QUASILINEAR ELLIPTIC PROBLEMS WITH SINGULAR AND HOMOGENEOUS LOWER ORDER TERMS

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Abstract. We deal with singular quasilinear elliptic equations, namely

$$\begin{cases} -\Delta u = \lambda u + \mu(x)\dfrac{|\nabla u|^q}{u^{q-1}} + f(x) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded smooth domain of $\mathbb{R}^N$ $(N \geq 3)$, $\lambda \in \mathbb{R}$, $1 < q \leq 2$, $0 \leq \mu \in L^\infty(\Omega)$ and $0 \leq f \in L^p(\Omega)$ for some $p > \frac{N}{2}$. We completely describe the set of values of the parameter λ for which the problem admits solution. Thus, we study existence, nonexistence and uniqueness of bounded weak solutions in both cases $f \gneqq 0$ and $f \equiv 0$.

1. Introduction

The present paper is devoted to the study of the following quasilinear elliptic problem:

$$(P_\lambda) \qquad \begin{cases} -\Delta u = \lambda u + \mu(x)\dfrac{|\nabla u|^q}{u^{q-1}} + f(x) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$ $(N \geq 3)$ with smooth boundary (say, of class $C^{1,1}$), $\lambda \in \mathbb{R}$, $0 \leq \mu \in L^\infty(\Omega)$, $0 \leq f \in L^p(\Omega)$ with $p > \frac{N}{2}$ and $1 < q \leq 2$.

Problem (P_λ) is a particular case of the following general model

$$(1.1) \qquad \begin{cases} -\Delta u = \lambda u + \mu(x)g(u)|\nabla u|^q + f(x) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

for some nontrivial real function g. We first observe that, for $\mu \equiv 0$, the equation above becomes linear. In fact, it is an eigenvalue problem if $f \equiv 0$ which admits solution if and only if $\lambda = \Lambda$ (the principal eigenvalue of the Laplacian in Ω with zero Dirichlet boundary condition), while if $f \not\equiv 0$, it is well known that there exists a solution to (1.1) for any f if and only if $\lambda < \Lambda$ (and in such a case, the solution is also unique).

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The picture changes drastically if $\mu \not\equiv 0$. Indeed, in such a case the equation becomes quasilinear and the above results are no longer true. In fact, when the gradient term is considered, existence and/or uniqueness of solutions may fail. For instance, the model problem, with $\mu \in L^\infty(\Omega)$ and $f \in L^p(\Omega)$, $p > N/2$,

$$\begin{cases} -\Delta u = \lambda u + \mu(x)|\nabla u|^2 + f(x) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

can be studied, for $\mu(x) \equiv \mu \in \mathbb{R}^+$, trough the Hopf-Cole transformation, and it is turned ($v = \frac{e^{\mu u}-1)}{\mu}$) into the following semilinear problem

$$\begin{cases} -\Delta v = (\mu v + 1)\left(f(x) + \dfrac{\lambda}{\mu}\log(1+\mu v)\right) & \text{in } \Omega, \\ v > 0 & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

Thus it is clear that the existence and (possibly) the uniqueness of a solution depends on the sizes of μ and f. Furthermore the nature of the problem is essentially different from the one of the linear problem. Indeed, it has been recently proved in [7] (see also [24]) that if problem (1.2) with $\mu(x) \geq \mu_0 > 0$ admits a solution with $\lambda = 0$, then there exist at least two different solutions to (1.1) for $0 < \lambda < \lambda^*$, for a suitable value $0 < \lambda^* < \Lambda$.

Our idea is that the threshold value λ^* is associated, in some sense, to the principal eigenvalue of the nonlinear differential operator that appears in the equation in (1.1) (with $f(x) \equiv 0$). Thus the lower order term has necessarily to satisfy a 1-homogeneous condition, that leads to the choice of a singular term of the form $g(u) = 1/u^{q-1}$ in (1.1) (see (P_λ)).

The study of singular Dirichlet problems with gradient terms having quadratic growth ($q = 2$) has raised considerable interest in recent years. Let us quote the main references [1, 2, 3, 4, 11, 15, 18, 19, 20], among others, dealing with existence (and nonexistence) results for equations with singular lower order terms, while we mention [5, 9, 14] for uniqueness results on this type of problems.

In contrast with the results of [7], when one considers a singular function as $g(u) = 1/u$ in problem (1.1), $q = 2$ and $f \gneq 0$, in [8] the authors prove the existence of solution for every $\lambda < \frac{\Lambda}{\|\mu\|_{L^\infty(\Omega)}+1}$. Moreover, if $\mu(x) \equiv \mu \in (0,1)$, they prove that there exists a solution if and only if $\lambda < \frac{\Lambda}{\mu+1}$, and in such a case, the solution is unique and $\frac{\Lambda}{\mu+1}$ is a bifurcation point from infinity.

Surprisingly, this phenomenon, analogous to the one observed in the the linear case, is not only due to the presence of a singularity at $u = 0$. Actually, the technique developed in [7] applies (with some small changes) to problem (1.1) with $\mu(x) \geq \mu_0 > 0$, $q = 2$, $g(s) = 1/s^\theta$, $\theta \in (0,1)$ and consequently if there exists a solution with $\lambda = 0$, then multiplicity occurs for $\lambda > 0$ small enough (as in the nonsingular case $g(s) \equiv 1$).

In the present paper we aim to provide a general method to deal with problem (P_λ) in the general framework $1 < q \leq 2$, depending only on the quasilinear nature of the problem and allowing the complete description of the set of values of the parameter λ such that (P_λ) admits a solution. Of course, the main difficulties in order to study such a problem are due to the superlinearity of the lower order term and the singularity as u approaches 0. In fact, we will notice that the key point is not (only) the singularity by itself, but the homogeneity that the singularity gives to the lower order term which allows us to look at (P_λ) through the following eigenvalue problem

$$(E_\lambda) \qquad \begin{cases} -\Delta u = \lambda u + \mu(x)\dfrac{|\nabla u|^q}{u^{q-1}} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

We provide this kind of eigenvalue existence result for problem (E_λ) making use of a precise characterization of the principal eigenvalue:

$$\lambda^* = \sup\left\{ \lambda \in \mathbb{R} \;\middle|\; \begin{array}{c} \text{there exists a supersolution } v \text{ to } (E_\lambda) \\ \text{such that } v \geq c \text{ in } \Omega \text{ for some } c > 0 \end{array} \right\} \tag{1.3}$$

(the precise meaning of supersolution used in (1.3) is specified in Section 2 below). This characterization has been inspired by the seminal paper [10] and allows us to study the (nonvariational) eigenvalue problem (E_λ) since it requires only working with supersolutions, but does not involve any variational structure of the problem.

However, the definition (1.3) will be useful only if we can compare subsolutions and supersolutions to problem (E_λ) that are appropriately ordered on the boundary of the domain. Indeed, we will be able to derive the required Comparison Principle (see Theorem 3.1 below) by adapting the ideas contained in [6].

Let us stress that, at least formally, the change of unknown $v = -\log(u)$ turns the solutions to (E_λ) into solutions to

$$(\widetilde{E}_\lambda) \qquad \begin{cases} -\Delta v + |\nabla v|^2 + \mu(x)|\nabla v|^q + \lambda = 0 & \text{in } \Omega, \\ v = +\infty & \text{on } \partial\Omega. \end{cases}$$

Quasilinear problems whose solutions blow-up at the boundary of the domain (known in literature as *large solutions*) have been widely studied (see for instance [26], [27], [29]). A particular feature of $(\widetilde{E}_\lambda)$ is that it is invariant under transformations of the type $v \mapsto v + t$ for all $t \in \mathbb{R}$. For problems of this class it has been proved in [26] [27] that there is a unique value of the parameter λ (the so called *ergodic constant*) for which the problem into consideration admits a (unique, up to additive constants) large solution.

We state now our first theorem about problem (E_λ), in which we show by an approximation and compactness argument that, in fact, λ^* is the principal eigenvalue to (E_λ).

Theorem 1.1. *Assume that $1 < q \leq 2$ and $0 \leq \mu \in L^\infty(\Omega)$, with $\|\mu\|_{L^\infty(\Omega)} < 1$ if $q = 2$. Then $\lambda^* \in (0, \Lambda]$ and problem (E_λ) admits a solution if and only if $\lambda = \lambda^*$. Moreover such a solution is unique up to multiplication by positive constants.*

As far as (P_λ) is concerned, some parts of the main result will require stronger hypotheses on the datum f, that we list here:

$$(f_0) \qquad \forall \omega \subset\subset \Omega \quad \exists c_\omega > 0 : \quad f(x) \geq c_\omega \quad \text{a.e. } x \in \omega;$$

$$(f_1) \qquad \exists \gamma \in \left(\frac{1}{2}, 1\right), C_1 > 0 : \quad f \geq C_1 \varphi_1^\gamma \quad \text{in } \Omega;$$

$$(f_2) \qquad \exists C_1 > 0 : \quad f \geq C_1 \varphi_1^\gamma \quad \text{in } \Omega, \quad \text{where } \gamma = \frac{1}{1 + \|\mu\|_{L^\infty(\Omega)}}.$$

Now we state our main result.

Theorem 1.2. *Assume that $1 < q \le 2$, $0 \le \mu \in L^\infty(\Omega)$, with $\|\mu\|_{L^\infty(\Omega)} < 1$ if $q = 2$, and $0 \lneq f \in L^p(\Omega)$ with $p > \frac{N}{2}$. Then (P_λ) has a unique solution if $\lambda \le 0$, has at least a solution if $\lambda < \lambda^*$, and has no solution if $\lambda > \lambda^*$. If, in addition, f satisfies condition (f_0), then (P_λ) has a unique solution for every $\lambda < \lambda^*$. Finally, if f satisfies condition (f_1) for $1 < q < 2$ and (f_2) for $q = 2$, then (P_λ) has no solution for any $\lambda \ge \lambda^*$ and moreover the set $\Sigma := \{(\lambda, u_\lambda) : u_\lambda$ is a solution to $(P_\lambda)\}$ is an unbounded continuum in $\mathbb{R} \times C(\overline{\Omega})$ which bifurcates from infinity at λ^* to the left.*

We stress that the previous theorem improves the existence result contained in [8] for μ nonconstant and $q = 2$. In fact, we determine that the set of $\lambda \in \mathbb{R}$ where problem (P_λ) admits a solution is either $(-\infty, \lambda^*)$ or possibly its closure. Moreover, we consider the whole range $1 < q \le 2$. The critical problem corresponding to $\lambda = \lambda^*$, and also the uniqueness for $\lambda > 0$, exhibit some difficulties. Nonetheless, we overcome them by imposing stronger hypothesis on f. Doing so, we prove that the interval $(-\infty, \lambda^*)$ is optimal for the existence of solution, and we even prove uniqueness in this interval.

It is worth to stress that one of the main contribution of this paper is the comparison principle. In fact it is not obvious, an indeed the literature on this topic is extremely poor, that a comparison principle holds true when we deal with positive values of λ in (1.2).

The plan of the paper is the following: we devote Section 2 to introduce the definitions of solution, supersolution and bifurcation point from infinity, and we also prove some regularity properties of the solutions; in Section 3 we state and prove some comparison principles and a uniqueness result to problem (P_λ); section 4 is devoted to prove that λ^* is well defined and positive, to give some alternative characterizations of it, and to prove some nonexistence results; in Section 5 we introduce the approximate problems, we prove some a priori estimates and a compactness result, and we give several existence and bifurcation results, and in Section 6 we collect the proofs of Theorem 1.1 and Theorem 1.2. Finally, in Appendix A we show that problem (P_λ) possesses two equivalent formulations and we also prove the regularity of the solution.

2. Definitions and preliminary results

In this section we make precise some definitions and we prove some results that we will use in the rest of the paper.

First of all we specify the meaning of solution to problem (P_λ), as well as the concept of supersolution used in (1.3).

Definition 2.1. For every $\lambda \in \mathbb{R}$, we say that $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ is a *solution* to (P_λ) if $u > 0$ a.e. in Ω, $\frac{|\nabla u|^q}{u^{q-1}} \in L^1_{\mathrm{loc}}(\Omega)$ and it holds

$$\int_\Omega \nabla u \nabla \phi = \lambda \int_\Omega u\phi + \int_\Omega \mu(x) \frac{|\nabla u|^q}{u^{q-1}} \phi + \int_\Omega f(x)\phi \qquad \forall \phi \in H_0^1(\Omega) \cap L^\infty(\Omega). \tag{2.1}$$

Similarly, we say that $u \in H^1(\Omega) \cap L^\infty(\Omega)$ is a *supersolution* to (P_λ) if $u > 0$ a.e. in Ω, $\frac{|\nabla u|^q}{u^{q-1}} \in L^1_{\mathrm{loc}}(\Omega)$ and the following inequality holds

$$\int_\Omega \nabla u \nabla \phi \ge \lambda \int_\Omega u\phi + \int_\Omega \mu(x) \frac{|\nabla u|^q}{u^{q-1}} \phi + \int_\Omega f(x)\phi \qquad \forall \phi \in H_0^1(\Omega) \cap L^\infty(\Omega), \quad \phi \ge 0\,. \tag{2.2}$$

Some remarks on about the formulation are in order.

Remark 2.2. *Let us observe that since the lower order term it is only locally integrable in Ω, there is a term above that, a priori, might not make sense. Actually, applying some density arguments we can show that, in spite of the presence of a singular lower order term, the above formulations are equivalent to the ones in which* *the test functions belong to $C_c^1(\Omega)$ both in (2.1) and (2.2). We collect the proof of such an equivalence in the Appendix.*

Remark 2.3. *In the model case $q = 2$ and μ constant, it is clear that the condition $\mu < 1$ is in fact necessary for the existence of solutions to problem (P_λ) with $\lambda > 0$. Indeed we can use u as test function in (2.1), so that we obtain $\int_\Omega |\nabla u|^2 = \lambda \int_\Omega u^2 + \mu \int_\Omega |\nabla u|^2 + \int_\Omega f(x)u$. Therefore, since $\lambda u^2 > 0$ in Ω, we have that $(1-\mu)\int_\Omega |\nabla u|^2 > 0$.*

The following three lemmata provide some properties of the solutions to (P_λ) which will be useful later.

