Research Article

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Regularity of solutions to a fractional elliptic problem with mixed Dirichlet–Neumann boundary data

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Abstract: In this work we study regularity properties of solutions to fractional elliptic problems with mixed Dirichlet–Neumann boundary data when dealing with the *spectral fractional Laplacian*.

Keywords: Fractional Laplacian, mixed boundary conditions, regularity

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1 Introduction

In this paper we study some regularity properties of the solutions to a fractional elliptic problem such as

$$\begin{cases} (-\Delta)^s u = f & \text{in } \Omega, \\ B(u) = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\frac{1}{2} < s < 1$, $f \in L^p(\Omega)$, $p > \frac{N}{2s}$ and Ω is a bounded domain of $\mathbb{R}^N$, $N \geq 1$. Here $(-\Delta)^s$ denotes the *spectral fractional Laplacian* defined through the spectral decomposition with mixed Dirichlet–Neumann boundary condition $B(u)$ (see Section 2 for further details) given by

$$B(u) = u\chi_{\Sigma_{\mathcal{D}}} + \frac{\partial u}{\partial \nu}\chi_{\Sigma_{\mathcal{N}}},$$

where ν is the outwards normal to $\partial\Omega$, χ_A stands for the characteristic function of the set $A \subset \partial\Omega$ and Ω satisfies the following.

Hypotheses 1.1. (1) $\Omega \subset \mathbb{R}^N$ is a bounded Lipschitz domain,
(2) $\Sigma_{\mathcal{D}}$ and $\Sigma_{\mathcal{N}}$ are smooth $(N-1)$-dimensional submanifolds of $\partial\Omega$,
(3) $\Sigma_{\mathcal{D}}$ is a closed manifold of positive $(N-1)$-dimensional Lebesgue measure,
(4) $|\Sigma_{\mathcal{D}}| = \alpha \in (0, |\partial\Omega|)$,
(5) $\Sigma_{\mathcal{D}} \cap \Sigma_{\mathcal{N}} = \emptyset$, $\Sigma_{\mathcal{D}} \cup \Sigma_{\mathcal{N}} = \partial\Omega$ and $\Sigma_{\mathcal{D}} \cap \overline{\Sigma}_{\mathcal{N}} = \Gamma$,
(6) Γ is a smooth $(N-2)$-dimensional submanifold of $\partial\Omega$.

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The main result we prove here is the following.

Theorem 1.2. *Assume Ω satisfies Hypotheses 1.1 and let u be the solution to problem* (1.1) *with $\frac{1}{2} < s < 1$, $f \in L^p(\Omega)$, $p > \frac{N}{2s}$. Then $u \in \mathcal{C}^\gamma(\overline{\Omega})$ for some $0 < \gamma < \frac{1}{2}$. Moreover, there exists a constant $\mathscr{H} = \mathscr{H}(N, s, f, p, |\Sigma_{\mathcal{D}}|) > 0$ such that*

$$|u(x) - u(y)| \le \mathscr{H}|x - y|^\gamma \quad \text{for all } x, y \in \overline{\Omega}.$$

The fact that solutions of elliptic problems with mixed boundary conditions cannot be more regular than Hölder continuous has been observed even for the Laplace operator. Indeed, Shamir (see [14]) observed that $\omega(x, y) = \operatorname{Im}(x + iy)^{1/2}$ solves $-\Delta\omega = 0$ in $\mathbb{R}^2_+$ and satisfies the mixed boundary conditions

$$\lim_{y\to 0} \omega(x, y) = 0, \quad x > 0; \qquad \lim_{y\to 0} \frac{\partial\omega}{\partial y} = 0, \quad x < 0.$$

Hence, the Hölder continuity of order 1/2 is the highest regularity one can expect.

This phenomenon also holds true for the spectral fractional Laplacian as we will show in the proof of Theorem 1.2.

Our approach consists in adapting the classical techniques developed by Stampacchia (see [15]) to (1.1). Due to the nonlocal nature of the problem, some difficulties naturally arise. In order to overcome them, we exploit some ideas contained in [3–5], based on the equivalence between (1.1) and a local degenerate problem set in a cylinder of $\mathbb{R}^{N+1}$. Thus, we use the results of [10] to adapt the procedures of [15] to the case of degenerate elliptic equations with weights in the Muckenhoupt class A_2 (see [10] for the precise definition as well as some useful properties of those weights).

In addition to Theorem 1.2, following some ideas in [8], in the last part of the work we study the behavior of problem (1.1) when we move the boundary condition in a regular way as follows. Given $I_\varepsilon = [\varepsilon, |\partial\Omega|]$ for some $\varepsilon > 0$, let us consider the family of closed sets $\{\Sigma_{\mathcal{D}}(\alpha)\}_{\alpha\in I_\varepsilon}$ satisfying

(B1) $\Sigma_{\mathcal{D}}(\alpha)$ has a finite number of connected components,
(B2) $\Sigma_{\mathcal{D}}(\alpha_1) \subset \Sigma_{\mathcal{D}}(\alpha_2)$ if $\alpha_1 < \alpha_2$,
(B3) $|\Sigma_{\mathcal{D}}(\alpha_1)| = \alpha_1 \in I_\varepsilon$.

We denote $\Sigma_{\mathcal{N}}(\alpha) = \partial\Omega \setminus \Sigma_{\mathcal{D}}(\alpha)$ and $\Gamma(\alpha) = \Sigma_{\mathcal{D}}(\alpha) \cap \overline{\Sigma}_{\mathcal{N}}(\alpha)$. For a family of this type we consider the corresponding family of mixed boundary value problems

$$\begin{cases} (-\Delta)^s u = f & \text{in } \Omega \subset \mathbb{R}^n, \\ B_\alpha(u) = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $B_\alpha(u)$ is the boundary condition associated to the parameter α in the previous hypotheses and the boundary manifolds $\Sigma_{\mathcal{D}}(\alpha)$ and $\Sigma_{\mathcal{N}}(\alpha)$ satisfy the corresponding Hypotheses 1.1. In this scenario, we prove the following result.

Theorem 1.3. *Given Ω and the family $\{\Sigma_{\mathcal{D}}(\alpha)\}_{\alpha\in I_\varepsilon}$ satisfying the corresponding Hypotheses 1.1, and* (B1)–(B3), *let u_α be the solution to* (1.2) *with $\frac{1}{2} < s < 1$, $f \in L^p(\Omega)$ and $p > \frac{N}{2s}$. Then there exist two constants $0 < \gamma < \frac{1}{2}$ and $\mathscr{H}_\varepsilon > 0$, both independent from $\alpha \in [\varepsilon, |\partial\Omega|]$, such that*

$$\|u_\alpha\|_{\mathcal{C}^\gamma(\overline{\Omega})} \le \mathscr{H}_\varepsilon.$$

As we will see in the proof of Theorem 1.3, when one takes $\alpha \to 0^+$, the control of the Hölder norm of such a family is lost. Hence, it is necessary to bound from below the measure of the family $\{\Sigma_{\mathcal{D}}(\alpha)\}_{\alpha\in I_\varepsilon}$, in order to guarantee the control on the Hölder norm for the family $\{u_\alpha\}_{\alpha\in I_\varepsilon}$.

Let us stress that problems related to the spectral fractional Laplacian with mixed boundary conditions are new and, to our knowledge, have been treated only in [6, 7]. We refer to [13] for general properties and several results concerning this spectral fractional Laplacian operator, as well as other different kind of fractional Laplacian operators with Dirichlet boundary conditions.

Since we are considering the spectral fractional Laplacian, the mixed boundary conditions are intrinsic in the functional space where we are working in (see Section 2). If one deals with the singular fractional Laplacian, things change drastically; the Neumann boundary condition has to be prescribed in (a subset of) the complementary of Ω, as it has been clearly explained in [9]. Let us recall, among others, the strong maximum principle [2] and a concave convex type result [1], both for the singular fractional Laplacian with mixed Dirichlet–Neumann boundary conditions.

2 Functional setting and preliminaries

As far as the fractional Laplace operator is concerned, we recall its definition given through the spectral decomposition. Let (φ_i, λ_i) be the eigenfunctions (normalized with respect to the $L^2(\Omega)$-norm) and the eigenvalues of $(-\Delta)$ equipped with homogeneous mixed Dirichlet–Neumann boundary data, respectively. Then (φ_i, λ_i^s) are the eigenfunctions and eigenvalues of the fractional operator $(-\Delta)^s$, where, given $u_i(x) = \sum_{j\geq 1}\langle u_i, \varphi_j\rangle\varphi_j$, $i = 1, 2$,

$$\langle (-\Delta)^s u_1, u_2\rangle = \sum_{j\geq 1} \lambda_j^s \langle u_1, \varphi_j\rangle\langle u_2, \varphi_j\rangle,$$

i.e., the action of the fractional operator on a smooth function u_1 is given by

$$(-\Delta)^s u_1 = \sum_{j\geq 1} \lambda_j^s \langle u_1, \varphi_j\rangle\varphi_j.$$

As a consequence, the fractional Laplace operator $(-\Delta)^s$ is well defined through its spectral decomposition in the following space of functions that vanish on $\Sigma_{\mathcal{D}}$:

$$H^s_{\Sigma_{\mathcal{D}}}(\Omega) = \Big\{ u = \sum_{j\geq 1} a_j\varphi_j \in L^2(\Omega) : \|u\|^2_{H^s_{\Sigma_{\mathcal{D}}}(\Omega)} = \sum_{j\geq 1} a_j^2\lambda_j^s < \infty \Big\}.$$

Observe that since $u \in H^s_{\Sigma_{\mathcal{D}}}(\Omega)$, it follows that

$$\|u\|_{H^s_{\Sigma_{\mathcal{D}}}(\Omega)} = \|(-\Delta)^{s/2}u\|_{L^2(\Omega)}.$$

As it is proved in [12, Theorem 11.1], if $0 < s \leq \frac{1}{2}$, then $H^s_0(\Omega) = H^s(\Omega)$, and therefore also $H^s_{\Sigma_{\mathcal{D}}}(\Omega) = H^s(\Omega)$, while for $\frac{1}{2} < s < 1$, $H^s_0(\Omega) \subsetneq H^s(\Omega)$. Hence, the range $\frac{1}{2} < s < 1$ guarantees that $H^s_{\Sigma_{\mathcal{D}}}(\Omega) \subsetneq H^s(\Omega)$ and it provides us the correct functional space to study the mixed boundary problem (1.1).

This definition of the fractional powers of the Laplace operator allows us to integrate by parts in the appropriate spaces, so that a natural definition of weak solution to problem (1.1) is the following.

Definition 2.1. We say that $u \in H^s_{\Sigma_{\mathcal{D}}}(\Omega)$ is a solution to (1.1) if

$$\int_\Omega (-\Delta)^{s/2}u(-\Delta)^{s/2}\psi\,dx = \int_\Omega f\psi\,dx \quad \text{for any } \psi \in H^s_{\Sigma_{\mathcal{D}}}(\Omega).$$

Due to the nonlocal nature of the fractional operator $(-\Delta)^s$ some difficulties arise when one tries to obtain an explicit expression of the action of the fractional Laplacian on a given function. In order to overcome this difficulty, we use the ideas by Caffarelli and Silvestre (see [5]) together with those of [3, 4] to give an equivalent definition of the operator $(-\Delta)^s$ by means of an auxiliary problem that we introduce next.

Given any domain $\Omega \subset \mathbb{R}^N$, we set the cylinder $\mathscr{C}_\Omega = \Omega \times (0, \infty) \subset \mathbb{R}^{N+1}_+$. We denote by (x, y) those points that belong to $\mathscr{C}_\Omega$ and by $\partial_L\mathscr{C}_\Omega = \partial\Omega \times [0, \infty)$ the lateral boundary of the cylinder. Let us also denote $\Sigma^*_{\mathcal{D}} = \Sigma_{\mathcal{D}} \times [0, \infty)$ and $\Sigma^*_{\mathcal{N}} = \Sigma_{\mathcal{N}} \times [0, \infty)$, as well as $\Gamma^* = \Gamma \times [0, \infty)$. It is clear that, by construction,

$$\Sigma^*_{\mathcal{D}} \cap \Sigma^*_{\mathcal{N}} = \emptyset, \quad \Sigma^*_{\mathcal{D}} \cup \Sigma^*_{\mathcal{N}} = \partial_L\mathscr{C}_\Omega \quad \text{and} \quad \Sigma^*_{\mathcal{D}} \cap \overline{\Sigma^*_{\mathcal{N}}} = \Gamma^*.$$

Given a function $u \in H^s_{\Sigma_{\mathcal{D}}}(\Omega)$, we define its s-harmonic extension function, denoted by $U(x, y) = E_s[u(x)]$, as the solution to the problem

$$\begin{cases} -\operatorname{div}(y^{1-2s}\nabla U(x, y)) = 0 & \text{in } \mathscr{C}_\Omega, \\ B(U(x, y)) = 0 & \text{on } \partial_L\mathscr{C}_\Omega, \\ U(x, 0) = u(x) & \text{on } \Omega \times \{y = 0\}, \end{cases}$$

where

$$B(U) = U\chi_{\Sigma_{\mathcal{D}}^*} + \frac{\partial U}{\partial \nu}\chi_{\Sigma_{\mathcal{N}}^*},$$

being ν, with an abuse of notation[1], the outwards normal to $\partial_L \mathscr{C}_\Omega$. Following the well known result by Caffarelli and Silvestre (see [5]), U is related to the fractional Laplacian of the original function through the formula

$$\frac{\partial U}{\partial \nu^s} := -\kappa_s \lim_{y\to 0^+} y^{1-2s}\frac{\partial U}{\partial y} = (-\Delta)^s u(x),$$

where κ_s is a suitable positive constant (see [3] for its exact value). The extension function belongs to the space

$$\mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathscr{C}_\Omega) := \overline{\mathcal{C}_0^\infty((\Omega\cup\Sigma_{\mathcal{N}})\times[0,\infty))}^{\|\cdot\|_{\mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathscr{C}_\Omega)}},$$

where we define

$$\|\cdot\|^2_{\mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathscr{C}_\Omega)} := \kappa_s \int_{\mathscr{C}_\Omega} y^{1-2s}|\nabla(\cdot)|^2\,dx\,dy.$$

Note that $\mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathscr{C}_\Omega)$ is a Hilbert space equipped with the norm $\|\cdot\|_{\mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathscr{C}_\Omega)}$ which is induced by the scalar product

$$\langle U, V\rangle_{\mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathscr{C}_\Omega)} = \kappa_s \int_{\mathscr{C}_\Omega} y^{1-2s}\langle \nabla U, \nabla V\rangle\,dx\,dy.$$

Moreover, the following inclusions are satisfied:

$$\mathcal{X}^s_0(\mathscr{C}_\Omega) \subset \mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathscr{C}_\Omega) \subsetneq \mathcal{X}^s(\mathscr{C}_\Omega), \tag{2.1}$$

with $\mathcal{X}^s_0(\mathscr{C}_\Omega)$ being the space of functions that belongs to $\mathcal{X}^s(\mathscr{C}_\Omega) \equiv H^1(\mathscr{C}_\Omega, y^{1-2s}\,dx\,dy)$ and vanish on the lateral boundary of $\mathscr{C}_\Omega$.

