \mathbb{C}$, we can deduce that

$$\|u \pm y_\theta\| = \max\{\|(1 \pm i)u_1\|, \|(1 \mp i)(u_2 - e)\|, \|(1 \pm i\cos(\theta))e \pm \sin(\theta)\tilde{e}\|\} = \sqrt{2}.$$

Since $y \neq y_\theta \to y$ for $\theta \to 0$, we conclude that y is non-isolated in $\mathcal{C}_u$ as claimed.

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(M. Cueto-Avellaneda, A.M. Peralta) Departamento de Análisis Matemático, Facultad de Ciencias, Universidad de Granada, 18071 Granada, Spain.

E-mail address: `mcueto@ugr.es, aperalta@ugr.es`

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CAN ONE IDENTIFY TWO UNITAL JB*-ALGEBRAS BY THE METRIC SPACES DETERMINED BY THEIR SETS OF UNITARIES?

MARÍA CUETO-AVELLANEDA, ANTONIO M. PERALTA

Abstract. Let M and N be two unital JB*-algebras and let $\mathcal{U}(M)$ and $\mathcal{U}(N)$ denote the sets of all unitaries in M and N, respectively. We prove that the following statements are equivalent:

(a) M and N are isometrically isomorphic as (complex) Banach spaces;
(b) M and N are isometrically isomorphic as real Banach spaces;
(c) There exists a surjective isometry $\Delta : \mathcal{U}(M) \to \mathcal{U}(N)$.

We actually establish a more general statement asserting that, under some mild extra conditions, for each surjective isometry $\Delta : \mathcal{U}(M) \to \mathcal{U}(N)$ we can find a surjective real linear isometry $\Psi : M \to N$ which coincides with Δ on the subset $e^{iM_{sa}}$. If we assume that M and N are JBW*-algebras, then every surjective isometry $\Delta : \mathcal{U}(M) \to \mathcal{U}(N)$ admits a (unique) extension to a surjective real linear isometry from M onto N. This is an extension of the Hatori–Molnár theorem to the setting of JB*-algebras.

1. Introduction

Every surjective isometry between two real normed spaces X and Y is an affine mapping by the Mazur–Ulam theorem. It seems then natural to ask whether the existence of a surjective isometry between two proper subsets of X and Y can be employed to identify metrically both spaces. By a result of P. Mankiewicz (see [34]) every surjective isometry between convex bodies in two arbitrary normed spaces can be uniquely extended to an affine function between the spaces. The so-called Tingley's problem, which ask if a surjective isometry between the unit spheres of two normed spaces can be also extended to a surjective linear isometry between the spaces, came out in the eighties (cf. [42]). To the best of our knowledge, Tingley's problem remains open even for two dimensional spaces (see [5] where it is solved for non-strictly convex two dimensional spaces). A full machinery has been developed in the different partial positive solutions to Tingley's problem in the case of classical Banach spaces, C*- and operator algebras and JB*-triples (see, for example the references [2, 5, 8, 10, 11, 15, 16, 18, 19, 20, 30, 32, 35, 36, 38, 39, 43] and the surveys [47, 37]).

The question at this stage is whether in Tingley's problem the unit spheres can be reduced to strictly smaller subsets. Even in the most favorable case of a finite dimensional normed space X, we cannot always conclude that every surjective isometry on the set of extreme points of the closed unit ball of X can be extended

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to a surjective real linear isometry on X (see [10, Remark 3.15]). So, the sets of extreme points is not enough to determine a surjective real linear isometry. The existence of an additional structure on X provides new candidates, this is the case of unital C*-algebras. In a unital C*-algebra A, the set $\mathcal{U}(A)$ of all unitary elements in A is, in general, strictly contained in the set of all extreme points of the closed unit ball of A. The symbol A_{sa} will stand for the set of self-adjoint elements in A. We recall that an element u in A is called *unitary* if $uu^* = \mathbf{1}_A = u^*u$, that is, u is invertible with inverse u^*. The set of all unitaries in A will be denoted by $\mathcal{U}(A)$. It is well known that $\mathcal{U}(A)$ is contained in the unit sphere of A and it is a subgroup of A which is also self-adjoint (i.e., u^* and uv lie in $\mathcal{U}(A)$ for all $u, v \in \mathcal{U}(A)$). However, the set $\mathcal{U}(A)$ is no longer stable under Jordan products of the form $a \circ b := \frac{1}{2}(ab + ba)$. Namely, let $u, v \in \mathcal{U}(A)$ the element $w = u \circ v$ is a unitary if and only if $\mathbf{1}_A = ww^* = w^*w$, that is,

$$\mathbf{1}_A = \frac{1}{4}(uv + vu)(v^*u^* + u^*v^*) = \frac{1}{4}(2 \cdot \mathbf{1}_A + uvu^*v^* + vuv^*u^*),$$

equivalently, $\mathbf{1}_A = \frac{uvu^*v^*+vuv^*u^*}{2}$ and thus $uvu^*v^* = vuv^*u^* = \mathbf{1}_A$, because $\mathbf{1}_A$ is an extreme point of the closed unit ball of A. In particular $uv = vu$. That is $u \circ v \in \mathcal{U}(A)$ if and only if u and v commute. Despite the instability of unitaries under Jordan products, expressions of the form uvu lie in $\mathcal{U}(A)$ for all $u, v \in \mathcal{U}(A)$, and they can be even expressed in terms of the Jordan product because $uvu = 2(u \circ v) \circ u - u^2 \circ v$.

O. Hatori and L. Molnár proved in [27, Theorem 1], that for each surjective isometry $\Delta : \mathcal{U}(A) \to \mathcal{U}(B)$, where A and B are unital C*-algebras, the identity $\Delta(e^{iA_{sa}}) = e^{iB_{sa}}$ holds, and there is a central projection $p \in B$ and a Jordan *-isomorphism $J : A \to B$ satisfying

$$\Delta(e^{ix}) = \Delta(1)(pJ(e^{ix}) + (1 - p)J(e^{ix})^*), \quad (x \in A_{sa}).$$

In particular A and B are Jordan *-isomorphic. Actually, every surjective isometry between the unitary groups of two von Neumann algebras admits an extension to a surjective real linear isometry between these algebras (see [27, Corollary 3]). These influencing results have played an important role in some of the recent advances on Tingley's problem and in several other problems.

Let us take a look at some historical precedents. S. Sakai proved in [41] that if M and N are AW*-factors, $\mathcal{U}(M)$, $\mathcal{U}(N)$ their respective unitary groups, and ρ a uniformly continuous group isomorphism from $\mathcal{U}(M)$ into $\mathcal{U}(N)$, then there is a unique map f from M onto N which is either a linear or conjugate linear *-isomorphism and which agrees with ρ on $\mathcal{U}(M)$. In the case of W*-factors not of type I_{2n} the continuity assumption was shown to be superfluous by H.A. Dye in [14, Theorem 2]. In the results by Hatori and Molnár, the mapping Δ is merely a distance preserving bijection between the unitary groups of two unital C*-algebras or two von Neumann algebras.

The proofs of the Hatori–Molnár theorems are based, among other things, on a study on isometries and maps compatible with inverted Jordan triple products on groups by O. Hatori, G. Hirasawa, T. Miura, L. Molnár [25]. Despite of the attractive terminology, the study of the surjective isometries between the sets of unitaries of two unital JB*-algebras has not been considered. There are different diffulties which are inherent to the Jordan setting. As we commented above, the set of unitary elements in a unital C*-algebra is not stable under Jordan products.

Motivated by the pioneering works of I. Kaplansky, JB*-algebras were introduced as a Jordan generalization of C*-algebras (see subsection 1.1 for the detailed definitions). For example every Jordan self-adjoint subalgebra of a C*-algebra is a JB*-algebra (these JB*-algebras are called JC*-algebras), but there exists exceptional JB*-algebras which cannot be represented as JC*-algebras.

Unitaries in unital C*-algebras and JB*-algebras have been intensively studied. They constitute the central notion in the Russo–Dye theorem [40] and its JB*-algebra-analogue in the Wright–Youngson–Russo–Dye theorem [45], which are milestone results in the field of functional analysis. The interest remains very active, for example, we recently obtained a metric characterization of unitary elements in a unital JB*-algebra (cf. [12]).

By the Gelfand–Naimark theorem, every unitary u in a unital C*-algebra A can be viewed as a unitary element in the algebra $B(H)$, of all bounded linear operators on a complex Hilbert space H, in such a way that u itself is a unitary on H. Consequently, one-parameter unitary groups in A are under the hypotheses of some well known results like Stone's one-parameter theorem. However, unitary elements in a unital JB*-algebra M cannot always be regarded as unitaries on some complex Hilbert space H. The lacking of a suitable Jordan version of Stone's one-parameter theorem for JB*-algebras leads us to establish an appropriate result for uniformly continuous one-parameter groups of unitaries in an arbitrary unital JB*-algebra in Theorem 3.1.

Let M and N denote two arbitrary unital JB*-algebras whose sets of unitaries are denoted by $\mathcal{U}(M)$ and $\mathcal{U}(N)$, respectively. In our first main result (Theorem 3.4) we prove that for each surjective isometry $\Delta : \mathcal{U}(M) \to \mathcal{U}(N)$ satisfying one of the following statements:

(1) $\|\mathbf{1}_{_N} - \Delta(\mathbf{1}_{_M})\| < 2$;
(2) there exists a unitary ω_0 in N such that $U_{\omega_0}(\Delta(\mathbf{1}_{_M})) = \mathbf{1}_{_N}$,

there exists a unitary ω in N satisfying

$$\Delta(e^{iM_{sa}}) = U_{\omega^*}(e^{iN_{sa}}),$$

Furthermore, we can find a central projection $p \in N$, and a Jordan *-isomorphism $\Phi : M \to N$ such that

$$\begin{aligned}\Delta(e^{ih}) &= U_{\omega^*}\left(p \circ \Phi(e^{ih})\right) + U_{\omega^*}\left((\mathbf{1}_{_N} - p) \circ \Phi(e^{ih})^*\right)\\ &= P_2(U_{\omega^*}(p))U_{\omega^*}(\Phi(e^{ih})) + P_2(U_{\omega^*}(\mathbf{1}_{_N} - p))U_{\omega^*}(\Phi((e^{ih})^*)),\end{aligned}$$

for all $h \in M_{sa}$. Consequently, the restriction $\Delta|_{e^{iM_{sa}}}$ admits a (unique) extension to a surjective real linear isometry from M onto N. We remark that Δ is merely a distance preserving bijection.

Among the consequences of the previous result we prove that the following statements are equivalent for any two unital JB*-algebras M and N:

(a) M and N are isometrically isomorphic as (complex) Banach spaces;
(b) M and N are isometrically isomorphic as real Banach spaces;
(c) There exists a surjective isometry $\Delta : \mathcal{U}(M) \to \mathcal{U}(N)$

(see Corollary 3.8).

Finally, in Theorem 3.9 we prove that any surjective isometry between the sets of unitaries of any two JBW*-algebras admits a (unique) extension to a surjective real linear isometry between these algebras.

