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Results in Mathematics

2019

This is the peer reviewed version of the following article: ***"Classical Sobolev orthogonal polynomials: eigenvalue problem"***, which has been published in final form at <https://doi.org/10.1007/s00025-019-1069-9> .

Please follow the links below for the published version and cite this paper as:

Juan F. Mañas-Mañas, Juan J. Moreno-Balcázar. *Classical Sobolev orthogonal polynomials: eigenvalue problem*, Results Math. **74** (2019), Art. 144.

<https://doi.org/10.1007/s00025-019-1069-9>

<https://link.springer.com/article/10.1007/s00025-019-1069-9>

# Classical Sobolev orthogonal polynomials: eigenvalue problem

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## Abstract

We consider the discrete Sobolev inner product

$$(f, g)_S = \int f(x)g(x)d\mu + Mf^{(j)}(c)g^{(j)}(c), \quad j \in \mathbb{N} \cup \{0\}, \quad c \in \mathbb{R}, \quad M > 0,$$

where  $\mu$  is a classical continuous measure with support on the real line (Jacobi, Laguerre or Hermite). The orthonormal polynomials with respect to this Sobolev inner product are eigenfunctions of a differential operator and obtaining the asymptotic behavior of the corresponding eigenvalues is the principal goal of this paper.

**Keywords:** Sobolev orthogonal polynomials · Differential operator · Eigenvalues · Asymptotics

**Mathematics Subject Classification (2010):** 33C47 · 42C05

## 1 Introduction

The classical continuous hypergeometric polynomials (CCHP) have been well known since the nineteenth century and they constitute a relevant class within the orthogonal polynomials. Thus, almost all the books devoted to orthogonal polynomials and their applications included chapters or sections about CHHP, see among others [3, 12]. CCHP can be defined as the polynomial solutions to the hypergeometric differential equation [5]

$$\sigma(x)y''(x) + \tau(x)y'(x) = \lambda_n y(x), \tag{1}$$

where  $\sigma$  and  $\tau$  are polynomials with  $\deg(\sigma) \leq 2$ ,  $\deg(\tau) = 1$ , and  $\lambda_n = n \left( \tau'(x) + \frac{n-1}{2} \sigma''(x) \right)$ . One can prove that these polynomial solutions are orthogonal polynomials with respect to a weight function  $w$ . In fact, on the real line they are orthogonal with respect to a measure  $\mu$  given by  $d\mu(x) = w(x)dx$  where, up to affine transformations,  $w$  corresponds to one of these situations: Jacobi case  $w(x) = (1-x)^\alpha(1+x)^\beta$  with  $\alpha, \beta > -1$  and  $x \in (-1, 1)$ , Laguerre case  $w(x) = x^\alpha e^{-x}$  with  $\alpha > -1$  and  $x \in (0, \infty)$ , and Hermite case  $w(x) = e^{-x^2}$  with  $x$  on the real line.

The equation (1) can be rewritten as

$$\mathbf{B}[y(x)] = \lambda_n y(x),$$

where  $\mathbf{B}$  is a differential operator defined as  $\mathbf{B} := \sigma(x)\mathcal{D}^2 + \tau(x)\mathcal{D}$ , being  $\mathcal{D}$  the usual derivative operator. In this way, the CCHP are the eigenfunctions of the operator  $\mathbf{B}$  and  $\lambda_n$  are the corresponding eigenvalues. Both  $\mathbf{B}$  and  $\lambda_n$  are explicitly known.

Since the second half of the last century an emergent theory of orthogonal polynomials in Sobolev spaces has risen, see for example the surveys [8] and [9]. The seminal papers on this topic linked Sobolev orthogonal polynomials (SOP) with the simultaneous approximation of a function and their derivatives, but now some recent applications have been found in [11, 13].

In this work we have considered a special case of SOP called discrete SOP which are orthogonal with respect to the Sobolev inner product

$$(f, g)_S = \int f(x)g(x)d\mu + Mf^{(j)}(c)g^{(j)}(c), \quad j \in \mathbb{N} \cup \{0\}, \quad M > 0, \quad (2)$$

where  $\mu$  is a classical measure, i.e.,  $d\mu(x) = w(x)dx$  being  $w$  one of the classical weights described previously. Notice that when  $j = 0$  we have the so-called Krall polynomials which were a first extension of the CCHP [6].