Using the above arguments, we can reformulate problem (1.1) in terms of the extension problem as follows:

$$\begin{cases} -\operatorname{div}(y^{1-2s}\nabla U) = 0 & \text{in } \mathscr{C}_\Omega, \\ B(U) = 0 & \text{on } \partial_L\mathscr{C}_\Omega, \\ \dfrac{\partial U}{\partial \nu^s} = f & \text{on } \Omega\times\{y=0\}, \end{cases} \tag{2.2}$$

and we have that $u(x) = U(x,0)$.

Next, we specify the meaning of solution to problem (2.2) and its relationship with the solutions to problem (1.1).

Definition 2.2. An energy solution to problem (2.2) is a function $U \in \mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathscr{C}_\Omega)$ such that

$$\kappa_s \int_{\mathscr{C}_\Omega} y^{1-2s}\langle \nabla U, \nabla\varphi\rangle\;dx\,dy = \int_\Omega f(x)\varphi(x,0)\,dx \quad \text{for all } \varphi \in \mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathscr{C}_\Omega). \tag{2.3}$$

If $U \in \mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathscr{C}_\Omega)$ is the solution to problem (2.2), we can associate the function $u(x) = \operatorname{Tr}[U(x,y)] = U(x,0)$, that belongs to $H^s_{\Sigma_{\mathcal{D}}}(\Omega)$, and solves problem (1.1). Moreover, also the converse is true: given the solution $u \in H^s_{\Sigma_{\mathcal{D}}}(\Omega)$ to (1.1), its s-harmonic extension $U = E_s[u(x)] \in \mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathscr{C}_\Omega)$ is the solution to (2.2). Thus, both formulations are equivalent and the *extension operator*

$$E_s : H^s_{\Sigma_{\mathcal{D}}}(\Omega) \to \mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathscr{C}_\Omega)$$

allows us to switch between each other.

According to [3, 5], due to the choice of the constant κ_s, the extension operator E_s is an isometry, i.e.,

$$\|E_s[\varphi](x,y)\|_{\mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathscr{C}_\Omega)} = \|\varphi(x)\|_{H^s_{\Sigma_{\mathcal{D}}}(\Omega)} \quad \text{for all } \varphi \in H^s_{\Sigma_{\mathcal{D}}}(\Omega). \tag{2.4}$$

1 Let ν be the outwards normal to $\partial\Omega$ and $\nu_{(x,y)}$ the outwards normal to $\mathscr{C}_\Omega$; then, by construction, $\nu_{(x,y)} = (\nu, 0)$, $y > 0$.

Let us also recall the *trace inequality*, that is a useful tool we exploit in many proofs in this paper (see [3]): there exists $C = C(N, s, r, |\Omega|)$ such that for all $z \in \mathcal{X}_0^s(\mathscr{C}_\Omega)$,

$$C\Big(\int_\Omega |z(x,0)|^r\,dx\Big)^{2/r} \le \int_{\mathscr{C}_\Omega} y^{1-2s}|\nabla z(x,y)|^2\,dx\,dy,$$

with $1 \le r \le 2_s^*$, $N > 2s$, and $2_s^* = \frac{2N}{N-2s}$.

Observe that such inequality turns out to be, in fact, equivalent to the fractional Sobolev inequality:

$$C\Big(\int_\Omega |v|^r\,dx\Big)^{2/r} \le \int_\Omega |(-\Delta)^{s/2}v|^2\,dx \quad \text{for all } v \in H_0^s(\Omega),\ 1 \le r \le 2_s^*,\ N > 2s.$$

When mixed boundary conditions are considered, the situation is quite similar, since the Dirichlet condition is imposed on a set $\Sigma_{\mathcal{D}} \subset \partial\Omega$ such that $|\Sigma_{\mathcal{D}}| = \alpha > 0$. Hence, thanks to (2.1), there exists a positive constant $C_{\mathcal{D}} = C_{\mathcal{D}}(N, s, |\Sigma_{\mathcal{D}}|)$ such that

$$0 < \inf_{\substack{u \in H^s_{\Sigma_{\mathcal{D}}}(\Omega)\\ u \not\equiv 0}} \frac{\|u\|^2_{H^s_{\Sigma_{\mathcal{D}}}(\Omega)}}{\|u\|^2_{L^{2_s^*}(\Omega)}} := C_{\mathcal{D}} < \inf_{\substack{u \in H^s_0(\Omega)\\ u \not\equiv 0}} \frac{\|u\|^2_{H^s_0(\Omega)}}{\|u\|^2_{L^{2_s^*}(\Omega)}}. \tag{2.5}$$

Remark 2.3. It is worth to observe (see [7]) that $C_{\mathcal{D}}(N, s, |\Sigma_{\mathcal{D}}|) \le 2^{-2s/N} C(N, s, 2_s^*)$. Moreover, having in mind the spectral definition of the fractional operator, by the Hölder inequality, it follows that $C_{\mathcal{D}} \le |\Omega|^{2s/N}\lambda_1^s(\alpha)$, where $\lambda_1(\alpha)$ denotes the first eigenvalue of the Laplace operator with mixed boundary conditions on the sets $\Sigma_{\mathcal{D}} = \Sigma_{\mathcal{D}}(\alpha)$ and $\Sigma_{\mathcal{N}} = \Sigma_{\mathcal{N}}(\alpha)$. Since $\lambda_1(\alpha) \to 0$ as $\alpha \to 0^+$, see [8, Lemma 4.3], we conclude that $C_{\mathcal{D}} \to 0$ as $\alpha \to 0^+$.

Gathering together (2.4) and (2.5), we obtain

$$C_{\mathcal{D}}\Big(\int_\Omega |\varphi(x,0)|^{2_s^*}\,dx\Big)^{2/2_s^*} \le \|\varphi(x,0)\|^2_{H^s_{\Sigma_{\mathcal{D}}}(\Omega)} = \|E_s[\varphi(x,0)]\|^2_{\mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathscr{C}_\Omega)}. \tag{2.6}$$

With this Sobolev-type inequality in hand, we can prove a trace inequality adapted to the mixed boundary data framework.

Lemma 2.4. *There exists a constant $C_{\mathcal{D}} = C_{\mathcal{D}}(N, s, |\Sigma_{\mathcal{D}}|) > 0$ such that*

$$C_{\mathcal{D}}\Big(\int_\Omega |\varphi(x,0)|^{2_s^*})\,dx\Big)^{2/2_s^*} \le \int_{\mathscr{C}_\Omega} y^{1-2s}|\nabla\varphi|^2\,dx\,dy \quad \text{for all } \varphi \in \mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathscr{C}_\Omega). \tag{2.7}$$

Proof. Thanks to (2.6), it is enough to prove that $\|E_s[\varphi(\cdot,0)]\|_{\mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathscr{C}_\Omega)} \le \|\varphi\|_{\mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathscr{C}_\Omega)}$. This inequality is satisfied, since, arguing as in [3], we find

$$\begin{aligned}
\|\varphi\|^2_{\mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathscr{C}_\Omega)} &:= \kappa_s \int_{\mathscr{C}_\Omega} y^{1-2s}|\nabla\varphi|^2\,dx\,dy\\
&= \kappa_s \int_{\mathscr{C}_\Omega} y^{1-2s}\big|\nabla\big(E_s[\varphi(x,0)] + \varphi(x,y) - E_s[\varphi(x,0)]\big)\big|^2\,dx\,dy\\
&= \|E_s[\varphi(x,0)]\|^2_{\mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathscr{C}_\Omega)} + \|\varphi(x,y) - E(\varphi(x,0))\|^2_{\mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathscr{C}_\Omega)}\\
&\qquad + 2\kappa_s \int_{\mathscr{C}_\Omega} y^{1-2s}\big\langle \nabla E_s[\varphi(x,0)], \nabla(\varphi(x,y) - E_s[\varphi(x,0)])\big\rangle\,dx\,dy\\
&= \|E_s[\varphi(x,0)]\|^2_{\mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathscr{C}_\Omega)} + \|\varphi(x,y) - E_s[\varphi(x,0)]\|^2_{\mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathscr{C}_\Omega)}\\
&\qquad + 2\int_\Omega (-\Delta)^s(\varphi(x,0))(\varphi(x,0) - \varphi(x,0))\,dx\\
&= \|E_s[\varphi(x,0)]\|^2_{\mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathscr{C}_\Omega)} + \|\varphi(x,y) - E_s[\varphi(x,0)]\|^2_{\mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathscr{C}_\Omega)}.
\end{aligned}$$

□

3 Hölder regularity

The principal result we prove in this section is Theorem 1.2, which deals with the Hölder regularity of the solution to problem (1.1). First we introduce the notation that we will follow along this section.

Notation. Given an open bounded set Ω, $x \in \overline{\Omega} \subset \mathbb{R}^N$ and $X \in \overline{\mathscr{C}}_\Omega \subset \mathbb{R}^{N+1}_+$, we define
- $\Omega(x,\rho) = \Omega \cap B_\rho(x)$,
- $\mathscr{C}_\Omega(X,\rho) = \mathscr{C}_\Omega \cap B_\rho(X)$.

Given $u(x) \in H^s_{\Sigma_{\mathcal{D}}}(\Omega)$ and $U(X) \in \mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathscr{C}_\Omega)$, let us also define
- $A_+(k) = \{x \in \Omega : u(x) > k\}$,
- $A^*_+(k) = \{X \in \mathscr{C}_\Omega : U(X) > k\}$,
- $A_+(k,\rho) = A_+(k) \cap \Omega(x,\rho)$,
- $A^*_+(k,\rho) = A^*_+(k) \cap \mathscr{C}_\Omega(X,\rho)$,
- $\{\cdot\}^k = \min(\cdot, k)$,
- $\{\cdot\}_k = \max(\cdot, k)$.

In a similar way, we may define the sets $A_-(k)$, $A^*_-(k)$, $A_-(k,\rho)$ and $A^*_-(k,\rho)$, replacing ">" with "<" in the latter definitions. We denote by
- $|A|_\omega$ the measure induced by a weight ω of the set A,
- $|A|_{y^{1-2s}}$ the measure induced by the weight y^{1-2s} of the set A,
- $|A|$ the usual Lebesgue measure of the set A.

On the regularity of Ω

Let us recall that Ω is assumed, in all the paper, to be Lipschitz and consequently also $\mathcal{C}_\Omega$ turns out to have the same regularity. In particular, among others, we use the following properties. There exists $\zeta \in (0,1)$ such that for $z \in \overline{\Omega}$, some $R > 0$ and any $\rho > 0$,

$$|\mathscr{C}_{\Omega(z,R)}(Z,\rho)| \geq \zeta |B_\rho(Z)| \quad \text{for all } Z \in \mathscr{C}_{\Omega(z,R)}. \tag{3.1}$$

Moreover, also the weighted counterpart is true, i.e., there exists $\zeta_s \in (0,1)$ such that for any $z \in \overline{\Omega}$ and any $\rho > 0$,

$$|\mathscr{C}_{\Omega(z,R)}(Z,\rho)|_{y^{1-2s}} \geq \zeta_s |B_\rho(Z)|_{y^{1-2s}} \quad \text{for all } Z \in \mathscr{C}_{\Omega(z,R)}. \tag{3.2}$$

Consequently, given $z \in \overline{\Omega}$, $R > 0$ and $0 < r < R$, there exists $\lambda > 0$ such that

$$|A^*_+(k,r)|_{y^{1-2s}} \leq \lambda |\mathscr{C}_{\Omega(z,R)}(Z,r)|_{y^{1-2s}} \quad \text{for all } Z \in \mathscr{C}_{\Omega(z,R)}. \tag{3.3}$$

It is worth to observe that all the results we prove in this paper might be proved for a larger class of open sets Ω. Indeed, following [15], this kind of results is true for the so called $\frac{1}{2}$-admissible domains. Here we decided to not deal with such domains for brevity and in order to not make the proofs much heavier.

Now we are ready to start with the statement and the proofs of several technical results.

