SEMILINEAR FRACTIONAL ELLIPTIC PROBLEMS WITH MIXED DIRICHLET-NEUMANN BOUNDARY CONDITIONS

José Carmona [1], **Eduardo Colorado** [2],
Tommaso Leonori [3], **Alejandro Ortega** [4]

Abstract

We study a nonlinear elliptic boundary value problem defined on a smooth bounded domain involving the fractional Laplace operator and a concave-convex term, together with mixed Dirichlet-Neumann boundary conditions.

MSC 2010: Primary 35J25; Secondary 35J61, 35J20

Key Words and Phrases: fractional Laplacian; mixed boundary conditions; concave-convex problem

1. Introduction

We study a nonlinear elliptic problem involving the fractional Laplace operator and a concave-convex power term together with mixed Dirichlet-Neumann boundary conditions. Namely,

$$\left\{\begin{array}{rcll} (-\Delta)^s u &=& \lambda u^q + u^r & \text{in } \Omega, \\ u &>& 0 & \text{in } \Omega, \\ u &=& 0 & \text{on } \Sigma_{\mathcal{D}}, \\ \dfrac{\partial u}{\partial \nu} &=& 0 & \text{on } \Sigma_{\mathcal{N}}, \end{array}\right. \tag{P_λ}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary, $N > 2s$, $(-\Delta)^s$, with $\frac{1}{2} < s < 1$, denotes the spectral fractional Laplace operator, $\lambda > 0$ is a real parameter and $0 < q \leq 1 < r < \frac{N+2s}{N-2s}$. In order to simplify the notation we denote the mixed boundary conditions as

$$B(u) = u\chi_{\Sigma_{\mathcal{D}}} + \frac{\partial u}{\partial \nu}\chi_{\Sigma_{\mathcal{N}}}, \tag{1.1}$$

where χ_A stands for the characteristic function of a set A and we assume that the boundary manifolds $\Sigma_{\mathcal{D}}$ and $\Sigma_{\mathcal{N}}$ are such that

pp. xxx–xxx, DOI: ....

$$(\mathfrak{B})\quad \begin{cases} \Sigma_{\mathcal{D}} \text{ and } \Sigma_{\mathcal{N}} \text{ are smooth } (N-1)\text{-dimensional submanifolds of } \partial\Omega. \\ \Sigma_{\mathcal{D}} \text{ is a closed manifold of positive } (N-1)\text{-dimensional Lebesgue} \\ \qquad \text{measure, } |\Sigma_{\mathcal{D}}| = \alpha \in (0, |\partial\Omega|). \\ \Sigma_{\mathcal{D}} \cap \Sigma_{\mathcal{N}} = \emptyset\,,\ \Sigma_{\mathcal{D}} \cup \Sigma_{\mathcal{N}} = \partial\Omega \text{ and } \Sigma_{\mathcal{D}} \cap \overline{\Sigma}_{\mathcal{N}} = \Gamma, \text{ where } \Gamma \text{ is a} \\ \qquad \text{smooth } (N-2)\text{-dimensional submanifold of } \partial\Omega. \end{cases}$$

Problems like (P_λ) have been studied in the last decades: with the classical Laplace operator and Dirichlet boundary condition, c.f. [8] or [3] for a deep study; with the Laplace operator and mixed Dirichlet-Neumann boundary conditions, cf. [1, 2, 16]; with the p-Laplace operator, cf. [8, 20, 21]; with fully nonlinear operators, cf. [13]; and more recently with the fractional Laplace operator and Dirichlet boundary conditions, cf. [6, 7, 9]. Up to our knowledge, this is the first work where the concave-convex problem is analyzed with the spectral fractional Laplace operator associated with mixed Dirichlet-Neumann boundary conditions.

The main result proven in this work is the following:

Theorem 1.1. *Assume that* $\frac{1}{2} < s < 1$, $N > 2s$ *and* $0 < q \le 1 < r < \frac{N+2s}{N-2s}$. *Then:*

1. *If* $q = 1$ *there exists at least one solution to* (P_λ) *for every* $0 < \lambda < \lambda_1^s$, *where* λ_1^s *denotes the first eigenvalue of the spectral fractional Laplacian with the boundary conditions* (1.1), *while there is no solution for* $\lambda \ge \lambda_1^s$. *Even more, there is a branch of solutions to* (P_λ) *bifurcating from* $(\lambda, u) = (\lambda_1^s, 0)$, *which cuts the axis* $\{\lambda = 0\}$.
2. *If* $0 < q < 1$ *there exists* $0 < \Lambda < \infty$ *such that:*
   - (a) *For* $0 < \lambda < \Lambda$ *there is a minimal solution to* (P_λ). *Moreover, the family of minimal solutions is increasing with respect to* λ.
   - (b) *For* $\lambda = \Lambda$ *there is at least one solution to* (P_λ).
   - (c) *For* $\lambda > \Lambda$ *there is no solution to* (P_λ).
   - (d) *Problem* (P_λ) *admits at least two solutions for every* $0 < \lambda < \Lambda$.

The following result deals with the sub-linear case $0 < q < 1$ and it provides a uniform $L^\infty(\Omega)$-bound for all the solutions to problems (P_λ) for any $0 < \lambda \le \Lambda$.

Theorem 1.2. *Assume that* $\frac{1}{2} < s < 1$, $N > 2s$, $0 < q < 1 < r < \frac{N+2s}{N-2s}$. *Then, there exists a constant* $C = C(N, s, \Omega, r, q) > 0$ *such that*

$$\sup_{x\in\Omega} u_\lambda(x) \le C,$$

for any solution u_λ to problems (P_λ) with $\lambda \in [0, \Lambda]$, and Λ defined in Theorem 1.1.

We also obtain uniform L^∞-estimates, in the case in which we move the boundary conditions. To be precise, we consider a family of sets $\{\Sigma_{\mathcal{D}}(\alpha)\}$, with $\alpha \in (0, |\partial\Omega|]$ and $|\cdot|$ denoting the Lebesgue measure in the appropriate dimension, such that:

(B$_1$) $\Sigma_{\mathcal{D}}(\alpha)$ is connected or has a finite number of connected components.
(B$_2$) $\Sigma_{\mathcal{D}}(\alpha_1) \subset \Sigma_{\mathcal{D}}(\alpha_2)$ if $\alpha_1 < \alpha_2$.
(B$_3$) $|\Sigma_{\mathcal{D}}(\alpha)| = \alpha$.

