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Ladder operators and a differential equation for varying generalized Freud–type orthogonal polynomials

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Abstract

In this paper we introduce varying generalized Freud–type polynomials which are orthogonal with respect to a varying discrete Freud–type inner product. Our main goal is to give ladder operators for this family of polynomials as well as finding a second order differential–difference equation that these polynomials satisfy. To reach this objective it is necessary to consider the standard Freud orthogonal polynomials and in the meanwhile we find new difference relations for the coefficients in the first order differential equations that this standard family satisfies.

Keywords: Orthogonal polynomials; Ladder operators; Freud weights.
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1 Introduction

Orthogonal polynomials with respect to a varying inner product have been considered in different frameworks, for instance, in general contexts related to different types of weights (see, among others, [4, 6, 14, 16, 25, 27, 28, 32, 35]) or in Sobolev orthogonality (see for instance [1, 3, 17, 30]). In this paper we consider a varying Krall–type inner product, that is, an inner product involving a measure with an absolutely continuous part that depends on a parameter t and a varying discrete part. Concretely,

$$(f,g)_n = \int_{-\infty}^{+\infty} f(x)g(x)w(x;t)dx + \frac{M_n}{2}\big(f(c)g(c)+f(-c)g(-c)\big), \qquad c\in\mathbb{R}, \tag{1}$$

where we assume that

$$w(x;t) = |x|^\gamma \exp(-v(x;t)), \quad \gamma \geq 1, \tag{2}$$

with $v(x;t)$ being an even and continuously differentiable function on $\mathbb{R}$ and $t \in \mathbb{R}$. In addition, we suppose that $\{M_n\}_{n=0}^\infty$ is a sequence of nonnegative real numbers. Thus, for every n, we have a square tableau of monic orthogonal polynomials with respect to (1), $\{Q_k^{(M_n,c)}(x;t)\}_{k=0}^\infty$, but we only deal with the diagonal of this tableau, $\{Q_n^{(M_n,c)}(x;t)\}_{n=0}^\infty$. To simplify the notation, we will denote them by $\{Q_n(x;t)\}_{n=0}^\infty$ and name them as *varying generalized Freud–type orthogonal polynomials*.

In the special case $c = 0$ we will use the notation $\{Q_n^{(0)}(x;t)\}_{n=0}^{+\infty}$ for the sequence of monic orthogonal polynomials with respect to

$$(f,g)_n = \int_{-\infty}^{+\infty} f(x)g(x)w(x;t)dx + M_n f(0)g(0). \tag{3}$$

Generalized Freud orthogonal polynomials have been extensively studied in the literature including books and articles (see, among others, [2, 12, 13, 14, 15, 16, 18, 19, 20, 23, 24, 26, 29, 33, 34, 36, 37]) and the references therein). In this paper we denote the sequence of monic generalized Freud orthogonal polynomials by $\{P_n(x;t)\}_{n=0}^\infty$. This family is orthogonal with respect to the inner product involving the weight function (2), i.e.

$$(f,g)_t = \int_{-\infty}^{+\infty} f(x)g(x)w(x;t)dx = \int_{-\infty}^{+\infty} f(x)g(x)|x|^\gamma \exp(-v(x;t))dx, \quad \gamma \geq 1. \tag{4}$$

Thus, the inner product (1) can be expressed as

$$(f,g)_n = (f,g)_t + \frac{M_n}{2}\big(f(c)g(c) + f(-c)g(-c)\big). \tag{5}$$

We note that the families of orthogonal polynomials $Q_n(x;t)$ and $P_n(x;t)$ are symmetric, that is,

$$Q_n(x;t) = (-1)^n Q_n(-x;t), \qquad P_n(x;t) = (-1)^n P_n(-x;t), \qquad n \geq 0. \tag{6}$$

Generalized Freud orthogonal polynomials play an essential role along the work. Thus, we are going to give a basic background about them. We denote the square of the norm of these polynomials by h_n, so:

$$h_n := h_n(t) = (P_n, P_n)_t = \int_{-\infty}^{+\infty} P_n^2(x;t)w(x;t)dx, \quad n \in \mathbb{N} \cup \{0\}.$$

Since the inner product (4) is standard, i.e. the property $(xf,g)_t = (f,xg)_t$ holds, and the corresponding orthogonal polynomials $P_n(x;t)$ are symmetric, then this family of polynomials satisfies a three–term recurrence relation such as

$$xP_n(x;t) = P_{n+1}(x;t) + \beta_n(t)P_{n-1}(x;t), \quad n \in \mathbb{N}, \tag{7}$$

where

$$\beta_n := \beta_n(t) = \frac{1}{h_{n-1}} \int_{-\infty}^{+\infty} x P_n(x;t) P_{n-1}(x;t) w(x;t) dx = \frac{h_n}{h_{n-1}}, \tag{8}$$

with the initial conditions $P_0(x;t) = 1$, $P_1(x;t) = x$.

The main goal of this paper is to obtain ladder operators (raising and lowering operators) for the varying Krall–type orthogonal polynomials with respect to (1).

Ladder operators are relevant in the theory of orthogonal polynomials. On the one hand, they have a natural connection with the coefficients of the recurrence relation and from the point of view of physics they are related to the harmonic oscillator (see [36]). On the other hand, they are a very useful tool to construct differential equations whose solutions are the corresponding orthogonal polynomials. For these reasons, among others, the literature about ladder operators in different frameworks is very wide, we cite some of them [7, 8, 9, 10, 11, 20, 21, 22, 36].

Ladder operators for the orthogonal polynomials $P_n(x;t)$ are known (see, for example, [8, 20]). Next result provides us with the the lowering operator. As usual, we denote the derivative with respect to the variable x by $\,'$.

Theorem 1.1 *([13, Corollary 1]) Let $\{P_n(x;t)\}_{n=0}^{\infty}$ be the sequence of orthogonal polynomials with respect to (4). Then, these polynomials satisfy the following differential–difference equation:*

$$x P_n'(x;t) = -B_n(x;t) P_n(x;t) + A_n(x;t) P_{n-1}(x;t), \qquad n \geq 1, \tag{9}$$

where

$$A_n(x;t) = \frac{x}{h_{n-1}} \int_{-\infty}^{+\infty} P_n^2(y;t) \frac{v'(x;t) - v'(y;t)}{x - y} w(y;t) dy, \tag{10}$$

$$B_n(x;t) = \frac{x}{h_{n-1}} \int_{-\infty}^{+\infty} P_n(y;t) P_{n-1}(y;t) \frac{v'(x;t) - v'(y;t)}{x - y} w(y;t) dy + \frac{\gamma}{2} \left(1 - (-1)^n\right). \tag{11}$$

Obviously, the lowering operator is given by relation (9) and can be rewritten as

$$\left(B_n(x;t) + x \frac{d}{dx} \right) P_n(x;t) = A_n(x;t) P_{n-1}(x;t).$$

The raising operator can be derived in a straightforward way using the lowering operator and the three–term recurrence relation (7),

$$\left(\frac{x A_{n-1}(x;t)}{\beta_{n-1}} - B_{n-1}(x;t) + x \frac{d}{dx} \right) P_{n-1}(x;t) = -\frac{A_{n-1}(x;t)}{\beta_{n-1}} P_n(x;t).$$

The coefficients $A_n(x;t)$ and $B_n(x;t)$ appearing in the ladder operators satisfy certain difference equations (see for instance [20, 26, 36]). These equations are usually known as *compatibility* *conditions* and they are useful to compute the coefficients in the corresponding three–term recurrence relation when we are working with a standard inner product. In the case of the Freud weight function $w(x) = \exp(-v(x))$ the difference equations for $A_n(x;t)$ and $B_n(x;t)$ were obtained in [8] and [22]. Thus, another objective of this paper is to deduce some difference equations for the coefficients $A_n(x;t)$ and $B_n(x;t)$ defined in (10) and (11), respectively.

The structure of the paper is the following: in Section 2 we obtain new difference equations for the coefficients in the first order differential–difference equation (9) satisfied by the polynomials $P_n(x;t)$; in section 3 we use a standard technique to obtain the algebraic relations between the polynomials $Q_n(x;t)$, their first derivatives and the standard polynomials $P_n(x;t)$. These relations are essential to tackle our main objective in Section 4 which is to give the ladder operators and a second order differential equation for the varying Freud–type orthogonal polynomials $Q_n(x;t)$. Along the paper we have illustrated the results with examples.

2 Difference equations for the coefficients $A_n(x;t)$ and $B_n(x;t)$

In case of the weight function $w(x;t) = |x|^\gamma \exp(-v(x;t))$ with $\gamma \geq 1$, one of these compatibility equations was already obtained in [13].

Theorem 2.1 *(**[13, Lemma 2]**) The functions $A_n(x;t)$ and $B_n(x;t)$ defined in (10) and (11), respectively, satisfy the following relation:*

$$B_{n+1}(x;t) + B_n(x;t) = \frac{xA_n(x;t)}{\beta_n} + \gamma - xv'(x;t). \tag{12}$$

Now, we will use the techniques given in [8] and [22] to obtain other compatibility equations for the coefficients $A_n(x;t)$ and $B_n(x;t)$ defined in (10) and (11), respectively. As far as we know, the difference equations that we have obtained in Theorems 2.2 and 2.3 are new.