The inner product (2) can be seen as a perturbation of the standard inner product  $\int f(x)g(x)d\mu$ . Thus, it is natural to wonder how this perturbation influences on the corresponding orthogonal polynomials, for example, about the asymptotic behavior of these SOP. In fact, there has been a wide literature about this so far (e.g. the previous surveys [8] and [9] and the references there in). On the one hand, it is obvious that the orthogonal polynomials with respect to (2) do not satisfy the hypergeometric equation (1). However, in [4] the authors impose conditions so that the polynomials  $q_n$  orthonormal with respect to (2) satisfy a (possibly infinite order) differential equation. Later, in [1] and [2] the differential operator is explicitly built as

$$\mathbf{L} := \sum_{i=1}^{\infty} r_i(x)\mathcal{D}^i,$$

where  $r_i$  is a polynomial with  $\deg(r_i) \leq i$ , and satisfying

$$\mathbf{L}[q_n(x)] = \tilde{\lambda}_n q_n(x).$$

In this way, the orthonormal polynomials  $q_n$  with respect to (2) are the eigenfunctions of the differential operator  $\mathbf{L}$  and  $\tilde{\lambda}_n$  are the corresponding eigenvalues.

Thus, we analyze the asymptotic behavior of the  $\tilde{\lambda}_n$ . We prove that this behavior is different of the one of  $\lambda_n$ . A first approach of this problem was done in [7] although on that occasion the authors focused their attention on computing a value related to the convergence of a series in a left-definite space. Here, we tackle the problem in a wider framework.

The structure of the paper is the following. In Section 2 we provide a brief background about eigenvalues of a differential operator related to discrete Sobolev orthonormal polynomials. Sections 3 and 4 are devoted to obtaining the asymptotic behavior of the eigenvalues  $\tilde{\lambda}_n$  distinguishing two cases: symmetric and nonsymmetric. In both cases the technique is the same although, as we will see, the computational details are different. Finally, in Section 5 we give a summary table of the results and we comment them.

## 2 Some known facts

We consider a classical nonsymmetric measure  $\mu$  and the corresponding sequence of orthonormal polynomials  $\{p_n\}_{n \geq 0}$ , then it was established in [1] and [2] that when  $p_n^{(j)}(c) \neq 0$  for all  $n = j, j+1, j+2, \dots$ , we get

$$\tilde{\lambda}_n = \lambda_n + M\alpha_n, \quad (3)$$

where  $\lambda_n$  are the eigenvalues of the differential operator  $\mathbf{B}$  and  $\alpha_n$  is a sequence of real numbers such that if they are chosen conveniently, then the differential operator  $\mathbf{L}$  is uniquely determined. To do this, it is enough to take  $\alpha_0 = 0$  and  $\{\alpha_i\}_{i=1}^j$  arbitrarily when  $j > 0$ .

Thus, we have the measure  $\mu$  such that  $d\mu(x) = w(x)dx$  with  $w$  corresponding to Laguerre or Jacobi case. We take  $c = 0$  for the Laguerre case and  $c \in \{-1, 1\}$  for the Jacobi one. Notice that the condition  $p_n^{(j)}(c) \neq 0$  is always satisfied when  $c$  is chosen in this way because  $\{p_n\}_{n \geq 0}$  is a family of CCHP. Then, using [1, f. (8-9)] we get

$$\alpha_n = \alpha_j + \sum_{i=j+1}^n (\lambda_i - \lambda_{i-1}) K_{i-1}^{(j,j)}(c, c), \quad n \geq j+1, \quad (4)$$

where  $K_n^{(j,j)}(c, c)$  denotes the partial derivatives of the  $n$ th kernel for the sequence of orthonormal polynomials  $\{p_n\}_{n \geq 0}$  with respect to  $\mu$ , i.e.,

$$K_n^{(r,s)}(x,y) = \sum_{i=0}^n p_i^{(r)}(x)p_i^{(s)}(y), \quad r,s \in \mathbb{N} \cup \{0\}.$$

Since  $\{\alpha_i\}_{i=1}^j$  can be chosen arbitrarily, we take  $\alpha_i = 0$  for  $i \in \{0, 1, 2, \dots, j\}$ . Thus, (4) is transformed into

$$\alpha_n = \sum_{i=j+1}^n (\lambda_i - \lambda_{i-1}) K_{i-1}^{(j,j)}(c, c), \quad n \geq j+1. \quad (5)$$

