MODES OF CONVERGENCE IN NETS, COUNTEREXAMPLES, AND LINEABILITY

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Abstract. In this work we continue with the ongoing search for what are often large algebraic structures of mathematical objects (functions, sequences, etc.) which enjoy certain special properties. This type of study belongs to the recent area of research known as lineability. On this occasion, and among several other results, we shall show that there are large algebraic structures within *(i)* the set of nets which are weakly convergent, but are not bounded, *(ii)* nets that are weakly convergent, but are not convergent in norm, or *(iii)* the set of nets of measurable functions which converge pointwise to a function that is not measurable and that are bounded in $[0, 1]$.

1. Introduction and preliminaries

Let X be any topological vector space and M any subset of X. We say that M is **spaceable** if $M \cup \{0\}$ contains a closed infinite dimensional subspace. The set M shall be called **lineable** if $M \cup \{0\}$ contains an infinite dimensional linear (not necessarily closed) space. At times, we shall be more specific, referring to the set M as κ-lineable if it contains a vector space of dimension κ (finite or infinite cardinality).

These notions of lineability and spaceability were originally coined by V. I. Gurariy and they first appeared in [4, 17, 21]. During the last decade, many authors have invested a lot of effort in studying *special* cases of lineable sets and *pathological* real-valued functions.

On the next level, we also have the following notions closely linked to that of lineability. If V is a topological vector space contained in a (not necessarily unital) algebra and if κ is any (finite or infinite) cardinal number, then a set A is called **κ-algebrable** if there exists an algebra M such that $M \setminus \{0\} \subseteq A$ and M is a κ-dimensional vector space. Here, by $S = \{s_\alpha : \alpha \in I\}$ is a minimal system of generators of M, we mean that M is the algebra generated by S and for every $\alpha_0 \in I$, s_{α_0} does not belong to the algebra generated by $S \setminus \{s_{\alpha_0}\}$.

Moreover, we shall also say that a set A is **strongly κ-algebrable** if there exists a κ-generated free algebra M such that $M \setminus \{0\} \subseteq A$. Recall that an algebra M is called a κ-generated free algebra if there exists a subset $X = \{x_\alpha < \kappa\}$ of M such that any function f from X to some algebra $\mathfrak{A}$ can be uniquely extended to a homomorphism from M into $\mathfrak{A}$. Then X is called a set of free generators of the algebra M. In a commutative algebra we have a simple criterion; namely, a subset $X = \{x_\alpha < \kappa\}$ in a commutative algebra B generates a free subalgebra M if and only if for any polynomial P without free term and

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any $x_{\alpha_i} \in X$, $1 \leq i \leq n$, we have $P(x_{\alpha_1}, \dots, x_{\alpha_n}) = 0$ if and only if $P = 0$. It should be noted that $X = \{x_\alpha < \kappa\} \subset B$ is a set of free generators of a free algebra $M \subset B$ if and only if the set of all elements of the form $x_{\alpha_1}^{k_1} x_{\alpha_2}^{k_2} \cdots x_{\alpha_n}^{k_n}$ is linearly independent and all linear combinations of these elements (called algebraic combination) are in $B \cup \{0\}$. The notion of strong algebrability is essentially stronger than the notion of algebrability.

We refer the interested reader to, e.g., [1–3, 5–7, 9, 10, 12, 14] and references therein, for a complete account on recent results in the theory of lineability, spaceability and algebrability. Let us recall a result from [13] that shall be very useful throughout this note.

Theorem 1.1. *Let X be a set, $\mathcal{K} \subset \mathbb{R}^X$ and $f \in \mathbb{R}^X$ such that*

- *(a) $f \in \mathcal{K}$ is so that $f[X]$ has, at least, one accumulation point.*
- *(b) The vector space generated by the functions of the form $f(u)^n e^{\alpha f(u)}$, where n is a positive integer and $\alpha > 0$, is contained in $\mathcal{K} \cup \{0\}$.*

Then $\mathcal{K}$ is strongly $\mathfrak{c}-$algebrable.

Also, and for the sake of the completeness of this note, let us recall the following well known results.

Theorem 1.2 (Dominated Convergence Theorem). *Let $\{f_n\}$ be a sequence of integrable functions on a measure space, $(\mu, \mathcal{A})$, such that it converges pointwise to a function f. If there exists an integrable function g satisfying that $\forall n$, $|f_n| \leq g$; then f is an integrable function with*

$$\int_A f d\mu = \lim_{n\to\infty} \int_A f_n d\mu \qquad \forall A \in \mathcal{A}.$$

Theorem 1.3 (Kronecker's Theorem). *If $\sigma_1, \sigma_2, ..., \sigma_k, 1$ are linearly independent over the rational numbers, then the set of points*

$$(n\sigma_1, n\sigma_2, ..., n\sigma_k) \text{ mod } 1$$

is dense in the unit cube, i.e. for every $(r_1, \dots, r_k) \in [0,1]^k$ and every $\varepsilon > 0$ there exist $n \in \mathbb{N}$ and $z_1, \dots, z_k \in \mathbb{Z}$ such that $|n\sigma_i - z_i - r_i| < \varepsilon$, $i = 1, \dots, k$.

The proof of this last result is showed in [18]. The following one can be found in [22, Theorem 3.25].

Theorem 1.4 (Heine-Cantor Theorem). *If $f : M \to N$ is a continuous function between two metric spaces, and M is compact, then f is uniformly continuous.*

Theorem 1.5 (Banach-Zarecki Theorem). *Let F be a real-valued function defined on a real bounded closed interval $[a, b]$. A necessary and sufficient condition for F to be absolutely continuous is that:*

- *(1) F is continuous and of bounded variation on $[a, b]$.*
- *(2) F satisfies Lusin's condition (it maps sets of Lebesgue measure zero into sets of Lebesgue measure zero).*

This last result can be found, for instance, in [20, Theorem 1.1].

For the sake of completeness of this work, let us briefly recall the notion of net. A net (also know as Moore-Smith sequence) is a generalization of the classical notion of sequence in Analysis. Recall that a **directed set** (or a directed preorder or a filtered set) is a non-empty set A together with a reflexive and transitive binary relation $\preceq$ (that is, a preorder), with the additional property that every pair of elements has an upper bound, that is, for any $a, b \in A$

there exists $c \in A$ with $a \preceq c$ and $b \preceq c$. Any function whose domain is a directed set is called a **net**. As it is done with the notion of subsequence, one could also define the notion of subnet in the same fashion. We refer the interested reader to the seminal book [11] for a complete account on nets and convergence theorems for nets.

The present work shall focus on studying the lineability within subsets of nets enjoying certain "unexpected" properties, for instance, we shall prove that there are large algebraic structures within *(i)* the set of nets which are weakly convergent, but are not bounded (Theorem 2.1), *(ii)* nets that are weakly convergent, but are not convergent in norm (Theorem 2.3), or *(iii)* the set of nets of measurable functions which converge pointwise to a function that is not measurable and that are bounded in $[0,1]$ (Theorem 2.13). The notation shall be (in general) rather usual and, when needed, we shall recall the necessary concepts.

2. The main results

Let $(X, \|\cdot\|)$ be a Banach space and X' its dual. For Y being any set, we define the index set $\mathbb{F}(Y)$ as the set whose elements are finite subsets of Y and $\mathbb{F}_1(Y) = \mathbb{F}(Y) \times]0, \infty[$

Next, let us consider $\mathbb{F}_1(X')$ endowed with the following order:

$$(\{\alpha_1, \ldots, \alpha_n\}, \varepsilon) \preceq (\{\alpha_1', \ldots, \alpha_m'\}, \varepsilon') \longleftrightarrow \{\alpha_1, \ldots, \alpha_n\} \subseteq \{\alpha_1', \ldots, \alpha_m'\} \text{ and } \varepsilon' \leq \varepsilon.$$

Let $(X, \|\cdot\|)$ be a Banach space, it is well known that weakly convergent sequences are bounded sequences but this is not longer true for nets. We shall begin by proving that the set of nets which are weakly convergent but are not bounded is $\mathfrak{c}-$lineable. More precisely we define $\mathcal{S}_1$ as the set of nets (whose indexes belong to $\mathbb{F}_1$) which are weakly convergent, but are not bounded.

