SINGULAR QUASILINEAR ELLIPTIC PROBLEMS WITH CHANGING SIGN DATUM: EXISTENCE AND HOMOGENIZATION

JOSÉ CARMONA, SALVADOR LÓPEZ-MARTÍNEZ, AND PEDRO J. MARTÍNEZ-APARICIO

ABSTRACT. We study singular quasilinear elliptic equations whose model is

$$\begin{cases} -\Delta u = \lambda u + \mu(x)\dfrac{|\nabla u|^q}{|u|^{q-1}} + f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded smooth domain of $\mathbb{R}^N$ ($N \geq 3$), $\lambda \in \mathbb{R}$, $1 < q < 2$, $0 \leq \mu \in L^\infty(\Omega)$ and the datum $f \in L^p(\Omega)$, for some $p > \frac{N}{2}$, may change sign. We prove existence of solution and we deal with the homogenization problem posed in a sequence of domains Ω^ε obtained by removing many small holes from a fixed domain Ω.

1 INTRODUCTION

In this paper we consider the following boundary value problem

$$(P_\lambda) \qquad \begin{cases} -\text{div}(M(x)\nabla u) = \lambda u + g(x,u)|\nabla u|^q + f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ ($N \geq 3$) is a bounded domain satisfying the boundary condition (A) below. Here, $f \in L^p(\Omega)$ for some $p > \frac{N}{2}$ and no assumption on its sign is imposed. Moreover $M(x)$ is an $N \times N$ matrix satisfying

$$(M_1) \qquad \begin{cases} M \in (L^\infty(\Omega) \cap W^{1,\infty}_{\text{loc}}(\Omega))^{N\times N} \text{ and for some } \eta > 0, \\ \eta|\xi|^2 \leq M(x)\xi\cdot\xi \quad \forall \xi \in \mathbb{R}^N, \quad \text{a.e. } x \in \Omega. \end{cases}$$

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We consider $1 < q < 2$, $\lambda \in \mathbb{R}$ and a wide class of Carathéodory functions $g : \Omega \times \mathbb{R} \setminus \{0\} \to [0, +\infty)$ satisfying that,

$$(g_1) \qquad \begin{cases} \text{for a.e. } x \in \Omega, \text{ the function } s \mapsto g(x,s)|s|^{q-1} \text{ is bounded} \\ \\ \text{and } \mu(x) \equiv \sup\limits_{s \in \mathbb{R}\setminus\{0\}} g(x,s)|s|^{q-1} \in L^\infty(\Omega). \end{cases}$$

Observe that hypothesis (g_1) includes the case in which g, at $s = 0$, admits a continuous extension but also the case in which the lower order term may have a singularity.

Our first goal is to study the existence of solution to problem (P_λ) under the previous hypotheses. Since the function $g(x, s)$ may be defined only for $|s| > 0$, having in mind the model problem where $g(x, s) = \frac{\mu(x)}{|s|^{q-1}}$, we have to clarify the meaning of solution. We say that a solution to problem (P_λ) is a function $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ such that $g(x, u)|\nabla u|^q \in L^1(\{|u| > 0\})$ and

$$\int_\Omega M(x)\nabla u \cdot \nabla\phi = \lambda \int_\Omega u\phi + \int_{\{|u|>0\}} g(x,u)|\nabla u|^q \phi + \int_\Omega f(x)\phi,$$

for every $\phi \in H_0^1(\Omega) \cap L^\infty(\Omega)$.

Let us note that, due to Stampacchia's Theorem, $\nabla u \equiv 0$ in the set $\{|u| = 0\}$, so this concept of solution coincides with the usual one when $g(x, s)$ is continuous at $s = 0$ or just g is bounded at $s = 0$. In the case g unbounded at $s = 0$ we remark that integrating in the set $\{|u| > 0\}$ does not avoid the singularity, to the contrary, the integrand is singular on $\partial\{|u| > 0\}$ and this set is nonempty if u is nontrivial.

Observe also that, if $f \gneq 0$, then the strong maximum principle implies that $u > 0$ a.e. in Ω for every solution u to problem (P_λ) with $\lambda < \lambda_1(M) \equiv \inf_{v \in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega M(x)\nabla v \cdot \nabla v}{\int_\Omega v^2}$, and therefore, $\{|u| > 0\} = \Omega$. This framework with nonnegative datum f and positive solutions is usual for this kind of singular problems (see [1], [3], [5] and references therein). In fact, it was adopted in [5], where the authors studied problem (P_λ) in the model case where M is the identity matrix and $g(x, s) = \frac{\mu(x)}{|s|^{q-1}}$.

Up to our knowledge, the first time a sign changing datum was considered in a singular quasilinear equation was in [14] (see also [15] and [16]). In [14] the authors studied a general problem whose simplest model is

$$\begin{cases} -\Delta u = \dfrac{|\nabla u|^2}{|u|^\theta} + f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\theta \in (0,1)$ and $f \in L^p(\Omega)$, with $p \geq \frac{N}{2}$. With a concept of solution very close to the one that we established above, they proved the existence of at least a solution to the problem.

In the present work, we aim to improve the existence results contained in [5] in several directions (in Section 2 we describe some concepts and results). On the one hand, we generalize the principal operator of the equation and the nonlinear term by imposing conditions (M_1) and (g_1) respectively. On the other hand, we will allow f to change sign. Hence, the solutions may vanish in a set of positive measure, in fact in Remark 3.2 we include two examples for which this actually happens.

Concerning the techniques that we use, we approximate the singular problem by a sequence of nonsingular ones. We prove that there exists a solution u_n to the approximated problems using the sub-supersolution method in [4, Théorème 3.1]. Then, we prove that, passing to a subsequence, u_n converges to u strongly in $H_0^1(\Omega)$ and also in $L^r(\Omega)$ for all $r \in [1,\infty)$ in order to pass to the limit in the approximated problems and to obtain a solution u to problem (P_λ).

The main interest of our proof by approximation lies on the way that the a priori estimates, needed for the compactness of the sequence $\{u_n\}$, are obtained. The greatest difficulty comes from the fact that f changes sign because, in this case, such a sequence is not uniformly bounded away from zero. This lower estimate represents a usual tool for proving, for instance, that the lower order term is bounded in $L^1_{\text{loc}}(\Omega)$. However, we will be able to prove a global L^1 estimate even if the lower estimate does not hold true (see Lemma 3.5 and Remark 3.6 below). It is also remarkable that an L^∞ estimate for $\{u_n\}$ can be obtained by using carefully the Comparison Principle [5, Theorem 3.2] (see also next section).

Another goal of the paper is the following homogenization problem

$$
\begin{cases} -\operatorname{div}(M(x)\nabla u^\varepsilon) = \lambda u^\varepsilon + g(x,u^\varepsilon)|\nabla u^\varepsilon|^q + f(x) & \text{in } \Omega^\varepsilon, \\ u^\varepsilon = 0 & \text{on } \partial\Omega^\varepsilon, \end{cases} \tag{1.1}
$$

where Ω^ε is a sequence of open sets which are included in a fixed bounded open set Ω of $\mathbb{R}^N$, $M(x)$ is an $N \times N$ matrix satisfying (M_1), g satisfies (g_1), $1 < q < 2$, $\lambda \in \mathbb{R}$ and $f \in L^p(\Omega)$, $p > \frac{N}{2}$.

More precisely, we study the asymptotic behaviour, as ε goes to zero, of a sequence of solutions to this problems posed in domains Ω^ε obtained by removing many small holes from a fixed domain Ω, following the framework of [10]. In such a paper it has been considered the linear homogenization problem

$$\begin{cases} -\Delta u^\varepsilon = f(x) & \text{in } \Omega^\varepsilon, \\ u^\varepsilon = 0 & \text{on } \partial\Omega^\varepsilon, \end{cases} \tag{1.2}$$

with $f \in L^2(\Omega)$ (see also [11], where this homogenization problem is studied in a more general framework). It is well known that problem (1.2) has a unique solution $u^\varepsilon \in H_0^1(\Omega)$. In [10] the authors showed that, if the holes satisfy certain hypotheses on their size and distribution, and if we denote as $\widetilde{u^\varepsilon}$ the extension of u^ε by zero in $\Omega \setminus \Omega^\varepsilon$, then $\widetilde{u^\varepsilon} \rightharpoonup u$ in $H_0^1(\Omega)$, where u is the unique solution to

$$\begin{cases} -\Delta u + \sigma u = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

with σ a positive constant. In fact, this case of σ constant is only a model example, but the hypotheses on the holes imposed in [10] are more general and σ can be proved to be, in the general framework, only a nonnegative finite Radon measure. It is widely remarked the presence of the "strange term" σu (which is the "asymptotic memory of the fact that $\widetilde{u^\varepsilon}$ was zero on the holes") appearing in the limit equation (1.3).

In [10], the authors proved also a corrector result, that is to say, a representation of $\nabla\widetilde{u^\varepsilon}$ in the strong topology of $L^2(\Omega)^N$. They showed that the corrector for the linear homogenization problem depends on the holes, and also depends on the limit u in a linear way.