When  $\mu$  is a classical symmetric measure with respect to the origin we take  $c = 0$ . According to [2, Sect. 2.3] to guarantee (3) it is necessary that  $p_n^{(j)}(0) \neq 0$  for all  $n = j, j+1, j+2, \dots$  with  $n-j$  even. But again this holds because  $\{p_n\}_n$  is a family of CCHP. Then, applying the results in [2, Sect. 2.3] and taking into account again that  $\{\alpha_i\}_{i=1}^j$  are chosen arbitrarily, we get

$$\alpha_{j+2n} = \sum_{i=1}^n (\lambda_{j+2i} - \lambda_{j+2i-2}) K_{j+2i-1}^{(j,j)}(0, 0), \quad n \geq 1.$$

We remark that in this case, when  $j$  is even the subsequence of orthonormal polynomials  $\{q_{2n+1}\}_{n \geq 0}$  with respect to the discrete Sobolev inner product (2) matches the one of standard orthonormal polynomials  $\{p_{2n+1}\}_{n \geq 0}$ . An analogous situation takes place when  $j$  is odd, i.e.,  $\{q_{2n}\}_{n \geq 0} \equiv \{p_{2n}\}_{n \geq 0}$ .

The Hermite case is an example of this situation, but we can also consider the Gegenbauer case which occurs when  $\alpha = \beta$  in the Jacobi case. Thus, for this last case we can take  $c$  as any value of the set  $\{-1, 0, 1\}$ .

With these results we are in condition to obtain the asymptotic behavior of the eigenvalues  $\tilde{\lambda}_n$  in the next section.

### 3 Asymptotic behavior of the eigenvalues: the non-symmetric case

First, we give a joint approach to cases related to Jacobi and Laguerre weights. As we have mentioned in the previous section, we denote by  $\{p_n\}_{n \geq 0}$  the orthonormal polynomials with respect to the classical weights. We also use the notation  $a_n \approx b_n$  meaning  $\lim_{n \rightarrow +\infty} a_n/b_n = 1$ . Then, it is easy to check that for these families of polynomials we have

$$p_n^{(k)}(c) \approx C_k (-1)^n n^{ak+b}, \quad 0 \leq k \leq n, \quad \text{with} \quad 2(ak+b)+1 > 0, \quad (6)$$

where  $C_k$  is a constant independent of  $n$ . When we consider the Laguerre case then  $c = 0$ , and  $c \in \{-1, 1\}$  for the Jacobi case. For other nonclassical families satisfying (6) see [10].

It is worth noting that the factor  $(-1)^n$  may appear or not, for example, it appears when  $c = -1$  in the Jacobi case and it does not when  $c = 1$  in the same case. However, for the results that we will obtain this factor will be not relevant from the asymptotic point of view.

**Lemma 1.** *Assuming the condition (6), we have*

$$\lim_{n \rightarrow +\infty} \frac{K_{n-1}^{(\ell, \ell)}(c, c)}{n^{2(al+b)+1}} = \frac{C_\ell^2}{2(al+b)+1}.$$

**Proof:** It is enough to use Stolz's criterion and (6) to get

$$\begin{aligned} \lim_{n \rightarrow +\infty} \frac{K_{n-1}^{(\ell, \ell)}(c, c)}{n^{2(al+b)+1}} &= \lim_{n \rightarrow +\infty} \frac{K_{n-1}^{(\ell, \ell)}(c, c) - K_{n-2}^{(\ell, \ell)}(c, c)}{n^{2(al+b)+1} - (n-1)^{2(al+b)+1}} \\ &= \lim_{n \rightarrow +\infty} \frac{(p_{n-1}^{(\ell)}(c))^2}{(2(al+b)+1)n^{2(al+b)}} = \frac{C_\ell^2}{2(al+b)+1}. \quad \square \end{aligned}$$

To obtain the asymptotic behavior of  $\tilde{\lambda}_n$  it is necessary to know the asymptotics of the sequence  $\{\alpha_n\}$  and use (3). Thus, we establish it in the next result.