We call $\Sigma_{\mathcal{N}}(\alpha) = \partial\Omega \backslash \Sigma_{\mathcal{D}}(\alpha)$ and we assume that $\Sigma_{\mathcal{D}}(\alpha) \cap \overline{\Sigma}_{\mathcal{N}}(\alpha) = \Gamma(\alpha)$ is a $(N-2)$-dimensional smooth submanifold. For a family of this type we consider the corresponding family of mixed boundary value problems,

$$\left\{ \begin{array}{ll} (-\Delta)^s u = \lambda u^q + u^r & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ B_\alpha(u) = 0 & \text{on } \partial\Omega, \end{array} \right. \tag{$P_{\alpha,\lambda}$}$$

where $B_\alpha(u)$ is defined as $B(u)$ with $\Sigma_{\mathcal{D}}$, $\Sigma_{\mathcal{N}}$ replaced by $\Sigma_{\mathcal{D}}(\alpha)$, $\Sigma_{\mathcal{N}}(\alpha)$ satisfying the corresponding hypotheses $(\mathfrak{B}_\alpha)$ and (B_1)-(B_3). In this scenario we prove the following result.

THEOREM 1.3. *Consider the family $\{\Sigma_{\mathcal{D}}(\alpha)\}_{\alpha \in (0, |\partial\Omega|]}$ satisfying the hypotheses $(\mathfrak{B}_\alpha)$ and (B_1)-(B_3). For every $0 < \varepsilon < |\partial\Omega|$, let us denote $I_\varepsilon = [\varepsilon, |\partial\Omega|]$ and let*

$$\mathcal{S}_\varepsilon = \{u : \Omega \to \mathbb{R} | \text{ such that } u \text{ is solution of } (P_{\alpha,\lambda}), \text{ with } \alpha \in I_\varepsilon\}.$$

Then, there exists a constant $\mathcal{M}_\varepsilon > 0$ such that

$$\|u\|_{L^\infty(\Omega)} \leq \mathcal{M}_\varepsilon, \quad \forall u \in \mathcal{S}_\varepsilon.$$

In addition, we will also prove the following behavior for the minimal solutions as we move the boundary conditions.

THEOREM 1.4. *Consider the family $\{\Sigma_{\mathcal{D}}(\alpha)\}_{\alpha \in (0, |\partial\Omega|]}$ satisfying the hypotheses $(\mathfrak{B}_\alpha)$ and (B_1)-(B_3). Then*

1. *the minimal solutions $\{\underline{u}(\alpha)\}$ are uniformly bounded for any $\alpha \in [0, |\partial\Omega|]$. Moreover,*
$$\|\underline{u}(\alpha)\|_{H^s(\Omega)}, \; \|\underline{u}(\alpha)\|_{L^\infty(\Omega)} \to 0 \text{ as } \alpha \to 0;$$
2. *the non minimal solutions (of mountain pass type) are bounded and they converge to zero in $H^s(\Omega)$ as $\alpha \to 0$.*

The paper is organized as follows: In Section 2, we introduce the appropriate functional framework for the spectral fractional Laplace operator. In that section we also recall the extension technique due to Caffarelli and Silvestre, see [11], that provides an equivalent definition of the fractional Laplace operator via an auxiliary problem. In Section 3 we study a half-space problem that will be useful in the proof of the main theorem; we make use of the moving planes method and we extend some results of [17] to the fractional setting. Section 4 is devoted to the concave-convex problem by means of certain limit problems, and we also prove Theorem 1.2 and Theorem 1.3 which are based on the blow-up method of [23]. To accomplish this step we need some compactness properties that requires to know precise Hölder estimates for the solutions to mixed boundary problems. We use the results of [12] where the Hölder regularity of such solutions is proven. Section 5 is devoted to the proof of Theorem 1.1 and the behavior when we move the boundary conditions of some class of solutions.

2. **Functional setting and preliminaries**

As far as the fractional Laplace operator is concerned, we recall its definition given through the spectral decomposition. We closely follow the notation and framework of [12]. Let (φ_i, λ_i), $i \in \mathbb{N}$, be the eigenfunctions (normalized with respect to the $L^2(\Omega)$-norm) and the eigenvalues of $(-\Delta)$ equipped with homogeneous mixed Dirichlet-Neumann boundary data, respectively. Then the pairs (φ_i, λ_i^s), $i \in \mathbb{N}$, turn out to be the eigenfunctions and eigenvalues of the fractional operator $(-\Delta)^s$. Consequently, given two smooth functions $u_i(x)$, $i = 1, 2$, we have that $u_i(x) = \sum_{j\geq 1} \langle u_i, \varphi_j\rangle \varphi_j$, and thus

$$\langle (-\Delta)^s u_1, u_2\rangle = \sum_{j\geq 1} \lambda_j^s \langle u_1, \varphi_j\rangle \langle u_2, \varphi_j\rangle,$$

i.e., the action of the fractional operator on a function u_1 is given by

$$(-\Delta)^s u_1 = \sum_{j\geq 1} \lambda_j^s \langle u_1, \varphi_j\rangle \varphi_j.$$

Hence the operator $(-\Delta)^s$ is well defined for functions that belong to the fractional Sobolev Space that vanish on $\Sigma_{\mathcal{D}}$. Indeed for any smooth function we consider its spectral decomposition as

$$u = \sum_{j\geq 1} a_j \varphi_j \qquad \text{with} \qquad a_j = \langle u, \varphi_j\rangle \in \ell^2$$

that allows us to define the following norm

$$\|u\|^2_{H^s(\Omega)} = \sum_{j\geq 1} a_j^2 \lambda_j^s\,.$$

Thus we define the Sobolev space as

$$H_{\Sigma_{\mathcal{D}}}^s(\Omega) = \overline{C_0^\infty(\Omega \cup \Sigma_{\mathcal{N}})}^{\|\cdot\|_{H^s(\Omega)}}.$$

Observe that for any $u \in H_{\Sigma_{\mathcal{D}}}^s(\Omega)$,

$$\|u\|_{H_{\Sigma_{\mathcal{D}}}^s(\Omega)} = \left\|(-\Delta)^{\frac{s}{2}} u\right\|_{L^2(\Omega)}.$$

As already stressed in [25, Theorem 11.1], if $0 < s \leq \frac{1}{2}$ then $H_0^s(\Omega) = H^s(\Omega)$ and, therefore, also $H_{\Sigma_{\mathcal{D}}}^s(\Omega) = H^s(\Omega)$, while for $\frac{1}{2} < s < 1$, $H_0^s(\Omega) \subsetneq H^s(\Omega)$. Hence, the range $\frac{1}{2} < s < 1$, for which we have $H_{\Sigma_{\mathcal{D}}}^s(\Omega) \subsetneq H^s(\Omega)$, provides the correct functional space to study the mixed boundary problem (P_λ).

