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Directional Dependence Orders of Random Vectors

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Abstract: In this paper, we define a multivariate order based on the concept of orthant directional dependence and study some of its properties. The relationships with other dependence orders given in the literature are also studied. We analyze the order between two random vectors in terms of their associated copulas and illustrate our results with several examples.

Keywords: copula; dependence concept; directional coefficient; stochastic order

MSC: 60E15; 62H05

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1. Introduction

There are various approaches to examine how random variables relate in terms of dependence. Jogdeo [1] highlights that this area stands as one of the most extensively researched subjects within the realms of probability and statistics. A multivariate model should be analyzed for the type of dependence structure that it covers so one can know whether a particular model might be usable for a given application or dataset. Among the types of dependence studied in the literature, we focus on negative or positive dependence. A positive dependence notion is any criterion which can mathematically describe the tendency of the components of a n-variate random vector to assume concordant values [2]. In this work, there is no attempt to be exhaustive in giving all dependence concepts studied in the literature. We restrict the attention to some dependence structures.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, where Ω is a non-empty set, $\mathcal{F}$ is a σ-algebra of subsets of Ω, and $\mathbb{P}$ is a probability measure on $\mathcal{F}$. Let n be a natural number such that $n \geq 2$, and let $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ be a random vector from Ω to $\overline{\mathbb{R}}^n = [-\infty, +\infty]^n$ with distribution function F and survival function $\overline{F}$, where $\overline{F}(x_1, x_2, \ldots, x_n) = \mathbb{P}[X_1 > x_1, X_2 > x_2, \ldots, X_n > x_n]$, for all $x_i \in \overline{\mathbb{R}}$. It is said that $\mathbf{X}$- or F- is *positive upper orthant-dependent* (PUOD) if

$$\overline{F}(x_1, x_2, \ldots, x_n) \geq \prod_{i=1}^{n} \mathbb{P}[X_i > x_i] \text{ for all } x_i \in \overline{\mathbb{R}} \tag{1}$$

and $\mathbf{X}$- or F- is *positive lower orthant-dependent* (PLOD) if

$$F(x_1, x_2, \ldots, x_n) \geq \prod_{i=1}^{n} \mathbb{P}[X_i \leq x_i] \text{ for all } x_i \in \overline{\mathbb{R}} \tag{2}$$

(see, e.g., [3]). If both (1) and (2) hold, then $\mathbf{X}$ or F is said to be *positive orthant-dependent* (POD). Note that, for the bivariate case, (1) and (2) are equivalent—this is not the case in higher dimensions: see [4] (Example 5.26)—and in this case, the dependence property is called *positive quadrant dependence* (PQD). If the inequalities (1) and (2) are reversed, then it is said that $\mathbf{X}$ is *negative upper orthant-dependent* (NUOD) and *negative lower orthant-dependent* (NUOD), respectively. For more details on these and other dependence concepts, see, e.g., [2,3,5–8].

In [9], the orthant dependence according to a direction is defined as follows: Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{R}^n$ such that $|\alpha_i| = 1$ for all $i = 1, 2, \ldots, n$. $\mathbf{X}$ or F is said to be *orthant positive-* (respectively, *negative-*) *dependent according to the direction* α—written PD(α) (respectively, ND (α))—if

$$\mathbb{P}\left[\bigcap_{i=1}^{n}(\alpha_i X_i > x_i)\right] \geq \prod_{i=1}^{n}\mathbb{P}[\alpha_i X_i > x_i] \text{ for all } x_i \in \overline{\mathbb{R}} \tag{3}$$

(respectively, we reverse the inequality in (3)). Directional coefficients which detect dependence in multivariate distributions are studied in [10].

If $\mathbf{X}$ is both PD(α) (respectively, ND(α)) and PD(-α) (respectively, ND(-α)), then $\mathbf{X}$ is said to be *strongly positive* (respectively, *negative*), and *dependent according to the direction* α, written SPD(α) (respectively, SND(α)).

Note that, for $\alpha = \mathbf{1} = (1, 1, \ldots, 1)$, the concepts of PD($\alpha$) (respectively, ND($\alpha$)) and PUOD (respectively, NUOD are the same; and for $\alpha = -\mathbf{1} = (-1, -1, \ldots, -1)$, the concepts of PD($\alpha$) (respectively, ND($\alpha$))) and PLOD (respectively, NLOD) are the same.

In the following, we will restrict our study based on the positive PLOD, PUOD, PD(α), and SPD(α) concepts. Similar results can be obtained if we base it on the respective negative concepts.

The positive dependence concepts defined above result from comparing a multivariate vector with a random vector of independent random variables with the same corresponding univariate margins. Of course, comparisons can be made via dependence orderings. Several dependence (partial) orderings, which compare the amount of dependence in two different random vectors of the same length and with the same marginal distributions, have been studied (see, e.g., [3,5,11–13]). Particularly, in a parametric family of multivariate distributions, the parameter is interpretable as a dependence parameter if the amount of dependence is increasing, or decreasing, as the parameter increases. It is the interest of comparing two multivariate distributions in the sense of some dependence concept.

The next definition recalls some dependence orderings, where $\Gamma_n(F_1, F_2 \ldots, F_n)$, $n \geq 2$, denotes the class of all the n-dimensional distributions with univariate marginals $F_1, F_2, \cdots, F_n$, that is, the Fréchet class, and the function $\phi : \mathbb{R}^n \to \mathbb{R}$ is *supermodular*, i.e., which satisfies

$$\phi(\mathbf{x}) + \phi(\mathbf{y}) \leq \phi(\mathbf{x} \wedge \mathbf{y}) + \phi(\mathbf{x} \vee \mathbf{y})$$

for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, where $\mathbf{x} \wedge \mathbf{y} = (x_1 \wedge y_1, x_2 \wedge y_2, \ldots, x_n \wedge y_n)$ and $\mathbf{x} \vee \mathbf{y} = (x_1 \vee y_1, x_2 \vee y_2, \ldots, x_n \vee y_n)$, where $\wedge$ and $\vee$ are the minimum and the maximum operators, respectively.

Definition 1. *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be two random vectors with respective distribution functions* F *and* G *in* $\Gamma_n(F_1, F_2, \ldots, F_n)$, $n \geq 2$, *and survival functions* $\overline{F}$ *and* $\overline{G}$. *It is said that:*

- *(i)* $\mathbf{X}$ *is smaller than* $\mathbf{Y}$ *in the positive upper orthant dependence order (denoted by* $\mathbf{X} \leq_{PUOD} \mathbf{Y}$*) if* $\overline{F}(\mathbf{x}) \leq \overline{G}(\mathbf{x})$ *for all* $\mathbf{x} \in \mathbb{R}^n$.
- *(ii)* $\mathbf{X}$ *is smaller than* $\mathbf{Y}$ *in the positive lower orthant dependence order (denoted by* $\mathbf{X} \leq_{PLOD} \mathbf{Y}$*) if* $F(\mathbf{x}) \leq G(\mathbf{x})$ *for all* $\mathbf{x} \in \mathbb{R}^n$.
- *(iii)* $\mathbf{X}$ *is smaller than* $\mathbf{Y}$ *in the positive orthant dependence order (denoted by* $\mathbf{X} \leq_{POD} \mathbf{Y}$*) if* $\overline{F}(\mathbf{x}) \leq \overline{G}(\mathbf{x})$ *and* $F(\mathbf{x}) \leq G(\mathbf{x})$ *for all* $\mathbf{x} \in \mathbb{R}^n$.
- *(iv)* $\mathbf{X}$ *is smaller than* $\mathbf{Y}$ *in the supermodular order (denoted by* $\mathbf{X} \leq_{sm} \mathbf{Y}$*) if* $E[\phi(\mathbf{X})] \leq E[\phi(\mathbf{Y})]$ *for any supermodular function* $\phi : \mathbb{R}^n \to \mathbb{R}$ *provided the expectations exit.*

Note that, for the bivariate case, the $\leq_{POD}$ order in Definition 1(iii) can be said to be the $\leq_{PQD}$ order, and from *(i)*, *(ii)* and *(iii)*, we have

$$\mathbf{X} \leq_{POD} \mathbf{Y} \Longleftrightarrow \mathbf{X} \leq_{PUOD} \mathbf{Y} \text{ and } \mathbf{X} \leq_{PLOD} \mathbf{Y}. \tag{4}$$

Moreover,

$$\mathbf{X} \leq_{sm} \mathbf{Y} \Longrightarrow \mathbf{X} \leq_{POD} \mathbf{Y} \tag{5}$$

(see [13] [Equation (9.A.17)]), and, for $n = 2$, $\leq_{sm}$ and $\leq_{PQD}$ are equivalent. For further details about these and others dependence orders, see, e.g., [2,3,5,13–15] and the references therein.