Theorem 2.2 *The functions $A_n(x;t)$ and $B_n(x;t)$ defined in (10) and (11), respectively, satisfy the following relation:*

$$B_{n+1}(x;t) - B_n(x;t) = \frac{A_{n+1}(x;t)}{x} - \frac{\beta_n A_{n-1}(x;t)}{x\beta_{n-1}} - 1. \tag{13}$$

Proof. To establish the result we will use (7), (8), and Theorem 1.1. We have

$$
\begin{aligned}
&x\left(B_{n+1}(x;t)-B_n(x;t)\right)\\
&=\frac{x^2}{h_n}\int_{-\infty}^{+\infty}P_{n+1}(y;t)P_n(y;t)\frac{v'(x;t)-v'(y;t)}{x-y}w(y;t)dy\\
&\quad-\frac{x^2}{h_n}\int_{-\infty}^{+\infty}\beta_nP_n(y;t)P_{n-1}(y;t)\frac{v'(x;t)-v'(y;t)}{x-y}w(y;t)dy+x\gamma(-1)^n\\
&=\frac{x}{h_n}\left(\mathcal{A}+\mathcal{B}+\mathcal{C}+\mathcal{D}\right)+x\gamma(-1)^n, \qquad (14)
\end{aligned}
$$

where $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ are the following integrals:

$$
\begin{aligned}
\mathcal{A} &= \int_{-\infty}^{+\infty}(x-y)P_{n+1}(y;t)P_n(y;t)\frac{v'(x;t)-v'(y;t)}{x-y}w(y;t)dy,\\
\mathcal{B} &= \int_{-\infty}^{+\infty}yP_{n+1}(y;t)P_n(y;t)\frac{v'(x;t)-v'(y;t)}{x-y}w(y;t)dy,\\
\mathcal{C} &= -\int_{-\infty}^{+\infty}(x-y)\beta_nP_n(y;t)P_{n-1}(y;t)\frac{v'(x;t)-v'(y;t)}{x-y}w(y;t)dy,\\
\mathcal{D} &= -\int_{-\infty}^{+\infty}y\beta_nP_n(y;t)P_{n-1}(y;t)\frac{v'(x;t)-v'(y;t)}{x-y}w(y;t)dy.
\end{aligned}
$$

To calculate the value of $\mathcal{A}$, we use relation $v'(x;t)w(x;t)=\frac{\gamma}{x}w(x;t)-w'(x;t)$, which easily follows from (2). We have

$$
\begin{aligned}
\mathcal{A} &= \int_{-\infty}^{+\infty}P_{n+1}(y;t)P_n(y;t)(v'(x;t)-v'(y;t))w(y;t)dy\\
&= v'(x;t)\int_{-\infty}^{+\infty}P_{n+1}(y;t)P_n(y;t)w(y;t)dy-\int_{-\infty}^{+\infty}P_{n+1}(y;t)P_n(y;t)v'(y;t)w(y;t)dy\\
&= -\int_{-\infty}^{+\infty}P_{n+1}(y;t)P_n(y;t)\frac{\gamma}{y}w(y;t)dy+\int_{-\infty}^{+\infty}P_{n+1}(y;t)P_n(y;t)w'(y;t)dy\\
&= -\frac{\gamma}{2}\left(1+(-1)^n\right)h_n+\int_{-\infty}^{+\infty}P_{n+1}(y;t)P_n(y;t)w'(y;t)dy,\\
&= -\frac{\gamma}{2}\left(1+(-1)^n\right)h_n-(n+1)h_n=-h_n\left(n+1+\frac{\gamma}{2}\left(1+(-1)^n\right)\right).
\end{aligned}
$$

where we have used ([13, f. (19)]):

$$
\int_{-\infty}^{\infty}\frac{P_n(y;t)P_{n-1}(y;t)}{y}w(y;t)dy=\frac{1}{2}(1-(-1)^n)h_{n-1},\quad n\in\mathbb{N}.
$$

Making the change $n \to n-1$ in $\mathcal{A}$, we obtain $\mathcal{C}$ up to the multiplicative factor $-\beta_n$. Therefore, we have

$$\mathcal{C} = \beta_n h_{n-1}\left(n + \frac{\gamma}{2}\left(1-(-1)^n\right)\right) = h_n\left(n + \frac{\gamma}{2}\left(1-(-1)^n\right)\right).$$

This yields

$$\mathcal{A} + \mathcal{C} = -h_n\left(1+\gamma(-1)^n\right).$$

To calculate $\mathcal{B}$ and $\mathcal{D}$, we will use (7) and (10):

$$\begin{aligned}
\mathcal{B} &= \int_{-\infty}^{+\infty}\left(P_{n+1}^2(y;t) + \beta_n P_{n+1}(y;t)P_{n-1}(y;t)\right)\frac{v'(x;t)-v'(y;t)}{x-y}w(y;t)dy,\\
\mathcal{D} &= -\int_{-\infty}^{+\infty}\beta_n P_{n+1}(y;t)P_{n-1}(y;t) - \beta_n^2 P_{n-1}^2(y;t)\frac{v'(x;t)-v'(y;t)}{x-y}w(y;t)dy.
\end{aligned}$$

Then,

$$\mathcal{B} + \mathcal{D} = \int_{-\infty}^{+\infty}\left(P_{n+1}^2(y;t) - \beta_n^2 P_{n-1}^2(y;t)\right)\frac{v'(x;t)-v'(y;t)}{x-y}w(y;t)dy.$$

Finally, substituting in (14) we obtain

$$\begin{aligned}
&x\left(B_{n+1}(x;t) - B_n(x;t)\right)\\
&= \frac{x}{h_n}\left(-h_n\left(1+\gamma(-1)^n\right) + \int_{-\infty}^{+\infty}\left(P_{n+1}^2(y;t) - \beta_n^2 P_{n-1}^2(y;t)\right)\frac{v'(x;t)-v'(y;t)}{x-y}w(y;t)dy\right)\\
&\quad + x\gamma(-1)^n\\
&= A_{n+1}(x;t) - \frac{\beta_n A_{n-1}(x;t)}{\beta_{n-1}} - x,
\end{aligned}$$

which proves the result. □

Theorem 2.3 *The functions $A_n(x;t)$ and $B_n(x;t)$ defined in (10) and (11), respectively, satisfy the following relation:*

$$B_n(x;t)\left(B_n(x;t) + xv'(x;t) - \gamma\right) + x\sum_{i=0}^{n-1}\frac{A_i(x;t)}{\beta_i} = \frac{A_n(x;t)A_{n-1}(x;t)}{\beta_{n-1}}. \tag{15}$$

Proof. The equation (13) is equivalent to

$$x + x\left(B_{k+1}(x;t) - B_k(x;t)\right) = A_{k+1}(x;t) - \frac{\beta_k A_{k-1}(x;t)}{\beta_{k-1}}.$$

Multiplying the above expression by $\frac{A_k(x;t)}{\beta_k}$, we get

$$x\frac{A_k(x;t)}{\beta_k} + x\frac{A_k(x;t)}{\beta_k}\left(B_{k+1}(x;t) - B_k(x;t)\right) = \frac{A_{k+1}(x;t)A_k(x;t)}{\beta_k} - \frac{A_k(x;t)A_{k-1}(x;t)}{\beta_{k-1}}. \qquad (16)$$

We can simplify the left side on formula (16) using (12). Thus, we get

$$\begin{aligned} &x\frac{A_k(x;t)}{\beta_k} + x\frac{A_k(x;t)}{\beta_k}\left(B_{k+1}(x;t) - B_k(x;t)\right) \\ &\quad = x\frac{A_k(x;t)}{\beta_k} + \left(B_{k+1}(x;t) - B_k(x;t)\right)\big(B_{k+1}(x;t) + B_k(x;t) + xv'(x;t) - \gamma\big) \\ &\quad = x\frac{A_k(x;t)}{\beta_k} + \mathfrak{B}_{k+1}(x;t) - \mathfrak{B}_k(x;t), \end{aligned}$$

where $\mathfrak{B}_k(x;t) = B_k^2(x;t) + B_k(x;t)(xv'(x;t) - \gamma)$. The right side of (16) can be rewritten as

$$\mathfrak{A}_{k+1}(x;t) - \mathfrak{A}_k(x;t),$$

where $\mathfrak{A}_k(x;t) = \frac{A_k(x;t)A_{k-1}(x;t)}{\beta_{k-1}}$.

Assuming $\mathfrak{A}_0(x;t) = \mathfrak{B}_0(x;t) = 0$, it remains to sum from $k = 0$ to $n-1$ in the previous expressions to obtain the result. $\square$

Next, we illustrate how Theorem 2.1, Theorem 2.2 and Theorem 2.3 are useful to obtain information about the coefficient β_n in the three–term recurrence relation (7). In addition, ladder operators are given explicitly.

2.1 Example 1

We consider a very simple case when $v(x)$ does not depend on t. We take $v(x) = x^2$. Then, the polynomials $P_n(x)$ are orthogonal with respect to the weight $w(x) = |x|^\gamma \exp(-x^2)$ with $\gamma \geq 1$. They are called *generalized Hermite orthogonal polynomials*. Using (10) and (11), we get

$$A_n(x) = 2x\beta_n, \quad B_n(x) = \frac{\gamma}{2}(1 - (-1)^n).$$

Thus, equation (9) is

$$xP_n'(x) = -\frac{\gamma}{2}(1 - (-1)^n)P_n(x) + 2x\beta_n P_{n-1}(x). \qquad (17)$$

Using equations (12), (13), and (15) we get, after some computations,

$$\beta_n = \frac{2n + \gamma(1 - (-1)^n)}{4}.$$

Using (17), we obtain the corresponding ladder operators.

- Lowering operator:

$$\Psi_n = \left[\frac{\gamma}{2}\left(1-(-1)^n\right) + x\frac{d}{dx}\right],$$

so, $\Psi_n\left[P_n(x)\right] = 2x\beta_n P_{n-1}(x)$.

- Raising operator. Making the transformation $n \to n-1$ in (17) and using (7), we deduce

$$\begin{aligned} xP'_{n-1}(x) &= \frac{\gamma}{2}\left(1+(-1)^n\right)P_{n-1}(x) + 2x\beta_{n-1}P_{n-2}(x) \\ &= \frac{\gamma}{2}\left(1+(-1)^n\right)P_{n-1}(x) + 2x\beta_{n-1}\left(\frac{xP_{n-1}(x)-P_n(x)}{\beta_{n-1}}\right) \\ &= \left(\frac{\gamma}{2}\left(1+(-1)^n\right) + 2x^2\right)P_{n-1}(x) - 2xP_n(x). \end{aligned}$$

Then, the raising operator is

$$\widehat{\Psi}_n = \left[\frac{\gamma}{2}\left(1+(-1)^n\right) + 2x^2 - x\frac{d}{dx}\right],$$

so, $\widehat{\Psi}_n\left[P_{n-1}(x)\right] = 2xP_n(x)$.