Theorem 2.1. *If $(X, \|\cdot\|)$ is an infinite-dimensional Banach space then $\mathcal{S}_1$ is $\mathfrak{c}-$lineable.*

Proof. Let us consider $X' = \{T_\alpha : \alpha \in \mathbb{J}\}$ (where $\mathbb{J}$ is an index set for the elements of X'). We have the following neighbourhood basis at 0 for the weak topology $\sigma(X, X')$:

$$V_{\alpha_1 \ldots \alpha_n, \varepsilon} := \{x \in X : |T_{\alpha_i}(x)| < \varepsilon, \ i = 1, \ldots, n\}. \tag{2.1}$$

Indeed, since $U_\varepsilon := (-\varepsilon, \varepsilon)$ is an open set in $\mathbb{R}$, then $T_{\alpha_i}^{-1}(U_\varepsilon)$ is an open set in the weak topology (weak open set) and consequently

$$V_{\alpha_1 \cdots \alpha_n, \varepsilon} = \bigcap_{\alpha_i = 1}^{n} T_{\alpha_i}^{-1}(U_\varepsilon)$$

is a weak open set. On the other hand, for every B neighbourhood of 0 in the weak topology $\sigma(X, X')$, there exists $n \in \mathbb{N}$ and $\alpha_1, \ldots, \alpha_n \in \mathbb{J}$ such that

$$W = \bigcap_{\alpha_i = 1}^{n} T_{\alpha_i}^{-1}(U_{\alpha_i}) \subset B,$$

with U_{α_i} a neighbourhood of 0 in $\mathbb{R}$. For some $\varepsilon > 0$ we have that $(-\varepsilon, \varepsilon) \subset U_{\alpha_i}$ for all $i \in \{1, \ldots, n\}$, then $0 \in V_{\alpha_1 \ldots \alpha_n, \varepsilon} \subset W \subset B$.

Since X is infinite dimensional, for each $n \in \mathbb{N}$ and $\alpha_1, \ldots, \alpha_n \in \mathbb{J}$ we may choose $x \in X \setminus \{0\}$ with $\|x\| = 1$ such that $T_{\alpha_i}(x) = 0$ for every $i = 1, \ldots, n$. We denote x as $x_{\alpha_1 \ldots \alpha_n}$ and for each $s \in [1, 2]$, we define the net $x_{\alpha_1 \ldots \alpha_n, \varepsilon}(s) = e^{\frac{s}{\varepsilon}} x_{\alpha_1 \ldots \alpha_n}$. Observe that $\|x_{\alpha_1 \ldots \alpha_n, \varepsilon}(s)\| = e^{\frac{s}{\varepsilon}} \to +\infty$ as $\varepsilon \to 0$ which implies that the net is unbounded.

We claim that the net $X_s = (x_{\alpha_1\cdots\alpha_n,\varepsilon}(s))$ with $(\{\alpha_1,\ldots,\alpha_n\},\varepsilon) \in \mathbb{F}_1$, weakly converges to zero. Indeed, since (2.1) is a neighbourhood basis of 0 for the weak topology, given B a neighbourhood of 0 in $\sigma(X,X')$ there exists $(\{\alpha_1,\ldots,\alpha_n\},\varepsilon) \in \mathbb{F}_1$ such that $V_{\alpha_1\ldots\alpha_n,\varepsilon} \subset B$. Moreover, whenever $(\{\alpha_1,\ldots,\alpha_n\},\varepsilon) \preceq (\{\alpha'_1,\ldots,\alpha'_m\},\varepsilon')$ one easily have that $V_{\alpha'_1\cdots\alpha'_m,\varepsilon'} \subset V_{\alpha_1\cdots\alpha_n,\varepsilon} \subset B$. Therefore, $x_{\alpha'_1\cdots\alpha'_m,\varepsilon'}(s) \in B$ (in fact this is true also for $\varepsilon' > \varepsilon$ since $\{\alpha_1,\ldots,\alpha_n\} \subseteq \{\alpha'_1,\ldots,\alpha'_m\}$ implies that $x_{\alpha'_1\ldots\alpha'_m,\varepsilon'}(s) \in V_{\alpha_1\ldots\alpha_n,\varepsilon}$, because $\left|T_{\alpha_i}\left(x_{\alpha'_1\ldots\alpha'_m,\varepsilon'}(s)\right)\right| = 0 < \varepsilon$ for every $i = 1,\ldots,n$). Thus the claim is proved.

In order to finish the proof we will show that the nets of the form X_s are linearly independent and its linear span is contained in $\mathcal{S}_1 \cup \{0\}$. Indeed, let us fix $m \in \mathbb{N}$, $\beta_1,\ldots,\beta_m \in \mathbb{R}\setminus\{0\}$, $s_1,\ldots,s_m \in [1,2]$ with $s_i \neq s_j$ if $i \neq j$, $(\{\alpha_1,\ldots,\alpha_n\},\varepsilon) \in \mathbb{F}_1$ and let us consider

$$g_{\alpha_1\ldots\alpha_n,\varepsilon} = \sum_{i=1}^{m} \beta_i X_{s_i}.$$

We can write $g_{\alpha_1\ldots\alpha_n,\varepsilon}$ as:

$$g_{\alpha_1\ldots\alpha_n,\varepsilon} = x_{\alpha_1\ldots\alpha_n} e^{\frac{s_j}{\varepsilon}} \sum_{i=1}^{m} \beta_i e^{\frac{s_i - s_j}{\varepsilon}},$$

where $s_j = \max\{s_i, i = 1,\ldots,m\}$. It is clear that

$$\lim_{\varepsilon\to 0} \|g_{\alpha_1\cdots\alpha_n,\varepsilon}\| = \lim_{\varepsilon\to 0} e^{\frac{s_j}{\varepsilon}} |\beta_j| = +\infty,$$

which implies that X_s are linearly independent. Moreover, although the net $g_{\alpha_1\cdots\alpha_n,\varepsilon}$ does not converge to zero in norm, as we have shown before, the net $X_s = (x_{\alpha_1\cdots\alpha_n,\varepsilon}(s))$ weakly converges to zero, then $g_{\alpha_1\cdots\alpha_n,\varepsilon}$ weakly converges to zero thanks to the linearity of the elements in X' (let us recall that a net (x_d) weakly converges to x_0 in a normed space X, if for each $T \in X'$ it is satisfied that $T(x_d) \to T(x_0)$), i.e., $g_{\alpha_1\cdots\alpha_n,\varepsilon} \in \mathcal{S}_1$. □

Remark 2.2. *Notice that, as a consequence of the above result, we infer that, in an infinite-dimensional Banach space, the set of nets indexed in* $\mathbb{F}_1$ *which are weakly convergent to zero but not convergent in norm is* $\mathfrak{c}-$*lineable.*

Due to the isomorphism between two separable Hilbert spaces, we are going to prove this result for l_2. Let $\mathbb{F}(\mathbb{N})$ (which we will denote as $\mathbb{F}$) be the subset of the power set of $\mathbb{N}$ formed by finite non-empty subsets, which is ordered by the inclusion. $\mathcal{S}_2$ shall stand for the set of nets which belong to l_2 indexed in $\mathbb{F}$, such that the net weakly converge, but does not converge in norm. The following result establishes the lineability of this set.

Theorem 2.3. $\mathcal{S}_2$ *is* $\mathfrak{c}-$*lineable.*

Proof. Let us consider $\{x_i : i \in \mathbb{N}\}$ an orthogonal Schauder basis for l_2 and, for $\alpha = \{\alpha_1,\ldots,\alpha_{n_\alpha}\} \in \mathbb{F}$, let us denote $x_\alpha = \left\{x_{\alpha_1},\ldots,x_{\alpha_{n_\alpha}}\right\}$. Let y_α be a vector indexed in $\mathbb{F}$ with norm n_α, such that it is orthogonal to the elements of x_α.