**Proposition 1.** *Assuming (6) and  $\lambda_n = \gamma n^2 + \delta n$  with  $\gamma, \delta \in \mathbb{R}$ , then we have*

$$\lim_{n \rightarrow +\infty} \frac{\alpha_n}{n^{2(aj+b)+3}} = \frac{2\gamma C_j^2}{(2(aj+b)+3)(2(aj+b)+1)}, \quad \text{if } \gamma \neq 0;$$

$$\lim_{n \rightarrow +\infty} \frac{\alpha_n}{n^{2(aj+b+1)}} = \frac{\delta C_j^2}{2(aj+b+1)(2(aj+b)+1)}, \quad \text{if } \gamma = 0,$$

where  $\alpha_n$  was defined in (5).

**Proof:** Observe that

$$\lambda_n - \lambda_{n-1} = \begin{cases} 2\gamma n - \gamma + \delta, & \text{if } \gamma \neq 0; \\ \delta, & \text{if } \gamma = 0. \end{cases}$$

Then, we need to distinguish two cases depending on  $\gamma$ . In both situations we use Lemma 1, (5), and again the Stolz's criterion to deduce the result:

- If  $\gamma \neq 0$ ,

$$\begin{aligned}
& \lim_{n \rightarrow +\infty} \frac{\alpha_n}{n^{2(aj+b)+3}} \\
&= \lim_{n \rightarrow +\infty} \frac{\sum_{i=j+1}^n (\lambda_i - \lambda_{i-1}) K_{i-1}^{(j,j)}(c, c) - \sum_{i=j+1}^{n-1} (\lambda_i - \lambda_{i-1}) K_{i-1}^{(j,j)}(c, c)}{n^{2(aj+b)+3} - (n-1)^{2(aj+b)+3}} \\
&= \lim_{n \rightarrow +\infty} \frac{(\lambda_n - \lambda_{n-1}) K_{n-1}^{(j,j)}(c, c)}{(2(aj+b)+3)n^{2(aj+b+1)}} = \frac{2\gamma C_j^2}{(2(aj+b)+3)(2(aj+b)+1)}.
\end{aligned}$$

- If  $\gamma = 0$ ,

$$\begin{aligned}
\lim_{n \rightarrow +\infty} \frac{\alpha_n}{n^{2(aj+b+1)}} &= \lim_{n \rightarrow +\infty} \frac{(\lambda_n - \lambda_{n-1}) K_{n-1}^{(j,j)}(c, c)}{2(aj+b+1)n^{2(aj+b+1)}} \\
&= \frac{\delta C_j^2}{2(aj+b+1)(2(aj+b)+1)}. \quad \square
\end{aligned}$$

**Theorem 1.** Let  $\tilde{\lambda}_n$  be the eigenvalues of the differential operator  $\mathbf{L}$  related to the orthonormal polynomials  $q_n$  with respect to (2). Under the hypothesis of Proposition 1, we get

$$\begin{aligned}
\lim_{n \rightarrow +\infty} \frac{\tilde{\lambda}_n}{n^{2(aj+b)+3}} &= \frac{2\gamma M C_j^2}{(2(aj+b)+3)(2(aj+b)+1)}, \quad \text{if } \gamma \neq 0; \\
\lim_{n \rightarrow +\infty} \frac{\tilde{\lambda}_n}{n^{2(aj+b+1)}} &= \frac{\delta M C_j^2}{2(aj+b+1)(2(aj+b)+1)}, \quad \text{if } \gamma = 0.
\end{aligned}$$

**Proof:** We only need to take limits in (3) and apply Proposition 1. We only show the proof when  $\gamma \neq 0$ , the other case is totally similar.

$$\begin{aligned}
\lim_{n \rightarrow +\infty} \frac{\tilde{\lambda}_n}{n^{2(aj+b)+3}} &= \lim_{n \rightarrow +\infty} \frac{\lambda_n + M\alpha_n}{n^{2(aj+b)+3}} \\
&= \lim_{n \rightarrow +\infty} \frac{\lambda_n}{n^{2(aj+b)+3}} + M \lim_{n \rightarrow +\infty} \frac{\alpha_n}{n^{2(aj+b)+3}} \\
&= \frac{2\gamma M C_j^2}{(2(aj+b)+3)(2(aj+b)+1)}.
\end{aligned}$$

The first limit is 0 because using (6) we have  $2(aj+b)+3 > 2$ .  $\square$

### 3.1 Discrete Jacobi–Sobolev case

We consider the discrete Sobolev inner product

$$(f, g)_{JS} = \int_{-1}^1 f(x)g(x)(1-x)^\alpha(1+x)^\beta dx + M f^{(j)}(1)g^{(j)}(1), \quad (7)$$

with  $\alpha, \beta > -1$  and  $j \in \mathbb{N} \cup \{0\}$ . We denote by  $\{p_n^{(\alpha, \beta)}\}_{n \geq 0}$  the sequence of the classical Jacobi orthonormal polynomials with respect to the weight function  $(1-x)^\alpha(1+x)^\beta$ . This inner product corresponds to (2) with  $c = 1$ .