2.2 Example 2

Now we consider $v(x;t)$ depending on t, concretely $v(x;t) = x^4 - tx^2$. Thus, $w(x;t) = |x|^\gamma \exp(-x^4 + tx^2)$ with $\gamma \geq 1$. We have the following expressions for the coefficients $A_n(x;t)$ and $B_n(x;t)$ (see [13, f. (56)]):

$$A_n(x;t) = 4x\beta_n\left(x^2 + \beta_{n+1} + \beta_n - \frac{t}{2}\right),$$

$$B_n(x;t) = 4x^2\beta_n + \frac{\gamma}{2}\left(1-(-1)^n\right).$$

Using (9), we get

$$\begin{aligned} xP'_n(x;t) = &-\left(4x^2\beta_n + \frac{\gamma}{2}\left(1-(-1)^n\right)\right)P_n(x;t) \\ &+ 4x\beta_n\left(x^2 + \beta_{n+1} + \beta_n - \frac{t}{2}\right)P_{n-1}(x;t). \end{aligned} \tag{18}$$

Therefore, the ladder operators are:

- Lowering operator:
$$\Psi_n = \left[4x^2\beta_n + \frac{\gamma}{2}\left(1-(-1)^n\right) + x\frac{d}{dx}\right],$$
so, $\Psi_n\left[P_n(x;t)\right] = 4x\beta_n\left(x^2+\beta_{n+1}+\beta_n-\frac{t}{2}\right)P_{n-1}(x;t)$.

- Raising operator: Making the change $n \to n-1$ in (18) and using (7), we get
$$\begin{aligned} xP'_{n-1}(x;t) &= -\left(4x^2\beta_{n-1}+\frac{\gamma}{2}\left(1+(-1)^n\right)\right)P_{n-1}(x;t) \\ &\quad +4x^2\beta_{n-1}\left(x^2+\beta_n+\beta_{n-1}-\frac{t}{2}\right)\left(\frac{xP_{n-1}(x;t)-P_n(x;t)}{\beta_{n-1}}\right). \end{aligned}$$
Thus, we obtain
$$\widehat{\Psi}_n = \left[4x^2\left(x^2+\beta_n+\beta_{n-1}-\frac{t}{2}\right) - 4x^2\beta_{n-1} - \frac{\gamma}{2}\left(1+(-1)^n\right) - x\frac{d}{dx}\right],$$
so, $\widehat{\Psi}_n\left[P_{n-1}(x;t)\right] = 4x\left(x^2+\beta_n+\beta_{n-1}-\frac{t}{2}\right)P_n(x;t)$.

We have the following nonlinear difference equation for β_n (see for example [13, f. (34)], [18, Th. 6.1] or [20] and the references therein),
$$\beta_{n+1}+\beta_n+\beta_{n-1} = \frac{t}{2} + \frac{2n+\gamma(1-(-1)^n)}{8\beta_n}, \tag{19}$$
which is called discrete Painlevé I ($\mathrm{P_I}(\mathrm{dP_I})$). We show how we can obtain this relation easily using Theorem 2.1, Theorem 2.2, and Theorem 2.3.

We make the change $n \to n-1$ in (13). Then, on one side we have
$$\begin{aligned} & \frac{A_n(x;t)}{x} - \frac{\beta_{n-1}A_{n-2}(x;t)}{\beta_{n-2}x} - 1 \\ = \;& 4\beta_n\left(x^2+\beta_{n+1}+\beta_n-\frac{t}{2}\right) - 4\beta_{n-1}\left(x^2+\beta_{n-1}+\beta_{n-2}-\frac{t}{2}\right) - 1 \\ = \;& 4\beta_n\left(x^2+\beta_{n+1}+\beta_n+\beta_{n-1}-\frac{t}{2}\right) - n \\ - \;& \left(4\beta_{n-1}\left(x^2+\beta_n+\beta_{n-1}+\beta_{n-2}-\frac{t}{2}\right) - (n-1)\right) \\ = \;& \mathbb{A}_n(x;t) - \mathbb{A}_{n-1}(x;t), \end{aligned}$$
where $\mathbb{A}_n(x;t) = 4\beta_n\left(x^2+\beta_{n+1}+\beta_n+\beta_{n-1}-\frac{t}{2}\right) - n$.

Thus, (13) becomes
$$\mathbb{A}_n(x;t) - \mathbb{A}_{n-1}(x;t) = B_n(x;t) - B_{n-1}(x;t).$$
Then, summing the above expression and taking into account the assumption $\mathbb{A}_0(x;t) = B_0(x;t) = 0$, we get the Painlevé equation $\mathrm{P_I}(\mathrm{dP_I})$ given by (19).

3 Relations for varying generalized Freud-type orthogonal polynomials

In this section we obtain some algebraic relations for the polynomials $Q_n(x;t)$ and their derivatives. In fact, these relationships play a relevant role in obtaining the corresponding ladder operators using a standard technique [5, 8, 9, 20]. Taking into account the relevance of the case $c = 0$, we particularize the results for this situation.

We use the very well–known kernel polynomials

$$K_n(x, y; t) = \sum_{k=0}^{n} \frac{P_k(x;t)P_k(y;t)}{h_k},$$

as well as the Christoffel–Darboux formula

$$K_n(x, y; t) = \frac{1}{h_n} \frac{P_{n+1}(x;t)P_n(y;t) - P_n(x;t)P_{n+1}(y;t)}{x - y}, \tag{20}$$

and its confluent form

$$K_n(x, x; t) = \frac{P'_{n+1}(x;t)P_n(x;t) - P'_n(x;t)P_{n+1}(x;t)}{h_n}.$$

First, we express the polynomials $Q_n(x;t)$ in terms of the kernel polynomials related to the polynomials $P_n(x;t)$.

Lemma 3.1 *Let $Q_n(x;t)$ be the n–th monic orthogonal polynomial with respect to (1). Then,*

$$Q_n(x;t) = P_n(x;t) - \frac{M_n}{2} Q_n(c;t) \left(K_{n-1}(c, x; t) + (-1)^n K_{n-1}(-c, x; t)\right), \quad n \geq 1, \tag{21}$$

with

$$Q_n(c;t) = \frac{P_n(c;t)}{1 + \frac{M_n}{2} \left(K_{n-1}(c, c; t) + (-1)^n K_{n-1}(-c, c; t)\right)}. \tag{22}$$

Proof. We follow the ideas in [31, Sect. 2], among others. It is well known that the sequence $\{P_k(x;t)\}_{k=0}^{n}$ forms a basis of the linear space $\mathbb{P}_n[x]$ of polynomials with real coefficients of degree at most n. So, we can write

$$Q_n(x;t) = \sum_{k=0}^{n} a_{n,k} P_k(x;t).$$

The coefficient $a_{n,n} = 1$ because $Q_n(x;t)$ and $P_n(x;t)$ are monic polynomials. For $0 \leq i \leq n-1$, using the orthogonality of $Q_n(x;t)$ we get

$$a_{n,k} = \frac{-\frac{M_n}{2} Q_n(c;t) \left(P_k(c;t) + (-1)^n P_k(-c;t)\right)}{h_k}.$$

Therefore,

$$
\begin{aligned}
Q_n(x;t) &= P_n(x;t) - \frac{M_n}{2} Q_n(c;t) \left(\sum_{k=0}^{n-1} \frac{P_k(c;t)P_k(x;t)}{h_k} + (-1)^n \frac{P_k(-c;t)P_k(x;t)}{h_k} \right) \\
&= P_n(x;t) - \frac{M_n}{2} Q_n(c;t) \left(K_{n-1}(c,x;t) + (-1)^n K_{n-1}(-c,x;t) \right).
\end{aligned}
$$

Finally, evaluating the above expression at $x = c$ we obtain

$$
Q_n(c;t) = P_n(c;t) - \frac{M_n}{2} Q_n(c;t) \left(K_{n-1}(c,c;t) + (-1)^n K_{n-1}(-c,c;t) \right),
$$

from where (22) follows. □

Corollary 3.1 *Let $Q_n^{(0)}(x;t)$ be the n–th monic orthogonal polynomial with respect to (3). Then,*

$$
Q_n^{(0)}(x;t) = \begin{cases} P_n(x;t), & \text{if } n \text{ is odd;} \\ P_n(x;t) - \frac{M_n P_n(0;t)}{1+M_n K_{n-1}(0,0;t)} K_{n-1}(0,x;t), & \text{if } n \text{ is even.} \end{cases} \qquad n \geq 1.
$$

Furthermore,

$$
Q_n^{(0)}(0;t) = \begin{cases} 0, & \text{if } n \text{ is odd;} \\ \frac{P_n(0;t)}{1+M_n K_{n-1}(0,0;t)}, & \text{if } n \text{ is even.} \end{cases} \qquad n \geq 1.
$$

Proof. It is an immediate consequence of Lemma 3.1. □

We will rewrite Lemma 3.1 using the following notation:

$$
\begin{aligned}
\kappa_{2n}(x,y;t) &:= \sum_{i=0}^{n} \frac{P_{2i}(x;t)P_{2i}(y;t)}{h_{2i}}, \\
\tilde{\kappa}_{2n}(x,y;t) &:= \sum_{i=0}^{n} \frac{P_{2i+1}(x;t)P_{2i+1}(y;t)}{h_{2i+1}}.
\end{aligned}
$$

Thus,

$$
K_{2n+1}(x,y;t) = \kappa_{2n}(x,y;t) + \tilde{\kappa}_{2n}(x,y;t),
$$

and

$$
K_{2n-1}(x,y;t) + K_{2n-1}(-x,y;t) = 2\kappa_{2(n-1)}(x,y;t), \tag{23}
$$

$$
K_{2n}(x,y;t) - K_{2n}(-x,y;t) = 2\tilde{\kappa}_{2(n-1)}(x,y;t). \tag{24}
$$

Thus, assuming that $Q_0(x;t) = 1$ and $Q_1(x;t) = x$, Lemma 3.1 becomes:

Corollary 3.2 *We have,*

$$\begin{aligned} Q_{2n}(x;t) &= P_{2n}(x;t) - \frac{M_{2n}P_{2n}(c;t)}{1+M_{2n}\kappa_{2(n-1)}(c,c;t)}\kappa_{2(n-1)}(c,x;t), \qquad n\geq 1, \\ Q_{2n+1}(x;t) &= P_{2n+1}(x;t) - \frac{M_{2n+1}P_{2n+1}(c;t)}{1+M_{2n+1}\tilde{\kappa}_{2(n-1)}(c,c;t)}\tilde{\kappa}_{2(n-1)}(c,x;t), \qquad n\geq 1. \end{aligned}$$

In the following result we establish a relation for the ratio of the square of the norm of $Q_n(x;t)$ and $h_n = (P_n, P_n)_t$.