Lemma 2.4. *Let $1 < q \leq 2$, $0 \leq \mu \in L^\infty(\Omega)$, with $\|\mu\|_{L^\infty(\Omega)} < 1$ if $q = 2$, $0 \lneq f \in L^p(\Omega)$ with $p > \frac{N}{2}$, and let u be a solution to (P_λ) for some $\lambda \in \mathbb{R}$. Then $u \in C^{0,\alpha}(\overline{\Omega}) \cap W^{1,2p}_{loc}(\Omega)$ for some $\alpha \in (0,1)$.*

The proof of the above lemma is given in its details in the Appendix. Anyway, we observe here that for the interior regularity we exploit that the solutions are strictly positive, as a consequence of the Strong Maximum Principle.

As far as the Hölder continuity up to the boundary is concerned, we need to strongly use the techniques developed in [25]: let us observe that since the singularity has the order of $1/u^{q-1}$ with $q < 2$ (in the case $q = 2$ it is also used that $\mu(x)$ is small), then it represents, in some sense, a "mild" singularity.

The Sobolev interior regularity is proved via an interpolation and bootstrap argument.

Remark 2.5. *Notice that Lemma 2.4 provides as much information about the regularity of the solutions to (P_λ) as the knowledge that one has about the regularity of the data. For instance, under the hypotheses of Lemma 2.4, we have in particular that any solution u to (E_λ) satisfies that $-\Delta u \in L^r_{loc}(\Omega)$ for any $r < \infty$. Hence, $u \in W^{2,r}_{loc}(\Omega)$ for any $r < \infty$. Even more, if $\mu \in W^{1,\infty}_{loc}(\Omega)$, we easily deduce that $-\Delta u \in W^{1,r}_{loc}(\Omega)$ for any $r < \infty$, and thus, $u \in W^{3,r}_{loc}(\Omega)$ for any $r < \infty$ (see* [22, Theorem 9.19]*). We may continue the bootstrap in this way so that, if $\mu \in W^{k,\infty}_{loc}(\Omega)$ for some $k \geq 1$, then $u \in W^{k+2,r}_{loc}(\Omega)$ for any $r < \infty$. Thus, if $\mu \in C^\infty(\Omega)$, then $u \in C^\infty(\Omega)$.*

Lemma 2.6. *Let $1 < q \leq 2$, $0 \leq \mu \in L^\infty(\Omega)$, with $\|\mu\|_{L^\infty(\Omega)} < 1$ if $q = 2$, $0 \lneq f \in L^p(\Omega)$ with $p > \frac{N}{2}$, and let u be a solution to (P_λ) for some $\lambda \in \mathbb{R}$. Then $u^\gamma \in H^1_0(\Omega)$ for every $\gamma > \gamma_0(q)$, given by*

$$\gamma_0(q) = \begin{cases} \dfrac{1}{2} & \text{if } 1 < q < 2, \\[2ex] \dfrac{1 + \|\mu\|_{L^\infty(\Omega)}}{2} & \text{if } q = 2. \end{cases}$$

Proof. We follow here the arguments of Theorem 3.1 in [8], which in turn come from the ideas of [3]. We claim that

$$\int_\Omega \frac{|\nabla u|^2}{u^{1-\beta}} < \infty, \qquad \forall \beta \in (\beta_0(q), 1], \tag{2.3}$$

where

$$\beta_0(q) = \begin{cases} 0 & \text{if } 1 < q < 2, \\ \|\mu\|_{L^\infty(\Omega)} & \text{if } q = 2. \end{cases}$$

Indeed, given $\beta \in (\beta_0(q), 1]$, observe that the function $(u+\varepsilon)^\beta - \varepsilon^\beta \in H^1_0(\Omega) \cap L^\infty(\Omega)$ for any $\varepsilon \in (0,1]$. Using it as test function in (P_λ) we obtain that

$$\beta \int_\Omega \frac{|\nabla u|^2}{(u+\varepsilon)^{1-\beta}} \leq C + \|\mu\|_{L^\infty(\Omega)} \int_\Omega \frac{|\nabla u|^q}{u^{q-1}}\left((u+\varepsilon)^\beta - \varepsilon^\beta\right)$$

for some constant $C > 0$, independent of ε, whose value may vary from line to line. Next, in the case $1 < q < 2$, using Young's inequality conveniently we easily derive that

$$\frac{\beta}{2}\int_\Omega \frac{|\nabla u|^2}{(u+\varepsilon)^{1-\beta}} \le C\left(1 + \int_\Omega \left((u+\varepsilon)^{(1-\beta)\frac{q}{2}}\,\frac{(u+\varepsilon)^\beta - \varepsilon^\beta}{u^{q-1}}\right)^{\frac{2}{2-q}}\right).$$

It is straightforward to check that the function $(s,t) \mapsto (s+t)^{(1-\beta)\frac{q}{2}}\frac{(s+t)^\beta - t^\beta}{s^{q-1}}$ is continuous in $[0, \|u\|_{L^\infty(\Omega)}] \times [0,1]$, which implies that

$$\frac{\beta}{2}\int_\Omega \frac{|\nabla u|^2}{(u+\varepsilon)^{1-\beta}} \le C. \tag{2.4}$$

On the other hand, if $q = 2$ and $\|\mu\|_{L^\infty(\Omega)} < 1$, we observe that

$$\begin{aligned}\frac{|\nabla u|^q}{u^{q-1}}((u+\varepsilon)^\beta - \varepsilon^\beta) &= \frac{|\nabla u|^2}{(u+\varepsilon)^{1-\beta}}\,\frac{(u+\varepsilon)^{1-\beta}((u+\varepsilon)^\beta - \varepsilon^\beta)}{u}\\ &= \frac{|\nabla u|^2}{(u+\varepsilon)^{1-\beta}}\left(1 + \varepsilon^\beta\,\frac{\varepsilon^{1-\beta} - (u+\varepsilon)^{1-\beta}}{u}\right) \le \frac{|\nabla u|^2}{(u+\varepsilon)^{1-\beta}}\end{aligned}$$

in Ω for any $\varepsilon \in (0,1]$. Hence, we deduce that

$$(\beta - \|\mu\|_{L^\infty(\Omega)})\int_\Omega \frac{|\nabla u|^2}{(u+\varepsilon)^{1-\beta}} \le C. \tag{2.5}$$

Finally, we apply Fatou's Lemma with respect to ε in (2.4) and in (2.5) to obtain (2.3). The Lemma follows by choosing $\gamma = \frac{\beta+1}{2}$. $\square$

Lemma 2.7. *Let $1 < q \le 2$, $0 \le \mu \in L^\infty(\Omega)$, with $\|\mu\|_{L^\infty(\Omega)} < 1$ if $q = 2$, $0 \lneq f \in L^r(\Omega)$ with $r > N$, and let u be a solution to (P_λ) for some $\lambda \in \mathbb{R}$. Then, if $1 < q < 2$, it holds that*

$$\forall \gamma \in \left(\frac{1}{2}, 1\right) \quad \exists C > 0 : \quad u \le C\varphi_1^\gamma \quad \text{in } \Omega.$$

Moreover, if $q = 2$, then

$$\exists C > 0 : \quad u \le C\varphi_1^\gamma \quad \text{in } \Omega, \qquad \text{where } \gamma = \frac{1}{1 + \|\mu\|_{L^\infty(\Omega)}} \in \left(\frac{1}{2}, 1\right).$$

Proof. Let $\gamma \in \left(\frac{1}{2}, 1\right)$. First of all observe that, if $q < 2$, we can use Young's inequality in such a way that

$$-\Delta u \le \left(\frac{1}{\gamma} - 1\right)\frac{|\nabla u|^2}{u} + (C_\gamma + \lambda)u + f(x) \tag{2.6}$$

for some $C_\gamma > 0$ large enough. If $q = 2$, we arrive to the same inequality directly with $\gamma = \frac{1}{1+\|\mu\|_{L^\infty(\Omega)}}$ and $C_\gamma = 0$.

Let now $g \equiv \frac{1}{\gamma}\left((C_\gamma + \lambda)u + f\right)$. Clearly, $0 \lneq g \in L^r(\Omega)$, so there exists $0 < z \in H_0^1(\Omega) \cap L^\infty(\Omega)$ a solution to

$$\begin{cases} -\Delta z = g(x) & \text{in } \Omega,\\ z = 0 & \text{on } \partial\Omega.\end{cases}$$

Since $r > N$, it is well-known that $z \in C^1(\overline{\Omega})$. This implies, by using Hopf's Lemma, that there is a constant $C > 0$ such that

$$z \le C\varphi_1 \quad \text{in } \Omega.$$

On the other hand, for every $k > 0$, the function $v = (kz)^\gamma \in H_0^1(\Omega) \cap L^\infty(\Omega)$ satisfies

$$-\Delta v = \left(\frac{1}{\gamma} - 1\right) \frac{|\nabla v|^2}{v} + \frac{\gamma k g(x)}{(kz)^{1-\gamma}}.$$

If we choose $k = \|z\|_{L^\infty(\Omega)}^{\frac{1}{\gamma}-1}$, then

$$-\Delta v \geq \left(\frac{1}{\gamma} - 1\right) \frac{|\nabla v|^2}{v} + \gamma g(x) = \left(\frac{1}{\gamma} - 1\right) \frac{|\nabla v|^2}{v} + (C_\gamma + \lambda) u + f(x). \tag{2.7}$$

Therefore, by (2.6) and (2.7), we can use Theorem 3.2 (see next section) and conclude that

$$u \leq v = (kz)^\gamma \leq C\varphi_1^\gamma.$$

□

We conclude this section by recalling the concept of bifurcation point from infinity.

Definition 2.8. A *bifurcation point from infinity* to problem (P_λ) is said to be a real number $\bar\lambda$ for which there exists a sequence $\{(\lambda_n, u_n)\}_{n\in\mathbb{N}}$ contained in the set

$$\Sigma := \{(\lambda, u) : u \text{ is a solution to } (P_\lambda)\},$$

such that $\lambda_n \to \bar\lambda$ and $\|u_n\|_{L^\infty(\Omega)} \to \infty$ as $n \to \infty$.

We say that the bifurcation occurs *to the left* if there exist $\varepsilon > 0$ and $M > 0$ such that for any $(\lambda, u) \in \Sigma$ with $\lambda \in (\bar\lambda - \varepsilon, \bar\lambda + \varepsilon)$ and $\|u\|_{L^\infty(\Omega)} \geq M$, it holds that $\lambda < \bar\lambda$.

3. Comparison principles

In this section we prove a Comparison Principle which allows us to compare suitable subsolutions and supersolutions to the equation

$$-\Delta u = \lambda u + g(x) \frac{|\nabla u|^q}{u^{q-1}} + h(x) \quad \text{in } \Omega$$

that are well ordered on the boundary.

Theorem 3.1. *Let $1 < q \leq 2$, $\lambda \in \mathbb{R}$, $g \in L^\infty(\Omega)$, $0 \leq h \in L^1_{loc}(\Omega)$ and assume that $u, v \in C(\Omega) \cap W^{1,N}_{loc}(\Omega)$ are such that $u, v > 0$ in Ω and they satisfy*

$$\limsup_{x \to x_0} \frac{u(x)}{v(x)} \leq 1 \quad \forall x_0 \in \partial\Omega, \tag{3.1}$$

$$\int_\Omega \nabla u \nabla \phi \leq \lambda \int_\Omega u\phi + \int_\Omega g(x) \frac{|\nabla u|^q}{u^{q-1}} \phi + \int_\Omega h(x)\phi, \qquad \forall \phi \in H_0^1(\Omega) \cap L^\infty(\Omega),\ \phi \geq 0,\ \operatorname{supp}(\phi) \subset \Omega \tag{3.2}$$

and

$$\int_\Omega \nabla v \nabla \phi \geq \lambda \int_\Omega v\phi + \int_\Omega g(x) \frac{|\nabla v|^q}{v^{q-1}} \phi + \int_\Omega h(x)\phi, \qquad \forall \phi \in H_0^1(\Omega) \cap L^\infty(\Omega),\ \phi \geq 0,\ \operatorname{supp}(\phi) \subset \Omega. \tag{3.3}$$

Then $u \leq v$ in Ω.