Using the properties of Jacobi polynomials (e.g., see [12, f. (4.1.1), (4.3.3), (4.21.7)]), we deduce

$$\left(p_n^{(\alpha, \beta)}\right)^{(k)}(1) \approx \frac{1}{2^{k+\frac{\alpha+\beta}{2}} \Gamma(\alpha+k+1)} n^{2k+\alpha+1/2},$$

so, (6) is satisfied with  $C_k = \frac{1}{2^{k+\frac{\alpha+\beta}{2}} \Gamma(\alpha+k+1)}$ ,  $a = 2$  and  $b = \alpha + 1/2$ . Since  $\alpha > -1$ , the condition  $2(ak+b)+1 > 0$  holds.

On the other hand, Jacobi polynomials  $p_n^{(\alpha, \beta)}$  satisfy the second-order differential equation (e.g., see [12, f. (4.2.1)]):

$$(x^2 - 1)y''(x) + (\alpha - \beta + (\alpha + \beta + 2)x)y'(x) = n(n + \alpha + \beta + 1)y(x),$$

thus, we deduce  $\lambda_n = n^2 + n(\alpha + \beta + 1)$ .

Now, we are ready to apply Theorem 1, getting

$$\lim_{n \rightarrow +\infty} \frac{\tilde{\lambda}_n}{n^{4j+2\alpha+4}} = \frac{M}{2^{2j+\alpha+\beta+1}(2j+\alpha+2)(2j+\alpha+1)\Gamma^2(\alpha+j+1)}.$$

A similar result can be obtained if we choose  $c = -1$  in (7) instead of  $c = 1$ .

### 3.2 Discrete Laguerre–Sobolev case

Now, we consider

$$(f, g)_{LS} = \int_0^{+\infty} f(x)g(x)x^\alpha e^{-x} dx + M f^{(j)}(0)g^{(j)}(0), \quad \alpha > -1, \quad j \in \mathbb{N} \cup \{0\}.$$

We denote by  $\{l_n^{(\alpha)}\}_{n \geq 0}$  the sequence of the classical Laguerre orthonormal polynomials with respect to the weight function  $x^\alpha e^{-x}$ . We have taken  $c = 0$  in (2). In this case we use the properties of Laguerre polynomials (e.g., see [12, f. (5.1.1), (5.1.7), (5.1.14)]) to obtain

$$\left(l_n^{(\alpha)}\right)^{(k)}(0) \approx \frac{(-1)^k}{\Gamma(\alpha+k+1)} n^{k+\alpha/2}.$$



Again, (6) is satisfied taking now  $C_k = \frac{(-1)^k}{\Gamma(\alpha+k+1)}$ ,  $a = 1$  and  $b = \alpha/2$ . Since classical Laguerre polynomials satisfy the hypergeometric differential equation (e.g., see [12, f. (5.1.2)]):

$$-xy''(x) + (x - \alpha - 1)y'(x) = ny(x),$$

we have  $\lambda_n = n$ . Therefore, we can apply Theorem 1 taking into account that in this case  $\gamma = 0$  and  $\delta = 1$ , getting

$$\lim_{n \rightarrow +\infty} \frac{\tilde{\lambda}_n}{n^{2j+\alpha+2}} = \frac{M}{(2j + \alpha + 2)(2j + \alpha + 1)\Gamma^2(\alpha + j + 1)}.$$

## 4 Asymptotic behavior of the eigenvalues: the symmetric case

We suppose that  $\mu$  is a symmetric measure and we take  $c = 0$ . Thus we can proceed like in the previous section, but bearing in mind that now both families of orthonormal polynomials,  $\{p_n\}$  and  $\{q_n\}$ , are symmetric. Therefore, we have to assume similar conditions to (6) for the subsequences of even and odd polynomials. Thus, we suppose that

$$p_{2n}^{(2k)}(0) \approx C_{k,1}(-1)^n n^{a_1k+b_1}, \quad p_{2n+1}^{(2k+1)}(0) \approx C_{k,2}(-1)^n n^{a_2k+b_2}, \quad (8)$$

with  $2(a_1k + b_1) + 1 > 0$  and  $2(a_2k + b_2) + 1 > 0$  for all  $k \in \{0, \dots, n\}$ .

