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On the measure induced by copulas that are invariant under univariate truncation

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Abstract

The probability mass distribution of a class of copulas that are invariant under univariate truncation is presented. In particular, it is shown how (differential) properties of the generator of the copula are able to identify the singular (respectively, absolutely continuous) component of the induced measure and its support.

Keywords: Copula; Doubly stochastic measure; Singular measure

1. Introduction

A stochastic system composed by several components can be conveniently represented in terms of the marginal behaviour and a suitable copula, as suggested by the celebrated *Sklar's theorem* [16] (see also [10,14]). Thus, having at disposal a variety of copula families may provide more realistic tools for expressing some particular features of multivariate random vectors such as asymmetries, heavy tails, and directional dependencies.

In the bivariate case, a family of copulas generated by a one-dimensional real-valued function $f\colon [0,+\infty] \longrightarrow [0,1]$ has been proposed in [8] and is recalled below. Notice that $f^{[-1]}$ denotes the right-inverse of f given by $f^{[-1]}(s) = \inf\{t \in [0,+\infty] : f(t) = s\}$.

Theorem 1. *Let $C_f\colon [0,1]^2 \longrightarrow [0,1]$ be the function defined by*

$$C_f(u,v) = \begin{cases} uf\left(\dfrac{f^{[-1]}(v)}{u}\right), & \text{if } u \neq 0, \\ 0, & \text{otherwise,} \end{cases} \tag{1}$$

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where $f: [0, +\infty] \longrightarrow [0, 1]$ *is surjective and monotone. If* f *is concave and non-decreasing (or, convex and non-increasing), then* C_f *is a copula.*

Copulas of the form (1) arise in the study of the preservation of the dependence structure under truncation. In fact, if C_f is the distribution function of a pair (U, V) of standard uniform random variables, then C_f is also the copula of the conditional distribution function of (U, V) given that $U \le \alpha$ for any $\alpha \in (0, 1)$. In words, we say that C_f is *invariant under univariate truncation*.

Various aspects that emphasize the possible use of copulas of type (1) in statistical estimation have been presented in [9,12] like measures of association, tail dependence, and stochastic simulation among others. In particular, [9] investigates whether copulas of type (1) are absolutely continuous (respectively, singular) by exploiting a link with Archimedean copulas (see Proposition 3.2 and Corollary 3.3 in [9]). These latter aspects are particularly relevant for (at least) two main reasons. First, knowing the density of the copula may be helpful to apply maximum likelihood techniques in the estimation procedures. Secondly, the singular component (if it exists) may suggest that there is a non-zero probability that transformations of the involved random variables may be equal each other – an aspect quite relevant in systemic risk and default models (see, e.g., [2,13]).

Motivated by these aspects, we complement here the study of measure-theoretic properties of copulas of type (1) considered in [9]. To this end, we exploit the disintegration of the copula measure (related to the existence of conditional probability measures), which is illustrated in the preliminary section 2. Then, in section 3, we introduce some auxiliary functions to determine the mass distribution of the measure induced by a copula of type (1) as well as to describe the singular and the absolutely continuous component of the measure.

2. Preliminaries

For basic definitions and properties about copulas, we refer to [10,14]. Here we recall that a (bivariate) *copula* is a function $C: [0, 1]^2 \longrightarrow [0, 1]$ which satisfies: (i) the boundary conditions $C(t, 0) = C(0, t) = 0$ and $C(t, 1) = C(1, t) = t$ for all t in $[0, 1]$, and (ii) the 2-increasing property, i.e. $V_C([u_1, u_2] \times [v_1, v_2]) := C(u_2, v_2) - C(u_2, v_1) - C(u_1, v_2) + C(u_1, v_1) \ge 0$ for all u_1, u_2, v_1, v_2 in $[0, 1]$ such that $u_1 \le u_2$ and $v_1 \le v_2$. $V_C(R)$ is usually called as the C*-volume* of the rectangle R.

Let $\mathcal{B}([0, 1])$ and $\mathcal{B}([0, 1]^2)$ denote the respective Borel σ-algebras in $[0, 1]$ and $[0, 1]^2$, and λ (respectively, λ_2) denotes the Lebesgue measure on $[0, 1]$ (respectively, $[0, 1]^2$). A measure μ on $\mathcal{B}([0, 1]^2)$ is *doubly stochastic* if $\mu(B \times [0, 1]) = \mu([0, 1] \times B) = \lambda(B)$ for every $B \in \mathcal{B}([0, 1])$. Each copula C induces a doubly stochastic measure μ_C by setting $\mu_C(R) := V_C(R)$ for every rectangle $R \subseteq [0, 1]^2$ and extending μ_C to $\mathcal{B}([0, 1]^2)$.