Proof. We follow the ideas contained in [6, Lemma 2.2] (see also the references therein). Let $u_1 = \log(u)$, $v_1 = \log(v)$, and denote $w = u_1 - v_1$. Observe that, using (3.1), we have that for every $k > 0$, the function $(w - k)^+$ has compact support in Ω and, in consequence, it belongs to $H_0^1(\Omega) \cap L^\infty(\Omega)$. This fact, together with the continuity of u and v (which implies that u and v are locally bounded away from zero), allows us to take $\frac{(w-k)^+}{u}$ as test function in (3.2) and $\frac{(w-k)^+}{v}$ in (3.3), obtaining

$$
\begin{aligned}
-\int_\Omega \frac{|\nabla u|^2}{u^2}(w-k)^+ + \int_\Omega \frac{\nabla u}{u}\nabla (w-k)^+ \leq & \lambda \int_\Omega (w-k)^+ \\
& + \int_\Omega g(x)\frac{|\nabla u|^q}{u^q}(w-k)^+ + \int_\Omega \frac{h(x)}{u}(w-k)^+
\end{aligned} \tag{3.4}
$$

and

$$
\begin{aligned}
-\int_\Omega \frac{|\nabla v|^2}{v^2}(w-k)^+ + \int_\Omega \frac{\nabla v}{v}\nabla (w-k)^+ \geq & \lambda \int_\Omega (w-k)^+ \\
& + \int_\Omega g(x)\frac{|\nabla v|^q}{v^q}(w-k)^+ + \int_\Omega \frac{h(x)}{v}(w-k)^+.
\end{aligned} \tag{3.5}
$$

Consider now the set

$$A_k = \{x \in \Omega : w(x) \geq k\} = \{x \in \Omega : u(x) \geq e^k v(x)\}.$$

Notice that $\operatorname{supp}(w-k)^+ \subseteq A_k$ and $h\left(\frac{1}{u}-\frac{1}{v}\right) \leq 0$ in A_k. Hence, subtracting (3.4) from (3.5) and using the definition of u_1, v_1 we have that

$$\int_\Omega \nabla w \nabla (w-k)^+ \leq \int_{A_k} \left(g(x)\left(|\nabla u_1|^q - |\nabla v_1|^q\right) + |\nabla u_1|^2 - |\nabla v_1|^2\right)(w-k)^+. \tag{3.6}$$

For every $j \in \mathbb{R}$, let us denote $\Omega_j = \{x \in \Omega : |w(x)| = j\}$, and consider also the set $J = \{j \in \mathbb{R} : |\Omega_j| \neq 0\}$. Since $|\Omega| < \infty$, then J is at most countable, which implies that the set $\bigcup_{j\in J}\Omega_j$ is measurable, and we also have that

$$\nabla w = 0 \quad \text{in } \bigcup_{j\in J}\Omega_j \implies |\nabla u_1| = |\nabla v_1| \quad \text{in } \bigcup_{j\in J}\Omega_j.$$

Hence, if we define the set $Z = \Omega \setminus \bigcup_{j\in J}\Omega_j$ and denote $\xi_t = t\nabla u_1 + (1-t)\nabla v_1$, with $0 < t < 1$, we deduce from (3.6) that

$$
\begin{aligned}
\int_\Omega \nabla w \nabla (w-k)^+ &\leq \int_{A_k\cap Z} \left(g(x)\left(|\nabla u_1|^q - |\nabla v_1|^q\right) + |\nabla u_1|^2 - |\nabla v_1|^2\right)(w-k)^+ \\
&= \int_{A_k\cap Z} \left(\int_0^1 \frac{d}{dt}(g(x)|\xi_t|^q + |\xi_t|^2)dt\right)(w-k)^+.
\end{aligned} \tag{3.7}
$$

Taking into account that $u_1, v_1 \in W^{1,N}_{\mathrm{loc}}(\Omega)$ and $A_k \subset\subset \Omega$, we have that

$$|\xi_t| \leq |\nabla u_1| + |\nabla v_1| + 1 \equiv \eta \in L^N(A_k \cap Z).$$

Hence, from (3.7) we derive that

$$
\begin{aligned}
\|(w-k)^+\|^2_{H^1_0(\Omega)} &\leq \int_{A_k\cap Z}\left(\int_0^1 (g(x)q|\xi_t|^{q-2}\xi_t + 2\xi_t)\nabla w dt\right)(w-k)^+ \\
&\leq \int_{A_k\cap Z}\left(\|g\|_{L^\infty(\Omega)} q\eta^{q-1} + 2\eta\right)|\nabla w|(w-k)^+ \leq C\int_{A_k\cap Z}\eta|\nabla (w-k)^+|(w-k)^+ \\
&\leq C\|\eta\|_{L^N(A_k\cap Z)}\|(w-k)^+\|_{H^1_0(\Omega)}\|(w-k)^+\|_{L^{2^*}(\Omega)} \leq C\|\eta\|_{L^N(A_k\cap Z)}\|(w-k)^+\|^2_{H^1_0(\Omega)}.
\end{aligned} \tag{3.8}
$$

Assume in order to achieve a contradiction that $w^+ \not\equiv 0$. For some $k_0 \in (0, \|w^+\|_{L^\infty(\Omega)})$, let us define the function $F : [k_0, \|w^+\|_{L^\infty(\Omega)}] \to \mathbb{R}$ by

$$F(k) = \|\eta\|_{L^N(A_k \cap Z)} = \||\nabla u_1| + |\nabla v_1| + 1\|_{L^N(A_k \cap Z)}, \quad \forall k \in [k_0, \|w^+\|_{L^\infty(\Omega)}),$$

and $F(\|w^+\|_{L^\infty(\Omega)}) = 0$. It is clear that F is nonincreasing and continuous in $[0, \|w^+\|_{L^\infty(\Omega)}]$.

Thus, choosing k close enough to $\|w^+\|_{L^\infty(\Omega)}$, we deduce from (3.8) that $(w-k)^+ \equiv 0$. That is to say, $w \le k$ in Ω. But this is not possible since $k < \|w^+\|_{L^\infty(\Omega)} = \sup_\Omega(w)$.

In conclusion, we have proved that $w^+ \equiv 0$, i.e., $w \le 0$ in Ω. □

The previous comparison principle does not guarantee uniqueness of $C(\Omega) \cap W^{1,N}_{\mathrm{loc}}(\Omega)$ solution to (P_λ) unless it is assured that any pair of such solutions satisfy (3.1). However, stronger hypotheses on h and g allow us to weaken (3.1) and derive another comparison result that provides uniqueness for (P_λ).

Theorem 3.2. *Let $1 < q \le 2$, $\lambda \in \mathbb{R}$, $0 \le g \in L^\infty(\Omega)$ and $0 \le h \in L^1_{loc}(\Omega)$. Assume that $u, v \in C(\Omega) \cap W^{1,N}_{loc}(\Omega)$, with $u, v > 0$ in Ω, and satisfy* (3.2) *and* (3.3) *respectively. Suppose also that, for every $\varepsilon > 0$, the following boundary condition holds:*

$$\limsup_{x\to x_0} \left(\frac{u(x)}{v(x)+\varepsilon}\right) \le 1 \quad \forall x_0 \in \partial\Omega. \tag{3.9}$$

Furthermore, if $\lambda > 0$, assume also that h satisfies condition (f_0). Then, $u \le v$ in Ω.

Proof. For every $\varepsilon > 0$, let us consider the function

$$w_\varepsilon = \log\left(\frac{u}{v+\varepsilon}\right).$$

We claim that $w_\varepsilon^+ \equiv 0$ for any $\varepsilon > 0$. Suppose by contradiction that there exists $\varepsilon_0 > 0$ such that $w_{\varepsilon_0}^+ \not\equiv 0$. Let us fix $k_0 \in \big(0, \|w_{\varepsilon_0}^+\|_{L^\infty(\Omega)}\big)$ and $\varepsilon \in (0, \varepsilon_0)$, the latter to be chosen small enough later. It is clear that $w_{\varepsilon_0} \le w_\varepsilon$ in Ω, so $w_\varepsilon^+ \not\equiv 0$, too.

For $k \in [k_0, \|w_\varepsilon^+\|_{L^\infty(\Omega)}]$, let us denote

$$A_k = \{x \in \Omega : w_\varepsilon(x) \ge k\} = \{x \in \Omega : u(x) \ge e^k(v(x)+\varepsilon)\}.$$

Notice that $\operatorname{supp}(w-k)^+ \subset A_k$. By hypothesis, we also have that $\limsup\limits_{x\to x_0} w_\varepsilon(x) \le 0$ for all $x_0 \in \partial\Omega$, which implies that $A_k \subset\subset \Omega$. Then, the function $(w_\varepsilon - k)^+$ has compact support, and in particular, $(w_\varepsilon-k)^+ \in H^1_0(\Omega) \cap L^\infty(\Omega)$. Therefore, we may take $\frac{(w_\varepsilon-k)^+}{u}$ as test function in (3.2), and $\frac{(w_\varepsilon-k)^+}{v+\varepsilon}$ in (3.3), obtaining

$$\begin{aligned}\int_\Omega \frac{\nabla u}{u}\nabla(w_\varepsilon-k)^+ \le{}& \int_\Omega \frac{|\nabla u|^2}{u^2}(w_\varepsilon-k)^+ + \lambda\int_\Omega (w_\varepsilon-k)^+ \\ &+ \int_\Omega g(x)\frac{|\nabla u|^q}{u^q}(w_\varepsilon-k)^+ + \int_\Omega \frac{h(x)}{u}(w_\varepsilon-k)^+\end{aligned} \tag{3.10}$$

and, using that $g \ge 0$,

$$\begin{aligned}\int_\Omega \frac{\nabla v}{v+\varepsilon}\nabla(w_\varepsilon-k)^+ \ge{}& \int_\Omega \frac{|\nabla v|^2}{(v+\varepsilon)^2}(w_\varepsilon-k)^+ + \lambda\int_\Omega \frac{v}{v+\varepsilon}(w_\varepsilon-k)^+ \\ &+ \int_\Omega g(x)\frac{|\nabla v|^q}{v^{q-1}(v+\varepsilon)}(w_\varepsilon-k)^+ + \int_\Omega \frac{h(x)}{v+\varepsilon}(w_\varepsilon-k)^+ \\ \ge{}& \int_\Omega \frac{|\nabla v|^2}{(v+\varepsilon)^2}(w_\varepsilon-k)^+ + \lambda\int_\Omega (w_\varepsilon-k)^+ - \int_\Omega \frac{\lambda\varepsilon}{v+\varepsilon}(w_\varepsilon-k)^+ \\ &+ \int_\Omega g(x)\frac{|\nabla v|^q}{(v+\varepsilon)^q}(w_\varepsilon-k)^+ + \int_\Omega \frac{h(x)}{v+\varepsilon}(w_\varepsilon-k)^+.\end{aligned} \tag{3.11}$$

Moreover, it is clear that

$$h\left(\frac{1}{u}-\frac{1}{v+\varepsilon}\right)+\frac{\lambda\varepsilon}{v+\varepsilon}\leq 0 \quad \text{in } A_k \text{ for every } k\in[k_0,\|w_\varepsilon^+\|_{L^\infty(\Omega)}] \tag{3.12}$$

whenever $\lambda\leq 0$. We claim that this is also true if $\lambda>0$, h satisfies (f_0) and ε is small enough.

Indeed, let $\omega\subset\subset\Omega$ be an open set such that $A_{k_0}\subset\omega$. Since $A_k\subset A_{k_0}$ for all $k\geq k_0$, there exists $c_\omega>0$ such that $h\geq c_\omega$ in A_k for all $k\geq k_0$. If we choose now

$$\varepsilon<\min\left\{\varepsilon_0,\frac{1-e^{-k_0}}{\lambda}\;c_\omega\right\},$$

we deduce easily that (3.12) holds.

Therefore, subtracting (3.10) and (3.11), and taking into account that $u,v\in W^{1,N}_{\mathrm{loc}}(\Omega)$ and also (3.12), we may argue as in the proof of Theorem 3.1 and achieve a contradiction taking k close enough to $\|w_\varepsilon^+\|_{L^\infty(\Omega)}$.

In conclusion, necessarily $w_\varepsilon^+\equiv 0$ for any $\varepsilon>0$, i.e., $u\leq v+\varepsilon$ in Ω for any $\varepsilon>0$. Letting $\varepsilon\to 0$ it follows that $u\leq v$ in Ω. □

4. The principal eigenvalue and nonexistence results

We devote this section to give some properties of λ^* defined by (1.3). In particular, we show that λ^* is the only possible value of the parameter λ for which (E_λ) admits a solution. This is a crucial fact on which are based the proofs of our main results, that exploit the existence of the principal eigenvalue associated to the nonlinear operator $-\Delta u-\mu(x)\frac{|\nabla u|^q}{u^{q-1}}$ (see Theorem 1.1). For the sake of clarity we collected such proofs in the last section.

Let us recall that $\lambda^*=\sup I^*$, where

$$I^*=\left\{\lambda\in\mathbb{R}\;\middle|\;\begin{array}{c}\text{there exists a supersolution } v \text{ to } (E_\lambda)\\ \text{such that } v\geq c \text{ in } \Omega \text{ for some } c>0\end{array}\right\}.$$

Firstly we point out some useful characterizations of λ^* as the supremum of the following sets:

$$I_1=\left\{\lambda\in\mathbb{R}\;\middle|\;\begin{array}{c}\text{there exists a supersolution } v \text{ to } (E_\lambda)\\ \text{such that } v-c\in H_0^1(\Omega) \text{ for some } c>0\end{array}\right\},$$

and

$$I_2=\left\{\lambda\in\mathbb{R}\;\middle|\;\begin{array}{c}\text{there exists a supersolution } v \text{ to } (E_\lambda)\\ \text{such that } v^\gamma\in H^1(\Omega)\ \forall\gamma>\gamma_0 \text{ for some } \gamma_0<1\end{array}\right\}.$$

Proposition 4.1. *Assume that $1<q\leq 2$ and $0\leq\mu\in L^\infty(\Omega)$. Then, the sets I^*, I_1 and I_2 are nonempty intervals, unbounded from below and they satisfy*

$$I^*=I_1, \tag{4.1}$$

$$\lambda^*=\sup I_2. \tag{4.2}$$

Moreover, $\lambda^*>0$ and we have that $\lambda^*\leq\Lambda\equiv\inf\limits_{w\in H_0^1(\Omega)\setminus\{0\}}\dfrac{\|w\|^2_{H_0^1(\Omega)}}{\|w\|^2_{L^2(\Omega)}}$.

Proof. We first observe that the sets under consideration are intervals. Moreover, taking $\varphi\equiv c$ for any constant $c>0$ in the definitions of I^*, I_1 and I_2, we deduce that $(-\infty,0]\subset I^*\cap I_1\cap I_2$.

We split the rest of the proof into several steps.

Step 1. We first prove (4.1). In order to prove that $I_1 \subseteq I^*$ we take $\lambda \in I_1$ and assume, without loss of generality, that $\lambda > 0$. Hence, there exist $0 < \varphi \in H^1(\Omega) \cap L^\infty(\Omega)$ and $c > 0$ with $\varphi > 0$ in Ω, $\varphi - c \in H_0^1(\Omega)$, and

$$-\Delta(\varphi - c) = -\Delta\varphi \geq \lambda\varphi + \mu(x)\frac{|\nabla\varphi|^q}{\varphi^{q-1}} \geq 0 \ \text{ in } \Omega.$$

Therefore, the maximum principle yields to $\varphi \geq c$ in Ω, and so $\lambda \in I^*$, too.