Joe [5] [p. 39] postulated a number of desirable axioms that a multivariate positive dependence order should satisfy. Later, Colangelo et al. [2] gave the following slight variation of these postulates:

P1. It should be a pre-order (reflexive and transitive).
P2. It should be antisymmetric.
P3. It should imply the PQD order of every (corresponding) bivariate marginal distribution.
P4. It should be closed under marginalization.
P5. It should be closed under limits in distribution.
P6. It should be closed under the permutation of the components.
P7. It should be closed under component-wise strictly increasing transformation.
P8. It should be maximal at the upper Fréchet bound; and, in the bivariate case, it should be minimal at the lower Fréchet bound.

Our main goal in this work is the study, in any dimension greater than or equal to 2, of the presence of orders that are not "detected" by the well-known PLOD and PUOD orders, and for this, we use the notion of orthant dependence according to the direction of a vector.

This paper is structured as follows. In Section 2, we define a new order based on the concept of positive dependence given in [9] and study some of its properties. Also, the relationship with other dependence orders given in the literature are studied. In Section 3, we study the comparison of two random vectors in terms of their associated copulas and provide several examples.

2. New Definitions and Basic Properties

Based on the PD(α) notion given in (3), and with the aim of comparing the strength of the positive dependence in a particular direction of two underlying multivariate distributions, we provide the following dependence orderings.

Definition 2 (The PD(α) order)**.** *Let* $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ *and* $\mathbf{Y} = (Y_1, Y_2, \ldots, Y_n)$ *be two random vectors with respective distribution functions* F *and* G *and survival functions* $\overline{F}$ *and* $\overline{G}$*. Let* $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{R}^n$ *such that* $|\alpha_i| = 1$ *for all* $i = 1, 2, \ldots, n$*.* $\mathbf{X}$ *is said to be smaller than* $\mathbf{Y}$ *in the positive dependence according to the direction* α *order, denoted by* $\mathbf{X} \leq_{PD(\alpha)} \mathbf{Y}$*, if, for every* $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \overline{\mathbb{R}}^n$*, we have*

$$\mathbb{P}\left[\bigcap_{i=1}^{n} (\alpha_i X_i > x_i)\right] \leq \mathbb{P}\left[\bigcap_{i=1}^{n} (\alpha_i Y_i > x_i)\right]. \tag{6}$$

Note that, from Equations (3) and (6), $\mathbf{X}$ is PD(α) if, and only if, $\mathbf{X}^{Ind} \leq_{PD(\alpha)} \mathbf{X}$, where $\mathbf{X}^{Ind}$ is a random vector with the same univariate marginals as $\mathbf{X}$ but with independent components. Also observe that $\mathbf{X} \leq_{PD(\alpha)} \mathbf{Y}$ is equivalent to $\alpha\mathbf{X} \leq_{PUOD} \alpha\mathbf{Y}$ or $-\alpha\mathbf{X} \leq_{PLOD} -\alpha\mathbf{Y}$, where $\alpha\mathbf{X} = (\alpha_1 X_1, \ldots, \alpha_n X_n)$ and similarly $\alpha\mathbf{Y}$.

From the SPD(α) notion, a stronger order than that of the PD(α) order can be defined as follows.

Definition 3 (The SPD(α) order)**.** *Let* $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ *and* $\mathbf{Y} = (Y_1, Y_2, \ldots, Y_n)$ *be two random vectors with respective distribution functions* F *and* G *and survival functions* $\overline{F}$ *and* $\overline{G}$*. Let* $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{R}^n$ *such that* $|\alpha_i| = 1$ *for all* $i = 1, 2, \ldots, n$*.* $\mathbf{X}$ *is said to be smaller than* $\mathbf{Y}$ *in the strongly positive dependence according to the direction* α *order, denoted by* $\mathbf{X} \leq_{SPD(\alpha)} \mathbf{Y}$*, if,* $\mathbf{X} \leq_{PD(\alpha)} \mathbf{Y}$ *and* $\mathbf{X} \leq_{PD(-\alpha)} \mathbf{Y}$*.*

Based on [2] (Example 2.15), the next example shows two vectors which are ordered in the sense of the PD(α) order but not in the SPD(α) one.

Example 1. *Let X_1, X_2, X_3 be three independent and identically distributed Bernoulli random variables with a common parameter 0.7, and let $\mathbf{Y} = (Y_1, Y_2, Y_3)$ be a random vector such that $\mathbb{P}[Y_1 = 0, Y_2 = 0, Y_3 = 1] = 0.2$, $\mathbb{P}[Y_1 = 1, Y_2 = 1, Y_3 = 1] = 0.5$ and $\mathbb{P}[Y_1 = 0, Y_2 = 1, Y_3 = 0] = \mathbb{P}[Y_1 = 1, Y_2 = 0, Y_3 = 0] = \mathbb{P}[Y_1 = 1, Y_2 = 1, Y_3 = 0] = 0.1$. Note that Y_1, Y_2 and Y_3 are Bernoulli distributed random variables with parameters 0.7. After some straightforward calculations, it can be proven that $\mathbf{X} \leq_{PD(1,1,1)} \mathbf{Y}$, given that $\mathbb{P}[X_1 > x_1, X_2 > x_2, X_3 > x_3] \leq \mathbb{P}[Y_1 > x_1, Y_2 > x_2, Y_3 > x_3]$ for all (x_1, x_2, x_3). However, $\mathbb{P}[X_1 \leq 0, X_2 \leq 0, X_3 \leq 0] = 0.3^3 \geq 0 = \mathbb{P}[Y_1 \leq 0, Y_2 \leq 0, Y_3 \leq 0]$, and thus, $\mathbf{X} \not\leq_{PD(-1,-1,-1)} \mathbf{Y}$. Therefore, $\mathbf{X} \not\leq_{SPD(1,1,1)} \mathbf{Y}$.*

Some closure properties of the PD(α) order are described in the next theorem, whose proof is straightforward and we omit it.

Theorem 1. *Let $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ and $\mathbf{Y} = (Y_1, Y_2, \ldots, Y_n)$ be two random vectors in the same Fréchet class.*

(a) *If $\mathbf{X} \leq_{PD(\alpha)} \mathbf{Y}$, then $\mathbf{X}_I \leq_{PD(\alpha^*)} \mathbf{Y}_I$, for each $I \subseteq \{1, 2, \ldots n\}$, and where α^* is the sub-vector of α whose components are in I. In other words, the PD(α) order is closed under marginalization.*

(b) *If $\mathbf{X} \leq_{PD(\alpha)} \mathbf{Y}$, then*

$$(g_1(X_1), g_2(X_2), \ldots, g_n(X_n)) \leq_{PD(\alpha)} (g_1(Y_1), g_2(Y_2), \ldots, g_n(Y_n)),$$

whenever g_i, $i = 1, 2, \ldots, n$, are n real-valued and increasing functions.