The *support* of a copula C is the complement of the union of all open subsets of $[0, 1]^2$ with μ_C-measure zero — we note that this definition can be extended to any probability distribution of any dimension. When we refer to the "mass" of a measure μ on a set S, we mean the value of μ for that set, i.e. $\mu(S)$. In particular, we say that the copula measure μ_C is concentrated on S, whenever $\mu_C(S) = 1$.

We also need to recall some concepts in measure theory [11]. A measure on the real line is *discrete* if it is concentrated on a set which is at most countable. If μ_1 and μ_2 are two finite measures on a σ-algebra $\mathcal{A}$, we say that μ_2 is *absolutely continuous* with respect to μ_1 if $\mu_2(A) = 0$ for any $A \in \mathcal{A}$ such that $\mu_1(A) = 0$. μ_1 and μ_2 are said to be *singular* if there are two sets $A, B \in \mathcal{A}$ such that $A \cap B = \emptyset$, $A \cup B = \mathcal{A}$ and $\mu_1(B) = \mu_2(A) = 0$. In particular, a measure defined on $\mathbb{R}^n$ is singular if it is singular with respect to the Lebesgue measure on this space. A copula C is said to be *singular* (respectively, *absolutely continuous*) if the measure μ_C induced by C is singular (respectively, absolutely continuous) with respect to λ_2 (see [5,6] for a detailed study).

In order to deal with the copula measure, we consider the following disintegration theorem (see e.g. [1]). First, let $P(\Omega)$ denote the collection of Borel probability measures on a metric space (Ω, d). We consider Ω as a Radon space, i.e. a topological space such that every Borel probability measure on Ω is inner regular, e.g. separable metric spaces on which every probability measure is a Radon measure.

Theorem 2 *(Disintegration theorem). Let* Ω_1 *and* Ω_2 *be two Radon spaces. Let* $\mu \in P(\Omega_1)$. *Let* $\pi: \Omega_1 \longrightarrow \Omega_2$ *be a Borel-measurable function, and let* $\gamma \in P(\Omega_2)$ *be the pushforward measure* $\gamma = \mu \circ \pi^{-1}$. *Then there exists a* γ*-almost everywhere uniquely determined Borel family of probability measures* $\{\mu_x\}_{x \in \Omega_2} \subseteq P(\Omega_1)$, *which provides a "disintegration" of* μ *into* $\{\mu_x\}_{x \in \Omega_2}$, *namely*

- *μ_x lives on the fiber $\pi^{-1}(x)$, i.e. for γ-almost all $x \in \Omega_2$, $\mu_x\left(\Omega_1 \backslash \pi^{-1}(x)\right) = 0$ and, hence, $\mu_x(E) = \mu_x(E \cap \pi^{-1}(x))$ for any $E \subseteq \Omega_1$;*
- *for every Borel-measurable function $f: \Omega_1 \to [0, \infty]$,*

$$\int_{\Omega_1} f(y)\,\mathrm{d}\mu(y) = \int_{\Omega_2} \left(\int_{\pi^{-1}(x)} f(y)\,\mathrm{d}\mu_x(y) \right) \mathrm{d}\gamma(x). \tag{2}$$

In particular, for any $E \subseteq \Omega_1$, taking f to be the indicator function of E, it holds

$$\mu(E) = \int_{\Omega_2} \mu_x(E)\,\mathrm{d}\gamma(x).$$

Now, we consider copulas of the form (1), where f is concave and non-decreasing (or, convex and non-increasing). We will use Theorem 2 to express the related copula measure μ_{C_f}. To this end, let $\pi\colon [0,1]^2 \longrightarrow [0,1]$ be the canonical projection with respect to the second variable and $\gamma = \lambda$, the Lebesgue measure. Set $\mu_{C_f,v} = \mu_v$. Then, for every Borel set $E \subseteq [0,1]^2$,

$$\mu_{C_f}(E) = \int_0^1 \mu_v(E_v)\,\mathrm{d}\gamma(v), \tag{3}$$

where $E_v = E \cap \pi^{-1}(v)$.

3. The probability mass distribution of invariant copulas under truncation

We consider copulas of the form (1) whose measure is expressed as in (3). In order to calculate the distribution function of μ_v (related to the Markov kernel representation of the copulas, see [17]), we need to recall some aspects about the generator of a copula of type (1).

We recall that, if f is a real-valued concave (respectively, convex) function, then f admits left and right derivatives — denoted by f'_- and f'_+, respectively — and these are monotonically non-increasing (respectively, non-decreasing) – see, e.g., [18]. In particular, if f generates a copula of type (1), then f is differentiable in $\{x \geq 0\colon x \notin S\}$, where S is a set of (at most) countably many points. In particular, for every $x \geq 0$ such that $x \notin S$ it holds that $f'_-(x) = f'_+(x)$, and

$$f'_-(x) = \lim_{\substack{y \to x \\ y < x}} f'_+(y).$$

With a slight abuse of notation, we set $f'_-(0) = 0$.