Let $z \in \overline{\Omega}$ and $R > 0$, and let u be a solution to problem (1.1): we write $u(x) = v(x) + w(x)$ for every $x \in \Omega(z,R)$, where the function $v(x)$ satisfies

$$\begin{cases} (-\Delta)^s v = f & \text{in } \Omega(z,R), \\ v = 0 & \text{on } \widetilde{\Sigma}_{\mathcal{D},R} := \partial\Omega(z,R) \setminus \Sigma_{\mathcal{N}}, \\ \dfrac{\partial v}{\partial \nu} = 0 & \text{on } \widetilde{\Sigma}_{\mathcal{N},R} := \partial\Omega(z,R) \cap \Sigma_{\mathcal{N}}, \end{cases} \tag{3.4}$$

and the function $w(x)$ is such that

$$\begin{cases} (-\Delta)^s w = 0 & \text{in } \Omega(z,R), \\ w = 0 & \text{on } \Sigma_{\mathcal{D},R} := \Sigma_{\mathcal{D}} \cap B_R(z), \\ \dfrac{\partial w}{\partial \nu} = 0 & \text{on } \Sigma_{\mathcal{N},R} := \Sigma_{\mathcal{N}} \cap B_R(z). \end{cases} \tag{3.5}$$

Using the extension technique, we can write $v(x) = V(x,0)$, with $V(x,y)$ solving the extended problem

$$\begin{cases} -\operatorname{div}(y^{1-2s}\nabla V) = 0 & \text{in } \mathscr{C}_{\Omega(z,R)}, \\ B(V) = 0 & \text{on } \partial_L \mathscr{C}_{\Omega(z,R)}, \\ \dfrac{\partial V}{\partial \nu^s} = f & \text{on } \Omega(z,R) \times \{y = 0\}, \end{cases} \tag{3.6}$$

where $B(V) = V\chi_{\widetilde{\Sigma}^*_{\mathcal{D},R}} + \frac{\partial V}{\partial \nu}\chi_{\widetilde{\Sigma}^*_{\mathcal{D},R}}$, with $\widetilde{\Sigma}^*_{\mathcal{D},R} = \widetilde{\Sigma}_{\mathcal{D},R} \times [0,\infty)$ and $\widetilde{\Sigma}^*_{\mathcal{N},R} = \widetilde{\Sigma}_{\mathcal{N},R} \times [0,\infty)$.

In the same way, we write $w(x) = W(x,0)$, with $W(x,y)$ satisfying the extended problem

$$\begin{cases} -\operatorname{div}(y^{1-2s}\nabla W) = 0 & \text{in } \mathscr{C}_{\Omega(z,R)}, \\ B(W) = 0 & \text{on } \Sigma^*_{\mathcal{D},R} \cup \Sigma^*_{\mathcal{N},R}, \\ \dfrac{\partial W}{\partial \nu^s} = 0 & \text{on } \Omega(z,R) \times \{y = 0\}, \end{cases} \tag{3.7}$$

where $B(V) = V\chi_{\Sigma^*_{\mathcal{D},R}} + \frac{\partial V}{\partial \nu}\chi_{\Sigma^*_{\mathcal{D},R}}$, with $\Sigma^*_{\mathcal{D},R} = \Sigma_{\mathcal{D},R} \times [0,\infty)$ and $\Sigma^*_{\mathcal{N},R} = \Sigma_{\mathcal{N},R} \times [0,\infty)$.

Let us observe that we have the following situations:

(i) If $z \in \Omega$, there exists $R > 0$ such that $\widetilde{\Sigma}_{\mathcal{D},R} = \partial\Omega(z,R)$ and $\Sigma_{\mathcal{D},R} = \Sigma_{\mathcal{N},R} = \emptyset$. Then $v \in H^s_0(\Omega(z,R))$ and it is solution to a Dirichlet problem. Moreover, w is an s-harmonic function, i.e., its extension $W = E_s[w] \in \mathcal{X}^s(\mathscr{C}_{\Omega(z,R)})$ satisfies

$$\int_{\mathscr{C}_{\Omega(z,R)}} y^{1-2s}\langle \nabla W, \nabla \Phi\rangle \, dx\, dy = 0 \quad \text{for all } \Phi \in \mathcal{X}^s_0(\mathscr{C}_{\Omega(z,R)}). \tag{3.8}$$

(ii) If $z \in \Sigma_{\mathcal{D}} \setminus \Gamma$, there exists $R > 0$ such that $\widetilde{\Sigma}_{\mathcal{D},R} = \partial\Omega(z,R)$ and $\Sigma_{\mathcal{N},R} = \emptyset$. Then $v \in H^s_0(\Omega(z,R))$ and it is a solution to a Dirichlet problem while $W \in \mathcal{X}^s_{\Sigma_{\mathcal{D},R}}(\mathscr{C}_{\Omega(z,R)})$ and, also in this case, it satisfies (3.8).

(iii) If $z \in \Sigma_{\mathcal{N}}$, there exists $R > 0$ such that $\Sigma_{\mathcal{D},R} = \emptyset$. Then the function $v \in H^s_{\widetilde{\Sigma}_{\mathcal{D},R}}(\Omega(z,R))$ and it is a solution to the mixed problem (3.4). Moreover, W belongs to $\mathcal{X}^s(\mathscr{C}_{\Omega(z,R)})$ and (3.8) holds for all $\Phi \in \mathcal{X}^s(\mathscr{C}_{\Omega(z,R)})$ vanishing on $\partial_L\mathscr{C}_{\Omega(z,R)} \setminus \Sigma^*_{\mathcal{N},R}$.

(iv) Finally, if $z \in \Gamma$, the sets $\widetilde{\Sigma}_{\mathcal{D},R}$, $\widetilde{\Sigma}_{\mathcal{N},R}$, $\Sigma_{\mathcal{D},R}$ and $\Sigma_{\mathcal{D},R}$ are nonempty for all $R > 0$. Then the function $v \in H^s_{\widetilde{\Sigma}_{\mathcal{D},R}}(\Omega(z,R))$ and it is a solution to the mixed problem (3.4); as far as w is concerned, $W \in \mathcal{X}^s_{\Sigma_{\mathcal{D},R}}(\mathscr{C}_{\Omega(z,R)})$ and fulfills (3.8) for any $\Phi \in \mathcal{X}^s(\mathscr{C}_{\Omega(z,R)})$ vanishing on $\partial_L\mathscr{C}_{\Omega(z,R)} \setminus \Sigma^*_{\mathcal{N},R}$.

We also define the following sets that will be useful in the sequel:

- $\mathscr{C}^\circ_{\Omega(z,R)} = \overline{\mathscr{C}}_{\Omega(z,R)} \setminus \{(x,y) \in \mathscr{C}_{\Omega(z,R)} : x \in \partial B_R(z)\}$,
- $\partial_0\mathscr{C}_{\Omega(z,R)} = \partial_L\mathscr{C}_{\Omega(z,R)} \setminus \Sigma^*_{\mathcal{N},R}$,
- $\partial_B\mathscr{C}_{\Omega(z,R)} = \partial_L\mathscr{C}_{\Omega(z,R)} \setminus (\Sigma^*_{\mathcal{D},R} \cup \Sigma^*_{\mathcal{N},R})$.

We continue by stating the definitions and results needed in what follows. The first definition is based on [15, Definition 2.1].

Definition 3.1. Given any $z_0 \in \overline{\Omega}$ and $Z \in \mathscr{C}^\circ_{\Omega(z_0,R)}$, let $\mathcal{K}^+(Z)$ (resp. $\mathcal{K}^-(Z)$) be the set of values $k \in \mathbb{R}$ such that there exists a number $\widetilde{\rho}(Z) > 0$ satisfying $\{U\}^k\eta \in \mathcal{X}^s_{\partial_0\mathscr{C}_{\Omega(z_0,R)}}(\mathscr{C}_{\Omega(z_0,R)})$ (resp. $\{U\}_k\,\eta \in \mathcal{X}^s_{\partial_0\mathscr{C}_{\Omega(z_0,R)}}(\mathscr{C}_{\Omega(z_0,R)})$) for any $U \in \mathcal{X}^s_{\Sigma_{\mathcal{D},R}}(\mathscr{C}_{\Omega(z_0,R)})$ and any function $\eta \in C^\infty(\mathbb{R}^{N+1}_+)$ such that $\operatorname{supp}(\eta) \subset B_{\widetilde{\rho}(Z)}(Z)$.

Remark 3.2. It is worth to observe the following:

- If $Z \in \Sigma^*_{\mathcal{D},R}$, then $\mathcal{K}^+(Z) = [0,\infty)$, $\mathcal{K}^-(Z) = (-\infty,0]$ and $\widetilde{\rho}(Z) = \operatorname{dist}(Z, \partial_B\mathscr{C}_{\Omega(z,R)})$.
- If $Z \in \mathscr{C}^\circ_{\Omega(z,R)} \setminus \Sigma^*_{\mathcal{D},R}$, then $\mathcal{K}^+(Z) = \mathcal{K}^-(Z) = (-\infty,\infty)$ and $\widetilde{\rho}(Z) = \operatorname{dist}(Z, \partial_0\mathscr{C}_{\Omega(z,R)})$.
- Thanks to the construction of the cylinder, it is immediate to notice that the number $\widetilde{\rho}(Z) > 0$ does not depend on the y variable.

The control of the oscillations of solutions of elliptic problems is usually carried out through integral estimates that mainly rely on a Sobolev-type inequality. Since the extension function solves a degenerate elliptic problem involving a weight (namely, y^{1-2s}) that belongs to the Muckenhoupt class A_2, it is necessary to establish a Sobolev-type inequality dealing with such a type of singular weights. To this end, we recall the following definition.

Definition 3.3. Given an open subset $D \subset \mathbb{R}^N$ and a function $\omega\colon D \to \mathbb{R}^+$, we say that ω belongs to the Muckenhoupt class A_p, with $p > 1$, if there exists a constant $C > 0$ such that

$$\sup_{B\subset D}\Big(\frac{1}{|B|}\int_B \omega\Big)\Big(\frac{1}{|B|}\int_B \omega^{-1/(p-1)}\Big)^{p-1} \le C.$$

Now we can recall the following result.

Theorem 3.4 ([10, Theorem 1.3 and Theorem 1.6]). *Let D be an open bounded Lipschitz set in $\mathbb{R}^N$ and consider $1 < p < \infty$ and a weight $\omega \in A_p$. Then there exist a positive constant $C(D)$ and $\delta > 0$ such that for all $u \in H_0^1(D, \omega)$ and any $1 \le \sigma \le \frac{N}{N-1} + \delta$, we have*

$$\|u\|_{L^{\sigma p}(D,\omega dx)} \le C(D)\|\nabla u\|_{L^p(D,\omega dx)}, \tag{3.9}$$

where $C(D) = c_\omega \operatorname{diam}(D)|D|_\omega^{1/p(1/\sigma-1)}$ for a positive constant c_ω depending on N, p and ω.

Moreover, for any $x_0 \in \partial D$, there exist a positive constant $C = C(B_\rho(x_0))$ and $\delta > 0$ such that for any $1 \le \sigma \le \frac{N}{N-1} + \delta$ and any $u \in H^1(D(x_0,\rho),\omega)$ vanishing on $\partial D \cap B_\rho(x_0)$, we have

$$\|u\|_{L^{\sigma p}(D(x_0,\rho),\omega dx)} \le C(B_\rho)\|\nabla u\|_{L^p((D(x_0,\rho),\omega dx)},$$

where $C(B_\rho) = c_\omega \rho |B_\rho|_\omega^{1/p(1/\sigma-1)}$ for a positive constant c_ω depending on ω, N, p and ξ.

We want to apply such a theorem to domains $D \subsetneq \mathscr{C}_\Omega \subset \mathbb{R}_+^{N+1}$ so that the correspondent exponent σ satisfies $1 \le \sigma \le \frac{N+1}{N}$.

As far as the weight is concerned, we set $\omega = y^{1-2s}$, that actually belongs to A_2. Let us observe that, according to [10], there exists $\varepsilon_0 > 0$ such that (3.9) holds true with $p \ge 2 - \varepsilon_0$.

As an immediate consequence of Theorem 3.4, we obtain the following result.

Lemma 3.5. *Let $Z \in \Sigma_{\mathcal{D}}^*$ and $p \ge 2 - \varepsilon_0$ for some $\varepsilon_0 > 0$. Then there exists $\overline{\rho} > 0$ such that for any $\rho < \overline{\rho}$ and any $U \in \mathcal{X}_{\Sigma_{\mathcal{D}}}^s(\mathscr{C}_\Omega)$, we have*

$$\|U\|_{L^{\sigma p}(\mathscr{C}_\Omega(Z,\rho),y^{1-2s}\,dx\,dy)} \le c_s \rho |B_\rho|_{y^{1-2s}}^{1/p(1/\sigma-1)} \|\nabla U\|_{L^p(\mathscr{C}_\Omega(Z,\rho),y^{1-2s}\,dx\,dy)}, \tag{3.10}$$

with $1 \le \sigma \le \frac{N+1}{N} + \delta$ for some $\delta > 0$ and c_s depending on N, p and the weight y^{1-2s}.

Next we establish inequality (3.10) for functions in $\mathcal{X}_{\Sigma_{\mathcal{D},R}}^s(\mathscr{C}_{\Omega(z,R)})$ and, given some point $Z \in \mathscr{C}_{\Omega(z,R)}^\circ \setminus \Sigma_{\mathcal{D},R}^*$, also for functions in $H^1(\mathscr{C}_\Omega(Z,\rho), y^{1-2s}\,dx\,dy)$ vanishing on suitable sets.

Definition 3.6. Given $p \ge 2 - \varepsilon_0$ for some $\varepsilon_0 \in (0,1)$ and an open bounded set A, we define $\mathcal{F}(\beta_s, A)$ as the family of sets $B \subset \overline{A}$ such that for any $U \in H^1(A, y^{1-2s}\,dx\,dy)$ vanishing on B,

$$\|U\|_{L^{\sigma p}(A,y^{1-2s}\,dx\,dy)} \le \beta_s \operatorname{diam}(A)|A|_{y^{1-2s}}^{1/p(1/\sigma-1)} \|\nabla U\|_{L^p(A,y^{1-2s}\,dx\,dy)} \tag{3.11}$$

for some $\beta_s > 0$ depending on N, p and the weight y^{1-2s}, and $1 \le \sigma \le \frac{N+1}{N} + \delta$ for some $\delta > 0$.