This definition of the fractional powers of the Laplace operator allows us to integrate by parts in the appropriate spaces, so that a natural definition of weak solution to problem (P_λ) is the following.

Definition 2.1. We say that a positive function $u \in H_{\Sigma_{\mathcal{D}}}^s(\Omega)$ is a solution to (P_λ) if

$$\int_\Omega (-\Delta)^{s/2} u \, (-\Delta)^{s/2} \psi dx = \int_\Omega (\lambda u^q + u^r) \, \psi dx, \quad \text{for all } \psi \in H_{\Sigma_{\mathcal{D}}}^s(\Omega).$$

Following the previous definition, we can associate to problem (P_λ) the following energy functional,

$$I_\lambda(u) = \frac{1}{2} \int_\Omega |(-\Delta)^{s/2} u|^2 dx - \frac{\lambda}{q+1} \int_\Omega |u|^{q+1} dx - \frac{1}{r+1} \int_\Omega |u|^{r+1} dx, \quad (2.1)$$

$u \in H_{\Sigma_{\mathcal{D}}}^s(\Omega)$, whose positive critical points correspond to solutions of (P_λ).

Working with the fractional operator $(-\Delta)^s$ it is well known that some difficulties arise when one tries to obtain explicit expressions of the action of the fractional Laplacian on, for example, products of functions. In order to overcome this difficulties, we use the ideas by Caffarelli and Silvestre, see [11], together with those of [9, 10] to give an equivalent definition of the operator $(-\Delta)^s$ by means of an auxiliary problem that we introduce next. Given a domain Ω, we set the cylinder $\mathcal{C}_\Omega = \Omega \times (0, \infty) \subset \mathbb{R}_+^{N+1}$. We denote by (x, y) points that belong to $\mathcal{C}_\Omega$ and with $\partial_L \mathcal{C}_\Omega = \partial\Omega \times [0, \infty)$ the lateral boundary of the cylinder.

Let us also denote by $\Sigma_{\mathcal{D}}^* = \Sigma_{\mathcal{D}} \times [0, \infty)$ and $\Sigma_{\mathcal{N}}^* = \Sigma_{\mathcal{N}} \times [0, \infty)$ as well as $\Gamma^* = \Gamma \times [0, \infty)$.

It is clear that, by construction,

$$\Sigma_{\mathcal{D}}^* \cap \Sigma_{\mathcal{N}}^* = \emptyset, \quad \Sigma_{\mathcal{D}}^* \cup \Sigma_{\mathcal{N}}^* = \partial_L \mathcal{C}_\Omega \quad \text{and} \quad \Sigma_{\mathcal{D}}^* \cap \overline{\Sigma_{\mathcal{N}}^*} = \Gamma^*.$$

Given a function $u \in H^s_{\Sigma_{\mathcal{D}}}(\Omega)$ we define its s-extension, denoted by $U = E_s[u]$, as the solution to the problem

$$\begin{cases} -\operatorname{div}(y^{1-2s}\nabla U) = 0 & \text{in } \mathcal{C}_\Omega, \\ B(U) = 0 & \text{on } \partial_L\mathcal{C}_\Omega, \\ U(x,0) = u(x) & \text{on } \Omega \times \{y=0\}, \end{cases}$$

where

$$B(U) = U\chi_{\Sigma^*_{\mathcal{D}}} + \frac{\partial U}{\partial \nu}\chi_{\Sigma^*_{\mathcal{N}}},$$

being ν, with an abuse of notation, the exterior normal to $\partial_L\mathcal{C}_\Omega$ (in fact, if ν denotes the outwards normal vector to $\partial\Omega$ and $\nu_{(x,y)}$ the outwards normal vector to $\mathcal{C}_\Omega$ then, by construction, $\nu_{(x,y)} = (\nu, y)$, $y > 0$). Following the well known result by Caffarelli and Silvestre (see [11]), U is related to the fractional Laplacian of the original function through the formula

$$\frac{\partial U}{\partial \nu^s} := -\kappa_s \lim_{y\to 0^+} y^{1-2s}\frac{\partial U}{\partial y} = (-\Delta)^s u(x),$$

where κ_s is a suitable positive constant (see [9] for its exact value). The extension function belongs to the space

$$H^1_{\Sigma^*_{\mathcal{D}}}(\mathcal{C}_\Omega, y^{1-2s}dxdy) := \overline{\mathcal{C}^\infty_0((\Omega\cup\Sigma_{\mathcal{N}})\times[0,\infty))}^{\|\cdot\|_{H^1_{\Sigma^*_{\mathcal{D}}}(\mathcal{C}_\Omega, y^{1-2s}dxdy)}},$$

that is a Hilbert space equipped with the norm induced by the scalar product

$$\langle U, V\rangle_{H^1_{\Sigma^*_{\mathcal{D}}}(\mathcal{C}_\Omega, y^{1-2s}dxdy)} = \kappa_s \int_{\mathcal{C}_\Omega} y^{1-2s}\langle\nabla U, \nabla V\rangle dxdy.$$

Moreover, the following inclusions are satisfied, for $\frac{1}{2} < s < 1$,

$$H^1_0(\mathcal{C}_\Omega, y^{1-2s}dxdy) \subset H^1_{\Sigma^*_{\mathcal{D}}}(\mathcal{C}_\Omega, y^{1-2s}dxdy) \subsetneq H^1(\mathcal{C}_\Omega, y^{1-2s}dxdy), \tag{2.2}$$

with $H^1_0(\mathcal{C}_\Omega, y^{1-2s}dxdy)$ the space of functions that belong to $H^1(\mathcal{C}_\Omega, y^{1-2s}dxdy)$ and vanish on the lateral boundary of $\mathcal{C}_\Omega$.

Consequently, we can reformulate problem (P_λ) in terms of the extension problem as follows:

$$\begin{cases} -\operatorname{div}(y^{1-2s}\nabla U) = 0 & \text{in } \mathcal{C}_\Omega, \\ B(U) = 0 & \text{on } \partial_L\mathcal{C}_\Omega, \\ U > 0 & \text{on } \Omega \times \{y=0\}, \\ \dfrac{\partial U}{\partial \nu^s} = \lambda U^q + U^r & \text{on } \Omega \times \{y=0\}. \end{cases} \tag{P^*_λ}$$

Hence we give a definition of energy solution of (P^*_λ) in the following way.