(c) *If $\mathbf{X} \leq_{PD(\alpha)} \mathbf{Y}$ and $\mathbf{U} \leq_{PD(\beta)} \mathbf{V}$, with $\mathbf{X}$ and $\mathbf{Y}$ independent of $\mathbf{U}$ and $\mathbf{V}$, respectively, then, $(\mathbf{X}, \mathbf{U}) \leq_{PD(\alpha,\beta)} (\mathbf{Y}, \mathbf{V})$.*

Proof. Let $J = \{1, 2, \ldots, n\}$ and $I \subseteq J$.

Firstly, since $\mathbf{X} \leq_{PD(\alpha)} \mathbf{Y}$, for any $(x_1, x_2, \ldots, x_n) \in \overline{\mathbb{R}}^n$, it follows that

$$\begin{aligned}
\mathbb{P}\left[\bigcap_{i\in I}(\alpha_i X_i > x_i)\right] &= \mathbb{P}\left[\bigcap_{i\in I}(\alpha_i X_i > x_i), \bigcap_{i\in J\setminus I}(\alpha_i X_i > y_i)\right] \\
&\leq \mathbb{P}\left[\bigcap_{i\in I}(\alpha_i Y_i > x_i), \bigcap_{i\in J\setminus I}(\alpha_i Y_i > y_i)\right] \\
&= \mathbb{P}\left[\bigcap_{i\in I}(\alpha_i Y_i > x_i)\right],
\end{aligned}$$

where y_i is the left endpoint in support of $\alpha_i X_i$ for every $i \in J \setminus I$, whence part (a) has been proven.

By considering $I = \{i \in J : \alpha_i > 0\}$, part (b) follows from the following:

$$\begin{aligned}
\mathbb{P}\left[\bigcap_{i=1}^{n}(\alpha_i g_i(X_i) > x_i)\right] &= \mathbb{P}\left[\bigcap_{i\in I}(g_i(X_i) > x_i), \bigcap_{i\in J\setminus I}(g_i(X_i) < x_i)\right] \\
&= \mathbb{P}\left[\bigcap_{i\in I}(X_i > g_i^{-1}(x_i)), \bigcap_{i\in J\setminus I}(X_i < g_i^{-1}(x_i))\right] \\
&\leq \mathbb{P}\left[\bigcap_{i\in I}(Y_i > g_i^{-1}(x_i)), \bigcap_{i\in J\setminus I}(Y_i < g_i^{-1}(x_i))\right] \\
&= \mathbb{P}\left[\bigcap_{i=1}^{n}(\alpha_i g_i(Y_i) > x_i)\right].
\end{aligned}$$

Finally, for part (c), let $\mathbf{U}$ and $\mathbf{V}$ be two random vectors with dimension m. It follows

$$\begin{aligned}
\mathbb{P}\left[\bigcap_{i=1}^{n}(\alpha_i X_i > x_i), \bigcap_{i=1}^{m}(\beta_i U_i > x_i)\right] &= \mathbb{P}\left[\bigcap_{i=1}^{n}(\alpha_i X_i > x_i)\right]\mathbb{P}\left[\bigcap_{i=1}^{m}(\beta_i U_i > x_i)\right] \\
&\leq \mathbb{P}\left[\bigcap_{i=1}^{n}(\alpha_i Y_i > x_i)\right]\mathbb{P}\left[\bigcap_{i=1}^{m}(\beta_i V_i > x_i)\right] \\
&= \mathbb{P}\left[\bigcap_{i=1}^{n}(\alpha_i Y_i > x_i), \bigcap_{i=1}^{m}(\beta_i V_i > y_i)\right],
\end{aligned}$$

completing the proof. □

Note that the properties in Theorem 1 are some of the desirable postulates that a multivariate positive dependence order should satisfy (specifically, P4 and P7). Moreover, the $PD(\alpha)$ order is also reflexive, transitive, and antisymmetric.

The following example shows that the $PD(\alpha)$ order does not imply the PQD order of every (corresponding) bivariate marginal, that is, postulate P3 is not satisfied.

Example 2. *Let X_1, X_2, X_3 be three independent and identically distributed Bernoulli random variables with respective parameters* 0.5*,* 0.9*, and* 0.8*, and let $\mathbf{Y} = (Y_1, Y_2, Y_3)$ be a random vector such that $\mathbb{P}[Y_1 = 0, Y_2 = 0, Y_3 = 1] = 0.1$, $\mathbb{P}[Y_1 = 1, Y_2 = 1, Y_3 = 1] = 0.5$ and $\mathbb{P}[Y_1 = 0, Y_2 = 1, Y_3 = 0] = \mathbb{P}[Y_1 = 0, Y_2 = 1, Y_3 = 1] = 0.2$. Note that Y_1, Y_2 and Y_3 are Bernoulli distributed random variables with parameters* 0.5*,* 0.9*, and* 0.8*, respectively. After some straightforward calculations, it can be proven that $\mathbf{Y} \leq_{PD(-1,1,1)} \mathbf{X}$, given that $\mathbb{P}[Y_1 \leq x_1, Y_2 > x_2, Y_3 > x_3] \leq \mathbb{P}[X_1 \leq x_1, X_2 > x_2, X_3 > x_3]$ for all (x_1, x_2, x_3). However, $\mathbb{P}[Y_1 > 0, Y_2 > 0] = 0.5 \geq 0.45 = \mathbb{P}[X_1 > 0, X_2 > 0]$, and thus, $(Y_1, Y_2) \leq_{PQD} (X_1, X_2)$ does not hold.*

Now, we prove that the PD(α) order is closed under weak convergence, where $\longrightarrow_{st}$ denotes convergence in distribution.

Theorem 2. *Let $\{\mathbf{X}_j\}_{j\in\mathbb{N}}$ and $\{\mathbf{Y}_j\}_{j\in\mathbb{N}}$ be two sequences of random vectors such that $\mathbf{X}_j \longrightarrow_{st} \mathbf{X}$ and $\mathbf{Y}_j \longrightarrow_{st} \mathbf{Y}$. If $\mathbf{X}_j \leq_{PD(\alpha)} \mathbf{Y}_j$ for all $j \in \mathbb{N}$, then $\mathbf{X} \leq_{PD(\alpha)} \mathbf{Y}$.*

Proof. If $\mathbf{X}_j \leq_{PD(\alpha)} \mathbf{Y}_j$ for all $j \in \mathbb{N}$, then $\alpha\mathbf{X}_j \leq_{PUOD} \alpha\mathbf{Y}_j$ for all $j \in \mathbb{N}$. Moreover, by using the continuous mapping theorem [16,17], it follows that $\alpha\mathbf{X}_j \longrightarrow_{st} \alpha\mathbf{X}$ and $\alpha\mathbf{Y}_j \longrightarrow_{st} \alpha\mathbf{Y}$. Thus, given that the PUOD order is closed under weak convergence, we have $\alpha\mathbf{X} \leq_{PUOD} \alpha\mathbf{Y}$, whence $\mathbf{X} \leq_{PD(\alpha)} \mathbf{Y}$, which completes the proof. □

In addition, some results related to the Fréchet upper bound for the bi- and trivariate cases are given. Recall that the Fréchet upper bound F^+ in the class $\Gamma_n(F_1, F_2, \ldots, F_n), n \geq 2$, is defined as $F^+(x_1, x_2, \ldots, x_n) = \min\{F_1, F_2, \ldots, F_n\}$.

Proposition 1. *Let $\mathbf{X}$ and $\mathbf{X}^+$ be two bivariate random vectors with respective distribution functions F and F^+ in $\Gamma_n(F_1, F_2)$, and let $\alpha = (\alpha_1, \alpha_2)$ such that $|\alpha_i| = 1, i = 1, 2$.*

(i) If $\alpha_1 \cdot \alpha_2 = 1$, then $\mathbf{X} \leq_{PD(\alpha)} \mathbf{X}^+$.

