A differential equation for varying Krall-type orthogonal polynomials

Galina Filipuk and Juan F. Mañas-Mañas

Random Matrices: Theory and Applications

2020

A Differential Equation for Varying Krall–Type Orthogonal Polynomials

Galina Filipuk[a], Juan F. Mañas–Mañas[b].

[a]Faculty of Mathematics, Informatics and Mechanics, University of Warsaw, Banacha 2, Warsaw, 02-097, Poland.
[b]Departamento de Matemáticas, Universidad de Almería, Spain.

E-mail addresses: (filipuk@mimuw.edu.pl) G. Filipuk, (jmm939@ual.es) J.F. Mañas–Mañas.

Abstract

In this contribution we consider varying Krall–type polynomials which are orthogonal with respect to a varying discrete Krall–type inner product. Our main goal is to give ladder operators for this family of polynomials as well as to find a second–order differential–difference equation that these polynomials satisfy. We generalize some results appeared recently in the literature.

Keywords: Orthogonal polynomials; Ladder operators; Holonomic equation; Varying weights.
Mathematics Subject Classification (2010): Mathematics Subject Classification 2010: 33C47, 42C05, 34A05.

1 Introduction

In this paper we are interested to know some properties of the sequence of monic polynomials orthogonal with respect to the varying discrete inner product

$$(f,g)_{n,m} = \int f(x)g(x)d\mu + \sum_{i=1}^{m} M_{n,i}f(c_i)g(c_i), \tag{1}$$

where μ is a finite positive Borel measure supported on an infinite subset of the real line, $\{M_{n,i}\}_{i=1}^{m}$ are sequences of nonnegative real numbers, $c_i \in \mathbb{R}$ for $i \in \{1, 2, \ldots, m\}$ and $c_i \neq c_j$ if $i \neq j$.

Polynomials orthogonal with respect to a varying inner product have been considered in different frameworks, for instance, in general contexts related to different types of weights (see, among others, [2, 5, 16, 17, 20, 29, 30, 31, 36, 38]) or in Sobolev orthogonality (see, for instance, [1, 18, 34]).

When we consider a varying inner product, for every n, we have a square tableau of monic orthogonal polynomials with respect to (1), i.e., $\left\{ Q_k^{(M_{n,1},M_{n,2},\ldots,M_{n,m})}(x; c_1, c_2, \ldots, c_m) \right\}_{k \geq 0}$, but we

only deal with the diagonal of this tableau, $\left\{Q_n^{(M_{n,1},M_{n,2},\ldots,M_{n,m})}(x;c_1,c_2,\ldots,c_m)\right\}_{n\geq 0}$. To simplify the notation, we will denote them by $\{Q_n(x)\}_{n\geq 0}$ and name them as *varying Krall–type orthogonal polynomials*.

In the special cases when $m=1$ and $m=2$ we denote the sequences of monic orthogonal polynomials with respect to (1) as $\{Q_n(x;c_1)\}_{n\geq 0}$ and $\{Q_n(x;c_1,c_2)\}_{n\geq 0}$, respectively. We will give some examples in these relevant cases.

Also, we define $\{P_n\}_{n\geq 0}$ to be the sequence of monic polynomials orthogonal with respect to the inner product

$$(f,g)_\mu = \int f(x)g(x)d\mu, \tag{2}$$

thus, we have

$$(P_n, P_k)_\mu = \int P_n(x)P_k(x)d\mu = h_n\delta_{n,k}, \qquad n,k \in \mathbb{N}\cup\{0\}, \tag{3}$$

where $\delta_{n,k}$ denotes the Kronecker delta and h_n is the square of the norm of these polynomials.

Since the inner product (1) is standard, i.e., the property $(xf,g)_\mu = (f,xg)_\mu$ holds, then it is known that the sequence of monic orthogonal polynomials $\{P_n\}_{n\geq 0}$ satisfies a three-term recurrence relation of the following form:

$$xP_n(x) = P_{n+1}(x) + \alpha_n P_n(x) + \beta_n P_{n-1}(x), \qquad n \geq 0, \tag{4}$$

with initial conditions $\beta_0 P_{-1}(x) = 0$ and $P_0(x) = 1$. In addition, we have $\beta_n = h_n/h_{n-1}$ for $n \geq 1$. If the measure μ is symmetric, then it is well known that $\alpha_n = 0$ (see, for example [14]).

Our objective is to obtain ladder operators and a linear second–order differential equation, which we will call *holonomic equation*, for the sequence of monic polynomials $\{Q_n\}_{n\geq 0}$ orthogonal with respect to (1).

Ladder operators are important in the theory of orthogonal polynomials. On the one hand, they have a natural connection with the coefficients of the recurrence relation and from the point of view of physics they are related to the harmonic oscillator (see [40]). On the other hand, they are a very useful tool to construct differential equations whose solutions are the corresponding orthogonal polynomials. For these reasons, among others, the literature about ladder operators in different frameworks is very wide, we cite some of them [7, 8, 10, 11, 12, 24, 25, 26, 40]. Moreover, the ladder operators approach has been successfully applied to show the connections between the solutions of the Painlevé equations and recurrence coefficients of certain orthogonal polynomials (see, for instance, [4] or [11] and the references therein).

We assume that the sequence of monic polynomials $\{P_n\}_{n\geq 0}$ orthogonal with respect to (2) satisfies the following relation:

$$A(x)P_n'(x) = B_n(x)P_n(x) + C_n(x)P_{n-1}(x), \qquad n \geq 1, \tag{5}$$

2

where $A(x)$ is a polynomial and, $B_n(x)$ and $C_n(x)$ are certain functions. It is important to remark that the relation (5) is very general, and as we will see later in the examples, all the classic families as Jacobi, Laguerre, Hermite or semi-classical families, for example the generalized Freud orthogonal polynomials (see for instance [15]) or the generalized Jacobi orthogonal polynomials (see for example [19]) or in more general framework (see [8]), satisfy this relation (5).