With this scheme in mind, we focus first on finding bounds for solutions to (3.4) in terms of the data of the problem.

Theorem 3.7. *Let u be a solution to (1.1), with $f \in L^p(\Omega)$, $p > \frac{N}{2s}$. Then there exists a positive constant $C = C(N, s, |\Sigma_{\mathcal{D}}|)$ such that*

$$\|u\|_{L^\infty(\Omega)} \le C\|f\|_{L^p(\Omega)}|\Omega|^{2s/N-1/p}.$$

In the proof of Theorem 3.7, we make use of the following technical result.

Lemma 3.8 ([11, Lemma B.1]). *Let $\varphi(k)$ be a nonnegative and nonincreasing function, defined for $k \ge k_0$, such that*

$$\varphi(h) \le \frac{C_0}{(h-k)^a}\varphi^b(k), \quad k < h,$$

where C_0, a, b are positive constants with $b > 1$. Then $\varphi(k_0 + d) = 0$, with $d^a = 2^{ab/(b-1)} C_0 |\varphi(k_0)|^{b-1}$.

Proof of Theorem 3.7. Here we just prove the upper bound, being the lower one completely analogous. Let us take $k \geq 0$, $U(x,y) = E_s[u(x)]$ and $\psi = (U-k)_+ \in \mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathscr{C}_\Omega)$ as a test function in (2.3). Using the trace inequality (2.7) together with the Hölder inequality, we get

$$\kappa_s \int_{\mathscr{C}_\Omega} y^{1-2s}\nabla U \nabla \psi \, dx\, dy = \kappa_s \int_{A_+^*(k)} y^{1-2s}|\nabla U|^2 \, dx\, dy = \int_{A_+(k)} (U(x,0)-k)f(x)\, dx$$
$$\leq \Big(\int_{A_+(k)} |f|^2 \, dx\Big)^{1/2}\Big(C_{\mathcal{D}}^{-1}|A_+(k)|^{2s/N}\int_{A_+^*(k)} y^{1-2s}|\nabla U|^2\, dx\, dy\Big)^{1/2}.$$

Thus,

$$\int_{A_+^*(k)} y^{1-2s}|\nabla U|^2\, dx\, dy \leq C_{\mathcal{D}}^{-1}\kappa_s^{-2}|A_+(k)|^{2s/N}\int_{A_+(k)} |f|^2\, dx \leq \frac{\|f\|^2_{L^p(\Omega)}|A_+(k)|^{1-2/p+2s/N}}{C_{\mathcal{D}}\kappa_s^2}, \tag{3.12}$$

and applying the trace inequality (2.7) to the left-hand side of (3.12), we get, for any $h > k$,

$$(h-k)^2|A_+(h)|^{2/2_s^*} \leq \Big(\int_{A_+(k)} |U(x,0)-k|^{2_s^*}\, dx\Big)^{2/2_s^*}.$$

Thus, we deduce

$$(h-k)^2|A_+(h)|^{2/2_s^*} \leq \frac{\|f\|^2_{L^p(\Omega)}}{(C_{\mathcal{D}}\kappa_s)^2}|A_+(k)|^{1-2/p+2s/N},$$

and setting $\varphi(h) = |A_+(h)|$, it follows that

$$\varphi(h) \leq \frac{\|f\|^{2_s^*}_{L^p(\Omega)}}{(C_{\mathcal{D}}\kappa_s)^2}\,\frac{\varphi^{(1-2/p+2s/N)2_s^*/2}(k)}{(h-k)^{2_s^*}}.$$

Applying now Lemma 3.8 with $a = 2_s^*$ and $b = (1-\frac{2}{p}+\frac{2s}{N})\frac{2_s^*}{2} > 1$, we find $|\varphi(k_0+d)| = 0$ with $d = C(N,s,|\Sigma_{\mathcal{D}}|)\|f\|_{L^p(\Omega)}|\varphi(k_0)|^{b-1/a}$, and $\frac{b-1}{a} = \frac{2s}{N}-\frac{1}{p}$, i.e.,

$$U(x,0) \leq k_0 + C(N,s,|\Sigma_{\mathcal{D}}|)\|f\|_{L^p(\Omega)}|A_+(k_0)|^{2s/N-1/p} \quad \text{a.e. in } \Omega,$$

for any $k_0 \geq 0$, and we conclude $u(x) \leq C(N,s,|\Sigma_{\mathcal{D}}|)\|f\|_{L^p(\Omega)}|\Omega|^{2s/N-1/p}$ a.e. in Ω. □

Let $v(x)$ be the solution to (3.4) and $V(x,y) = E_s[v(x)]$ the solution to (3.6). Since $(V-k)_+ \in \mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathscr{C}_\Omega)$ for any $k \geq 0$, repeating the proof above we deduce that for all $z \in \Omega$,

$$\|v(x)\|_{L^\infty(\Omega(z,R))} \leq C(N,s,|\Sigma_{\mathcal{D}}|)\|f\|_{L^p(\Omega)}|\Omega(z,R)|^{2s/N-1/p}. \tag{3.13}$$

Now we turn our attention to the study of the behavior of solutions to the homogeneous problem (3.7).

Lemma 3.9 (Caccioppoli inequality). *Assume that $z_0 \in \overline{\Omega}$ and $R > 0$, and suppose that the function $W \in \mathcal{X}^s_{\Sigma_{\mathcal{D}},R}(\mathscr{C}_{\Omega(z_0,R)})$ is a solution to problem* (3.7). *Then, for any $Z \in \mathscr{C}^\circ_{\Omega(z_0,R)}$ and $0 < \rho < r < \tilde{\rho}(Z)$, we have that there exists $C > 0$ such that*

$$\int_{\mathscr{C}_{\Omega(z_0,R)}(Z,\rho)} y^{1-2s}|\nabla W|^2\, dx\, dy \leq \frac{C}{(r-\rho)^2}\int_{\mathscr{C}_{\Omega(z_0,R)}(Z,r)} y^{1-2s}|W|^2\, dx\, dy.$$

Proof. We use $\psi = \eta^2 W$ as a test function in (3.8), with $\eta \in C^1(\mathscr{C}_{\Omega(z_0,R)})$ vanishing on $\partial_B\mathscr{C}_{\Omega(z_0,R)}$; observe that, in particular, $\psi \equiv 0$ on $\partial_0\mathscr{C}_{\Omega(z_0,R)}$, so that we have

$$\int_{\mathscr{C}_{\Omega(z_0,R)}} y^{1-2s}\eta^2|\nabla W|^2\, dx\, dy = -2\int_{\mathscr{C}_{\Omega(z_0,R)}} y^{1-2s}\langle \eta\nabla W, W\nabla\eta\rangle\, dx\, dy$$
$$\leq 2\Big(\frac{1}{2\varepsilon}\int_{\mathscr{C}_{\Omega(z_0,R)}} y^{1-2s}|\nabla\eta|^2W^2\, dx\, dy + \frac{\varepsilon}{2}\int_{\mathscr{C}_{\Omega(z_0,R)}} y^{1-2s}\eta^2|\nabla W|^2\, dx\, dy\Big) \tag{3.14}$$

for any $0 < \varepsilon < 1$. To complete the proof, given $Z \in \mathscr{C}^{\circ}_{\Omega(z_0,R)}$ and $\rho < r < \tilde{\rho}(Z)$, it is enough to set η such that

$$\eta \equiv 1 \quad \text{in } B_\rho(Z), \qquad \eta \equiv 0 \quad \text{in } B_r^c(Z) \qquad \text{and} \qquad |\nabla\eta| \leq \frac{c}{(r-\rho)},$$

and plug it into (3.14). □

Next we prove the following two lemmas.

Lemma 3.10. *Let $p \geq 2 - \varepsilon_0$ for some $0 < \varepsilon_0 < 1$ and $U \in \mathcal{X}^s(\mathscr{C}_\Omega)$ such that $\{U = 0\} \in \mathcal{F}(\beta, A)$ for $A \subset \overline{\mathscr{C}}_\Omega$. Then there exists $\beta_s = \beta_s(N, p, y^{1-2s}) > 0$ such that*

$$\int_A y^{1-2s}|U|^p\,dx\,dy \leq \beta_s^p[\operatorname{diam}(A)]^p|A|^{(1/\sigma-1)}_{y^{1-2s}}\big|\{(x,y) \in A : U \neq 0\}\big|^{1/\sigma'}_{y^{1-2s}} \int_A y^{1-2s}|\nabla U|^p\,dx\,dy, \tag{3.15}$$

with $\frac{1}{\sigma} + \frac{1}{\sigma'} = 1$, and

$$\int_{A_+^*(k,r)} y^{1-2s}|U-k|^2\,dx\,dy \leq \beta_s^2 r^2|B_r|^{1/\sigma-1}_{y^{1-2s}}|A_+^*(k,r)|^{1/\sigma'}_{y^{1-2s}} \int_{A_+^*(k,r)} y^{1-2s}|\nabla U|^2\,dx\,dy, \tag{3.16}$$

with $1 \leq \sigma \leq \frac{N+1}{N} + \delta$ for some $\delta > 0$.

Proof. In fact, (3.15) is consequence of (3.11) and the Hölder inequality.

As far as (3.16) is concerned, we follow [15, Theorem 6.1]: given $U \in \mathcal{X}^s(\mathscr{C}_{\Omega(z_0,R)})$, let us consider the function $t_k^+(U) = (U-k)_+$ that belongs to $\mathcal{X}^s(\mathscr{C}_{\Omega(z_0,R)})$ for any $k \in \mathbb{R}$. Moreover, if $U \in \mathcal{X}^s_{\Sigma_{\mathcal{D},R}}(\mathscr{C}_{\Omega(z_0,R)})$, then $t_k^+(U) \in \mathcal{X}^s_{\Sigma_{\mathcal{D},R}}(\mathscr{C}_{\Omega(z_0,R)})$ for any $k \geq 0$. Then, applying (3.11) to $(U-k)_+$ with $p = 2$, (3.16) follows. □

Lemma 3.11. *Given $z_0 \in \overline{\Omega}$ and $R > 0$, let $U \in \mathcal{X}^s(\mathscr{C}_{\Omega(z_0,R)})$. Then, for any $Z \in \mathscr{C}^{\circ}_{\Omega(z_0,R)}$ and $0 < r < \overline{\rho}(Z)$, there exist $\varepsilon_0 \in (0,1)$ and $\beta_s = \beta_s(N, p, y^{1-2s}) > 0$ such that*

$$(h-k)^2|A_+^*(h,r)|^{2/q}_{y^{1-2s}} \leq \beta_s^2 r^2|B_r|^{2(\frac{1}{q}-1/p)}_{y^{1-2s}}|A_+^*(k,r) - A_+^*(h,r)|^{2/p-1}_{y^{1-2s}} \int_{A_+^*(k,r)} y^{1-2s}|\nabla U|^2\,dx\,dy, \tag{3.17}$$

with $h > k$, $q = \frac{N+1}{N}(2-\varepsilon_0)$ and $p = 2 - \varepsilon_0$.

Proof. Given $U \in \mathcal{X}^s(\mathscr{C}_{\Omega(z_0,R)})$ and $h > k$, let $t^+_{h,k}(U) = \{U\}^h - \{U\}^k$. Note that $t^+_{h,k}(U) \in \mathcal{X}^s(\mathscr{C}_{\Omega(z_0,R)})$ for any $k \in \mathbb{R}$. Moreover, if $U \in \mathcal{X}^s_{\Sigma_{\mathcal{D},R}}(\mathscr{C}_{\Omega(z_0,R)})$, then $t^+_{h,k}(U) \in \mathcal{X}^s_{\Sigma_{\mathcal{D},R}}(\mathscr{C}_{\Omega(z_0,R)})$ for any $h > k \geq 0$. Thus, using (3.11) with $\sigma = \frac{N+1}{N}$ and $p = 2 - \varepsilon_0$, so that, taking $q = \sigma p = \frac{N+1}{N}(2-\varepsilon_0)$, we obtain

$$\Big(\int_{\mathscr{C}_{\Omega(z_0,R)}(Z,r)} y^{1-2s}|t^+_{h,k}(U)|^q\,dx\,dy\Big)^{1/q} \leq \beta_s r|B_r|^{1/q-1/p}_{y^{1-2s}}\Big(\int_{A_+^*(k,r)-A_+^*(h,r)} y^{1-2s}|\nabla U|^p\,dx\,dy\Big)^{1/p}. \tag{3.18}$$

On the one hand, it is immediate that

$$(h-k)^2|A_+^*(h,r)|^{2/q}_{y^{1-2s}} \leq \Big(\int_{\mathscr{C}_{\Omega(z_0,R)}(Z,r)} y^{1-2s}|t^+_{h,k}(U)|^q\,dx\,dy\Big)^{2/q}. \tag{3.19}$$

On the other hand, thanks to the Hölder inequality,

$$\Big(\int_{A_+^*(k,r)-A_+^*(h,r)} y^{1-2s}|\nabla U|^p\,dx\,dy\Big)^{2/p} \leq |A_+^*(k,r) - A_+^*(h,r)|^{2/p-1}_{y^{1-2s}} \int_{A_+^*(k,r)} y^{1-2s}|\nabla U|^2\,dx\,dy. \tag{3.20}$$

Thus, (3.17) follows by gathering together (3.18), (3.19) and (3.20). □

Following [15, Theorem 8.1], we show the next result.