On the other hand, l_2 is a reflexive space, $l'_2 = l_2$, and as we have done earlier, let us denote $l'_2 = \{T_i : i \in \mathfrak{c}\}$, with $T_i(x) = \langle z_i, x\rangle$ for some $z_i \in l_2$. Then for $\varepsilon > 0$, and $\alpha \in \mathbb{F}$ we can consider the weak open set $V_{\alpha_1\ldots\alpha_{n_\alpha},\varepsilon} = \{x \in l_2 : \|x\| < 2n_\alpha, |T_{\alpha_i}(x)| < \varepsilon, i = 1,...,n_\alpha\}$ and this family constitute a neighbourhood basis at 0 for the weak topology $\sigma(l_2, l'_2)$. Indeed, for every B neighbourhood of 0 in that weak topology, there exists $n \in \mathbb{N}$ and $i_1,\ldots,i_n \in \mathfrak{c}$ such

that
$$W = \bigcap_{k=1}^{n} T_{i_k}^{-1}(U_{i_k}) \subset B,$$
with U_{i_k} a neighbourhood of 0 in $\mathbb{R}$. Moreover, using that $\{x_i\}_{i\in\mathbb{N}}$ is an orthogonal basis, for every $\varepsilon > 0$ there exists $\alpha(\varepsilon) := \alpha = \{\alpha_1, \dots, \alpha_n\} \in \mathbb{F}$ such that
$$\langle z_{i_k} - x_{\alpha_k}, z_{i_k} - x_{\alpha_k}\rangle < \frac{\varepsilon^2}{16n^2}, \quad k = 1, \dots, n.$$
If we assume that $(-\varepsilon, \varepsilon) \subset U_{i_k}$ for all $k \in \{1, \dots, n\}$, then for every $x \in V_{\alpha_1 \dots \alpha_n, \frac{\varepsilon}{2}}$
$$|T_{i_k}(x)| = |\langle z_{i_k}, x\rangle| \leq |\langle z_{i_k} - x_{\alpha_k}, x\rangle| + |\langle x_{\alpha_k}, x\rangle| < \frac{\varepsilon}{4n} 2n + \frac{\varepsilon}{2} = \varepsilon.$$
Therefore $0 \in V_{\alpha_1 \dots \alpha_n, \frac{\varepsilon}{2}} \subset W \subset B$.

Arguing as in the proof of Theorem 2.1 we deduce that the net $\{y_\alpha\}_{\alpha\in\mathbb{F}}$ weakly converges to 0 since y_α is orthogonal to the elements of x_α, i.e. , $\langle y_\alpha, x_{\alpha_i}\rangle = 0$, for every $i = 1, ..., n_\alpha$ or equivalently $T_{\alpha_i}(y_\alpha) = 0$, for every $i \in \{1, \dots, n\}$.

Moreover, $\{\|y_\alpha\|\}_{\alpha\in\mathbb{F}} = \{n_\alpha\}_{\alpha\in\mathbb{F}}$ tends to infinity due to the fact that for $l \in \mathbb{R}$ we have a set $\alpha = \{\alpha_1, \dots, \alpha_{n_\alpha}\} \in \mathbb{F}$ with $n_\alpha > l > 0$. Thus, for each $\alpha' = \left\{\alpha'_1, \dots, \alpha'_{n_{\alpha'}}\right\} \in \mathbb{F}$ such that $\{\alpha_1, \dots, \alpha_{n_\alpha}\} \subseteq \left\{\alpha'_1, \dots, \alpha'_{n_{\alpha'}}\right\}$, we have that $n_{\alpha'} > l$.

Let us define, for every $c \in]0, 1[$, the net $y(c) \in \mathcal{S}_2$, with $y(c) \equiv y_\alpha(c) = e^{cn_\alpha} y_\alpha$. We will finish the proof by showing that the nets of the form $y(c)$ are linearly independent and its linear span is contained in $\mathcal{S}_2 \cup \{0\}$. Indeed, let us consider $k \in \mathbb{N}$ and for every $i \in \{1, \dots, k\}$, $c_i \in]0, 1[$, $a_i \in \mathbb{R} \setminus \{0\}$ and the net $\sum_{i=1}^k a_i y(c_i) = \left\{\sum_{i=1}^k a_i y_\alpha(c_i)\right\}_{\alpha\in\mathbb{F}}$. This net weakly converges to zero, due to the linearity of the elements of l'_2, in addition it diverges in norm:
$$\left\|\sum_{i=1}^k a_i y_\alpha(c_i)\right\| = \left\|\sum_{i=1}^k a_i e^{c_i n_\alpha} y_\alpha\right\| = \left|\sum_{i=1}^k a_i e^{c_i n_\alpha}\right| \|y_\alpha\| = \left|\sum_{i=1}^k a_i e^{(c_i - c_j) n_\alpha}\right| e^{c_j n_\alpha} n_\alpha,$$
where $c_j = \max\{c_i : i = 1, \dots, k\}$. As we have seen before, $\|y_\alpha\|$ diverges and, for n_α big enough,
$$\left\|\sum_{i=1}^k a_i y_\alpha(c_i)\right\| = \left|\sum_{i=1}^k a_i e^{(c_i - c_j) n_\alpha}\right| e^{c_j n_\alpha} n_\alpha > \frac{|a_j|}{2} e^{c_j n_\alpha} n_\alpha.$$

Therefore, $\left\{\left\|\sum_{i=1}^k a_i y_\alpha(c_i)\right\|\right\}_{\alpha\in\mathbb{F}}$ diverges and, thus, we have that it belongs to $\mathcal{S}_2$. Even more, we also have the linear independence of the nets $y(c)$ (since $\{0\}_{\alpha\in\mathbb{F}}$ does not diverge). This concludes the proof. □

Now we deal with the differences between the class sequentially continuous functions and the class of continuous functions. Let us recall the following definition.

Definition 2.4. *A function between two topological spaces $f : X \to Y$ is sequentially continuous if $\{f(x_n)\}_{n\in\mathbb{N}}$ converges to $f(x)$ in Y whenever $\{x_n\}_{n\in\mathbb{N}}$ converges to x in X.*

In every normed space, the notion of continuity and sequential continuity are both the same. This does not occur in non-metrizable topological spaces, in our case we are going to study the case of Schur spaces.

Definition 2.5. *A Banach space is said to be a Schur space if it has the Schur property, which means that the weak convergence of a sequence entails convergence in norm. Therefore, for these spaces, sequences that converge in norm are the same sequences that are weakly convergent.*

In the following theorem we prove that, in a Schur space X endowed with the weak topology, the set $\mathcal{S}_3$ formed by the sequentially continuous functions of $\mathbb{R}^X$ that are not continuous is algebraically generic.

Theorem 2.6. *$\mathcal{S}_3$ is strongly $\mathfrak{c}-$algebrable.*

Proof. Recall that a function is not weakly continuous if the image of some weakly convergent net is not convergent. Since $\mathcal{S}_1$ is nonempty we have that the norm is not weakly continuous. We are going to use Theorem 1.1 to show that $\mathcal{S}_3 \subset \mathbb{R}^X$ is strongly $\mathfrak{c}-$algebrable. First of all, we need to ensure that $\|\cdot\|$ satisfies the hypotheses of Theorem 1.1.

(1) Let us see that $\|\cdot\| \in \mathcal{S}_3$. Since $\mathcal{S}_1$ is nonempty we have that the norm is not weakly continuous. However $\|\cdot\|$ it is sequentially weakly continuous. Indeed, recall that X is a Schur space and if a sequence $\{x_n\}_{n\in\mathbb{N}}$ is weakly convergent to x then it will converge in norm, in particular $\{\|x_n\|\}_{n\in\mathbb{N}}$ converges to $\|x\|$.

(2) $\|X\| := \{\|x\| : x \in X\}$ has at least one accumulation point. Indeed, 0 is an accumulation point of $\|X\|$ since for every $x \in X \setminus \{0\}$ we have that $\left\{\left\|\frac{x}{n}\right\|\right\}_{n\in\mathbb{N}} \subset \|X\| \setminus \{0\}$ converges to zero.

(3) The vector space generated by $\|u\|^t e^{r\|u\|}$, where t is a positive integer and $r > 0$, is contained in $\mathcal{S}_3 \bigcup \{0\}$.

Let us consider the function

$$f(u) = \sum_{i=1}^{k} a_i \|u\|^{t_i} e^{r_i\|u\|},$$

for some $k \in \mathbb{N}$, $a_1, \ldots, a_k \in \mathbb{R} \setminus \{0\}$ and $(t_1, r_1), \ldots, (t_k, r_k) \in \mathbb{N}\times]0, +\infty[$ with $(t_i, r_i) \neq (t_j, r_j)$ for $i \neq j$. We shall see that $f \in \mathcal{S}_3$. We have that f is sequentially continuous, because $\|\cdot\|$, $e^{(\cdot)}$ and $(\cdot)^n$ are too. In order to prove that the function is not continuous, let us take the net $\{x_\alpha\}_{\alpha\in\mathbb{F}_1(X)} \in \mathcal{S}_1$ weakly convergent to zero. We claim that the net $\{f(x_\alpha)\}_{\alpha\in\mathbb{F}_1(X)}$ does not converge to zero. Indeed,

$$|f(x_\alpha)| = \left|\sum_{i=1}^{k} a_i \|x_\alpha\|^{t_i} e^{r_i\|x_\alpha\|}\right| = \|x_\alpha\|^{t_j} e^{r_j\|x_\alpha\|} \left|\sum_{i=1}^{k} a_i \|x_\alpha\|^{t_i - t_j} e^{(r_i - r_j)\|x_\alpha\|}\right|,$$

with $j \in \{1, \ldots, k\}$ such that $r_j = \max\{r_i : i = 1, \ldots, k\}$ and $t_j = \max\{t_i : i = 1, \ldots, k \text{ with } r_i = r_j\}$. Since $\{\|x_\alpha\|\}_{\alpha\in\mathbb{F}_1}$ diverges we have that for $\|x_\alpha\|$ big enough

$$|f(x_\alpha)| > \frac{|a_i|}{2} \|x_\alpha\|^{t_j} e^{r_j\|x_\alpha\|}.$$

Thus, $|f(x_\alpha)|$ diverges and f is not continuous. Thus, $f \in \mathcal{S}_3$.