In [8] the author studied the quasilinear homogenization problem

$$\begin{cases} -\Delta u^\varepsilon + \lambda u^\varepsilon = \gamma|\nabla u^\varepsilon|^2 + f(x) & \text{in } \Omega^\varepsilon, \\ u^\varepsilon = 0 & \text{on } \partial\Omega^\varepsilon, \end{cases}$$

where γ is a real constant, $\lambda > 0$ and $f \in L^\infty(\Omega)$. He used a suitable change of unknown function that turns the equation into a semilinear one, a careful analysis of this semilinear homogenization problem allowed the author to pass to the limit as in the linear case. Undoing the change of variables, he proved that the limit u satisfies that

$$\begin{cases} -\Delta u + \lambda u + \dfrac{\sigma(e^{\gamma u} - 1)}{\gamma e^{\gamma u}} = \gamma|\nabla u|^2 + f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

As in the linear case, a new term appears in the equation that satisfies u, but in this case the new term is nonlinear (σ is the same constant that appears in the linear problem). As the author remarked, this means that the perturbation of the linear problem (1.2) by a nonlinear term, namely $\gamma|\nabla u^\varepsilon|^2$, changes the structure of the new term in the limit equation. This perturbation affects also the corrector corresponding to this problem, since it turns out to be nonlinear in u as well. Similar results were proved in [7] in which the nonlinear perturbation of (1.2) is a general function of the form $H(x,u,\nabla u)$, where H has (at most) natural growth in the gradient.

We remark that in all the previous cases the lower order term is locally bounded with respect to u. Up to our knowledge, the first time it was considered a singular term was in [6]. The authors studied the homogenization of the model problem

$$\begin{cases} -\Delta u^\varepsilon + \dfrac{|\nabla u^\varepsilon|^2}{|u^\varepsilon|^\theta} = f(x) & \text{in } \Omega^\varepsilon, \\ u^\varepsilon = 0 & \text{on } \partial\Omega^\varepsilon, \end{cases} \tag{1.4}$$

with $\theta \in (0,1)$ and f a nonnegative datum (positive solutions) in a suitable space of Lebesgue. There, since the lower order term is positive, following [18] and [20], it is easy to prove that $\widetilde{u^\varepsilon}$ is bounded in $H_0^1(\Omega)$ and in $L^\infty(\Omega)$ respectively, thus the main difficulty resides in avoiding the singularity when passing to the limit. Their main result, written here only in the case σ constant, is that for every $f \in L^{\frac{2N}{N+2}}(\Omega)$, $f \gneq 0$, the unique solution u^ε to problem (1.4) satisfies $\widetilde{u^\varepsilon} \rightharpoonup u$ in $H_0^1(\Omega)$, where u is the unique solution to problem

$$\begin{cases} -\Delta u + g(u)|\nabla u|^2 + \sigma\Psi(u)e^{G(u)} = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

and $G(s) = \int_1^s g(t)dt$, $\Psi(s) = \int_0^s e^{-G(t)}dt$ for every $s > 0$. Thus, the strange term turns out to be again nonlinear in u, as well as the corrector, as it is shown in [6].

The above results describe the general questions we are concerned with. We will prove that, also for our problem (1.1) there is a limit u which is a solution to a new problem. We will show that, unlike in the cases mentioned above, the strange term is linear even if the equation is not, and the Radon measure depends only on the holes. Furthermore, the corrector is also linear. The reason for this unexpected phenomenon to occur is that the lower order term is bounded in $L^1(\Omega)$, so it represents a *mild* perturbation for the linear equation. As for the existence result explained above, the proof of this estimate is not trivial since the functions $\widetilde{u^\varepsilon}$ vanish on the holes, so the usual local lower estimate does not hold true neither in this case. The L^1 estimate will allow us to prove that $\{\widetilde{u^\varepsilon}\}$ converges strongly in $W_0^{1,r}(\Omega)$ for all $r \in [1,2)$, which is essential for passing to the limit in the equation.

The plan of the paper is the following. We collect some preliminary results in the second section. We prove that the problem (P_λ) has solution in a suitable sense in Section 3. We dedicate Section 4 to the homogenization of problem (1.1). In Subsection 4.1 we give the precise assumptions of the perforated domains, following the framework of [10]. In Subsection 4.2 we prove the existence of solution to problem (1.1). We enunciate our homogenization result in Subsection 4.3. In Subsection 4.4 we prove the main tool in order to pass to the limit as ε tends to zero, the L^r-strong convergence of the gradients for $r < 2$. Our homogenization result for the singular quasilinear problem (1.1) is studied in Subsection 4.5 and a corrector result is proved in Subsection 4.6.

2 Preliminary results

As we announced, in this paper we will improve some existence results contained in [5]. In order to make a simpler exposition, we will include in this section some concepts and results from [5] that we will need in our proofs.

Recall that in such a paper the authors studied the singular problem

$$\begin{cases} -\operatorname{div}(M(x)\nabla u) = \lambda u + \mu(x)\dfrac{|\nabla u|^q}{u^{q-1}} + f(x) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

in the special case in which $M(x)$ is the identity matrix and $f \gneq 0$. They dealt with the problem by taking advantage of the homogeneous structure of the equation. Thus, they studied first the eigenvalue problem

$$\begin{cases} -\operatorname{div}(M(x)\nabla u) = \lambda u + \mu(x)\dfrac{|\nabla u|^q}{|u|^{q-1}} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{E_λ}$$

again in the case $M = I$. The authors proved the existence and main properties of the principal eigenvalue, that can be characterized by

$$\lambda^* = \sup\left\{ \lambda \in \mathbb{R} \;\middle|\; \begin{array}{c} \text{there exists a supersolution } v \text{ to } (E_\lambda) \\ \text{such that } v \geq c \text{ in } \Omega \text{ for some } c > 0 \end{array} \right\}. \tag{2.2}$$

If necessary, we will write $\lambda^*(\Omega)$ to make explicit the dependence on the domain.

Arguing as in [5] without relevant changes, it is possible to prove that, assuming (M_1), then $\lambda^* \in (0, \lambda_1(M)]$ and, if $\partial\Omega$ is smooth enough, problem (E_λ) admits a positive solution if and only if $\lambda = \lambda^*$.

Concerning the smoothness of the domain, we introduce the following definition.

Definition 2.1. Let $D \subset \mathbb{R}^N$ be an open set. We say that D satisfies condition (A) if there exist $r_0, \theta_0 > 0$ such that, if $x \in \partial D$ and $0 < r < r_0$, then

$$|D_r| \le (1 - \theta_0)|B_r(x)|$$

for every connected component D_r of $D \cap B_r(x)$, where $B_r(x)$ denotes the ball centered at x with radius r.

We remark that a sufficient condition for Ω to satisfy condition (A) is that $\partial\Omega$ is Lipschitz (see [2]).

The existence result from [5], adapted to our needs, reads as follows.

Theorem 2.2. *Let $1 < q < 2$, $0 \le \mu \in L^\infty(\Omega)$ and assume that M and Ω satisfy conditions (M_1) and (A) respectively. Then, there exists at least a solution to* (2.1) *for every $\lambda < \lambda^*$, where λ^* is given by* (2.2).

We will also use the following comparison principle proved as in [5].

Theorem 2.3. *Let $1 < q < 2$, $\lambda \in \mathbb{R}$, $0 \le \mu \in L^\infty(\Omega)$ and $0 \le h \in L^1_{\mathrm{loc}}(\Omega)$. Assume that $u, v \in C(\Omega) \cap W^{1,N}_{\mathrm{loc}}(\Omega)$ are such that $u, v > 0$ in Ω and satisfy*

$$\int_\Omega M(x)\nabla u \cdot \nabla\phi \le \lambda \int_\Omega u\phi + \int_\Omega \mu(x)\frac{|\nabla u|^q}{u^{q-1}}\phi + \int_\Omega h(x)\phi, \tag{2.3}$$

and

$$\int_\Omega M(x)\nabla v \cdot \nabla\phi \ge \lambda \int_\Omega v\phi + \int_\Omega \mu(x)\frac{|\nabla v|^q}{v^{q-1}}\phi + \int_\Omega h(x)\phi, \tag{2.4}$$

for all $0 \le \phi \in H^1_0(\Omega) \cap L^\infty(\Omega)$ with compact support. Suppose also that, for every $\varepsilon > 0$, the following boundary condition holds

$$\limsup_{x \to x_0} \left(\frac{u(x)}{v(x) + \varepsilon}\right) \le 1 \quad \forall x_0 \in \partial\Omega. \tag{2.5}$$

Furthermore, if $\lambda > 0$, assume also that $\inf_\Omega(h) > 0$. Then, $u \le v$ in Ω.

Some remarks are now in order.

Remark 2.4. Even though the principal operator considered in [5] is the Laplacian, we observe here that a perturbation with a bounded elliptic matrix $M(x)$ satisfying (M_1) does not involve any additional difficulty in the proofs of the previous results. We remark that the fact that the coefficients of $M(x)$ are locally Lipschitz is needed in order to apply elliptic regularity (see [21, Theorem 3.8] and also problem 3.3, p. 202, in that book).