Our proofs, which are completely independent from the results for C*-algebras, undoubtedly benefit from results in JB*-triple theory. This beautiful subject (meaning the theory of JB*- and JBW*- triples) makes simpler and more accessible our arguments.

1.1. **Definitions and background.** A complex (respectively, real) *Jordan algebra* M is a (non-necessarily associative) algebra over the complex (respectively, real) field whose product is abelian and satisfies the so-called *Jordan identity*: $(a \circ b) \circ a^2 = a \circ (b \circ a^2)$ $(a, b \in M)$. A *normed Jordan algebra* is a Jordan algebra M equipped with a norm, $\|.\|$, satisfying $\|a \circ b\| \leq \|a\| \, \|b\|$ $(a, b \in M)$. A *Jordan Banach algebra* is a normed Jordan algebra whose norm is complete. Every real or complex associative Banach algebra is a real Jordan Banach algebra with respect to the product $a \circ b := \frac{1}{2}(ab + ba)$.

Let M be a Jordan Banach algebra. Given $a, b \in M$, we shall write $U_{a,b} : M \to M$ for the bounded linear operator defined by

$$U_{a,b}(x) = (a \circ x) \circ b + (b \circ x) \circ a - (a \circ b) \circ x,$$

for all $x \in M$. The mapping $U_{a,a}$ will be simply denoted by U_a. One of the fundamental identities in Jordan algebras assures that

$$U_a U_b U_a = U_{U_a(b)}, \text{ for all } a, b \text{ in a Jordan algebra } M \tag{1}$$

(see [24, 2.4.18]). The multiplication operator by an element $a \in M$ will be denoted by M_a, that is, $M_a(b) = a \circ b$ $(b \in M)$.

Henceforth, the powers of an element a in a Jordan algebra M will be denoted as follows:

$$a^1 = a; \quad a^{n+1} = a \circ a^n, \quad n \geq 1.$$

If M is unital, we set $a^0 = \mathbf{1}_M$. An algebra $\mathcal{B}$ is called *power associative* if the subalgebras generated by single elements of $\mathcal{B}$ are associative. In the case of a Jordan algebra M this is equivalent to say that the identity $a^m \circ a^n = a^{m+n}$, holds for all $a \in M$, $m, n \in \mathbb{N}$. It is known that any Jordan algebra is power associative ([24, Lemma 2.4.5]).

By analogy with the associative case, if M is a unital Jordan Banach algebra, the closed subalgebra generated by an element $x \in M$ and the unit is always associative, and hence we can always consider the elements of the form e^x in M, defined by $e^x = \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$ (cf. [6, § 1.1.29]). Let us suppose that M is a unital Jordan Banach subalgebra of an associative unital Banach algebra, and let x be an element in M. Take a and b in M such that $a = e^{itx}$ and $b = e^{isx}$, where $t, s \in \mathbb{R}$. From now on, it will be useful to keep in mind that

$$\begin{aligned} a \circ b = e^{itx} \circ e^{isx} &= \frac{1}{2}(e^{itx} e^{isx} + e^{isx} e^{itx}) \\ &= \frac{1}{2}(e^{i(t+s)x} + e^{i(t+s)x}) = e^{i(t+s)x} = e^{itx} e^{isx} = ab. \end{aligned}$$

An element a in a unital Jordan Banach algebra M is called *invertible* whenever there exists $b \in M$ satisfying $a \circ b = 1$ and $a^2 \circ b = a$. The element b is unique and it will be denoted by a^{-1} (cf. [24, 3.2.9] and [6, Definition 4.1.2]). We know from [6, Theorem 4.1.3] that an element $a \in M$ is invertible if and only if U_a is a bijective mapping, and in such a case $U_a^{-1} = U_{a^{-1}}$.

A *JB*-algebra* is a complex Jordan Banach algebra M equipped with an algebra involution * satisfying $\| \{a, a, a\} \| = \|a\|^3$, $a \in M$ (where $\{a, a, a\} = U_a(a^*) = 2(a \circ a^*) \circ a - a^2 \circ a^*$). We know from a result by M.A. Youngson that the involution of every JB*-algebra is an isometry (cf. [48, Lemma 4]).

A *JB-algebra* is a real Jordan Banach algebra J in which the norm satisfies the following two axioms for all $a, b \in J$:

(i) $\|a^2\| = \|a\|^2$;
(ii) $\|a^2\| \leq \|a^2 + b^2\|$.

The hermitian part, M_{sa}, of a JB*-algebra, M, is always a JB-algebra. A celebrated theorem due to J.D.M. Wright asserts that, conversely, the complexification of every JB-algebra is a JB*-algebra (see [44]). We refer to the monographs [24] and [6] for the basic notions and results in the theory of JB- and JB*-algebras.

Every C*-algebra A is a JB*-algebra when equipped with its natural Jordan product $a \circ b = \frac{1}{2}(ab + ba)$ and the original norm and involution. Norm-closed Jordan *-subalgebras of C*-algebras are called *JC*-algebras*. JC*-algebras which are also dual Banach spaces are called *JW*-algebras*. Any JW*-algebra is a weak*-closed Jordan *-subalgebra of a von Neumann algebra.

We recall that an element u in a unital JB*-algebra M is a *unitary* if it is invertible and its inverse coincides with u^*. As in the associative setting, we shall denote by $\mathcal{U}(M)$ the set of all unitary elements in M. Let us observe that if a unital C*-algebra is regarded as a JB*-algebra both notions of unitaries coincide. An element s in a unital JB-algebra J is called a *symmetry* if $s^2 = \mathbf{1}_J$. If M is a JB*-algebra, the symmetries in M are defined as the symmetries in its self-adjoint part M_{sa}.

A celebrated result in the theory of JB*-algebras is the so-called Shirshov-Cohn theorem, which affirms that the JB*-subalgebra of a JB*-algebra generated by two self-adjoint elements (and the unit element) is a JC*-algebra, that is, a JB*-subalgebra of some $B(H)$ (cf. [24, Theorem 7.2.5] and [44, Corollary 2.2]).

Two elements a, b in a Jordan algebra M are said to *operator commute* if

$$(a \circ c) \circ b = a \circ (c \circ b),$$

for all $c \in M$ (cf. [24, 4.2.4]). By the *centre* of M (denoted by $Z(M)$) we mean the set of all elements of M which operator commute with any other element in M. Any element in the center is called *central*.

A JB*-algebra may admit two different Jordan products compatible with the same norm. However, when JB*-algebras are regarded as JB*-triples, any surjective linear isometry between them is a triple isomorphism (see [33, Proposition 5.5]). This fact produces a certain uniqueness of the triple product (see next section for more details). We recall the definition of JB*-triples.

A JB*-triple is a complex Banach space E equipped with a continuous triple product $\{.,.,.\} : E \times E \times E \to E$, $(a, b, c) \mapsto \{a, b, c\}$, which is bilinear and symmetric in (a, c) and conjugate linear in b, and satisfies the following axioms for all $a, b, x, y \in E$:

(a) $L(a, b)L(x, y) = L(x, y)L(a, b) + L(L(a, b)x, y) - L(x, L(b, a)y)$, where $L(a, b) : E \to E$ is the operator defined by $L(a, b)x = \{a, b, x\}$;
(b) $L(a, a)$ is a hermitian operator with non-negative spectrum;
(c) $\|\{a, a, a\}\| = \|a\|^3$.

The definition presented here dates back to 1983 and it was introduced by W. Kaup in [33].

Examples of JB*-triples include all C*-algebras and JB*-algebras with the triple products of the form

$$\{x, y, z\} = \frac{1}{2}(xy^*z + zy^*x), \tag{2}$$

and

$$\{x, y, z\} = (x \circ y^*) \circ z + (z \circ y^*) \circ x - (x \circ z) \circ y^*, \tag{3}$$

respectively.

The triple product of every JB*-triple is a non-expansive mapping, that is,

$$\|\{a, b, c\}\| \leq \|a\|\|b\|\|c\| \text{ for all } a, b, c \text{ (see [22, Corollary 3])}. \tag{4}$$

Let E be a JB*-triple. Each element e in E satisfying $\{e, e, e\} = e$ is called a *tripotent*. Each tripotent $e \in E$, determines a decomposition of E, known as the *Peirce decomposition* associated with e, in the form

$$E = E_2(e) \oplus E_1(e) \oplus E_0(e),$$

where $E_j(e) = \{x \in E : \{e, e, x\} = \frac{j}{2}x\}$ for each $j = 0, 1, 2$.

Triple products among elements in Peirce subspaces satisfy the following *Peirce arithmetic*:

$$\{E_i(e), E_j(e), E_k(e)\} \subseteq E_{i-j+k}(e) \text{ if } i - j + k \in \{0, 1, 2\},$$
$$\{E_i(e), E_j(e), E_k(e)\} = \{0\} \text{ if } i - j + k \notin \{0, 1, 2\},$$

and $\{E_2(e), E_0(e), E\} = \{E_0(e), E_2(e), E\} = 0$. Consequently, each Peirce subspace $E_j(e)$ is a JB*-subtriple of E.

The projection $P_k(e)$ of E onto $E_k(e)$ is called the *Peirce k-projection*. It is known that Peirce projections are contractive (cf. [21, Corollary 1.2]) and determined by the following identities $P_2(e) = Q(e)^2$, $P_1(e) = 2(L(e, e) - Q(e)^2)$, and $P_0(e) = \mathrm{Id}_E - 2L(e, e) + Q(e)^2$, where $Q(e) : E \to E$ is the conjugate or real linear map defined by $Q(e)(x) = \{e, x, e\}$.

It is worth remarking that if e is a tripotent in a JB*-triple E, the Peirce 2-subspace $E_2(e)$ is a unital JB*-algebra with unit e, product $x \circ_e y := \{x, e, y\}$ and involution $x^{*_e} := \{e, x, e\}$, respectively (cf. [6, Theorem 4.1.55]).

Following standard notation, a tripotent e in in a JB*-triple E is called *unitary* if $E_2(e) = E$.

Remark 1.1. *The reader should be warned that if a unital JB*-algebra M is regarded as a JB*-triple we have two, a priori, different uses of the word "unitary". However, there is no conflict between these two notions because unitary elements in a unital JB*-algebra M are precisely the unitary tripotents in M when the latter is regarded as a JB*-triple (cf.* [3, Proposition 4.3] *or* [6, Theorem 4.2.24, Definition 4.2.25 and Fact 4.2.26]*).*

2. Unitaries in JB*-algebras and inverted Jordan triple products

Unitary elements in JB*-algebras have been intensively studied for many geometric reasons. As in the setting of C*-algebras, they play a protagonist role in the Wright–Youngson extension of the Russo-Dye theorem for JB*-algebras [46] (see also [6, Corollary 3.4.7 and Fact 4.2.39]). Different applications can be found on the study of surjective isometries between JB- and JB*-algebras (see [46, 29] and [6, Proposition 4.2.44]).