Therefore we can now apply Theorem 1.1 and $\mathcal{S}_3$ is strongly $\mathfrak{c}-$algebrable. □

Remark 2.7. *Theorem 2.6 is not an exclusive result for spaces with the Schur property. We can obtain the same result for other topological spaces, for example, the set $\beta\mathbb{N}$ (the Stone-Čech* *compactification of* $\mathbb{N}$, *see* [16]) *with usual topology. We define the function* $f \in \mathbb{R}^{\beta\mathbb{N}}$:

$$f(x) := \begin{cases} x & \text{if } x \in 2\mathbb{N}, \\ 1/x & \text{if } x \in 2\mathbb{N}+1, \\ 0 & \text{otherwise.} \end{cases}$$

Due to the fact that the only convergent sequences in $\beta\mathbb{N}$ *are the trivial ones, all functions in* $\mathbb{R}^{\beta\mathbb{N}}$ *are sequentially continuous. Since* $\beta\mathbb{N}$ *is compact, if* f *is continuous,* $f[\beta\mathbb{N}]$ *must be bounded, but this is certainly not possible. On the other hand,* 0 *is an accumulation point of* $f[\beta\mathbb{N}]$. *Finally,* $\sum_{i=1}^{k} f(u)^{n_i} e^{\alpha_i f(u)}$ *is not continuous since it is not bounded.*

A canonical, and typical, example of a space with this property is l_1. The rest of the classical l_p spaces with $1 < p < \infty$ do not satisfy the Schur property. Let us see that we can find big subsets of $\mathcal{S}_{4,p}$, a set formed by the sequences of elements which belong to l_p, such that they are weakly convergent but they do not converge in norm.

Theorem 2.8. *For* $p \in]1, +\infty[$, *the set* $\mathcal{S}_{4,p}$ *is* $\mathfrak{c}$*−lineable.*

Proof. Let us consider a Hamel basis $\mathcal{H}$ of $\mathbb{R}$ over $\mathbb{Q}$ whose elements are positive and $1 \in \mathcal{H}$. If $h \in \mathcal{H}\setminus\{1\}$ we denote the sequence $\rho_{n,h} \in \mathbb{R}^{\mathbb{N}}$, whose elements are 0 except the n−th element which is $\sin(2\pi nh)$. Let $\rho_h \in l_p^{\mathbb{N}}$ be the sequence $\{\rho_{n,h}\}_{n\in\mathbb{N}}$.

We claim that the sequences ρ_h are weakly convergent to 0, but they do not converge in norm. Indeed, we have that $\|\rho_{n,h}\|_{l_p} = |\sin(2\pi nh)|$. Kronecker's Theorem in one dimension, Theorem 1.3, says that $\{2\pi nh \mod 2\pi : n \in \mathbb{N}\}$ is dense in $[0, 2\pi]$, so $\{\sin(2\pi nh) : n \in \mathbb{N}\}$ is dense in $[-1, 1]$. Then $\{\rho_{n,h}\}_{n\in\mathbb{N}}$ does not converge in norm.

On the other hand, every $T \in l_p'$ has the form $T(x) = \sum_i x_i y_i$ with $y \in l_q$, such that $\frac{1}{q} + \frac{1}{p} = 1$. Particularly, $T(\rho_{n,h}) = y_n \sin(2\pi nh)$ tends to 0 when n tends to ∞. Then, $\{\rho_{n,h}\}_{n\in\mathbb{N}}$ weakly converges to 0.

The sequences ρ_h are linearly independent. Let consider $h_i \in \mathcal{H}\setminus\{1\}$ pairwise different, for $i = 1, ..., N$ which are rationally independent, $\alpha_1, ..., \alpha_N \in \mathbb{R}\setminus\{0\}$ and the sequence $\sum_{i=1}^{N} \alpha_i \rho_{h_i}$.

The weak convergence of $\{\rho_{n,h}\}_{n\in\mathbb{N}}$ to 0 implies that $\left\{\sum_{i=1}^{N} \alpha_i \rho_{n,h_i}\right\}_{n\in\mathbb{N}}$ weakly converges to 0.

On the other hand, we have that

$$\left\|\sum_{i=1}^{N} \alpha_i \rho_{n,h_i}\right\| = \left|\sum_{i=1}^{N} \alpha_i \sin(2\pi nh_i)\right|.$$

Then by using Kronecker's Theorem, Theorem 1.3, $(2\pi nh_1 \mod 2\pi, ..., 2\pi nh_N \mod 2\pi)$ is dense in $[0, 2\pi]^N$. Then we have that

$$\{(\sin(2\pi nh_1), ..., \sin(2\pi nh_N)) : n \in \mathbb{N}\}$$

is dense in $[-1, 1]^N$ so we can find infinitely $n \in \mathbb{N}$ such that $\sin(2\pi nh_1) > 1 - \varepsilon$ and $|\sin(2\pi nh_i)| < \varepsilon$ when $i \in \{2, \ldots, N\}$, so

$$\left|\sum_{i=1}^{N} \alpha_i \sin(2\pi nh_i)\right| \geq (1-\varepsilon)|\alpha_1| - (N-1)\,\varepsilon \max\{|a_i| : i = 2, ..., N\}.$$

Choosing

$$2\varepsilon < \frac{|\alpha_1|}{|\alpha_1| + (N-1)\,\varepsilon \max\{|a_i| : i = 2, ..., N\}},$$

we have that
$$\left|\sum_{i=1}^{N} \alpha_i \sin(2\pi n h_i)\right| \geq \frac{|\alpha_1|}{2}.$$
This implies that the sum can not be zero, so the ρ_h's are linearly independent. Thus, we have that $\mathcal{S}_{4,p}$ is $\mathfrak{c}-$lineable. □

Next, we shall see that, in the set $\mathbb{R}^{[0,1]}$ (with the product topology), the set $\mathcal{S}_5$ formed by the convergent nets contained in a compact set and having no convergent subsequence is "big" from the lineability viewpoint. In order to achieve this, we shall use the class $\mathcal{T}$ whose elements have the form: $T = \{t_i : 0 \leq t_1 < \cdots < t_{n-1} < t_n \leq 1\}$. In $\mathcal{T}$ we define the inclusion order; then for $T_1, T_2 \in \mathcal{T}$, $T_1 \leq T_2$ if $T_1 \subseteq T_2$.

Consider a function $f \in \mathbb{R}^{[0,1]}$, then we define the following associated functions:

$$f_T(x) := \begin{cases} 0 & \text{if } x \in T, \\ f(x) & \text{otherwise.} \end{cases} \tag{2.2}$$

In the next lemma we prove the main properties of the net $\{f_T\}_{T\in\mathcal{T}}$ when f has an uncountable set of non-zero values.

Lemma 2.9. *Assume that $f \in \mathbb{R}^{[0,1]}$ and the set $\{x : f(x) \neq 0\}$ is not countable,*
a) $\{f_T\}_{T\in\mathcal{T}}$ converges to 0 (with product topology).
b) There exists no sequence $\{f_{T_n}\}_{n\in\mathbb{N}}$ that converges to 0 (with product topology).