Remark 2.5. Originally, in [5] condition (A) is replaced by a more restrictive hypothesis, i.e., it is imposed that $\partial\Omega$ is of class $C^{1,1}$. This last smoothness condition is used only for proving a nonexistence result for $\lambda > \lambda^*$. In our context, we do not expect that a similar nonexistence result holds true because our solutions are not necessarily positive. Hence, condition (A) is enough for our purposes since suffices to prove that the solutions are Hölder continuous up the boundary.

3 Existence of solution for the quasilinear problem

In this section we will prove existence of solution to (P_λ) for every $\lambda < \lambda^*$, generalizing thus Theorem 2.2 above. As was pointed out at the Introduction our concept of solution is the following.

Definition 3.1. We say that $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ is a solution to problem (P_λ) if $g(x,u)|\nabla u|^q \in L^1(\{|u|>0\})$ and

$$\int_\Omega M(x)\nabla u \cdot \nabla\phi = \lambda\int_\Omega u\phi + \int_{\{|u|>0\}} g(x,u)|\nabla u|^q\phi + \int_\Omega f(x)\phi,$$

for every $\phi \in H_0^1(\Omega) \cap L^\infty(\Omega)$.

Remark 3.2. The following examples show that, when f changes sign, a solution to (P_λ) may vanish in a set of positive measure either in a neighbourhood of the boundary or even far away from the boundary. Indeed, standard computations show that, for convenient data f_1 and f_2, the functions

$$u_1(x) = \begin{cases} e^{\frac{1}{|x|^2-1}} & |x| \le 1, \\ 0 & 1 < |x| \le 2, \end{cases}$$

and

$$u_2(x) = \begin{cases} 0 & |x| < 1, \\ (|x|-1)^2(2-|x|)^2 & 1 \le |x| < 2, \end{cases}$$

satisfy $-\Delta u_i = \frac{|\nabla u_i|^q}{|u_i|^{q-1}} + f_i(x)$ in $B_2(0)$.

The statement of the main result of this section is as follows.

Theorem 3.3. *Assume that* Ω *satisfies condition* (A), $1 < q < 2$, $f \in L^p(\Omega)$ *for some* $p > \frac{N}{2}$ *and conditions* (M_1) *and* (g_1) *are satisfied. Then there exists at least a solution to problem* (P_λ) *for all* $\lambda < \lambda^*$.

We will find the solution of Theorem 3.3 as the limit of a sequence of solutions to nonsingular problems that approximate (P_λ). More precisely, we consider, for every $n \in \mathbb{N}$, the following problem

$$(Q_n) \quad \begin{cases} -\mathrm{div}(M(x)\nabla u_n) = \lambda u_n + g_n(x, u_n)|\nabla u_n|^q + f_n(x) & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega, \end{cases}$$

where

$$g_n(x,s) \stackrel{\text{def}}{=} \begin{cases} g(x,s) & |s| \geq \dfrac{1}{n}, \\ g(x,s)|s|^q n^q & 0 < |s| \leq \dfrac{1}{n}, \\ 0 & s = 0, \end{cases}$$

and $f_n(x) = \max\{-n, \min\{f(x), n\}\}$. Observe that $g_n : \Omega \times \mathbb{R} \to [0, +\infty)$ is continuous in the second variable and

$$g_n(x,s) \leq g(x,s) \quad \text{a.e. } x \in \Omega, \ \forall s \in \mathbb{R} \setminus \{0\}.$$

In the next lemma we prove the existence of solution to (Q_n) by means of the subsolution and supersolution method in [4].

Lemma 3.4. *Let* $1 < q < 2$, $\lambda < \lambda^*$, $f \in L^1(\Omega)$, *and assume that conditions* (M_1) *and* (g_1) *are satisfied. Then there exists a solution* u_n *to problem* (Q_n) *for all* n.

Proof. Let $\overline{\lambda} \in (\lambda, \lambda^*)$, and let $\varphi \in H^1(\Omega) \cap L^\infty(\Omega)$ be such that

$$\varphi \geq c > 0 \quad \text{and} \quad -\mathrm{div}(M(x)\nabla\varphi) \geq \overline{\lambda}\varphi + \mu(x)\frac{|\nabla\varphi|^q}{\varphi^{q-1}} \quad \text{in } \Omega.$$

For some constant $k > 0$, let $\overline{\psi} = k\varphi$. Then,

$$\begin{aligned} &\mathrm{div}(M(x)\nabla\overline{\psi}) + \lambda\overline{\psi} + g_n(x,\overline{\psi})|\nabla\overline{\psi}|^q + f_n(x) \\ &\quad \leq k\left(\mathrm{div}(M(x)\nabla\varphi) + \overline{\lambda}\varphi + \mu(x)\frac{|\nabla\varphi|^q}{\varphi^{q-1}}\right) + n - (\overline{\lambda} - \lambda)kc \leq 0, \end{aligned}$$

if k is chosen large enough.

On the other hand, let $\underline{\psi} = -k\varphi$. Then,

$$\begin{aligned} &\mathrm{div}(M(x)\nabla\underline{\psi}) + \lambda\underline{\psi} + g_n(x,\underline{\psi})|\nabla\underline{\psi}|^q + f_n(x) \\ &\quad \geq \mathrm{div}(M(x)\nabla\underline{\psi}) + \lambda\underline{\psi} - k\mu(x)\frac{|\nabla\varphi|^q}{\varphi^{q-1}} + f_n(x) \\ &\quad \geq -k\left(\mathrm{div}(M(x)\nabla\varphi) + \overline{\lambda}\varphi + \mu(x)\frac{|\nabla\varphi|^q}{\varphi^{q-1}}\right) + (\overline{\lambda} - \lambda)kc - n \geq 0. \end{aligned}$$

Obviously, $\underline{\psi} \le 0 \le \overline{\psi}$ in $\overline{\Omega}$. Therefore, by virtue of [4, Théorème 3.1], there exists a solution u_n to problem (Q_n) such that $\underline{\psi} \le u_n \le \overline{\psi}$. $\square$

In the following lemma we prove the a priori estimates and the compactness needed for passing to the limit.

Lemma 3.5. *Let $1 < q < 2$, $\lambda \in \mathbb{R}$, $f \in L^1(\Omega)$ and assume that conditions (M_1) and (g_1) are satisfied. Assume also that $\{u_n\}$ is a sequence of solutions to problem (Q_n) bounded in $L^\infty(\Omega)$. Then there exists $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ such that*

i) *up to a subsequence, $u_n \to u$ strongly in $L^r(\Omega)$, $r \in [1, \infty)$,*
ii) *$\{|u_n|^\alpha\}$ is bounded in $H_0^1(\Omega)$ for all $\alpha > \frac{1}{2}$,*
iii) *up to a subsequence, $u_n \to u$ strongly in $H_0^1(\Omega)$.*

Remark 3.6. Observe that, for any $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ and any $\delta > 0$, the chain rule for weak derivatives implies that

$$\frac{|\nabla u|^q}{|u|^{q-1}} = q^q \left|\nabla |u|^{\frac{1}{q}}\right|^q$$

in the set $\{|u| > \delta\}$. In particular, the equality holds in $\{|u| > 0\} = \bigcap_{\delta>0}\{|u| > \delta\}$. Thus, by (g_1),

$$g(x,u)|\nabla u|^q \le \mu(x) q^q \left|\nabla |u|^{\frac{1}{q}}\right|^q$$

in $\{|u| > 0\}$. Therefore, if $|u|^{\frac{1}{q}} \in W^{1,q}(\Omega)$ (which is precisely a consequence of Lemma 3.5) one has that $g(x,u)|\nabla u|^q \in L^1(\{|u| > 0\})$.

Proof of Lemma 3.5. Let us take u_n as test function in (Q_n). Then, using the $L^\infty(\Omega)$ bound we immediately obtain that $\{u_n\}$ is bounded in $H_0^1(\Omega)$. Hence, passing to a subsequence, there exists $u \in H_0^1(\Omega)$ such that $u_n \rightharpoonup u$ weakly in $H_0^1(\Omega)$ and $u_n \to u$ a.e. in Ω. Furthermore, again the $L^\infty(\Omega)$ estimate clearly implies that $u \in L^\infty(\Omega)$ and also that $u_n \to u$ strongly in $L^r(\Omega)$ for all $r \in [1, \infty)$. This completes the proof of item i).

Now we deal with the proof of item ii) which is straightforward, using the $L^\infty(\Omega)$ estimate, in the case $\alpha \ge 1$. When $\frac{1}{2} < \alpha < 1$ we take $\beta = 2\alpha - 1 \in (0,1)$ and we first prove a uniform bound in $L^1(\Omega)$ for $\frac{|\nabla u_n|^2}{(|u_n|+\delta)^{1-\beta}}$ with $n \in \mathbb{N}$, $\delta \in (0,1)$. Then, passing to the limit as $\delta \to 0$ we show that $|u_n|^{\frac{\beta+1}{2}} = |u_n|^\alpha$ is bounded in $H_0^1(\Omega)$.