The definition of unitary in a JB*-algebra and its natural connection with the notion of unitary (tripotent) in the setting of JB*-triples has been recalled at the introduction. We shall next revisit some basic properties with the aim of clarifying and make accessible our subsequent arguments.

The first result, which has been almost outlined in the introduction, has been borrowed from [6].

Lemma 2.1. [6, Lemma 4.2.41, Theorem 4.2.28, Corollary 3.4.32], [46], [29] *Let M be a unital JB*-algebra, and let u be a unitary element in M. Then the following statements hold:*

(a) *The Banach space of M becomes a unital JB*-algebra with unit u for the (Jordan) product defined by $x \circ_u y := U_{x,y}(u^*) = \{x, u, y\}$ and the involution $*_u$ defined by $x^{*_u} := U_u(x^*) = \{u, x, u\}$. (This JB*-algebra $M(u) = (M, \circ_u, *_u)$ is called the* u-isotope *of M.)*

(b) *The unitary elements of the JB*-algebras M and $(M, \circ_u, *_u)$ are the same, and they also coincide with the unitary tripotents of M when the latter is regarded as a JB*-triple.*

(c) *The triple product of M satisfies*

$$\begin{aligned}\{x, y, z\} &= (x \circ y^*) \circ z + (z \circ y^*) \circ x - (x \circ z) \circ y^* \\ &= (x \circ_u y^{*_u}) \circ_u z + (z \circ_u y^{*_u}) \circ_u x - (x \circ_u z) \circ_u y^{*_u},\end{aligned}$$

for all $x, y, z \in M$. Actually, the previous identities hold when $\circ$ is replaced with any Jordan product on M making the latter a JB-algebra with the same norm.*

(d) *The mapping $U_u : M \to M$ is a surjective isometry and hence a triple isomorphism. Consequently, $U_u(\mathcal{U}(M)) = \mathcal{U}(M)$. Furthermore, the operator $U_u : (M, \circ_{u^*}, *_{u^*}) \to (M, \circ_u, *_u)$ is a Jordan *-isomorphism.*

Proof. Statements (a) and (b) can be found in [6, Lemma 4.2.41] (see also [3, Proposition 4.3]). Moreover, Theorem 4.2.28 (vii) in [6] assures that the mapping U_u is a surjective linear isometry. The remaining statements are consequences of the fact that a linear bijection between JB*-triples is an isometry if and only if it is a triple isomorphism (cf. [33, Proposition 5.5] and [7, Theorem 5.6.57]). Furthermore each unital triple isomorphism between unital JB*-algebras must be a Jordan *-isomorphism. □

Let u be a unitary element in a unital C*-algebra A. It is known that $\|\mathbf{1} - u\| < 2$ implies that $u = e^{ih}$ for some $h \in A_s$ (see [31, Exercise 4.6.6]). In our next lemma we combine this fact with the Shirshov-Cohn theorem.

Lemma 2.2. *Let u, v be two unitaries in a unital JB*-algebra M. Let us suppose that $\|u - v\| = \eta < 2$. Then the following statements hold:*

(*a*) *There exists a self-adjoint element h in the u-isotope JB*-algebra $M(u) = (M, \circ_u, *_u)$ such that $v = e^{ih}$, where the exponential is computed in the JB*-algebra $M(u)$.*
(*b*) *There exists a unitary w in M satisfying $U_w(u^*) = v$.*

Moreover, if $\|u - v\| = \eta = \left|1 - e^{it_0}\right| = \sqrt{2}\sqrt{1 - \cos(t_0)}$ for some $t_0 \in (-\pi, \pi)$, we can further assume that $\|w - u\|, \|w - v\| \leq \sqrt{2}\sqrt{1 - \cos(\frac{t_0}{2})}$.

Proof. We consider the unital JB*-algebra $M(u) = (M, \circ_u, *_u)$. Let $\mathcal{C}$ denote the JB*-subalgebra of $M(u)$ generated by v and its unit –i.e. u–. Let us observe that the product and involution on $\mathcal{C}$ are precisely $\circ_u|_{\mathcal{C}\times\mathcal{C}}$ and $*_u|_{\mathcal{C}}$, respectively. Since v is unitary in $M(u)$ (cf. Lemma 2.1(b)), the JB*-subalgebra $\mathcal{C}$ must be isometrically Jordan *-isomorphic to a unital commutative C*-algebra, that is, to some $C(\Omega)$ for an appropriate compact Hausdorff space Ω, and under this identification, u corresponds to the unit (cf. [24, 3.2.4. The spectral theorem] or [6, Proposition 3.4.2 and Theorem 4.1.3(v)]).

Since $\|u - v\| < 2$, v is a unitary in $\mathcal{C}$ and u is the unit element of the C*-algebra $\mathcal{C} \equiv C(\Omega)$, we can find a self-adjoint element $h \in \mathcal{C}_{sa}$ such that $v = e^{ih}$ (see [31, Exercise 4.6.6]), where the exponential is, of course, computed with respect to the structure of $\mathcal{C}$, that is with the product and involution of $M(u)$. This finishes the proof of (a).

By setting $w = e^{i\frac{h}{2}}$ we get a unitary element in $\mathcal{C}$ satisfying $w \cdot u^{*_u} \cdot w = w \cdot u \cdot w = v$ (let us observe that the involution of $\mathcal{C}$ is precisely the restriction of $*_u$ to $\mathcal{C}$). Let U_a^u denote the U operator on the unital JB*-algebra $M(u)$ associated with the element a. Since $\mathcal{C}$ is a unital JB*-subalgebra of $M(u)$, we deduce that

$$v = w \cdot u^{*_u} \cdot w = U_w^u(u^{*_u}) = \{w, u, w\} = U_w(u^*),$$

and clearly w is a unitary in M because it is a unitary in $M(u)$ (cf. Lemma 2.1(b) and (c)). We have therefore concluded the proof of (b).

The final statement is a clear consequence of the identification of $\mathcal{C}$ with $C(\Omega)$ in which u corresponds to the unit and v and w with e^{ih} and $e^{i\frac{h}{2}}$, respectively. □

The next lemma can be deduced by similar arguments to those given in the previous result.

Lemma 2.3. *Let u and w be unitary elements in a unital JB*-algebra M. Suppose $U_w(u^*) = u$ and $\|u - w\| < 2$. Then $w = u$.*

Proof. Let $M(w) = (M, \circ_w, *_w)$ and let $\mathcal{C}$ denote the JB*-subalgebra of $M(w)$ generated by u and its unit. By applying the identification of $\mathcal{C}$ with an appropriate $C(\Omega)$ space as the one given in the proof of the previous lemma, we can identify w with the unit of $C(\Omega)$ and u with a unitary in this commutative unital C*-algebra with $u^{*_w} = U_w(u^*) = u$ (self-adjoint in $\mathcal{C}$) and $\|u - w\| < 2$, which implies that $w = u$. □

We shall gather next some results on isometries between metric groups due to O. Hatori, G. Hirasawa, T. Miura and L Molnár [25]. The conclusions in the just quoted paper provided the tools applied in the study of surjective isometries between the unitary groups of unital C*-algebras in subsequent references [26] and [27].

Henceforth, let $\mathcal{G}$ be a group and let (X, d) be a non-trivial metric space such that X is a subset of $\mathcal{G}$ and

$$yx^{-1}y \in X \text{ for all } x, y \in X \tag{5}$$

(note that we are not assuming that X is a subgroup of $\mathcal{G}$).

Definition 2.4. *Let us fix a, b in X. We shall say that condition $B(a, b)$ holds for (X, d) if the following properties hold:*
$(B.1)$ For all $x, y \in X$ we have $d(bx^{-1}b, by^{-1}b) = d(x, y)$.
$(B.2)$ There exists a constant $K > 1$ satisfying

$$d(bx^{-1}b, x) \geq Kd(x, b),$$

for all $x \in L_{a,b} = \{x \in X : d(a, x) = d(ba^{-1}b, x) = d(a, b)\}$.

Definition 2.5. *Let us fix $a, b \in X$. We shall say that condition $C_1(a, b)$ holds for (X, d) if the following properties hold:*
$(C.1)$ For every $x \in X$ we have $ax^{-1}b, bx^{-1}a \in X$;
$(C.2)$ $d(ax^{-1}b, ay^{-1}b) = d(x, y)$, for all $x, y \in X$.

We shall say that condition $C_2(a, b)$ holds for (X, d) if there exists $c \in X$ such that $ca^{-1}c = b$ and $d(cx^{-1}c, cy^{-1}c) = d(x, y)$ for all $x, y \in X$.

An element $x \in X$ is called *2-divisible* if there exists $y \in X$ such that $y^2 = x$. X is called *2-divisible* if every element in X is 2-divisible. Furthermore, X is a called *2-torsion free* if it contains the unit of $\mathcal{G}$ and the condition $x^2 = 1$ with $x \in X$ implies $x = 1$.

We shall need the following result taken from [25].

Theorem 2.6. [25, Theorem 2.4] *Let (X, d_X) and (Y, d_Y) be two metric spaces. Pick two points $a, c \in X$. Suppose that $\varphi : X \to X$ is a distance preserving map such that $\varphi(c) = c$ and $\varphi \circ \varphi$ is the identity map on X. Let*

$$L_{a,c} = \{x \in X : d_X(a, x) = d_X(\varphi(a), x) = d_X(a, c)\}.$$

Suppose that that there exists a constant $K > 1$ such that $d_X(\varphi(x), x) \geq Kd_X(x, c)$ holds for every $x \in L_{a,c}$. If δ is a bijective distance preserving map from X onto Y, and ψ is a bijective distance preserving map from Y onto itself such that $\psi(\Delta(a)) = \Delta(\varphi(a))$ and $\psi(\Delta(\varphi(a))) = \Delta(a)$, then we have $\psi(\Delta(c)) = \Delta(c)$. □

Conditions $B(a, b)$, $C_1(a, b)$ and $C_2(a, b)$ are perfectly applied in [25] (and subsequently in [26]) to establish a generalization of the Mazur-Ulam theorem for commutative groups [25, Corollary 5.1], and to present a metric characterization of normed real-linear spaces among commutative metric groups [25, Corollary 5.4]. Despite of the tempting title of [25] for the audience on Jordan structures–i.e. *"Isometries and maps compatible with inverted Jordan triple products on groups"*–, the results in the just quoted reference have not been applied in a proper Jordan setting yet. There are so many handicaps reducing its potential applicability. We are aimed to present a first application in this paper.