Proof. a) We are going to show that the net $\{f_T\}_{T\in\mathcal{T}}$ converges to 0. Observe that for $T \in \mathcal{T}$ and $\varepsilon > 0$, we have that the family:
$$V_{T,\varepsilon} = \left\{ g \in \mathbb{R}^{[0,1]} : |g(t)| < \varepsilon \,,\, t \in T \right\},$$
is a neighbourhood basis at 0 for the product topology. Given B a neighborhood of zero in the product topology there exists $T \in \mathcal{T}$ and $\varepsilon > 0$ such that $V_{T,\varepsilon} \subset B$. Thus, for each partition T^*, such that $T^* > T$, then $f_{T^*}(t_i) = 0$ for every $t_i \in T^*$, in particular $f_{T^*} \in V_{T^*,\varepsilon} \subset V_{T,\varepsilon} \subset B$. Thus, $\{f_T\}_{T\in\mathcal{T}}$ converges to 0.

b) Let us suppose that there exists a sequence $\{f_{T_n}\}_{n\in\mathbb{N}}$ such that it converges to 0. We denote it as $\{f_n\}_{n\in\mathbb{N}}$ and we have that $f_n(x) \to 0$ for every $x \in [0,1]$. On the other hand, due to the fact that $\{x : f(x) \neq 0\}$ is uncountable, there exists $x_0 \in [0,1] \setminus \bigcup_{n\in\mathbb{N}} T_n$ with $f_n(x_0) \neq 0$. Then, $0 \neq f_n(x_0) = f(x_0)$ for every $n \in \mathbb{N}$. Hence $\{f_{T_n}\}_{n\in\mathbb{N}}$ does not converge to 0. □

Now we deal with the set $\mathcal{S}_5$ of every net that converges to 0 in $\mathbb{R}^{[0,1]}$ (with product topology), whose elements are contained in a compact set and without any convergent subnet being a sequence.

Theorem 2.10. *$\mathcal{S}_5$ is strongly $\mathfrak{c}-$algebrable.*

Proof. As index set we choose the family $\mathcal{T}$ previously defined with the inclusion order. Let $\alpha \in \mathbb{R}$ we denote e_α the function $e_\alpha(x) = e^{\alpha x}$. Let us consider a Hamel basis $\mathcal{H}$ of $\mathbb{R}$ over $\mathbb{Q}$ with positive elements. First we see that the family of nets $\{e_{h,T}\}_{T\in\mathcal{T}}$ with $h \in \mathcal{H}$ is algebraically independent. Suppose, on the contrary, that there exists a polynomial P with n variables such that $P(0,\ldots,0) = 0$ and $P\left(\{e_{h_1,T}\}_{T\in\mathcal{T}}, \ldots, \{e_{h_n,T}\}_{T\in\mathcal{T}}\right) = \{0\}_{T\in\mathcal{T}}$ for some $h_1, \ldots, h_n \in \mathcal{H}$. Equivalently, $\{P\left(e_{h_1,T}, \ldots, e_{h_n,T}\right)\}_{T\in\mathcal{T}} = \{0\}_{T\in\mathcal{T}}$, i.e.,

$P\left(e_{h_1,T},\ldots,e_{h_n,T}\right)(x)=0$ for every $x\in[0,1]$. Using that $\mathcal{H}$ is a Hamel basis, we have that there exist $m\in\mathbb{N}$, $a_1,\ldots,a_m\in\mathbb{R}\setminus\{0\}$ and $\alpha_1,\ldots,\alpha_m$ with $\alpha_i\neq\alpha_j$ when $i\neq j$ such that

$$P\left(e_{h_1,T},\ldots,e_{h_n,T}\right)(x)=\left(\sum_{i=1}^{m}a_ie^{\alpha_i x}\right)_T=0.$$

Then have that $\sum_{i=1}^{m}a_ie^{\alpha_i x_0}=0$ for some $x_0\in]0,1[$ which is not possible because e_α are linearly independent in any interval. Thus $P\left(e_{h_1,T},\ldots,e_{h_n,T}\right)$ can not be 0 and we have the nets $\{e_{h,T}\}_{T\in\mathcal{T}}$ are algebraically independent.

Note that, since $e_\alpha(x)$ is bounded for $x\in[0,1]$ and fixed $\alpha\in\mathbb{R}$ then $P\left(e_{h_1},\ldots,e_{h_n}\right)$ is also bounded. Moreover, since $P(0,\ldots,0)=0$ implies that $P\left(e_{h_1,T},\ldots,e_{h_n,T}\right)=P\left(e_{h_1},\ldots,e_{h_n}\right)_T$, we have that $P\left(e_{h_1,T},\ldots,e_{h_n,T}\right)$ is contained in a compact set. If the image of $[0,1]$ through $P\left(e_{h_1},\ldots,e_{h_n}\right)$ is contained in $[-a,a]$, the same occurs for $P\left(e_{h_1,T},\ldots,e_{h_n,T}\right)$. This means that the net $\{P\left(e_{h_1,T},\ldots,e_{h_n,T}\right)\}_{T\in\mathcal{T}}$ is contained in $[-a,a]^{[0,1]}$ that is compact (by Tychonoff's Theorem).

Since $P\left(e_{h_1},\ldots,e_{h_n}\right)$ is continuous and different from 0 we can use Lemma 2.9-a) to deduce that $P\left(e_{h_1,T},\ldots,e_{h_n,T}\right)=P\left(e_{h_1},\ldots,e_{h_n}\right)_T$ converges to 0. On the other hand, thanks to Lemma 2.9-b) there is no subnet being a sequence and convergent to 0. In particular $P\left(e_{h_1,T},\ldots,e_{h_n,T}\right)$ belongs to $\mathcal{S}_5$. □

This results will help us to show another difference between sequences and nets. Particularly every net contained in a compact set have a convergent subnet, but the set $\mathcal{S}_6\subset\mathbb{R}^{[0,1]}$, formed by the sequences that are in a compact set and does not have any convergent subsequence is "big".

Corollary 2.11. *$\mathcal{S}_6$ is strongly $\mathfrak{c}-$algebrable.*

Proof. Let consider the sets $\mathcal{T}\ni T_n=\{1,1/2,\ldots,1/n\}$ and the sequences $\{e_{h,T_n}\}_{n\in\mathbb{N}}$ with $h\in\mathcal{H}$. $\{e_{h,T_n}\}_{n\in\mathbb{N}}$ is a subnet of $\{e_{h,T}\}_{T\in\mathcal{T}}$ which is a sequence. Arguing as in the proof of Theorem 2.10, we have that the sequences $\{e_{h,T_n}\}_{n\in\mathbb{N}}$ are algebraically independent. Moreover, if P is a polinomial of r variables such that $P(0,\ldots,0)=0$, we have that

$$P\left(\{e_{h_1,T_n}\}_{n\in\mathbb{N}},\ldots,\{e_{h_r,T_n}\}_{n\in\mathbb{N}}\right)$$

is a subnet of $P\left(\{e_{h_1,T}\}_{T\in\mathcal{T}},\ldots,\{e_{h_n,T}\}_{T\in\mathcal{T}}\right)$. Thus, $P\left(\{e_{h_1,T_n}\}_{n\in\mathbb{N}},\ldots,\{e_{h_r,T_n}\}_{n\in\mathbb{N}}\right)$ is contained in a compact set. Finally, we have that $P\left(\{e_{h_1,T_n}\}_{n\in\mathbb{N}},\ldots,\{e_{h_r,T_n}\}_{n\in\mathbb{N}}\right)$ does not have any convergent subsequence, otherwise, it will have a convergent subnet such that is a subsequence. This is a contradiction with Lemma 2.9-b). □

Now, let us briefly revisit the *Dominated Convergence Theorem*. We shall see that, if we use the pointwise convergence of nets instead of the pointwise convergence of sequences, then the dominated convergence theorem does not hold. We shall need the following result, of easy proof, for our purposes (see, also, [8,15]).

Lemma 2.12. *The set of the continuous functions defined in* $[0,1]$ *which only have a finite number of zeros is strongly $\mathfrak{c}-$algebrable.*

Proof. We can apply the Theorem 1.1 to the identity function in $[0,1]$, $I:[0,1]\to[0,1]$, because:

(1) It is clear that I is a continuous function in $[0,1]$ with finite number of zeros.
(2) $I([0,1])=[0,1]$ and has, at least, one accumulation point.

(3) The vector space generated by $u^n e^{\alpha u}$ with n a positive integer and $\alpha > 0$, is clearly contained in the set of the continuous function defined in $[0,1]$ with a finite number of zeros up to the zero function.

Then we have that the set is strongly $\mathfrak{c}-$algebrable. □

First of all we are going to see that the Dominated Convergence Theorem with nets might fail because the limit of the function might not be measurable.

$\mathcal{S}_7$ is the set of nets of measurable functions which converge pointwise to a function that is not measurable and that are bounded in $[0,1]$.

Theorem 2.13. *$\mathcal{S}_7$ is strongly $\mathfrak{c}-$algebrable.*

Proof. Let $A \subset [0,1]$ be a non-measurable set. We consider the following set of indexes $\Lambda = \{\beta \subset A : \beta \text{ is finite}\}$ ordered by inclusion. Let us denote $\mathcal{A}$ the algebra we obtained from the Lemma 2.12. Since A have an uncountable cardinal we can also consider a bijection $\varphi : A \to [0,1]$ and define the set

$$\mathcal{A}_\varphi = \{g : g = f \circ \varphi \text{ with } f \in \mathcal{A}\}.$$

There exists a bijection between $\mathcal{A}_\varphi$ and $\mathcal{A}$. Moreover, we may consider the class $\mathcal{F}$ of functions in $\mathcal{A}_\varphi$ extended by zero to $[0,1]$. This set $\mathcal{F}$ is an algebra that contain $\mathfrak{c}$ algebraically independent generators, due to the fact that φ is a bijection and that $\mathcal{A}$ is strongly $\mathfrak{c}-$algebrable.