(ii) If $\alpha_1 \cdot \alpha_2 = -1$, then $\mathbf{X}^+ \leq_{PD(\alpha)} \mathbf{X}$.

Proof. First, note that, for $\alpha = (1, 1)$ or $\alpha = (-1, -1)$, the PD(α) order is equivalent to the PQD order between random vectors, and it is well known (see [13, p. 390]) that $\mathbf{X} \leq_{PQD} \mathbf{X}^+$. Thus, the result in (i) holds.

Now, consider $\alpha = (-1, 1)$. It follows

$$\begin{aligned} \mathbb{P}(X_1 \leq x_1, X_2 > x_2) &= F_1(x_1) - P[X_1 \leq x_1, X_2 \leq x_2] \\ &\geq F_1(x_1) - \mathbb{P}[X_1^+ \leq x_1, X_2^+ \leq x_2] \\ &= \mathbb{P}[X_1^+ \leq x_1, X_2^+ > x_2], \end{aligned}$$

where the inequality follows from (i), that is, $F(x_1, x_2) \leq F^+(x_1, x_2)$ for all (x_1, x_2). Thus, $\mathbf{X}^+ \leq_{PD(-1,1)} \mathbf{X}$. The proof for $\alpha = (1, -1)$ is analogously obtained and therefore, (ii) holds. □

The following example shows that, for the trivariate case, the results in Proposition 1 do not hold.

Example 3. *Let $\mathbf{Y} = (Y_1, Y_2, Y_3)$ be a random vector defined as in Example 2, and let $\mathbf{Y}^+ = (Y_1^+, Y_2^+, Y_3^+)$ be a random vector with the same univariate marginals of $\mathbf{Y}$ and joint distribution function given that $F_{\mathbf{Y}^+}(y_1, y_2, y_3) = \min\{F_1(y_1), F_2(y_2), F_3(y_3)\}$, that is, the trivariate upper Fréchet bound. It is easy to show that, for $\alpha = (-1, -1, 1)$, we have $0.1 = \mathbb{P}[Y_1 \leq 0.5, Y_2 \leq 0.5, Y_3 > 0.5] > \mathbb{P}[Y_1^+ \leq 0.5, Y_2^+ \leq 0.5, Y_3^+ > 0.5] = 0$, but $0.1 = \mathbb{P}[Y_1 \leq 0.5, Y_2 \leq 1, Y_3 > 0.5] < \mathbb{P}[Y_1^+ \leq 0.5, Y_2^+ \leq 1, Y_3^+ > 0.5] = 0.3$. Moreover, for $\alpha = (-1, -1, -1)$, we obtain $0 = \mathbb{P}[Y_1 \leq 0.5, Y_2 \leq 0.5, Y_3 \leq 0.5] < \mathbb{P}[Y_1^+ \leq 0.5, Y_2^+ \leq 0.5, Y_3^+ \leq 0.5] = 0.1$, but $0.2 = \mathbb{P}[Y_1 \leq 0.5, Y_2 \leq 1, Y_3 \leq 0.5] > \mathbb{P}[Y_1^+ \leq 0.5, Y_2^+ \leq 1, Y_3^+ \leq 0.5] = 0.1$.*

To conclude this section, regarding the relationship with other stochastic orders, we summarize some straightforward results:

(a) For $n = 2$, and $\alpha = (1, 1)$ or $\alpha = (-1, -1)$, then

$$\mathbf{X} \leq_{PD(\alpha)} \mathbf{Y} \text{ is equivalent to } \mathbf{X} \leq_{PQD} \mathbf{Y}.$$

(b) If $n > 2$ and $\alpha = \mathbf{1}$, then

$$\mathbf{X} \leq_{PD(\alpha)} \mathbf{Y} \text{ is equivalent to } \mathbf{X} \leq_{PUOD} \mathbf{Y}. \tag{7}$$

(c) If $n > 2$ and $\alpha = -\mathbf{1}$, then

$$\mathbf{X} \leq_{PD(\alpha)} \mathbf{Y} \text{ is equivalent to } \mathbf{X} \leq_{PLOD} \mathbf{Y}. \tag{8}$$

(d) If $\alpha = \mathbf{1}$, from (4), (7) and (8),

$$\mathbf{X} \leq_{SPD(\alpha)} \mathbf{Y} \text{ if and only if } \mathbf{X} \leq_{PQD} \mathbf{Y}.$$

(e) From (5), if $\mathbf{X} \leq_{sm} \mathbf{Y}$, then $\mathbf{X} \leq_{SPD(\alpha)} \mathbf{Y}$ with $\alpha = \mathbf{1}$.

(f) In the general case, for $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ with $|\alpha_i| = 1$, $i = 1, 2, \ldots, n$, from (4), it follows that

$$\mathbf{X} \leq_{SPD(\alpha)} \mathbf{Y} \Longrightarrow \alpha\mathbf{X} \leq_{POD} \alpha\mathbf{Y} \text{ and } -\alpha\mathbf{X} \leq_{POD} -\alpha\mathbf{Y}.$$

(g) Finally, $\mathbf{X} \leq_{PD(\alpha)} \mathbf{Y}$ does not generally imply that $\alpha\mathbf{X} \leq_{sm} \alpha\mathbf{Y}$. For instance, for $\alpha = \mathbf{1}$, it can be seen by using Example 1 and taking into account that $\mathbf{X} \leq_{sm} \mathbf{Y}$ implies $\mathbb{P}[\mathbf{X} \leq \mathbf{x}] \leq \mathbb{P}[\mathbf{Y} \leq \mathbf{x}]$ for all $\mathbf{x}$.

3. Directional Dependence Orders and Copulas

Copulas are a very useful tool for studying the positive dependence property of a random vector—since it contains the dependence properties of the corresponding multivariate distribution function, independently of the marginal distributions—and scale-free measures of dependence, and they represent a starting point for constructing families of distributions (see [18]). Our goal now is the study of some of the orders given in the previous section in terms of copulas.

For $n \geq 2$, an n-dimensional *copula* (n-copula, for short) is the restriction to $[0,1]^n$ of a continuous n-dimensional distribution function whose univariate margins are uniform on $[0,1]$. The importance of copulas in statistics is described in the following result due to Abe Sklar [19]: let $\mathbf{X}$ be a random vector with a joint distribution function F and one-dimensional marginal distributions $F_1, F_2, \ldots, F_n$, respectively. Then, there exists an n-copula C (which is uniquely determined on $\times_{i=1}^{n} \mathrm{Range} F_i$) such that

$$F(\mathbf{x}) = C(F_1(x_1), F_2(x_2), \ldots, F_n(x_n)) \quad \text{for all } \mathbf{x} \in \overline{\mathbb{R}}^{\mathrm{n}}$$

(for a complete proof of this result, see [20]). Thus, copulas link joint distribution functions to their one-dimensional margins. For a survey on copulas, see [4,21] and some results about positive dependence properties and ordering by using copulas can be seen, for instance, in [4,5,9,22–25].

Let $\mathbf{X}$ be a random vector with associated n-copula C, and let X_{ij} denote the pair of random variables with components i and j of $\mathbf{X}$. $C_{X_{ij}}$ denotes the (i,j)-*margin* of C, i.e., $C_{ij}(u_i, u_j) = C(1, \ldots, 1, u_i, 1, \ldots, 1, u_j, 1, \ldots, 1)$, for every $1 \leq i < j \leq n$, which is the 2-copula associated with the pair X_{ij}.