For the discussion in this paragraph, let $\mathcal{A}$ be a unital JC*-algebra which will be regarded as a JB*-subalgebra of some $B(H)$. Let us observe that the unit of $\mathcal{A}$ must be a projection $\mathbf{1}_{\mathcal{A}}$ in $B(H)$, and thus by replacing H with $\mathbf{1}_{\mathcal{A}}(H)$, we can always assume that $\mathcal{A}$ and $B(H)$ share the same unit. We shall denote the product of $B(H)$ by mere juxtaposition. The set $\mathcal{U}(\mathcal{A})$ of all unitaries in $\mathcal{A}$ is not in general a subgroup of $\mathcal{U}(B(H))$ –the latter is not even stable under Jordan products–, however $U_u(v) = uvu$, and u^* lie in $\mathcal{U}(\mathcal{A})$ for all $u, v \in \mathcal{U}(\mathcal{A})$ (cf. Lemma 2.1). The set $\mathcal{U}(B(H))$ is a group for its usual product and will be equipped with the distance provided by the operator norm. Conditions of the type $C_1(a, b)$ do not hold for $(\mathcal{U}(\mathcal{A}), \|.\|)$ because products of the form $ax^{-1}b$ do not necessarily lie in $\mathcal{U}(\mathcal{A})$ for all $a, b, x \in \mathcal{U}(\mathcal{A})$. The set $\mathcal{U}(\mathcal{A})$ is not 2-torsion free since $-1 \in \mathcal{U}(\mathcal{A})$. Furthermore, the identity $yx^{-1}y = y^2x^{-1}$ does not necessarily hold for $x, y \in \mathcal{U}(\mathcal{A})$. We have therefore justified that [25, Corollaries 3.9, 3.10 and 3.11] cannot be applied in the Jordan setting, even under more favorable hypothesis of working with a JC*-algebra.

Let $\text{Iso}(Z)$ denote the group of all surjective linear isometries on a Banach space Z. Hidden within the proof of [26, Theorem 6], it is shown that for a complex Banach spaces Z, condition $B(a, b)$ is satisfied for elements a, b in $\text{Iso}(Z)$ which are at distance strictly smaller than $\frac{1}{2}$. Let us concretize the exact statement.

Lemma 2.7. [26, Proof of Theorem 6] *Let Z be a complex Banach space, let u, v be two elements in $Iso(Z)$ with $\|u - v\| < \frac{1}{2}$. Then for $K = 2 - 2\|u - v\| > 1$, the inequality*

$$\|vw^{-1}v - w\| \geq K\|w - v\|$$

holds for every w in the set

$$L_{u,v} = \left\{w \in Iso(Z) : \|u - w\| = \|vu^{-1}v - w\| = \|u - v\|\right\}.$$

The next result is a consequence of the previous lemma in the case in which M is a JC*-algebra by just regarding M as a unital Jordan *-subalgebra of some $B(H)$ with the same unit (we can always see $\mathcal{U}(M)$ inside $\text{Iso}(H)$). The existence of exceptional JB*-algebras which cannot be embedded as Jordan *-subalgebras of $B(H)$ (see [24, Corollary 2.8.5], [6, Example 3.1.56]), forces us to develop a new argument.

Lemma 2.8. *Let u, v be two elements in $\mathcal{U}(M)$, where M is a unital JB*-algebra. Suppose $\|u - v\| < 1/2$. Then the Jordan version of condition $B(u, v)$ holds for $\mathcal{U}(M)$, that is,*

(*a*) *For all $x, y \in \mathcal{U}(M)$ we have $\left\|U_v(x^{-1}) - U_v(y^{-1})\right\| = \|x^* - y^*\| = \|x - y\|$.*
(*b*) *The constant $K = 2 - 2\|u - v\| > 1$ satisfies that*

$$\|U_v(w^*) - w\| = \left\|U_v(w^{-1}) - w\right\| \geq K\|w - v\|,$$

for all w in the set

$$L_{u,v} = \{w \in \mathcal{U}(M) : \|u - w\| = \|U_v(u^{-1}) - w\| = \|U_v(u^*) - w\| = \|u - v\|\}.$$

Proof. Statement (a) is clear from Lemma 2.1(d) and the fact that the involution on M is an isometry.

Let us consider the u-isotope JB*-algebra $M(u) = (M, \circ_u, *_u)$ of M. The U-operator on $M(u)$ will be denoted by U^u. We fix an element $w \in L_{u,v}$. Since $\|u-w\| = \|u-v\| < \frac{1}{2}$, we deduce from Lemma 2.2 the existence of two self-adjoint elements $h_1, h_2 \in M(u)$ such that $v = e^{ih_1}$ and $w = e^{ih_2}$. Let $\mathcal{B}$ denote the JB*-subalgebra of $M(u)$ generated by u, h_1, h_2. The Shirshov-Cohn theorem assures the existence of a complex Hilbert space H such that $\mathcal{B}$ is a JB*-subalgebra of $B(H)$ and both share the same unit u (the product of $B(H)$ will be denoted by mere juxtaposition and the involution by $\sharp$). Obviously $u, v, w \in \mathcal{U}(\mathcal{B}) \subseteq \mathcal{U}(B(H)) \subseteq \operatorname{Iso}(H)$ with $\|u-w\|_{\mathcal{B}} = \|u-w\|_M = \|u-v\|_M = \|u-v\|_{\mathcal{B}}$. Let us compute

$$U_v^u(u) = 2(v \circ_u u) \circ_u v - (v \circ_u v) \circ_u u$$
$$= 2v \circ_u v - v \circ_u v = v \circ_u v = \{v, u, v\} = U_v(u^*),$$

and

$$\|vu^{-1}v - w\|_{B(H)} = \|vu^{-1}v - w\|_{\mathcal{B}} = \|U_v^u(u^\sharp) - w\|_{\mathcal{B}} = \|U_v^u(u^{*_u}) - w\|_{\mathcal{B}}$$
$$= \|U_v^u(u) - w\|_{\mathcal{B}} = \|U_v(u^*) - w\|_{\mathcal{B}} = \|U_v(u^*) - w\|_M.$$

Lemma 2.7 proves that for $K = 2 - 2\|u-v\| > 1$ we have

$$\|U_v^u(w^{*_u}) - w\|_{\mathcal{B}} = \left\|U_v^u(w^{-1}) - w\right\|_{\mathcal{B}} = \|vw^{-1}v - w\|_{B(H)} \geq K\|w-v\|_{\mathcal{B}} = K\|w-v\|.$$

On the other hand, by the uniqueness of the triple product (see [33, Proposition 5.5] or Lemma 2.1(c)) we have

$$U_v^u(w^{*_u}) = \{v, w, v\}_{\mathcal{B}} = \{v, w, v\}_{M(u)} = \{v, w, v\}_M = U_v(w^*).$$

All together gives

$$\left\|U_v(w^{-1}) - w\right\| = \|U_v(w^*) - w\|_M = \|U_v(w^*) - w\|_{\mathcal{B}}$$
$$= \|U_v^u(w^{*_u}) - w\|_{\mathcal{B}} \geq K\|w-v\|,$$

which completes the proof. □

We can now establish a key result for our goals.

Theorem 2.9. *Let $\Delta : \mathcal{U}(M) \to \mathcal{U}(N)$ be a surjective isometry, where M and N are unital JB*-algebras. Suppose $u, v \in \mathcal{U}(M)$ with $\|u-v\| < \frac{1}{2}$. Then the following statements are true:*

1. *The Jordan version of condition $B(u, v)$ holds for $\mathcal{U}(M)$;*
2. *The Jordan version of condition $C_2(\Delta(u), \Delta(U_v(u^*)))$ holds for $\mathcal{U}(N)$;*
3. *The identity $\Delta(U_v(u^*)) = \Delta(U_v(u^{-1})) = U_{\Delta(v)}(\Delta(u)^*) = U_{\Delta(v)}(\Delta(u)^{-1})$ holds.*

Proof. Statement (1) follows from Lemma 2.8.

(2) By hypotheses $\|\Delta(u) - \Delta(v)\| = \|u-v\| < \frac{1}{2}$. Let $\mathcal{B}$ denote the JB*-subalgebra of the u-isotope $M(u)$ generated by u and v. We shall denote by U^u the U-operator in $M(u)$. Again, by the Shirshov-Cohn theorem we can find a complex Hilbert space H such that $\mathcal{B}$ is a JB*-subalgebra of $B(H)$ and u is the unit of $B(H)$. The product of $B(H)$ will be denoted by mere juxtaposition and the involution by $\sharp$. In this case we have

$$\|\Delta(u) - \Delta(U_v(u^*))\| = \|u - U_v(u^*)\| = \|u - \{v, u, v\}\| = \|u - \{v, u, v\}_{\mathcal{B}}\|_{\mathcal{B}}$$
$$= \|u - U_v^u(u^{*_u})\|_{\mathcal{B}} = \|u - U_v^u(u^{*_u})\|_{B(H)} = \|u - vu^\sharp v\|_{B(H)}$$
$$= \|u - vv\|_{B(H)} = \|v^\sharp - v\|_{B(H)} \leq \|v^\sharp - u\|_{B(H)} + \|u-v\|_{B(H)}$$
$$= 2\|u-v\|_{B(H)} = 2\|u-v\|_{\mathcal{B}} = 2\|u-v\| < 1$$

(cf. Lemma 2.1(c)).

Lemma 2.2(b) assures the existence of $w \in \mathcal{U}(N)$ satisfying

$$U_w(\Delta(u)^*) = \Delta\left(U_v(u^*)\right), \text{ and } \|w - \Delta(v)\| < 1 \text{ (or smaller).} \tag{6}$$

This shows that the Jordan version of $C_2(\Delta(u), \Delta(U_u(v^*)))$ holds for $\mathcal{U}(N)$ because the remaining requirement, i.e.,

$$\|U_w(x^{-1}) - U_w(y^{-1})\| = \|x^{-1} - y^{-1}\| = \|x^* - y^*\| = \|x - y\|$$

holds for all $x, y \in \mathcal{U}(N)$ (cf. Lemma 2.1(d)).

(3) Let $w \in \mathcal{U}(N)$ be the element found in the proof of (2). We define a couple of mappings $\varphi : \mathcal{U}(M) \to \mathcal{U}(M)$ and $\psi : \mathcal{U}(N) \to \mathcal{U}(N)$ given by $\varphi(x) := U_v(x^{-1}) = U_v(x^*)$ and $\psi(y) := U_w(y^{-1}) = U_w(y^*)$, respectively. Clearly, φ and ψ are distance preserving bijections (cf. Lemma 2.1(d)).

It is clear that $\varphi(v) = v$ and $\varphi\circ\varphi$ is the identity mapping on $\mathcal{U}(M)$. Furthermore, by (6) we have

$$\psi(\Delta(u)) = U_w(\Delta(u)^*) = \Delta\left(U_v(u^*)\right) = \Delta\left(\varphi(u)\right)$$

$$\psi\left(\Delta\left(\varphi(u)\right)\right) = U_w\left(\Delta\left(U_v(u^*)^*\right)\right) = U_{w^*}\left(\Delta\left(U_v(u^*)\right)\right)^* = \Delta(u)^{**} = \Delta(u).$$

Since by (1) the Jordan version of condition $B(u, v)$ holds for $\mathcal{U}(M)$, and by (2) the Jordan version of $C_2(\Delta(u), \Delta(U_v(u^*)))$ holds for $\mathcal{U}(N)$, we can see that all the hypotheses of Theorem 2.6 [25, Theorem 2.4] are satisfied. We deduce from the just quoted theorem that $U_w(\Delta(v)^*) = \Delta(v)$, and since $\|w - \Delta(v)\| < 2$ (cf. (6)), Lemma 2.3 guarantees that $w = \Delta(v)$ and hence $\Delta(U_v(u^*)) = U_{\Delta(v)}(\Delta(u)^*)$ as desired (see (6)). □

3. Surjective isometries between sets of unitaries

In this section we shall try to find a precise description of the surjective isometries between the sets of unitaries in two unital JB*-algebras. Our first goal is to find conditions under which any such surjective isometry can be extended to a surjective real linear isometry between these JB*-algebras.