Assuming (8), we can obtain similar results to the ones obtained in the previous section. Since the techniques are the same we only state the main outcome.

**Theorem 2.** *Let  $\tilde{\lambda}_n$  be the eigenvalues of the differential operator  $\mathbf{L}$  related to the orthonormal polynomials  $q_n$  with respect to (2). We assume (8) and  $\lambda_n = \gamma n^2 + \delta n$  with  $\gamma, \delta \in \mathbb{R}$ . Then,*

- If  $j = 2r$ , we get

$$\lim_{n \rightarrow +\infty} \frac{\tilde{\lambda}_{2r+2n}}{n^{2(a_1r+b_1)+3}} = \frac{8\gamma MC_{r,1}^2}{(2(a_1r + b_1) + 3)(2(a_1r + b_1) + 1)}, \quad \text{if } \gamma \neq 0;$$

$$\lim_{n \rightarrow +\infty} \frac{\tilde{\lambda}_{2r+2n}}{n^{2(a_1r+b_1+1)}} = \frac{\delta MC_{r,1}^2}{(a_1r + b_1 + 1)(2(a_1r + b_1) + 1)}, \quad \text{if } \gamma = 0.$$

- If  $j = 2r + 1$ , we get

$$\lim_{n \rightarrow +\infty} \frac{\tilde{\lambda}_{2r+1+2n}}{n^{2(a_2r+b_2)+3}} = \frac{8\gamma MC_{r,2}^2}{(2(a_2r+b_2)+3)(2(a_2r+b_2)+1)}, \quad \text{if } \gamma \neq 0;$$

$$\lim_{n \rightarrow +\infty} \frac{\tilde{\lambda}_{2r+1+2n}}{n^{2(a_2r+b_2+1)}} = \frac{\delta MC_{r,2}^2}{(a_2r+b_2+1)(2(a_2r+b_2)+1)}, \quad \text{if } \gamma = 0.$$

#### 4.1 Discrete Hermite–Sobolev case

We take the Hermite weight function and  $c = 0$ , then the inner product (2) is transformed into

$$(f, g)_{HS} = \int_{-\infty}^{+\infty} f(x)g(x)e^{-x^2}dx + Mf^{(j)}(0)g^{(j)}(0), \quad j \in \mathbb{N} \cup \{0\}.$$

We denote by  $\{h_n\}_{n \geq 0}$  the sequence of the classical Hermite orthonormal polynomials with respect to the weight function  $e^{-x^2}$ . Using the properties of Hermite polynomials (e.g., see [12, f. (5.5.1), (5.5.5), (5.5.10)]) we deduce

$$h_{2n}^{(2k)}(0) \approx (-1)^n \frac{(-1)^k 2^{2k}}{\sqrt{\pi}} n^{k-1/4}, \quad h_{2n+1}^{(2k+1)}(0) \approx (-1)^n \frac{(-1)^k 2^{2k+1}}{\sqrt{\pi}} n^{k+1/4}.$$

Therefore, (8) holds with  $C_{k,1} = \frac{(-1)^k 2^{2k}}{\sqrt{\pi}}$ ,  $a_1 = 1$ ,  $b_1 = -1/4$ ,  $C_{k,2} = \frac{(-1)^k 2^{2k+1}}{\sqrt{\pi}}$ ,  $a_2 = 1$  and  $b_2 = 1/4$ .