We define the lowering operator from (5) by

$$\Psi_n := A(x)\frac{d}{dx} - B_n(x), \qquad n \geq 1, \tag{6}$$

hence

$$\Psi_n[P_n(x)] = C_n(x)P_{n-1}(x), \qquad n \geq 1. \tag{7}$$

We can obtain the raising operator. Indeed, using (4) we have:

$$P_{n-2}(x) = \frac{(x - \alpha_{n-1})P_{n-1}(x) - P_n(x)}{\beta_{n-1}}, \qquad n \geq 2, \tag{8}$$

so, by changing $n \to n - 1$ in (5) and inserting the previous expression (8), we get

$$
\begin{aligned}
& A(x)P'_{n-1}(x) \\
&= B_{n-1}(x)P_{n-1}(x) + C_{n-1}(x)P_{n-2}(x) \\
&= \left(B_{n-1}(x) + \frac{C_{n-1}(x)(x - \alpha_{n-1})}{\beta_{n-1}}\right)P_{n-1}(x) - \frac{C_{n-1}(x)}{\beta_{n-1}}P_n(x).
\end{aligned}
\tag{9}
$$

Similarly to the previous case, from (9) we define the raising operator by

$$\Phi_n := B_{n-1}(x) + \frac{C_{n-1}(x)(x - \alpha_{n-1})}{\beta_{n-1}} - A(x)\frac{d}{dx}, \qquad n \geq 2,$$

hence,

$$\Phi_n[P_{n-1}(x)] = \frac{C_{n-1}(x)}{\beta_{n-1}}P_n(x), \qquad n \geq 2. \tag{10}$$

Now, we can obtain the holonomic equation for the sequence of monic polynomials $\{P_n\}_{n\geq 0}$ orthogonal with respect to (2). We get

$$\rho_{n,1}(x)P''_n(x) + \rho_{n,2}(x)P'_n(x) + \rho_{n,3}P_n(x) = 0, \qquad n \geq 2, \tag{11}$$

where

$$\rho_{n,1}(x) = \beta_{n-1}A^2(x)C_n(x),$$

$$\rho_{n,2}(x) = \beta_{n-1}A(x)\Big(A'(x)C_n(x) - A(x)C_n'(x) - B_n(x)C_n(x)\Big)$$
$$-A(x)C_n(x)\Big(\beta_{n-1}B_{n-1}(x) + C_{n-1}(x)(x-\alpha_{n-1})\Big),$$

$$\rho_{n,3}(x) = C_n(x)\Big(C_{n-1}(x)C_n(x) + B_n(x)\big(\beta_{n-1}B_{n-1}(x) + C_{n-1}(x)(x-\alpha_{n-1})\big)\Big)$$
$$+\beta_{n-1}A(x)\Big(B_n(x)C_n'(x) - B_n'(x)C_n(x)\Big).$$

In order to obtain this result, we follow the typical proof (see, for example, [3, 8, 9, 13, 20, 24, 26] and the references therein). From raising operator we have that

$$\Phi_n[P_{n-1}(x)] = \frac{C_{n-1}(x)}{\beta_{n-1}}P_n(x).$$

Then, using (7), we get

$$\Phi_n[P_{n-1}(x)]$$
$$=\Phi_n\left[\frac{1}{C_n(x)}\Psi_n[P_n(x)]\right]$$
$$=\left(B_{n-1}(x) + \frac{C_{n-1}(x)(x-\alpha_{n-1})}{\beta_{n-1}} - A(x)\frac{d}{dx}\right)\left(\frac{1}{C_n(x)}\Psi_n[P_n(x)]\right)$$
$$=\left(B_{n-1}(x) + \frac{C_{n-1}(x)(x-\alpha_{n-1})}{\beta_{n-1}}\right)\left(\frac{1}{C_n(x)}\left(A(x)\frac{d}{dx} - B_n(x)\right)P_n(x)\right)$$
$$\quad - A(x)\left(\frac{1}{C_n(x)}\left(A(x)\frac{d}{dx} - B_n(x)\right)P_n(x)\right)'$$
$$=\left(B_{n-1}(x) + \frac{C_{n-1}(x)(x-\alpha_{n-1})}{\beta_{n-1}}\right)\frac{A(x)P_n'(x) - B_n(x)P_n(x)}{C_n(x)}$$
$$\quad - A(x)\frac{(A'(x)P_n'(x) + A(x)P_n''(x))C_n(x) - A(x)P_n'(x)C_n'(x)}{C_n^2(x)}$$
$$\quad + A(x)\frac{(B_n'(x)P_n(x) + B_n(x)P_n'(x))C_n(x) - B_n(x)P_n(x)C_n'(x)}{C_n^2(x)}.$$

Now, equating with the expression on the right side of (10) and multiplying everything by $\beta_{n-1}C_n^2(x)$, we obtain the required second order ordinary differential equation (11).

4

We will follow a similar technique to obtain the holonomic equation for the polynomials $Q_n(x)$ from the corresponding ladder operators. This paper is structured as follows. In Section 2 we obtain some connection formulas that are satisfied by the polynomials $Q_n(x)$. In Section 3 we use a standard technique to obtain the algebraic relations between the polynomials $Q_n(x)$, their first derivatives and polynomials $P_n(x)$. These relations are essential to tackle our main objective in Section 4 which is devoted to the derivation of the ladder operators and a second–order differential equation for the orthogonal polynomials $Q_n(x)$. Finally, we give explicit examples.

2   Connection formula for $\mathbf{Q_n(x)}$

In this section we obtain a connection formula between the polynomials $\{Q_n\}_{n\geq0}$, orthogonal with respect to (1), and the sequence of polynomials $\{P_n\}_{n\geq0}$, orthogonal with respect to (2). We follow a similar technique as in [23, Sect. 2] or [35, Sect. 2].