Theorem 3.12. *Let $z_0 \in \overline{\Omega}$, $R > 0$, and let $W \in \mathcal{X}^s_{\Sigma_{\mathcal{D},R}}(\mathscr{C}_{\Omega(z_0,R)})$ be a solution to the homogeneous problem (3.7). Then, for any $Z \in \mathscr{C}^{\circ}_{\Omega(z_0,R)}$, $0 < \ell < 1$ and $0 < r < \min\{\tilde{\rho}(Z), \overline{\rho}(Z)\}$, there exists a positive constant $\Lambda = \Lambda(\ell)$ such that*

$$|A_+^*(k+\ell d, r-\ell r)| = 0, \quad \text{with } k \in \mathcal{K}^+(Z), \qquad \text{and} \qquad |A_-^*(k-\ell d, r-\ell r)| = 0, \quad \text{with } k \in \mathcal{K}^-(Z),$$

where

$$d^2 \geq \frac{1}{\Lambda(\ell)\,|B_r|_{y^{1-2s}}} \int\limits_{A_+^*(k,r)} y^{1-2s}|W-k|^2\,dx\,dy. \tag{3.21}$$

In the proof of Theorem 3.12, we make use of the following technical result.

Lemma 3.13 ([11, Lemma C.7]). *Assume that $\varphi(k,\rho)$ is a nonnegative function defined for $k \geq k_0$ and $0<\rho\leq r_0$, which is nonincreasing with respect to k, nondecreasing with respect to ρ and such that*

$$\varphi(h,\rho) \leq \frac{C_0}{(h-k)^\alpha (r-\rho)^\gamma}\varphi^\mu(k,r), \quad k<h,\ \rho<r\leq r_0,$$

where C, α, β, γ are positive constants with $\mu>1$. Then there exist $\ell \in (0,1)$ and $d>0$ such that

$$\varphi(k_0+\ell d, r_0(1-\ell)) = 0, \quad \text{with } d^\alpha = C_0 \frac{2^{(\alpha+\gamma)\mu/(\mu-1)}[\varphi(k_0,r_0)]^{\mu-1}}{\ell^{\alpha+\gamma} r_0^\gamma}.$$

Proof of Theorem 3.12. Given $z_0 \in \overline{\Omega}$, $k_0 \in \mathcal{K}^+(Z)$ and $k \geq k_0$, let us define

$$i(k,\rho) = \int\limits_{A_+^*(k,\rho)} y^{1-2s}|W-k|^2\,dx\,dy \quad \text{and} \quad a(k,\rho) = |A_+^*(k,\rho)|_{y^{1-2s}}.$$

Observe that for $h>k$, we have

$$(h-k)^2|A_+^*(h,\rho)|_{y^{1-2s}} \leq \int\limits_{A_+^*(k,r)} y^{1-2s}|W-k|^2\,dx\,dy. \tag{3.22}$$

Assume that $Z \in \Sigma^*_{\mathcal{D},R} \cap \mathscr{C}_{\Omega(z_0,R)}$ and let $0<r_0<\min\{\tilde{\rho}(Z), \overline{\rho}(Z)\}$. Then, due to Lemmas 3.9 and 3.10, for any $r_0(1-\ell)\leq\rho<r\leq r_0$ and $h>k$, we have

$$\begin{aligned}
\int\limits_{A_+^*(h,\rho)} y^{1-2s}|W-h|^2\,dx\,dy &\leq K_{\mathscr{C}_\Omega(\rho)}\Big(\int\limits_{A_+^*(h,\rho)} y^{1-2s}|\nabla W|^2\,dx\,dy\Big)|A_+^*(h,\rho)|^{1/\sigma'}_{y^{1-2s}}\\
&\leq K_{\mathscr{C}_\Omega(\rho)}\Big(\int\limits_{A_+^*(k,\rho)} y^{1-2s}|\nabla W|^2\,dx\,dy\Big)|A_+^*(k,\rho)|^{1/\sigma'}_{y^{1-2s}}\\
&\leq K_{\mathscr{C}_\Omega(\rho)}\Big(\frac{1}{(r-\rho)^2}\int\limits_{A_+^*(k,r)} y^{1-2s}|W-k|^2\,dx\,dy\Big)|A_+^*(k,r)|^{1/\sigma'}_{y^{1-2s}},
\end{aligned} \tag{3.23}$$

where $K_{\mathscr{C}_\Omega(r)} = \beta_s^2 r^2 |B_r|^{1/\sigma-1}_{y^{1-2s}}$, with $\beta_s = \beta_s(N, y^{1-2s}, \partial\Omega)>0$ and $1\leq\sigma\leq\frac{N+1}{N}+\delta$ for some $\delta>0$.

Assume, on the contrary, that $Z_0 \in \mathscr{C}^\circ_{\Omega(z_0,R)} \setminus \Sigma^*_{\mathcal{D},R}$. Recalling (3.2), let $\Lambda = \Lambda(\ell)>0$, satisfying

$$\frac{\Lambda}{\zeta_s(1-\ell)^{N+2(1-s)}} \leq (1-\lambda) \quad \text{for some } \lambda \in (0,1).$$

Therefore, given $h\geq k_0$ and $(1-\ell)r_0\leq\rho\leq r_0$, we find

$$\begin{aligned}
|A_+^*(h,\rho)|_{y^{1-2s}} &\leq |A_+^*(k_0,r_0)|_{y^{1-2s}} \leq |\mathscr{C}_{\Omega(z_0,R)}(Z,r_0)|_{y^{1-2s}} \leq |B_{r_0}(Z)|_{y^{1-2s}}\\
&\leq \frac{|B_\rho(Z)|_{y^{1-2s}}}{(1-\ell)^{N+2(1-s)}} \leq \frac{\Lambda\,|\mathscr{C}_{\Omega(z_0,R)}(Z,\rho)|_{y^{1-2s}}}{\zeta_s(1-\ell)^{N+2(1-s)}} \leq (1-\lambda)|\mathscr{C}_{\Omega(z_0,R)}(Z,\rho)|_{y^{1-2s}}.
\end{aligned}$$

Using Lemma 3.9 and Lemma 3.10, we deduce that (3.23) holds true.

As a consequence, for any $Z \in \mathscr{C}^\circ_{\Omega(z_0,R)}$,

$$i(h,\rho) \leq \frac{K_{\mathscr{C}_\Omega(\rho)}}{(r-\rho)^2} i(k,r)[a(k,r)]^{1/\sigma'}, \quad r_0(1-\ell)\leq\rho<r\leq r_0 \text{ and } h>k\geq k_0, \tag{3.24}$$

with $k_0 \in \mathcal{K}^+(Z)$ satisfying (3.3). Moreover, since $|B_{\mu r}|_{y^{1-2s}} = \mu^{N+2(1-s)}|B_r|_{y^{1-2s}}$, we have $K_{\mathscr{C}_\Omega(\mu r)} = \mu^\varsigma K_{\mathscr{C}_\Omega(r_0)}$, where $\varsigma = 2+(\frac{1}{\sigma}-1)(N+2(1-s))$.

If we let $1 < \sigma \le 1 + \frac{2}{N-2s}$ (so that $\varsigma > 0$), then $K_{\mathscr{C}_\Omega(r)} \le K_{\mathscr{C}_\Omega(r_0)}$ for any $0 < r < r_0$. Hence, from (3.24), we obtain

$$i(h,\rho) \le \frac{K_{\mathscr{C}_\Omega(r_0)}}{(r-\rho)^2} i(k,r)[a(k,r)]^{1/\sigma'}, \quad \rho < r \le r_0,\ h > k \ge k_0, \tag{3.25}$$

with $K_{\mathscr{C}_\Omega(r_0)} = \beta_s^2 r_0^2 |B_{r_0}|_{y^{1-2s}}^{1/\sigma-1}$. We set now $\xi + 1 = \theta\xi$ and $\frac{\xi}{\sigma'} = \theta$, so that $\theta = \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{1}{\sigma'}} > 1$ turns out to be the unique positive solution to the equation $\theta^2 - \theta - \frac{1}{\sigma'} = 0$. Assume, in addition, that the constant Λ satisfies

$$\Lambda^{\theta/2} \le \frac{\ell^{\xi+1}}{\beta_s^{\xi} 2^{(\xi+1)\theta/(\theta-1)}}. \tag{3.26}$$

From (3.22) and (3.25), we obtain

$$|i(h,\rho)|^{\xi}|a(h,\rho)| \le \frac{K^{\xi}_{\mathscr{C}_\Omega(r_0)}}{(r-\rho)^{2\xi}(h-k)^2} |i(k,r)|^{\xi+1}|a(k,r)|^{\xi/\sigma'}.$$

Then, taking $\varphi(k,\rho) = |i(k,\rho)|^{\xi}|a(k,\rho)|$, it follows that φ satisfies

$$\varphi(h,\rho) \le \frac{K^{\xi}_{\mathscr{C}_\Omega(r_0)}}{(r-\rho)^{2\xi}(h-k)^2}\varphi^{\theta}(k,r), \quad h > k \ge k_0,\ \rho < r \le r_0.$$

Using Lemma 3.13 with $\alpha = 2$, $\mu = \theta$, $\gamma = 2\xi$, we deduce that exist $d_0 > 0$ and $\ell \in (0,1)$ such that

$$\varphi(k_0 + \ell d_0, r_0(1-\ell)) = 0,$$

for any $k_0 \in \mathcal{K}^+(Z)$ satisfying (3.3), $0 < r_0 < \min\{\tilde{\rho}(Z), \overline{\rho}(Z)\}$ and d_0 such that

$$d_0 = \frac{2^{(\xi+1)\theta/\theta-1}}{\ell^{\xi+1}} \frac{K^{\xi/2}_{\mathscr{C}_\Omega(r_0)}[\varphi(k_0,r_0)]^{(\theta-1)/2}}{r_0^{\xi}} \le \bigg(\frac{1}{\Lambda|B_{r_0}|_{y^{1-2s}}} \int_{A_+^*(k,r_0)} y^{1-2s}|W-k_0|^2\,dx\,dy\bigg)^{1/2}.$$

Since $|A_+^*(k_0+\ell d_0, r_0(1-\ell))|_{y^{1-2s}} = 0$ implies $|A_+^*(k_0+\ell d_0, r_0(1-\ell))| = 0$, the proof is complete.

The proof on the lower bound follows using the same inequalities on $(W+k)^-$ and getting the bounds on $|A_-^*(k_0-\ell d, r_0(1-\ell))|_{y^{1-2s}}$. □

As a consequence of the above theorem, we get the L^∞ bound on W.

Corollary 3.14. *Let $z_0 \in \overline{\Omega}$, $R > 0$, and let $W \in \mathcal{X}^s_{\Sigma_{\mathcal{D}},R}(\mathscr{C}_{\Omega(z_0,R)})$ be a solution to the homogeneous problem* (3.7)*; consider the set $\mathscr{C}^m_{\Omega(z_0,R/2)} = \mathscr{C}_{\Omega(z_0,R/2)} \cap \{y < m\}$ with $m > 0$. Then $W \in L^\infty(\mathscr{C}^m_{\Omega(z_0,R/2)})$ for any $m > 0$. In particular, any solution $w \in H^s_{\Sigma_{\mathcal{D}},R}(\Omega(z_0,R))$ of problem* (3.5) *satisfies $w \in L^\infty(\Omega(z_0,R/2))$.*

Proof. First, let us prove that $w \in L^\infty(\Omega(z_0,R/2))$, with w satisfying problem (3.5). Let $W \in \mathcal{X}^s_{\Sigma_{\mathcal{D}},R}(\mathscr{C}_{\Omega(z_0,R)})$ be a solution to problem (3.7) and since $\Omega(z_0,R/2)$ is a bounded set, there exists $Z_i = (z_i, 0) \in \mathscr{C}^{\circ}_{\Omega(z_0,R)}$, $i = 1,2,\dots,M$, such that

$$\overline{\Omega(z_0,R/2)} = \bigg(\bigcup_{i=1}^{M} \mathscr{C}^{\circ}_{\Omega(z_0,R)}(Z_i, r_i/2)\bigg) \cap \{y = 0\}, \tag{3.27}$$

with $0 < r_i < \{\tilde{\rho}(Z_i), \overline{\rho}(Z_i)\}$. Let $\overline{k} > 0$ and $\hat{k} < 0$ be such that

$$|A_+^*(\overline{k}, r_i)| \le \Lambda|\mathscr{C}_{\Omega(z_0,R)}(Z_i,r_i)| \quad \text{and} \quad |A_-^*(\hat{k}, r_i)| \le \Lambda|\mathscr{C}_{\Omega(z_0,R)}(Z_i,r_i)|$$

for any $i = 1,2,\dots,M$. Then, by applying Theorem 3.12, we conclude that, given $X \in \mathscr{C}_{\Omega(z_0,R)}(Z_i,r_i)$ for some $i = 1,2,\dots,M$, we have

$$\kappa_m := \hat{k} - \ell d \le W(x,y) \le \kappa_M := \overline{k} + \ell d, \tag{3.28}$$

with

$$d^2 \ge \frac{1}{\Lambda|B_r|_{y^{1-2s}}} \int_{\mathscr{C}_{\Omega(z_0,R)}} y^{1-2s}|W|^2\,dx\,dy$$

for any $0 < r < \min_{i=1,\dots,M} r_i$. In particular, by (3.27), the former inequality holds for any point $X = (x, 0)$ with $x \in \overline{\Omega(z_0, R/2)}$, and we are done.