We recall that a *one-parameter group* of bounded linear operators on a Banach space Z is a mapping $\mathbb{R} \to B(Z)$, $t \mapsto E(t)$ satisfying $E(0) = I$ and $E(t+s) = E(s)E(t)$, for all $s, t \in \mathbb{R}$. A one-parameter group $\{E(t) : t \in \mathbb{R}\}$ is uniformly continuous at the origin if $\lim\limits_{t\to 0} \|E(t) - I\| = 0$. It is known that being uniformly continuous at zero is equivalent to the existence of a bounded linear operator $R \in B(Z)$ such that $E(t) = e^{tR}$ for all $t \in \mathbb{R}$, where the exponential is computed in the Banach algebra $B(Z)$ (see, for example, [4, Proposition 3.1.1]). A one-parameter group on $\{E(t) : t \in \mathbb{R}\}$ on a complex Hilbert space H is called *strongly continuous* if for each ξ in H the mapping $t \mapsto E(t)(\xi)$ is continuous ([9, Definition 5.3, Chapter X]). A *one-parameter unitary group* on H is a one-parameter group on H such that $E(t)$ is a unitary element for each $t \in \mathbb{R}$.

The celebrated Stone's one-parameter theorem affirms that for each strongly continuous one-parameter unitary group $\{E(t) : t \in \mathbb{R}\}$ on a complex Hilbert space H there exists a self-adjoint operator $h \in B(H)$ such that $E(t) = e^{ith}$, for every $t \in \mathbb{R}$ ([9, 5.6, Chapter X]).

We recall that a *triple derivation* on a JB*-triple E is a linear mapping $\delta : E \to E$ satisfying a ternary version of Leibniz' rule

$$\delta\{a, b, c\} = \{\delta(a), b, c\} + \{a, \delta(b), c\} + \{a, b, \delta(c)\}, \quad (a, b, c \in E).$$

We shall apply that every triple derivation is automatically continuous (see [1, Corollary 2.2]). If $\delta : M \to M$ is a triple derivation on a unital JB*-algebra, it is known that $\delta(\mathbf{1}_M)^* = -\delta(\mathbf{1}_M)$, that is, $i\delta(\mathbf{1}_M) \in M_{sa}$ (cf. [28, Proof of Lemma 1]).

The existence of exceptional JB*-algebra which cannot be represented inside a $B(H)$ space, adds extra difficulties to apply Stone's one-parameter theorem. The study of uniformly continuous one-parameter groups of surjective isometries (i.e. triple isomorphisms), Jordan *-isomorphisms and orthogonality preserving operators on JB*-algebras has been recently initiated in [23]. We complement the results in the just quoted reference with the next result, which is a Jordan version of Stone's theorem for uniformly continuous unitary one-parameter groups.

Theorem 3.1. *Let M be a unital JB*-algebra. Suppose $\{u(t) : t \in \mathbb{R}\}$ is a family in $\mathcal{U}(M)$ satisfying $u(0) = \mathbf{1}$, and $U_{u(t)}(u(s)) = u(2t + s)$, for all $t, s \in \mathbb{R}$. We also assume that the mapping $t \mapsto u(t)$ is continuous. Then there exists $h \in M_{sa}$ such that $u(t) = e^{ith}$ for all $t \in \mathbb{R}$.*

Proof. We shall first prove that

$$u(s + t) = u(t) \circ u(s) \text{ for all } t, s \in \mathbb{R}. \tag{7}$$

Fix a real t, it follows from the hypothesis that

$$u(t)^2 = U_{u(t)}(u(0)) = u(2t), \text{ and } u(t)^3 = U_{u(t)}(u(t)) = u(3t).$$

Arguing by induction on n, it can be established that $u(t)^n = u(nt)$ for all $n \in \mathbb{N}$, $t \in \mathbb{R}$. Indeed, for any integer $n \geq 4$, by the induction hypothesis

$$u(t)^n = U_{u(t)}(u(t)^{n-2}) = U_{u(t)}(u((n - 2)t)) = u(nt).$$

The identity $U_{u(t)}U_{u(-t)}U_{u(t)} = U_{u(t)}$ (see (1)) together with the fact that $U_{u(t)}$ is invertible in $B(M)$ proves that $U_{u(t)^*} = U^{-1}_{u(t)} = U_{u(-t)}$ and thus $u(t)^* = u(t)^{-1} = u(-t)$ for all $t \in \mathbb{R}$ (cf. [6, Theorem 4.1.3]).

Given an integer $n \leq 0$ we observe that $u(t)^n = (u(t)^{-n})^{-1} = (u(t)^{-n})^*$ for every $t \in \mathbb{R}$, and thus

$$u(t)^n = (u(t)^{-n})^* = u((-n)t)^* = u(nt).$$

We have therefore shown that

$$u(t)^n = u(nt), \text{ for all } t \in \mathbb{R} \text{ and } n \in \mathbb{Z}. \tag{8}$$

By the continuity of the mapping $t \mapsto u(t)$, in order to prove (7) it suffices to show that the identity $u(r + r') = u(r) \circ u(r')$ holds for any rational numbers r and r'. Therefore, let us take $r = n/m$ and $r' = n'/m'$, with $n, n' \in \mathbb{Z}$ and $m, m' \in \mathbb{N}$. By a couple of applications of (8) and the power associativity of M (see [24, Lemma 2.4.5]) we have

$$\begin{aligned} u(r + r') &= u\left(\frac{nm' + mn'}{mm'}\right) = u\left(\frac{1}{mm'}\right)^{nm'+mn'} = u\left(\frac{1}{mm'}\right)^{nm'} \circ u\left(\frac{1}{mm'}\right)^{mn'} \\ &= u\left(\frac{nm'}{mm'}\right) \circ u\left(\frac{mn'}{mm'}\right) = u(r) \circ u(r'), \end{aligned}$$

as desired.

Let us define a mapping $\Phi : \mathbb{R} \to \operatorname{Iso}(M)$, $t \mapsto \Phi(t) = U_{u(t)}$. Clearly, Φ is continuous with $\Phi(0) = Id_M$. By applying the fundamental identity (1) one sees that

$$\Phi(t)\Phi(s)\Phi(t) = U_{u(t)}U_{u(s)}U_{u(t)} = U_{U_{u(t)}(u(s))} = U_{u(2t+s)} = \Phi(2t + s),$$

for all $s, t \in \mathbb{R}$. It then follows that $\Phi(t)^2 = \Phi(t)\Phi(0)\Phi(t) = \Phi(2t)$, $\Phi(3t) = \Phi(t)^3$, and by induction on $n (\geq 3)$, $\Phi(nt) = \Phi(2t + (n-2)t) = \Phi(t)\Phi((n-2)t)\Phi(t) = \Phi(t)\Phi(t)^{n-2}\Phi(t) = \Phi(t)^n$ for all $t \in \mathbb{R}$, $n \in \mathbb{N}$. Therefore $\Phi(nt) = \Phi(t)^n$ for all $t \in \mathbb{R}$, $n \in \mathbb{Z}$.

Since, for each real t, $\Phi(t)\Phi(-t)\Phi(t) = \Phi(t)$ and $\Phi(t) \in \mathrm{Iso}(M)$ we can deduce that $\Phi(-t) = \Phi(t)^{-1}$ for all $t \in \mathbb{R}$. It follows that, for a negative integer n and each real t, we have

$$\Phi(nt) = \Phi((-n)(-t)) = \Phi(-t)^{-n} = \Phi(t)^n,$$

an identity which then holds for all $n \in \mathbb{Z}$.

We claim that $t \mapsto \Phi(t)$ is a one-parameter group of surjective isometries on M. Let us fix two rational numbers $\frac{n}{m}, \frac{n'}{m'}$ with $m, m' \in \mathbb{N}$, $n, n' \in \mathbb{Z}$. It follows from the above properties that

$$\Phi\left(\frac{n}{m} + \frac{n'}{m'}\right) = \Phi\left(\frac{nm' + n'm}{mm'}\right) = \Phi\left(\frac{1}{mm'}\right)^{nm'+n'm}$$

$$= \Phi\left(\frac{1}{mm'}\right)^{nm'} \Phi\left(\frac{1}{mm'}\right)^{n'm} = \Phi\left(\frac{n}{m}\right)\Phi\left(\frac{n'}{m'}\right).$$

It follows from the continuity of Φ that

$$\Phi(t+s) = \Phi(t)\Phi(s), \text{ for all } t, s \in \mathbb{R},$$

that is, $\Phi(t)$ is a uniformly continuous one-parameter group of surjective isometries on M. By [23, Lemma 3.1] there exists a triple derivation $\delta : M \to M$ satisfying $\Phi(t) = e^{t\delta} = \displaystyle\sum_{n=0}^{\infty} \frac{t^n}{n!}\delta^n$ for all $t \in \mathbb{R}$, where the exponential is computed in $B(M)$.

Let us observe that $w(t) := \Phi(t)(\mathbf{1}) = U_{u(t)}(\mathbf{1}) = u(t)^2 = U_{u(t)}(u(0)) = u(2t)$ for all $t \in \mathbb{R}$, and hence by (7)

$$w(t+s) = \Phi(t+s)(\mathbf{1}) = u(2(t+s)) = u(2t) \circ u(2s)$$

$$= \Phi(t)(\mathbf{1}) \circ \Phi(s)(\mathbf{1}) = w(t) \circ w(s),$$

for all $s, t \in \mathbb{R}$. We shall next show that

$$\delta(\mathbf{1})^n = \delta^n(\mathbf{1}), \text{ for all natural } n.$$

Since $w(t+s) = w(t) \circ w(s)$, by taking derivatives in t at $t = 0$ we get

$$\sum_{n=1}^{\infty} \frac{s^{n-1}}{(n-1)!}\delta^n(\mathbf{1}) = \frac{\partial}{\partial t}_{|t=0} w(t+s) = \frac{\partial}{\partial t}_{|t=0} w(t) \circ w(s) = \delta(\mathbf{1}) \circ w(s),$$

for all $s \in \mathbb{R}$. Taking a new derivative in s at $s = 0$ we have

$$\delta^2(\mathbf{1}) = \frac{\partial}{\partial s}_{|s=0} \frac{\partial}{\partial t}_{|t=0} w(t+s) = \delta(\mathbf{1}) \circ \frac{\partial}{\partial s}_{|s=0} w(s)$$

$$= \delta(\mathbf{1}) \circ (\delta(\mathbf{1}) \circ w(0)) = M^2_{\delta(\mathbf{1})}(w(0)) = \delta(\mathbf{1})^2.$$

Similarly,

$$\delta^n(\mathbf{1}) = \frac{\partial^{n-1}}{\partial s^{n-1}}_{|s=0} \frac{\partial}{\partial t}_{|t=0} w(t+s) = M^n_{\delta(\mathbf{1})}(w(0)) = \delta(\mathbf{1})^n,$$

which gives the desired statement.