Reciprocally, assume that $0 < \lambda \in I^*$, and let $\varphi \in H^1(\Omega) \cap L^\infty(\Omega)$ and $c > 0$ with $\varphi \geq c$ and $-\Delta\varphi \geq \lambda\varphi + \mu(x)\frac{|\nabla\varphi|^q}{\varphi^{q-1}}$ in Ω. Clearly, thanks to Remark 2.2 we have that $\overline{\psi} = \varphi - c \geq 0$ is an $H^1(\Omega) \cap L^\infty(\Omega)$ supersolution to the nonsingular problem

$$\begin{cases} -\Delta\psi = \lambda\psi + \mu(x)\dfrac{|\nabla\psi|^q}{|\psi + c|^{q-1}} + \lambda c & \text{in } \Omega, \\ \psi = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.3}$$

On the other hand, $\underline{\psi} \equiv 0$ is obviously a subsolution. Therefore, [13, Théorème 3.1] (see also [23]) implies that there exists a solution $\psi \in H_0^1(\Omega) \cap L^\infty(\Omega)$ to (4.3) satisfying that $0 \leq \psi \leq \varphi - c$ in Ω. Thus, the function $\psi + c$ satisfies: $(\psi + c) \in H^1(\Omega) \cap L^\infty(\Omega)$, $\psi + c > 0$ in Ω, $(\psi + c) - c \in H_0^1(\Omega)$ and

$$-\Delta(\psi + c) = \lambda(\psi + c) + \mu(x)\frac{|\nabla(\psi + c)|^q}{(\psi + c)^{q-1}} \ \text{ in } \Omega.$$

This proves that $\lambda \in I_1$.

Step 2. We deduce now (4.2). Let $\{\lambda_n\}_{n\in\mathbb{N}} \subseteq I_2$ be an increasing sequence of real numbers such that $\lambda_n < \sup I_2$ for any n, satisfying $\lambda_n \to \sup I_2$. In particular, for every n there exists $u_n \in H^1(\Omega) \cap L^\infty(\Omega)$ and $\widetilde{\gamma}_n \in (0, 1)$ satisfying

$$u_n > 0 \text{ in } \Omega, \quad -\Delta u_n \geq \lambda_n u_n + \mu(x)\frac{|\nabla u_n|^q}{u_n^{q-1}} \text{ in } \Omega, \quad u_n^\gamma \in H^1(\Omega) \quad \forall \gamma > \widetilde{\gamma}_n.$$

Let $\varphi_1 > 0$ be the principal eigenfunction (normalized in $L^\infty(\Omega)$) to the $-\Delta$ operator in Ω with zero Dirichlet boundary conditions. Let us fix $n > 1$, and consider $\varepsilon = \varepsilon_n > 0$ (to be chosen small enough later) and $\gamma = \gamma_n \in \left(\max\left\{\frac{1}{2}, \widetilde{\gamma}_n, \frac{\lambda_{n-1}}{\lambda_n}\right\}, 1\right)$. Since $\gamma > \frac{1}{2}$ and $\gamma > \widetilde{\gamma}_n$, we have, using Lemma 2.6, that the function

$$\psi_n = \varepsilon(\varphi_1^\gamma + 1) + u_n^\gamma \in H^1(\Omega) \cap L^\infty(\Omega)$$

and, clearly, $\psi_n \geq \varepsilon$ in Ω.

Let $\phi \in H_0^1(\Omega) \cap L^\infty(\Omega)$ be such that $\phi \geq 0$ in Ω and has compact support. Observe that the function $\gamma\varphi_1^{\gamma-1}\phi \in H_0^1(\Omega) \cap L^\infty(\Omega)$, so it may be chosen as test function in

$$\begin{cases} -\Delta\varphi_1 = \Lambda\varphi_1 & \text{in } \Omega, \\ \varphi_1 = 0 & \text{on } \partial\Omega. \end{cases}$$

Similarly, $\gamma u_n^{\gamma-1}\phi \in H_0^1(\Omega) \cap L^\infty(\Omega)$ and has compact support, so it may be taken as test function in the inequality satisfied by u_n. Therefore, direct computations yield to

$$\int_\Omega \nabla\varphi_1^\gamma \nabla\phi = \gamma(1 - \gamma)\int_\Omega \frac{|\nabla\varphi_1|^2}{\varphi_1^{2-\gamma}}\phi + \gamma\Lambda\int_\Omega \varphi_1^\gamma\phi \tag{4.4}$$

and

$$\int_\Omega \nabla u_n^\gamma \nabla\phi \geq \gamma(1 - \gamma)\int_\Omega \frac{|\nabla u_n|^2}{u_n^{2-\gamma}}\phi + \gamma\lambda_n\int_\Omega u_n^\gamma\phi + \gamma\int_\Omega \mu(x)\frac{|\nabla u_n|^q}{u_n^{q-\gamma}}\phi. \tag{4.5}$$

Recalling that

$$\int_\Omega \nabla\psi_n\nabla\phi = \varepsilon\int_\Omega \nabla\varphi_1^\gamma\nabla\phi + \int_\Omega \nabla u_n^\gamma\nabla\phi\,,$$

using both (4.4) and (4.5) we easily deduce that

$$\begin{aligned}(4.6)\qquad &\int_\Omega\left(-\nabla\psi_n\nabla\phi + \lambda_{n-1}\psi_n\phi + \mu(x)\frac{|\nabla\psi_n|^q}{\psi_n^{q-1}}\phi\right)\\ &\le \varepsilon\int_\Omega\left(-\gamma(1-\gamma)\frac{|\nabla\varphi_1|^2}{\varphi_1^{2-\gamma}} + (\lambda_{n-1}-\gamma\Lambda)\varphi_1^\gamma + \lambda_{n-1}\right)\phi +\\ &\int_\Omega\left(-\gamma(1-\gamma)\frac{|\nabla u_n|^2}{u_n^{2-\gamma}} - (\gamma\lambda_n-\lambda_{n-1})u_n^\gamma - \gamma\mu(x)\frac{|\nabla u_n|^q}{u_n^{q-\gamma}} + \mu(x)\frac{|\nabla\psi_n|^q}{\psi_n^{q-1}}\right)\phi.\end{aligned}$$

Since $\gamma<1<q$, there exists a constant $C_1>0$ (that depends only on q and γ) such that

$$\begin{aligned}(4.7)\qquad \frac{|\nabla\psi_n|^q}{\psi_n^{q-1}} &\le \frac{1}{\psi_n^{q-1}}\left(C_1|\nabla(\varepsilon\varphi_1^\gamma)|^q + \frac{1}{\gamma^{q-1}}|\nabla(u_n^\gamma)|^q\right)\\ &\le C_1\frac{|\nabla(\varepsilon\varphi_1^\gamma)|^q}{\varepsilon^{q-1}} + \frac{|\nabla(u_n^\gamma)|^q}{(\gamma u_n^\gamma)^{(q-1)}} = C_1\varepsilon\frac{|\nabla\varphi_1|^q}{\varphi_1^{q(1-\gamma)}} + \gamma\frac{|\nabla u_n|^q}{u_n^{q-\gamma}}\end{aligned}$$

in Ω. Hence, combining (4.6) and (4.7) we deduce that

$$\begin{aligned}(4.8)\qquad &\int_\Omega\left(-\nabla\psi_n\nabla\phi + \lambda_{n-1}\psi_n\phi + \mu(x)\frac{|\nabla\psi_n|^q}{\psi_n^{q-1}}\phi\right) \le -(\gamma\lambda_n-\lambda_{n-1})\int_\Omega u_n^\gamma\phi +\\ &\varepsilon\int_\Omega\left(-\gamma(1-\gamma)\frac{|\nabla\varphi_1|^2}{\varphi_1^{2-\gamma}} + (\lambda_{n-1}-\gamma\Lambda)\varphi_1^\gamma + \lambda_{n-1} + \|\mu\|_{L^\infty(\Omega)}C_1\frac{|\nabla\varphi_1|^q}{\varphi_1^{q(1-\gamma)}}\right)\phi.\end{aligned}$$

Denoting $d(x)=\mathrm{dist}(x,\partial\Omega)$, since $\varphi_1\in C^1(\overline{\Omega})$, Hopf's Lemma yields that there exist two constants $\delta_0, C_2>0$ such that $|\nabla\varphi_1|^2\ge C_2$ in the set $\Omega_\delta=\{x\in\Omega : d(x)\le\delta\}$ for every $\delta\in(0,\delta_0)$. Hence, using now that $\varphi_1\in C(\overline{\Omega})$ and $\varphi_1=0$ on $\partial\Omega$, we have that, for every $\kappa>0$, there exists $\delta\in(0,\delta_0)$ such that $\frac{|\nabla\varphi_1|^2}{\varphi_1^{2-\gamma}}\ge\kappa$ in Ω_δ. Using also that $\gamma\lambda_n-\lambda_{n-1}>0$ and $q(1-\gamma)<2-\gamma$, we choose δ sufficiently small, but independent of ε, such that

$$(4.9)\qquad \varepsilon\left(-\gamma(1-\gamma)\frac{|\nabla\varphi_1|^2}{\varphi_1^{2-\gamma}} + (\lambda_{n-1}-\gamma\Lambda)\varphi_1^\gamma + \lambda_{n-1} + \|\mu\|_{L^\infty(\Omega)}C_1\frac{|\nabla\varphi_1|^q}{\varphi_1^{q(1-\gamma)}}\right) \le 0 \le (\gamma\lambda_n-\lambda_{n-1})u_n^\gamma$$

in Ω_δ. Consequently, we take ε small enough in order to have

$$\begin{aligned}(4.10)\qquad &\varepsilon\left(-\gamma(1-\gamma)\frac{|\nabla\varphi_1|^2}{\varphi_1^{2-\gamma}} + (\lambda_{n-1}-\gamma\Lambda)\varphi_1^\gamma + \lambda_{n-1} + \|\mu\|_{L^\infty(\Omega)}C_1\frac{|\nabla\varphi_1|^q}{\varphi_1^{q(1-\gamma)}}\right)\\ &\le \varepsilon C_3 \le (\gamma\lambda_n-\lambda_{n-1})\inf_{\Omega\setminus\Omega_\delta}(u_n^\gamma) \le (\gamma\lambda_n-\lambda_{n-1})u_n^\gamma\end{aligned}$$

in $\Omega\setminus\Omega_\delta$, where $C_3>0$ is a constant independent of ε. Gathering (4.8), (4.9) and (4.10) together we conclude that

$$\int_\Omega\nabla\psi_n\nabla\phi \ge \lambda_{n-1}\int_\Omega\psi_n\phi + \int_\Omega\mu(x)\frac{|\nabla\psi_n|^q}{\psi_n^{q-1}}\phi.$$

In short, we have proved that $\lambda_{n-1}\in I^*$ for any $n>1$, and thus, $\lambda_{n-1}\le\lambda^*$ for any $n>1$. Therefore, letting $n\to\infty$ we get that $\sup I_2\le\lambda^*$. Finally, the reverse inequality is trivial since $I^*\subseteq I_2$.

Step 3. We show now that $\lambda^* > 0$. Indeed, given the constants $c, \delta >$, let us consider the problem

$$\begin{cases} -\Delta u = \dfrac{\mu(x)}{c^{q-1}} |\nabla u|^q + \delta & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.11}$$

If $q < 2$, by using Young's inequality, we obtain that

$$\frac{\mu(x)}{c^{q-1}} |\xi|^q + \delta \leq \mu(x)|\xi|^2 + \left(1 - \frac{q}{2}\right)\left(\frac{q}{2}\right)^{\frac{q}{2-q}} \frac{\|\mu\|_{L^\infty(\Omega)}}{c^{\frac{2(q-1)}{2-q}}} + \delta$$

for a.e. $x \in \Omega$ and for every $\xi \in \mathbb{R}^N$. Then, taking c large enough and δ small enough, [17, Theorem 3.4] implies that there exists a solution $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ to (4.11). If $q = 2$, then the same result provides a solution $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ if δ is small enough. In both cases, by the Maximum Principle, $u \geq 0$ in Ω.

Let $v = u + c \in H^1(\Omega) \cap L^\infty(\Omega)$. It is clear that $v \geq c$ in Ω and, for some $\lambda > 0$,

$$\begin{aligned} -\Delta v = -\Delta u &= \frac{\mu(x)}{c^{q-1}} |\nabla u|^q + \delta \geq \mu(x) \frac{|\nabla u|^q}{(u+c)^{q-1}} + \delta \\ &= \mu(x)\frac{|\nabla v|^q}{v^{q-1}} + \lambda v + (\delta - \lambda v) \geq \mu(x)\frac{|\nabla v|^q}{v^{q-1}} + \lambda v + (\delta - \lambda \|v\|_{L^\infty(\Omega)}). \end{aligned}$$

Taking now λ sufficiently small, we conclude that v is a supersolution to (E_λ). This means that $0 < \lambda \leq \lambda^*$, as we wished to prove.

Step 4 We prove here that $\lambda^* \leq \Lambda$. Let $0 < \lambda \in I^* = I_1$. We already know, from Step 1, that there exists a solution $\psi \geq 0$ to (4.3) for some $c > 0$. Then, taking φ_1 as test function in (4.3) we have

$$\Lambda \int_\Omega \varphi_1 \psi = \int_\Omega \nabla\varphi_1 \nabla \psi = \lambda \int_\Omega \psi\varphi_1 + \int_\Omega \mu(x) \frac{|\nabla \psi|^q}{(\psi + c)^{q-1}} \varphi_1 + \lambda c \int_\Omega \varphi_1.$$

In particular

$$(\Lambda - \lambda) \int_\Omega \psi\varphi_1 = \int_\Omega \mu(x) \frac{|\nabla \psi|^q}{(\psi + c)^{q-1}} \varphi_1 + c\lambda \int_\Omega \varphi_1 > 0.$$

Thus, necessarily $\lambda < \Lambda$, which implies that $\lambda^* \leq \Lambda$. ☐

Remark 4.2. *We point out that in Step 1 of the previous proof it has been shown that one can equivalently define I_1 in terms of solutions instead of supersolutions. That is to say,*

$$I_1 = \left\{ \lambda \in \mathbb{R} \,\middle|\, \begin{array}{c} \text{there exists a solution } v \text{ to the equation in } (E_\lambda) \\ \text{such that } v - c \in H_0^1(\Omega) \text{ for some } c > 0 \end{array} \right\}.$$

In order to prove that λ^* is the only possible eigenvalue to (E_λ) we need to use the comparison principle proved in the previous section. Indeed, it allows us to prove nonexistence of solutions to (E_λ) when $\lambda < \lambda^*$. On the other hand, we use the characterization of λ^* given by (4.2) to prove nonexistence for $\lambda > \lambda^*$; this latter nonexistence result is valid for (P_λ), even with $f \gneq 0$. Summarizing, we have the following result.