It is clear that $v_{n,\delta} = (-u_n^- + \delta)^\beta - \delta^\beta \in H_0^1(\Omega) \cap L^\infty(\Omega)$, where $u_n^- = \min\{u_n, 0\}$. Therefore, it can be taken as test function in (Q_n), and using (M_1), the $L^\infty(\Omega)$ bound and the fact that g_n is nonnegative, we obtain that

$$
\begin{aligned}
-\beta\eta\int_{\{u_n\le 0\}}\frac{|\nabla u_n|^2}{(-u_n+\delta)^{1-\beta}} &\ge \int_\Omega M(x)\nabla u_n\cdot\nabla v_{n,\delta}\\
&=\lambda\int_\Omega u_n v_{n,\delta}+\int_\Omega g_n(x,u_n)|\nabla u_n|^q v_{n,\delta}+\int_\Omega f_n(x)v_{n,\delta} \qquad (3.1)\\
&\ge\lambda\int_\Omega u_n v_{n,\delta}+\int_\Omega f_n(x)v_{n,\delta}\ge -C.
\end{aligned}
$$

On the other hand, $w_{n,\delta}=(u_n^+ +\delta)^\beta-\delta^\beta\in H_0^1(\Omega)\cap L^\infty(\Omega)$, so we can use it as test function in (Q_n). Hence, using the $L^\infty(\Omega)$ bound and Young inequality conveniently we deduce that

$$
\begin{aligned}
\beta\eta\int_{\{u_n>0\}}\frac{|\nabla u_n|^2}{(u_n+\delta)^{1-\beta}} &\le \int_\Omega M(x)\nabla u_n\cdot\nabla w_{n,\delta}\\
&=\lambda\int_\Omega u_n w_{n,\delta}+\int_\Omega g_n(x,u_n)|\nabla u_n|^q w_{n,\delta}+\int_\Omega f_n(x)w_{n,\delta}\\
&\le C+\int_{\{u_n>0\}}\mu(x)\frac{|\nabla u_n|^q}{u_n^{q-1}}[(u_n+\delta)^\beta-\delta^\beta]\\
&\le C+\frac{\beta\eta}{2}\int_{\{u_n>0\}}\frac{|\nabla u_n|^2}{(u_n+\delta)^{1-\beta}}\\
&\quad +C\int_\Omega\left[\frac{(u_n+\delta)^\beta-\delta^\beta}{u_n^{q-1}}(u_n+\delta)^{(1-\beta)\frac{q}{2}}\right]^{\frac{2}{2-q}}.
\end{aligned}
$$

It is straightforward to prove that the function

$$
(s,t)\mapsto\left[\frac{(s+t)^\beta-t^\beta}{s^{q-1}}(s+t)^{(1-\beta)\frac{q}{2}}\right]^{\frac{2}{2-q}},\ (0,t)\mapsto 0
$$

is continuous in $[0,B]\times[0,1]$ for any $B>0$. Thus, choosing $B>0$ such that $\|u_n\|_{L^\infty(\Omega)}\le B$ for all n, we deduce that

$$
\int_{\{u_n>0\}}\frac{|\nabla u_n|^2}{(u_n+\delta)^{1-\beta}}\le C, \qquad (3.2)
$$

where $C>0$ is independent of n and δ.

From (3.1) and (3.2) we conclude that

$$
\int_\Omega\frac{|\nabla u_n|^2}{(|u_n|+\delta)^{1-\beta}}\le C.
$$

In other words,

$$\frac{4}{(1+\beta)^2}\int_\Omega \left|\nabla\left[(|u_n|+\delta)^{\frac{\beta+1}{2}}-\delta^{\frac{\beta+1}{2}}\right]\right|^2 \le C. \tag{3.3}$$

Denoting $z_{n,\delta}=(|u_n|+\delta)^{\frac{\beta+1}{2}}-\delta^{\frac{\beta+1}{2}}$, we have proved that there exists $C>0$ such that $\|z_{n,\delta}\|_{H_0^1(\Omega)}\le C$ for all $\delta>0$. Hence, there exists $z_n\in H_0^1(\Omega)$ such that, passing to a subsequence, $z_{n,\delta}\rightharpoonup z_n$ weakly in $H_0^1(\Omega)$ as $\delta\to 0$. On the other hand, $z_{n,\delta}\to |u_n|^{\frac{\beta+1}{2}}$ a.e. in Ω as $\delta\to 0$. This implies that $z_n=|u_n|^{\frac{\beta+1}{2}}$, so $|u_n|^{\frac{\beta+1}{2}}\in H_0^1(\Omega)$.

Since the $H_0^1(\Omega)$ norm is weakly lower semicontinuous, the inequality (3.3) yields

$$\int_\Omega |\nabla|u_n|^\alpha|^2=\int_\Omega \left|\nabla|u_n|^{\frac{1+\beta}{2}}\right|^2\le C,$$

for $C>0$ independent of n. This concludes the proof of item ii).

Regarding item iii) let us take u_n-u as test function in (Q_n). We obtain that

$$\int_\Omega M(x)\nabla u_n\cdot\nabla(u_n-u)=\lambda\int_\Omega u_n(u_n-u) + \int_\Omega g_n(x,u_n)|\nabla u_n|^q(u_n-u)+\int_\Omega f_n(x)(u_n-u).$$

It is clear that the first and the third terms on the right hand side of the last equality converge to zero as n tends to infinity. Concerning the nonlinear term, observe first that we can use item ii) with $\alpha=1/q$ so that $\{|u_n|^{\frac{1}{q}}\}$ is bounded in $H_0^1(\Omega)$. Hence, using Remark 3.6 together with the facts that $g_n(x,0)=0$ and $g_n(x,s)\le g(x,s)$, we deduce that

$$\begin{aligned}\left|\int_\Omega g_n(x,u_n)|\nabla u_n|^q(u_n-u)\right| &\le C\int_\Omega \left|\nabla|u_n|^{\frac{1}{q}}\right|^q|u_n-u|\\ &\le C\left(\int_\Omega\left|\nabla|u_n|^{\frac{1}{q}}\right|^2\right)^{\frac{q}{2}}\left(\int_\Omega|u_n-u|^{\frac{2}{2-q}}\right)^{1-\frac{q}{2}}\\ &\le C\left(\int_\Omega|u_n-u|^{\frac{2}{2-q}}\right)^{1-\frac{q}{2}}.\end{aligned}$$

This sequence converges to zero, using item i) with $r=2/(2-q)$. Thus it is clear now that

$$\int_\Omega M(x)\nabla u_n\cdot\nabla(u_n-u)\to 0.$$

Therefore, the weak convergence in $H_0^1(\Omega)$ yields

$$\eta \int_\Omega |\nabla(u_n - u)|^2 \leq \int_\Omega M(x)\nabla(u_n - u)\cdot\nabla(u_n - u)$$
$$= \int_\Omega M(x)\nabla u_n \cdot \nabla(u_n - u) - \int_\Omega M(x)\nabla u\cdot\nabla(u_n - u) \to 0,$$

finishing the proof of item iii). □

Proof of Theorem 3.3. Let $\{u_n\}$ be the sequence of solutions to problems (Q_n) given by Lemma 3.4, i.e. given $\phi \in H_0^1(\Omega)\cap L^\infty(\Omega)$,

$$\int_\Omega M(x)\nabla u_n\cdot\nabla\phi = \lambda\int_\Omega u_n\phi + \int_\Omega g_n(x,u_n)|\nabla u_n|^q\phi + \int_\Omega f_n(x)\phi. \tag{3.4}$$

We will obtain a solution to problem (P_λ) as a limit of this sequence. We divide the proof into two steps, in the first one we prove a uniform $L^\infty(\Omega)$ estimate which allows us to take limits easily in all the terms of the previous equality, except the nonlinear one which will be treated in the second step.

First of all observe that we can argue as in [17, Theorem 1.1] at Section 4 (p. 249-251) to deduce, thanks to condition (A), that $u_n \in C^{0,\alpha}(\overline{\Omega})$ for some $\alpha\in(0,1)$ (see also [5, Appendix]).

Step 1. $\{u_n\}$ is bounded in $L^\infty(\Omega)$ and pass to the limit in some terms of (3.4).

In order to find a uniform upper bound on u_n we observe that it is immediately deduced if the open set

$$\omega_n = \{x\in\Omega : u_n(x) > 0\}$$

is empty. Assuming that ω_n is not empty, u_n satisfies that

$$\begin{cases} -\mathrm{div}(M(x)\nabla u_n) = \lambda u_n + g_n(x,u_n)|\nabla u_n|^q + f_n(x) & \text{in } \omega_n,\\ u_n > 0 & \text{in } \omega_n,\\ u_n = 0 & \text{on } \partial\omega_n.\end{cases}$$

Now, since $u_n \in C^{0,\alpha}(\omega_n)$, then we deduce that $u_n \in W^{1,N}_{\mathrm{loc}}(\omega_n)$ arguing as in [5, Lemma 2.4] and using condition (M_1) for the elliptic regularity (see [21, Theorem 3.8] and also problem 3.3, p. 202, in that book). Moreover, u_n is a subsolution to the following problem

$$\begin{cases} -\mathrm{div}(M(x)\nabla\zeta) = \lambda\zeta + \mu(x)\dfrac{|\nabla\zeta|^q}{\zeta^{q-1}} + |f(x)| + 1 & \text{in } \omega_n,\\ \zeta > 0 & \text{in } \omega_n,\\ \zeta = 0 & \text{on } \partial\omega_n,\end{cases} \tag{3.5}$$

in the sense that $0 < u_n \in C(\omega_n) \cap W^{1,N}_{\mathrm{loc}}(\omega_n)$ satisfies (2.3) with $h = |f| + 1$.