Archimedean copulas are an important class of copulas because they are easily constructed and possess many nice properties; there is a great variety of families of copulas in this class and they have important applications in different areas. Let ϕ be a continuous and non-increasing function from $[0, +\infty]$ to $[0,1]$ such that $\phi(0) = 1$ and $\phi(+\infty) = 0$, and let ϕ^{-1} be the inverse of ϕ. Then, the function given by

$$C_\phi(\mathbf{u}) = \phi\left(\sum_{i=1}^{n} \phi^{-1}(u_i)\right), \quad \mathbf{u} \in [0,1]^n,$$

is an n-copula if and only if $(-1)^k \phi^{(k)}(t) \geq 0$ for $k = 0, 1, \ldots, n-2$, where $\phi^{(k)}$ denotes the *k-th* derivative of ϕ, and $(-1)^{n-2}\phi^{(n-2)}$ is non-increasing and convex. In such a case, we say that C_ϕ is an *Archimedean n-copula*, and the function ϕ is called a *generator* of C_ϕ. For more details, see [4,26].

In this section, we deal with the study of n-copulas associated with random vectors which are ordered in the sense of the PD(α) order.

3.1. The Bivariate Case

By simplicity, we start our study with the bivariate case.

Theorem 3. *Let $\mathbf{X} = (X_1, X_2)$ and $\mathbf{Y} = (Y_1, Y_2)$ be two random vectors with respective associated 2-copulas $C_{\mathbf{X}}$ and $C_{\mathbf{Y}}$. Let $\alpha = (\alpha_1, \alpha_2)$ such that $|\alpha_i| = 1, i = 1, 2$.*

(i) *If* $\alpha_1 \cdot \alpha_2 = 1$, *then* $\mathbf{X} \leq_{PD(\alpha)} \mathbf{Y}$ *if and only if* $C_{\mathbf{X}}(u,v) \leq C_{\mathbf{Y}}(u,v)$ *for all* $(u,v) \in [0,1]^2$.

(ii) *If* $\alpha_1 \cdot \alpha_2 = -1$, *then* $\mathbf{X} \leq_{PD(\alpha)} \mathbf{Y}$ *if and only if* $C_{\mathbf{Y}}(u,v) \leq C_{\mathbf{X}}(u,v)$ *for all* $(u,v) \in [0,1]^2$.

Proof. Consider the random vectors $\mathbf{X}^* = (\alpha_1 X_1, \alpha_2 X_2)$ and $\mathbf{Y}^* = (\alpha Y_1, \alpha_2 Y_2)$ with $\alpha = (\alpha_1, \alpha_2)$ and $|\alpha_i| = 1, i = 1,2$, and assume that $\mathbf{X} \leq_{PD(\alpha)} \mathbf{Y}$. Then, from Definition 2, it holds that

$$\overline{F}_{\mathbf{X}^*}(x_1, x_2) \leq \overline{G}_{\mathbf{Y}^*}(x_1, x_2) \tag{9}$$

for all $(x_1, x_2) \in \mathbb{R}^2$, where $\overline{F}_{\mathbf{X}^*}$ and $\overline{G}_{\mathbf{Y}^*}$ are the respective survival functions of $\mathbf{X}^*$ and $\mathbf{Y}^*$. By using the relationship between the survival function of a random vector and its associated *survival copula* (see, [4] [p. 32]), (9) is equivalent to

$$\hat{C}_{\mathbf{X}^*}(\overline{F}_{\alpha_1 X_1}(x_1), \overline{F}_{\alpha_2 X_2}(x_2)) \leq \hat{C}_{\mathbf{Y}^*}(\overline{F}_{\alpha_1 X_1}(x_1), \overline{F}_{\alpha_2 X_2}(x_2)), \tag{10}$$

where $\hat{C}_{\mathbf{X}^*}$ and $\hat{C}_{\mathbf{Y}^*}$ are the respective survival copulas associated with $\mathbf{X}^*$ and $\mathbf{Y}^*$. Moreover, given that $\mathbf{X}$ and $\mathbf{Y}$ are in the same Fréchet class and by considering the relationship between the copula and the corresponding survival copula, (10) is equivalent to

$$C_{\mathbf{X}^*}(F_{\alpha_1 X_1}(x_1), F_{\alpha_2 X_2}(x_2)) \leq C_{\mathbf{Y}^*}(F_{\alpha_1 X_1}(x_1), F_{\alpha_2 X_2}(x_2)).$$

Therefore, $\mathbf{X} \leq_{PD(\alpha)} \mathbf{Y}$ is equivalent to $C_{\mathbf{X}^*}(u,v) \leq C_{\mathbf{Y}^*}(u,v)$, for all $u, v \in [0,1]$.

Since copulas are invariant under the strictly increasing transformation of their components (see [4] (Theorem 2.4.3)), we have that, for $\alpha = (1,1)$, $\mathbf{X} \leq_{PD(\alpha)} \mathbf{Y}$ is equivalent to $C_{\mathbf{X}}(u,v) \leq C_{\mathbf{Y}}(u,v)$, for all $u, v \in [0,1]$. Furthermore, using [4] (Theorem 2.4.4), it follows that, for $\alpha = (-1,-1,)$, $C_{\mathbf{X}^*}(u,v) = u + v - 1 + C_{\mathbf{X}}(1-u, 1-v)$, and analogously for $C_{\mathbf{Y}^*}(u,v)$. So, for this case, $\mathbf{X} \leq_{PD(\alpha)} \mathbf{Y}$ is equivalent to $C_{\mathbf{X}}(1-u,1-v) \leq C_{\mathbf{Y}}(1-u,1-v)$ for all $u, v \in [0,1]$, that is, *(i)* is obtained.

By using [4] (Theorem 2.4.4), the result in *(ii)* is obtained following the same steps as above, which completes the proof. □

In the sequel, with the use of copulas, for the PD(α) order, we will use both the notations $\mathbf{X} \leq_{PD(\alpha)} \mathbf{Y}$ and $C_{\mathbf{X}} \leq_{PD(\alpha)} C_{\mathbf{Y}}$ interchangeably.

Example 4. *Let* $C_\theta^{\mathbf{CA}}$ *be the parametric family of Cuadras–Augé two-copulas given by* $C_\theta^{\mathbf{CA}}(u,v) = (uv)^{1-\theta} \min\{u,v\}^\theta$ *for every* $(u,v) \in [0,1]^2$, *where* $\theta \in [0,1]$ *(see [4,27]). In [4], (Example 2.19), it is shown that* $C_{\theta_1}^{\mathbf{CA}} \leq_{PD(-1,-1)} C_{\theta_2}^{\mathbf{CA}}$ *for* $\theta_1 \leq \theta_2$. *Furthermore, if* $\alpha_1 \cdot \alpha_2 = 1$ *(respectively,* $\alpha_1 \cdot \alpha_2 = -1$*), we have* $C_{\theta_1}^{\mathbf{CA}} \leq_{PD(\alpha)} C_{\theta_2}^{\mathbf{CA}}$ *if and only if* $\theta_1 \leq \theta_2$ *(respectively,* $\theta_2 \leq \theta_1$*).*

Example 5. *Let* $C_{\phi,\delta}^{\mathbf{AMH}}$ *be the Ali–Mikhail–Haq (AMH) Archimedean 2-copula [28] given by*

$$C_{\phi,\delta}^{\mathbf{AMH}}(u,v) = \frac{uv}{1 + \delta(1-u)(1-v)}$$

for all $(u,v) \in [0,1]^2$, *with* $\delta \in [-1,1]$ *and generator* $\phi(t) = \frac{1-\delta}{e^t - \delta}$. *In [4] (Exercise 2.32), it is stated that* $C_{\phi,\delta_1}^{\mathbf{AMH}} \leq_{PD(-1,-1)} C_{\phi,\delta_2}^{\mathbf{AMH}}$ *for* $\delta_1 \leq \delta_2$. *Furthermore, if* $\alpha_1 \cdot \alpha_2 = 1$ *(respectively,* $\alpha_1 \cdot \alpha_2 = -1$*), we have* $C_{\phi,\delta_1}^{\mathbf{AMH}} \leq_{PD(\alpha)} C_{\phi,\delta_2}^{\mathbf{AMH}}$ *if, and only if,* $\delta_1 \leq \delta_2$ *(respectively,* $\delta_2 \leq \delta_1$*).*

3.2. The Trivariate Case

Usually, the properties and results obtained for two-copulas become more difficult to develop in higher dimensions. Next, we show this fact, focusing on the trivariate case, for the sake of simplicity.