It is well known that the sequence of polynomials $\{P_n\}_{n\geq0}$ orthogonal with respect to (2) forms a basis of the linear space $\mathbb{P}_n[x]$ of polynomials with real coefficients of degree at most n. So, we can write

$$Q_n(x) = \sum_{k=0}^{n} a_{n,k} P_k(x).$$

The coefficient $a_{n,n} = 1$ because $Q_n(x)$ and $P_n(x)$ are monic polynomials. For $0 \leq k \leq n-1$, we use the orthogonality of $Q_n(x)$ and $P_n(x)$, and we get

$$
\begin{aligned}
0 &= (Q_n(x), P_k(x))_{n,m} \\
&= \left(\sum_{j=0}^{n} a_{n,j} P_j(x), P_k(x) \right)_{n,m} \\
&= \sum_{j=0}^{n} a_{n,j} \left(P_j(x), P_k(x) \right)_{\mu} + \sum_{i=1}^{m} M_{n,i} Q_n(c_i) P_k(c_i) \\
&= a_{n,k} h_k + \sum_{i=1}^{m} M_{n,i} Q_n(c_i) P_k(c_i).
\end{aligned}
$$

Thus,

$$a_{n,k} = - \frac{\sum_{i=1}^{m} M_{n,i} Q_n(c_i) P_k(c_i)}{h_k}. \tag{12}$$

Finally, we define the kernel polynomials by

$$K_n(x, y) = \sum_{i=0}^{n} \frac{P_i(x)P_i(y)}{h_i}. \tag{13}$$

It is well established that the kernel polynomials (13) satisfy the Christoffel–Darboux formula (see, for example, [14] or [39])

$$K_n(x, y) = \frac{1}{h_n} \frac{P_{n+1}(x)P_n(y) - P_n(x)P_{n+1}(y)}{x - y}, \tag{14}$$

and its confluent form

$$K_n(x, x) = \frac{1}{h_n} \left(P'_{n+1}(x)P_n(x) - P'_n(x)P_{n+1}(x) \right).$$

Using (12), (13), we get

$$
\begin{aligned}
Q_n(x) &= P_n(x) - \sum_{k=0}^{n-1} \frac{\sum_{i=1}^{m} M_{n,i}Q_n(c_i)P_k(c_i)}{h_k} P_k(x) \\
&= P_n(x) - \sum_{i=1}^{m} M_{n,i}Q_n(c_i) \sum_{k=0}^{n-1} \frac{P_k(c_i)P_k(x)}{h_k} \\
&= P_n(x) - \sum_{i=1}^{m} M_{n,i}Q_n(c_i)K_{n-1}(x, c_i). \tag{15}
\end{aligned}
$$

In order to obtain the value of $Q_n(c_k)$ with $k \in \{1, 2, \ldots, m\}$, we substitute $x = c_k$ into (15) and we write the following system of m linear equations with m unknowns. For $1 \leq k \leq m$, we have

$$P_n(c_k) = (1 + M_{n,k}K_{n-1}(c_k, c_k)) Q_n(c_k) + \sum_{\substack{i=1 \\ i \neq k}}^{m} M_{n,i}Q_n(c_i)K_{n-1}(c_k, c_i).$$

We follow the notation of [23] and get

$$\mathcal{P}_{n,m} = \mathcal{A}_{n,m}\mathcal{X}_{n,m},$$

where

$$
\mathcal{A}_{n,m} = \begin{pmatrix}
1 + M_{n,1}K_{n-1}(c_1, c_1) & M_{n,2}K_{n-1}(c_1, c_2) & \ldots & M_{n,m}K_{n-1}(c_1, c_m) \\
M_{n,1}K_{n-1}(c_2, c_1) & 1 + M_{n,2}K_{n-1}(c_2, c_2) & \ldots & M_{n,m}K_{n-1}(c_2, c_m) \\
\vdots & \vdots & \ddots & \vdots \\
M_{n,1}K_{n-1}(c_m, c_1) & M_{n,2}K_{n-1}(c_m, c_2) & \ldots & 1 + M_{n,m}K_{n-1}(c_m, c_m)
\end{pmatrix},
$$

6

$$\mathcal{P}_{n,m} = \begin{pmatrix} P_n(c_1) \\ P_n(c_2) \\ \vdots \\ P_n(c_m) \end{pmatrix} \quad \text{and} \quad \mathcal{X}_{n,m} = \begin{pmatrix} Q_n(c_1) \\ Q_n(c_2) \\ \vdots \\ Q_n(c_m) \end{pmatrix}.$$

Using Cramer's rule we can obtain the values of $Q_n(c_i)$ for all $i \in \{1, 2, \ldots, m\}$.

Corollary 2.1 *In the particular cases when $m = 1$ and $m = 2$, we have*

$$Q_n(x; c_1) = P_n(x) - \frac{M_{n,1} P_n(c_1)}{1 + M_{n,1} K_{n-1}(c_1, c_1)} K_{n-1}(x, c_1), \tag{16}$$

$$Q_n(x; c_1, c_2) = P_n(x) - M_{n,1} Q_n(c_1; c_1, c_2) K_{n-1}(x, c_1) - M_{n,2} Q_n(c_2; c_1, c_2) K_{n-1}(x, c_2),$$

where

$$Q_n(c_1; c_1, c_2) = \frac{P_n(c_1)(1 + M_{n,2} K_{n-1}(c_2, c_2)) - P_n(c_2) M_{n,2} K_{n-1}(c_1, c_2)}{\det(\mathcal{A}_{n,2})}, \tag{17}$$

$$Q_n(c_2; c_1, c_2) = \frac{P_n(c_2)(1 + M_{n,1} K_{n-1}(c_1, c_1)) - P_n(c_1) M_{n,1} K_{n-1}(c_1, c_2)}{\det(\mathcal{A}_{n,2})}, \tag{18}$$

with

$$\begin{aligned} \det(\mathcal{A}_{n,2}) &= 1 + M_{n,1} K_{n-1}(c_1, c_1) + M_{n,2} K_{n-1}(c_2, c_2) \\ &\quad + M_{n,1} M_{n,2} \big(K_{n-1}(c_1, c_1) K_{n-1}(c_2, c_2) - K_{n-1}^2(c_1, c_2) \big). \end{aligned}$$