Definition 2.2. An energy solution to problem (P^*_λ) is a function U belonging to $H^1_{\Sigma^*_{\mathcal{D}}}(\mathcal{C}_\Omega, y^{1-2s}dxdy)$, with $U > 0$ on $\Omega \times \{y = 0\}$, such that

$$\kappa_s \int_{\mathcal{C}_\Omega} y^{1-2s}\langle \nabla U, \nabla \varphi\rangle \, dxdy = \int_\Omega \left(\lambda\, U^q(x,0) + U^r(x,0)\right)\varphi(x,0)dx,$$

for all $\varphi \in H^1_{\Sigma^*_{\mathcal{D}}}(\mathcal{C}_\Omega, y^{1-2s}dxdy)$.

To any energy solution $U \in H^1_{\Sigma^*_{\mathcal{D}}}(\mathcal{C}_\Omega, y^{1-2s}dxdy)$ to problem (P^*_λ) we can associate the function $u(x) = Tr[U(x,y)] = U(x,0)$, that belongs to $H^s_{\Sigma_{\mathcal{D}}}(\Omega)$, and solves problem (P_λ). Moreover, also the viceversa is true: given a solution $u \in H^s_{\Sigma_{\mathcal{D}}}(\Omega)$ we can define its s-extension $U(x,y)$ as a solution of (P^*_λ) with $U \in H^1_{\Sigma^*_{\mathcal{D}}}(\mathcal{C}_\Omega, y^{1-2s}dxdy)$. Thus, both formulations are equivalent and the *Extension operator*

$$E_s : H^s_{\Sigma_{\mathcal{D}}}(\Omega) \to H^1_{\Sigma^*_{\mathcal{D}}}(\mathcal{C}_\Omega, y^{1-2s}dxdy),$$

allows us to switch from (P_λ) to (P^*_λ).

According with [11, 9], due to the choice of the constant κ_s, the extension operator E_s is an isometry, i.e.,

$$\|E_s[\varphi](x,y)\|_{H^1_{\Sigma^*_{\mathcal{D}}}(\mathcal{C}_\Omega, y^{1-2s}dxdy)} = \|\varphi(x)\|_{H^s_{\Sigma_{\mathcal{D}}}(\Omega)}, \ \forall\ \varphi \in H^s_{\Sigma_{\mathcal{D}}}(\Omega).$$

It has also been proved in [9] that there exists $C_0 = C_0(N, s, r, |\Omega|)$ such that the *trace inequality*,

$$\int_{\mathcal{C}_\Omega} y^{1-2s}|\nabla z(x,y)|^2 dxdy \geq C_0 \left(\int_\Omega |z(x,0)|^r dx\right)^{\frac{2}{r}},$$

for any $z \in H^1_0(\mathcal{C}_\Omega, y^{1-2s}dxdy)$, provided $1 \leq r \leq 2^*_s$, $N > 2s$, where $2^*_s = \frac{2N}{N-2s}$ is the critical fractional Sobolev exponent. Such inequality turns out to be very useful and it is in fact equivalent to the fractional Sobolev inequality,

$$\int_\Omega |(-\Delta)^{\frac{s}{2}} v|^2 dx \geq C_0 \left(\int_\Omega |v|^r dx\right)^{\frac{2}{r}}, \quad \forall v \in H^s_0(\Omega),\ 1 \leq r \leq 2^*_s,\ N > 2s.$$

When mixed boundary conditions are considered, the situation is quite similar since the Dirichlet condition is imposed on a set $\Sigma_{\mathcal{D}} \subset \partial\Omega$ such that $|\Sigma_{\mathcal{D}}| = \alpha > 0$. Hence, thanks to (2.2), there exists a positive constant $S(\Sigma_{\mathcal{D}}) = S(N, s, \Sigma_{\mathcal{D}}, \Omega)$ such that

$$0 < S(\Sigma_{\mathcal{D}}) := \inf_{\substack{u \in H^s_{\Sigma_{\mathcal{D}}}(\Omega) \\ u \not\equiv 0}} \frac{\|u\|^2_{H^s_{\Sigma_{\mathcal{D}}}(\Omega)}}{\|u\|^2_{L^{2^*_s}(\Omega)}} \leq \inf_{\substack{u \in H^s_0(\Omega) \\ u \not\equiv 0}} \frac{\|u\|^2_{H^s_0(\Omega)}}{\|u\|^2_{L^{2^*_s}(\Omega)}}.$$

Remark 2.1. Actually, $S(\Sigma_{\mathcal{D}}) \leq 2^{-\frac{2s}{N}} C_0(N,s)$, see [15]. Moreover, taking in mind the spectral definition of the fractional operator and making use of the Hölder inequality, it follows that $S(\Sigma_{\mathcal{D}}) \leq |\Omega|^{\frac{2s}{N}} \lambda_1^s(\alpha)$, with $\lambda_1(\alpha)$

the first eigenvalue of the Laplace operator with mixed boundary conditions on the sets $\Sigma_{\mathcal{D}} = \Sigma_{\mathcal{D}}(\alpha)$ and $\Sigma_{\mathcal{N}} = \Sigma_{\mathcal{N}}(\alpha)$. Under geometrical assumptions (B_1)-(B_3) one has that, by [16, Lemma 4.3], $\lambda_1(\alpha) \to 0$ as $\alpha \searrow 0$ which shows that $S(\Sigma_{\mathcal{D}}) \to 0$ as $\alpha \searrow 0$.

Then, in analogy with the Dirichlet boundary data case, the following mixed trace inequality holds (see [12]).

LEMMA 2.1. *There exists a constant $C = C(N, s, r, \Sigma_{\mathcal{D}}, \Omega) > 0$ such that,*

$$\int_{\mathcal{C}_\Omega} y^{1-2s} |\nabla \varphi|^2 dxdy \geq C \left(\int_\Omega |\varphi(x,0)|^r dx \right)^{\frac{2}{r}}, \tag{2.3}$$

for any $\varphi \in H^1_{\Sigma^*_{\mathcal{D}}}(\mathcal{C}_\Omega, y^{1-2s}dxdy)$ and $1 \leq r \leq 2^*_s$, $N > 2s$, where $2^*_s = \frac{2N}{N-2s}$.

As a consequence,

$$\int_\Omega |(-\Delta)^{\frac{s}{2}} v|^2 dx \geq \kappa_s C \left(\int_\Omega |v|^r dx \right)^{\frac{2}{r}}, \quad \forall v \in H^s_{\Sigma_{\mathcal{D}}}(\Omega),\ 1 \leq r \leq 2^*_s,\ N > 2s.$$

Note that in case $r = 2^*_s$, then $\kappa_s C = S(\Sigma_{\mathcal{D}})$.