It follows from the above identities that

$$u(t) = \Phi(\frac{t}{2}) = e^{\frac{t}{2}\delta}(\mathbf{1}) = \sum_{n=0}^{\infty} \frac{t^n}{2^n n!}\delta^n(\mathbf{1}) = \sum_{n=0}^{\infty} \frac{t^n}{2^n n!}\delta(\mathbf{1})^n = e^{t\frac{\delta(\mathbf{1})}{2}}, \text{ for all } t \in \mathbb{R},$$

where, as we commented before this proposition, $\delta(\mathbf{1})^* = -\delta(\mathbf{1})$ in M (cf. [28, Proof of Lemma 1]). □

We continue by enunciating a variant of an argument which has been applied in several cases before.

Remark 3.2. *Suppose u is a unitary element in a unital JB*-algebra M such that $\|\mathbf{1}-u\| < 2$. By Lemma 2.2(a) we can find a self-adjoint element $h \in M_{sa}$ satisfying $u = e^{ih}$. Let us consider the unitary $\omega = e^{-i\frac{h}{2}} \in \mathcal{U}(M)$ and the mapping $U_\omega : M \to M$. Let us observe that $U_\omega(u) = \mathbf{1}$ (just apply that u and ω operator commute by definition). Since ω is unitary in M, and hence $U_\omega^{-1} = U_{\omega^*}$ (cf.* [6, Theorem 4.1.3]*) we can conclude from* (4) *that $U_\omega : M(u) = (M, \circ_u, *_u) \to M$ is a unital surjective isometry (see also Lemma 2.1(d) or* [6, Theorem 4.2.28] *for a direct argument). We can therefore conclude from Theorem 6 in* [46] *that $U_\omega : M(u) = (M, \circ_u, *_u) \to M$ is a Jordan *-isomorphism.*

As in the case of unital C-algebras (cf. the discussion preceding Proposition 4.4.10 in* [31]*), not each unitary element of a unital JB*-algebra M is of the form e^{ih} for some $h \in M_{sa}$. However, if we assume that M is a JBW*-algebra the conclusion is different. Let u be a unitary element in a JBW*-algebra M. Let $\mathcal{W}$ denote the JBW*-subalgebra of M generated by u, u^* and the unit of M. Clearly $\mathcal{W}$ is an associative JBW*-algebra (cf.* [24, Theorem 3.2.2, Remark 3.2.3 and Theorem 4.4.16]*), and we can therefore assume that $\mathcal{W}$ is a commutative von Neumann algebra. Theorem 5.2.5 in* [31] *implies the existence of an element $h \in \mathcal{W}_{sa} \subseteq M_{sa}$ such that $e^{ih} = u$ (in $\mathcal{W}$ and also in M). We therefore have*

$$\mathcal{U}(M) = \{e^{ih} : h \in M_{sa}\} \tag{9}$$

for all JBW-algebra M.*

The next lemma, which is a Jordan version of [26, Lemma 7], will be required later.

Lemma 3.3. *Let M and N be two unital JB*-algebras. Let $\{u_k : 0 \le k \le 2^n\}$ be a subset of $\mathcal{U}(M)$ and let $\Phi : \mathcal{U}(M) \to \mathcal{U}(N)$ be a mapping such that $U_{u_{k+1}}(u_k^*) = u_{k+2}$ and*

$$\Phi(U_{u_{k+1}}(u_k^*)) = U_{\Phi(u_{k+1})}(\Phi(u_k)^*),$$

for all $0 \le k \le 2^n - 2$. Then $U_{u_{2^n-1}}(u_0^*) = u_{2^n}$ and

$$\Phi\left(U_{u_{2^n-1}}(u_0^*)\right) = U_{\Phi(u_{2^n-1})}\Phi(u_0)^*.$$

Proof. We shall argue by induction on n. The statement is clear for $n = 1$. Suppose that our statement is true for every family with $2^n + 1$ elements satisfying the conditions above. Let $\{w_k : 0 \le k \le 2^{n+1}\}$ be a subset of $\mathcal{U}(M)$ such that $U_{w_{k+1}}(w_k^*) = w_{k+2}$ and

$$\Phi(U_{w_{k+1}}(w_k^*)) = U_{\Phi(w_{k+1})}(\Phi(w_k)^*),$$

for all $0 \leq k \leq 2^{n+1}-2$. Set $u_k = w_{2k}$. We shall next show that we can apply the induction hypothesis to the family $\{u_k : 0 \leq k \leq 2^n\} \subseteq \mathcal{U}(M)$. Fix $0 \leq k \leq 2^n - 2$,

$$\begin{aligned} u_{k+2} = w_{2k+4} &= U_{w_{2k+3}}(w_{2k+2}^*) = U_{U_{w_{2k+2}}(w_{2k+1}^*)}(w_{2k+2}^*) \\ &= U_{w_{2k+2}} U_{w_{2k+1}^*} U_{w_{2k+2}}(w_{2k+2}^*) = U_{w_{2k+2}} U_{w_{2k+1}^*}(w_{2k+2}) \\ &= U_{w_{2k+2}} U_{w_{2k+1}^*} U_{w_{2k+1}}(w_{2k}^*) = U_{w_{2k+2}}(w_{2k}^*) = U_{u_{k+1}}(u_k^*), \end{aligned}$$

where in the fourth equality we applied the identity (1).

On the other hand, the previous identities and the induction hypothesis also give

$$\begin{aligned} \Phi(U_{u_{k+1}}(u_k^*)) &= \Phi(U_{w_{2k+2}}(w_{2k}^*)) = \Phi(U_{w_{2k+3}}(w_{2k+2}^*)) = U_{\Phi(w_{2k+3})}(\Phi(w_{2k+2})^*) \\ &= U_{\Phi(U_{w_{2k+2}}(w_{2k+1}^*))}(\Phi(w_{2k+2})^*) = U_{U_{\Phi(w_{2k+2})}(\Phi(w_{2k+1})^*)}(\Phi(w_{2k+2})^*) \\ &= U_{\Phi(w_{2k+2})} U_{\Phi(w_{2k+1})^*} U_{\Phi(w_{2k+2})}(\Phi(w_{2k+2})^*) \\ &= U_{\Phi(w_{2k+2})} U_{\Phi(w_{2k+1})^*}(\Phi(w_{2k+2})) \\ &= U_{\Phi(w_{2k+2})} U_{\Phi(w_{2k+1})^*}(\Phi(U_{w_{2k+1}}(w_{2k}^*))) \\ &= U_{\Phi(w_{2k+2})} U_{\Phi(w_{2k+1})^*} U_{\Phi(w_{2k+1})}(\Phi(w_{2k})^*) \\ &= U_{\Phi(w_{2k+2})}(\Phi(w_{2k})^*) = U_{\Phi(u_{k+1})}(\Phi(u_k)^*), \end{aligned}$$

where in the sixth equality we applied the identity (1).

It follows from the induction hypothesis that

$$U_{w_{2^n}}(w_0^*) = U_{u_{2^{n-1}}}(u_0^*) = u_{2^n} = w_{2^{n+1}}$$

and

$$\Phi\left(U_{w_{2^n}}(w_0^*)\right) = \Phi\left(U_{u_{2^{n-1}}}(u_0^*)\right) = U_{\Phi(u_{2^{n-1}})}\Phi(u_0)^* = U_{\Phi(w_{2^n})}\Phi(w_0)^*,$$

which finishes the induction argument. □

We are now ready to establish our first main result, which asserts that, under some mild conditions, for each surjective isometry Δ between the unitary sets of two unital JB*-algebras M and N we can find a surjective real linear isometry $\Psi : M \to N$ which coincides with Δ on the subset $e^{iM_{sa}}$.

Theorem 3.4. *Let $\Delta : \mathcal{U}(M) \to \mathcal{U}(N)$ be a surjective isometry, where M and N are two unital JB*-algebras. Suppose that one of the following holds:*

(1) $\|\mathbf{1}_N - \Delta(\mathbf{1}_M)\| < 2$;
(2) *There exists a unitary ω_0 in N such that $U_{\omega_0}(\Delta(\mathbf{1}_M)) = \mathbf{1}_N$.*

Then there exists a unitary ω in N satisfying

$$\Delta(e^{iM_{sa}}) = U_{\omega^*}(e^{iN_{sa}}).$$

Furthermore, there exists a central projection $p \in N$ and a Jordan *-isomorphism $\Phi : M \to N$ such that

$$\begin{aligned} \Delta(e^{ih}) &= U_{\omega^*}\left(p \circ \Phi(e^{ih})\right) + U_{\omega^*}\left((\mathbf{1}_N - p) \circ \Phi(e^{ih})^*\right) \\ &= P_2(U_{\omega^*}(p)) U_{\omega^*}(\Phi(e^{ih})) + P_2(U_{\omega^*}(\mathbf{1}_N - p)) U_{\omega^*}(\Phi((e^{ih})^*)), \end{aligned}$$

for all $h \in M_{sa}$. Consequently, the restriction $\Delta|_{e^{iM_{sa}}}$ admits a (unique) extension to a surjective real linear isometry from M onto N.

Proof. If Δ satisfies (1), by Remark 3.2 (see also Lemma 2.2(a)) there exists a unitary ω in N such that $U_\omega(\Delta(\mathbf{1}_M)) = \mathbf{1}_N$, and U_ω is an isometric Jordan *-isomorphism from the $\Delta(\mathbf{1})$-isotope $N(\Delta(\mathbf{1}))$ onto N. Since $\mathcal{U}\left(N, \circ_{\Delta(\mathbf{1}_M)}, *_{\Delta(\mathbf{1}_M)}\right)$ $= \mathcal{U}(N)$ (see Lemma 2.1) we have $U_\omega(\mathcal{U}(N)) = \mathcal{U}(N)$. In case that (2) holds we take $\omega = \omega_0$.

The surjective isometry $\Delta_0 : \mathcal{U}(M) \to \mathcal{U}(N)$, $\Delta_0(u) = U_\omega(\Delta(u))$, satisfies $\Delta_0(\mathbf{1}_M) = \mathbf{1}_N$, that is, Δ_0 is a unital surjective isometry between the unitary sets of M and N.