3 Relations between $\mathbf{Q_n(x)}$ and $\mathbf{P_n(x)}$

In order to obtain the holonomic equation for $Q_n(x)$, we need several technical relations between $Q_n(x)$ and $P_n(x)$. We have introduced a perturbation in the inner product (2) adding masses located at the points c_i, with $1 \leq i \leq m$. Then, the polynomial $(x - c_1)(x - c_2) \ldots (x - c_m)$ can be used to *eliminate* this perturbation. The following results show how this polynomial helps us to obtain a useful relation between both families of orthogonal polynomials $P_n(x)$ and $Q_n(x)$. We follow a similar ideas as in [3], [20] or [23].

Lemma 3.1 *For the sequences $\{Q_n\}_{n \geq 0}$ and $\{P_n\}_{n \geq 0}$ we get*

$$\begin{aligned} w_m(x) Q_n(x) &= f_1(x, n) P_n(x) + g_1(x, n) P_{n-1}(x), & n \geq 2, & \tag{19} \\ A(x) \left(w_m(x) Q_n(x) \right)' &= f_2(x, n) P_n(x) + g_2(x, n) P_{n-1}(x), & n \geq 2, & \tag{20} \end{aligned}$$

where

$$w_m(x) = \prod_{i=1}^{m}(x - c_i),$$

$$w_{m,k}(x) = \begin{cases} 1, & \text{if } m = 1, \\ \displaystyle\prod_{\substack{i=1 \\ i \neq k}}^{m}(x - c_i), & \text{if } m > 1, \end{cases}$$

$$f_1(x, n) = w_m(x) - \sum_{i=1}^{m} \frac{M_{n,i}Q_n(c_i)P_{n-1}(c_i)w_{m,i}(x)}{h_{n-1}},$$

$$g_1(x, n) = \sum_{i=1}^{m} \frac{M_{n,i}Q_n(c_i)P_n(c_i)w_{m,i}(x)}{h_{n-1}},$$

$$f_2(x, n) = f_1'(x, n)A(x) + f_1(x, n)B_n(x) - g_1(x, n)\frac{C_{n-1}(x)}{\beta_{n-1}},$$

$$g_2(x, n) = f_1(x, n)C_n(x) + g_1'(x, n)A(x) + g_1(x, n)\left(B_{n-1}(x) + \frac{C_{n-1}(x)(x - \alpha_{n-1})}{\beta_{n-1}}\right),$$

with $A(x)$, $B_n(x)$ and $C_n(x)$ defined in (5).

Proof: To prove (19), in (15) we use (14) and multiply by $w_n(x)$:

$$\begin{aligned} Q_n(x) &= P_n(x) - \sum_{i=1}^{m} M_{n,i}Q_n(c_i)K_{n-1}(x, c_i) \\ &= P_n(x) - \sum_{i=1}^{m} M_{n,i}Q_n(c_i)\frac{1}{h_{n-1}}\frac{P_n(x)P_{n-1}(c_i) - P_n(c_i)P_{n-1}(x)}{x - c_i} \\ &= \left(1 - \sum_{i=1}^{m} \frac{M_{n,i}Q_n(c_i)P_{n-1}(c_i)}{h_{n-1}(x - c_i)}\right)P_n(x) + \left(\sum_{i=1}^{m} \frac{M_{n,i}Q_n(c_i)P_n(c_i)}{h_{n-1}(x - c_i)}\right)P_{n-1}(x). \end{aligned}$$

Multiplying by already mentioned $w_n(x)$ we obtain the result. To prove (20), we differentiate (19):

$$(w_m(x)Q_n(x))' = f_1'(x, n)P_n(x) + f_1(x, n)P_n'(x) + g_1'(x, n)P_{n-1}(x) + g_1(x, n)P_{n-1}'(x).$$

Now, we multiply by $A(x)$ in the previous expression and use (5) and (9) to get the result. □

Lemma 3.2 *For the sequences of monic polynomials $\{Q_n\}_{n\geq 0}$ and $\{P_n\}_{n\geq 0}$ we have*

$$w_m(x)Q_{n-1}(x) = f_3(x,n)P_n(x) + g_3(x,n)P_{n-1}(x), \qquad n \geq 3, \tag{21}$$

$$A(x)\left(w_m(x)Q_{n-1}(x)\right)' = f_4(x,n)P_n(x) + g_4(x,n)P_{n-1}(x), \qquad n \geq 3, \tag{22}$$

where

$$f_3(x,n) = \frac{-g_1(x,n-1)}{\beta_{n-1}},$$

$$g_3(x,n) = f_1(x,n-1) + g_1(x,n-1)\frac{x-\alpha_{n-1}}{\beta_{n-1}},$$

$$f_4(x,n) = \frac{-g_2(x,n-1)}{\beta_{n-1}},$$

$$g_4(x,n) = f_2(x,n-1) + g_2(x,n-1)\frac{x-\alpha_{n-1}}{\beta_{n-1}},$$

with $A(x)$, $B_n(x)$ and $C_n(x)$ defined in (5), and $f_1(x,n)$, $g_1(x,n)$, $f_2(x,n)$ and $g_2(x,n)$ defined in Lemma 3.1.