3. Moving planes and monotonicity

In this section we establish a monotonicity result for bounded solutions to $(-\Delta)^s u = u^r$ in $\mathbb{R}^N_+ \equiv \mathbb{R}^{N-1} \times \mathbb{R}_+$ satisfying the boundary conditions:

- $u = 0$ on $\Sigma_{\mathcal{D}}(\tau) = \{(x_1, \cdots, x_N) \in \mathbb{R}^N : x_N = 0, x_1 \leq \tau\}$, for some $\tau \in \mathbb{R}$.
- $\frac{\partial u}{\partial x_N} = 0$ on $\Sigma_{\mathcal{N}}(\tau) = \{(x_1, \cdots, x_N) \in \mathbb{R}^N : x_N = 0, x_1 > \tau\}$, for some $\tau \in \mathbb{R}$.

The principal result proven in this section is the following.

THEOREM 3.1. *Assume that $1 < r < \frac{N+2s}{N-2s}$, $N > 2s$, and $\tau \in \mathbb{R}$. Let $u \in H^s_{loc}(\mathbb{R}^N_+) \cap \mathcal{C}^0(\overline{\mathbb{R}^N_+})$ be a weak solution to*

$$\begin{cases} (-\Delta)^s u = u^r, \quad u > 0, & \text{in } \mathbb{R}^N_+, \\ u = 0 & \text{on } \Sigma_{\mathcal{D}}(\tau), \\ \frac{\partial u}{\partial x_N} = 0 & \text{on } \Sigma_{\mathcal{N}}(\tau). \end{cases} \tag{3.1}$$

Then, u is nondecreasing with respect to the x_1-direction.

REMARK 3.1. We make the proof assuming $\tau = 0$. For $\tau \neq 0$ the proof is analogous through a translation with respect to the variable x_1.

The proof of Theorem 3.1 is based on the moving planes method, introduced by Alexandrov and first exploited in the context of Partial Differential Equations by J. Serrin [27], see also [22] for more details.

Let us introduce some notations in order to apply the moving planes method. We denote by $\mathbb{R}^{N+1}_{++} \equiv \mathbb{R}^N_+ \times \mathbb{R}_+$, i.e., the set of points $X = (x, y)$ with $x = (x_1, \ldots, x_N)$ and $x_N, y > 0$. For a fixed $\rho \in \mathbb{R}$, we define the sets

$$\Upsilon_\rho = \{x \in \mathbb{R}^N_+ : \ x_1 < \rho\}, \quad \Upsilon^*_\rho = \Upsilon_\rho \times \mathbb{R}_+,$$

$$T_\rho = \{X \in \overline{\mathbb{R}^{N+1}_{++}} : \ x_1 = \rho\}.$$

For any $X \in \mathbb{R}^{N+1}_{++}$ the reflection with respect to the hyperplane T_ρ is denoted by

$$X^\rho = (x^\rho, y) = X + 2(\rho - x_1)e_1 = (2\rho - x_1, x_2, \ldots, x_N, y).$$

Let us define the point $O_\rho = (2\rho, 0, \ldots, 0, 0) \in \mathbb{R}^{N+1}$, whose reflection is the origin, and $o_\rho = (2\rho, 0, \ldots, 0) \in \mathbb{R}^N$. We also recall that the Kelvin transform of a nontrivial point $x \in \mathbb{R}^N$ is given by $\mathcal{K}(x) = \frac{x}{|x|^2}$. It is easy to see that $\mathcal{K}(\mathbb{R}^N_+) = \mathbb{R}^N_+$ and $\mathcal{K}\left(\Upsilon^*_\rho\right) = (\mathbb{R}^{N+1}_{++}) \cap B_{\frac{1}{-4\rho}}(O_{\frac{1}{4\rho}})$ for any $\rho < 0$.

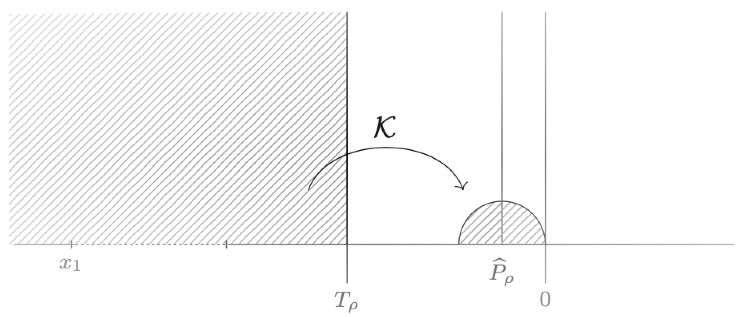

Fig. 1: The Kelvin Transform acting on the set Υ^*_ρ, with $\rho < 0$.

Next, we follow an approach similar to the one in [9] based on the fractional Kelvin transform, $\mathcal{K}_s(u)$, which acts on functions defined in a subset of $\mathbb{R}^N$, in the following way:

$$\mathcal{K}_s(u) = \frac{1}{|x|^{N-2s}} u\left(\mathcal{K}(x)\right) = \frac{1}{|x|^{N-2s}} u\left(\frac{x}{|x|^2}\right).$$

As proven in [9], if $(-\Delta)^s u = f(u)$, then the action of the fractional laplacian on the fractional Kelvin transform of u is given by

$$(-\Delta)^s \mathcal{K}_s(u) = \frac{1}{|x|^{N+2s}} f\left(u(\mathcal{K}(x))\right).$$

Let $u(x)$ be a solution to problem (3.1) and define $f(t) = t^r$ and $g(t) = \frac{f(t)}{t^{\frac{N+2s}{N-2s}}}$. Then, the Kelvin transform $v = \mathcal{K}_s(u)$ satisfies the following mixed BVP,

$$\begin{cases} (-\Delta)^s v = g(|x|^{N-2s} v) v^{\frac{N+2s}{N-2s}}, \quad v > 0, & \text{in } \mathbb{R}^N_+, \\ v = 0 & \text{on } \Sigma_{\mathcal{D}}(0), \\ \dfrac{\partial v}{\partial x_N} = 0 & \text{on } \Sigma_{\mathcal{N}}(0), \end{cases}$$

since on $\{x_N = 0\}$, we have

$$\frac{\partial v}{\partial x_N}(x) = (2s - N) \frac{x_N}{|x|^{N+2(1-s)}} u\left(\mathcal{K}(x)\right) + \frac{1}{|x|^{N-2s}} \frac{\partial u}{\partial x_N}\left(\mathcal{K}(x)\right) = 0.$$