Proposition 3.1 *It holds,*

$$\begin{aligned} \frac{(Q_{2n},Q_{2n})_{2n}}{h_{2n}} &= \frac{1+M_{2n}\kappa_{2n}(c,c;t)}{1+M_{2n}\kappa_{2(n-1)}(c,c;t)}, \\ \frac{(Q_{2n+1},Q_{2n+1})_{2n+1}}{h_{2n+1}} &= \frac{1+M_{2n+1}\tilde{\kappa}_{2n}(c,c;t)}{1+M_{2n+1}\tilde{\kappa}_{2(n-1)}(c,c;t)}. \end{aligned}$$

Proof. Using Lemma 3.1, (5) and (23) we get

$$\begin{aligned} (Q_{2n},Q_{2n})_{2n} &= (Q_{2n},P_{2n})_{2n} \\ &= (Q_{2n},P_{2n})_t + \frac{M_{2n}}{2}\left(Q_{2n}(c;t)P_{2n}(c;t) + Q_{2n}(-c;t)P_{2n}(-c;t)\right) \\ &= h_{2n} + M_{2n}Q_{2n}(c;t)P_{2n}(c;t) = h_{2n} + M_{2n}\frac{P_{2n}^2(c;t)}{1+M_{2n}\kappa_{2(n-1)}(c,c;t)} \\ &= h_{2n}\frac{1+M_{2n}\kappa_{2n}(c,c;t)}{1+M_{2n}\kappa_{2(n-1)}(c,c;t)}. \end{aligned}$$

The result for the odd case is obtained in an analogous way but now using (24). $\square$

We have introduced a perturbation in the inner product $(\cdot,\cdot)_t$ adding masses located at the points c and $-c$. Then, the quadratic polynomial x^2-c^2 can be used to *eliminate* that perturbation. Technical Lemmas 2–5 show how this polynomial helps obtain a useful relation between both families of orthogonal polynomials $P_n(x;t)$ and $Q_n(x;t)$.

Lemma 3.2 *Let* $\{Q_n(x;t)\}_{n=0}^{\infty}$ *and* $\{P_n(x;t)\}_{n=0}^{\infty}$ *be the sequences of monic orthogonal polynomials with respect to (1) and (4), respectively. Then,*

$$(x^2-c^2)Q_n(x;t) = f_1(n,x,c;t)P_n(x;t) + g_1(n,x,c;t)P_{n-1}(x;t), \qquad n\geq 1, \tag{25}$$

where

$$\begin{aligned} f_1(n,x,c;t) &= \left(x^2-c^2-c\rho_{n,c}P_{n-1}(c;t)\right), &(26a) \\ g_1(n,x,c;t) &= x\rho_{n,c}P_n(c;t), &(26b) \end{aligned}$$

with

$$\rho_{n,c} = \frac{M_n Q_n(c;t)}{h_{n-1}}, \qquad n \geq 1.$$

Proof. We will use (6), (20) and (21). Multiplying the expression (21) by $(x^2 - c^2)$, and using (6) and (20), we obtain

$$\begin{aligned}
&(x^2 - c^2)Q_n(x;t) \\
&= (x^2 - c^2)P_n(x;t) - \frac{M_n}{2}Q_n(c;t) \\
&\quad \times \left(\frac{x+c}{h_{n-1}}\left(P_n(x;t)P_{n-1}(c;t) - P_n(c;t)P_{n-1}(x;t)\right)\right. \\
&\qquad \left. + \frac{x-c}{h_{n-1}}(-1)^n\left(P_n(x;t)P_{n-1}(-c;t) - P_n(-c;t)P_{n-1}(x;t)\right)\right) \\
&= (x^2 - c^2)P_n(x;t) - \frac{M_n}{2}Q_n(c;t)\frac{2cP_n(x;t)P_{n-1}(c;t) - 2xP_n(c;t)P_{n-1}(x;t)}{h_{n-1}} \\
&= \left(x^2 - c^2 - M_n Q_n(c;t)\frac{cP_{n-1}(c;t)}{h_{n-1}}\right)P_n(x;t) + M_n Q_n(c;t)\frac{xP_n(c;t)}{h_{n-1}}P_{n-1}(x;t).
\end{aligned}$$

which proves the result. $\square$

Corollary 3.3 *Let $\{Q_n^{(0)}(x;t)\}_{n=0}^{\infty}$ be the sequence of monic orthogonal polynomials with respect to (3). Then,*

$$xQ_n^{(0)}(x;t) = f_{1,0}(x)P_n(x;t) + g_{1,0}(n;t)P_{n-1}(x;t), \qquad n \geq 1, \tag{27}$$

where

$$\begin{aligned}
f_{1,0}(x) &= x, \\
g_{1,0}(n;t) &= \frac{M_n P_n^2(0;t)}{h_{n-1}(1 + M_n K_{n-1}(0,0;t))} = \rho_{n,0}P_n(0;t).
\end{aligned}$$

Proof. Taking $c = 0$ in Lemma 3.2, using Corollary 3.1 and simplifying, the result follows straightforwardly. $\square$

Remark 3.1 *Note that the functions $f_1(n,x,0;t)$ and $f_{1,0}(x)$ are different as it also happens with $g_1(n,x,0;t)$ and $g_{1,0}(n;t)$. In fact,*

$$f_1(n,x,0;t) = xf_{1,0}(x), \quad g_1(n,x,0;t) = xg_{1,0}(n;t).$$

It is worth observing that when n is odd we have $g_{1,0}(n;t) = 0$.

Lemma 3.3 *Let* $\{Q_n(x;t)\}_{n=0}^{\infty}$ *and* $\{P_n(x;t)\}_{n=0}^{\infty}$ *be the sequences of monic orthogonal polynomials with respect to (1) and (4), respectively. Then,*

$$(x^2-c^2)Q_n'(x;t) = f_2(n,x,c;t)P_n(x;t) + g_2(n,x,c;t)P_{n-1}(x;t), \quad n \geq 2, \tag{28}$$

where

$$\begin{aligned} f_2(n,x,c;t) = 2x &- f_1(n,x,c;t)\left(\frac{2x}{x^2-c^2}+\frac{B_n(x;t)}{x}\right) \\ &- \frac{\rho_{n,c}P_n(c;t)A_{n-1}(x;t)}{\beta_{n-1}}, \end{aligned} \tag{29a}$$

$$\begin{aligned} g_2(n,x,c;t) = &\frac{-2x^2\rho_{n,c}P_n(c;t)}{x^2-c^2} + f_1(n,x,c;t)\frac{A_n(x;t)}{x} \\ &+ \rho_{n,c}P_n(c;t)\left(1+\frac{xA_{n-1}(x;t)}{\beta_{n-1}} - B_{n-1}(x;t)\right). \end{aligned} \tag{29b}$$

The functions $A_n(x;t)$, $B_n(x;t)$, $f_1(n,x,c;t)$ *and the value* $\rho_{n,c}$ *are defined in (10), (11) and Lemma 3.2, respectively.*

Proof. First, we take derivatives in (25) obtaining

$$\begin{aligned} 2xQ_n(x;t) &+ (x^2-c^2)Q_n'(x;t) \\ &= 2xP_n(x;t) + f_1(n,x,c;t)P_n'(x;t) \\ &\quad + \rho_{n,c}P_n(c;t)P_{n-1}(x;t) + \rho_{n,c}P_n(c;t)xP_{n-1}'(x;t). \end{aligned} \tag{30}$$

Now, we consider the differential–difference equation (9) given in Theorem 1.1 and using (7), we then get

$$\begin{aligned} xP_{n-1}'(x;t) &= -B_{n-1}(x;t)P_{n-1}(x;t) + A_{n-1}(x;t)P_{n-2}(x;t) \\ &= -\frac{A_{n-1}(x;t)}{\beta_{n-1}}P_n(x;t) + \left(\frac{xA_{n-1}(x;t)}{\beta_{n-1}} - B_{n-1}(x;t)\right)P_{n-1}(x;t). \end{aligned}$$

We use the previous expression in an adequate way in (30), so we deduce

$$
\begin{aligned}
&(x^2-c^2)Q_n'(x;t)\\
&= \frac{-2x}{x^2-c^2}(x^2-c^2)Q_n(x;t) + 2xP_n(x;t) + f_1(n,x,c;t)P_n'(x;t)\\
&\quad + \rho_{n,c}P_n(c;t)P_{n-1}(x;t) + \rho_{n,c}P_n(c;t)xP_{n-1}'(x;t)\\
&= \frac{-2x}{x^2-c^2}\left(f_1(n,x,c;t)P_n(x;t) + g_1(n,x,c;t)P_{n-1}(x;t)\right) + 2xP_n(x;t)\\
&\quad + f_1(n,x,c;t)P_n'(x;t) + \rho_{n,c}P_n(c;t)P_{n-1}(x;t) + \rho_{n,c}P_n(c;t)xP_{n-1}'(x;t)\\
&= \frac{-2x}{x^2-c^2}\left(f_1(n,x,c;t)P_n(x;t) + g_1(n,x,c;t)P_{n-1}(x;t)\right) + 2xP_n(x;t)\\
&\quad + f_1(n,x,c;t)\left(\frac{-B_n(x;t)P_n(x;t) + A_n(x;t)P_{n-1}(x;t)}{x}\right)\\
&\quad + \rho_{n,c}P_n(c;t)P_{n-1}(x;t)\\
&\quad + \rho_{n,c}P_n(c;t)\left(\frac{-A_{n-1}(x;t)}{\beta_{n-1}}P_n(x;t) + \left(\frac{xA_{n-1}(x;t)}{\beta_{n-1}} - B_{n-1}(x;t)\right)P_{n-1}(x;t)\right).
\end{aligned}
$$