Proof: To prove (21) and (22) it is only necessary to change $n \to n-1$ in (19) and (20) and apply (8). □

Lemma 3.3 *For the sequences of monic polynomials $\{Q_n\}_{n\geq 0}$ and $\{P_n\}_{n\geq 0}$ we have*

$$P_n(x) = \frac{w_m(x)}{\Delta(x;n)}\left(g_3(x,n)Q_n(x) - g_1(x,n)Q_{n-1}(x)\right), \qquad n \geq 3, \tag{23}$$

$$P_{n-1}(x) = \frac{w_m(x)}{\Delta(x;n)}\left(-f_3(x,n)Q_n(x) + f_1(x,n)Q_{n-1}(x)\right), \qquad n \geq 3, \tag{24}$$

where $\Delta(x,n) = f_1(x,n)g_3(x,n) - g_1(x,n)f_3(x,n)$, and $f_1(x,n)$, $g_1(x,n)$, $f_3(x,n)$ and $g_3(x,n)$ are defined in Lemma 3.1 and Lemma 3.2.

Proof: From (19) and (21) we have a system of two linear equations with two unknowns $P_n(x)$ and $P_{n-1}(x)$, and the lemma follows by applying Cramer's rule. □

4 Ladder operators and holonomic equation

In this section we obtain the second–order linear differential equation satisfied by the varying Krall–type orthogonal polynomials. The first step is to obtain the ladder operators for this family of polynomials. Lemmas 3.1–3.3 obtained in the previous section are the key to deduce these ladder operators.

Theorem 4.1 *Let $\{Q_n\}_{n=0}^{\infty}$ be the sequence of monic polynomials orthogonal with respect to (1). Then, there exit a lowering differential operator, $\mathcal{L}_n$, and a raising differential operator, $\mathcal{R}_n$, defined by*

$$\mathcal{L}_n \quad := \quad \Psi_1(x,n) + \Phi(x,n)\frac{d}{dx}, \qquad n \geq 3,$$

$$\mathcal{R}_n \quad := \quad \Psi_3(x,n) + \Phi(x,n)\frac{d}{dx}, \qquad n \geq 3,$$

and satisfying

$$\mathcal{L}_n\left[Q_n(x)\right] \quad = \quad \Psi_2(x,n)Q_{n-1}(x), \qquad n \geq 3, \tag{25}$$

$$\mathcal{R}_n\left[Q_{n-1}(x)\right] \quad = \quad \Psi_4(x,n)Q_n(x), \qquad n \geq 3, \tag{26}$$

with

$$\Phi(x,n) \quad = \quad A(x)w_m(x)\Delta(x,n),$$

$$\Psi_1(x,n) \quad = \quad A(x)w'_m(x)\Delta(x,n) + w_m(x)(f_3(x,n)g_2(x,n) - f_2(x,n)g_3(x,n)),$$

$$\Psi_2(x,n) \quad = \quad w_m(x)(f_1(x,n)g_2(x,n) - g_1(x,n)f_2(x,n)),$$

$$\Psi_3(x,n) \quad = \quad A(x)w'_m(x)\Delta(x,n) + w_m(x)(f_4(x,n)g_1(x,n) - f_1(x,n)g_4(x,n)),$$

$$\Psi_4(x,n) \quad = \quad w_m(x)(f_4(x,n)g_3(x,n) - g_4(x,n)f_3(x,n)).$$

Proof: Replacing (23)–(24) in (20) and (22) we obtain the result. □

Theorem 4.2 *(**Holonomic equation**) The varying Krall–type orthogonal polynomial satisfy the following second–order linear differential equation:*

$$\sigma_{n,1}(x,n)Q''_n(x) + \sigma_{n,2}(x,n)Q'_n(x) + \sigma_{n,3}(x,n)Q_n(x) = 0, \qquad n \geq 3, \tag{27}$$

with

$$\sigma_{n,1}(x,n) \quad = \quad \Psi_2(x,n)\Phi^2(x,n),$$

$$\sigma_{n,2}(x,n) \quad = \quad \Psi_2(x,n)\Phi(x,n)\left(\Psi_3(x,n) + \Psi_1(x,n) + \Phi'(x,n)\right) - \Phi^2(x,n)\Psi'_2(x,n),$$

$$\sigma_{n,3}(x,n) \quad = \quad \Psi_2(x,n)\left(\Psi_1(x,n)\Psi_3(x,n) + \Psi'_1(x,n)\Phi(x,n) - \Psi_2(x,n)\Psi_4(x,n)\right)$$
$$\qquad\qquad -\Psi_1(x,n)\Psi'_2(x,n)\Phi(x,n).$$

Proof: The proof is straightforward once we know the corresponding ladder operators given in Theorem 4.1. We follow similar steps that we used to prove (11).

On the one hand, from (26) we have $\mathcal{R}_n\left[Q_{n-1}(x)\right] = \Psi_4(x,n)Q_n(x)$. On the other hand, from (25), we have

10

$$\mathcal{R}_n\left[Q_{n-1}(x)\right]$$

$$= \mathcal{R}_n\left[\frac{\mathcal{L}_n\left[Q_n(x)\right]}{\Psi_2(x,n)}\right]$$

$$= \mathcal{R}_n\left[\frac{\Psi_1(x,n)Q_n(x)+\Phi(x,n)Q'_n(x)}{\Psi_2(x,n)}\right]$$

$$= \mathcal{R}_n\left[\frac{\Psi_1(x,n)Q_n(x)}{\Psi_2(x,n)}\right]+\mathcal{R}_n\left[\frac{\Phi(x,n)Q'_n(x)}{\Psi_2(x,n)}\right]$$

$$= \frac{\Psi_3(x,n)\Psi_1(x,n)Q_n(x)}{\Psi_2(x,n)}$$

$$+\Phi(x,n)\frac{\Psi_2(x,n)\left(\Psi'_1(x,n)Q_n(x)+\Psi_1(x,n)Q'_n(x)\right)-\Psi'_2(x,n)\Psi_1(x,n)Q_n(x)}{\Psi_2^2(x,n)}$$

$$+\frac{\Psi_3(x,n)\Phi(x,n)Q'_n(x)}{\Psi_2(x,n)}$$

$$+\Phi(x,n)\frac{\Psi_2(x,n)\left(\Phi'(x,n)Q'_n(x)+\Phi(x,n)Q''_n(x)\right)-\Psi'_2(x,n)\Phi(x,n)Q'_n(x)}{\Psi_2^2(x,n)}.$$