Fix $h \in M_{sa}$. We consider the continuous mapping $\mathbb{R} \to \mathcal{U}(M)$, $E_h(t) = e^{ith}$. Let u and v be two arbitrary unitary elements in $E_h(\mathbb{R})$ (that is, $u = e^{ith}$ and $v = e^{ish}$, for some $t, s \in \mathbb{R}$). Choose now a positive integer m such that $e^{\frac{\|i(s-t)h\|_M}{2^m}} - 1 < \frac{1}{2}$. It follows from this assumption that

$$\|e^{\frac{i(s-t)h}{2^m}} - \mathbf{1}_M\|_M \leq e^{\frac{\|i(s-t)h\|_M}{2^m}} - 1 < \frac{1}{2}. \tag{10}$$

Let us define the family $\{u_k : 0 \leq k \leq 2^{m+1}\}$, where

$$u_k = u \circ e^{\frac{ik(s-t)h}{2^m}}, \quad 0 \leq k \leq 2^{m+1}.$$

Since u and $e^{\frac{ik(s-t)h}{2^m}}$ operator commute in M, the element u_k is a unitary in M for every $k \in \{0, \ldots, 2^{m+1}\}$. Clearly, $u_0 = u$, $u_{2^m} = v$ and $u_{2^{m+1}} = U_v(u^*)$.

Any two elements in the family $\{u_k\}_{k=0}^{2^{m+1}}$ operator commute, thus it is not hard to see that for any $0 \leq k \leq 2^{m+1} - 2$,

$$U_{u_{k+1}}(u_k^{-1}) = U_{u_{k+1}}(u_k^*) = u_{k+2}.$$

On the other hand, by our election of m in (10), we have

$$\begin{aligned} \|u_{k+1} - u_k\|_M &= \|u \circ e^{\frac{i(k+1)(s-t)h}{2^m}} - u \circ e^{\frac{ik(s-t)h}{2^m}}\|_M \\ &\leq \|u\|_M \|e^{\frac{ik(s-t)h}{2^m}} \circ e^{\frac{i(s-t)h}{2^m}} - e^{\frac{ik(s-t)h}{2^m}}\|_M \\ &= \|e^{\frac{ik(s-t)h}{2^m}} \circ \left(e^{\frac{i(s-t)h}{2^m}} - \mathbf{1}_M\right)\|_M \\ &\leq \|e^{\frac{ik(s-t)h}{2^m}}\|_M \|e^{\frac{i(s-t)h}{2^m}} - \mathbf{1}_M\|_M \leq e^{\frac{\|i(s-t)h\|_M}{2^m}} - 1 < \frac{1}{2}, \end{aligned}$$

for any $0 \leq k \leq 2^{m+1} - 1$. Theorem 2.9(3) affirms that the identity

$$\Delta(U_{u_{k+1}}(u_k^{-1})) = \Delta(U_{u_{k+1}}(u_k^*)) = U_{\Delta(u_{k+1})}(\Delta(u_k)^*) = U_{\Delta(u_{k+1})}(\Delta(u_k)^{-1}),$$

holds for every $0 \leq k \leq 2^{m+1} - 1$.

Lemma 3.3 applied to Δ and the family $\{u_k : 0 \leq k \leq 2^{m+1}\}$ proves that $U_{u_{2^m}}(u_0^*) = u_{2^{m+1}} = U_v(u^*)$ and

$$\Delta(U_v(u^*)) = \Delta\left(U_{u_{2^m}}(u_0^*)\right) = U_{\Delta(u_{2^m})}(\Delta(u_0)^*) = U_{\Delta(v)}(\Delta(u)^*), \tag{11}$$

for any u, v arbitrary elements in the one-parameter unitary group $\{E_h(t) : t \in \mathbb{R}\}$. By similar arguments applied to Δ_0, or by applying U_ω at the previous identity, we deduce that

$$U_{\Delta_0(v)}(\Delta_0(u)^*) = \Delta_0(U_v(u^*)), \tag{12}$$

for every u, v in $\{E_h(t) : t \in \mathbb{R}\}$. Taking $v = \mathbf{1} = E_h(0)$ and u arbitrary in (12) and having in mind that $\Delta_0(\mathbf{1}) = \mathbf{1}$ we get

$$\Delta_0(u)^* = \Delta_0(u^*), \text{ for all } u \in \{E_h(t) : t \in \mathbb{R}\}. \tag{13}$$

Furthermore, for any two $u, v \in \{E_h(t) : t \in \mathbb{R}\}$ their adjoined u^*, v^* also lie in the set $\{E_h(t) : t \in \mathbb{R}\}$ and thus by (12) and (13) we derive

$$U_{\Delta_0(v)}(\Delta_0(u)) = U_{\Delta_0(v)}(\Delta_0(u^*)^*) = \Delta_0(U_v(u^{**})) = \Delta_0(U_v(u)),$$

for every u, v in $\{E_h(t) : t \in \mathbb{R}\}$, that is,

$$U_{\Delta_0(E_h(t))}(\Delta_0(E_h(s))) = \Delta_0(U_{E_h(t)}(E_h(s))) = \Delta_0(E_h(2t + s)),$$

for all $t, s \in \mathbb{R}$.

By applying Theorem 3.1 to $\{\Delta_0(E_h(t)) : t \in \mathbb{R}\}$ we deduce the existence (as well as the uniqueness) of a self-adjoint element y in N such that $\Delta_0(E_h(t)) = e^{ity} \in \mathcal{U}(N)$ for every $t \in \mathbb{R}$.

We can therefore define a mapping $f : M_{sa} \to N_{sa}$ as the one which maps h into y, that is, $f(h) = y$ (where y is the unique element in N_{sa} such that $\Delta_0(E_h(t)) = e^{ity} \in \mathcal{U}(N)$ for every $t \in \mathbb{R}$). Thus, f satisfies

$$\Delta_0(e^{ith}) = e^{itf(h)}, \tag{14}$$

for each $t \in \mathbb{R}$, and each $h \in M_{sa}$. We shall show that f is actually a surjective isometry.

Let us first observe that the injectivity of Δ_0 implies that f also is injective. On the other hand, replacing Δ_0 by Δ_0^{-1} in the previous arguments, we can deduce the existence of an injective mapping $g : N_{sa} \to M_{sa}$ such that

$$\Delta_0^{-1}(e^{ity}) = e^{itg(y)}, \tag{15}$$

for any $y \in N_{sa}$, and any $t \in \mathbb{R}$. Therefore, by combining the properties of f and g, we derive that $y = f(g(y))$ ($y \in N_{sa}$), and hence f is surjective.

The bijectivity of f allows us to assure that Δ_0 maps any one-parameter unitary group $\{e^{ith}\}_{t\in\mathbb{R}}$ in $\mathcal{U}(M)$ (for any self-adjoint h in M) onto the one-parameter unitary group $\{e^{ity}\}_{t\in\mathbb{R}}$ in $\mathcal{U}(N)$, with $y \in N_{sa}$. Consequently, for any $y \in N_{sa}$, and any real number t,

$$U_{\omega^*}(e^{ity}) = U_{\omega^*}(\Delta_0(e^{ith})) = U_{\omega^*}U_\omega(\Delta(e^{ith})) = \Delta(e^{ith}).$$

That proves the first statement, namely,

$$\Delta(e^{iM_{sa}}) = U_{\omega^*}(e^{iN_{sa}}).$$

Our next goal is to prove that f is an isometry. To this end, given $h, h' \in M_{sa}$ and a real number t, let us compute the following limits (with respect to the norm topology) as $t \to 0$:

$$\frac{e^{ith} - e^{ith'}}{t} = \frac{e^{ith} - \mathbf{1}_N}{t} - \frac{e^{ith'} - \mathbf{1}_N}{t} \longrightarrow ih - ih',$$

and

$$\frac{e^{itf(h)} - e^{itf(h')}}{t} = \frac{e^{itf(h)} - \mathbf{1}_N}{t} - \frac{e^{itf(h')} - \mathbf{1}_N}{t} \longrightarrow if(h) - if(h').$$

On the other hand, by (14), we have

$$\|e^{itf(h)} - e^{itf(h')}\|_N = \|\Delta_0(e^{ith}) - \Delta_0(e^{ith'})\|_N = \|e^{ith} - e^{ith'}\|_N.$$

Therefore, by uniqueness of the limits above, $\|f(h) - f(h')\|_N = \|h - h'\|_N$. The arbitrariness of h and h' in M_{sa} gives the expected conclusion, that is, $f : M_{sa} \to N_{sa}$ is a surjective isometry. Moreover, since $\Delta_0(\mathbf{1}_M) = \mathbf{1}_N$, we deduce that $f(0) = 0$, and hence the Mazur–Ulam theorem implies that f is a surjective real linear isometry from M_{sa} onto N_{sa}.

It is known since the times of Kaplansky that the self-adjoint part of any JB*-algebra is a JB-algebra. Thus, $f : M_{sa} \to N_{sa}$ can be regarded as a linear surjective isometry between JB-algebras. Theorem 1.4 and Corollary 1.11 in [29] guarantee the existence of a central symmetry $f(\mathbf{1}_M)$ in M_{sa} and a unital surjective linear isometry $\Phi : M \to N$ such that

$$f(h) = f(\mathbf{1}_M) \circ \Phi(h), \tag{16}$$

for every $h \in M_{sa}$. Therefore Φ is a unital triple isomorphism between unital JB*-algebras, and hence Φ must be a Jordan *-isomorphism (cf. [33, Proposition 5.5]).

By construction there exists a central projection p in N such that $f(\mathbf{1}_M) = 2p - \mathbf{1}_N = p - (\mathbf{1}_M - p)$, where p and $(\mathbf{1}_N - p)$ clearly are orthogonal projections in N, and for any $n > 0$,

$$(2p - \mathbf{1}_N)^n = (p - (\mathbf{1}_N - p))^n = p + (-1)^n (\mathbf{1}_N - p).$$

Finally, we shall describe Δ_0 in terms of p and Φ. To achieve this goal, we shall employ the equalities obtained in (14), (16), and the definition of the exponential in a Jordan algebra. According to this, given an arbitrary $h \in M_{sa}$, we have

$$\Delta_0(e^{ih}) = e^{if(h)} = e^{if(\mathbf{1}_M)\circ\Phi(h)} = e^{i(2p-\mathbf{1}_N)\circ\Phi(h)} = \sum_{n=0}^{\infty} \frac{(i(2p - \mathbf{1}_N) \circ \Phi(h))^n}{n!}.$$