Note that, if f generates a copula of type (1), then $g(x) = f(\alpha x)$, with $\alpha > 0$, determines the same copula via (1). For this reason, without loss of generality, we may consider that either f is bijective or $f(1) = 1$ and $f(x) < 1$ for $x < 1$. If f is not bijective, f^{-1} represents the inverse of the restriction of f to $[0,1]$.

Now, we present some results about the measure of C_f when f is concave and non-decreasing. Notice that, if the generating function f is surjective, monotone convex and non-increasing, then the related copula C_f obtained from (1) can be represented as $C_f(u,v) = u - C_g(u, 1-v)$ where $g = 1 - f$ is concave and non-decreasing (see [9]). Thus, the results can be translated from one case to the other one with suitable modifications.

Theorem 3. *Let C_f be the copula of type (1) generated by $f\colon [0,+\infty] \longrightarrow [0,1]$ being surjective, concave and non-decreasing. Suppose that μ_{C_f} can be written as in (3). Then, for every $v \notin \{0,1\}$, the distribution function of the measure μ_v is given by*

$$F_v(u) = \begin{cases} 0, & \text{if } u \leq 0, \\ f'_-\left(\dfrac{f^{-1}(v)}{u}\right) \dfrac{1}{f'_-\left(f^{-1}(v)\right)}, & \text{if } 0 < u < 1, \\ 1, & \text{if } 1 \leq u. \end{cases} \tag{4}$$

Proof. Let v be in $]0, 1[$. Since the function f is monotone and concave, then f'_- is non-increasing and positive, and hence the function F_v in (4) is monotonically non-decreasing. Moreover, since the map $u \mapsto \frac{f^{-1}(v)}{u}$ is non-increasing and f'_- is left-continuous, the function $F_v(u)$ is right-continuous.

Let u be fixed in $[0, 1]$. Since f is concave, the function

$$f'_-\left(\frac{f^{-1}(v)}{u}\right)\frac{1}{f'_-\left(f^{-1}(v)\right)} \tag{5}$$

is continuous except, maybe, on a countable set of points and has 1 as upper bound; thus the function (5) is measurable and it is possible to calculate its integral. When $u = 0$ we have

$$\int_0^v F_s(0)\,\mathrm{d}s = 0;$$

otherwise, we define

$$D(u, v) := \int_0^v F_s(u)\,\mathrm{d}s = \int_0^v f'_-\left(\frac{f^{-1}(s)}{u}\right)\frac{1}{f'_-\left(f^{-1}(s)\right)}\,\mathrm{d}s.$$

Since $D(u, v)$ and $C_f(u, v)$ are two absolutely continuous functions and have the same λ-almost everywhere derivative with respect to the second variable, we have $D(u, v) = C_f(u, v)$ for all $(u, v) \in [0, 1]^2$. Thus, the measure induced by the distribution functions F_v coincides with μ_{C_f}. It follows that the measures μ_v given by the disintegration procedure of Theorem 2 are those that have the distribution functions given in (4). □

Remark 4. Observe that, under the same conditions of Theorem 3, the distribution function $F_v(u)$ can be also written as

$$F_v(u) = \begin{cases} 0, & \text{if } v \le 0, \\ f'_+\left(\frac{f^{-1}(v)}{u}\right)\frac{1}{f'_+\left(f^{-1}(v)\right)}, & \text{if } 0 < v < 1, \\ 1, & \text{if } 1 \le v. \end{cases} \tag{6}$$

In the sequel, we use Theorem 3, taking into account that its results can be equivalently formulated in terms of (6).

Remark 5. Notice that, if the function f from Theorem 3 is not bijective, then the support of C_f cannot be $[0, 1]^2$. In fact, in such a case, if we take $(u_0, v_0) \in (0, 1)^2$ such that $u_0 < f^{-1}(v_0)$, then there exists an open ball $B_r(u_0, v_0)$ of radius r and center (u_0, v_0) satisfying $u < f^{-1}(v)$ for all $(u, v) \in B_r(u_0, v_0)$. Thus, $\mu_{C_f}(R) = 0$ for a suitable non-empty rectangle $R \subseteq B_r(u_0, v_0)$ such that $(u_0, v_0) \in R$. Hence, μ_{C_f} cannot be of full support.

Now, by using Theorem 3, we provide several examples of copulas of type (1) by considering a function f with different properties.