For $\beta \in \Lambda$ and $f \in \mathcal{F}$ we define the measurable function $f_\beta = f\mathbf{1}_\beta$, where $\mathbf{1}_\beta$ denotes the characteristic function of β. The nets $\{f_\beta\}_{\beta\in\Lambda}$ are a free algebra with $\mathfrak{c}$ generators. Moreover, for every $f \in \mathcal{F}$ the limit of the net $\{f_\beta\}_{\beta\in\Lambda}$ in the product topology (pointwise convergence) is f which is not measurable because the set of values which make the function f be zero is $[0,1]\backslash A$ attached to a set at most countable. This set is not measurable due to the fact that A is non-measurable and, thus, neither is f. □

Given the partition $T = \{t_i : 0 \le t_1 < \cdots < t_{n-1} < t_n \le 1\}$, we consider now another partition of $[0,1]$ different from T, denoted by T', a partition of $[0,1]$ that is formed by means of adding to T the elements of the form $\frac{3t_i+t_{i+1}}{4}$ and $\frac{t_i+3t_{i+1}}{4}$. For any continuous function g we define the function

$$g_T(x) := \begin{cases} g(x) & \text{if } x \in \left[\frac{3t_i+t_{i+1}}{4}, \frac{t_i+3t_{i+1}}{4}\right], \\ 0 & \text{otherwise.} \end{cases} \tag{2.3}$$

We will represent the Riemann sum of g_T with the partition T' as $S(g_T, T')$, $d(T) = sup\{t_{i+1} - t_i\}$, and A_T the set formed by the union of $\left(\frac{3t_i+t_{i+1}}{4}, \frac{t_i+3t_{i+1}}{4}\right)$.

Lemma 2.14. *If f is a continuous function in $[0,1]$, $\{T_m\}$ a sequence of partitions of $[0,1]$ such that $\{d(T_m)\}$ converges to 0, then $\int f_{T_m} \to \frac{1}{2}\int f$.*

Proof. Let us consider the following Riemann sums

$$S(f,T) = \sum_i f(\alpha_i)(t_{i+1} - t_i),$$

$$S(f_T, T') = \sum_i f(\beta_i)\left(\frac{t_{i+1} - t_i}{2}\right).$$

If we fix $\varepsilon > 0$ and we use Theorem 1.4, we have that there exits $\delta > 0$, such that $d(T) < \delta$ implies that $|f(\alpha_i) - f(\beta_i)| < \frac{\varepsilon}{2}$. Thus, for large m, we have that $|2S(f_{T_m}, T'_m) - S(f, T_m)| < \varepsilon$, i.e. this sequence converges to zero. Then, due to the fact that $S(f, T_m)$ converges to $\int f$ we have that $S(f_{T_m}, T'_m) \to \frac{1}{2}\int f$. Now, if we consider the upper and lower Darboux sums, $\overline{S}, \underline{S}$, we have that $\underline{S}(f_{T_m}, T'_m) \leq S(f_{T_m}, T'_m) \leq \overline{S}(f_{T_m}, T'_m)$ and $\underline{S}(f_{T_m}, T'_m) \leq \int f_{T_m} \leq \overline{S}(f_{T_m}, T'_m)$. Now, thanks to Theorem 1.4, $|\overline{S}(f_{T_m}, T'_m) - \underline{S}(f_{T_m}, T'_m)| < \frac{\varepsilon}{2}$. Therefore, for big enough values of m, we have that

$$\begin{aligned} \left|\int f_{T_m} - \frac{1}{2}\int f\right| &\leq \left|\int f_{T_m} - S(f_{T_m}, T'_m)\right| + \left|S(f_{T_m}, T'_m) - \frac{1}{2}\int f\right| \\ &\leq \left|\overline{S}(f_{T_m}, T'_m) - \underline{S}(f_{T_m}, T'_m)\right| + \left|S(f_{T_m}, T'_m) - \frac{1}{2}\int f\right| < \varepsilon. \end{aligned}$$

This concludes the proof. □

Now, we define the set $\mathcal{S}_8$ of functions $g \in \mathbb{R}^{[0,1]}$ such that g is bounded and the net $\{g_T\}_{T\in\mathcal{T}}$, being g_T is defined by equation (2.3), is pointwise convergent to $\mathbf{0}$ but the Dominated Convergence Theorem fail (the limit of the integrals does not match the integral of the limit in every measurable set J).

Theorem 2.15. $\mathcal{S}_8$ *is strongly* $\mathfrak{c}-$*algebrable.*

Proof. In order to show that $\mathcal{S}_8$ is strongly $\mathfrak{c}-$algebrable, we shall need to apply Theorem 1.1 to $\mathcal{S}_8$, $X = [0, 1]$, and with the function

$$h(t) = \sum_{j=1}^{k} \rho_j t^{n_j} e^{\alpha_j t},$$

with $k \in \mathbb{N}$, $\rho_1, \ldots, \rho_k \in \mathbb{R} \setminus \{0\}$, $(n_1, \alpha_1), \ldots, (n_k, \alpha_k) \in \mathbb{N}\times]0, +\infty[$ pairwise different.

(1) $h \in \mathcal{S}_8$.
   (a) $\{h_T\}_{T\in\mathcal{T}}$ is pointwise convergent to 0. It is enough to show that the net $\{f_T\}_{T\in\mathcal{T}}$ converges to 0 with $f(t) = t$. Observe that the family

$$V_{T,\varepsilon} = \left\{g \in \mathbb{R}^{[0,1]} : |g(t)| < \varepsilon \text{ when } t \in T\right\}$$

   with $T \in \mathcal{T}$ and $\varepsilon > 0$ i a neighborhood basis at 0 for the product topology. In addition, since $f_T(t_i) = 0$ for every $t_i \in T$ we have that $f_T \in V_{T,\varepsilon}$ for every $\varepsilon > 0$ and using that $V_{T^*,\varepsilon} \subset V_{T,\varepsilon}$ for every $\varepsilon > 0$ whenever $T \leq T^*$ we conclude that $\{f_T\}_{T\in\mathcal{T}}$ converges to 0.
   (b) It is clear that h is bounded in $[0, 1]$.
   (c) Let us see that zero is not the limit of $\{\int_J h_T\}_{T\in\mathcal{T}}$ for some measurable set J, i.e. the thesis of Theorem 1.2 is not satisfied for this net. Given $T \in \mathcal{T}$ we can consider the partition T_n, obtained from the union of T with $\left\{\frac{i}{2^n} : 0 \leq i \leq 2^n\right\}$ and denote A_{T_n} the set of the points in which h_{T_n} is different from 0. When n tends to infinity, thanks to the Lemma 2.14, we have that

$$\int_{[0,1]} \sum_{j=1}^{k} \rho_j t^{n_j} e^{\alpha_j t}\, \mathbf{1}_{A_{T_n}}(t)dt \longrightarrow \frac{1}{2}\int_{[0,1]} \sum_{j=1}^{k} \rho_j t^{n_j} e^{\alpha_j t}\, dt.$$

Since $T \in \mathcal{T}$ is arbitrary, this implies that the net $\{\int_{[0,1]} h_T\}_{T\in\mathcal{T}}$ converges to $\frac{1}{2}\int_{[0,1]} h(t)dt$. We can argue similarly in any other interval $J \subset [0,1]$ and, due to the fact that $h(t)$ is not identically zero, we can choose the interval J such that $\int_J h(t)dt \neq 0$ and we have

$$\int_J h_{T_n} \longrightarrow \frac{1}{2}\int_J h(t)dt \neq 0.$$

Therefore Theorem 1.2 is not satisfied for the net $\{\int_J h_T\}_{T\in\mathcal{T}}$.

(2) Also, notice that $h([0,1])$ has, at least, one accumulation point since it is a compact non-trivial interval.