Moreover, Hermite polynomials satisfy the second-order differential equation (e.g., see [12, f. (5.5.2)])

$$-y''(x) + 2xy'(x) = 2ny(x).$$

Then, we have  $\lambda_n = 2n$ . In this way, we can apply Theorem 2 with  $\gamma = 0$  and  $\delta = 2$ , obtaining the corresponding asymptotic behavior of the eigenvalues  $\tilde{\lambda}_n$ , i.e.,

- If  $j = 2r$ , then

$$\lim_{n \rightarrow +\infty} \frac{\tilde{\lambda}_{2r+2n}}{n^{2r+3/2}} = \frac{M2^{4r+1}}{\pi(r+3/4)(2r+1/2)}.$$

- If  $j = 2r + 1$ , then

$$\lim_{n \rightarrow +\infty} \frac{\tilde{\lambda}_{2r+1+2n}}{n^{2r+5/2}} = \frac{M2^{4r+3}}{\pi(r+5/4)(2r+3/2)}.$$

## 4.2 Discrete Gegenbauer–Sobolev case

For this case, we consider the discrete Sobolev inner product

$$(f, g)_{GS} = \int_{-1}^1 f(x)g(x)(1-x^2)^\alpha dx + M f^{(j)}(0)g^{(j)}(0), \quad \alpha > -1, \quad j \in \mathbb{N} \cup \{0\}.$$

We denote by  $\{c_n^{(\alpha)}\}_{n \geq 0}$  the sequence of the classical Gegenbauer orthonormal polynomials with respect to the weight function  $(1-x^2)^\alpha$ . Using some properties of these polynomials (e.g., [12, f. (4.7.1), (4.7.14), (4.7.15), (4.7.30)] and the relations in [12, p. 60] for  $\alpha = -1/2$ ; notice that in [12] the author works with  $c_n^{(\lambda-1/2)}$  with  $\lambda > -1/2$ ), we get

$$\begin{aligned} \left(c_{2n}^{(\alpha)}\right)^{(2k)}(0) &\approx (-1)^n \frac{(-1)^k 2^{2k+1/2}}{\sqrt{\pi}} n^{2k}, \\ \left(c_{2n+1}^{(\alpha)}\right)^{(2k+1)}(0) &\approx (-1)^n \frac{(-1)^k 2^{2k+3/2}}{\sqrt{\pi}} n^{2k+1}. \end{aligned}$$

Then, (8) holds with  $C_{k,1} = \frac{(-1)^k 2^{2k+1/2}}{\sqrt{\pi}}$ ,  $a_1 = 2$ ,  $b_1 = 0$ ,  $C_{k,2} = \frac{(-1)^k 2^{2k+3/2}}{\sqrt{\pi}}$ ,  $a_2 = 2$ , and  $b_2 = 1$ . This family of polynomials satisfies the hypergeometric differential equation (e.g., see [5, f. (9.8.23)])

$$(x^2 - 1) \left(c_n^{(\alpha)}\right)''(x) + 2(\alpha + 1)x \left(c_n^{(\alpha)}\right)'(x) = n(n + 2\alpha + 1)c_n^{(\alpha)}(x),$$

so, we have  $\lambda_n = n^2 + n(2\alpha + 1)$ . Finally, we apply Theorem 2 with  $\gamma = 1$  and  $\delta = 2\alpha + 1$ , getting

- If  $j = 2r$ ,

$$\lim_{n \rightarrow +\infty} \frac{\tilde{\lambda}_{2r+2n}}{n^{4r+3}} = \frac{2^{4(r+1)}M}{\pi(4r+3)(4r+1)},$$

- If  $j = 2r + 1$ ,

$$\lim_{n \rightarrow +\infty} \frac{\tilde{\lambda}_{2r+1+2n}}{n^{4r+5}} = \frac{2^{4r+6}M}{\pi(4r+5)(4r+3)}.$$

## 5 Conclusions

Finally, we provide a summary table of the results obtained for SOP and we compare them with those ones known for classical polynomials.

Eigenvalues \ Case	Jacobi	Laguerre
Asymptotics of $\lambda_n$	$n^2$	$n$
Asymptotics of $\tilde{\lambda}_n$	$\mathcal{C}_1 n^{4j+2\alpha+4}$	$\mathcal{C}_2 n^{2j+\alpha+2}$

	Eigenvalues \ Case	Hermite	Gegenbauer
	Asymptotics of $\lambda_n$	$2n$	$n^2$
If $j = 2r$	Asymptotics of $\tilde{\lambda}_{2n}$	$\mathcal{C}_3 n^{2r+3/2}$	$\mathcal{C}_4 n^{4r+3}$
	Asymptotics of $\tilde{\lambda}_{2n+1}$	$4n$	$4n^2$
If $j = 2r + 1$	Asymptotics of $\tilde{\lambda}_{2n}$	$4n$	$4n^2$
	Asymptotics of $\tilde{\lambda}_{2n+1}$	$\mathcal{C}_5 n^{2r+5/2}$	$\mathcal{C}_6 n^{4r+5}$

The constants  $\mathcal{C}_i$ ,  $i = 1, \dots, 6$ . can be found explicitly in the previous sections.