On the other hand, Theorem 2.2 implies that there exists a solution v to

$$\begin{cases} -\mathrm{div}(M(x)\nabla v) = \lambda v + \mu(x)\dfrac{|\nabla v|^q}{v^{q-1}} + |f(x)| + 1 & \text{in } \Omega, \\ v > 0 & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

Moreover, $v \in C(\overline{\Omega}) \cap W^{1,N}_{\mathrm{loc}}(\Omega)$ reasoning as before. Then, v is a supersolution to (3.5) in the sense it satisfies (2.4) with $h = |f| + 1$. Furthermore, condition (2.5) is clearly satisfied in $\partial\omega_n$. Therefore, applying Theorem 2.3 we deduce that

$$u_n \le v \le \|v\|_{L^\infty(\Omega)} \quad \text{in } \omega_n.$$

Thus, $u_n \le \|v\|_{L^\infty(\Omega)}$ in Ω, and this is a uniform upper bound on u_n.

A similar (actually simpler) argument by comparison provides us an analogue lower bound. Indeed, as was pointed out in Section 2, we know that $\lambda < \lambda^* \le \lambda_1(M)$, and we have from the maximum principle that $u_n \ge z$ with $z \in H^1_0(\Omega) \cap L^\infty(\Omega)$ and $-\mathrm{div}(M(x)\nabla z) = \lambda z - |f(x)|$ in Ω. In conclusion, $\{u_n\}$ is bounded in $L^\infty(\Omega)$ and the proof of Step 1 is concluded.

Our aim now is to use this a priori estimate to pass to the limit in (Q_n). In order to do that, recall that Lemma 3.5 implies that there exists $u \in H^1_0(\Omega) \cap L^\infty(\Omega)$ such that $u_n \to u$ strongly in $H^1_0(\Omega)$ and in $L^r(\Omega)$ for all $r \in [1, \infty)$. Hence, given $\phi \in H^1_0(\Omega) \cap L^\infty(\Omega)$,

$$\lim_{n\to\infty} \int_\Omega g_n(x,u_n)|\nabla u_n|^q \phi = \lim_{n\to\infty} \left(\int_\Omega M(x)\nabla u_n \cdot \nabla\phi - \lambda \int_\Omega u_n\phi - \int_\Omega f_n(x)\phi \right) = \int_\Omega M(x)\nabla u \cdot \nabla\phi - \lambda \int_\Omega u\phi - \int_\Omega f(x)\phi.$$

Step 2. $\displaystyle\lim_{n\to\infty} \int_\Omega g_n(x,u_n)|\nabla u_n|^q\phi = \int_{\{|u|>0\}} g(x,u)|\nabla u|^q\phi.$

Let us fix a decreasing sequence (to be specified later) $\{\delta_m\}$ of positive real numbers such that $\delta_m \to 0$. We have that

$$\int_\Omega g_n(x,u_n)|\nabla u_n|^q\phi = \int_{\{|u_n|>\delta_m\}} g_n(x,u_n)|\nabla u_n|^q\phi + \int_{\{|u_n|\le\delta_m\}} g_n(x,u_n)|\nabla u_n|^q\phi. \tag{3.6}$$

We will pass to the limit in both terms, first with respect to n, and after that with respect to m.

Concerning the first term, we know that there exists $h \in L^1(\Omega)$ such that, passing to a subsequence, $|\nabla u_n|^q \leq h$ in Ω, for all n. Then,

$$|g_n(x,u_n)|\nabla u_n|^q \phi \chi_{\{|u_n|>\delta_m\}}| \leq \frac{Ch}{\delta_m^{q-1}} \in L^1(\Omega),$$

so we have domination.

In order to prove the almost everywhere convergence, consider the set

$$\mathcal{N} = \{\delta \geq 0 : |\{x \in \Omega : |u(x)| = \delta\}| > 0\}.$$

It is well known that $\mathcal{N}$ is countable, so the sequence $\{\delta_m\}$ can be chosen in $\mathbb{R} \setminus \mathcal{N}$. Thus, since $u_n \to u$ a.e. in Ω, it is straightforward to check that also $\chi_{\{|u_n|>\delta_m\}} \to \chi_{\{|u|>\delta_m\}}$ a.e. in Ω as $n \to \infty$.

On the other hand, if $|u_n| > \delta_m$, we can take n large enough such that $|u_n| \geq \frac{1}{n}$, so $g_n(x,u_n) = g(x,u_n)$. Hence, using the continuity of $g(x,\cdot)$ in the set $(-\infty,-\delta_m] \cup [\delta_m,\infty)$ and the fact that $u_n \to u$ a.e. in Ω, we deduce that

$$g_n(x,u_n) \to g(x,u) \text{ a.e. in } \Omega \text{ as } n \to \infty.$$

In sum, using also that $\nabla u_n \to \nabla u$ a.e. in Ω, we obtain that

$$g_n(x,u_n)|\nabla u_n|^q \phi \chi_{\{|u_n|>\delta_m\}} \to g(x,u)|\nabla u|^q \phi \chi_{\{|u|>\delta_m\}} \text{ a.e. in } \Omega \text{ as } n \to \infty.$$

Therefore, the Dominated Convergence Theorem implies that

$$\int_{\{|u_n|>\delta_m\}} g_n(x,u_n)|\nabla u_n|^q \phi \to \int_{\{|u|>\delta_m\}} g(x,u)|\nabla u|^q \phi \quad \text{as } n \to \infty.$$

In order to pass to the limit with respect to m we recall that Lemma 3.5 gives also that $\{|u_n|^{\frac{1}{q}}\}$ is bounded in $H_0^1(\Omega)$. This, in particular, implies that $|u|^{\frac{1}{q}} \in H_0^1(\Omega)$. Now, taking into account Remark 3.6, we have a uniform domination with respect to m:

$$\begin{aligned}\left|g(x,u)|\nabla u|^q \phi \chi_{\{|u|>\delta_m\}}\right| \leq & C\frac{|\nabla u|^q}{|u|^{q-1}} \chi_{\{|u|>0\}} \\ = & C|\nabla |u|^{\frac{1}{q}}|^q \in L^1(\{|u|>0\}),\end{aligned}$$

and also almost everywhere convergence

$$g(x,u)|\nabla u|^q \phi \chi_{\{|u|>\delta_m\}} \to g(x,u)|\nabla u|^q \phi \quad \text{a.e. in } \{|u|>0\}.$$

Therefore, the Dominated Convergence Theorem yields

$$\int_{\{|u|>\delta_m\}} g(x,u)|\nabla u|^q \phi \to \int_{\{|u|>0\}} g(x,u)|\nabla u|^q \phi.$$

To conclude the proof, we will show that the last term in (3.6) vanishes as n and m tend to infinity. Notice first that such a term has a limit with respect to n (because the remaining two terms do). Furthermore, by virtue of Lemma 3.5 we derive, taking $\varepsilon > 0$ with $\frac{1-\varepsilon}{q} > \frac{1}{2}$, that

$$\left|\int_{\{|u_n|\le\delta_m\}} g_n(x,u_n)|\nabla u_n|^q\phi\right| \le C\int_{\{0<|u_n|\le\delta_m\}} \frac{|\nabla u_n|^q}{|u_n|^{q-1}}$$
$$= C\int_{\{0<|u_n|\le\delta_m\}} \frac{|\nabla u_n|^q}{|u_n|^{q+\varepsilon-1}}|u_n|^\varepsilon = C\int_{\{0<|u_n|\le\delta_m\}} \left(\frac{|\nabla u_n|}{|u_n|^{1+\frac{\varepsilon-1}{q}}}\right)^q |u_n|^\varepsilon$$
$$\le C\delta_m^\varepsilon \int_\Omega \left|\nabla |u_n|^{\frac{1-\varepsilon}{q}}\right|^q \le C\delta_m^\varepsilon \left(\int_\Omega \left|\nabla |u_n|^{\frac{1-\varepsilon}{q}}\right|^2\right)^{\frac{q}{2}} \le C\delta_m^\varepsilon.$$

In consequence,

$$\lim_{m\to\infty}\left(\lim_{n\to\infty}\int_{\{|u_n|\le\delta_m\}} g_n(x,u_n)|\nabla u_n|^q\phi\right) = 0.$$

In conclusion, we have proved that

$$\int_\Omega g_n(x,u_n)|\nabla u_n|^q\phi \to \int_{\{|u|>0\}} g(x,u)|\nabla u|^q\phi.$$

That is to say, u is a solution to (P_λ). □

4 Homogenization of problem (1.1)

The existence result of the previous section allows us to consider the homogenization problem associated to (1.1).

4.1 The perforated domains

In this subsection, following [10], we describe the geometry of the domains Ω^ε in which we study our homogenization result.