Example 6. Consider the function

$$f(x) = \begin{cases} x^\alpha, & \text{if } 0 \le x \le 1, \\ 1, & \text{if } 1 < x, \end{cases} \tag{7}$$

where $0 < \alpha < 1$. Since f is concave and non-decreasing it generates a copula of type (1). The related measure μ_v from (3) has distribution function given by

$$F_v(u) = \begin{cases} 0, & \text{if } u < v^{1/\alpha}, \\ u^{1-\alpha}, & \text{if } v^{1/\alpha} \le u \le 1, \\ 1, & \text{if } 1 < u. \end{cases}$$

The related copula C_f is given by $C_f(u, v) = \min(u, vu^{1-\alpha})$ for every $(u, v) \in [0, 1]^2$. Thus, the support of the copula C_f is the set $\{(u, v) \in [0, 1]^2 : v \le u^\alpha\}$ and

$$\mu_{C_f}\left(\left\{(u,v)\in[0,1]^2 : v=u^\alpha\right\}\right)=\alpha.$$

These copulas belong to the Marshall-Olkin family as considered in [7].

In the previous example the generator f is not bijective and is not differentiable at the point $x=1$. In the following example, f is still not a bijective function, but it is differentiable at the point $x=1$.

Example 7. We consider the function

$$f(x)=\begin{cases}\sqrt{2x-x^2}, & \text{if } 0\le x\le 1,\\ 1 & \text{if } 1<x,\end{cases} \tag{8}$$

that generates a copula of type (1). The distribution function of the related measure μ_v from (3) is given by

$$F_v(u)=\begin{cases}0, & \text{if } u<1-\sqrt{1-v^2},\\ \dfrac{v\left(u+\sqrt{1-v^2}-1\right)}{\sqrt{(1-v^2)\left(-2+2u+v^2+2(1-u)\sqrt{1-v^2}\right)}}, & \text{if } 1-\sqrt{1-v^2}\le u\le 1,\\ 1 & \text{if } 1<u.\end{cases}$$

Thus, the support of the copula C_f is the set given by $\left\{(u,v)\in[0,1]^2 : v\le\sqrt{2u-u^2}\right\}$ and

$$\mu_{C_f}\left(\left\{(u,v)\in[0,1]^2 : v=\sqrt{2u-u^2}\right\}\right)=0.$$

A sample from this copula is visualized in Fig. 1.

Finally, in the next example, we consider a generating function f that is bijective.

Example 8. Consider the function

$$f(x)=\frac{x}{x+1},\quad \text{for all } x\in[0,+\infty]. \tag{9}$$

The distribution function of the measure μ_v is given by

$$F_v(u)=\begin{cases}0, & \text{if } u<0,\\ \dfrac{u^2}{(u+v-uv)^2}, & \text{if } 0\le u\le 1,\\ 1, & \text{if } 1<u.\end{cases}$$

The related copula C_f is given by $C_f(u,v)=uv/(u+v-uv)$ for every $(u,v)\in[0,1]^2$, which corresponds to the Gumbel's bivariate logistic distribution. Hence, the support of the copula C_f is $[0,1]^2$.

Now, we define a (univariate) distribution function which will help us to study the distribution of the mass of the copula C_f.

Let f be a function satisfying the hypotheses of Theorem 3, and let C_f be the copula given by (1). For $r\in[0,1]$, we define the distribution function

$$R_f(r):=\mu_{C_f}\left(E_{f,r}\right), \tag{10}$$

where $E_{f,r}:=\left\{(u,v)\in[0,1]^2 : v\le f(f^{-1}(r)u)\right\}$. The following result holds.

Theorem 9. *The distribution function $R_f(r)$ defined by (10) is given by*

$$R_f(r)=r-f'_-\left(f^{-1}(r)\right)f^{-1}(r) \tag{11}$$

for every $r\in[0,1]$.