Proposition 4.3. *Assume that $1 < q \leq 2$, $0 \leq \mu \in L^\infty(\Omega)$, with $\|\mu\|_{L^\infty(\Omega)} < 1$ if $q = 2$, and $0 \lneq f \in L^p(\Omega)$ with $p > \frac{N}{2}$. Then, there is no solution to (P_λ) for any $\lambda > \lambda^*$. Moreover, there is no solution to (E_λ) for any $\lambda \neq \lambda^*$.*

Proof. Arguing by contradiction, assume that there exists a solution u to (P_λ) for some $\lambda > \lambda^*$. Then, it is in particular a supersolution to (E_λ), and Lemma 2.6 implies that $u^\gamma \in H^1(\Omega)$ for every $\gamma > \gamma_0(q)$. Since $\gamma_0(q) < 1$, then this contradicts (4.2) in Proposition 4.1. In conclusion, there is no solution to (P_λ) for any $\lambda > \lambda^*$. Observe that, in particular, we have nonexistence of solutions to (E_λ) for $\lambda > \lambda^*$.

On the other hand, assume now that there exists a solution u to (E_λ) for some $\lambda < \lambda^*$. By virtue of Proposition 4.1 we have that $\lambda \in I_1$, so there exist $c > 0$ and $\varphi \in H^1(\Omega)\cap L^\infty(\Omega)$ satisfying that $\varphi - c \in H_0^1(\Omega)$ and (see also Remark 4.2) $-\Delta\varphi = \lambda\varphi + \mu(x)\frac{|\nabla\varphi|^q}{\varphi^{q-1}}$ in Ω. Hence, arguing as in Lemma 2.4 we deduce that $\varphi \in C(\overline{\Omega}) \cap W^{1,N}_{\mathrm{loc}}(\Omega)$.

Observe also that tu is also a solution to (E_λ) for every $t > 0$. Then, Lemma 2.4 implies that $tu \in C(\overline{\Omega}) \cap W^{1,N}_{\mathrm{loc}}(\Omega)$, and in particular,

$$\limsup_{x\to x_0} \frac{tu(x)}{\varphi(x)} = \lim_{x\to x_0} \frac{tu(x)}{\varphi(x)} = 0 \le 1 \quad \forall x_0 \in \partial\Omega,\, \forall t \ge 0.$$

Consequently, using also that tu and φ satisfy respectively (3.2) and (3.3) with the choices $g \equiv \mu$ and $h \equiv 0$, an application of Theorem 3.1 gives that $tu \le \varphi$ in Ω. But this is impossible if t is taken large enough. Therefore, there is no solution to (E_λ) for any $\lambda < \lambda^*$. ☐

5. Existence and bifurcation results

We turn now to the problem of finding sufficient conditions on λ for the existence of solutions to (P_λ). The proofs of our results are based on an approximation procedure and make use of the main results of the previous sections.

Consider for every $n \in \mathbb{N}$ the family of approximate problems

$$(Q_n) \qquad \begin{cases} -\Delta u_n = \lambda u_n + \mu(x)\dfrac{|\nabla u_n|^q}{\left|u_n + \frac{1}{n}\right|^{q-1}} + T_n(f(x)) & \text{in } \Omega,\\ u_n = 0 & \text{on } \partial\Omega, \end{cases}$$

where $T_n(s) = \min\{n, \max\{-n, s\}\}$ for $s \in \mathbb{R}$. The following result is devoted to show that, below λ^*, the approximate problems (Q_n) admit a positive solution for any n. We also prove that such a sequence of solutions is locally bounded away from zero. Finally, we prove that an a priori bound in $L^\infty(\Omega)$ implies compactness of the approximate sequence.

Lemma 5.1. *Assume that $1 < q \le 2$, $0 \le \mu \in L^\infty(\Omega)$, with $\|\mu\|_{L^\infty(\Omega)} < 1$ if $q = 2$, and $0 \lneq f \in L^p(\Omega)$ with $p > \frac{N}{2}$ and let $\lambda < \lambda^*$. Then there exists a solution $u_n \in H_0^1(\Omega) \cap L^\infty(\Omega)$ to (Q_n) for any n. In addition, the following local lower bound, uniform with respect to n, holds:*

$$(5.1) \qquad \forall \omega \subset\subset \Omega\ \exists c_\omega > 0 :\ u_n \ge c_\omega \ \text{ in } \omega, \quad \forall n.$$

Moreover, if the sequence $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $L^\infty(\Omega)$, there exists a function $0 < u \in H_0^1(\Omega)\cap L^\infty(\Omega)$ such that, passing to a subsequence, $u_n \rightharpoonup u$ weakly in $H_0^1(\Omega)$ and $u_n \to u$ uniformly in $\overline{\Omega}$.

Proof. Let us fix $n \in \mathbb{N}$, and let $\bar\lambda \in I^*$ be such that $\lambda < \bar\lambda < \lambda^*$. Then, there exist a constant $c > 0$ and a function $\varphi \in H^1(\Omega) \cap L^\infty(\Omega)$ satisfying $\varphi \ge c$ in Ω and $-\Delta\varphi \ge \bar\lambda\varphi + \mu(x)\frac{|\nabla\varphi|^q}{\varphi^{q-1}}$ in Ω. Taking $M > 0$ large enough, the function $\overline{\psi} := M\varphi$ turns out to be an $H^1(\Omega)\cap L^\infty(\Omega)$ supersolution for (Q_n), since

$$\Delta\overline{\psi} + \lambda\overline{\psi} + \mu(x)\frac{|\nabla\overline{\psi}|^q}{|\overline{\psi} + \frac{1}{n}|^{q-1}} + T_n(f(x)) \le n - Mc(\bar\lambda - \lambda) < 0 \qquad \text{in } \Omega\,.$$

Clearly, $\underline{\psi} \equiv 0$ is a subsolution to (Q_n) and $\underline{\psi} \equiv 0 \le \overline{\psi}$ in Ω. Therefore, [13, Théorème 3.1] (see also [23]) implies that there exists a solution u_n to (Q_n) such that $0 \le u_n \le \overline{\psi}$ in Ω.

In order to prove (5.1), we use an argument by comparison. Indeed, we first observe that

$$\begin{cases} -\Delta u_n \ge \lambda u_n + T_1(f) & \text{in } \Omega,\\ u_n = 0 & \text{on } \partial\Omega. \end{cases}$$

Since $\lambda < \lambda^* \leq \Lambda$, then the operator $-(\Delta + \lambda)$ verifies the maximum principle, so that we compare u_n with the solution ζ to the problem

$$\begin{cases} -\Delta\zeta = \lambda\zeta + T_1(f) & \text{in } \Omega, \\ \zeta = 0 & \text{on } \partial\Omega, \end{cases}$$

and thus, we obtain that $u_n \geq \zeta$ in Ω. Now, since $f \gneq 0$ in Ω, the strong maximum principle (which holds since Ω is connected and, again, because $\lambda < \Lambda$, see [22]) implies that ζ satisfies (5.1), and then, so does u_n.

In order to prove the compactness of the sequence $\{u_n\}_{n\in\mathbb{N}}$, we choose now u_n as test function in (Q_n), and using that $T_n(f) \leq f$ in Ω for any n together with the $L^\infty(\Omega)$ bound, we easily deduce that $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $H_0^1(\Omega)$. This implies that there exists a function $u \in H_0^1(\Omega)$ such that, passing to a subsequence, $u_n \rightharpoonup u$ weakly in $H_0^1(\Omega)$. Using that, in particular, $u_n \to u$ a.e. in Ω, we deduce that $u > 0$ and $u \in L^\infty(\Omega)$.

On the other hand, by Lemma A.8 we deduce that $\{u_n\}_{n\in\mathbb{N}}$ is uniformly bounded in $C^{0,\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1)$. Hence, the compact embedding $C^{0,\alpha}(\overline{\Omega}) \hookrightarrow C(\overline{\Omega})$ yields that $u_n \to u$ uniformly in $\overline{\Omega}$. □

Using the compactness provided by the previous result, we prove now the existence of a solution to (P_λ) for $f \gneq 0$ and for every $\lambda < \lambda^*$.

Proposition 5.2. *Assume that $1 < q \leq 2$, $0 \leq \mu \in L^\infty(\Omega)$, with $\|\mu\|_{L^\infty(\Omega)} < 1$ if $q = 2$, and $0 \lneq f \in L^p(\Omega)$ with $p > \frac{N}{2}$. Then, there exists a solution to (P_λ) for every $\lambda < \lambda^*$.*

Proof. Consider the sequence $\{u_n\}_{n\in\mathbb{N}}$ of solutions to (Q_n) given by Lemma 5.1. We claim that $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $L^\infty(\Omega)$. Indeed, arguing by contradiction, assume that it is unbounded in $L^\infty(\Omega)$, and take a (not relabelled) subsequence such that $\|u_n\|_{L^\infty(\Omega)} \to \infty$. Then, we have that the function $z_n \equiv \frac{u_n}{\|u_n\|_{L^\infty(\Omega)}} \in H_0^1(\Omega) \cap L^\infty(\Omega)$ satisfies, for every n, that

$$\begin{cases} -\Delta z_n = \lambda z_n + \mu(x) \dfrac{|\nabla z_n|^q}{\left(z_n + \frac{1}{n\|u_n\|_{L^\infty(\Omega)}}\right)^{q-1}} + \dfrac{T_n(f(x))}{\|u_n\|_{L^\infty(\Omega)}} & \text{in } \Omega, \\ z_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.2}$$

Since $\|z_n\|_{L^\infty(\Omega)} = 1$ for any n, then $\{z_n\}_{n\in\mathbb{N}}$ is clearly bounded in $L^\infty(\Omega)$, so following the arguments of the proof of Lemma 5.1 we deduce that there exists $0 \leq z \in H_0^1(\Omega) \cap L^\infty(\Omega)$ such that, passing to a subsequence, $z_n \rightharpoonup z$ weakly in $H_0^1(\Omega)$ and $z_n \to z$ uniformly in $\overline{\Omega}$. However, we can not argue as in Lemma 5.1 to prove neither the local lower bound to the sequence $\{z_n\}_{n\in\mathbb{N}}$, nor that the limit $z > 0$ in Ω, since one does not have a lower bound for $\left\{\frac{T_n(f(x))}{\|u_n\|_{L^\infty(\Omega)}}\right\}_{n\in\mathbb{N}}$ independent of n. Hence, we need to use a different argument.

Indeed, observe first that the uniform convergence implies that $\|z\|_{L^\infty(\Omega)} = 1$, so $z \gneq 0$ in Ω. Fix now $0 \leq \phi \in H_0^1(\Omega) \cap L^\infty(\Omega)$. We know by the weak $H_0^1(\Omega)$ convergence that

$$\lim_{n\to+\infty} \int_\Omega \nabla z_n \nabla\phi - \lambda \int_\Omega z_n\phi = \int_\Omega \nabla z \nabla\phi - \lambda \int_\Omega z\phi$$

and, since $\displaystyle\int_\Omega \nabla z_n \nabla\phi - \lambda \int_\Omega z_n\phi \geq 0$ for any n, we have that $\displaystyle\int_\Omega \nabla z \nabla\phi - \lambda \int_\Omega z\phi \geq 0$, too.

On the other hand, since $\lambda < \lambda^* \leq \Lambda$, the strong maximum principle holds for the operator $-(\Delta + \lambda)$. Thus, since Ω is connected and z is not constant (the only constant in $H_0^1(\Omega)$ is the null function), the strong maximum principle implies that, for every $\omega \subset\subset \Omega$, there exists a constant $\widetilde{c}_\omega > 0$ such that $z \geq \widetilde{c}_\omega$ in ω for any n (in particular, $z > 0$ in Ω and $\frac{|\nabla z|^q}{z^{q-1}} \in L^1_{\mathrm{loc}}(\Omega)$). Furthermore, the uniform convergence yields that z_n

satisfies $z_n \geq c_\omega > 0$, $\forall \omega \subset\subset \Omega$, $\forall n \in \mathbb{N}$. This implies that $\{-\Delta z_n\}_{n\in\mathbb{N}}$ is bounded in $L^1_{\rm loc}(\Omega)$, that combined with the H^1 bound implies that

$$\nabla z_n \to \nabla z \ \text{ strongly in } L^r(\Omega)^N \text{ for any } r < 2$$

(see [12]). The local lower bound and the convergence of the gradients will allow us to pass to the limit in (5.2).

Indeed, assume first that $1 < q < 2$, and let $\phi \in C^1_c(\Omega)$. We know that there exists a function $h \in L^1(\Omega)$ such that, passing to a subsequence if necessary, $|\nabla z_n|^q \leq h(x)$ in Ω, and we also have that $|\nabla z_n|^q \to |\nabla z|^q$ a.e. in Ω. Therefore, choosing an open set $\omega \subset\subset \Omega$ such that $\mathrm{supp}(\phi) \subset \omega$, we have that

$$\frac{\mu(x)|\nabla z_n|^q \phi}{\left(z_n + \frac{1}{n\|u_n\|_{L^\infty(\Omega)}}\right)^{q-1}} \leq \frac{\mu(x)h(x)\phi}{\widetilde{c}_\omega^{\,q-1}} \qquad \text{in } \Omega,$$

where $\dfrac{\mu h \phi}{\widetilde{c}_\omega^{\,q-1}} \in L^1(\Omega)$. On the other hand, we also have that

$$\frac{\mu(x)|\nabla z_n|^q \phi}{\left(z_n + \frac{1}{n\|u_n\|_{L^\infty(\Omega)}}\right)^{q-1}} \to \frac{\mu(x)|\nabla z|^q \phi}{z^{q-1}} \qquad \text{a.e. in } \Omega \qquad \text{as } n \to +\infty.$$

Hence, we may pass to the limit in the second term of the right hand side of the equation in (5.2). It is straightforward to verify that the rest of the terms also converge, so that we conclude that z is a solution (see Lemma A.5 in the Appendix below) to (E_λ), but this is a contradiction with Theorem 4.3 since $\lambda < \lambda^*$.