Finally, equating this with $\Psi_4(x,n)Q_n(x)$ and multiplying by $\Psi_2^2(x,n)$ we obtain the result.
□

Remark 4.1 *It is important to remark that all functions $\Phi(x,n)$, $\Delta(x,n)$, $f_i(x,n)$, $g_i(x,n)$ and $\Psi_i(x,n)$, with $1 \leq i \leq 4$, that were defined in Lemmas 3.1–3.3 and Theorems 4.1–4.2, are given explicitly and, hence, they can easily be calculated using any computer algebra system. Thus, knowing the functions $A(x)$, $B_n(x)$ and $C_n(x)$ in the expression (5) we can deduce the lowering and raising operators (see (25) and (26), respectively) as well as the holonomic equation (27).*

4.1   Example 1

We consider the inner product (1) with $m = 1$ and involving the Jacobi measure, i.e., $\mu(x) = (1 - x)^\alpha (1 + x)^\beta$ with $\alpha, \beta > -1$. We take $\{M_{n,1}\}_{n \geq 0}$ a sequence of nonnegative real numbers and $c_1 = 1$. Then, the inner product (1) is transformed into

$$(f, g)_{n,1} = \int_{-1}^{1} f(x)g(x)(1 - x)^\alpha (1 + x)^\beta dx + M_{n,1}f(1)g(1). \tag{28}$$

This inner product was used, for example, in [37] or in more general framework in [33], where the authors obtained asymptotic properties of polynomials $\left\{Q_n^{(\alpha,\beta)}(x;1)\right\}_{n \geq 0}$ orthogonal with respect to (28).

We denote by $\left\{P_n^{(\alpha,\beta)}(x)\right\}_{n\geq 0}$ the sequence of monic Jacobi polynomials orthogonal with respect to the measure $\mu(x) = (1-x)^\alpha(1+x)^\beta$ and we denote by $\left\{Q_n^{(\alpha,\beta)}(x;1)\right\}_{n\geq 0}$ the sequence of monic polynomials ortogonal with respect to (28).

The sequence of monic Jacobi orthogonal polynomials satisfies (4) with (see, for example, [14, p. 211] or [9, f. (2.15)] and [9, f. (2.18)])

$$\alpha_n = \frac{\beta^2 - \alpha^2}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)}, \tag{29}$$

$$\beta_n = \frac{4n(n+\alpha)(n+\beta)(n+\alpha+\beta)}{(2n+\alpha+\beta)^2(2n+\alpha+\beta+1)(2n+\alpha+\beta-1)}. \tag{30}$$

The norm is deduced using (3) and (4). We have

$$h_n = h_0\beta_1\beta_2\ldots\beta_n,$$

and using that

$$h_0 = 2^{\alpha+\beta+1}\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)},$$

we get

$$h_n = \frac{2^{2n+\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)\Gamma(n+\alpha+\beta+1)\Gamma(n+1)}{\Gamma(2n+\alpha+\beta+1)\Gamma(2n+\alpha+\beta+2)}. \tag{31}$$

The value at $x = 1$ is given by (see, for example, [9, f. (3.3)])

$$P_n^{(\alpha,\beta)}(1) = 2^n\frac{\Gamma(n+\alpha+1)\Gamma(n+\alpha+\beta+1)}{\Gamma(\alpha+1)\Gamma(2n+\alpha+\beta+1)}. \tag{32}$$

Also, using [9, Sect. 3], we can deduce that $P_n^{(\alpha,\beta)}(x)$ satisfies (5) with

$$A(x) = x^2 - 1, \tag{33}$$

$$B_n(x) = (x+1)n + 2r_n, \tag{34}$$

$$C_n(x) = -2\beta_n R_n, \tag{35}$$

where the quantities r_n and R_n are given by (see [9, f. (2.9)] and [9, f. (2.12)])

$$r_n = r_n(\alpha,\beta) := \frac{\beta - \alpha - 2n - (\alpha+\beta+2n+2)\alpha_n}{4}, \tag{36}$$

$$R_n = R_n(\alpha,\beta) := \frac{2n+\alpha+\beta+1}{2}. \tag{37}$$

12

With the expressions (29)–(37) we can obtain the ladder operators and the holonomic equation for the polynomials $Q_n^{(\alpha,\beta)}(x;1)$. It is only important to remark that the quantity $Q_n^{(\alpha,\beta)}(1;1)$ can be obtained from (16).

If we take $c = -1$ in the inner product (28) we can obtain similar results using that (see, for example, [39, f. (4.1.3)]) $P_n^{(\alpha,\beta)}(x) = (-1)^n P_n^{(\beta,\alpha)}(-x)$, so we deduce that

$$P_n^{(\alpha,\beta)}(-1) = (-1)^n 2^n \frac{\Gamma(n+\beta+1)\Gamma(n+\alpha+\beta+1)}{\Gamma(\beta+1)\Gamma(2n+\alpha+\beta+1)}. \tag{38}$$

4.2   Example 2

We consider the inner product (1) with $m = 2$ and involving the Jacobi measure, i.e., $\mu(x) = (1-x)^\alpha(1+x)^\beta$ with $\alpha, \beta > -1$. We take $\{M_{n,1}\}_{n\geq0}$ and $\{M_{n,2}\}_{n\geq0}$ sequences of nonnegative real numbers and $c_1 = -1$ and $c_2 = 1$. Then, the inner product (1) is transformed into

$$(f,g)_{n,2} = \int_{-1}^{1} f(x)g(x)(1-x)^\alpha(1+x)^\beta dx + M_{n,1}f(-1)g(-1) + M_{n,2}f(1)g(1). \tag{39}$$

This inner product (39) was used, for example in [21], [27] or [28], in the nonvarying case (when $M_{n,1}$ and $M_{n,2}$ are constant) where the authors studied another properties. Using (29)–(38) we get the holonomic equation for the orthogonal polynomials with respect to (39). If we denote by $\left\{Q_n^{(\alpha,\beta)}(x;1,-1)\right\}_{n\geq0}$ the sequence of monic polynomials orthogonal with respect to (39), it is important to remark that the quantities $Q_n^{(\alpha,\beta)}(1;1,-1)$ and $Q_n^{(\alpha,\beta)}(-1;1,-1)$ are given by (17) and (18), respectively.