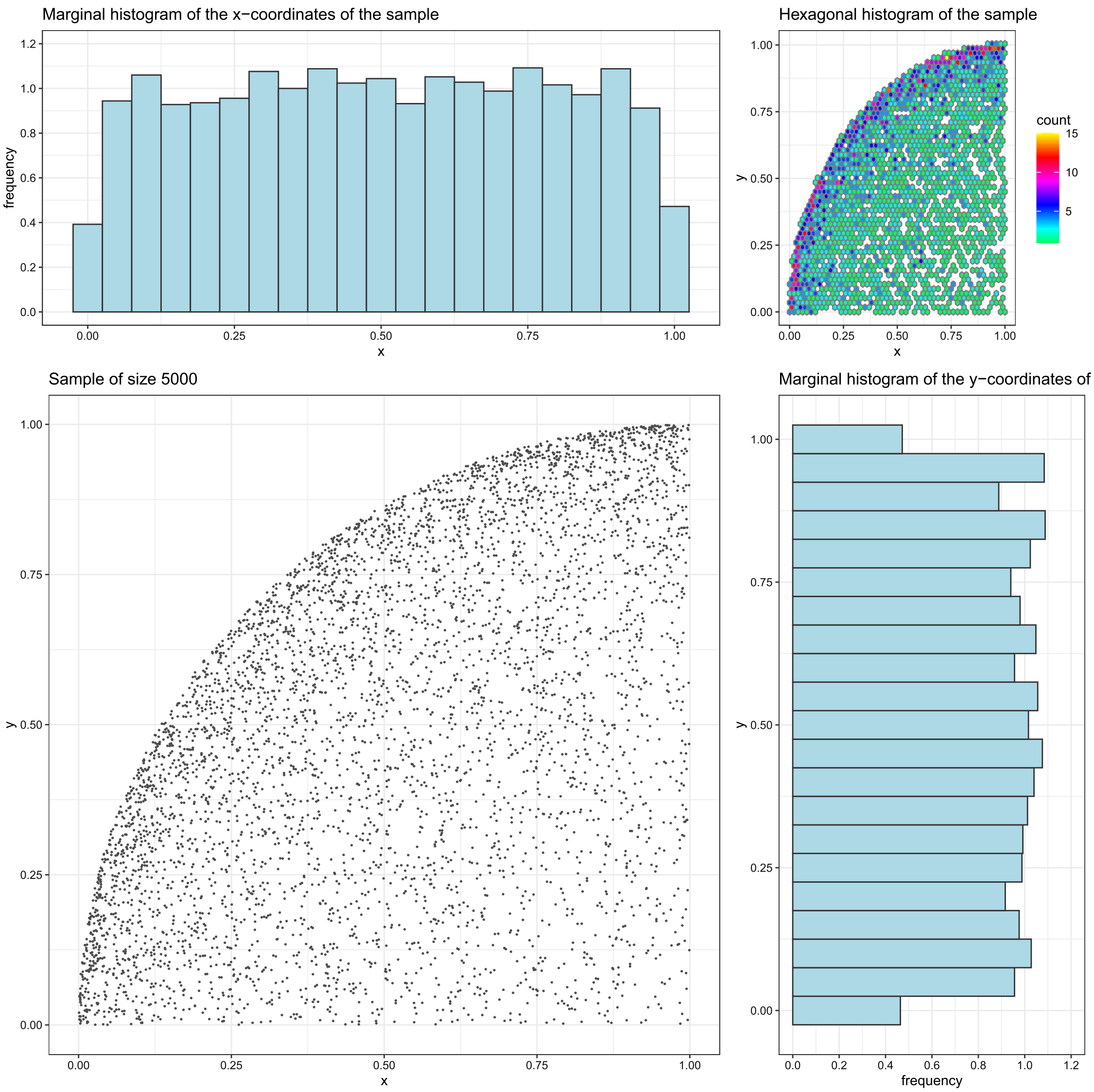

Fig. 1. Sample of 5000 points from the copula in Example 7.

Proof. To prove the result, we apply the disintegration given by Theorem 2 to the measure μ_{C_f}. Thus, by disintegrating with respect to the second variable v and considering that $E_{f,r} \subseteq [0,1] \times [0,r]$, we obtain the following chain of equalities:

$$
\begin{aligned}
R_f(r) &= \mu_{C_f}\left(\left\{(u,v) \in [0,1]^2 : v \le f(f^{-1}(r)u)\right\}\right) \\
&= \int_0^1 \mu_v\left((E_{f,r})_v\right) \mathrm{d}v \\
&= \int_0^r \left(1 - \frac{f'_-\left(f^{-1}(r)\right)}{f'\left(f^{-1}(v)\right)}\right) \mathrm{d}v
\end{aligned}
$$

$$= r - f'_-\left(f^{-1}(r)\right)\int_0^r \frac{1}{f'\left(f^{-1}(v)\right)}\,\mathrm{d}v$$
$$= r - f'_-\left(f^{-1}(r)\right) f^{-1}(r),$$

where the third equality is obtained by using (4). This completes the proof. □

Remark 10. Suppose that (U, V) is a random pair distributed according to a copula C_f of type (1). According to [9, Proposition 4.3], R_f is the distribution function of the random variable $C_f(U, V)/U$. See also [3].

Now we compute the distribution function R_f related to previous examples.

Example 11. Consider the copula C_f generated by a function f that satisfies the assumptions of Theorem 3.

- If f is given by (7), then
$$R_f(r) = \begin{cases} r(1-\alpha), & \text{if } 0 \le r < 1, \\ 1, & \text{if } r = 1. \end{cases}$$
- If f is given by (8), then
$$R_f(r) = \frac{1-\sqrt{1-r^2}}{r}, \quad \text{for all } r \in]0,1].$$
Note that
$$\lim_{r\to 0^+} R_f(r) = \lim_{r\to 0^+} \frac{1-\sqrt{1-r^2}}{r} = 0.$$
- If f is given by (9), then $R_f(r) = r^2$ for every $r \in [0, 1]$.