On the other hand, assume now that $q = 2$ and $\|\mu\|_{L^\infty(\Omega)} < 1$, and let $\phi \in C^1_c(\Omega)$. We may assume without loss of generality that $\phi \geq 0$ in Ω. In this case we argue as in [3] (see also [8]). Thus, using that

$$\begin{aligned} z_n &\to z \qquad \text{a.e. in } \Omega, \text{ weakly in } H^1_0(\Omega), \text{ strongly in } L^2(\Omega), \\ \nabla z_n &\to \nabla z \qquad \text{a.e. in } \Omega, \end{aligned}$$

we obtain, by virtue of Fatou's Lemma, the inequality

$$\int_\Omega \nabla z \nabla \phi \geq \lambda \int_\Omega z\phi + \int_\Omega \mu(x)\frac{|\nabla z|^2}{z}\phi.$$

In order to prove the reverse inequality, let us take $\dfrac{z_n}{z}\phi \in H^1_0(\Omega) \cap L^\infty(\Omega)$ as test function in (5.2). It follows that

$$\int_\Omega \frac{|\nabla z_n|^2\phi}{z} - \int_\Omega \frac{z_n\phi}{z^2}\nabla z_n \nabla z + \int_\Omega \frac{z_n}{z}\nabla z_n \nabla\phi = \lambda\int_\Omega \frac{z_n^2\phi}{z} + \int_\Omega \frac{\mu(x)|\nabla z_n|^2 z_n\phi}{\left(z_n + \frac{1}{n\|u_n\|_{L^\infty(\Omega)}}\right)z} + \int_\Omega \frac{T_n(f(x))z_n\phi}{\|u_n\|_{L^\infty(\Omega)}z}.$$

Since $\|\mu\|_{L^\infty(\Omega)} < 1$, we deduce that

$$\frac{|\nabla z_n|^2\phi}{z} - \frac{\mu(x)|\nabla z_n|^2 z_n\phi}{\left(z_n + \frac{1}{n\|u_n\|_{L^\infty(\Omega)}}\right)z} \geq 0 \qquad \text{a.e. in } \Omega,$$

therefore, Fatou's Lemma yields to

$$\int_\Omega \frac{|\nabla z|^2\phi}{z} - \int_\Omega \frac{\mu(x)|\nabla z|^2\phi}{z} \leq \liminf_{n\to\infty}\left(\int_\Omega \frac{z_n\phi}{z^2}\nabla z_n\nabla z - \int_\Omega \frac{z_n}{z}\nabla z_n\nabla\phi + \lambda\int_\Omega \frac{z_n^2\phi}{z} + \int_\Omega \frac{T_n(f(x))z_n\phi}{\|u_n\|_{L^\infty(\Omega)}z}\right).$$

Finally, using that $z_n \rightharpoonup z$ weakly in $H^1_0(\Omega)$ and $z_n \to z$ strongly in $L^2(\Omega)$, we obtain

$$\int_\Omega \nabla z \nabla \phi \leq \lambda \int_\Omega z\phi + \int_\Omega \mu(x)\frac{|\nabla z|^2}{z}\phi.$$

In conclusion, z is a solution (see again Lemma A.5) to problem (E_λ), so that we get again a contradiction.

Thus, we have proved that the sequence $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $L^\infty(\Omega)$. We conclude the proof of the result by applying Lemma 5.1 and passing to the limit in (Q_n) as we did for $\{z_n\}_{n\in\mathbb{N}}$, with the only difference that this time the local lower bound for $\{u_n\}_{n\in\mathbb{N}}$ is provided directly by (5.1) in Lemma 5.1. □

We are ready now to prove our bifurcation result.

Proposition 5.3. *Assume that* $1 < q \leq 2$, $0 \leq \mu \in L^\infty(\Omega)$, *with* $\|\mu\|_{L^\infty(\Omega)} < 1$ *if* $q = 2$, *and* $0 \lneq f \in L^p(\Omega)$ *with* $p > \frac{N}{2}$. *Then, if* $\bar{\lambda} \in \mathbb{R}$ *is a bifurcation point from infinity of* (P_λ), *necessarily* $\bar{\lambda} = \lambda^*$. *Moreover, if* f *satisfies condition* (f_0), *then the set*

$$\Sigma := \{(\lambda, u_\lambda) \in \mathbb{R} \times C(\overline{\Omega}) : u_\lambda \text{ is a solution to } (P_\lambda)\}$$

is a continuum.

If in addition (P_λ) *has no solution for* $\lambda = \lambda^*$, *then the continuum is unbounded and it bifurcates from infinity at* λ^* *to the left.*

Proof. Assume that $\bar{\lambda} \in \mathbb{R}$ is a bifurcation point from infinity to (P_λ), i.e., there exists a sequence of real numbers $\{\lambda_n\}_{n\in\mathbb{N}}$ with $\lambda_n \to \bar{\lambda}$ such that there exists a solution u_n to (P_{λ_n}) for any n satisfying $\|u_n\|_{L^\infty(\Omega)} \to \infty$. Proceeding as in the proof of Proposition 5.2, we may pass to the limit in the equation satisfied by $z_n \equiv \frac{u_n}{\|u_n\|_{L^\infty(\Omega)}}$, so that we obtain a solution z to problem (E_λ). Therefore, by virtue of Proposition 4.3, we have necessarily that $\lambda = \lambda^*$.

Assume now that f satisfies (f_0). We will prove that the set Σ is a continuum. In other words, we will show that the function

$$\begin{gathered}(-\infty, \lambda^*) \to C(\overline{\Omega}) \\ \lambda \mapsto u_\lambda\end{gathered}$$

is continuous, where u_λ denotes the unique solution to (P_λ) given by Proposition 5.2 (the uniqueness of $u_\lambda \in C(\overline{\Omega})$ follows from Lemma 2.4 and Theorem 3.2). Indeed, let us fix $\lambda \in (-\infty, \lambda^*)$, and choose a sequence $\{\lambda_n\}_{n\in\mathbb{N}} \subset (-\infty, \lambda^*)$ such that $\lambda_n \to \lambda$ as n diverges. Arguing again as in the proof of Proposition 5.2, if one assumes that $\{u_{\lambda_n}\}_{n\in\mathbb{N}}$ is unbounded in $L^\infty(\Omega)$, then a solution to problem (E_λ) can be found, but this is impossible because $\lambda < \lambda^*$. Thus, necessarily $\{u_{\lambda_n}\}_{n\in\mathbb{N}}$ is bounded in $L^\infty(\Omega)$, so that we deduce as in Lemma 5.1 that $u_{\lambda_n} \to u_\lambda$ uniformly in $\overline{\Omega}$, i.e., in the space $C(\overline{\Omega})$.

To conclude we prove that the continuum is unbounded by showing that λ^* is a bifurcation point from infinity to the left of the axis $\lambda = \lambda^*$. Indeed, assuming that $\{u_{\lambda_n}\}_{n\in\mathbb{N}}$ is bounded in $L^\infty(\Omega)$ for some sequence $\{\lambda_n\}_{n\in\mathbb{N}} \subset (-\infty, \lambda^*)$ with $\lambda_n \to \lambda^*$ as n diverges, we can pass to the limit in (P_{λ_n}) and we find a solution to (P_{λ^*}), but this is a contradiction.

In conclusion, λ^* is a bifurcation point from infinity to the left of $\lambda = \lambda^*$. □

6. Proofs of the main results and final remarks

Proof of Theorem 1.2. We deduce from Proposition 5.2 the existence of at least one solution to (P_λ) if $\lambda < \lambda^*$. Moreover, the nonexistence for $\lambda > \lambda^*$ in deduced by Proposition 4.3.

As far as uniqueness is concerned, we observe that if u, v are two solutions to (P_λ), then Lemma 2.4 implies that $u, v \in C(\overline{\Omega}) \cap W^{1,N}_{\text{loc}}(\Omega)$. In particular, using the continuity up to the boundary of u, v and the fact that $u(x_0) = 0$ for any $x_0 \in \partial\Omega$, we have that u, v satisfy (3.9) for any $\varepsilon > 0$. Moreover, they obviously satisfy (3.2) and (3.3) respectively. Therefore, Theorem 3.2 implies that $u \leq v$ in Ω. The reverse inequality follows by interchanging the roles of u and v.

We give now a proof for the nonexistence of solutions to (P_{λ^*}). Thus, assume by contradiction that there exists a solution u to (P_{λ^*}). Then, we can find a solution v to

$$\begin{cases} -\Delta v = \lambda^* v + \mu(x)\dfrac{|\nabla v|^q}{v^{q-1}} + C\varphi_1^\gamma & \text{in } \Omega, \\ v > 0 & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{6.1}$$

The proof of this fact follows basically the same steps as Proposition 5.2: the only difference is the way of proving the L^∞ estimate, which does not work in this case. However, since we are assuming that there is a solution u to (P_{λ^*}), then, by comparison, any solution to the approximate problems (Q_n) is smaller than $\|u\|_{L^\infty(\Omega)}$, which gives the a priori estimate.

Furthermore, using Lemma 2.7 we deduce that for $\varepsilon > 0$ small enough, the following holds:

$$C_1\varphi_1^\gamma - \varepsilon v \geq (C_1 - \varepsilon C))\varphi_1^\gamma \geq 0.$$

Therefore,

$$-\Delta v = (\lambda^* + \varepsilon)v + \mu(x)\frac{|\nabla v|^q}{v^{q-1}} + (C_1\varphi_1^\gamma - \varepsilon v) \geq (\lambda^* + \varepsilon)v + \mu(x)\frac{|\nabla v|^q}{v^{q-1}}.$$

That is to say, v is a supersolution to $(E_{\lambda^*+\varepsilon})$. Moreover, by Proposition 2.6, $v^\eta \in H_0^1(\Omega)$ for every $\eta \in (\eta_0, 1)$ for some $\eta_0 < 1$. This is a contradiction with the characterization (4.2) in Proposition 4.1. So we have proved the nonexistence result.

Finally, the claim about bifurcation follows from Proposition 5.3.

□

Proof of Theorem 1.1. We have shown in Proposition 4.1 that $\lambda^* \in (0, \Lambda]$. Moreover, as a consequence of Proposition 4.3, if (E_λ) admits a solution then $\lambda = \lambda^*$. Thus, for the first part of the theorem it only remains to prove the existence of solution to (E_λ) for $\lambda = \lambda^*$. In order to do that, by virtue of Proposition 5.3, we may choose $\lambda_n \to \lambda^*$ such that $\|u_{\lambda_n}\|_{L^\infty(\Omega)} \to \infty$, where u_{λ_n} denotes, for any n, the unique solution to the problem

$$\begin{cases} -\Delta u = \lambda_n u + \mu(x)\dfrac{|\nabla u|^q}{u^{q-1}} + 1 & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Hence, arguing again as in Proposition 5.2, we may pass to the limit in the equation satisfied by $z_n \equiv \dfrac{u_{\lambda_n}}{\|u_{\lambda_n}\|_{L^\infty(\Omega)}}$ using that $\|z_n\|_{L^\infty(\Omega)} = 1$ for any n, concluding that the limit z is a solution to (E_{λ^*}).

Regarding the uniqueness of the solution up to multiplicative constants, it follows by adapting the uniqueness result proved in [26] to $v_i = \log(u_i)$, $i = 1, 2$, being u_1 and u_2 two solutions to (E_{λ^*}). □

We conclude the section with some remarks concerning the principal eigenvalue and some possible extensions of our results.

Remark 6.1. *Let us remark that the global behavior of the continuum in Proposition 5.3 corresponds to the one obtained in* [8] *for* $q = 2$. *That is to say,* $\lambda^* > 0$ *is the only possible bifurcation point from infinity.*

However, as it was pointed out in the introduction, there are similar singular problems that exhibit a completely different behavior. For instance, whenever bifurcation occurs for the problem

$$\begin{cases} -\Delta u = \lambda u + \mu \dfrac{|\nabla u|^2}{u^\theta} + f(x) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

with $\theta \in [0,1)$, it is possible to prove (see [7]) that $\lambda = 0$ is the only possible bifurcation point from infinity. Hence the qualitative behavior of the continuum of solutions is different if the problem above possesses or not a solution for $\lambda = 0$.

Remark 6.2. *We show here that, in the case $q = 2$, there exists $0 \le \mu \in L^\infty(\Omega)$ with $\|\mu\|_{L^\infty(\Omega)} < 1$ such that $\lambda^* > \dfrac{\Lambda}{\|\mu\|_{L^\infty(\Omega)} + 1}$. This proves that, if μ is not a constant, then the condition $\lambda < \dfrac{\Lambda}{\|\mu\|_{L^\infty(\Omega)} + 1}$ is not necessary in general for the existence of solutions to (P_λ).*

Indeed, by contradiction, assume that $\lambda^* = \dfrac{\Lambda}{\|\mu\|_{L^\infty(\Omega)} + 1}$ for any $0 \le \mu \in L^\infty(\Omega)$ with $\|\mu\|_{L^\infty(\Omega)} < 1$. Fix a point $x_0 \in \Omega$, and consider a sequence of balls $\{B_{\frac{1}{n}}(x_0)\}_{n\in\mathbb{N}} \subset \Omega$. For any n, let us define in Ω the functions

$$\mu_n(x) = \frac{1}{2}\chi_{B_{\frac{1}{n}}(x_0)}$$

Since $\|\mu_n\|_{L^\infty(\Omega)} = 1/2 < 1$ for any n, we may consider a solution u_n to

$$\begin{cases} -\Delta u_n = \dfrac{\Lambda}{\|\mu_n\|_{L^\infty(\Omega)} + 1} u_n + \mu_n(x) \dfrac{|\nabla u_n|^2}{u_n} & \text{in } \Omega, \\ u_n > 0 & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega. \end{cases}$$

Observe that $\dfrac{\Lambda}{\|\mu_n\|_{L^\infty(\Omega)} + 1} = \dfrac{2\Lambda}{3}$ for any n, and that $\mu_n \to 0$ a.e. in Ω.