Moreover, v is a continuous and positive function in $\mathbb{R}^N \backslash \{0\}$, with a possible singularity at the origin and it decays at infinity as $\frac{1}{|x|^{N-2s}} u(0)$, thus $v \in L^{2^*_s}(\mathbb{R}^N_+) \cap L^\infty(\mathbb{R}^N_+ \backslash B_r(0))$ for any $r > 0$. Finally, we consider $V = E_s[v]$ the extension function of the Kelvin transform $v = \mathcal{K}_s(u)$ and the corresponding extension problem,

$$\begin{cases} -\text{div}(y^{1-2s} \nabla V) = 0 & \text{in } \mathbb{R}^{N+1}_{++} \subset \mathbb{R}^{N+1}_+, \\ B(V) = 0 & \text{on } (\Sigma_{\mathcal{D}}(0) \cup \Sigma_{\mathcal{N}}(0)) \times \mathbb{R}_+, \\ \dfrac{\partial U}{\partial \nu^s} = g(|x|^{N-2s} v) v^{\frac{N+2s}{N-2s}} & \text{on } \Omega \times \{y = 0\}. \end{cases} \tag{3.2}$$

Observe that, since $v \in L^{2^*_s}(\mathbb{R}^N_+ \backslash B_r(0))$ for any $r > 0$ and the extension operator E_s is an isometry, by [19], the extension function $V \in L^{\overline{2}^*}(\Upsilon^*_\rho, y^{1-2s} dX)$ for any $\rho < 0$, where $\overline{2}^* = \frac{2(N+1)}{N-1}$ denotes to the Sobolev conjugate exponent in dimension $N + 1$.

The following lemma, which extends to the fractional framework [17, Lemma 2.1], provides us with a key-point inequality in order to obtain monotonicity in the x_1-direction for the function V defined in (3.2).

Here we use the notation $V_\rho(X) = V(X^\rho)$ and $v_\rho(x) = v(x^\rho)$ for the reflected functions that are singular at the point O_ρ and o_ρ respectively. Moreover we denote by $\mathcal{A}_\rho = \{x \in \Upsilon_\rho \backslash O_\rho : v \geq v_\rho\}$.

Lemma 3.1. *Assume that* $u \in H^s_{loc}(\mathbb{R}^N_+) \cap \mathcal{C}^0(\overline{\mathbb{R}^N_+})$ *is a weak solution of* (3.1) *and let* $v = \mathcal{K}_s(u)$. *Then, for any* $\rho < 0$, $(v - v_\rho)^+ \in H^s_{\Sigma_{\mathcal{D}}}(\Upsilon_\rho) \cap L^\infty(\Upsilon_\rho)$. *Moreover, there exists* $C_\rho > 0$, *increasing with respect to* ρ, *such that*

$$\int_{\Upsilon^*_\rho} y^{1-2s} |\nabla (V - V_\rho)^+|^2 dxdy \leq C_\rho \left(\int_{\mathcal{A}_\rho} \frac{1}{|x|^{2N}} dx \right)^{\frac{2s}{N}} \int_{\Upsilon^*_\rho} y^{1-2s} |\nabla (V - V_\rho)^+|^2 dxdy. \tag{3.3}$$

P r o o f. Since for a given $\rho < 0$ there exists $r > 0$ such that $\Upsilon_\rho \subset \mathbb{R}^N_+ \backslash B_r(0)$, the functions v and $(v - v_\rho)^+ \leq v$ belong to $L^{2^*_s}(\Upsilon_\rho) \cap L^\infty(\Upsilon_\rho)$ and the function $\frac{1}{|x|^{2N}}$ is integrable in Υ_ρ. The assertion $(v-v_\rho)^+ \in H^s_{\Sigma_\mathcal{D}}(\Upsilon_\rho)$ follows from (3.3) taking in mind that the extension operator E_s is an isometry.

In order to prove inequality (3.3) we test conveniently the equations

$$(-\Delta)^s v = g(|x|^{N-2s} v) v^{\frac{N+2s}{N-2s}}, \quad (-\Delta)^s v_\rho = g(|x^\rho|^{N-2s} v_\rho) v_\rho^{\frac{N+2s}{N-2s}},$$

in the set $\Upsilon_\rho \backslash O_\rho$. At this point, we make full use of the extension technique, so that we consider the extension functions $V = E_s[v]$ and $V_\rho = E_s[v_\rho] = V(X^\rho)$ and we set the nonnegative function $\varphi = \varphi_\varepsilon = \eta_\varepsilon^2 (V - V_\rho)^+$ as a test function in the corresponding extended problem for a convenient function η_ε. More precisely, for $\varepsilon > 0$ small enough we take $\eta_\varepsilon \in \mathcal{C}^1_0(\mathbb{R}^{N+1})$ with $0 \leq \eta_\varepsilon \leq 1$ and such that:

$$\begin{cases} \eta_\varepsilon \equiv 1 & \text{for } 2\varepsilon \leq |X - O_\rho| \leq \dfrac{1}{\varepsilon} \\ \eta_\varepsilon \equiv 0 & \text{for } |X - O_\rho| \leq \varepsilon \quad \text{or} \quad \dfrac{2}{\varepsilon} \leq |X - O_\rho|, \\ |\nabla \eta_\varepsilon| \leq \dfrac{c}{\varepsilon} & \text{for } \varepsilon < |X - O_\rho| < 2\varepsilon \\ |\nabla \eta_\varepsilon| \leq c\varepsilon & \text{for } \dfrac{1}{\varepsilon} < |X - O_\rho| < \dfrac{2}{\varepsilon}. \end{cases}$$

Observe that in the set Υ^*_ρ the function $(V - V_\rho)^+$ vanishes where the Dirichlet condition holds for V but also where the Dirichlet condition holds for the reflected function and, therefore, it is allowed to take $\varphi = \eta_\varepsilon^2 (V - V_\rho)^+$ as a test function in the corresponding extended problem.

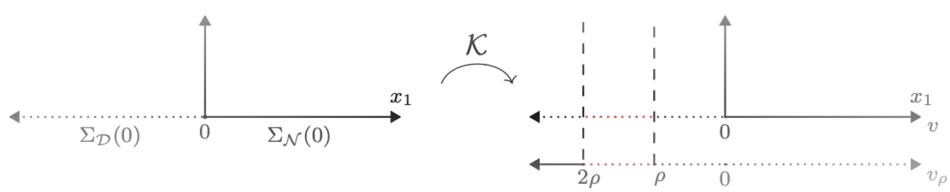

Fig. 2: The Kelvin transform centered at 0 acting on $\Sigma_\mathcal{D}(0)$ (doted line) and $\Sigma_\mathcal{N}(0)$ for the functions v and v_ρ.