We can observe that in all the cases the presence of the discrete part  $Mf^{(j)}(c)g^{(j)}(c)$  in the Sobolev inner product leads to important changes in the asymptotic behavior of the eigenvalues. For example, in the Laguerre case the growing orden of the eigenvalues increases  $2j+\alpha+1 > 0$ , i.e., when  $M = 0$  the eigenvalues have a linear growth that changes to a growth of  $O(n^{2j+\alpha+2})$  if  $M > 0$ . Similar situations occur in the rest of the cases. We can observe that this change is bigger in the bounded cases: Jacobi and Gegenbauer.

**Acknowledgments.** The authors are partially supported by the Ministry of Science, Innovation and Universities of Spain and the European Regional Development Fund (ERDF), grant MTM2017-89941-P and by Research Group FQM-0229 (belonging to Campus of International Excellence CEIMAR). The author JFMM is funded by a grant of Plan Propio de la Universidad de Almería. The author JJMB is also partially supported by the research centre CDTIME of Universidad de Almería and by Junta de Andalucía and ERDF, ref. SOMM17/6105/UGR.

## References

- [1] H. BAVINCK, *Differential and difference operators having orthogonal polynomials with two linear perturbations as eigenfunctions*, J. Comput. Appl. Math. **92** (1998), 85–95.
- [2] H. BAVINCK, J. KOEKOEK, *Differential operators having symmetric orthogonal polynomials as eigenfunctions*, J. Comput. Appl. Math. **106** (1999), 369–393.

- [3] M. E. H. ISMAIL, Classical and Quantum Orthogonal Polynomials in one Variable. Enciclopedia of Mathematics and its Applications Vol. **98** Cambridge University Press, Cambridge U.K. 2005.
- [4] I. H. JUNG, K. H. KWON, G. J. YOON, *Differential equations of infinite order for Sobolev-type orthogonal polynomials*, J. Comput. Appl. Math. **78** (1997), 277–293.
- [5] R. KOEKOEK, P. A. LESKY, R. F. SWARTTOUW, *Hypergeometric Orthogonal Polynomials and Their  $q$ -Analogues*, Springer Monographs in Mathematics, Springer, Berlin, 2010.
- [6] A. M. KRALL, *Orthogonal polynomials satisfying fourth order differential equations*, Proc. Roy. Soc. Edinburgh Sect. A **87** (3-4) (1980/81), 271–288.
- [7] L. L. LITTLEJOHN, J. F. MAÑAS–MAÑAS, J. J. MORENO–BALCÁZAR, R. WELLMAN, *Differential operator for discrete Gegenbauer-Sobolev orthogonal polynomials: Eigenvalues and asymptotics*, J. Approx. Theory, **230** (2018), 32–49.
- [8] F. MARCELLÁN, J. J. MORENO–BALCÁZAR, *Asymptotics and zeros of Sobolev orthogonal polynomials on unbounded supports*, Acta Appl. Math. **94** (2) (2006), 163–192.
- [9] F. MARCELLÁN, Y. XU, *On Sobolev Orthogonal Polynomials*, Expo. Math. **33** (3) (2015), 308–352.
- [10] A. PEÑA, M. L. REZOLA, *Connection formulas for general discrete Sobolev polynomials: Mehler-Heine asymptotics*, Appl. Math. Comput. **261** (2015), 216–230.
- [11] Y. REN, X. YU, Z. WANG, *Diagonalized Chebyshev Rational Spectral Methods for Second-Order Elliptic Problems on Unbounded Domains*, Numer. Math. Theor. Meth. Appl. **12** (1) (2019), 265–284.
- [12] G. SZEGŐ, Orthogonal Polynomials, Amer. Math. Soc. Colloq. Publ. vol. **23**, Fourth Edition, Amer. Math. Soc. Providence, RI, 1975.
- [13] K. TAKEMURA, A. NAGAI, Y. KAMETAKA, K. WATANABE, H. YAMAGISHI, *Sobolev inequality of free boundary value problem for  $(-1)^M(d/dx)^{2M}$* , J. Inequal. Appl. **54** (2015), 20 pp.