If we choose u_n so that $\|u_n\|_{L^\infty(\Omega)} = 1$ for any n, then, arguing as in the proof of Proposition 5.2, we may pass to the limit and find a solution u to

$$\begin{cases} -\Delta u = \dfrac{2\Lambda}{3} u & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

But this is a contradiction since $\dfrac{2\Lambda}{3} < \Lambda$.

Remark 6.3. *It is worth to highlight that we can prove the strict inequality $\lambda^* < \Lambda$ provided $\mu > 0$ in Ω. Indeed, let u be a solution to (E_{λ^*}). Then, taking φ_1 as test function in (E_{λ^*}), we obtain*

$$(\Lambda - \lambda^*) \int_\Omega u\varphi_1 = \int_\Omega \mu(x) \frac{|\nabla u|^q}{u^{q-1}} \varphi_1 > 0,$$

which clearly implies what we claimed.

Remark 6.4. *We also point out that Theorem 1.1 yields to*

$$I^* = I_1 = (-\infty, \lambda^*) \quad \text{and} \quad I_2 = (-\infty, \lambda^*].$$

Indeed, if u is a solution to (E_{λ^*}), then it follows trivially from Lemma 2.6 that $\lambda^* \in I_2$. On the other hand, assume by contradiction that $\lambda^* \in I^*$. Then, there exists a supersolution $\varphi \in H^1(\Omega) \cap L^\infty(\Omega)$ to (E_{λ^*}) with $\varphi \geq c$ in Ω for some $c > 0$. Hence, we may argue as in Proposition 4.3 to obtain that $tu \leq \varphi$ in Ω for every $t > 0$, which is obviously impossible.

Remark 6.5. *In the whole paper we have confined ourselves to the case $q > 1$ in order to deal with the singular problems (P_λ) and (E_λ). Nevertheless, our results hold true also for $q = 1$ by following the same approach (with some small difference in the proof of the positivity of λ^*).*

Remark 6.6. *The hypotheses made on the smoothness of $\partial\Omega$ deserve also some comments. For the sake of clarity we have assumed in the whole paper that the boundary is of class $C^{1,1}$. Actually, such a regularity of the boundary is needed only in order to obtain $C^1(\overline{\Omega})$ regularity of solutions to linear problems (which is provided by Calderon-Zygmund regularity theory).*

Apart from those results, it suffices to impose a weaker regularity assumption on $\partial\Omega$ in order to prove the rest of our results. Indeed, one needs to suppose that Ω satisfies the following condition:

(6.2) *Let $\Omega \subset \mathbb{R}$ be an open set and suppose that there exist r_0, $\theta_0 > 0$ such that if $x \in \partial\Omega$ and $0 < r < r_0$, then $|\Omega_r| \leq (1-\theta_0)\ |B_r(x)|$, for every connected component Ω_r of $\Omega \cap B_r(x)$.*

Such a condition is specifically needed to prove $C^{0,\alpha}(\overline{\Omega})$ regularity (and also uniform estimates in this space) of the solutions.

Appendix A

This section consists of five lemmata that prove that the formulation given in Definition 2.1 is totally meaningful and actually can be changed into an equivalent one in which the test functions have compact support.

Lemma A.1. *Let $\phi \in H_0^1(\Omega) \cap L^\infty(\Omega)$ be such that $\operatorname{supp}(\phi) \subset \Omega$. Then, there exist an open set $\omega \subset\subset \Omega$ and a sequence $\{\phi_n\}_{n\in\mathbb{N}} \subset C_c^1(\Omega)$ such that it is bounded in $L^\infty(\Omega)$, $\operatorname{supp}(\phi_n) \subset \omega$ for any n, and $\phi_n \to \phi$ strongly in $H_0^1(\Omega)$.*

Proof. Take an open set ω such that $\operatorname{supp}(\phi) \subset \omega \subset\subset \Omega$. Then, $\phi \in H_0^1(\omega) \cap L^\infty(\omega)$. Let $\psi_n \in C_c^1(\omega)$ be such that $\psi_n \to \phi$ strongly in $H_0^1(\omega)$, then

$$\psi_n \to \phi \quad \text{a.e. in } \Omega, \qquad \nabla\psi_n \to \nabla\phi \quad \text{a.e. in } \Omega,$$

Consider now a function $G : \mathbb{R} \to \mathbb{R}$ satisfying the following properties:

(i) $G \in C^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$,
(ii) $G(s) = s \quad \forall s \in \left[-\|\phi\|_{L^\infty(\omega)}, \|\phi\|_{L^\infty(\omega)}\right]$.

Clearly, we have that $\{G(\psi_n)\}_{n\in\mathbb{N}} \subset C_c^1(\omega)$, it is bounded in $L^\infty(\omega)$ and, in addition,

$$\nabla G(\psi_n) = G'(\psi_n)\nabla\psi_n \to G'(\phi)\nabla\phi = \nabla G(\phi) = \nabla\phi \quad \text{a.e. in } \omega.$$

Moreover,

$$|\nabla G(\psi_n) - \nabla\phi|^2 \leq |\nabla G(\psi_n)|^2 + |\nabla\phi|^2 + 2|\nabla G(\psi_n)||\nabla\phi|$$

and therefore, the Vitali's Theorem yields that $G(\psi_n) \to \phi$ strongly in $H_0^1(\omega)$.

For any n, let us define the function ϕ_n in Ω by

$$\phi_n = \begin{cases} G(\psi_n) & \text{in } \omega, \\ 0 & \text{in } \Omega \setminus \omega. \end{cases}$$

Thus, we have that $\{\phi_n\}_{n\in\mathbb{N}} \subset C_c^1(\Omega)$, it is bounded in $L^\infty(\Omega)$, $\phi_n \to \phi$ strongly in $H_0^1(\Omega)$ and, in addition, $\mathrm{supp}(\phi_n) \subset \omega$ for any n. $\square$

Lemma A.2. *Let $\phi \in H_0^1(\Omega)$ be such that $\phi \geq 0$ a.e. in Ω, and let $\{\phi_n\}_{n\in\mathbb{N}}$ be a sequence in $C_c^1(\Omega)$ such that $\phi_n \to \phi$ strongly in $H_0^1(\Omega)$. Then, $\phi_n^+ \to \phi$ strongly in $H_0^1(\Omega)$.*

Proof. We know, passing to a subsequence, that

$$\phi_n \to \phi \quad \text{a.e. in } \Omega, \text{ and } \nabla\phi_n \to \nabla\phi \quad \text{a.e. in } \Omega.$$

Observe that

$$\begin{aligned} \int_\Omega |\nabla(\phi_n^+ - \phi)|^2 &= \int_\Omega |\nabla(\phi_n - \phi)|^2 + 2\int_\Omega \nabla\phi_n^- \nabla\phi - \int_\Omega |\nabla\phi_n^-|^2 \\ &\leq \int_\Omega |\nabla(\phi_n - \phi)|^2 + 2\int_\Omega \nabla\phi_n^- \nabla\phi \end{aligned} \tag{A.1}$$

for any n.

Now, by continuity we deduce that $\phi_n^- \to \phi^- = 0$ a.e. in Ω. Therefore, passing to a new subsequence, we infer that $\{\phi_n^-\}_{n\in\mathbb{N}}$ weakly converges in $H_0^1(\Omega)$ to some limit ψ, and then $\phi_n^- \to \psi$ a.e. in Ω. Hence, necessarily $\psi \equiv 0$, that is to say, $\phi_n^- \rightharpoonup 0$ weakly in $H_0^1(\Omega)$.

Finally, from (A.1) we conclude that $\phi_n^+ \to \phi$ strongly in $H_0^1(\Omega)$, where $\{\phi_n\}_{n\in\mathbb{N}}$ is a subsequence of the original sequence. Actually we have prove that any subsequence of $\{\phi_n\}_{n\in\mathbb{N}}$ admits a subsequence such that $\phi_n^+ \to \phi$ strongly in $H_0^1(\Omega)$. Then we have necessarily that, indeed, the positive part of the original sequence strongly converges to ϕ in $H_0^1(\Omega)$. $\square$

Lemma A.3. *Let $\phi \in H_0^1(\Omega)$ be such that $\phi \geq 0$ a.e. in Ω. Then, there exists a sequence $\{\phi_n\}_{n\in\mathbb{N}} \subset H_0^1(\Omega)$ such that $\mathrm{supp}(\phi_n) \subset \Omega$ for any n, $0 \leq \phi_n \leq \phi$ a.e. in Ω for any n, and $\phi_n \to \phi$ strongly in $H_0^1(\Omega)$. In particular, if $\phi \in L^\infty(\Omega)$, then $\{\phi_n\}_{n\in\mathbb{N}}$ is bounded in $L^\infty(\Omega)$.*

Proof. Let $\{\psi_n\}_{n\in\mathbb{N}} \subset C_c^1(\Omega)$ be such that $\psi_n \to \phi$ strongly in $H_0^1(\Omega)$. By virtue of Lemma A.2, we have that $\psi_n^+ \to \phi$ strongly in $H_0^1(\Omega)$. For any n, let us define now the function $\phi_n = \phi + (\psi_n^+ - \phi)^-$ in Ω. Clearly, $\phi_n \in H_0^1(\Omega)$ and $0 \leq \phi_n \leq \phi$ a.e. in Ω for any n. Observe also that, for any n, it holds that $\phi_n \leq \psi_n^+$ a.e. in Ω and $\psi_n^+ = 0$ a.e. in $\Omega\backslash\mathrm{supp}(\psi_n^+)$, so $\phi_n = 0$ a.e. in $\Omega\backslash\mathrm{supp}(\psi_n^+)$. Hence, by the definition of the essential support of ϕ_n, we have that $\Omega\backslash\mathrm{supp}(\psi_n^+) \subset \Omega\backslash\mathrm{supp}(\phi_n)$, and thus, $\mathrm{supp}(\phi_n) \subset \mathrm{supp}(\psi_n^+) \subset \Omega$.

Finally, we have that

$$\int_\Omega |\nabla(\phi_n - \phi)|^2 = \int_\Omega |\nabla(\psi_n^+ - \phi)^-|^2 \leq \int_\Omega |\nabla(\psi_n^+ - \phi)|^2,$$

and therefore, $\phi_n \to \phi$ strongly in $H_0^1(\Omega)$. $\square$

Lemma A.4. *Assume that $1 < q \leq 2$, $0 \leq \mu \in L^\infty(\Omega)$, $0 \leq f \in L^p(\Omega)$ with $p > \frac{N}{2}$ and $\lambda \in \mathbb{R}$. Let $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ be such that $u > 0$ a.e. in Ω, $\frac{|\nabla u|^q}{u^{q-1}} \in L^1_{loc}(\Omega)$ and satisfies*

$$\int_\Omega \nabla u \nabla\phi = \lambda\int_\Omega u\phi + \int_\Omega \mu(x)\frac{|\nabla u|^q}{u^{q-1}}\phi + \int_\Omega f(x)\phi$$

for any $\phi \in C_c^1(\Omega)$. Then, the same equality holds for every $\phi \in H_0^1(\Omega) \cap L^\infty(\Omega)$ with compact support.

Proof. Let $\phi \in H_0^1(\Omega) \cap L^\infty(\Omega)$ be such that $\mathrm{supp}(\phi) \subset \Omega$, and let ω and $\{\phi_n\}_{n\in\mathbb{N}}$ be the open set and the sequence given by Lemma A.1, respectively. This lemma gives also that

$$\mu(x)\frac{|\nabla u|^q}{u^{q-1}}\phi_n \leq \|\mu\|_{L^\infty(\Omega)}\frac{|\nabla u|^q}{u^{q-1}}\|\phi_n\|_{L^\infty(\Omega)} \leq C\frac{|\nabla u|^q}{u^{q-1}}$$

a.e. in Ω, where $C > 0$ is a constant independent of n. Therefore, since $\frac{|\nabla u|^q}{u^{q-1}} \in L^1_{\rm loc}(\Omega)$, we can use the Lebesgue's Theorem to derive

$$\int_\Omega \mu(x)\frac{|\nabla u|^q}{u^{q-1}}\phi_n = \int_\omega \mu(x)\frac{|\nabla u|^q}{u^{q-1}}\phi_n \to \int_\omega \mu(x)\frac{|\nabla u|^q}{u^{q-1}}\phi = \int_\Omega \mu(x)\frac{|\nabla u|^q}{u^{q-1}}\phi.$$

The conclusion of the lemma is now straightforward. □

Lemma A.5. *Assume that $1 < q \le 2$, $0 \le \mu \in L^\infty(\Omega)$, $0 \le f \in L^p(\Omega)$ with $p > \frac{N}{2}$ and $\lambda \in \mathbb{R}$. Let $u \in H^1_0(\Omega) \cap L^\infty(\Omega)$ be such that $u > 0$ a.e. in Ω, $\frac{|\nabla u|^q}{u^{q-1}} \in L^1_{loc}(\Omega)$ and satisfies*

$$\int_\Omega \nabla u \nabla \phi = \lambda \int_\Omega u\phi + \int_\Omega \mu(x)\frac{|\nabla u|^q}{u^{q-1}}\phi + \int_\Omega f(x)\phi$$

for any $\phi \in C^1_c(\Omega)$. Then, u is a solution to (P_λ) in the sense of Definition 2.1.