Consider for every $\varepsilon > 0$ a finite number, $n(\varepsilon) \in \mathbb{N}$, of closed subsets $T_i^\varepsilon \subset \mathbb{R}^N$, $1 \le i \le n(\varepsilon)$, which are the holes. Let us denote $D^\varepsilon = \mathbb{R}^N \setminus \bigcup_{i=1}^{n(\varepsilon)} T_i^\varepsilon$. The domain Ω^ε is defined by removing the holes T_i^ε from Ω, that is

$$\Omega^\varepsilon = \Omega - \bigcup_{i=1}^{n(\varepsilon)} T_i^\varepsilon = \Omega \cap D^\varepsilon.$$

Hypotheses on the holes. We suppose that the sequence of domains Ω^ε is such that there exist a sequence of functions $\{w^\varepsilon\}$ and $\sigma \in H^{-1}(\Omega)$ such that

$$w^\varepsilon \in H^1(\Omega) \cap L^\infty(\Omega), \tag{4.1}$$

$$0 \le w^\varepsilon \le 1 \text{ a.e. } x \in \Omega, \tag{4.2}$$

$$w^\varepsilon \phi \in H_0^1(\Omega^\varepsilon) \cap L^\infty(\Omega^\varepsilon) \ \forall \phi \in H_0^1(\Omega) \cap L^\infty(\Omega), \tag{4.3}$$

$$w^\varepsilon \rightharpoonup 1 \text{ weakly in } H^1(\Omega), \tag{4.4}$$

and given $z^\varepsilon, \phi, z \in H^1(\Omega) \cap L^\infty(\Omega)$ such that $z^\varepsilon \phi \in H_0^1(\Omega^\varepsilon) \cap L^\infty(\Omega^\varepsilon)$ and $z^\varepsilon \rightharpoonup z$ weakly in $H^1(\Omega)$ it is satisfied that

$$\int_\Omega M(x)^T \nabla w^\varepsilon \cdot \nabla(z^\varepsilon \phi) \to \langle \sigma, z\phi \rangle_{H^{-1}(\Omega), H_0^1(\Omega)}. \tag{4.5}$$

The model example for Ω^ε

The prototype of the examples where assumptions (4.1), (4.2), (4.3), (4.4) and (4.5) are satisfied is the case where the matrix $M(x)$ is the identity (and where therefore the operator is the Laplace's operator $-div(M(x)D) = -\Delta$), where $\Omega \subset \mathbb{R}^N$, $N \geq 2$, and where the holes T_i^ε are balls of radius r^ε with r^ε given by

$$\begin{cases} r^\varepsilon = C_0 \varepsilon^{N/(N-2)} & \text{if } N \geq 3, \\ \varepsilon^2 \log r^\varepsilon \to -C_0 & \text{if } N = 2, \end{cases}$$

for some $C_0 > 0$ (taking $r^\varepsilon = \exp(-C_0/\varepsilon^2)$ is the model case for $N = 2$) which are periodically distributed at the vertices of an N-dimensional lattice of cubes of size 2ε; in this case the measure σ is given by

$$\begin{cases} \sigma = \dfrac{S_{N-1}(N-2)}{2^N} C_0^{N-2} & \text{if } N \geq 3, \\ \sigma = \dfrac{2\pi}{4} \dfrac{1}{C_0} & \text{if } N = 2, \end{cases}$$

where S_{N-1} is the surface of the unit sphere in $\mathbb{R}^{N-1}$, see e.g. [10] and [19] for more details, and for other examples, in particular for the case where the holes have a different form and/or are distributed on a manifold.

Remark 4.1. In dimension $N = 1$, there is no sequence w^ε which satisfies (4.3) and (4.4) whenever for every ε there exists at least one hole $T_{i_\varepsilon}^\varepsilon$ with $T_{i_\varepsilon}^\varepsilon \cap \overline{\Omega} \neq \emptyset$, see Remark 5.1 of [12] for more details. □

4.2 Existence of solution to problem (1.1)

We study in this subsection the existence of solution to problem (1.1) in order to deal with the homogenization result.

Proposition 4.2. *Let $1 < q < 2$, $\lambda < \lambda^*(\Omega)$, $f \in L^p(\Omega)$ with $p > \frac{N}{2}$ and assume that conditions (M_1) and (g_1) are satisfied. Consider the open set $D^\varepsilon = \mathbb{R}^N \setminus \bigcup_{i=1}^{n(\varepsilon)} T_i^\varepsilon$ for all $\varepsilon > 0$. If both Ω and D^ε satisfy condition (A) from Definition 2.1, then there exists a solution to* (1.1) *for all $\varepsilon > 0$.*

Proof. Let us fix $\varepsilon > 0$. First of all notice that $\lambda < \lambda^*(\Omega) \leq \lambda^*(\Omega^\varepsilon)$. Thus, if Ω^ε satisfies condition (A), then this result is a mere consequence of Theorem 3.3. Let us show that, in fact, Ω^ε has the required regularity. Let $r_0, \theta_0 > 0$ be small enough so that they correspond to condition (A) for both Ω and D^ε. Fix $x \in \partial\Omega^\varepsilon$, and assume first that $x \in \partial\Omega \cap \partial\Omega^\varepsilon$. For $0 < r < r_0$, let Ω_r^ε be any connected component of $\Omega^\varepsilon \cap B_r(x)$, and let Ω_r be the connected component of $\Omega \cap B_r(x)$ which contains Ω_r^ε. Then,

$$|\Omega_r^\varepsilon| \leq |\Omega_r| \leq (1-\theta_0)|B_r|.$$

The same idea is valid if $x \in \partial D^\varepsilon \cap \partial\Omega^\varepsilon$. Therefore, Ω^ε satisfies condition (A) with parameters r_0, θ_0, and the proof is concluded. □

4.3 The homogenization result

Now, we can state our homogenization result.

Theorem 4.3. *Assume that the sequence of perforated domains Ω^ε satisfies* (4.1)*,* (4.2)*,* (4.3)*,* (4.4) *and* (4.5)*. Suppose also that conditions (M_1) and (g_1) are satisfied for $1 < q < 2$, that $f \in L^p(\Omega)$ for some $p > \frac{N}{2}$, that $\lambda < \lambda^*(\Omega)$ and that both Ω and D^ε satisfy condition (A), where $D^\varepsilon = \mathbb{R}^N \setminus \bigcup_{i=1}^{n(\varepsilon)} T_i^\varepsilon$. Then, there exists a sequence $\{u^\varepsilon\}$ of solutions to problem* (1.1) *such that $\{\widetilde{u^\varepsilon}\}$ is bounded in $L^\infty(\Omega)$ and $\widetilde{u^\varepsilon} \rightharpoonup u$ weakly in $H_0^1(\Omega)$, being u a solution to*

$$(4.6) \qquad \begin{cases} -\mathrm{div}(M(x)\nabla u) + \sigma u = \lambda u + g(x,u)|\nabla u|^q + f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

in the sense that $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$, $g(x,u)|\nabla u|^q \in L^1(\{|u| > 0\})$ and

$$(4.7) \qquad \int_\Omega M(x)\nabla u \cdot \nabla\phi + \langle \sigma, u\phi \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} = \lambda \int_\Omega u\phi + \int_{\{|u|>0\}} g(x,u)|\nabla u|^q \phi + \int_\Omega f(x)\phi$$

for all $\phi \in H_0^1(\Omega) \cap L^\infty(\Omega)$.

Remark 4.4. Under the hypotheses of Theorem 4.3, assume also that f satisfies that

$$\forall \omega \subset\subset \Omega,\ \exists c_\omega > 0: \quad f \geq c_\omega \text{ in } \omega.$$

Then, it is easy to prove that every solution u to problem (4.6) satisfies that $u \geq 0$. If we further assume that the holes are "good enough" so that $\sigma \in L^r(\Omega)$ for some $r > \frac{N}{2}$, then the strong maximum principle holds (see [22, Corollary 5.1]), so that $u > 0$. With this hypothesis, it can also be proved, following the arguments in [5], that $u \in C^{0,\alpha}(\overline{\Omega}) \cap W^{1,N}_{\text{loc}}(\Omega)$. Having this regularity and the strict positivity, the proof of the comparison principle [5, Theorem 3.2] can be reproduced with no relevant changes. In conclusion, we have uniqueness of solution to problem (4.6) for a right choice of the holes.

However, if σ is a general measure, the strong maximum principle does not hold in general. In [13] the authors have given two counterexamples which prove it. Hence, the uniqueness of solution to problem (4.6) is still open in the general case.

4.4 Strong convergence of the gradients

In the present subsection we prove some properties of the solutions u^ε to the problems (1.1) which allow us to prove our homogenization result Theorem 4.3 and a corrector result Theorem 4.6.