Finally, simplifying these expressions we prove the result. □

Corollary 3.4 *Let $\{Q_n^{(0)}(x;t)\}_{n=0}^{\infty}$ be the sequence of monic orthogonal polynomials with respect to (3). Then,*

$$x\left(Q_n^{(0)}\right)'(x;t) = f_{2,0}(n,x;t)P_n(x;t) + g_{2,0}(n,x;t)P_{n-1}(x;t), \qquad n \geq 2,$$

where

$$
\begin{aligned}
f_{2,0}(n,x;t) &= -B_n(x;t) - \frac{\rho_{n,0}P_n(0;t)A_{n-1}(x;t)}{x\beta_{n-1}},\\
g_{2,0}(n,x;t) &= A_n(x;t) + \frac{\rho_{n,0}P_n(0;t)}{x}\left(-1 + \frac{xA_{n-1}(x;t)}{\beta_{n-1}} - B_{n-1}(x;t)\right).
\end{aligned}
$$

In Lemmas 3.2 and 3.3 we have provided formulae for $(x^2-c^2)Q_n(x;t)$ and $(x^2-c^2)Q_n'(x;t)$ in terms of the standard polynomials $P_n(x;t)$ and $P_{n-1}(x;t)$. In the following two lemmas we will give other different formulae applying the previous results. All of them will be very useful to construct the ladder operators for the polynomials $Q_n(x;t)$ in the next section.

Lemma 3.4 *Let $\{Q_n(x;t)\}_{n=0}^{\infty}$ and $\{P_n(x;t)\}_{n=0}^{\infty}$ be the sequences of monic orthogonal polynomials with respect to (1) and (4), respectively. Then,*

$$(x^2-c^2)Q_{n-1}(x;t) = f_3(n,x,c;t)P_n(x;t) + g_3(n,x,c;t)P_{n-1}(x;t), \qquad n \geq 2,$$

where

$$
\begin{aligned}
f_3(n,x,c;t) &= -\frac{x\rho_{n-1,c}P_{n-1}(c;t)}{\beta_{n-1}},\\
g_3(n,x,c;t) &= \left(x^2-c^2-c\rho_{n-1,c}P_{n-2}(c;t)+\frac{x^2\rho_{n-1,c}P_{n-1}(c;t)}{\beta_{n-1}}\right).
\end{aligned}
$$

Proof. It is enough to use (25) in Lemma 3.2 and the three–term recurrence relation given by (7). Thus, we get

$$
\begin{aligned}
&(x^2-c^2)Q_{n-1}(x;t)\\
&= f_1(n-1,x,c;t)P_{n-1}(x;t)+g_1(n-1,x,c;t)P_{n-2}(x;t)\\
&= f_1(n-1,x,c;t)P_{n-1}(x;t)+g_1(n-1,x,c;t)\left(\frac{xP_{n-1}(x;t)-P_n(x;t)}{\beta_{n-1}}\right)\\
&= -\frac{g_1(n-1,x,c;t)}{\beta_{n-1}}P_n(x;t)+\left(f_1(n-1,x,c;t)+\frac{xg_1(n-1,x,c;t)}{\beta_{n-1}}\right)P_{n-1}(x;t).
\end{aligned}
$$

Using (26) the statement follows. □

Corollary 3.5 *Let $\{Q_n^{(0)}(x;t)\}_{n=0}^{\infty}$ be a sequence of monic orthogonal polynomials with respect to (3). Then,*

$$
xQ_{n-1}^{(0)}(x;t)=f_{3,0}(n;t)P_n(x;t)+g_{3,0}(n,x;t)P_{n-1}(x;t),\qquad n\geq 2, \tag{31}
$$

where

$$
\begin{aligned}
f_{3,0}(n;t) &= -\frac{\rho_{n-1,0}P_{n-1}(0;t)}{\beta_{n-1}},\\
g_{3,0}(n,x;t) &= x\left(1+\frac{\rho_{n-1,0}P_{n-1}(0;t)}{\beta_{n-1}}\right).
\end{aligned}
$$

Remark 3.2 *The situation in this corollary is very similar to the one described in Remark 3.1. Now, we note that $f_{3,0}(n;t)=0$ when n is even.*

Lemma 3.5 *Let $\{Q_n(x;t)\}_{n=0}^{\infty}$ and $\{P_n(x;t)\}_{n=0}^{\infty}$ be the sequences of monic orthogonal polynomials with respect to (1) and (4), respectively. Then,*

$$
(x^2-c^2)Q'_{n-1}(x;t)=f_4(n,x,c;t)P_n(x;t)+g_4(n,x,c;t)P_{n-1}(x;t),\qquad n\geq 3,
$$

where

$$
\begin{aligned}
f_4(n,x,c;t) &= -\frac{g_2(n-1,x,c;t)}{\beta_{n-1}},\\
g_4(n,x,c;t) &= f_2(n-1,x,c;t)+x\frac{g_2(n-1,x,c;t)}{\beta_{n-1}}.
\end{aligned}
$$

The functions $f_2(n,x,c;t)$ and $g_2(n,x,c;t)$ are defined in (29).

Proof. We proceed as in the previous lemma but now using Lemma 3.3. $\square$

Corollary 3.6 *Let $\{Q_n^{(0)}(x;t)\}_{n=0}^{\infty}$ be the sequence of monic polynomials orthogonal with respect to (3). Then,*

$$x\left(Q_{n-1}^{(0)}\right)'(x;t) = f_{4,0}(n,x;t)P_n(x;t) + g_{4,0}(n,x;t)P_{n-1}(x;t), \qquad n \geq 3,$$

where

$$\begin{aligned} f_{4,0}(n,x;t) &= -\frac{g_{2,0}(n-1,x;t)}{\beta_{n-1}}, \\ g_{4,0}(n,x;t) &= f_{2,0}(n-1,x;t) + x\frac{g_{2,0}(n-1,x;t)}{\beta_{n-1}}. \end{aligned}$$

Proof: It is only necessary to use Corollary 3.4 and (7). $\square$

With these Lemmas we have the polynomials $P_n(x;t)$ as the solutions of linear systems of two equations. Thus, these standard polynomials are expressed in terms of the varying Freud–type orthogonal polynomials.

Lemma 3.6 *For $n \geq 2$, we have*

$$P_n(x;t) = \frac{(x^2-c^2)\left(g_3(n,x,c;t)Q_n(x;t) - g_1(n,x,c;t)Q_{n-1}(x;t)\right)}{f_1(n,x,c;t)g_3(n,x,c;t) - g_1(n,x,c;t)f_3(n,x,c;t)}, \tag{32}$$

$$P_{n-1}(x;t) = \frac{(x^2-c^2)\left(-f_3(n,x,c;t)Q_n(x;t) + f_1(n,x,c;t)Q_{n-1}(x;t)\right)}{f_1(n,x,c;t)g_3(n,x,c;t) - g_1(n,x,c;t)f_3(n,x,c;t)}, \tag{33}$$

where the functions $f_1(n,x,c;t)$, $g_1(n,x,c;t)$, $f_3(n,x,c;t)$ and $g_3(n,x,c;t)$ are defined in Lemma 3.2 and Lemma 3.4.

Proof. From Lemma 3.2 and Lemma 3.4, we can write

$$\begin{cases} f_1(n,x,c;t)P_n(x;t) + g_1(n,x,c;t)P_{n-1}(x;t) = (x^2-c^2)Q_n(x;t), \\ f_3(n,x,c;t)P_n(x;t) + g_3(n,x,c;t)P_{n-1}(x;t) = (x^2-c^2)Q_{n-1}(x;t). \end{cases}$$

It is enough to apply Cramer's rule to get the result. $\square$

For the case $c = 0$ we can obtain simpler expressions.

Corollary 3.7 *For $c = 0$ and $n \geq 2$, we have*

$$\begin{aligned} g_{3,0}(n,x;t)P_n(x;t) &= g_{3,0}(n,x;t)Q_n^{(0)}(x;t) - g_{1,0}(n;t)Q_{n-1}^{(0)}(x;t), \\ g_{3,0}(n,x;t)P_{n-1}(x;t) &= -f_{3,0}(n,x;t)Q_n^{(0)}(x;t) + f_{1,0}(n,x;t)Q_{n-1}^{(0)}(x;t), \end{aligned}$$

where the functions $f_{1,0}(x)$, $g_{1,0}(n;t)$, $f_{3,0}(n;t)$, and $g_{3,0}(n,x;t)$ are given in Corollary 3.3 and Corollary 3.5.