Following the same development as that of [9] (Theorem 2), the next result holds.

Theorem 4. *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be two trivariate random vectors with respective associated 3-copulas* $C_{\mathbf{X}}$ *and* $C_{\mathbf{Y}}$*. Let* $C_{\mathbf{X}_{ij}}$ *and* $C_{\mathbf{Y}_{ij}}$ *denote the* $(i,j)-$*margin of* $C_{\mathbf{X}}$ *and* $C_{\mathbf{Y}}$*, respectively, for* $1 \leq i < j \leq 3$*. Then, for all* $(u,v,w) \in [0,1]^3$*, we have:*

(i) $\mathbf{X} \leq_{PD(-1,-1,-1)} \mathbf{Y}$ *if and only if*

$$C_{\mathbf{X}}(u,v,w) \leq C_{\mathbf{Y}}(u,v,w).$$

(ii) $\mathbf{X} \leq_{PD(-1,-1,1)} \mathbf{Y}$ *if and only if*

$$C_{\mathbf{X}_{12}}(u,v) - C_{\mathbf{X}}(u,v,w) \leq C_{\mathbf{Y}_{12}}(u,v) - C_{\mathbf{Y}}(u,v,w).$$

(iii) $\mathbf{X} \leq_{PD(-1,1,-1)} \mathbf{Y}$ *if and only if*

$$C_{\mathbf{X}_{13}}(u,w) - C_{\mathbf{X}}(u,v,w) \leq C_{\mathbf{Y}_{13}}(u,w) - C_{\mathbf{Y}}(u,v,w).$$

(iv) $\mathbf{X} \leq_{PD(1,-1,-1)} \mathbf{Y}$ *if and only if*

$$C_{\mathbf{X}_{23}}(v,w) - C_{\mathbf{X}}(u,v,w) \leq C_{\mathbf{Y}_{23}}(v,w) - C_{\mathbf{Y}}(u,v,w).$$

(v) $\mathbf{X} \leq_{PD(-1,1,1)} \mathbf{Y}$ *if and only if*

$$C_{\mathbf{X}}(u,v,w) - C_{\mathbf{X}_{12}}(u,v) - C_{\mathbf{X}_{13}}(u,w) \leq C_{\mathbf{Y}}(u,v,w) - C_{\mathbf{Y}_{12}}(u,v) - C_{\mathbf{Y}_{13}}(u,w).$$

(vi) $\mathbf{X} \leq_{PD(1,-1,1)} \mathbf{Y}$ *if and only if*

$$C_{\mathbf{X}}(u,v,w) - C_{\mathbf{X}_{12}}(u,v) - C_{\mathbf{X}_{23}}(v,w) \leq C_{\mathbf{Y}}(u,v,w) - C_{\mathbf{Y}_{12}}(u,v) - C_{\mathbf{Y}_{23}}(u,w).$$

(vii) $\mathbf{X} \leq_{PD(1,1,-1)} \mathbf{Y}$ *if and only if*

$$C_{\mathbf{X}}(u,v,w) - C_{\mathbf{X}_{13}}(u,w) - C_{\mathbf{X}_{23}}(v,w) \leq C_{\mathbf{Y}}(u,v,w) - C_{\mathbf{Y}_{13}}(u,w) - C_{\mathbf{Y}_{23}}(v,w).$$

(viii) $\mathbf{X} \leq_{PD(1,1,1)} \mathbf{Y}$ *if and only if*

$$\begin{aligned} C_{\mathbf{X}_{12}}(u,v) + C_{\mathbf{X}_{13}}(u,w) + C_{\mathbf{X}_{23}}(v,w) - C_{\mathbf{X}}(u,v,w) \quad &\leq \\ C_{\mathbf{Y}_{12}}(u,v) + C_{\mathbf{Y}_{13}}(u,w) + C_{\mathbf{Y}_{23}}(v,w) - C_{\mathbf{Y}}(u,v,w)&. \end{aligned}$$

Proof. Let $\mathbf{X}$ be the random vector with the joint distribution function $F_{\mathbf{X}}$ and associated three-copula $C_{\mathbf{X}}$. For $\alpha = (-1,-1,-1)$, we have

$$\begin{aligned} \mathbb{P}[\alpha_1 X_1 > x_1, \alpha_2 X_2 > x_2, \alpha_3 X_3 > x_3] &= \mathbb{P}[X_1 \leq x_1, X_2 \leq x_2, X_3 \leq x_3] \\ &= F_{\mathbf{X}}(x_1,x_2,x_3) \end{aligned}$$

for all $(x_1,x_2,x_3) \in \overline{\mathbb{R}}^3$. From Sklar's theorem, we have

$$F_{\mathbf{X}}(x_1,x_2,x_3) = C_{\mathbf{X}}(F_1(x_1),F_2(x_2),F_3(x_3)) = C_{\mathbf{X}}(u_1,u_2,u_3)$$

for all $(u_1,u_2,u_3) \in [0,1]^3$, whence part (i) easily follows.

To prove part (ii), note that, for $\alpha = (-1,-1,1)$, we have

$$\begin{aligned} \mathbb{P}[\alpha_1 X_1 > x_1, \alpha_2 X_2 > x_2, \alpha_3 X_3 > x_3] &= \mathbb{P}[X_1 \leq x_1, X_2 \leq x_2, X_3 > x_3] \\ &= \mathbb{P}[X_1 \leq x_1, X_2 \leq x_2] - \mathbb{P}[X_1 \leq x_1, X_2 \leq x_2, X_3 \leq x_3] \\ &= F_{1,2}(x_1,x_2) - F_{\mathbf{X}}(x_1,x_2,x_3) \end{aligned}$$

for all $(x_1,x_2,x_3) \in \overline{\mathbb{R}}^3$, where $F_{i,j}$ denotes the (i,j)-margin of $F_{\mathbf{X}}$ for $1 \leq i < j \leq n$. Parts (iii) and (iv) can be proved in a similar way.

Part (v)—and similarly, parts (vi) and (vii)—follows from the fact that, for $\alpha = (-1,1,1)$ and for all $(x_1,x_2,x_3) \in \overline{\mathbb{R}}^3$, we have

$$\begin{aligned}\mathbb{P}[\alpha_1 X_1 > x_1, \alpha_2 X_2 > x_2, \alpha_3 X_3 > x_3] &= \mathbb{P}[X_1 \le x_1, X_2 > x_2, X_3 > x_3] \\ &= \mathbb{P}[X_1 \le x_1, X_2 > x_2] - \mathbb{P}[X_1 \le x_1, X_2 > x_2, X_3 \le x_3] \\ &= \mathbb{P}[X_1 \le x_1] - \mathbb{P}[X_1 \le x_1, X_2 \le x_2] \\ &\quad -\mathbb{P}[X_1 \le x_1, X_3 \le x_3] \\ &\quad +\mathbb{P}[X_1 \le x_1, X_2 \le x_2, X_3 \le x_3] \\ &= x_1 - F_{1,2}(x_1,x_2) - F_{1,3}(x_1,x_3) + F_{\mathbf{X}}(x_1,x_2,x_3). \end{aligned} \tag{11}$$