4.3   Example 3

Another relevant case happens when we introduce the Laguerre measure, i.e., $\mu(x) = x^\alpha e^{-x}$ with $\alpha > -1$. Thus, the inner product (1) is transformed into

$$(f,g)_{n,m} = \int_0^{+\infty} f(x)g(x)x^\alpha e^{-x}dx + \sum_{i=1}^{m} M_{n,i}f(c_i)g(c_i). \tag{40}$$

The particular case with $c = 0$ and $m = 1$ was introduced in [6] and more generally in [32] where the authors obtained asymptotic properties of orthogonal polynomials with respect to (40) and properties about their zeros. It is important to remark that the holonomic equation for the orthogonal polynomials with respect to (40) was obtained in [22] in the constant case.

In order to obtain the holonomic equation and ladder operators, it is necessary to use the following properties (see, for example [14] or [39, Ch. 5]).

We denote by $\left\{ L_n^{(\alpha)}(x) \right\}_{n \geq 0}$ the sequence of monic classical Laguerre orthogonal polynomials. Then, $L_n^{(\alpha)}(x)$ satisfy (4) with

$$
\begin{aligned}
\alpha_n &= 2n + \alpha + 1, & (41)\\
\beta_n &= n(n + \alpha). & (42)
\end{aligned}
$$

The value in $x = 0$ are given by

$$
L_n^{(\alpha)}(0) = \frac{(-1)^n \Gamma(n + \alpha + 1)}{\Gamma(\alpha + 1)}. \tag{43}
$$

The norm is

$$
h_n = \Gamma(n + 1)\Gamma(n + \alpha + 1). \tag{44}
$$

The classical Laguerre polynomials $L_n^{(\alpha)}$ also satisfy (6) with

$$
\begin{aligned}
A(x) &= x, & (45)\\
B_n(x) &= n, & (46)\\
C_n(x) &= n(n + \alpha). & (47)
\end{aligned}
$$

With the expressions (41)–(47) and applying Lemmas 3.1–3.3 and Theorems 4.1–4.2 we can deduce explicitly the ladders operators and the holonomic equation for the orthogonal polynomials with respect to the inner product (40).

Acknowledgements

References

[1] M. Alfaro, A. Martínez–Finkelshtein, M. L. Rezola, *Asymptotic properties of balanced extremal Sobolev polynomials: coherent case*, J. Approx. Theory **100** (1) (1999), 44–59.

[2] A. I. APTEKAREV, J. S. GERONIMO, W. VAN ASSCHE, *Varying weights for orthogonal polynomials with monotonically varying recurrence coefficients*, J. Approx. Theory **150** (2) (2008), 214–238.

[3] A. ARCEO, E. J. HUERTAS-CEJUDO, F. MARCELLÁN, *On polynomials associated with an Uvarov modification of a quartic potential Freud–like weight*, Appl. Math. Comput. **281** (2016), 102–120.

[4] E. BASOR, Y. CHEN, T. EHRHARDT, *Painlevé V and time–dependent Jacobi polynomials*, J. Phys. A **43** (2010), 15204 (25pp.).

[5] C. BOSBACH, W. GAWRONSKI, *Strong asymptotics for Jacobi polynomials with varying weights*, Methods Appl. Anal. **6** (1) (1999), 39–54.

[6] L. CASTAÑO–GARCÍA, J. J. MORENO–BALCÁZAR, *Zeros of varying Laguerre–Krall orthogonal polynomials*, Proc. Amer. Math. Soc. **141** (6) (2013), 2051–2060.

[7] Y. CHEN, J. GRIFFIN, *Krall–type polynomials via the Heine formula*, J. Phys. A: Math. Gen. **35** (2002), 637–656.

[8] Y. CHEN, M. E. H. ISMAIL, *Ladder operators and differential equations for orthogonal polynomials*, J. Phys. A: Math. Gen. **30** (1997), 7817–7829.

[9] Y. CHEN, M. E. H. ISMAIL, *Jacobi polynomials from compatibility conditions*, Proc. Amer. Math. Soc. **133** (2) (2004), 465–472.

[10] Y. CHEN, M. E. H. ISMAIL, *Ladder operators for q-orthogonal polynomials*, J. Math. Anal. Appl. **345** (2008), 1–10.

[11] Y. CHEN, A. ITS, *Painlevé III and a singular linear statistics in Hermitian random matrix ensembles, I*, J. Approx. Theory **162** (2010), 270–297.

[12] Y. CHEN, G. PRUESSNER, *Orthogonal polynomials with discontinuous weights*, J. Phys. A: Math. Gen. **38** (2005), L191–L198.

[13] Y. CHEN, L. ZHANG, *Painlevé VI and the unitary Jacobi ensembles*, Stud. Appl. Math. **125** (1998), 91–112.

[14] T. S. CHIHARA, An Introduction to Orthogonal Polynomials, Gordon and Breach, New York, 1978.

[15] P. A. CLARKSON, K. JORDAAN, A. KELIL, *A generalized Freud weight*, Stud. Appl. Math. **136** (2016), 288–320.