Thus, using the definition of weak solution for the extended problem satisfied by V and V_ρ respectively and subtracting those expressions, we obtain

$$\kappa_s \int_{\Upsilon_\rho^*} y^{1-2s}\nabla(V-V_\rho)\nabla\varphi\, dxdy$$
$$= \int_{\Upsilon_\rho}\left(g(|x|^{N-2s}v)v^{\frac{N+2s}{N-2s}} - g(|x^\rho|^{N-2s}v_\rho)v_\rho^{\frac{N+2s}{N-2s}}\right)\varphi(x,0)dx.$$

On the other hand,

$$\kappa_s \int\limits_{\Upsilon_\rho^*\cap[2\varepsilon\le|X-O_\rho|\le\frac{1}{\varepsilon}]} y^{1-2s}|\nabla(V-V_\rho)^+|^2 dxdy \le \kappa_s \int_{\Upsilon_\rho^*} y^{1-2s}|\nabla(\eta_\varepsilon(V-V_\rho)^+)|^2 dxdy$$
$$= \kappa_s \int_{\Upsilon_\rho^*} y^{1-2s}\nabla(V-V_\rho)\nabla\varphi\, dxdy + \kappa_s \int_{\Upsilon_\rho^*} y^{1-2s}[(V-V_\rho)^+]^2|\nabla\eta_\varepsilon|^2 dxdy$$
$$= \kappa_s \int_{\Upsilon_\rho^*} y^{1-2s}\nabla(V-V_\rho)\nabla\varphi\, dxdy + I_\varepsilon$$
$$= \int_{\Upsilon_\rho}\left(g(|x|^{N-2s}v)v^{\frac{N+2s}{N-2s}} - g(|x^\rho|^{N-2s}v_\rho)v_\rho^{\frac{N+2s}{N-2s}}\right)\varphi(x,0)dx + I_\varepsilon.$$

Since g is a nonincreasing function, $|x| \ge |x^\rho|$ in Υ_ρ and $v \ge v_\rho$ in the set where $\varphi(\cdot,0)\ne 0$, it follows that $-g(|x^\rho|^{N-2s}v_\rho) \le -g(|x|^{N-2s}v)$ and therefore,

$$\kappa_s \int\limits_{\Upsilon_\rho^*\cap[2\varepsilon\le|X-O_\rho|\le\frac{1}{\varepsilon}]} y^{1-2s}|\nabla(V-V_\rho)^+|^2 dxdy$$
$$\int_{\Upsilon_\rho} g(|x|^{N-2s}v)\left(v^{\frac{N+2s}{N-2s}} - v_\rho^{\frac{N+2s}{N-2s}}\right)\varphi(x,0)dx + I_\varepsilon$$
$$\le \int_{\mathcal{A}_\rho} g(|x|^{N-2s}v)\left(v^{\frac{N+2s}{N-2s}} - v_\rho^{\frac{N+2s}{N-2s}}\right)\varphi(x,0)dx + I_\varepsilon. \tag{3.4}$$

Now, if $0 \le v_\rho \le v$ from the Mean Value Theorem, we find

$$v^{\frac{N+2s}{N-2s}} - v_\rho^{\frac{N+2s}{N-2s}} \le \frac{N+2s}{N-2s} v^{\frac{4s}{N-2s}}(v-v_\rho).$$

Using that $f(t) = t^r$ with $1 < r < \frac{N+2s}{N-2s}$, it follows that

$$g(t)t^{\frac{4s}{N-2s}} = \frac{f(t)}{t^{\frac{N+2s}{N-2s}}}t^{\frac{4s}{N-2s}} = \frac{f(t)}{t} = t^{r-1},$$

and $g(t)t^{\frac{4s}{N-2s}}$ is bounded in any interval $(0,t_0)$. Moreover, since $|x|^{N-2s}v(x)$ $= u\left(\frac{x}{|x|^2}\right)$ is bounded from above for $x \in \Upsilon_\rho$ and $\rho < 0$, we conclude

$$g(|x|^{N-2s}v)\left(v^{\frac{N+2s}{N-2s}}-v_\rho^{\frac{N+2s}{N-2s}}\right)\le \frac{N+2s}{N-2s}g(|x|^{N-2s}v)v^{\frac{4s}{N-2s}}(v-v_\rho)$$
$$\le \frac{N+2s}{N-2s}\frac{g(|x|^{N-2s}v)(|x|^{N-2s}v)^{\frac{4s}{N-2s}}}{|x|^{4s}}(v-v_\rho)\le \widetilde{C}_\rho\frac{1}{|x|^{4s}}(v-v_\rho),$$

for a positive constant $\widetilde{C}_\rho$, increasing with respect to ρ. Then, inequality (3.4) takes the form

$$\kappa_s\int\limits_{\Upsilon_\rho^*\cap[2\varepsilon\le|X-O_\rho|\le\frac{1}{\varepsilon}]}y^{1-2s}|\nabla(V-V_\rho)^+|^2dxdy$$
$$\le \widetilde{C}_\rho\int_{\mathcal{A}_\rho}\frac{1}{|x|^{4s}}(v-v_\rho)\varphi(x,0)dx+I_\varepsilon\le \widetilde{C}_\rho\int_{\mathcal{A}_\rho}\frac{1}{|x|^{4s}}\eta_\varepsilon^2(x,0)[(v-v_\rho)^+]^2dx+I_\varepsilon$$
$$\le \widetilde{C}_\rho\int_{\mathcal{A}_\rho}\frac{1}{|x|^{4s}}[(v-v_\rho)^+]^2dx+I_\varepsilon.$$

Using Hölder's inequality with $p=\frac{N}{2s}$ and $q=\frac{2_s^*}{2}$ we conclude

$$\kappa_s\int\limits_{\Upsilon_\rho^*\cap[2\varepsilon\le|X-O_\rho|\le\frac{1}{\varepsilon}]}y^{1-2s}|\nabla(V-V_\rho)^+|^2dxdy$$
$$\le \widetilde{C}_\rho\left(\int_{\mathcal{A}_\rho}\frac{1}{|x|^{2N}}dx\right)^{\frac{2s}{N}}\left(\int_{\Upsilon_\rho}[(v-v_\rho)^+]^{2_s^*}dx\right)^{\frac{2}{2_s^*}}+I_\varepsilon.$$