Finally, for part (viii), note that, for every $(x_1,x_2,x_3) \in \overline{\mathbb{R}}^3$ and using (11), we have

$$\begin{aligned}\mathbb{P}[\alpha_1 X_1 > x_1, \alpha_2 X_2 > x_2, \alpha_3 X_3 > x_3] &= \mathbb{P}[X_2 > x_2, X_3 > x_3] \\ &\quad -\mathbb{P}[X_1 \le x_1, X_2 > x_2, X_3 > x_3] \\ &= \mathbb{P}[X_2 > x_2] - \mathbb{P}[X_2 > x_2, X_3 \le x_3] \\ &\quad -\mathbb{P}[X_1 \le x_1, X_2 > x_2, X_3 > x_3] \\ &= 1 - x_2 - \mathbb{P}[X_3 \le x_3] + \mathbb{P}[X_2 \le x_2, X_3 \le x_3] \\ &\quad -\mathbb{P}[X_1 \le x_1, X_2 > x_2, X_3 > x_3] \\ &= 1 - x_1 - x_2 - x_3 + -F_{1,2}(x_1,x_2) + F_{1,3}(x_1,x_3) \\ &\quad +F_{2,3} - F_{\mathbf{X}}(x_2,x_3), \end{aligned}$$

which completes the proof. □

Example 6. *Let $C_\theta^{\mathbf{FGM}}$ be the one-parameter three-copula given by*

$$C_\theta^{\mathbf{FGM}}(\mathbf{u}) = \left(\prod_{i=1}^{3} u_i\right)\left[1 + \theta \prod_{i=1}^{3}(1-u_i)\right], \quad \mathbf{u} \in [0,1]^3, \tag{12}$$

with θ in $[0,1]$. $C_\theta^{\mathbf{FGM}}$ belongs to the Farlie–Gumbel–Morgenstern family of 3-copulas (see [4,21]). Consider two members of this family, say, $C_{\theta_1}^{\mathbf{FGM}}$ and $C_{\theta_2}^{\mathbf{FGM}}$. For $\prod_{i=1}^3 \alpha_i = -1$ (respectively, $\prod_{i=1}^3 \alpha_i = 1$), we have that $C_{\theta_1}^{\mathbf{FGM}} \le_{PD(\alpha)} C_{\theta_2}^{\mathbf{FGM}}$ if and only if $\theta_1 \le \theta_2$ (respectively, $\theta_2 \le \theta_1$).

An additional example involving a bi-parametric family of three-copulas and the three-copula for three independent random variables is given in [9] (Example 3).

The following result, in which $=_{st}$ denotes equality in distribution, shows that if two (trivariate) random vectors are ordered in all directions, then they have the same distribution.

Theorem 5. *Given the two trivariate random vectors $\mathbf{X}$ and $\mathbf{Y}$, we have that $\mathbf{X} \le_{PD(\alpha)} \mathbf{Y}$ for every direction $\alpha \in [0,1]^3$, with $|\alpha_i| = 1$, $i = 1,2,3$, if and only if $\mathbf{X} =_{st} \mathbf{Y}$.*

Proof. Assume that $\mathbf{X} \le_{PD(\alpha)} \mathbf{Y}$, for all directions $\alpha \in [0,1]^3$. From items *(i)* and *(iii)* in Theorem 4 and for every $(u,v,w) \in [0,1]^3$, it follows that $C_{\mathbf{Y}}(u,v,w) \ge C_{\mathbf{X}}(u,v,w) \ge C_{\mathbf{Y}}(u,v,w) + C_{\mathbf{X}_{13}}(u,w) - C_{\mathbf{Y}_{13}}(u,w)$, and therefore, $C_{\mathbf{X}_{13}}(u,w) \le C_{Y_{13}}(u,w)$. On the other hand, from *(ii)* and *(v)*, we obtain $C_{\mathbf{Y}}(u,v,w) + C_{\mathbf{X}_{12}}(u,v) - C_{\mathbf{Y}_{12}}(u,v) + C_{\mathbf{X}_{13}}(u,w) - C_{\mathbf{Y}_{13}}(u,w) \ge C_{\mathbf{X}}(u,v,w) \ge C_{\mathbf{Y}}(u,v,w) + C_{\mathbf{X}_{12}}(u,v) - C_{\mathbf{Y}_{12}}(u,v)$, and therefore $C_{\mathbf{X}_{13}}(u,w) \ge C_{\mathbf{Y}_{13}}(u,w)$. Combining both results, it follows that $C_{\mathbf{X}_{1,3}}(u,w) = C_{\mathbf{Y}_{13}}(u,w)$. Similarly, using the different items, we obtain $C_{\mathbf{X}_{12}}(u,v) = C_{\mathbf{Y}_{12}}(u,v)$ and $C_{\mathbf{X}_{23}}(v,w) = C_{\mathbf{Y}_{23}}(v,w)$. From *(iii)*, we have $C_{\mathbf{Y}}(u,v,w) \le C_{\mathbf{X}}(u,v,w) + C_{\mathbf{Y}_{13}}(u,w) - C_{\mathbf{X}_{13}}(u,w) = C_{\mathbf{X}}(u,v,w)$, and hence, we conclude $C_{\mathbf{Y}}(u,v,w) = C_{\mathbf{X}}(u,v,w)$. □

3.3. The $PD(-\mathbf{1})$ Order for Archimedean n-Copulas

The next result—whose proof can be found in [29] for the bivariate case, and in [30] for the general case—shows, under some conditions, the $\mathrm{PD}(-\mathbf{1})$ order for two Archimedean n-copulas. For that, we recall that a function f defined on $[0,+\infty]$ is *super-additive* if $f(x+y) \geq f(x) + f(y)$ for all $x, y \in [0,+\infty]$.

Proposition 2. *For two Archimedean n-copulas C_1 and C_2 with respective generators ϕ_1 and ϕ_2, if $\phi_2^{-1} \circ \phi_1$ is super-additive, and then $C_1 \leq_{PD(-\mathbf{1})} C_2$.*

As an application of Proposition 2, we provide an example.

Example 7. *For all $t \in [0,+\infty]$, given the generators $\phi_1(t) = \frac{1-\delta}{e^t-\delta}$, with $\delta \in [0,1]$, and $\phi_2(t) = (1+\gamma t)^{-1/\gamma}$, with $\gamma > 0$, we consider the generalizations to n-dimensions of the AMH family of Archimedean two-copulas—denoted by $C^{\mathbf{AMH}}_{n,\phi_1,\delta}$— given in Example 5 (see [31]) and a Clayton subfamily of Archimedean two-copulas—denoted by $C^{\mathbf{C}}_{n,\phi_2,\gamma}$— (see [31,32]). For the sake of simplicity, we consider $\gamma = 1$. Since $(\phi_2^{-1} \circ \phi_1)(t) = \frac{e^t-\delta}{1-\delta} - 1$ for all $t \in [0,+\infty]$, we have that, for all $x, y \in [0,+\infty]$,*

$$\left(\phi_2^{-1} \circ \phi_1\right)(x+y) \geq \left(\phi_2^{-1} \circ \phi_1\right)(x) + \left(\phi_2^{-1} \circ \phi_1\right)(y)$$

if and only if

$$\frac{e^{x+y}-\delta}{1-\delta} - 1 \geq \frac{e^x-\delta}{1-\delta} - 1 + \frac{e^y-\delta}{1-\delta} - 1,$$

which is equivalent to

$$e^{x+y} \geq e^x + e^y - 1,$$

i.e.,

$$(e^x-1)(e^y-1) \geq 0,$$

whence $\phi_2^{-1} \circ \phi_1$ is super-additive, and hence, from Proposition 2, we have $C^{\mathbf{AMH}}_{n,\phi_1,\delta} \leq_{PD(-\mathbf{1})} C^{\mathbf{C}}_{n,\phi_2,1}$.