Proof. Gathering (27) and (31) in Corollary 3.3 and Corollary 3.5, respectively, we have a linear system of two equations. Solving it, we obtain:

$$
\begin{aligned}
P_n(x;t) &= \frac{x\left(g_{3,0}(n,x;t)Q_n^{(0)}(x;t) - g_{1,0}(n;t)Q_{n-1}^{(0)}(x;t)\right)}{f_{1,0}(x)g_{3,0}(n,x;t) - g_{1,0}(n;t)f_{3,0}(n;t)}, \\
P_{n-1}(x;t) &= \frac{x\left(-f_{3,0}(n;t)Q_n^{(0)}(x;t) + f_{1,0}(x)Q_{n-1}^{(0)}(x;t)\right)}{f_{1,0}(x)g_{3,0}(n,x;t) - g_{1,0}(n;t)f_{3,0}(n;t)}.
\end{aligned}
$$

Using Corollaries 3.3 and 3.5 we obtain

$$
\begin{aligned}
& f_{1,0}(x)g_{3,0}(n,x;t) - g_{1,0}(n;t)f_{3,0}(n;t) \\
= \; & x^2\left(1 + \frac{\rho_{n-1,0}P_{n-1}(0;t)}{\beta_{n-1}}\right) + \rho_{n,0}P_n(0;t)\frac{\rho_{n-1,0}P_{n-1}(0;t)}{\beta_{n-1}} \\
= \; & x^2\left(1 + \frac{\rho_{n-1,0}P_{n-1}(0;t)}{\beta_{n-1}}\right) \\
= \; & x g_{3,0}(n,x;t),
\end{aligned}
$$

where we have used the fact that $P_n(0;t)P_{n-1}(0;t)$ is always zero. $\square$

4 Ladder operators and second order differential equation

In this section we obtain the second order linear differential equation satisfied by the varying generalized Freud–type orthogonal polynomials. The first step is to obtain the ladder operators for this family of polynomials. Lemmas 3.2–3.6 obtained in the previous section are the key to deduce these ladder operators.

Theorem 4.1 ***(Ladder Operators)*** *Let* $\{Q_n(x;t)\}_{n=0}^{\infty}$ *be the sequence of monic orthogonal polynomials with respect to (1). Then, there exit a lowering differential operator* Φ_n *and a raising differential operator* $\widehat{\Phi}_n$ *defined as*

$$
\begin{aligned}
\Phi_n &:= \varphi_{3,2}(n,x,c;t) + \varphi_{1,3}(n,x,c;t)\frac{d}{dx}, \qquad n \geq 2, \\
\widehat{\Phi}_n &:= \varphi_{1,4}(n,x,c;t) - \varphi_{1,3}(n,x,c;t)\frac{d}{dx}, \qquad n \geq 3,
\end{aligned}
$$

and satisfying

$$
\Phi_n\left[Q_n(x;t)\right] = \varphi_{1,2}(n,x,c;t)Q_{n-1}(x;t), \tag{34}
$$

$$
\widehat{\Phi}_n\left[Q_{n-1}(x;t)\right] = \varphi_{3,4}(n,x,c;t)Q_n(x;t), \tag{35}
$$

with

$$\varphi_{i,j}(n,x,c;t) = \begin{vmatrix} f_i(n,x,c;t) & f_j(n,x,c;t) \\ g_i(n,x,c;t) & g_j(n,x,c;t) \end{vmatrix}, \qquad i,j \in \{1,2,3,4\},$$

where the functions $f_i(n,x,c;t)$ and $g_i(n,x,c;t)$ with $i \in \{1,2,3,4\}$ are defined in the Lemmas 3.2–3.5.

Proof. To prove (34), we substitute the relations (32) and (33) into (28) and simplify. Then, we get

$$\begin{aligned}
&(x^2-c^2)Q_n'(x;t) \\
&= f_2(n,x,c;t)P_n(x;t) + g_2(n,x,c;t)P_{n-1}(x;t) \\
&= \frac{f_2(n,x,c;t)(x^2-c^2)\left(g_3(n,x,c;t)Q_n(x;t) - g_1(n,x,c;t)Q_{n-1}(x;t)\right)}{\varphi_{1,3}(n,x,c;t)} \\
&\quad + \frac{g_2(n,x,c;t)(x^2-c^2)\left(-f_3(n,x,c;t)Q_n(x;t) + f_1(n,x,c;t)Q_{n-1}(x;t)\right)}{\varphi_{1,3}(n,x,c;t)}.
\end{aligned}$$

Then,

$$\begin{aligned}
&\varphi_{1,3}(n,x,c;t)Q_n'(x;t) \\
&= \left(f_2(n,x,c;t)g_3(n,x,c;t) - g_2(n,x,c;t)f_3(n,x,c;t)\right)Q_n(x;t) \\
&\quad + \left(-f_2(n,x,c;t)g_1(n,x,c;t) + g_2(n,x,c;t)f_1(n,x,c;t)\right)Q_{n-1}(x;t) \\
&= \varphi_{2,3}(n,x,c;t)Q_n(x;t) + \varphi_{1,2}(n,x,c;t)Q_{n-1}(x;t).
\end{aligned}$$

Taking into account the obvious fact that $\varphi_{3,2}(n,x,c;t) = -\varphi_{2,3}(n,x,c;t)$ we deduce (34). The proof of (35) is completely similar. □

Corollary 4.1 *The ladder operators in the case $c = 0$ are given by*

$$\begin{aligned}
\Phi_n^{(0)} &:= \Upsilon_{3,2}(n,x;t) + \Upsilon_{1,3}(n,x;t)\frac{d}{dx}, \qquad n \geq 2, \\
\widehat{\Phi}_n^{(0)} &:= \Upsilon_{1,4}(n,x;t) - \Upsilon_{1,3}(n,x;t)\frac{d}{dx}, \qquad n \geq 3,
\end{aligned}$$

which satisfy

$$\begin{aligned}
\Phi_n^{(0)}\left[Q_n^{(0)}(x;t)\right] &= \Upsilon_{1,2}(n,x;t)Q_{n-1}^{(0)}(x;t), \\
\widehat{\Phi}_n^{(0)}\left[Q_{n-1}^{(0)}(x;t)\right] &= \Upsilon_{3,4}(n,x;t)Q_n^{(0)}(x;t),
\end{aligned}$$

with

$$\Upsilon_{i,j}(n,x;t) = \begin{vmatrix} f_{i,0} & f_{j,0} \\ g_{i,0} & g_{j,0} \end{vmatrix}, \qquad i,j \in \{1,2,3,4\}, \tag{36}$$

where the functions $f_{i,0}$ and $g_{i,0}$ with $i \in \{1,2,3,4\}$ are defined in the Corollaries 3.3–3.6.

Proof. It is enough to use Corollaries 3.3–3.7 in the same way as in the proof of Theorem 4.1. $\square$

We are ready to establish one of the main results.

Theorem 4.2 ***(Holonomic equation)*** *The varying generalized Freud–type orthogonal polynomial satisfy the following second order linear differential equation:*

$$\alpha(n,x,c;t)Q_n''(x;t)+\sigma(n,x,c;t)Q_n'(x;t)+\tau(n,x,c;t)Q_n(x;t)=0, \tag{37}$$

with $n\geq 3$, where

$$\begin{aligned}
\alpha(n,x,c;t) &= \varphi_{1,3}^2(n,x,c;t)\varphi_{1,2}(n,x,c;t),\\
\sigma(n,x,c;t) &= \varphi_{1,3}(n,x,c;t)\Big(\varphi_{1,2}(n,x,c;t)(\varphi_{3,2}(n,x,c;t)+\varphi_{1,3}'(n,x,c;t)\\
&- \varphi_{1,4}(n,x,c;t))-\varphi_{1,2}'(n,x,c;t)\varphi_{1,3}(n,x,c;t)\Big),\\
\tau(n,x,c;t) &= \varphi_{1,2}(n,x,c;t)(\varphi_{1,2}(n,x,c;t)\varphi_{3,4}(n,x,c;t)-\varphi_{1,4}(n,x,c;t)\varphi_{3,2}(n,x,c;t))\\
&+ \varphi_{1,3}(n,x,c;t)\left(\varphi_{3,2}'(n,x,c;t)\varphi_{1,2}(n,x,c;t)-\varphi_{1,2}'(n,x,c;t)\varphi_{3,2}(n,x,c;t)\right).
\end{aligned}$$

Proof. The technique is standard once we know the corresponding ladder operators given in Theorem 4.1. By (35) we have

$$\widehat{\Phi}_n[Q_{n-1}(x;t)]=\varphi_{3,4}(n,x,c;t)Q_n(x;t).$$

Taking into account that by (34)

$$Q_{n-1}(x;t)=\frac{1}{\varphi_{1,2}(n,x,c;t)}\Phi_n[Q_n(x;t)],$$

then

$$\widehat{\Phi}_n\left[\frac{1}{\varphi_{1,2}(n,x,c;t)}\Phi_n[Q_n(x;t)]\right]=\varphi_{3,4}(n,x,c;t)Q_n(x;t). \tag{38}$$

Now, we deal with the left hand side in the above expression getting

$$\begin{aligned}
&\widehat{\Phi}_n\left[\frac{1}{\varphi_{1,2}(n,x,c;t)}\Phi_n[Q_n(x;t)]\right]\\
=\ &\frac{\varphi_{1,4}(n,x,c;t)}{\varphi_{1,2}(n,x,c;t)}\Phi_n[Q_n(x;t)]-\varphi_{1,3}(n,x,c;t)\frac{d}{dx}\left(\frac{\Phi_n[Q_n(x;t)]}{\varphi_{1,2}(n,x,c;t)}\right)\\
=\ &\frac{\varphi_{1,4}(n,x,c;t)}{\varphi_{1,2}(n,x,c;t)}\left(\varphi_{3,2}(n,x,c;t)Q_n(x;t)+\varphi_{1,3}(n,x,c;t)Q_n'(x;t)\right)\\
-\ &\varphi_{1,3}(n,x,c;t)\frac{d}{dx}\left(\frac{\varphi_{3,2}(n,x,c;t)Q_n(x;t)}{\varphi_{1,2}(n,x,c;t)}+\frac{\varphi_{1,3}(n,x,c;t)Q_n'(x;t)}{\varphi_{1,2}(n,x,c;t)}\right).
\end{aligned} \tag{39}$$