Since $f(\mathbf{1}_N)$ (and hence p) is central, it operator commutes with any element in N. Additionally, Φ is a Jordan *-isomorphism, and we can thus conclude that

$$\begin{aligned}
\Delta_0(e^{ih}) &= \sum_{n=0}^{\infty} \frac{i^n (2p - \mathbf{1}_N)^n \circ \Phi(h)^n}{n!} = \sum_{n=0}^{\infty} \frac{i^n\,(p + (-1)^n (\mathbf{1}_N - p)) \circ \Phi(h^n)}{n!} \\
&= \sum_{n=0}^{\infty} \frac{i^n p \circ \Phi(h^n)}{n!} + \sum_{n=0}^{\infty} \frac{i^n (-1)^n (\mathbf{1}_N - p) \circ \Phi(h^n)}{n!} \\
&= p \circ \sum_{n=0}^{\infty} \frac{i^n \Phi(h^n)}{n!} + (\mathbf{1}_N - p) \circ \sum_{n=0}^{\infty} \frac{i^n (-1)^n \Phi(h^n)}{n!} \\
&= p \circ \Phi\left(\sum_{n=0}^{\infty} \frac{i^n h^n}{n!}\right) + (\mathbf{1}_N - p) \circ \Phi\left(\sum_{n=0}^{\infty} \frac{i^n (-1)^n h^n}{n!}\right) \\
&= p \circ \Phi(e^{ih}) + (\mathbf{1}_N - p) \circ \Phi(e^{-ih}) = p \circ \Phi(e^{ih}) + (\mathbf{1}_N - p) \circ \Phi(e^{ih})^*.
\end{aligned}$$

The arbitrariness of the self-adjoint element h in M gives the following statement

$$\Delta_0(e^{ih}) = U_\omega(\Delta(e^{ih})) = p \circ \Phi(e^{ih}) + (\mathbf{1}_N - p) \circ \Phi(e^{ih})^*, \quad (h \in M_{sa}),$$

and consequently,

$$\begin{aligned}\Delta(e^{ih}) &= U_{\omega^*}(\Delta(e^{ih})) = U_{\omega^*}\left(p\circ\Phi(e^{ih}) + (\mathbf{1}_N - p)\circ\Phi(e^{ih})^*\right)\\ &= U_{\omega^*}\left(P_2(p)\Phi(e^{ih})\right) + U_{\omega^*}\left(P_2(\mathbf{1}_N - p))\Phi((e^{ih})^*)\right)\\ &= P_2(U_{\omega^*}(p))U_{\omega^*}(\Phi(e^{ih})) + P_2(U_{\omega^*}(\mathbf{1}_N - p))U_{\omega^*}(\Phi((e^{ih})^*))\end{aligned} \tag{17}$$

because U_{ω^*} is a triple isomorphism. □

It should be remarked that the idea of employing one-parameter unitary groups was already employed by O. Hatori and L. Molnár in [27], where they were motivated by previous results on uniformly continuous group isomorphisms of unitary groups in AW*-factors due to Sakai (see [41]). In our proof this idea is combined with the Jordan version of Stone's one-parameter theorem developed in Theorem 3.1.

Corollary 3.5. *Let $\Delta : \mathcal{U}(M) \to \mathcal{U}(N)$ be a surjective isometry, where M and N are two unital JB*-algebras. Then there exist a central projection p in the $\Delta(\mathbf{1}_M)$-isotope $N(\Delta(\mathbf{1}_M))$ and an isometric Jordan *-isomorphism $\Phi : M \to N(\Delta(\mathbf{1}_M))$ such that*

$$\begin{aligned}\Delta(e^{ih}) &= p\circ_{\Delta(\mathbf{1}_M)} \Phi(e^{ih}) + (\mathbf{1}_N - p)\circ_{\Delta(\mathbf{1}_M)} \Phi(e^{ih})^{*_{\Delta(\mathbf{1}_M)}}\\ &= p\circ_{\Delta(\mathbf{1}_M)} \Phi(e^{ih}) + (\mathbf{1}_N - p)\circ_{\Delta(\mathbf{1}_M)} \Phi((e^{ih})^*),\end{aligned}$$

for all $h \in M_{sa}$.

Proof. The desired statement follows from Theorem 3.4 by just observing that $\Delta(\mathbf{1}_M)$ is the unit of the $\Delta(\mathbf{1}_M)$-isotope $N(\Delta(\mathbf{1}_M))$, and $\mathcal{U}(N) = \mathcal{U}(N(\Delta(\mathbf{1}_M)))$ (cf. Lemma 2.1(*b*)). □

Remark 3.6. *Let $\Delta : \mathcal{U}(M) \to \mathcal{U}(N)$ be a surjective isometry, where M and N are two unital JB*-algebras. We have shown in the proof of Theorem 3.4 (see also Remark 3.2) that the assumption $\|\mathbf{1}_N - \Delta(\mathbf{1}_M)\| < 2$ implies the existence of a unitary ω in N such that $U_\omega(\Delta(\mathbf{1}_M)) = \mathbf{1}_N$. So, condition* (1) *seems stronger, but* (2) *is all what is needed in the proof of this theorem.*

Remark 3.7. *From the point of view of JB*-triples, the conclusion of the previous Theorem 3.4 can be also stated in the following terms: There exist two orthogonal tripotents u_1 and u_2 in M and two orthogonal tripotents $\widetilde{u}_1$ and $\widetilde{u}_2$ in N, a linear surjective isometry (i.e. triple isomorphism) $\Psi_1 : M_2(u_1) \to N_2(\widetilde{u}_1)$ and a conjugate linear surjective isometry (i.e. triple isomorphism) $\Psi_2 : M_2(u_2) \to N_2(\widetilde{u}_2)$ such that $M = M_2(u_1) \oplus^\infty M_2(u_2)$, $N = N_2(\widetilde{u}_1) \oplus^\infty N_2(\widetilde{u}_2)$, and the surjective real linear isometry $\Psi = \Psi_1 + \Psi_2 : M_2(u_1) \oplus^\infty M_2(u_2) \to N_2(\widetilde{u}_1) \oplus^\infty N_2(\widetilde{u}_2)$ restricted to $e^{iM_{sa}}$ coincides with Δ.*

To see this conclusion, we resume the proof of Theorem 3.4 from its final paragraph. Set $\widetilde{\Psi} = U_{\omega^*}\Phi$, $\widetilde{u}_1 = U_{\omega^*}(p)$, $\widetilde{u}_2 = U_{\omega^*}(\mathbf{1}_N - p)$, $u_1 = \widetilde{\Psi}^{-1}(\widetilde{u}_1)$ and $u_2 = \widetilde{\Psi}^{-1}(\widetilde{u}_2)$. Since $N = N_2(p) \oplus^{\ell_\infty} N_2(\mathbf{1}_N - p)$, U_{ω^*} and $\widetilde{\Psi}$ are triple isomorphisms, we deduce that $N = N_2(\widetilde{u}_1) \oplus^\infty N_2(\widetilde{u}_2)$ and $M = M_2(u_1) \oplus^\infty M_2(u_2)$. The identity in (17) *actually proves that*

$$\begin{aligned}\Delta(e^{ih}) &= P_2(U_{\omega^*}(p))U_{\omega^*}(\Phi(e^{ih})) + P_2(U_{\omega^*}(\mathbf{1}_N - p))U_{\omega^*}(\Phi((e^{ih})^*))\\ &= P_2(\widetilde{u}_1)\widetilde{\Psi}P_2(u_1)(e^{ih}) + P_2(\widetilde{u}_2)\widetilde{\Psi}P_2(u_2)((e^{ih})^*),\end{aligned}$$

for all $h \in M_{sa}$. *It can be easily checked that the maps* $\Psi_1(x) = P_2(\widetilde{u}_1)\widetilde{\Psi}P_2(u_1)(x)$ *and* $\Psi_2(x) = P_2(\widetilde{u}_2)\widetilde{\Psi}P_2(u_2)(x^*)$ $(x \in M)$ *give the desired statement.*

The next corollary asserts that the Banach spaces underlying two unital JB*-algebras are isometrically isomorphic if and only if the metric spaces determined by the unitary sets of these algebras are isometric.

Corollary 3.8. *Two unital JB***-algebras* M *and* N *are Jordan* **-isomorphic if and only if there exists a surjective isometry* $\Delta : \mathcal{U}(M) \to \mathcal{U}(N)$ *satisfying one of the following*

(1) $\|\mathbf{1}_N - \Delta(\mathbf{1}_M)\| < 2$;
(2) *There exists a unitary* ω *in* N *such that* $U_\omega(\Delta(\mathbf{1}_M)) = \mathbf{1}_N$.

Furthermore, the following statements are equivalent for any two unital JB**-algebras* M *and* N*:*

(a) *M and N are isometrically isomorphic as (complex) Banach spaces;*
(b) *M and N are isometrically isomorphic as real Banach spaces;*
(c) *There exists a surjective isometry* $\Delta : \mathcal{U}(M) \to \mathcal{U}(N)$.

Proof. The first equivalence follows from Theorem 3.4.

We deal next with the second statement. The implication $(a) \Rightarrow (b)$ is clear. $(b) \Rightarrow (c)$ Suppose we can find a surjective real linear isometry $\Phi : M \to N$. It follows from [13, Corollary 3.2] (or even from [17, Corollary 3.4]) that Φ is a triple homomorphism, that is, Φ preserves triple products of the form $\{x, y, z\} = (x \circ y^*) \circ z + (z \circ y^*) \circ x - (x \circ z) \circ y^*$. In particular Φ maps unitaries in M to unitaries in N, and hence $\Phi(\mathcal{U}(M)) = \mathcal{U}(N)$. Therefore $\Delta = \Phi|_{\mathcal{U}(M)} : \mathcal{U}(M) \to \mathcal{U}(N)$ is a surjective isometry.

The implication $(c) \Rightarrow (a)$ is given by Corollary 3.5. □

As we have seen in Remark 3.2(9) for each JBW*-algebra M the set of all unitaries in M is precisely the set $e^{iM_{sa}}$. We are now ready to establish the main result of this paper in which we relax some of the hypotheses in Theorem 3.4 at the cost of considering surjective isometries between the unitary sets of two JBW*-algebras.

Theorem 3.9. *Let* $\Delta : \mathcal{U}(M) \to \mathcal{U}(N)$ *be a surjective isometry, where* M *and* N *are two JBW***-algebras. Then there exist a unitary* ω *in* N*, a central projection* $p \in N$*, and a Jordan* **-isomorphism* $\Phi : M \to N$ *such that*

$$\begin{aligned}\Delta(u) &= U_{\omega^*}\,(p \circ \Phi(u)) + U_{\omega^*}\,((\mathbf{1}_N - p) \circ \Phi(u)^*)\\ &= P_2(U_{\omega^*}(p))U_{\omega^*}(\Phi(u)) + P_2(U_{\omega^*}(\mathbf{1}_N - p))U_{\omega^*}(\Phi(u^*)),\end{aligned}$$

for all $u \in \mathcal{U}(M)$*. Consequently,* Δ *admits a (unique) extension to a surjective real linear isometry from* M *onto* N*.*

Proof. We only need to appeal to the identities $\mathcal{U}(M) = e^{iM_{sa}}$ and $\mathcal{U}(N) = e^{iN_{sa}}$, and to the arguments in Remark 3.2 to find a unitary $\omega \in N$ such that $U_\omega(\Delta(\mathbf{1}_M)) = \mathbf{1}_N$. The rest is a consequence of Theorem 3.4. □

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(M. Cueto-Avellaneda, A.M. Peralta) Departamento de Análisis Matemático, Facultad de Ciencias, Universidad de Granada, 18071 Granada, Spain.
E-mail address: `mcueto@ugr.es, aperalta@ugr.es`