(3) The vector space generated by $h(u)^n e^{\alpha h(u)}$, where n is an integer and $\alpha > 0$, is contained in $\mathcal{S}_8 \bigcup\{0\}$. Let us consider

$$\varphi(u) = \sum_i^m a_i h(u)^{n_i} e^{\alpha_i h(u)}.$$

It is clear that $\varphi(u) \in \mathcal{S}_8$. Indeed,

(a) $\{\varphi_T\}_{T\in\mathcal{T}}$ is pointwise convergent to 0 since $\{h_T\}_{T\in\mathcal{T}}$ is pointwise convergent to 0.

(b) φ is bounded, because h is bounded too.

(c) We can argue, as earlier, with the function $f(t) = t$ replaced by h and, using that $\{h_T\}_{T\in\mathcal{T}}$ does not satisfy Theorem 1.2, we obtain the same for $\{\varphi_T\}_{T\in\mathcal{T}}$.

□

Recall that, as it happens in every topological space, if we have in $\mathbb{R}$ (with the usual topology) a sequence $\{x_n\}$ converging to some $x \in \mathbb{R}$ then we have that the set formed by the union of the elements of this sequence and its limit is compact. However, if we consider a net instead of a sequence, this is not necessarily true in general.

Let us consider the index set $X = \mathbb{Q} \cap [0,1[$ with the usual order and the set $\mathcal{S}_9 \subset \mathbb{R}^X$ of each net $\{x_\beta\}_{\beta\in X}$ converging to $x \in \mathbb{R}$ in such a way that $\{x_\beta\} \cup \{x\}$ is not compact. We have the following result.

Theorem 2.16. *$\mathcal{S}_9$ is strongly $\mathfrak{c}-$algebrable.*

Proof. We apply Theorem 1.1 to the set $X = \mathbb{Q} \cap [0,1[$ and to $g(\beta) = \beta$ for every $\beta \in \mathbb{Q} \cap [0,1[$. Observe that the net $x_\beta = \beta$ converges to 1.

(1) $\{x_\beta\} \in \mathcal{S}_9$ since $\{x_\beta\} \cup \{1\} = \mathbb{Q} \cap [0,1]$ and this is not a closed set.

(2) It is clear that $g(\mathbb{Q} \cap [0,1[) = \mathbb{Q} \cap [0,1[$ and it has accumulation points.

(3) The vector space generated by $x_\beta^n e^{\alpha x_\beta}$, where n is an integer and $\alpha > 0$, is contained in $\mathcal{S}_9 \bigcup\{0\}$. Let us consider the function $\sum_{j=1}^k \rho_j x_\beta^{n_j} e^{\alpha_j x_\beta}$, now we show that it belongs to $\mathcal{S}_9$.

We know that $\{x_\beta\}$ converges to 1, then $\sum_{j=1}^k \rho_j x_\beta^{n_j} e^{\alpha_j x_\beta}$ converges to $\sum_{j=1}^k \rho_j e^{\alpha_j}$. Let us show that the set

$$\left\{\sum_{j=1}^k \rho_j x_\beta^{n_j} e^{\alpha_j x_\beta}\right\} \cup \left\{\sum_{j=1}^k \rho_j e^{\alpha_j}\right\}$$

is not compact.

Notice that the image of $[0,1]$ through the function $f(x)=\sum_{j=1}^{k}\rho_j x^{n_j}e^{\alpha_j x}$, is an interval $[a,b]$ whose interior is not empty and $f(\mathbb{Q}\cap[0,1[)$ is dense in $[a,b]$. Thus,

$$\left\{\sum_{j=1}^{k}\rho_j x_\beta^{n_j}e^{\alpha_j x_\beta}\right\}\bigcup\left\{\sum_{j=1}^{k}\rho_j e^{\alpha_j}\right\}$$

would be closed if it is equal to $[a,b]$. This does not occur since f is absolutely continuous, which implies that $f(\mathbb{Q}\cap[0,1[)$ is a null set (by Theorem 1.5.)

□

We are going to see now the case of linear operators in a dual space such that are sequentially continuous, but they are not continuous. In order to develop this part, we are going to follow the ideas that appear in [19]. Let us recall these ideas briefly for the sake of completion.

Let $B\subset\mathbb{R}^{[0,1]}$ be the set of bounded functions that are null except, possibly, in a countable set points. This space, endowed with the supremum norm, is a Banach space. Moreover,

a) If $T\in B'$, we can find a countable set Λ_T such that $T(x)\neq 0\Rightarrow x(\Lambda_T)\neq 0$. We denote by S_T the class of sets Λ with this property.
b) If $\Lambda_1,\Lambda_2\in S_T$, we have that $T(\mathbf{1}_{\Lambda_1})=T(\mathbf{1}_{\Lambda_2})$. Therefore, we can define the linear functional $\Phi:B'\to\mathbb{R}$ as $\Phi(T)=T(\mathbf{1}_{\Lambda})$ with $\Lambda\in S_T$.
c) Φ is weak-$*$ sequentially continuous, but it is not continuous.
d) For $s\in[0,1]$, $T_s:B\to\mathbb{R}$ defined as $T_s(x)=x(s)$ belongs to B' with $\Phi(T_s)=1$

If we denote by $\mathcal{S}_{10}$ to the set of nets that converges weakly-$*$ to zero, such that its image through the function Φ does not converge to zero, we have the following result:

Theorem 2.17. *$\mathcal{S}_{10}$ is $\mathfrak{c}-$lineable.*

Proof. Let us consider $\mathbb{F}_1(B)$ as the index set, i.e., finite sets $\{x_1,\ldots,x_j,\varepsilon\}$, with $x_i\in B$, $\varepsilon>0$ with the order $\{x_1,\ldots,x_j,\varepsilon\}\leq\left\{x_1',\ldots,x_{j'}',\varepsilon'\right\}$ whenever $\{x_1,\ldots,x_j\}\subseteq\left\{x_1',\ldots,x_{j'}'\right\}$ and $\varepsilon'\leq\varepsilon$. For every $\alpha\in]0,1[$, we define the net

$$T_{\alpha\{x_1,\ldots,x_j,\varepsilon\}}=e^{\alpha s}T_s,$$

where $s\in[0,1]$ is chosen in such a way that

$$x_1(s)=\ldots=x_j(s)=0.$$

In fact, since for every fixed $x_1,\ldots,x_j$, we can choose s in $[0,1]$ except for a countable set $A_{\{x_1,\ldots,x_j\}}$, we choose

$$s=s(x_1,\ldots,x_j,\varepsilon)=:s(\varepsilon)$$

such that $s((a,b))=[0,1]\setminus A_{\{x_1,\ldots,x_j\}}$ for every $0<a<b$, i.e., s is everywhere surjective (see [4,7]) in $[0,1]\setminus A_{\{x_1,\ldots,x_j\}}$.

Let us prove that $\left\{T_{\alpha\{x_1,\ldots,x_j,\varepsilon\}}\right\}$ converges weakly-$*$ to the functional 0. For the weak-$*$ topology we can consider the following neighbourhood basis at the functional zero:

$$V_{\{x_1,\ldots,x_j,\varepsilon\}}=\left\{T\in B':|T(x_i)|<\varepsilon,i=1,\ldots,j\right\}.$$

Observe that for $\{x_1, \ldots, x_j, \varepsilon\} \leq \left\{x'_1, \ldots, x'_{j'}, \varepsilon'\right\}$ we have that $V_{\left\{x'_1,\ldots,x'_{j'},\varepsilon'\right\}} \subset V_{\{x_1,\ldots,x_j,\varepsilon\}}$. Moreover, $T_{\alpha\left\{x'_1,\ldots,x'_j,\varepsilon'\right\}} \in V_{\left\{x'_1,\ldots,x'_{j'},\varepsilon'\right\}}$ since

$$\left|T_{\alpha\left\{x'_1,\ldots,x'_j,\varepsilon'\right\}}(x'_i)\right| = \left|e^{\alpha s} T_s(x'_i)\right| = e^{\alpha s}\left|x'_i(s)\right| = 0 < \varepsilon'.$$

This implies that the net $T_{\alpha\{x_1,\ldots,x_j,\varepsilon\}}$ converges weakly-$*$ to 0.