We calculate the derivatives in (39):

$$
\begin{aligned}
&\frac{d}{dx}\left(\frac{\varphi_{3,2}(n,x,c;t)Q_n(x;t)}{\varphi_{1,2}(n,x,c;t)}\right)\\
&\quad=\frac{\varphi'_{3,2}(n,x,c;t)\varphi_{1,2}(n,x,c;t)-\varphi'_{1,2}(n,x,c;t)\varphi_{3,2}(n,x,c;t)}{\varphi_{1,2}(n,x,c;t)^2}Q_n(x;t)\\
&\qquad+\frac{\varphi_{3,2}(n,x,c;t)}{\varphi_{1,2}(n,x,c;t)}Q'_n(x;t), \qquad (40)\\
&\frac{d}{dx}\left(\frac{\varphi_{1,3}(n,x,c;t)Q'_n(x;t)}{\varphi_{1,2}(n,x,c;t)}\right)\\
&\quad=\frac{\varphi'_{1,3}(n,x,c;t)\varphi_{1,2}(n,x,c;t)-\varphi'_{1,2}(n,x,c;t)\varphi_{1,3}(n,x,c;t)}{\varphi^2_{1,2}(n,x,c;t)}Q'_n(x;t)\\
&\qquad+\frac{\varphi_{1,3}(n,x,c;t)}{\varphi_{1,2}(n,x,c;t)}Q''_n(x;t). \qquad (41)
\end{aligned}
$$

Substituting (40) and (41) in (39) and using (38), we obtain

$$
\begin{aligned}
&\varphi_{3,4}(n,x,c;t)Q_n(x;t)\\
=\;&\frac{\varphi_{1,4}(n,x,c;t)}{\varphi_{1,2}(n,x,c;t)}\left(\varphi_{3,2}(n,x,c;t)Q_n(x;t)+\varphi_{1,3}(n,x,c;t)Q'_n(x;t)\right)\\
-\;&\varphi_{1,3}(n,x,c;t)\left(\frac{\varphi'_{3,2}(n,x,c;t)\varphi_{1,2}(n,x,c;t)-\varphi'_{1,2}(n,x,c;t)\varphi_{3,2}(n,x,c;t)}{\varphi_{1,2}(n,x,c;t)^2}Q_n(x;t)\right.\\
+\;&\frac{\varphi_{3,2}(n,x,c;t)}{\varphi_{1,2}(n,x,c;t)}Q'_n(x;t)\\
+\;&\frac{\varphi'_{1,3}(n,x,c;t)\varphi_{1,2}(n,x,c;t)-\varphi'_{1,2}(n,x,c;t)\varphi_{1,3}(n,x,c;t)}{\varphi_{1,2}(n,x,c;t)^2}Q'_n(x;t)\\
+\;&\left.\frac{\varphi_{1,3}(n,x,c;t)}{\varphi_{1,2}(n,x,c;t)}Q''_n(x;t).\right)
\end{aligned}
$$

To get the second order differential equation for the polynomials $Q_n(x;t)$ it only remains to multiply the above expression by $\varphi^2_{1,2}(n,x,c;t)$ and simplify . □

As it is usual when $c=0$ this differential equation looks simpler.

Corollary 4.2 *For $c=0$ and $n\geq 3$, we have*

$$
\alpha_0(n,x;t)\left(Q_n^{(0)}\right)''(x;t)+\sigma_0(n,x;t)\left(Q_n^{(0)}\right)'(x;t)+\tau_0(n,x;t)Q_n^{(0)}(x;t)=0, \qquad (42)
$$

where

$$\begin{aligned}
\alpha_0(n,x;t) &= \Upsilon_{1,3}^2(n,x;t)\Upsilon_{1,2}(n,x;t),\\
\sigma_0(n,x;t) &= \Upsilon_{1,3}(n,x;t)\Big(\Upsilon_{1,2}(n,x;t)(\Upsilon_{3,2}(n,x;t)+\Upsilon'_{1,3}(n,x;t)-\Upsilon_{1,4}(n,x;t))\\
&- \Upsilon'_{1,2}(n,x;t)\Upsilon_{1,3}(n,x;t)\Big),\\
\tau_0(n,x;t) &= \Upsilon_{1,2}(n,x;t)(\Upsilon_{1,2}(n,x;t)\Upsilon_{3,4}(n,x;t)-\Upsilon_{1,4}(n,x;t)\Upsilon_{3,2}(n,x;t))\\
&+ \Upsilon_{1,3}(n,x;t)\left(\Upsilon'_{3,2}(n,x;t)\Upsilon_{1,2}(n,x;t)-\Upsilon'_{1,2}(n,x;t)\Upsilon_{3,2}(n,x;t)\right).
\end{aligned}$$

The functions $\Upsilon_{i,j}$ with $i,j\in\{1,2,3,4\}$ are defined in (36).

Finally, we illustrate the results obtained in this section with two examples: ladder operators and second order differential equations for the varying Freud–type orthogonal polynomials. In these examples we consider the weight functions like the ones in subsections 2.1 and 2.2.

4.1 Example 3

We consider the orthogonal polynomials $Q_n^{(0)}(x)$ with respect to (3) with $v(x)=x^2$ and $c=0$. We have seen in Example 1 that

$$A_n(x)=2x\beta_n,\qquad B_n(x)=\frac{\gamma}{2}(1-(-1)^n).$$

To calculate the lowering operator, we use (36) in Corollary 4.1. Then, we have for $n\geq 2$,

$$\Upsilon_{1,3}(n,x)\left(Q_n^{(0)}\right)'(x)+\Upsilon_{3,2}(n,x)Q_n^{(0)}(x)=\Upsilon_{1,2}(n,x)Q_{n-1}^{(0)}(x).$$

We can compute all the coefficients in the above differential equation. Taking into account that $P_n(0)P_{n-1}(0)$ and $\rho_{n,0}\rho_{n-1,0}$ are always zero, and after tedious computations we obtain

$$\begin{aligned}
\Upsilon_{1,3}(n,x) &= x^2\left(1+\frac{\rho_{n-1,0}P_{n-1}(0)}{\beta_{n-1}}\right),\\
\Upsilon_{3,2}(n,x) &= -\frac{2x\beta_n\rho_{n-1,0}P_{n-1}(0)}{\beta_{n-1}}+\frac{\gamma}{2}(1-(-1)^n)x\left(1+\frac{\rho_{n-1,0}P_{n-1}(0)}{\beta_{n-1}}\right)\\
&\quad +2x\rho_{n,0}P_n(0),\\
\Upsilon_{1,2}(n,x) &= 2x^2\beta_n+\rho_{n,0}P_n(0)\left(-1+2x^2-\gamma(-1)^n\right)+2\rho_{n,0}^2P_n^2(0).
\end{aligned}$$

On the other hand, the lowering operator is given by

$$\begin{aligned}
\Phi_n^{(0)} &= \Bigg[x\left(\frac{\gamma}{2}(1-(-1)^n)\left(1+\frac{\rho_{n-1,0}P_{n-1}(0)}{\beta_{n-1}}\right)+2\rho_{n,0}P_n(0)-\frac{2\beta_n\rho_{n-1,0}P_{n-1}(0)}{\beta_{n-1}}\right)\\
&\quad + x^2\left(1+\frac{\rho_{n-1,0}P_{n-1}(0)}{\beta_{n-1}}\right)\frac{d}{dx}\Bigg],
\end{aligned}$$

and it acts as

$$\Phi_n^{(0)}\left[Q_n^{(0)}(x)\right] = \left(2x^2\beta_n + \rho_{n,0}P_n(0)\left(-1+2x^2-\gamma(-1)^n\right) + 2\rho_{n,0}^2P_n^2(0)\right)Q_{n-1}^{(0)}(x).$$

Obviously when $\rho_{n,0}=0$ for all n, then the above lowering operator is the same as the one obtained in Example 1 as it was expectable. It is equivalent to $M_n=0$ for all n.

Now, we calculate the raising operator. Using (36) in Corollary 4.1 we have for $n\geq 3$,

$$\Upsilon_{1,4}(n,x)Q_{n-1}^{(0)}(x) - \Upsilon_{1,3}(n,x)\left(Q_{n-1}^{(0)}\right)'(x) = \Upsilon_{3,4}(n,x)Q_n^{(0)}(x).$$

Using the fact that $P_n(0)P_{n-1}(0)$ and $\rho_{n,0}\rho_{n-1,0}$ are always zero, and after some computations we obtain the coefficients in the previous differential equation.

$$\begin{aligned}
\Upsilon_{1,4}(n,x) &= 2x^3 + 2x\rho_{n,0}P_n(0) - x\frac{\gamma}{2}(1-(-1)^{n-1}) - 2x\rho_{n-1,0}P_{n-1}(0) \\
&+ \frac{x\rho_{n-1,0}P_{n-1}(0)}{\beta_{n-1}}\left(-1+2x^2-\frac{\gamma}{2}(1-(-1)^{n-2})\right), \\
\Upsilon_{1,3}(n,x) &= x^2\left(1+\frac{\rho_{n-1,0}P_{n-1}(0)}{\beta_{n-1}}\right), \\
\Upsilon_{3,4}(n,x) &= 2x^2 + \frac{\rho_{n-1,0}P_{n-1}(0)}{\beta_{n-1}}\left(-1+2x^2-\gamma(-1)^{n-1}+2\rho_{n-1,0}P_{n-1}(0)\right).
\end{aligned}$$

Thus, the raising operator is given by

$$\begin{aligned}
\widehat{\Phi}_n^{(0)} &= \Bigg[x\Bigg(2x^2 + 2x\rho_{n,0}P_n(0) - \frac{\gamma}{2}(1+(-1)^n) - 2\rho_{n-1,0}P_{n-1}(0) \\
&+ \frac{\rho_{n-1,0}P_{n-1}(0)}{\beta_{n-1}}\left(-1+2x^2-\frac{\gamma}{2}(1-(-1)^n)\right)\Bigg) - x^2\left(1+\frac{\rho_{n-1,0}P_{n-1}(0)}{\beta_{n-1}}\right)\frac{d}{dx}\Bigg],
\end{aligned}$$

and it acts as

$$\begin{aligned}
&\widehat{\Phi}_n^{(0)}\left[Q_{n-1}^{(0)}\right] \\
&= \left(2x^2 + \frac{\rho_{n-1,0}P_{n-1}(0)}{\beta_{n-1}}\left(-1+2x^2-\gamma(-1)^{n-1}+2\rho_{n-1,0}P_{n-1}(0)\right)\right)Q_n^{(0)}(x).
\end{aligned}$$