In view of the Lebesgue decomposition theorem (see, e.g., [18]) R_f can be decomposed into the sum of an absolutely continuous function, a singular function (i.e. a continuous function of bounded variation whose classical derivative vanishes almost everywhere) and a jump function (i.e. a right continuous non-decreasing function with countable many jumps). Moreover, the measure induced by R_f can be also decomposed as the sum of a measure that is absolutely continuous with respect to the Lebesgue measure λ, a measure that is singular with respect to λ and has no point masses, and a discrete measure (that has only point masses). Such aspects can be recovered from the analogous properties of the function f'_-.

Theorem 12. *The distribution function R_f given by (11) corresponds to a discrete, singular or absolutely continuous measure if, and only if, the function f'_- is, respectively, a jump, a singular or an absolutely continuous function.*

Proof. First, consider that f'_- is a monotone non-increasing function on $]0, +\infty[$ that is of bounded variation. As such, in view of the Lebesgue decomposition (see, e.g., [18]) it can be decomposed as the sum of three functions $(f'_-)_d$, $(f'_-)_s$ and $(f'_-)_{ac}$ that are, respectively, a jump, a singular and an absolutely continuous function. Now, we distinguish three cases.

- Suppose that f'_- has a jump of size $\alpha_s > 0$ in the point s. If $\frac{f^{-1}(v)}{s} = u < 1$, then it follows from (4) that
$$\mu_v\left(\left\{\frac{f^{-1}(v)}{s}\right\}\right) = \frac{\alpha_s}{f'_-(f^{-1}(v))}.$$
Moreover, it holds

$$\begin{aligned}\mu_{R_f}(f(s)) &= \mu_{C_f}\left(\{(u,v)\in[0,1]^2\colon v=f(su)\}\right)\\ &= \mu_{C_f}\left(\{(u,v)\in[0,1]^2\colon \tfrac{f^{-1}(v)}{s}=u\}\right)\\ &= \alpha_s\int_0^{f(s)}\frac{1}{f'_-(f^{-1}(v))}dv>0.\end{aligned}$$

Therefore, μ_{R_f} has a point mass in $f(s)$.
Conversely, if $\mu_{R_f}(f(s))>0$, then

$$\mu_{R_f}(f(s))=\alpha_s\int_0^{f(s)}\frac{1}{f'_-(f^{-1}(v))}dv>0,$$

where α_s is the size of the jump of f'_- in the point s. thus, $\alpha_s>0$. Summarizing, each discontinuity point (with finite jump) of f'_- corresponds to a discontinuity point in R_f. Thus, the discrete components of $\mu_{-f'_-}$ and μ_{R_f} are in one-to-one correspondence.

- Suppose that f'_- is singular in $S\subseteq[a,b]$ with $a>0$, where $\mu_{-f'_-}(S)>0$ and f'_- has no jumps in S. If $v<f(a)$ and $s\in S$, then $\frac{f^{-1}(v)}{s}<1$. Thus

$$\mu_v\left(\left\{\frac{f^{-1}(v)}{s}\colon s\in S\right\}\right)>0$$

Set $S_v=\left\{\frac{f^{-1}(v)}{s}\colon s\in S\right\}$ and $S_f=\bigcup\limits_{v<f(a)}S_v\times\{v\}$. It follows that $\mu_{C_f}(S_f)>0$. Moreover,

$$\begin{aligned}\mu_{R_f}(f(S)) &= \mu_{C_f}\left(\{(u,v)\in[0,1]^2\colon v=f(su),\ s\in S\}\right)\\ &= \mu_{C_f}\left(\{(u,v)\in[0,1]^2\colon u\in S_v\}\right)\\ &= \mu_{C_f}(S_f)>0.\end{aligned}$$

Since f is concave, it is absolutely continuous and, hence, it maps sets of λ-measure zero into sets of λ-measure zero. It follows that $\lambda(f(S))=0$. Thus, R_f has a singular component on $f(S)$. Analogously, we can prove that, if there exists a set S of λ-measure equal to zero such that $\mu_{-f'_-}(S)=\mu_{-f'_-}(]0,+\infty[)$, then $\mu_{R_f}(f(S))=\mu_{C_f}(S_f)=1$. It follows that, if f'_- is singular, then R_f is singular (and vice versa).
- Analogously, if f'_- has a non-vanishing absolutely continuous component, then the absolutely continuous component of R_f is not vanishing too. Moreover, if f'_- is absolutely continuous, then R_f is absolutely continuous.

Finally, notice that, if μ_{R_f} is discrete, then f'_- can have neither a non-zero absolutely continuous component nor a non-zero singular component. The same aspect occurs in the singular and in the absolutely continuous case. □

Interestingly, the properties of the measure induced by R_f allow to characterize some measure-theoretic aspects of the copula measure μ_{C_f}. For the sake of simplicity, given a function f as in Theorem 3, we denote by f_x the mapping $u\mapsto f(xu)$ for a fixed $x\in[0,1]$. Moreover, if f is differentiable, then the function $-f'_-$ is a continuous and non-decreasing function and its associated measure is denoted by $\mu_{-f'_-}$.