Since $\mathscr{C}_{\Omega(z_0,R/2)}$ is an unbounded domain, if we repeat the steps above in order to prove that $W \in L^\infty(\overline{\mathscr{C}}_{\Omega(z_0,R/2)})$ from (3.28), the numbers $\hat{k}, \overline{k}$ do diverge when considering a covering sequence $\{Z_i\}_{i\in\mathbb{N}}$. Nevertheless, it is clear that given any finite truncation of the extension cylinder, $\mathscr{C}^m_{\Omega(z_0,R/2)} = \mathscr{C}_{\Omega(z_0,R/2)} \cap \{y < m\}$, there exists a finite covering sequence, and hence we conclude $W \in L^\infty(\overline{\mathscr{C}^m_{\Omega(z_0,R/2)}})$ for all finite $m > 0$. □

We focus now on the oscillation of the solutions $W \in \mathcal{X}^s_{\Sigma_{\mathcal{D},R}}(\mathscr{C}_{\Omega(z_0,R)})$ to problem (3.7). Let us set

$$m(\rho) = \inf_{X\in\overline{\mathscr{C}}_{\Omega(z_0,R)}(Z,\rho)} W(X) \quad \text{and} \quad M(\rho) = \sup_{X\in\overline{\mathscr{C}}_{\Omega(z_0,R)}(Z,\rho)} W(X),$$

and define the oscillation function as

$$\omega(\rho) := M(\rho) - m(\rho).$$

Our aim is to give some estimates on $\omega(\rho)$ through the following result.

Theorem 3.15. *Given $z_0 \in \overline{\Omega}$ and $R > 0$, let $Z \in \mathscr{C}^\circ_{\Omega(z_0,R)}$ and let $W \in \mathcal{X}^s_{\Sigma_{\mathcal{D},R}}(\mathscr{C}_{\Omega(z_0,R)})$ be a solution to the homogeneous problem* (3.7). *Moreover, given $0 < 4\rho < \min\{\tilde{\rho}(Z), \overline{\rho}(Z)\}$, let $0 < \eta < 1$ be such that*
(i) $(M(4\rho) - \eta\omega(4\rho), +\infty) \subset \mathcal{K}^+(Z)$,
(ii) $|A^*_+(M(4\rho) - \eta\omega(4\rho), 2\rho)|_{y^{1-2s}} \le \Lambda|\mathscr{C}_{\Omega(z_0,R)}(Z, 2\rho)|_{y^{1-2s}}$,
where Λ is determined by (3.21) *with $\ell = \frac{1}{2}$. Then there exists $0 < \overline{\eta} < 1$ independent from Z and ρ such that*

$$\omega(\rho) \le \overline{\eta}\omega(4\rho). \tag{3.29}$$

Proof. Let $Z \in \mathscr{C}^\circ_{\Omega(z_0,R)}$ and $0 < 4\rho < \min\{\tilde{\rho}(Z), \overline{\rho}(Z)\}$; let us define the sequence

$$k_j = M(4\rho) - \eta_j\omega(4\rho), \quad \text{with } \eta_j = \frac{1}{2^{j+1}},\ j \in \mathbb{N}.$$

Assume first that $Z \in \mathscr{C}^\circ_{\Omega(z_0,R)} \setminus \Sigma^*_{\mathcal{D},R}$, so that $\mathcal{K}^+(Z) = (-\infty, \infty)$, and observe that one of the following conditions is satisfied:

$$|A^*_+(k_0, 2\rho)|_{y^{1-2s}} \le \frac{1}{2}|\mathscr{C}_{\Omega(z_0,R)}(Z, 2\rho)|_{y^{1-2s}} \quad \text{or} \quad |A^*_-(k_0, 2\rho)|_{y^{1-2s}} \le \frac{1}{2}|\mathscr{C}_{\Omega(z_0,R)}(Z, 2\rho)|_{y^{1-2s}}.$$

Assume without loss of generality that $|A^*_+(k_0, 2\rho)| \le \frac{1}{2}|\mathscr{C}_{\Omega(z_0,R)}(Z, 2\rho)|$. As a consequence,

$$|A^*_+(k_j, 2\rho)| \le \frac{1}{2}|\mathscr{C}_{\Omega(z_0,R)}(Z, 2\rho)| \quad \text{for } j \ge 1.$$

On the other hand, if $Z \in \Sigma^*_{\mathcal{D},R}$, we can assume that at least one between $M(4\rho)$ and $-m(4\rho)$ is greater than $\frac{1}{2}\omega(4\rho)$; suppose that $M(4\rho) > \frac{1}{2}\omega(4\rho)$. Therefore, we have that $k_j > 0$ for $j \ge 0$.

Then, using Lemma 3.11 with $h = k_{j+1}$ and $k = k_j$, we obtain

$$(k_{j+1} - k_j)^2|A^*_+(k_{j+1}, 2\rho)|^{2/q}_{y^{1-2s}} \le \beta_s^2(2\rho)^2|B_{2\rho}|^{2(1/q-1/p)}_{y^{1-2s}} \int\limits_{A^*_+(k_j,2\rho)} y^{1-2s}|\nabla W|^2\,dx\,dy,$$

with p, q such that $q = \frac{N+1}{N}(2 - \varepsilon_0)$ and $p = 2 - \varepsilon_0$ for a suitable $\varepsilon_0 > 0$.

Moreover, applying Lemma 3.9 to the function $t^+_{k_j}(W) \in \mathcal{X}^s_{\Sigma_{\mathcal{D},R}}(\mathscr{C}_{\Omega(z,R)})$, $j \ge 0$, we find

$$\int\limits_{A^*_+(k_j,2\rho)} y^{1-2s}|\nabla W|^2\,dx\,dy \le \frac{C}{4\rho^2} \int\limits_{A^*_+(k_j,4\rho)} y^{1-2s}|W - k_j|^2\,dx\,dy \le \frac{C}{4\rho^2}[M(4\rho) - k_j]^2|B_{4\rho}(Z)|_{y^{1-2s}}.$$

Gathering together the above inequalities, we have that

$$(k_{j+1} - k_j)^2|A^*_+(k_{j+1}, 2\rho)|^{2/q}_{y^{1-2s}} \le C\beta_s|B_{2\rho}|^{2(1/q-1/p)+1}_{y^{1-2s}}[M(4\rho) - k_j]^2|A^*_+(k_j, 2\rho) - A^*_+(k_{j+1}, 2\rho)|^{2/p-1}_{y^{1-2s}}, \tag{3.30}$$

where the constant $C > 0$ is the one appearing in the Caccioppoli inequality. Let us define

$$\varphi(k) = \frac{|A^*_+(k, 2\rho)|_{y^{1-2s}}}{|\mathscr{C}_{\Omega(z,R)}(Z, 2\rho)|_{y^{1-2s}}},$$

and note that, by (3.1) and (3.2), we have $|B_{2\rho}|_{y^{1-2s}} \le \frac{1}{\zeta_s}|\mathscr{C}_{\Omega(z,R)}(Z,2\rho)|_{y^{1-2s}}$. Then, since $2(\frac{1}{q}-\frac{1}{p})+1>0$, taking into account that

$$k_{j+1}-k_j=\eta_{j+1}\omega(4\rho) \quad \text{and} \quad M(4\rho)-k_j=\eta_j\omega(4\rho),$$

from (3.30), we find

$$|\varphi(k_{j+1})|^{2/q} \le \vartheta[\varphi(k_j)-\varphi(k_{j+1})]^{2/p-1}, \quad \text{with } \vartheta=\frac{4C\beta_s}{\zeta_s^{2(1/q-1/p)+1}}.$$

Let us set $\mu=\frac{2}{q}\frac{1}{2/p-1}>0$ and $a=\frac{p}{2-p}$, so that the above inequality turns into

$$\varphi^\mu(k_n) \le \vartheta^a[\varphi(k_j)-\varphi(k_{j+1})], \quad j \ge 0.$$

Summing up the above inequality for $j=0,1,\dots,n$ and noticing that $\varphi(k_j)\ge\varphi(k_n)$, we get

$$n\varphi^\mu(k_n) \le \vartheta^a[\varphi(k_0)-\varphi(k_{n+1})]$$

and, by (3.30), we conclude that

$$\varphi(k_n) \le \Big(\frac{\vartheta^a\varphi(k_0)}{n}\Big)^{1/\mu}. \tag{3.31}$$

Let us set $\overline{n}>0$ such that

$$\overline{n} \ge \Big\lceil \frac{(4C\beta_s)^a\varphi(k_0)}{\zeta_s^{\mu-1}\Lambda^\mu}\Big\rceil, \tag{3.32}$$

where Λ is determined by (3.3) with $\ell=\frac{1}{2}$, ζ_s depends on ζ in (3.1) and the A_2-*constant* (see (3.2)), the constant β_s depends on N and the weight y^{1-2s}, and $C>0$ is a universal constant coming from the Caccioppoli inequality.

Consequently, $\overline{n}$ is independent of Z and ρ. Then, by inequality (3.31), we find

$$\frac{|A_+^*(k_n,2\rho)|_{y^{1-2s}}}{|\mathscr{C}_{\Omega(z,R)}(Z,2\rho)|_{y^{1-2s}}} \le \Lambda \quad \text{for all } n \ge \overline{n}.$$

Applying Theorem 3.12 with $k_{\overline{n}}=M(4\rho)-\eta_{\overline{n}}\omega(4\rho)$, $r=2\rho$ and $\ell=\frac{1}{2}$, so that

$$\frac{1}{\Lambda|B_{2\rho}(Z)|_{y^{1-2s}}}\int\limits_{A_+^*(M(4\rho)-\eta_{\overline{n}}\omega(4\rho),2\rho)} y^{1-2s}|W-(M(4\rho)-\eta_{\overline{n}}\omega(4\rho))|^2\,dx\,dy \le (\eta_{\overline{n}}\omega(4\rho))^2=d^2,$$

we obtain

$$W(X)\le k+\ell d \le [M(4\rho)-\eta_{\overline{n}}\omega(4\rho)]+\frac{1}{2}\eta_{\overline{n}}\omega(4\rho) \le M(4\rho)-\frac{1}{2}\eta_{\overline{n}}\omega(4\rho) \quad \text{a.e. in } \mathscr{C}_{\Omega(z,R)(Z,\rho)}.$$

As a consequence,

$$\omega(\rho)=M(\rho)-m(\rho)\le M(\rho)-m(4\rho)\le\Big[M(4\rho)-\frac{1}{2}\eta_{\overline{n}}\omega(4\rho)\Big]-m(4\rho)\le\Big(1-\frac{1}{2}\eta_{\overline{n}}\Big)\omega(4\rho),$$

and we deduce (3.29), by choosing $\overline{\eta}=(1-\eta_{\overline{n}+1})$. □

The next result gives an estimate on the growth of the oscillation.

Theorem 3.16. *Given $z_0\in\overline{\Omega}$ and $R>0$, let $W\in\mathcal{X}^s_{\Sigma_{\mathcal{D}},R}(\mathscr{C}_{\Omega(z_0,R)})$ be a solution to the homogeneous problem* (3.7). *Then there exist $0<\mathcal{H}<1$ and $0<\tau<\frac{1}{2}$ such that for any $Z\in\mathscr{C}^\circ_{\Omega(z_0,R)}$, there exists $\delta(Z)>0$ such that*

$$\omega(\rho)=\sup_{X\in\overline{\mathscr{C}}_{\Omega(z_0,R)}(Z,\rho)}W(X)-\inf_{X\in\overline{\mathscr{C}}_{\Omega(z_0,R)}(Z,\rho)}W(X)\le\mathcal{H}\rho^\tau$$

for any $0<\rho<\delta(Z)$.

Proof. Let $r(Z)=\min\{\widetilde{\rho}(Z),\overline{\rho}(Z)\}$, by Theorem 3.15, inequality (3.29) holds true for any $\rho<r(Z)/4$. Take τ and M positive such that $4^\tau\overline{\eta}=a<1$ and $\omega(\rho)\le M\rho^\tau$ for $r(Z)/4\le\rho<r(Z)$. Then, again by (3.29), we have that

$$\omega(\rho)\le\overline{\eta}4^\tau M\rho^\tau$$

for $r(Z)/4^2 \le \rho < r(Z)/4$. In general, if $r(Z)/4^{i+1} \le \rho < r(Z)/4^i$ for some $i \in \mathbb{N}$, we deduce that $\omega(\rho) \le (\overline{\eta}4^\tau)^i M\rho^\tau$. Letting $\bar{i}$ be large enough such that $\mathcal{H} = Ma^{\bar{i}} < 1$, we obtain $\omega(\rho) \le \mathcal{H}\rho^\tau$ for any $\rho < \delta(Z) = r(Z)/4^{\bar{i}}$. On the other hand, since we have chosen $\tau > 0$ such that $4^\tau\overline{\eta} < 1$ and, by Theorem 3.15, $\overline{\eta} = 1 - \eta_{\overline{n}+1}$ for some $\overline{n} \ge 0$ independent from Z and ρ, it follows that

$$\tau < \frac{1}{2}\log_2\Big(\frac{2^{\overline{n}+2}}{2^{\overline{n}+2}-1}\Big) < \frac{1}{2}. \tag{3.33}$$

□

Before proving Theorem 1.2, let us observe the following:

(i) If $z_0 \in \Omega$, then there exist $R > 0$ sufficiently small such that $\Sigma_{\mathcal{D},R} = \Sigma_{\mathcal{N},R} = \emptyset$ and $\widetilde{\rho}(Z) = \operatorname{dist}(Z, \partial_L\mathscr{C}_{\Omega(z_0,R)})$ for any $z \in \mathscr{C}_{\Omega(z_0,R)}$.

(ii) If $z_0 \in \Sigma_{\mathcal{D}} \setminus \Gamma$, then there exist $R > 0$ such that $\Sigma_{\mathcal{N},R} = \emptyset$. Hence, $\widetilde{\rho}(Z) = \operatorname{dist}(Z, \partial_B\mathscr{C}_{\Omega(z_0,R)})$ for any $Z \in \Sigma^*_{\mathcal{D},R}$ and $\widetilde{\rho}(Z) = \operatorname{dist}(Z, \partial_0\mathscr{C}_{\Omega(z_0,R)})$ for any $Z \in \mathscr{C}^\circ_{\Omega(z_0,R)} \setminus \Sigma^*_{\mathcal{D},R}$.