Similarly, if $u \in H^1(\Omega) \cap L^\infty(\Omega)$ is such that $u > 0$ a.e. in Ω, $\frac{|\nabla u|^q}{u^{q-1}} \in L^1_{loc}(\Omega)$ and satisfies

$$\int_\Omega \nabla u \nabla \phi \ge \lambda \int_\Omega u\phi + \int_\Omega \mu(x)\frac{|\nabla u|^q}{u^{q-1}}\phi + \int_\Omega f(x)\phi$$

for any $0 \le \phi \in H^1_0(\Omega) \cap L^\infty(\Omega)$ with compact support, then u is a supersolution to (P_λ) in the sense of Definition 2.1.

Proof. Let $\phi \in H^1_0(\Omega) \cap L^\infty(\Omega)$, and let $\{\phi_n\}_{n\in\mathbb{N}}$ be the sequence in $H^1_0(\Omega) \cap L^\infty(\Omega)$ given by Lemma A.3 such that $\phi_n \to \phi^+$ strongly in $H^1_0(\Omega)$. By virtue of Lemma A.4, we have that

$$\int_\Omega \mu(x)\frac{|\nabla u|^q}{u^{q-1}}\phi_n = \int_\Omega \nabla u \nabla \phi_n - \lambda \int_\Omega u\phi_n - \int_\Omega f(x)\phi_n \quad \forall n \in \mathbb{N}. \tag{A.2}$$

Hence, by Fatou's Lemma and by the weak convergence $\phi_n \rightharpoonup \phi^+$ in $H^1_0(\Omega)$, we obtain that

$$\int_\Omega \mu(x)\frac{|\nabla u|^q}{u^{q-1}}\phi^+ \le \int_\Omega \nabla u \nabla \phi^+ - \lambda \int_\Omega u\phi^+ - \int_\Omega f(x)\phi^+.$$

That means that $\dfrac{|\nabla u|^q}{u^{q-1}}\phi^+ \in L^1(\Omega)$. This fact allows us to pass to the limit in (A.2) by using the Lebesgue's Theorem for the left hand side (recall that $\phi_n \le \phi^+$ for any n), and again the weak convergence for the right hand side, so that we obtain

$$\int_\Omega \mu(x)\frac{|\nabla u|^q}{u^{q-1}}\phi^+ = \int_\Omega \nabla u \nabla \phi^+ - \lambda \int_\Omega u\phi^+ - \int_\Omega f(x)\phi^+.$$

An analogous procedure provides us the same identity but replacing ϕ^+ by ϕ^-. The proof of the first part of the lemma concludes by simply adding both identities.

The last part of the lemma about supersolutions can be proved in a similar way. □

Regularity of the solutions

Proof of Lemma 2.4: Hölder regularity.

Here we prove that any $H^1_0(\Omega) \cap L^\infty(\Omega)$ solution to (P_λ) actually belongs to $C^{0,\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1)$. For this purpose, we make use of the regularity theory developed by Ladyzenskaya and Ural'tseva in [25].

We denote an open ball with radius $\rho > 0$ as B_ρ, and for $v : \Omega \to \mathbb{R}$, $k \in \mathbb{R}$, we also write

$$A_{k,\rho}(v) = \{x \in \Omega \cap B_\rho : v(x) \ge k\}.$$

Definition A.6 ([25], p. 90)**.** Let $\Omega \subset \mathbb{R}^N$ be an open domain, and let $M, \gamma, \delta > 0$, $r \in (N, +\infty]$, $m > 1$. We say that a function $u : \Omega \to \mathbb{R}$ belongs to the class $\mathcal{B}_m\left(\overline{\Omega}, M, \gamma, \delta, \frac{1}{r}\right)$ if $u \in W^{1,m}(\Omega) \cap L^\infty(\Omega)$ with $\|u\|_{L^\infty(\Omega)} \leq M$, and the following holds for $v = u$ and also for $v = -u$:

$$\int_{A_{k,\rho-\sigma\rho}(v)} |\nabla v|^m \leq \gamma \left(\frac{1}{\sigma^m \rho^{m\left(1-\frac{N}{r}\right)}} \|v-k\|^m_{L^\infty(A_{k,\rho}(v))} + 1 \right) |A_{k,\rho}(v)|^{1-\frac{m}{r}}, \tag{A.3}$$

for any $\rho > 0$ and all B_ρ such that $\Omega \cap B_\rho \neq \emptyset$, for all $\sigma \in (0,1)$ and for all $k \geq k_\rho$, where $k_\rho = \max\left\{ \sup_{\Omega\cap B_\rho}(v) - \delta, \sup_{\partial\Omega\cap B_\rho}(v) \right\}$ if $\partial\Omega \cap B_\rho \neq \emptyset$, while $k_\rho = \sup_{B_\rho}(v) - \delta$ otherwise.

We will use the following result (see [25, Theorem 7.1, p. 91]).

Theorem A.7. *Let $\Omega \subset \mathbb{R}^N$ be a $C^{1,1}$ domain and let $M, \gamma, \delta > 0$, $r \in (N, +\infty]$, $m > 1$, $\beta \in (0,1)$, $L > 0$. Then, there exist $\alpha \in (0,1)$ and $K > 0$ such that, if $u \in \mathcal{B}_m\left(\overline{\Omega}, M, \gamma, \delta, \frac{1}{r}\right)$ satisfies*

$$\sup_{\partial\Omega\cap B_\rho}(u) - \inf_{\partial\Omega\cap B_\rho}(u) \leq L\rho^\beta \tag{A.4}$$

for every B_ρ centered at $\partial\Omega$ with $0 < \rho < a_0$, then $u \in C^{0,\alpha}(\overline{\Omega})$ and, moreover, $\|u\|_{C^{0,\alpha}(\overline{\Omega})} \leq K$.

Thus the core of our result is proving that any solution of (P_λ) belongs to $\mathcal{B}_m\left(\overline{\Omega}, M, \gamma, \delta, \frac{1}{r}\right)$.

Lemma A.8. *Let $\Omega \subset \mathbb{R}^N$ be a $C^{1,1}$ domain, $\lambda \in \mathbb{R}$, $1 < q \leq 2$, $0 \leq \mu \in L^\infty(\Omega)$, with $\|\mu\|_{L^\infty(\Omega)} < 1$ if $q = 2$, and $0 \lneq f \in L^p(\Omega)$ for $p > \frac{N}{2}$. Then, for every $M > 0$, there exist $\alpha \in (0,1)$ and $K > 0$ such that every solution u to (P_λ) satisfying $\|u\|_{L^\infty(\Omega)} \leq M$ belongs to $C^{0,\alpha}(\overline{\Omega})$ with $\|u\|_{C^{0,\alpha}(\overline{\Omega})} \leq K$.*

Remark A.9. *The condition on the regularity of the boundary can be relaxed, assuming that $\partial\Omega$ satisfies* (6.2)*.*

Proof of Lemma A.8. Let $M > 0$ and let u be a solution to (P_λ) such that $\|u\|_{L^\infty(\Omega)} \leq M$. We will show that $u \in \mathcal{B}_2\left(\overline{\Omega}, M, \gamma, M, \frac{1}{2p}\right)$ for some $\gamma > 0$.

Indeed, fix $\rho > 0$ and B_ρ such that $\Omega \cap B_\rho \neq \emptyset$, fix also $\sigma \in (0,1)$, and consider a function $\zeta \in C_c^\infty(\mathbb{R}^N)$, compactly supported in B_ρ, satisfying that $0 \leq \zeta \leq 1$ in B_ρ, $\zeta \equiv 1$ in the concentric ball $B_{\rho-\sigma\rho}$, and $|\nabla\zeta| < \frac{a}{\sigma\rho}$ in B_ρ for some constant $a > 0$ independent of ρ, σ. We start by showing that inequality (A.3) is satisfied for $v = u$.

Thus, let $k \geq k_\rho$. If $\partial\Omega \cap B_\rho \neq \emptyset$, then $k_\rho = 0$ (since $v = 0$ on $\partial\Omega$), so $(v-k)^+ \in H_0^1(\Omega) \cap L^\infty(\Omega)$, and in particular $\zeta^2(v-k)^+ \in H_0^1(\Omega) \cap L^\infty(\Omega)$. On the other hand, if $B_\rho \subset \Omega$, then $\zeta \in C_c^\infty(\Omega)$, so again $\zeta^2(v-k)^+ \in H_0^1(\Omega) \cap L^\infty(\Omega)$. In both cases, we can take $\zeta^2(v-k)^+ \in H_0^1(\Omega) \cap L^\infty(\Omega)$ as test function in the weak formulation of (P_λ), so that we obtain

$$\begin{aligned} \int_{A_{k,\rho}(v)} \zeta^2 |\nabla v|^2 \leq{}& 2\int_{A_{k,\rho}(v)} \zeta(v-k)|\nabla\zeta||\nabla v| + |\lambda| \int_{A_{k,\rho}(v)} v(v-k) \\ &+ \|\mu\|_{L^\infty(\Omega)} \int_{A_{k,\rho}(v)} \zeta^2 |\nabla v|^q \frac{v-k}{|v|^{q-1}} + \int_{A_{k,\rho}(v)} f(x)(v-k). \end{aligned} \tag{A.5}$$

By the definition of k_ρ, it is clear that $v-k\le M$ in $A_{k,\rho}(v)$. Using also that $v\le M$, we deduce that

$$\int_{A_{k,\rho}(v)}\zeta^2|\nabla v|^2\le 2\int_{A_{k,\rho}(v)}\zeta(v-k)|\nabla\zeta||\nabla v|+|\lambda|M^2|A_{k,\rho}(v)| + \|\mu\|_{L^\infty(\Omega)}M^{2-q}\int_{A_{k,\rho}(v)}\zeta^2|\nabla v|^q+M\int_{A_{k,\rho}(v)}f(x).$$

If $q<2$ we use Young's inequality conveniently in the first and the third terms of the right hand side of the last inequality, so we derive

$$\frac{1}{C}\int_{A_{k,\rho}(v)}\zeta^2|\nabla v|^2\le(|\lambda|+C)M^2|A_{k,\rho}(v)| + M\|f\|_{L^p(\Omega)}|A_{k,\rho}(v)|^{\frac{1}{p'}}+C\int_{A_{k,\rho}(v)}(v-k)^2|\nabla\zeta|^2,$$

for some $C=C(q,\mu)>0$ large enough. Similarly, if $q=2$ and $\|\mu\|_{L^\infty(\Omega)}<1$, we arrive at the same inequality in a similar way, but Young's inequality is not needed in the second term of the right hand side.

Noticing that

$$|A_{k,\rho}(v)|=|A_{k,\rho}(v)|^{\frac{1}{p'}}|A_{k,\rho}(v)|^{\frac{1}{p}}\le|A_{k,\rho}(v)|^{\frac{1}{p'}}C(N,p)\rho^{\frac{N}{p}},$$

and recalling the properties of ζ, we finally arrive at

$$\int_{A_{k,\rho-\sigma\rho}(v)}|\nabla v|^2\le\gamma\left(\frac{1}{\sigma^2\rho^{2-\frac{N}{p}}}\|v-k\|^2_{L^\infty(A_{k,\rho}(v))}+1\right)|A_{k,\rho}(v)|^{\frac{1}{p'}}, \tag{A.6}$$

for some $\gamma>0$ which depends on M but not on v,k,ρ,σ.

Let us now prove that (A.3) holds for $v=-u$. First of all, notice that v satisfies

$$\begin{cases}-\Delta v\le\lambda v & \text{in }\Omega,\\ v=0 & \text{on }\partial\Omega.\end{cases} \tag{A.7}$$

Let $k\ge k_\rho$; if, on the contrary, $\partial\Omega\cap B_\rho\neq\emptyset$, then $k_\rho=0$, so $A_{k,\rho}(v)=\emptyset$ and (A.3) is trivially satisfied.

On the other hand, if $B_\rho\subset\Omega$, then $\zeta^2(v-k)^+\in H_0^1(\Omega)\cap L^\infty(\Omega)$, and it can be used as test function in (A.7). In particular, (A.5) also holds, so the same computations above can be reproduced up to (A.6), and the proof of our claim is done.

In conclusion, we have proved that $u\in\mathcal{B}_2\left(\overline{\Omega},M,\gamma,M,\frac{1}{2p}\right)$. Since (A.4) is satisfied being $u=0$ on $\partial\Omega$, then Theorem A.7 implies that $u\in C^{0,\alpha}(\overline{\Omega})$ for some $\alpha\in(0,1)$, and, in addition, $\|u\|_{C^{0,\alpha}(\overline{\Omega})}\le K$, where K,α are positive constants independent of u.

□

Proof of Lemma 2.4: Local Sobolev regularity. As far as the local Sobolev regularity is concerned, we use a classical bootstrap argument (see [16] for more details). We first observe that since $u>0$, by virtue of the strong maximum principle we have that $u\ge\inf_\omega(u)>0$ in ω for every smooth open set $\omega\subset\subset\Omega$. Hence, we may apply [21, Chapter V, Proposition 2.1] to derive that $u\in W^{1,t_0}(\omega)$ for some $t_0>2$, and by standard regularity theory we have that $u\in W^{2,\frac{t_0}{2}}(\omega)$. Now, since $u\in C^{0,\alpha}(\omega)\cap W^{2,\frac{t_0}{2}}(\omega)$, then [28, Teorema IV] implies that $u\in W^{1,t_1}(\omega)$, where $t_1=\frac{\frac{t_0}{2}(2-\alpha)-\alpha}{1-\alpha}>t_0$.

We may continue the bootstrap argument as in [6, Lemma 2.1] to obtain that $u \in W^{1,t_n}(\omega)$ as long as $t_{n-1} < 2p$, where

$$t_n = \frac{\frac{t_{n-1}}{2}(2-\alpha) - \alpha}{1-\alpha} \quad \forall n \in \mathbb{N}.$$

Observe that the sequence $\{t_n\}_{n\in\mathbb{N}}$ is increasing. Assume by contradiction that $t_n < 2p$ for any n. Then the sequence $\{t_n\}_{n\in\mathbb{N}}$ is convergent, and $t_n \to 2$. But this contradicts the fact that $t_0 > 2$. Hence, necessarily $t_n \geq 2p$ for some n, and the proof is done. □

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