4.2 Example 4

Now, we choose the same weight function that we have considered in Example 2, i.e. we take $w(x;t)=|x|^\gamma\exp(-x^4+tx^2)$ with $v(x)=x^4-2tx^2$ and $\gamma\geq 1$. As we know (see [13, f. (56)])

$$A_n(x;t) = 4x\beta_n\left(x^2+\beta_{n+1}+\beta_n-\frac{t}{2}\right), \qquad B_n(x;t) = 4x^2\beta_n + \frac{\gamma}{2}\left(1-(-1)^n\right).$$

We take $c = 0$ for simplicity. As in the Example 3, we are going to calculate explicitly the functions $\Upsilon_{1,3}(n,x;t)$, $\Upsilon_{3,2}(n,x;t)$, $\Upsilon_{1,4}(n,x;t)$, $\Upsilon_{1,2}(n,x;t)$, $\Upsilon_{3,4}(n,x;t)$ of formula (36) in Corollary 4.1 which are necessary to deduce the corresponding ladder operators. Even for this case $c = 0$ the calculations are tedious, and after these computations we get

$$
\begin{aligned}
\Upsilon_{1,3}(n,x;t) &= x^2\left(1+\frac{\rho_{n-1,0}P_{n-1}(0;t)}{\beta_{n-1}}\right).\\
\Upsilon_{3,2}(n,x;t) &= -4x\beta_n\left(x^2+\beta_{n+1}+\beta_n-\frac{t}{2}\right)\frac{\rho_{n-1,0}P_{n-1}(0;t)}{\beta_{n-1}}\\
&+ x\left(4x^2\beta_n+\frac{\gamma}{2}\left(1-(-1)^n\right)\right)\left(1+\frac{\rho_{n-1,0}P_{n-1}(0;t)}{\beta_{n-1}}\right)\\
&+ 4x\left(x^2+\beta_n+\beta_{n-1}-\frac{t}{2}\right)\rho_{n,0}P_n(0;t).
\end{aligned}
$$

$$
\begin{aligned}
&\Upsilon_{1,4}(n,x;t)\\
=\;& x\left(-\left(4x^2\beta_{n-1}+\frac{\gamma}{2}\left(1+(-1)^n\right)\right)-4\rho_{n-1,0}P_{n-1}(0;t)\left(x^2+\beta_{n-1}+\beta_{n-2}-\frac{t}{2}\right)\right)\\
+\;& 4x^3\left(x^2+\beta_n+\beta_{n-1}-\frac{t}{2}\right)\\
+\;& \frac{x\rho_{n-1,0}P_{n-1}(0;t)\left(-1+4x^2\left(x^2+\beta_{n-1}+\beta_{n-2}-\frac{t}{2}\right)-\left(4x^2\beta_{n-2}+\frac{\gamma}{2}\left(1-(-1)^n\right)\right)\right)}{\beta_{n-1}}\\
+\;& 4x\rho_{n,0}P_n(0;t)\left(x^2+\beta_n+\beta_{n-1}-\frac{t}{2}\right).
\end{aligned}
$$

$$
\begin{aligned}
&\Upsilon_{1,2}(n,x;t)\\
=\;& 4x^2\beta_n\left(x^2+\beta_{n+1}+\beta_n-\frac{t}{2}\right)\\
+\;& \rho_{n,0}P_n(0;t)\left(-1+4x^2\left(x^2+\beta_n+\beta_{n-1}-\frac{t}{2}\right)-\left(4x^2\beta_{n-1}+\frac{\gamma}{2}\left(1+(-1)^n\right)\right)\right)\\
+\;& \rho_{n,0}P_n(0;t)\left(4x^2\beta_n+\frac{\gamma}{2}\left(1-(-1)^n\right)+4\rho_{n,0}P_n(0;t)\left(x^2+\beta_n+\beta_{n-1}-\frac{t}{2}\right)\right).
\end{aligned}
$$

$$
\begin{aligned}
& \Upsilon_{3,4}(n,x;t) \\
= \;& 4x^2\left(x^2+\beta_n+\beta_{n-1}-\frac{t}{2}\right) \\
+ \;& \frac{\rho_{n-1,0}P_{n-1}(0;t)}{\beta_{n-1}} \\
\times \;& \left(4x^2\beta_{n-1}+\frac{\gamma}{2}\left(1+(-1)^n\right)+4\rho_{n-1,0}P_{n-1}(0;t)\left(x^2+\beta_{n-1}+\beta_{n-2}-\frac{t}{2}\right)\right) \\
+ \;& \frac{\rho_{n-1,0}P_{n-1}(0;t)}{\beta_{n-1}}\left(-1+4x^2\left(x^2+\beta_{n-1}+\beta_{n-2}-\frac{t}{2}\right)-4x^2\beta_{n-2}-\frac{\gamma}{2}\left(1-(-1)^n\right)\right).
\end{aligned}
$$

Now, we are ready to give the ladder operators. To start with, the lowering operator is given by

$$
\begin{aligned}
\Phi_n^{(0)} \;=\;& \left[-4x\beta_n\left(x^2+\beta_{n+1}+\beta_n-\frac{t}{2}\right)\frac{\rho_{n-1,0}P_{n-1}(0;t)}{\beta_{n-1}}\right. \\
+\;& x\left(4x^2\beta_n+\frac{\gamma}{2}\left(1-(-1)^n\right)\right)\left(1+\frac{\rho_{n-1,0}P_{n-1}(0;t)}{\beta_{n-1}}\right) \\
+\;& 4x\left(x^2+\beta_n+\beta_{n-1}-\frac{t}{2}\right)\rho_{n,0}P_n(0;t) \\
+\;& \left. x^2\left(1+\frac{\rho_{n-1,0}P_{n-1}(0)}{\beta_{n-1}}\right)\frac{d}{dx}\right],
\end{aligned}
$$

and which verifies

$$
\begin{aligned}
& \Phi_n^{(0)}\left[Q_n^{(0)}(x;t)\right] \\
= \;& \left(4x^2\beta_n\left(x^2+\beta_{n+1}+\beta_n-\frac{t}{2}\right)\right. \\
+ \;& \rho_{n,0}P_n(0;t)\left(-1+4x^2\left(x^2+\beta_n+\beta_{n-1}-\frac{t}{2}\right)-\left(4x^2\beta_{n-1}+\frac{\gamma}{2}\left(1+(-1)^n\right)\right)\right) \\
+ \;& \left.\left(4x^2\beta_n+\frac{\gamma}{2}\left(1-(-1)^n\right)+4\rho_{n,0}P_n(0;t)\left(x^2+\beta_n+\beta_{n-1}-\frac{t}{2}\right)\right)\rho_{n,0}P_n(0;t)\right) \\
\times \;& Q_{n-1}^{(0)}(x;t).
\end{aligned}
$$

The raising operator is given by

$$
\begin{aligned}
&\widehat{\Phi}_n^{(0)} \\
= \;& \Bigg[x\left(-\left(4x^2\beta_{n-1}+\frac{\gamma}{2}\left(1+(-1)^n\right)\right)-4\rho_{n-1,0}P_{n-1}(0;t)\left(x^2+\beta_{n-1}+\beta_{n-2}-\frac{t}{2}\right)\right) \\
+ \;& 4x^3\left(x^2+\beta_n+\beta_{n-1}-\frac{t}{2}\right) \\
+ \;& \frac{x\rho_{n-1,0}P_{n-1}(0;t)\left(-1+4x^2\left(x^2+\beta_{n-1}+\beta_{n-2}-\frac{t}{2}\right)-\left(4x^2\beta_{n-2}+\frac{\gamma}{2}\left(1-(-1)^n\right)\right)\right)}{\beta_{n-1}} \\
+ \;& 4x\rho_{n,0}P_n(0;t)\left(x^2+\beta_n+\beta_{n-1}-\frac{t}{2}\right) \\
- \;& x^2\left(1+\frac{\rho_{n-1,0}P_{n-1}(0)}{\beta_{n-1}}\right)\frac{d}{dx}\Bigg],
\end{aligned}
$$

and which satisfies,

$$
\begin{aligned}
&\widehat{\Phi}_n^{(0)}\left[Q_{n-1}^{(0)}(x;t)\right] \\
= \;& \left(\frac{\rho_{n-1,0}P_{n-1}(0;t)}{\beta_{n-1}}\left(4x^2\beta_{n-1}+\frac{\gamma}{2}\left(1+(-1)^n\right)+4\rho_{n-1,0}P_{n-1}(0;t)\left(x^2+\beta_{n-1}+\beta_{n-2}-\frac{t}{2}\right)\right)\right. \\
+ \;& 4x^2\left(x^2+\beta_n+\beta_{n-1}-\frac{t}{2}\right) \\
+ \;& \left.\frac{\rho_{n-1,0}P_{n-1}(0;t)}{\beta_{n-1}}\left(-1+4x^2\left(x^2+\beta_n+\beta_{n-2}-\frac{t}{2}\right)-4x^2\beta_{n-2}-\frac{\gamma}{2}\left(1-(-1)^n\right)\right)\right)Q_n^{(0)}(x;t).
\end{aligned}
$$

Remark 4.1 *In the previous examples we can give explicitly the coefficients of the second order differential equation that the polynomials $Q_n^{(0)}(x;t)$ satisfy. However, the expressions are very cumbersome, long, and can be deduced in an easy way from the expressions of $\Upsilon_{i,j}$ with $i,j\in\{1,2,3,4\}$ given in (36). Thus, we omit them.*

We could have shown examples where the point c is different from 0, but again the expressions are huge and they do not contribute to give more clarity to the theoretical results.

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