(iii) If $z_0 \in \Sigma_{\mathcal{N}}$, then there exist $R > 0$ such that $\Sigma_{\mathcal{D},R} = \emptyset$. Hence, $\widetilde{\rho}(Z) = \operatorname{dist}(Z, \partial_B\mathscr{C}_{\Omega(z_0,R)})$ for any $Z \in \mathscr{C}^\circ_{\Omega(z_0,R)}$.

(iv) If $z_0 \in \Gamma$, then for all $R > 0$ both $\Sigma_{\mathcal{D},R} \ne \emptyset$ and $\Sigma_{\mathcal{N},R} \ne \emptyset$, and hence $\widetilde{\rho}(Z) = \operatorname{dist}(Z, \partial_B\mathscr{C}_{\Omega(z_0,R)})$ for any $Z \in \Sigma^*_{\mathcal{D},R}$ and $\widetilde{\rho}(Z) = \operatorname{dist}(Z, \partial_0\mathscr{C}_{\Omega(z_0,R)})$ for any $Z \in \mathscr{C}^\circ_{\Omega(z_0,R)} \setminus \Sigma^*_{\mathcal{D},R}$.

Now, consider $\overline{\mathscr{C}}_{\Omega(z_0,R/2)} \subset \mathscr{C}_{\Omega(z,R)}$ if $z \in \Omega$ and $\overline{\mathscr{C}}_{\Omega(z_0,R/2)} \subset \mathscr{C}^\circ_{\Omega(z,R)}$ if $z \in \partial\Omega$.

Thus, we deduce:

(i) If $z \in \Omega$, then $\widetilde{\rho}(Z) = \operatorname{dist}(Z, \partial_L\mathscr{C}_{\Omega(z,R)}) \ge \widetilde{\rho} > 0$ for any $Z \in \overline{\mathscr{C}}_{\Omega(z,R/2)}$ and some positive $\widetilde{\rho}$.

(ii) If $z \in \Sigma_{\mathcal{D}} \setminus \Gamma$, then $\widetilde{\rho}(Z) = \widetilde{\rho} > 0$ for some positive $\widetilde{\rho}$ for any $Z \in \Sigma^*_{\mathcal{D},R/2}$ and $\widetilde{\rho}(Z) = \operatorname{dist}(Z, \Sigma^*_{\mathcal{D},R/2})$ for any $Z \in \overline{\mathscr{C}}_{\Omega(z,R/2)} \setminus \Sigma^*_{\mathcal{D},R/2}$.

(iii) If $z \in \Sigma_{\mathcal{N}}$, then $\widetilde{\rho}(Z) = \operatorname{dist}(Z, \partial_B\mathscr{C}_{\Omega(z,R)}) \ge \widetilde{\rho} > 0$ for any $Z \in \overline{\mathscr{C}}_{\Omega(z,R/2)}$ and some positive $\widetilde{\rho}$.

(iv) If $z \in \Gamma$, then $\widetilde{\rho}(Z) = \widetilde{\rho} > 0$ for some positive $\widetilde{\rho}$ for any $Z \in \Sigma^*_{\mathcal{D},R/2}$ and $\widetilde{\rho}(Z) = \operatorname{dist}(Z, \Sigma_{\mathcal{D},R/2})$ for any $Z \in \overline{\mathscr{C}}_{\Omega(z,R/2)} \setminus \Sigma^*_{\mathcal{D},R/2}$.

Observe that if either (i) or (iii) holds true, then the number $0 < \delta(Z)$ in Theorem 3.16 has an infimum value, namely, $0 < \delta < \delta(Z)$ for any $Z \in \overline{\mathscr{C}}_{\Omega(z_0,R/2)}$ and we deduce that solutions W to problem (3.7) are Hölder continuous up to the boundary of $\mathscr{C}_{\Omega(z_0,R/2)}$. In fact, let us consider two points Z_1 and Z_2 in $\overline{\mathscr{C}^m_{\Omega(z_0,R)}}$ with $m > 0$. Then, by Corollary 3.14 and Theorem 3.16, we find:

- If $|Z_1 - Z_2| \ge \delta$, we have

$$\frac{|W(Z_1) - W(Z_2)|}{|Z_1 - Z_2|^\tau} \le \frac{2}{\delta^\tau}\max_{\mathscr{C}^m_{\Omega(z_0,R/2)}} W = \frac{2}{\delta^\tau}\|W\|_{L^\infty(\mathscr{C}^m_{\Omega(z_0,R/2)})}.$$

- If $|Z_1 - Z_2| < \delta$, by Theorem 3.16, $\frac{|W(Z_1)-W(Z_2)|}{|Z_1-Z_2|^\tau} \le \mathcal{H}$, $0 < \mathcal{H} < 1$.

We conclude the Hölder regularity with a constant

$$\mathcal{T} = \max\Big\{\mathcal{H}, \frac{2}{\delta^\tau}\|W\|_{L^\infty(\mathscr{C}^m_{\Omega(z,R/2)})}\Big\}. \tag{3.34}$$

Now we deal with the situation described in items (ii) and (iv).

Theorem 3.17. *For any $z_0 \in \Sigma_{\mathcal{D}}$ and $R > 0$, let $W \in \mathcal{X}^s_{\Sigma_{\mathcal{D},R}}(\mathscr{C}_{\Omega(z_0,R)})$ be a solution to the homogeneous problem* (3.7). *Then $W \in \mathcal{C}^\tau_{\mathrm{loc}}(\overline{\mathscr{C}}_{\Omega(z_0,R/2)})$ for some $0 < \tau < \frac{1}{2}$.*

Proof. Observe that the number $0 < \delta(Z)$ in Theorem 3.16 is bounded from below by some $0 < \delta_{\mathrm{H}}$ for $Z \in \Sigma^*_{\mathcal{D},R/2}$, and we can assume that $\delta(Z) \ge \min\{\delta_{\mathrm{H}}, \operatorname{dist}(Z, \Sigma^*_{\mathcal{D},R/2})\}$ for $Z \in \Sigma^*_{\mathcal{N},R/2}$. Moreover, by the construction of the lateral boundary of the extension cylinder, the numbers $\delta(Z)$ do not depend on the y variable. Hence, such an infimum $\delta_{\mathrm{H}} > 0$ is attained at those points of the type $Z = (z, 0)$ in $\partial\Omega \times \{0\}$. Consider the set

$$\mathscr{C}^\delta_{\Omega(z_0,R/2)} = \{Z \in \overline{\mathscr{C}^m_{\Omega(z,R/2)}} : \operatorname{dist}(Z, \Sigma^*_{\mathcal{D},R/2}) \ge \delta_{\mathrm{H}}\}.$$

As above, we only need to study the case $|Z_1 - Z_2| < \delta_{\mathrm{H}}$. Suppose that $Z_1 \in \mathscr{C}^\delta_{\Omega(z_0,R/2)}$. Then $|Z_1 - Z_2| \le \delta_{\mathrm{H}} < \operatorname{dist}(Z_1, \Sigma^*_{\mathcal{D},R/2}) = \delta(Z_1)$, and thus, by Theorem 3.16, we have

$$\frac{|W(Z_1) - W(Z_2)|}{|Z_1 - Z_2|^\tau} \le \mathcal{H}.$$

If neither Z_1 nor Z_2 belongs to $\mathscr{C}^{\delta}_{\Omega(z_0,R/2)}$ but one of them, say $Z_1 \in \Sigma^*_{\mathcal{D},R/2}$, we have $|Z_1 - Z_2| \le \delta_{\mathrm{H}} = \delta(Z_1)$, and the results follows as before. If, instead, none of them belongs neither to $\mathscr{C}^{\delta}_{\Omega(z_0,R/2)}$ nor to $\Sigma^*_{\mathcal{D},R/2}$, we have two cases:

- $|Z_1 - Z_2| \le \max\{\operatorname{dist}(Z_1, \Sigma^*_{\mathcal{D},R/2}), \operatorname{dist}(Z_2, \Sigma^*_{\mathcal{D},R/2})\}$,
- $|Z_1 - Z_2| > \max\{\operatorname{dist}(Z_1, \Sigma^*_{\mathcal{D},R/2}), \operatorname{dist}(Z_2, \Sigma^*_{\mathcal{D},R/2})\}$.

In the first case, at least one of the two points, say Z_1, satisfies the inequality $|Z_1 - Z_2| \le \delta_{\mathrm{H}} < \operatorname{dist}(Z_1, \Sigma^*_{\mathcal{D},R/2}) = \delta(Z_1)$, and we have the result as before. In the second case, there exists at least one $\overline{Z} \in \Sigma^*_{\mathcal{D},R/2}$ such that $|\overline{Z} - Z_1| \le |Z_1 - Z_2|$, and using the triangle inequality, it follows that $|\overline{Z} - Z_2| \le 2|Z_1 - Z_2|$. Since the result has been proved for the case when at least one point belongs to $\Sigma^*_{\mathcal{D},R/2}$, we find

$$|W(Z_1) - W(Z_2)| \le |W(Z_1) - W(\overline{Z})| + |W(\overline{Z}) - W(Z_2)| \le 3\mathcal{H}|Z_1 - Z_2|^{\tau}, \tag{3.35}$$

and we conclude the Hölder regularity with constant $\mathcal{T} = \max\{3\mathcal{H}, 2\delta_{\mathrm{H}}^{-\tau}\|W\|_{L^\infty(\mathscr{C}_{\Omega(z,R/2)})}\}$, with $0 < \mathcal{H} < 1$ given by Theorem 3.16, see (3.34). □

Corollary 3.18. *Assume Hypotheses 1.1 and let w be the solution to problem* (3.5) *with $z \in \overline{\Omega}$ and $R > 0$. Then the function $w \in \mathcal{C}^{\tau}(\overline{\Omega(z, R/2)})$ for some $0 < \tau < \frac{1}{2}$.*

Proof. Since Ω satisfies Hypotheses 1.1, there exists $0 < \delta_{\mathrm{H}} < \delta(Z)$ for $Z \in \Sigma^*_{\mathcal{D},R/2}$, and we can assume that $\delta(Z) \ge \min\{\delta_{\mathrm{H}}, \operatorname{dist}(Z, \Sigma^*_{\mathcal{D},R/2})\}$ for $Z \in \Sigma^*_{\mathcal{N},R/2}$, with $\delta(Z)$ given in Theorem 3.16.

Suppose that $z_1, z_2 \in (\overline{\Omega(z, R/2)})$.

- If $|z_1 - z_2| \ge \delta_{\mathrm{H}}$, then, due to Corollary 3.14, we have $\|w\|_{L^\infty(\Omega(z,R/2))} < \infty$, and therefore
$$\frac{|w(z_1) - w(z_2)|}{|z_1 - z_2|^{\tau}} \le \frac{2}{\delta_{\mathrm{H}}^{\tau}} \max_{\Omega(z,R/2)} w.$$
- While for $|z_1 - z_2| < \delta_{\mathrm{H}}$, let us set $Z_1 = (z_1, 0)$ and $Z_2 = (z_2, 0)$, $Z_1, Z_2 \in \overline{\mathscr{C}}_{\Omega(z,R/2)}$, such that $|Z_1 - Z_2| < \delta_{\mathrm{H}}$. Then, as in (3.35) in Theorem 3.17,
$$\frac{|w(z_1) - w(z_2)|}{|z_1 - z_2|^{\tau}} = \frac{|W(Z_1) - W(Z_2)|}{|Z_1 - Z_2|^{\tau}} \le 3\mathcal{H}, \quad 0 < \mathcal{H} < 1.$$

Hence, we conclude
$$|w(z_1) - w(z_2)| \le \mathcal{T}|z_1 - z_2|^{\tau} \quad \text{for all } z_1, z_2 \in \overline{\Omega(z, R/2)},$$
with $\mathcal{T} = \max\{3\mathcal{H}, 2\delta_{\mathrm{H}}^{-\tau}\|w\|_{L^\infty(\Omega(z,R/2))}\}$ and $\delta_{\mathrm{H}} > 0$ given as above. □

We prove now the main result of this work.