If we have a linear combination $\left\{\sum a_\alpha T_{\alpha\{x_1,\ldots,x_j,\varepsilon\}}\right\}$ with a finite number of non-trivial coefficients a_α, we have that:

$$\Phi\left(\sum a_\alpha T_{\alpha\{x_1,\ldots,x_j,\varepsilon\}}\right) = \Phi\left(\sum a_\alpha e^{\alpha s(x_1,\ldots,x_j,\varepsilon)} T_{s(x_1,\ldots,x_j,\varepsilon)}\right) = \sum a_\alpha e^{\alpha s(x_1,\ldots,x_j,\varepsilon)}.$$

Due to the fact that $s(x_1, \ldots, x_j, \varepsilon)$ is everywhere surjective in $[0,1] \setminus A_{\{x_1,\ldots,x_j\}}$ we have that $\left\{\sum a_\alpha e^{\alpha s(x_1,\ldots,x_j,\varepsilon)}\right\}$ does not converge to zero. Thus $\left\{\sum a_\alpha T_{\alpha\{x_1,\ldots,x_j,\varepsilon\}}\right\} \in \mathcal{S}_{10}$. Moreover $\left\{\sum a_\alpha T_{\alpha\{x_1,\ldots,x_j,\varepsilon\}}\right\}$ is not equal to zero, because its image does not tend to zero. In particular, $\left\{T_{\alpha\{x_1,\ldots,x_j,\varepsilon\}}\right\}$ are linearly independent. □

To conclude this work, we would like to consider the space $\mathcal{C}([0,1])$ of real valued continuous functions. For our purposes we shall use two different topologies, on one had the one inherited from the product topology in $\mathbb{R}^{[0,1]}$ (denoted by $\mathcal{T}$) and, on the other hand, the one induced by the distance d,

$$d(f,g) = \int_0^1 \frac{|f(x) - g(x)|}{1 + |f(x) - g(x)|} dx$$

which is a distance when restricted to the class of measurable functions.

Recall that the convergence of a sequence in this last metric is equivalent to the weak convergence within the class of measurable functions (random variables within the probability theory framework) on $[0,1]$ with the Borel $\sigma-$algebra. Since we are dealing with continuous functions, we can dismiss the usage of equivalence classes. Within the set of operators from $(\mathbb{R}^{[0,1]}, \mathcal{T})$ to $(\mathbb{R}^{[0,1]}, d)$ we define the standard addition and multiplication by scalars. The product of operators F and G, FG, is defined in a such a way that $FG(f)$ is the function that maps x to $F(f)(x)G(f)(x)$. If we denote by $\mathcal{S}_{11}$ to the set of sequentially continuous operators from $(\mathbb{R}^{[0,1]}, \mathcal{T})$ to $(\mathbb{R}^{[0,1]}, d)$ that are not continuous, we have the following result:

Theorem 2.18. *$\mathcal{S}_{11}$ is strongly $\mathfrak{c}$-algebrable*

Proof. Let us define the operators $T_\alpha : (\mathbb{R}^{[0,1]}, \mathcal{T}) \to (\mathbb{R}^{[0,1]}, d)$ as

$$T_\alpha(f) = f e^{\alpha f}.$$

Let us study the algebra generated by

$$\{T_h : h \in \mathcal{H}\}$$

with $\mathcal{H}$ being a Hamel basis of $\mathbb{R}$ as $\mathbb{Q}$-vector space, whose elements are positive. First, let us see that the T_h's are algebraically independent.

A polynomial P without constant term, $P(T_{h_1}, \ldots, T_{h_n})$, is a finite sum of the form

$$\sum \rho_i f^{n_i} e^{\beta_i f}$$

with $\beta_i \neq \beta_j$ if $i \neq j$.

Let us suppose that this finite sum is such that $\sum \rho_i f^{n_i} e^{\beta_i f} = 0$ for any f. If we take f as the identity, we would have that $\sum \rho_i x^{n_i} e^{\beta_i x} = 0 \ \forall x \in [0,1]$. This cannot happen, since the β_i's are all different (this follows from the Identity Principle).

Let us now see that

$$P(T_{h_1}, \ldots, T_{h_n}) : (\mathbb{R}^{[0,1]}, \mathcal{T}) \to (\mathbb{R}^{[0,1]}, d)$$

is sequentially continuous. Take $\{f_n\}$ converging to a f, that is, $\{f_n(x)\} \to f(x)$. The latter, together with the Dominated Convergence Theorem, tells us that the distances

$$d(f, f_n) = \int_0^1 \frac{|f(x) - f_n(x)|}{1 + |f(x) - f_n(x)|} \mathrm{dx}$$

tend to 0.

Next, we see that it is not continuous. In order to do this, we shall show that the preimage of the open disk of center 0 and radius ϵ is not open for a certain value of ϵ, that shall be decided later on. Let us denote by A_ϵ to this preimage. Since $P(T_{h_1}, \ldots, T_{h_n})(0) = 0$, we have that $0 \in A_\epsilon$ always.

Let us check that A_ϵ does not contain any basic open set containing 0. Assume that A_ϵ actually contains some

$$V(x_0, \ldots, x_m, \delta) := \{f : |f(x_i)| < \delta \text{ with } i = 0, \ldots, m\}.$$

In particular, it would contain all functions vanishing at x_i with $i = 0, \ldots, m$. Suppose that $0 = x_0 \leq \ldots \leq x_m = 1$ and let us define

$$f_{\varepsilon,k}(x) := \begin{cases} \frac{k}{\varepsilon}x - \frac{k}{\varepsilon}x_i & \text{if } x \in [x_i, x_i + \varepsilon] \\ k & \text{if } x \in]x_i + \varepsilon, x_{i+1} - \varepsilon[\\ -\frac{k}{\varepsilon}x + \frac{k}{\varepsilon}x_{i+1} & \text{if } x \in [x_{i+1} - \varepsilon, x_{i+1}] \\ 0 & \text{otherwise.} \end{cases}$$

We have that $P(T_{h_1}, \ldots, T_{h_n})(f_{\varepsilon,k}) \in V(x_0, \ldots, x_m, \delta)$ for every δ and

$$\begin{aligned} d\left(P(T_{h_1}, \ldots, T_{h_n})(f_{\varepsilon,k}), 0\right) &= \int_0^1 \frac{|P(T_{h_1}, \ldots, T_{h_n})(f_{\varepsilon,k})(x)|}{1 + |P(T_{h_1}, \ldots, T_{h_n})(f_{\varepsilon,k})(x)|} \mathrm{dx} \\ &= \int_0^1 \frac{\left|\sum_{i=1}^l \rho_i f_{\varepsilon,k}^{n_i} e^{\beta_i f_{\varepsilon,k}(x)}\right|}{1 + \left|\sum_{i=1}^l \rho_i f_{\varepsilon,k}^{n_i} e^{\beta_i f_{\varepsilon,k}(x)}\right|} \mathrm{dx} \\ &\geq \sum_{s=1}^m \int_{x_i+\varepsilon}^{x_{i+1}-\varepsilon} \frac{\left|\sum_{i=1}^l \rho_i k^{n_i} e^{\beta_i k}\right|}{1 + \left|\sum_{i=1}^l \rho_i k^{n_i} e^{\beta_i k}\right|} \mathrm{dx}. \end{aligned}$$

Since $\left|\sum_{i=1}^l \rho_i k^{n_i} e^{\beta_i k}\right|$ goes to infinity as $k \to \infty$, we have that, from some k_0 on, it is

$$\frac{\left|\sum_{i=1}^l \rho_i k^{n_i} e^{\beta_i k}\right|}{1 + \left|\sum_{i=1}^l \rho_i k^{n_i} e^{\beta_i k}\right|} > 1 - \varepsilon.$$

Therefore, $d\left(P(T_{h_1}, \ldots, T_{h_n})(f_{\varepsilon,k}), 0\right) > (1 - 2m\varepsilon)(1 - \varepsilon)$. That is, if $\epsilon < (1 - 2m\varepsilon)(1 - \varepsilon)$ then $V(x_0, \ldots, x_m, \delta)$ cannot be contained in A_ϵ.

Now, whenever $V(x'_0, \ldots, x'_{m'}, \delta) \subseteq A_\epsilon$ we may take $\{x_0, \ldots, x_m\} = \left\{x'_0, \ldots, x'_{m'}\right\} \cup \{0, 1\}$ and, since $\{x_0, \ldots, x_m\} \supseteq \left\{x'_0, \ldots, x'_{m'}\right\}$ we have $V(x_0, \ldots, x_m, \delta) \subseteq V(x'_0, \ldots, x'_{m'}, \delta) \subset A_\epsilon$

and this is not possible since $P(T_{h_1}, \dots, T_{h_n})(f_{\varepsilon,k}) \in V(x_0, \dots, x_m, \delta) \setminus A_\epsilon$ for ε small enough to assure

$$\epsilon < (1 - 2m\varepsilon)(1 - \varepsilon) = 1 - (1 + 2m)\varepsilon + 2m\varepsilon^2 \quad \text{(for example } \varepsilon < \frac{1 - \epsilon}{1 + 2m}\text{)}.$$

Thus, $P(T_{h_1}, \dots, T_{h_n})$ is not a continuous operator (although it is sequentially continuous). □

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