Lemma 4.5. *Let $1 < q < 2$, assume that conditions (M_1) and (g_1) are satisfied, and that $f \in L^1(\Omega)$. Let $\{\Omega^\varepsilon\}$ be any sequence of domains such that $\Omega^\varepsilon \subset \Omega$ for all $\varepsilon > 0$, and let $\{u^\varepsilon\}$ be a sequence of solutions to problem (1.1) with $\{\widetilde{u^\varepsilon}\}$ bounded in $L^\infty(\Omega)$. Then, $\{|\widetilde{u^\varepsilon}|^\alpha\}$ is bounded in $H^1_0(\Omega)$ for all $\alpha > \frac{1}{2}$. Moreover, there exists $u \in H^1_0(\Omega) \cap L^\infty(\Omega)$ such that, passing to a subsequence, $\widetilde{u^\varepsilon} \rightharpoonup u$ weakly in $H^1_0(\Omega)$ and $\widetilde{u^\varepsilon} \to u$ strongly in $W^{1,r}_0(\Omega) \cap L^s(\Omega)$ for all $r \in [1, 2)$ and $s \in [1, \infty)$.*

Proof. The proof of this result is analogous to the one of Lemma 3.5, except for the strong convergence in $W^{1,r}_0(\Omega)$. In this case, it is not possible to take $u^\varepsilon - u$ as test function in (1.1) since in general $u^\varepsilon - u \notin H^1_0(\Omega^\varepsilon)$. Thus, the proof for the strong convergence in $H^1_0(\Omega)$ does not work here. However, [9, Lemma 4.8] can be applied to obtain that $\widetilde{u^\varepsilon} \to u$ strongly in $W^{1,r}_0(\Omega)$ for all $r \in [1, 2)$. Since the proof of this fact is simple in our context, we include it here for completeness.

Indeed, for given $\delta > 0$ observe that

$$\begin{aligned} &\int_\Omega |\nabla(\widetilde{u^\varepsilon} - u)|^r = \\ &= \int_{\{|\widetilde{u^\varepsilon} - u| \geq \delta\}} |\nabla(\widetilde{u^\varepsilon} - u)|^r + \int_{\{|\widetilde{u^\varepsilon} - u| < \delta\}} |\nabla(\widetilde{u^\varepsilon} - u)|^r \leq \\ &\leq |\{|\widetilde{u^\varepsilon} - u| \geq \delta\}|^{1-\frac{r}{2}} \left(\int_\Omega |\nabla(\widetilde{u^\varepsilon} - u)|^2 \right)^{\frac{r}{2}} + \\ &+ |\Omega|^{1-\frac{r}{2}} \left(\int_{\{|\widetilde{u^\varepsilon} - u| < \delta\}} |\nabla(\widetilde{u^\varepsilon} - u)|^2 \right)^{\frac{r}{2}}. \end{aligned} \tag{4.8}$$

Clearly, $|\{|\widetilde{u^\varepsilon}-u|\geq\delta\}|\to 0$ as $\varepsilon\to 0$, and $\int_\Omega|\nabla(\widetilde{u^\varepsilon}-u)|^2$ is bounded uniformly in ε. Hence, the first term of (4.8) vanishes as $\varepsilon\to 0$. Let us focus on the second term.

Let us define $f^\varepsilon:\Omega\to\mathbb{R}$ by $f^\varepsilon(x)=\lambda u^\varepsilon+g(x,u^\varepsilon)|\nabla u^\varepsilon|^q+f(x)$ when $x\in\{|u^\varepsilon|>0\}$ and $f^\varepsilon(x)=f(x)$ otherwise. Consider also $T_\delta(t)=\max\{-\delta,\min\{\delta,t\}\}$ for $t\in\mathbb{R},\delta>0$.

Using the function $T_\delta(\widetilde{u^\varepsilon}-u)+T_\delta(u)\in H_0^1(\Omega^\varepsilon)\cap L^\infty(\Omega^\varepsilon)$ as test in the weak formulation of (1.1), we obtain that

$$\begin{aligned}\int_\Omega f^\varepsilon(x)(T_\delta(\widetilde{u^\varepsilon}-u)+T_\delta(u)) &=\\ =\int_\Omega M(x)\nabla\widetilde{u^\varepsilon}\cdot\nabla(T_\delta(\widetilde{u^\varepsilon}-u)+T_\delta(u)) &\geq\\ \geq\eta\int_{\{|\widetilde{u^\varepsilon}-u|<\delta\}}|\nabla(\widetilde{u^\varepsilon}-u)|^2+\int_{\{|\widetilde{u^\varepsilon}-u|<\delta\}}M(x)\nabla u\cdot\nabla(\widetilde{u^\varepsilon}-u)&+\\ +\int_{\{|u|<\delta\}}M(x)\nabla\widetilde{u^\varepsilon}\cdot\nabla u&.\end{aligned}$$

And this implies that

$$\begin{aligned}C\int_{\{|\widetilde{u^\varepsilon}-u|<\delta\}}|\nabla(\widetilde{u^\varepsilon}-u)|^2 &\leq\\ \leq\int_\Omega|f^\varepsilon(x)||T_\delta(\widetilde{u^\varepsilon}-u)+T_\delta(u)|+\int_{\{|u|<\delta\}}|\nabla\widetilde{u^\varepsilon}||\nabla u|&+\\ +\left|\int_\Omega M(x)\nabla u\cdot\nabla(\widetilde{u^\varepsilon}-u)\right|&\end{aligned}\tag{4.9}$$

for some constant $C>0$ dependent on η and M but independent of ε.

On the one hand, since $\{|\widetilde{u^\varepsilon}|^{\frac{1}{q}}\}$ is bounded in $H_0^1(\Omega)\cap L^\infty(\Omega)$, we have that $\{f^\varepsilon\}$ is bounded in $L^1(\Omega)$. Then, we deduce that

$$\int_\Omega|f^\varepsilon(x)||T_\delta(\widetilde{u^\varepsilon}-u)+T_\delta(u)|\leq C\delta,$$

for another constant $C>0$ independent of ε. Therefore,

$$\lim_{\delta\to 0}\left(\limsup_{\varepsilon\to 0}\int_\Omega|f^\varepsilon(x)||T_\delta(\widetilde{u^\varepsilon}-u)+T_\delta(u)|\right)=0.\tag{4.10}$$

On the other hand, since $\{\widetilde{u^\varepsilon}\}$ is bounded in $H_0^1(\Omega)$, we have that

$$\begin{aligned}\int_{\{|u|<\delta\}} |\nabla\widetilde{u^\varepsilon}||\nabla u| \leq &\|\widetilde{u^\varepsilon}\|_{H_0^1(\Omega)} \left(\int_{\{|u|<\delta\}} |\nabla u|^2\right)^{\frac{1}{2}} \\ \leq &C\left(\int_\Omega |\nabla u|^2 \chi_{\{|u|<\delta\}}\right)^{\frac{1}{2}},\end{aligned}$$

again for a constant $C>0$ independent of ε. Thus,

$$\lim_{\delta\to 0}\left(\limsup_{\varepsilon\to 0}\int_{\{|u|<\delta\}} |\nabla\widetilde{u^\varepsilon}||\nabla u|\right)=0. \tag{4.11}$$

Finally, the weak convergence in $H_0^1(\Omega)$ yields to

$$\left|\int_\Omega M(x)\nabla u\cdot\nabla(\widetilde{u^\varepsilon}-u)\right|\to 0 \quad \text{as } \varepsilon\to 0. \tag{4.12}$$

In conclusion, from (4.9), (4.10), (4.11) and (4.12), we deduce that

$$\lim_{\delta\to 0}\left(\limsup_{\varepsilon\to 0}\int_{\{|\widetilde{u^\varepsilon}-u|<\delta\}} |\nabla(\widetilde{u^\varepsilon}-u)|^2\right)=0,$$

and the proof finishes by applying this last convergence to (4.8). □

4.5 Proof of Theorem 4.3

We dedicate this subsection in order to prove Theorem 4.3 in two steps.

Step 1. $\widetilde{u^\varepsilon}$ is bounded in $L^\infty(\Omega)$.

Let $\{u^\varepsilon\}$ be the sequence of solutions to (1.1) given by Proposition 4.2. One can easily follow the arguments by comparison in the proof of Theorem 3.3 to deduce that $\widetilde{u^\varepsilon}$ is bounded in $L^\infty(\Omega)$. Therefore, Lemma 4.5 implies that there exists $u\in H_0^1(\Omega)\cap L^\infty(\Omega)$ such that $\widetilde{u^\varepsilon}\rightharpoonup u$ weakly in $H_0^1(\Omega)$ and $\widetilde{u^\varepsilon}\to u$ strongly in $W_0^{1,r}(\Omega)\cap L^s(\Omega)$ for all $r\in[1,2)$ and $s\in[1,\infty)$.

Step 2. $u\in H_0^1(\Omega)\cap L^\infty(\Omega)$ is solution of (4.6) and $g(x,u)|\nabla u|^q\in L^1(\{|u|>0\})$.

The idea is to take $w^\varepsilon\phi$ as test function in (1.1) for some $\phi\in H_0^1(\Omega)\cap L^\infty(\Omega)$, and then pass to the limit as $\varepsilon\to 0$ (observe that, thanks to (4.3) $w^\varepsilon\phi\in H_0^1(\Omega^\varepsilon)\cap L^\infty(\Omega^\varepsilon)$ for every $\phi\in H_0^1(\Omega)\cap L^\infty(\Omega)$).