[16] P. Deift, T. Kriecherbauer, K. T.-R. McLaughlin, S. Venakides, X. Zhou, *Asymptotics for polynomials orthogonal with respect to varying exponential weights*, Internat. Math. Res. Notices **16** (1997), 759–782.

[17] P. Deift, T. Kriecherbauer, K. T.-R. McLaughlin, S. Venakides, X. Zhou, *Uniform asymptotics for polynomials orthogonal with respect to varying exponential weights and applications to universality questions in random matrix theory*, Commun. Pure Appl. Math. **52** (11) (1999), 1335–1425.

[18] C. Díaz Mendoza, R. Orive, H. Pijeira Cabrera, *Asymptotically extremal polynomials with respect to varying weights and application to Sobolev orthogonality*, J. Math. Anal. Appl. **346** (2008), 480–488.

[19] B. X. Fejzullahu, *Mehler–Heine formulas for orthogonal polynomials with respect to the modified Jacobi weight*, Proc. Amer. Math. Soc. **142** (6) (2014), 2035–2045.

[20] G. Filipuk, J. F. Mañas–Mañas, J. J. Moreno–Balcázar, *Ladder operators and a differential equation for varying generalized Freud–type orthogonal polynomials*, Random Matrices Theory Appl. **7** (4) (2018), 1840005 (28pp.).

[21] A. Foulquié–Moreno, F. Marcellán, B. P. Osilenker, *Estimates for polynomials orthogonal with respect to some Gegenbauer–Sobolev type inner product*, J. Inequal. Appl. **3** (4) (1999), 401–419.

[22] C. Hermoso, E. J. Huertas-Cejudo, A. Lastra, Determinantal form for ladder operators in a problem concerning a convex linear combination of discrete and continuous measures in *Formal and analytic solutions of diff. equations*, 263—274, Springer Proc. Math. Stat. **256**, Springer, Cham, 2018.

[23] E. J. Huertas-Cejudo, F. Marcellán, H. Pijeira-Cabrera, *An electrostatic model for zeros of perturbed Laguerre polynomials*, Proc. Amer. Math. Soc. **142** (5) (2014), 1733–1747.

[24] M. E. H. Ismail, Classical and Quantum Orthogonal Polynomials in one Variable. Enciclopedia of Mathematics and its Applications Vol. **98** Cambridge University Press, Cambridge U.K. 2005.

[25] M. E. H. Ismail, I. Nikolova, P. Simeonov, *On difference equations and discriminants for discrete orthogonal polynomials*, Ramanujan J. **8** (2004), 475–502.

[26] M. E. H. Ismail, J. Wimp, *On differential equations for orthogonal polynomials*, Methods Appl. Anal. **5** (4) (1998), 439–452.

[27] T. H. Koornwinder, *Orthogonal polynomials with weight functions* $(1-x)^{\alpha}(1+x)^{\beta}+M\delta(x+1)+N\delta(x-1)$, Canad. Math. Bull. **27** (2) (1984), 205–214.

[28] A. M. KRALL, *Orthogonal polynomials satisfying fourth order differential equations*, Proc. Roy. Soc. Edinburgh Sect. A **87** (3-4) (1980/81), 271–288.

[29] A. B. J. KUIJLAARS, P. M. J. TIBBOEL, *The asymptotic behaviour of recurrence coefficients for orthogonal polynomials with varying exponential weights*, J. Comput. Appl. Math. **233** (2009), 775–785.

[30] E. LEVIN, D. S. LUBINSKY, *Bounds and Asymptotics for Orthogonal Polynomials for Varying Weights*, Springer Briefs in Mathematics, Springer, Cham (Switzerland), 2018.

[31] K. T.-R. MCLAUGHLIN, P. D. MILLER, *The $\overline{\partial}$ steepest descent method for orthogonal polynomials on the real line with varying weights*, Int. Math. Res. Not. IMRN ID rnn 075 (2008), 66 pp.

[32] J. F. MAÑAS–MAÑAS, F. MARCELLÁN, J. J. MORENO–BALCÁZAR, *Varying discrete Laguerre–Sobolev orthogonal polynomials: asymptotic behavior and zeros*, Appl. Math. Comput. **222** (2013), 612–618.

[33] J. F. MAÑAS–MAÑAS, F. MARCELLÁN, J. J. MORENO–BALCÁZAR, *Asymptotic behavior of varying discrete Jacobi–Sobolev orthogonal polynomials*, J. Comput. Appl. Math. **300** (2016), 341–353.

[34] J. F. MAÑAS–MAÑAS, F. MARCELLÁN, J. J. MORENO–BALCÁZAR, *Asymptotics for varying discrete Sobolev orthogonal polynomials*, Appl. Math. Comput. **314** (2017), 65–79.

[35] F. MARCELLÁN, A. RONVEAUX, *On a class of polynomials orthogonal with respect to a discrete Sobolev inner product*, Indag. Math. (N. S.), **1** (4) (1990), 451–464.

[36] A. MARTÍNEZ–FINKELSHTEIN, P. MARTÍNEZ–GONZÁLEZ, R. ORIVE, *On asymptotic zero distribution of Laguerre and generalized Bessel polynomials with varying parameters*, J. Comput. Appl. Math. **133** (2001), 477–487.

[37] J. J. MORENO–BALCÁZAR, *Varying Jacobi–Krall orthogonal polynomials: local asymptotic behaviour and zeros*, Ramanujan J. **28** (2012), 79–88.

[38] H. STALH, *Strong asymptotics for orthonormal polynomials with varying weights*, Acta Sci. Math. (Szeged) **66** (1-2) (2000), 147–192.

[39] G. SZEGŐ, Orthogonal Polynomials, Amer. Math. Soc. Colloq. Publ. vol. **23**, Fourth Edition, Amer. Math. Soc. Providence, RI, 1975.

[40] W. VAN ASSCHE, Orthogonal Polynomials and Painlevé Equations, Australian Mathematical Society Lecture Series **27**, Cambridge University Press, Cambridge, 2017.