Theorem 13. *Let f be a function satisfying the hypotheses of Theorem 3, and let C_f be the copula given by (1). Let R_f be given by (11).*

(a) *The measure μ_{C_f} is singular if, and only if, R_f does not admit a non-zero absolutely continuous component. Moreover, if R_f is discrete, then the mass of μ_{C_f} is concentrated on the graphs of the functions f_x, where x is a jump point for f'_-.*

(b) *The measure μ_{C_f} is absolutely continuous if, and only if, R_f is absolutely continuous. Moreover, the mass of μ_{C_f} is concentrated on the graphs of the functions f_x, where x belongs to a full measure set of $\mu_{f'_-}$.*

Proof. We prove part (a) since part (b) can be done analogously.
Since f'_- is monotone, it is almost everywhere differentiable. Let A be the set of points in which f admits a non-zero derivative. Suppose that $\lambda(A) > 0$, which is equivalent to the fact that R_f has a non-zero absolutely continuous component in view of Theorem 12. With a slight abuse of notation, we denote by $f''(a)$ the derivative of f'_- at each point $a \in A$.

For a fixed $v \in]0, 1[$, the distribution function F_v is differentiable in $\frac{f^{-1}(v)}{a} = u < 1$ and it holds

$$F'_v(u) = -\frac{f''(a)a^2}{f^{-1}(v) f'_-(f^{-1}(v))} > 0$$

By similar steps as in the proof of Theorem 12 it follows that

$$\mu_{C_f}\left(\{(u, v) \in [0, 1]^2 \colon v = f(au),\, a \in A\}\right) > 0$$

Thus, C_f has an absolutely continuous component with density given by

$$(c_f)_{ac}(u, v) = \begin{cases} F'_v(u), & v = f(au),\, a \in A, \\ 0, & \text{otherwise.} \end{cases}$$

Thus, if R_f has a non-zero absolutely continuous component, then C_f is not singular. Equivalently, for C_f being singular it is necessary that the absolutely continuous component of R_f is zero. Conversely, if C_f is singular, then the absolutely continuous component of R_f is zero. In fact, on the contrary, f'_- has a non-zero absolutely continuous component and a similar way of thinking as above will lead to a contradiction.

To conclude part (a), notice that R_f is a jump function if, and only if, f'_- is a jump function (see Theorem 12). Moreover, the previous reasoning ensures that the mass of μ_{C_f} is concentrated on the graphs of the functions $u \mapsto f(au)$, where a is a jump point for f'_-. □

As a consequence of Theorems 12 and 13, we obtain the following result.

Corollary 14. *Let f be a function satisfying the hypotheses of Theorem 3, and let C_f be the copula given by (1). The following statements hold:*

(a) *μ_{C_f} is singular if, and only if, f'_- does not have an absolutely continuous component;*
(b) *μ_{C_f} is absolutely continuous if, and only if, f'_- is absolutely continuous.*

Note that the previous result is equivalent to Proposition 3.2 and Corollary 3.3 from [9], but these latter proofs exploit a one-to-one correspondence among truncation invariant copulas and Archimedean copulas.

Finally, while the copula given in Example 8 is absolutely continuous, below we present two examples of a singular copula.

Example 15. Consider the piecewise linear function $f \colon [0, +\infty] \longrightarrow [0, 1]$ given by

$$f(x) = \begin{cases} 2x, & \text{if } 0 \le x \le 1/4, \\ x + 1/4, & \text{if } 1/4 < x \le 1/2, \\ x/2 + 1/2, & \text{if } 1/2 < x \le 1, \\ 1, & \text{if } 1 < x. \end{cases} \tag{12}$$

Since f is concave and non-decreasing it generates a copula of type (1). In order to determine the measure μ_v from (3) we need to study different cases, which depend on the values of v.

1. If $v = 0$, then μ_v is the atomic measure that concentrates all the mass in $u = 0$.

2. If $0 < v \le 1/2$, then $f'_-\left(f^{-1}(v)\right) = f'_-\left(\frac{v}{2}\right) = 2$, and we need to distinguish four sub-cases, which depend on the values of u:
   (a) If $0 \le u < v/2$, then $\frac{v}{2u} \in]1, +\infty]$, and hence $f'_-\left(\frac{v}{2u}\right) = 0$.
   (b) If $v/2 \le u < v$, then $\frac{v}{2u} \in]1/2, 1]$, and hence $f'_-\left(\frac{v}{2u}\right) = \frac{1}{2}$.
   (c) If $v \le u < 2v$, then $\frac{v}{2u} \in]1/4, 1/2]$, and hence $f'_-\left(\frac{v}{2u}\right) = 1$.
   (d) If $2v \le u < 1$, then $\frac{v}{2u} \in]0, 1/4]$, and hence $f'_-\left(\frac{v}{2u}\right) = 2$.
   Thus, we obtain