Taking $w^\varepsilon\phi$ as test function in (1.1) we have, using (4.1),

$$\int_{\Omega^\varepsilon}(M(x)\nabla u^\varepsilon\cdot\nabla\phi)w^\varepsilon+\int_{\Omega^\varepsilon}(M(x)\nabla u^\varepsilon\cdot\nabla w^\varepsilon)\phi=$$
$$=\lambda\int_{\Omega^\varepsilon}u^\varepsilon w^\varepsilon\phi+\int_{\{|u^\varepsilon|>0\}}g(x,u^\varepsilon)|\nabla u^\varepsilon|^q w^\varepsilon\phi+\int_{\Omega^\varepsilon}f(x)w^\varepsilon\phi$$

or equivalently

$$\int_{\Omega}(M(x)\nabla\widetilde{u^\varepsilon}\cdot\nabla\phi)w^\varepsilon+\int_{\Omega}(M(x)\nabla\widetilde{u^\varepsilon}\cdot\nabla w^\varepsilon)\phi=$$
$$=\lambda\int_{\Omega}\widetilde{u^\varepsilon}w^\varepsilon\phi+\int_{\{|\widetilde{u^\varepsilon}|>0\}}g(x,\widetilde{u^\varepsilon})|\nabla\widetilde{u^\varepsilon}|^q w^\varepsilon\phi+\int_{\Omega}f(x)w^\varepsilon\phi. \tag{4.13}$$

Now we pass to the limit as $\varepsilon\to 0$ in each term of the previous equality. For the first term of the left hand side we use (4.4) and that $\widetilde{u^\varepsilon}\rightharpoonup u$ weakly in $H_0^1(\Omega)$ to obtain that

$$\int_{\Omega}(M(x)\nabla\widetilde{u^\varepsilon}\cdot\nabla\phi)w^\varepsilon\to\int_{\Omega}M(x)\nabla u\cdot\nabla\phi.$$

In order to pass to the limit as $\varepsilon\to 0$ in the second term of the left hand side of (4.13) we use (4.3), (4.4) and (4.5) and we get

$$\int_{\Omega}(M(x)\nabla\widetilde{u^\varepsilon}\cdot\nabla w^\varepsilon)\phi=\int_{\Omega}M(x)\nabla(\widetilde{u^\varepsilon}\phi)\cdot\nabla w^\varepsilon$$
$$-\int_{\Omega}(M(x)\nabla\phi\cdot\nabla w^\varepsilon)\widetilde{u^\varepsilon}=\int_{\Omega}M(x)^T\nabla w^\varepsilon\cdot\nabla(\widetilde{u^\varepsilon}\phi)$$
$$-\int_{\Omega}(M(x)\nabla\phi\cdot\nabla w^\varepsilon)\widetilde{u^\varepsilon}\to\langle\sigma,u\phi\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}.$$

With respect to the first and third terms of the right hand side of (4.13), we use the estimate in the Sobolev space and (4.4). Passing to the limit as $\varepsilon\to 0$ we get

$$\lambda\int_{\Omega}\widetilde{u^\varepsilon}w^\varepsilon\phi\to\lambda\int_{\Omega}u\phi$$

and

$$\int_{\Omega}f(x)w^\varepsilon\phi\to\int_{\Omega}f(x)\phi.$$

The pass to the limit of the second term of the right hand side of (4.13) is more delicated, but one can argue as in the proof of Theorem 3.3 to prove that $g(x,u)|\nabla u|^q\in L^1(\{|u|>0\})$ and

$$\int_{\{|\widetilde{u^\varepsilon}|>0\}}g(x,\widetilde{u^\varepsilon})|\nabla\widetilde{u^\varepsilon}|^q w^\varepsilon\phi\to\int_{\{|u|>0\}}g(x,u)|\nabla u|^q\phi.$$

Therefore u satisfies (4.7) and we conclude the proof. □

4.6 The corrector result

Finally, in this subsection we prove a corrector result.

Theorem 4.6. *Assume the hypotheses of Theorem 4.3, and suppose also that the matrix M is symmetric. Let $\{u^\varepsilon\}$ and u be the sequence of solutions to* (1.1) *and its limit, respectively, given by Theorem 4.3. Then,*

$$\widetilde{u^\varepsilon} - uw^\varepsilon \to 0 \text{ strongly in } H_0^1(\Omega).$$

Remark 4.7. In [7] it is proved that, for general problems with gradient-dependent lower order terms, the simple representation given by Theorem 4.6, does not hold in general. However, the nature of our problem allows us to prove a corrector result analogous to the linear case in spite of the presence of the gradient term.

Proof of Theorem 4.6. First of all observe that

$$\begin{aligned} \eta \int_\Omega |\nabla(\widetilde{u^\varepsilon} - uw^\varepsilon)|^2 &\leq \int_\Omega M(x)\nabla(\widetilde{u^\varepsilon} - uw^\varepsilon) \cdot \nabla(\widetilde{u^\varepsilon} - uw^\varepsilon) \\ &= \int_\Omega M(x)\nabla\widetilde{u^\varepsilon} \cdot \nabla(\widetilde{u^\varepsilon} - uw^\varepsilon) - \int_\Omega M(x)\nabla(uw^\varepsilon) \cdot \nabla(\widetilde{u^\varepsilon} - uw^\varepsilon). \end{aligned} \tag{4.14}$$

We will now pass to the limit in each term of the right hand side of the equality (4.14).

Indeed, for the first one, we take $\widetilde{u^\varepsilon} - uw^\varepsilon$ as test function in (1.1) and obtain that

$$\begin{aligned} \int_\Omega M(x)\nabla\widetilde{u^\varepsilon} \cdot \nabla(\widetilde{u^\varepsilon} - uw^\varepsilon) &= \lambda \int_\Omega \widetilde{u^\varepsilon}(\widetilde{u^\varepsilon} - uw^\varepsilon) \\ &+ \int_{\{|\widetilde{u^\varepsilon}|>0\}} g(x,\widetilde{u^\varepsilon})|\nabla\widetilde{u^\varepsilon}|^q(\widetilde{u^\varepsilon} - uw^\varepsilon) + \int_\Omega f(x)(\widetilde{u^\varepsilon} - uw^\varepsilon). \end{aligned} \tag{4.15}$$

By Theorem 4.3, Lemma 4.5 and also by (4.2) and (4.4), we know that $\widetilde{u^\varepsilon} - uw^\varepsilon \to 0$ strongly in $L^s(\Omega)$ for all $s \in [1, \infty)$. Moreover, Lemma 4.5 also implies that $\{|u^\varepsilon|^{\frac{1}{q}}\}$ is bounded in $W_0^{1,q}(\Omega)$. Therefore, we can pass to the limit in (4.15) arguing as in the proof of the strong convergence in Lemma 3.5. In sum, we deduce that

$$\int_\Omega M(x)\nabla\widetilde{u^\varepsilon} \cdot \nabla(\widetilde{u^\varepsilon} - uw^\varepsilon) \to 0.$$

Concerning the second term of (4.14), we derive, using the symmetry of M, that

$$\int_\Omega M(x)\nabla(uw^\varepsilon)\cdot\nabla(\widetilde{u^\varepsilon}-uw^\varepsilon)=\int_\Omega uM(x)\nabla w^\varepsilon\cdot\nabla(\widetilde{u^\varepsilon}-uw^\varepsilon)$$
$$+\int_\Omega w^\varepsilon M(x)\nabla u\cdot\nabla(\widetilde{u^\varepsilon}-uw^\varepsilon)=\int_\Omega M(x)^T\nabla w^\varepsilon\cdot\nabla(u(\widetilde{u^\varepsilon}-uw^\varepsilon))$$
$$-\int_\Omega(\widetilde{u^\varepsilon}-uw^\varepsilon)M(x)\nabla w^\varepsilon\cdot\nabla u+\int_\Omega w^\varepsilon M(x)\nabla u\cdot\nabla(\widetilde{u^\varepsilon}-uw^\varepsilon)$$

Observe now that (4.5) implies that

$$\int_\Omega M(x)^T\nabla w^\varepsilon\cdot\nabla(u(\widetilde{u^\varepsilon}-uw^\varepsilon))\to 0.$$

Moreover, the remaining terms

$$\int_\Omega(\widetilde{u^\varepsilon}-uw^\varepsilon)M(x)\nabla w^\varepsilon\cdot\nabla u,\quad \int_\Omega w^\varepsilon M(x)\nabla u\cdot\nabla(\widetilde{u^\varepsilon}-uw^\varepsilon)$$

are both products in $L^2(\Omega)^N$ of a strongly convergent sequence times a weakly convergent one. Therefore both terms converge and the limits are clearly zero.

In conclusion, we have proved that we can pass to the limit in (4.14), and the limit is zero. The proof of the result is now finished. □

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Departamento de Matemáticas, Universidad de Almería, Ctra. Sacramento s/n, La Cañada de San Urbano, 04120 - Almería, Spain
Email address: jcarmona@ual.es

Departamento de Análisis Matemático, Universidad de Granada, Facultad de Ciencias, Avenida Fuentenueva s/n, 18071, Granada, Spain
Email address: salvadorlopez@ugr.es

Departamento de Matemáticas, Universidad de Almería, Ctra. Sacramento s/n, La Cañada de San Urbano, 04120 - Almería, Spain
Email address: pedroj.ma@ual.es