Remark 1. *As Nelsen [4] notes, verifying the super-additivity of $\phi_2^{-1} \circ \phi_1$ is not easy, but there exist several results that give sufficient conditions for that super-additivity—in principle, for the bivariate case— and can be generalized to n dimensions. We refer to [4,29,33] for more details.*

However, in general, the Archimedean copulas are not ordered in the sense of the $PD(\boldsymbol{\alpha})$ order for $\boldsymbol{\alpha} \neq -\mathbf{1}$, as the following example shows.

Example 8. *For all $t \in [0,+\infty]$, given the generator $\phi_{3,\beta}(t) = exp\left(-t^{1/\beta}\right)$, with $\beta \in [1,+\infty]$, we consider the Gumbel–Hougaard family of Archimedean two-copulas (see [4,34,35]). A generalization to n dimensions of this family, which we denote by $C^{\mathbf{GH}}_{n,\phi_{3,\beta}}$, can be found in [4] (Example 4.25). We consider two members of this family, i.e., $C^{\mathbf{GH}}_{n,\phi_{3,\beta_i}}$ for $i = 1,2$. In [4], (Example 4.12), it is shown that $C^{\mathbf{GH}}_{2,\phi_{3,\beta_2}} \leq_{PD(-\mathbf{1})} C^{\mathbf{GH}}_{2,\phi_{3,\beta_1}}$ if and only if $\beta_2 \leq \beta_1$. From Proposition 2, it is easy to prove that "$C^{\mathbf{GH}}_{n,\phi_{3,\beta_2}} \leq_{PD(-\mathbf{1})} C^{\mathbf{GH}}_{n,\phi_{3,\beta_1}}$ if, and only if, $\beta_2 \leq \beta_1$" is also satisfied.*

Now, let us consider the direction $\boldsymbol{\alpha} = (-1,-1,1)$. For $(u,v,w) = (0.43, 0.52, 0.43)$, by using Theorem 4(ii), we have $C^{\mathbf{GH}}_{3,\phi_{3,3}}(0.43, 0.52, 1) - C^{\mathbf{GH}}_{3,\phi_{3,3}}(0.43, 0.52, 0.43) = 0.06 > 0.02 = C^{\mathbf{GH}}_{3,\phi_{3,8}}(0.43, 0.52, 1) - C^{\mathbf{GH}}_{3,\phi_{3,8}}(0.43, 0.52, 0.43)$; however, for $(u,v,w) = (0.29, 0.26, 0.1)$, we obtain $C^{\mathbf{GH}}_{3,\phi_{3,3}}(0.29, 0.26, 1) - C^{\mathbf{GH}}_{3,\phi_{3,3}}(0.29, 0.26, 0.1) = 0.11 < 0.14 = C^{\mathbf{GH}}_{3,\phi_{3,8}}(0.29, 0.26, 1) - C^{\mathbf{GH}}_{3,\phi_{3,8}}(0.29, 0.26, 0.1)$; therefore, these three-copulas are not ordered in this direction $\boldsymbol{\alpha}$.

3.4. Directional Coefficients

One of the most important nonparametric measures of association between the components of a continuous random pair (X, Y) is Spearman's rho, which we denote by $\rho(C)$, where C is the two-copula associated with the pair (X, Y), being

$$\rho(C) = 12 \int_{[0,1]^2} C(u, v) \, \mathrm{d}u\mathrm{d}v - 3$$

(see [4] and the references therein for more details). This measure of association—in fact, a measure of concordance [36]—provides information about the magnitude and direction of the association between the random variables.

As an immediate consequence of Theorem 3, we have the following result.

Corollary 1. *Let $\mathbf{X} = (X_1, X_2)$ and $\mathbf{Y} = (Y_1, Y_2)$ be two random vectors with respective associated two-copulas $C_{\mathbf{X}}$ and $C_{\mathbf{Y}}$. Let $\alpha = (\alpha_1, \alpha_2)$ such that $|\alpha_i| = 1, i = 1, 2$.*

(i) If $\alpha_1 \cdot \alpha_2 = 1$, then $\mathbf{X} \leq_{PD(\alpha)} \mathbf{Y}$ implies $\rho(C_{\mathbf{X}}) \leq \rho(C_{\mathbf{Y}})$.

(ii) If $\alpha_1 \cdot \alpha_2 = -1$, then $\mathbf{X} \leq_{PD(\alpha)} \mathbf{Y}$ implies $\rho(C_{\mathbf{Y}}) \leq \rho(C_{\mathbf{X}})$.

Now, we consider the trivariate case (given the complexity in higher dimensions). For a trivariate random vector (X_1, X_2, X_3) of continuous random variables uniform on $[0, 1]$, whose distribution function is the 3-copula C, the *directional ρ-coefficients* are defined for each $(\alpha_1, \alpha_2, \alpha_3)$, with $|\alpha_i| = 1$ for $i = 1, 2, 3$, as

$$\rho_3^{(\alpha_1,\alpha_2,\alpha_3)}(C) = 8 \int_{[0,1]^3} Q_{\alpha_1\alpha_2\alpha_3}(x_1, x_2, x_3) \, \mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3,$$

where

$$Q_{\alpha_1\alpha_2\alpha_3}(x_1, x_2, x_3) = \mathbb{P}\left[\bigcap_{i=1}^{3}(\alpha_i X_i > \alpha_i x_i)\right] - \prod_{i=1}^{3} \mathbb{P}[\alpha_i X_i > \alpha_i x_i]$$

(see [10]). Unlike the measure Spearman's rho, the coefficient $\rho_3^{(\alpha_1,\alpha_2,\alpha_3)}$ is not a multivariate measure of association.

Example 9. *Consider the subfamily of FGM three-copulas given by (12). Then, it is easy to show that: (i) For $\prod_{i=1}^{3} \alpha_i = -1$, we have $\rho_3^{(\alpha_1,\alpha_2,\alpha_3)}(C_\theta^{\mathbf{FGM}}) = \frac{\theta}{27}$; and (ii) for $\prod_{i=1}^{3} \alpha_i = 1$, we have $\rho_3^{(\alpha_1,\alpha_2,\alpha_3)}(C_\theta^{\mathbf{FGM}}) = -\frac{\theta}{27}$.*

As a result of our findings, we have the following outcome.

Corollary 2. *Let $\mathbf{X}$ and $\mathbf{Y}$ be two trivariate vectors of continuous random variables uniform on $[0, 1]$ whose respective distribution functions are the three-copulas $C_{\mathbf{X}}$ and $C_{\mathbf{Y}}$. Let $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ be a direction such that $|\alpha_i| = 1, i = 1, 2, 3$. If $\mathbf{X} \leq_{PD(\alpha)} \mathbf{Y}$, then $\rho_3^{(\alpha_1,\alpha_2,\alpha_3)}(C_{\mathbf{X}}) \leq \rho_3^{(\alpha_1\alpha_2\alpha_3)}(C_{\mathbf{Y}})$.*

We note that Corollary 2 generalizes that, for the bivariate case, the *POD* order between two vectors implies the order between their corresponding Spearman's ρ coefficients (see, for instance, [5]).

4. Conclusions

In this paper, we establish a multivariate order by leveraging the principle of orthant directional dependence and delve into an in-depth exploration of its inherent properties. Our investigation extends to scrutinizing the connections it shares with alternative dependence orders expounded in existing literature. Furthermore, we engage in a comprehensive analysis by comparing two random vectors through their respective associated copulas. To provide a tangible and illustrative dimension to our findings, we incorporate a diverse set of examples that serve to underscore and elucidate the nuances of our results.

The study of certain outcomes regarding the Baire category for the stochastic orders introduced within this work (similarly to those investigated in [37]) stands as a focal point for future research.

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