Proof of Theorem 1.2. Given $z \in \overline{\Omega}$ and $0 < R < 1$, let v be the solution to (3.4) and $w = u - v$ a function satisfying (3.5). Thus, using (3.13) and Corollary 3.18, we conclude that, for any $x, y \in \overline{\Omega(z, R/2)}$,

$$\omega(u, R/2) \le \omega(w, R/2) + 2 \max_{x \in \Omega(z,R/2)} v(x) \le \mathcal{T}R^{\tau} + C(N, s, |\Sigma_{\mathcal{D}}|)\|f\|_{L^p(\Omega(z,R))} R^{2s-N/p} \le \mathcal{C}R^{\gamma},$$

where $\gamma = \min\{\tau, 2s - \frac{N}{p}\} < \frac{1}{2}$ and $\mathcal{C} = \max\{\mathcal{T}, 2C(N, s, |\Sigma_{\mathcal{D}}|)\|f\|_{L^p(\Omega(z,R))}\}$, with

$$\mathcal{T} = \max\{3\mathcal{H}, 2\delta_{\mathrm{H}}^{-\tau}\|w\|_{L^\infty(\Omega(z,R/2))}\} = \max\{3\mathcal{H}, 2\delta_{\mathrm{H}}^{-\tau}\|u - v\|_{L^\infty(\Omega(z,R/2))}\}.$$

Moreover, by Theorem 3.7,

$$\|u - v\|_{L^\infty(\Omega(z,R/2))} \le \|u\|_{L^\infty(\Omega(z,R))} + \|v\|_{L^\infty(\Omega(z,R))} \le 2C(N, s, |\Sigma_{\mathcal{D}}|)\|f\|_{L^p(\Omega(z,R))}.$$

Hence, we obtain
$$\mathcal{T} \le \max\{3\mathcal{H}, 4\delta_{\mathrm{H}}^{-\tau}C(N, s, |\Sigma_{\mathcal{D}}|)\|f\|_p\}.$$

Therefore, $\mathcal{C} = \max\{3\mathcal{H}, 4\delta_{\mathrm{H}}^{-\tau}C(N, s, |\Sigma_{\mathcal{D}}|)\|f\|_{L^p(\Omega(z,R))}\}$. Repeating the steps above in Theorem 3.17, we conclude
$$|u(x) - u(y)| \le \mathscr{H}|x - y|^{\gamma} \quad \text{for any } x, y \in \overline{\Omega(z, R/2)}, \tag{3.36}$$

where

$$\mathscr{H} = \max\Big\{9\mathcal{H}, \frac{C(N, s, |\Sigma_{\mathcal{D}}|)\|f\|_{L^p(\Omega(z,R))}}{\delta_{\mathrm{H}}^{\gamma}}\Big\},$$

and $\gamma = \min\{\tau, 2s - \frac{N}{p}\} < \frac{1}{2}$. Since the constants $\mathscr{H}$ and γ do not depend neither on z nor on R, to complete the proof, set $z_i \in \overline{\Omega}$, $i = 1, 2, \ldots, m$ and $R_i > 0$, small enough such that

$$\overline{\Omega} = \bigcup_{i=1}^{m} \Omega(z_i, R_i/4).$$

Then (3.36) follows by using a suitable recovering argument. □

4 Moving the boundary conditions

In this last part, we study the behavior of the solutions to problem (1.1) when we move the boundary conditions. First, let us describe this mixed moving boundary data framework. As introduced above, given $I_\varepsilon = [\varepsilon, |\partial\Omega|]$, let us consider the family of closed sets $\{\Sigma_{\mathcal{D}}(\alpha)\}_{\alpha\in I_\varepsilon}$, satisfying (B1)–(B3). We define $\Sigma_{\mathcal{N}}(\alpha) = \partial\Omega\setminus\Sigma_{\mathcal{D}}(\alpha)$ and $\Gamma(\alpha) = \Sigma_{\mathcal{D}}(\alpha) \cap \overline{\Sigma}_{\mathcal{N}}(\alpha)$. Observe that, under hypotheses (B1)–(B3), the limit sets $\Sigma_{\mathcal{D}}(\alpha)$, $\Sigma_{\mathcal{N}}(\alpha)$ as $\alpha \to \varepsilon^+$ are not degenerated sets (for instance, a Cantor-like set).

For a family of this type, we consider the corresponding family of mixed boundary value problems

$$\begin{cases} (-\Delta)^s u = f & \text{in } \Omega, \\ B_\alpha(u) = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

where $B_\alpha(u)$ means $B(u)$ with $\Sigma_{\mathcal{D}}$, $\Sigma_{\mathcal{N}}$, and Γ replaced by $\Sigma_{\mathcal{D}}(\alpha)$, $\Sigma_{\mathcal{N}}(\alpha)$, and $\Gamma(\alpha)$, respectively.

Our main aim here is to prove Theorem 1.3.

The key point in order to obtain it, is to prove that we can choose $\beta_s > 0$ in (3.11) independent of the measure of the Dirichlet part. Nevertheless, as we will see below, when one takes $\alpha \to 0^+$, the control of the Hölder norm of such a family is lost. Hence, it is necessary to fix a positive minimum $\varepsilon > 0$ on the measure of the family $\{\Sigma_{\mathcal{D}}(\alpha)\}_{\alpha\in I_\varepsilon}$ in order to guarantee the control on the Hölder norm for the family $\{u_\alpha\}_{\alpha\in I_\varepsilon}$.

Proof of Theorem 1.3. By the corresponding Hypotheses 1.1, there exists $\delta > 0$ such that $\overline{\rho}(Z) \geq \delta$ for all $Z \in \partial_L\mathscr{C}_\Omega$. Then the following hold:

(1) If $Z \in \overline{\mathscr{C}}_\Omega \setminus \Sigma_{\mathcal{D}}^*(\alpha)$, inequality (3.11) holds true with $\beta_s = c_s/\zeta\lambda$ independent of α, for all $0 < \rho < \delta$.

(2) If $Z \in \Sigma_{\mathcal{D}}^*(\alpha) \setminus \Gamma^*(\alpha)$, we can set $0 < \rho < \min\{\delta, \operatorname{dist}(Z, \Gamma^*(\alpha))\}$ such that for all $X \in \mathscr{C}_\Omega(Z, \rho)$,

$$\Pi(X, \Sigma_{\mathcal{D}}^* \cap B_\rho(Z), \mathscr{C}_\Omega(Z, \rho)) \geq \varphi > 0,$$

with φ independent from α, recalling that (according to [15, Section 4])

$$\Pi(x_0, E, A) = |\mathcal{V}_{x_0}(E) \cap \mathbb{S}_{N-1}(x_0)| = |\mathscr{S}_{x_0}|,$$

with $\mathcal{V}$ defined as follows: Given $x_0 \in A$ and a closed set $E \subset A$, let us consider the cone $\mathcal{V}_{x_0}(E) \subset A$ consisting on all rays starting at x_0 and ending at some point $P \in E$.

Hence, inequality (3.11) holds true with $\beta_s \leq \frac{c_s}{\varphi}$ also independent from α.

(3) If $Z \in \Gamma^*(\alpha)$, we can assume, without loss of generality, that for some neighborhood of radius $0 < \rho < \min\{\delta, \delta_\Gamma\}$ of the point $Z = (Z_1, \ldots, Z_{N+1})$, $\partial_L\mathscr{C}_\Omega$ coincides with the hyperplane $\mathbb{R}^{N+1} \cap \{x_N = 0\}$ and $\Gamma^*(\alpha) \subset \mathbb{R}_+^{N+1} \cap \{x_N = 0, x_{N-1} = 0\}$, in such a way that in $\Sigma_{\mathcal{D}}^*(\alpha)$, we have $x_{N-1} \geq 0$, and in $\Sigma_{\mathcal{N}}^*(\alpha)$, we have $x_{N-1} < 0$. Now, $\mathscr{C}_\Omega(Z, \rho)$ is transformed by the bi-Lipschitz transform (that in fact keeps the extension variable unchanged)

$$x_i = \xi_i, \quad i = 1, 2, \ldots, N-1, \qquad x_N = \begin{cases} \xi_N & \text{if } \xi_{N-1} < 0, \\ \xi_N - \xi_{N-1} & \text{if } \xi_{N-1} \geq 0, \end{cases}$$

into a set $\mathcal{O}_\rho(Z) = \mathcal{O}^1_\rho(Z) \cup \mathcal{O}^2_\rho(Z)$, with

$$\mathcal{O}^1_\rho(Z) = \Big\{\xi_N \geq 0,\ \xi_{N-1} < 0,\ \sum_{i=1}^{N}(\xi_i - Z_i)^2 + (y - Z_{N+1})^2 \leq \rho^2\Big\},$$

$$\mathcal{O}^2_\rho(Z) = \Big\{\xi_{N-1} \geq 0,\ \sum_{i=1}^{N-1}(\xi_i - Z_i)^2 + (y - Z_{N+1})^2 \leq \rho^2,$$
$$\xi_{N-1} \leq \xi_N \leq \xi_{N-1} + \Big(\rho^2 - \sum_{i=1}^{N-1}(\xi_i - Z_i)^2 - (y - Z_{N+1})^2\Big)^{1/2}\Big\}.$$

Moreover, $\Sigma^*_{\mathcal{D}} \cap B_\rho(Z)$ is transformed into the set

$$\mathcal{D}_\rho(Z) = \Big\{\xi_N = \xi_{N-1}, \xi_{N-1} \geq 0,\ \sum_{i=1}^{N-1}(\xi_i - Z_i)^2 + (y - Z_{N+1})^2 \leq \rho^2\Big\}.$$

Given $X_0 \in \mathcal{O}_\rho(Z)$, we use again the representation (see [15, cfr. 13.1]):

$$\Pi(X_0, \mathcal{D}_\rho(Z), \mathcal{O}_\rho(Z)) = \frac{1}{|\mathbb{S}_N(X_0)|}\int\limits_{\mathcal{D}_\rho(Z)} \frac{1}{|X_0 - Y|^N}\cos(\psi)\,d\sigma,$$

where $\cos(\psi) = \langle \frac{X_0-Y}{|X_0-Y|}, \vec{\nu}\rangle$, with $\vec{\nu}$ the normal vector to $\{\xi_N = \xi_{N-1}\} \cap \mathbb{R}^{N+1}_+$. Since $\cos(\psi)$ vanish only when $X_0 \in \mathcal{D}_\rho(Z)$, we conclude that $\Pi(X_0, \mathcal{D}_\rho(Z), \mathbb{R}^{N+1}_+) \geq \varphi > 0$ for all $X_0 \in \mathcal{O}_\rho(Z)$ and some $\varphi > 0$ independent of α. On the other hand, it is immediate that φ is independent of ρ. Hence, inequality (3.11) holds true with $\beta_s \leq \frac{c_s}{\varphi}$ also independent of α.

Let us define

$$\overline{\rho}_\alpha(Z) := \begin{cases} \min\{\delta, \operatorname{dist}(Z, \Sigma^*_{\mathcal{D}})\} & \text{if } Z \in \overline{\mathscr{C}}_\Omega \setminus \Sigma^*_{\mathcal{D}}(\alpha), \\ \min\{\delta, \operatorname{dist}(Z, \Gamma^*)\} & \text{if } Z \in \Sigma^*_{\mathcal{D}}(\alpha) \setminus \Gamma^*(\alpha), \\ \min\{\delta, \delta_\Gamma\} & \text{if } Z \in \Gamma^*(\alpha). \end{cases} \tag{4.2}$$

As a consequence of (1)–(3) above, we deduce the following:

(i) By (3.26), the constant Λ appearing in Theorem 3.12 and Theorem 3.15 is independent of α. Hence, inequality (3.28) does not depends on α, and also the number $0 < \mathcal{H} < 1$ in Theorem 3.16 is independent from α.

(ii) By (3.32), the constant $\overline{\eta}$ in Theorem 3.15 is independent from α and, by (3.33), also $0 < \gamma < \frac{1}{2}$ is independent from α.

Then, given u_α a solution to problem (4.1) with $\alpha \in I_\varepsilon$, by Theorem 1.2, we deduce

$$\|u_\alpha\|_{\mathcal{C}^\gamma(\Omega)} \leq \mathscr{H}_\alpha,$$

with $\gamma = \min\{\tau, 2s - \frac{N}{p}\} < \frac{1}{2}$ independent of α and $\mathscr{H}_\alpha = \max\{9\mathcal{H}, C(N,s,\alpha)\|f\|_p/\delta^\tau_{\mathrm{H},\alpha}\}$, with the constants $0 < \tau < \frac{1}{2}$ and $\delta_{\mathrm{H},\alpha}$ given as in Corollary 3.18. Now, if we consider the family $\{u_\alpha\}_{\alpha\in I_\varepsilon}$, since $\overline{\rho}_{\alpha_1}(Z) \leq \overline{\rho}_{\alpha_2}(Z)$, it is clear that $\delta_{\mathrm{H},\alpha_1} \leq \delta_{\mathrm{H},\alpha_2}$ and, therefore, $\mathscr{H}_{\alpha_1} \geq \mathscr{H}_{\alpha_2}$ for all $\alpha_1, \alpha_2 \in [\varepsilon, |\partial\Omega|]$, $\alpha_1 \leq \alpha_2$. Therefore, we can take $0 < \gamma < \frac{1}{2}$ and $\mathscr{H}_\varepsilon = \max\{9\mathcal{H}, C(N,s,\varepsilon)\|f\|_p/\delta^\tau_{\mathrm{H},\varepsilon}\}$ independent from α such that

$$\|u_\alpha\|_{\mathcal{C}^\gamma(\Omega)} \leq \mathscr{H}_\varepsilon.$$

To conclude, we observe that the condition $\alpha \in [\varepsilon, |\partial\Omega|]$ is necessary in order to control the Hölder norm of the family $\{u_\alpha\}_{\alpha\in I_\varepsilon}$. If we let $\alpha = |\Sigma_{\mathcal{D}}(\alpha)| \to 0^+$, then it is clear that $|\Sigma^*_{\mathcal{D}}(\alpha) \cap \overline{\mathscr{C}}_\Omega(Z,\rho)| \to 0$ for any $Z \in \overline{\mathscr{C}}_\Omega$ and $\rho > 0$. Thus, if $\alpha \to 0^+$, we conclude from (4.2) that $\overline{\rho}_\alpha(Z) \to 0$ for any $Z \in \Sigma^*_{\mathcal{D}}$, and hence $\delta_{\mathrm{H},\alpha} \to 0$ while $\mathscr{H}_\alpha \to +\infty$. □

Remark 4.1. Given an interphase point $Z \in \Gamma^*$, it is clear from (4.2) that we can choose a uniform $\rho_\varepsilon > 0$ in the lines of [8, Corollary 6.1]. In fact, it is enough to choose δ_Γ in (4.2) in such a way that $\Sigma^*_{\mathcal{D}}(\varepsilon) \cap \overline{\mathscr{C}}_\Omega(Z,\rho)$ is contained in some hyperplane (see (3) in the proof of Theorem 1.3). Clearly, this Dirichlet boundary part, say $(\{x_N = 0, x_{N-1} \geq 0\} \cap \mathbb{R}^{N+1}_+) \cap B_{\rho_\varepsilon}(Z)$, converges to an empty set as $\rho_\varepsilon \to 0$.

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