$$F_v(u) = f'_-\left(\frac{f^{-1}(v)}{u}\right) \cdot \frac{1}{f'_-\left(f^{-1}(v)\right)} = f'\left(\frac{v}{2u}\right) \cdot \frac{1}{f'\left(\dfrac{v}{2}\right)}$$
$$= \begin{cases} 0, & \text{if } 0 \le u < v/2, \\ 1/4, & \text{if } v/2 \le u < v, \\ 1/2, & \text{if } v \le u < 2v, \\ 1, & \text{if } 2v \le u < 1, \end{cases}$$

   i.e. μ_v is the atomic measure that concentrates its mass in three points, namely

$$\mu_v(u) = \begin{cases} 1/4, & \text{if } u = v/2, \\ 1/4, & \text{if } u = v, \\ 1/2, & \text{if } u = 2v, \\ 0, & \text{otherwise.} \end{cases}$$

3. If $1/2 < v \le 3/4$, then μ_v is the atomic measure that concentrates its mass in two points, namely

$$\mu_v(u) = \begin{cases} 1/2, & \text{if } u = v - 1/4, \\ 1/2, & \text{if } u = 2v - 1/2, \\ 0, & \text{otherwise.} \end{cases}$$

4. If $3/4 < v \le 1$, then μ_v is the atomic measure that concentrates its mass in the point $u = 2v - 1$; so that $\mu_v(2v - 1) = 1$, while $\mu_v(u) = 0$, otherwise.

The function R_f associated with f via (11) is given by

$$R_f(r) = \begin{cases} 0, & \text{if } 0 \le r < 1/2, \\ 1/4, & \text{if } 1/2 \le r < 3/4, \\ 1/2, & \text{if } 3/4 \le r < 1, \\ 1, & \text{if } r = 1. \end{cases}$$

It follows that the measure μ_{C_f} is concentrated on the graphs of the following functions

$$f_{1/4}(u) = f(u/4), \qquad f_{1/2}(u) = f(u/2), \qquad f_1(u) = f(u)$$

defined for every $u \in [0, 1]$. See Fig. 2.

Example 16. Here, we revisit [9, Example 3.2]. Let $f \colon [0, +\infty] \longrightarrow [0, 1]$ be defined by

$$f(x) = \sum_{p,q=1}^{+\infty} 2^{-\pi(p,q)} \max\left(0, 1 - \frac{q}{p}x\right),$$

where $\pi : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}$ is the bijection given by $\pi(p, q) = \frac{1}{2}(p + q - 2)(p + q - 1) + q$. Then f is singular. The related copula C_f is singular and has mass concentrated on the graphs of the functions $f_{p/q}$ for any natural p and q.

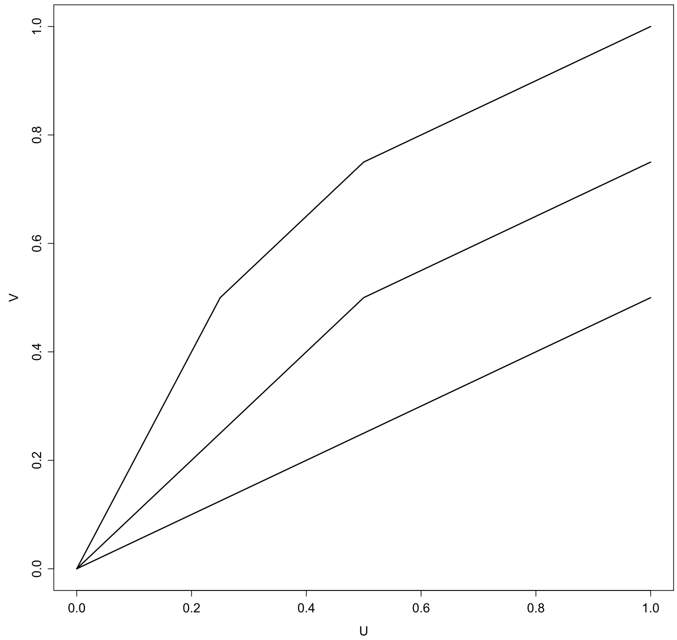

Fig. 2. Support of the copula of Example 15.

4. Conclusions

We have considered a class of copulas that are invariant under univariate truncation and that can be expressed in the form (1). For such copulas, we have determined how the singular and the absolutely component of the copula measure can be expressed in terms of the properties of the generating functions f.

As known from [8] any copula that is invariant under univariate truncation can be represented as gluing ordinal sum of copulas of type (1). Given the stochastic representation of gluing construction [15] and its interpretation in terms of patchwork methods [4], the properties of the measure presented here can be easily extended to the copulas belonging to this more general class.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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