Next, we focus on the term $I_\varepsilon=\displaystyle\int_{\Upsilon_\rho^*}y^{1-2s}[(V-V_\rho)^+]^2|\nabla\eta_\varepsilon|^2dxdy$. Define the set

$$\mathcal{W}_\varepsilon=\left\{X\in\Upsilon_\rho^*:\ \varepsilon<|X-O_\rho|<2\varepsilon \text{ or } \frac{1}{\varepsilon}<|X-O_\rho|<\frac{2}{\varepsilon}\right\},$$

so that $supp(|\nabla\eta_\epsilon|^2)\subseteq\overline{\mathcal{W}_\varepsilon}$. Since $\left||\nabla\eta_\varepsilon|^{N+1}\chi_{\mathcal{W}_\varepsilon}\right|\le c(\frac{1}{\varepsilon^{N+1}}\varepsilon^{N+1}+\varepsilon^{N+1}\frac{1}{\varepsilon^{N+1}})$ $=c'$ and $(V-V_\rho)^+\in L^{\overline{2}^*}(\Upsilon_\rho^*,y^{1-2s}dxdy)$, applying Hölder's inequality with $p=\frac{N+1}{2}$ and $q=\frac{\overline{2}^*}{2}$, we find

$$I_\varepsilon\le\left(\int_{\mathcal{W}_\varepsilon}y^{1-2s}[(V-V_\rho)^+]^{\overline{2}^*}dxdy\right)^{\frac{2}{\overline{2}^*}}\left(\int_{\mathcal{W}_\varepsilon}y^{1-2s}|\nabla\eta_\varepsilon|^{N+1}dxdy\right)^{\frac{2}{N+1}}$$
$$\le C\left(\int_{\mathcal{W}_\varepsilon}y^{1-2s}[(V-V_\rho)^+]^{\overline{2}^*}dxdy\right)^{\frac{2}{\overline{2}^*}}\to 0 \text{ as } \varepsilon\to 0.$$

Letting ε go to 0 and applying the trace inequality (2.3), we conclude

$$\int_{\Upsilon_\rho^*}y^{1-2s}|\nabla(V-V_\rho)^+|^2dxdy$$

$$\leq \kappa_s^{-1}\widetilde{C}_\rho \left(\int_{\mathcal{A}_\rho} \frac{1}{|x|^{2N}}dx\right)^{\frac{2s}{N}} \left(\int_{\Upsilon_\rho} [(v-v_\rho)^+]^{2_s^*}dx\right)^{\frac{2}{2_s^*}}$$
$$\leq C_\rho \left(\int_{\mathcal{A}_\rho} \frac{1}{|x|^{2N}}dx\right)^{\frac{2s}{N}} \int_{\Upsilon_\rho^*} y^{1-2s}|\nabla(V-V_\rho)^+|^2 dxdy,$$

for a positive constant C_ρ increasing with respect to ρ. □

Proof of Theorem 3.1. The proof follows the lines of [17, Proposition 2.1] adapted to our framework. First, we establish a starting plane that delimits a hyperspace in which the monotonicity in the x_1-direction holds. Next we extend to such a region progressively until we reach the half-space, and in a second step, to the whole space having a special care to the singularity of the Kelvin transform at the origin. Since

$$\int_{\mathcal{A}_\rho} \frac{1}{|x|^{2N}}dx \leq \int_{\Upsilon_\rho} \frac{1}{|x|^{2N}}dx \to 0, \text{ as } \rho \to -\infty,$$

then there exists $-\infty < \rho_0 < 0$ such that

$$C_\rho \left(\int_{\mathcal{A}_\rho} \frac{1}{|x|^{2N}}dx\right)^{\frac{2s}{N}} < 1, \quad \text{for all} \quad \rho \in (-\infty, \rho_0).$$

From (3.3) we deduce that $(V-V_\rho)^+ \equiv 0$ in Υ_ρ^*, and therefore $V \leq V_\rho$ in Υ_ρ^* for all $\rho \in (-\infty, \rho_0)$. Consequently $v \leq v_\rho$ in Υ_ρ for any $\rho \in (-\infty, \rho_0)$.

Assume now that $\rho_0 < 0$ is maximal. By the Maximum Principle, $v < v_{\rho_0}$ in Υ_{ρ_0}. Then $\chi_{\mathcal{A}_\rho} \cdot \frac{1}{|x|^{2N}} \to 0$ point-wisely as $\rho \to \rho_0$ in $\mathbb{R}^N_+ \setminus \{T_{\rho_0} \cup O_{\rho_0}\}$.

Thus, if $\rho < \rho_0 + \delta < 0$ then $\chi_{\mathcal{A}_\rho} \cdot \frac{1}{|x|^{2N}} \leq \chi_{\Upsilon_{\rho_0+\delta}} \cdot \frac{1}{|x|^{2N}} \in L^1(\mathbb{R}^N_+)$ so that applying the Dominated Convergence Theorem

$$\int_{\mathcal{A}_\rho} \frac{1}{|x|^{2N}}dx \to 0, \quad \text{as } \rho \to \rho_0,$$

and we conclude

$$C_\rho \left(\int_{\mathcal{A}_\rho} \frac{1}{|x|^{2N}}dx\right)^{\frac{2s}{N}} < 1, \ \forall \rho \in (\rho_0, \rho_0+\delta),$$

for some $\delta > 0$ sufficiently small. Therefore $(V-V_\rho)^+ \equiv 0$ in Υ_ρ^* for $\rho \in (\rho_0, \rho_0+\delta)$ in contradiction with the maximality of ρ_0. As a consequence $V < V_\rho$ in Υ_ρ^* provided $\rho < 0$ and by continuity $V \leq V_0$ in Υ_0^*, so that $v \leq v_0$ in Υ_0. Noticing that $|x| = |x^\rho|$ for $\rho = 0$ we conclude $u \leq u_0$ in Υ_0.

The above argument works for the Kelvin transform centered at a point $P = P_\mu = (\mu, 0, \dots, 0) \in \mathbb{R}^N_+$, namely, $v^\mu(x) = \frac{1}{|x|^{N-2s}} u(P_\mu + \frac{x}{|x|^2})$ with $\mu \leq 0$ (see Figure 3).

This fractional Kelvin transform v^μ satisfies a Dirichlet condition in the part of the boundary with $x_N = 0$ and $x_1 < 0$ so we can prove as before that for any $\rho < 0$ the inequality $v^\mu \leq v^\mu_\rho$ holds in Υ_ρ. Since $\rho < 0$ is arbitrary, it follows that $v^\mu \leq v_0^\mu$ in Υ_0. Thus $u \leq u_\mu$ in Υ_μ for $\mu \leq 0$, so u is nondecreasing in the x_1-direction provided $x_1 < 0$.

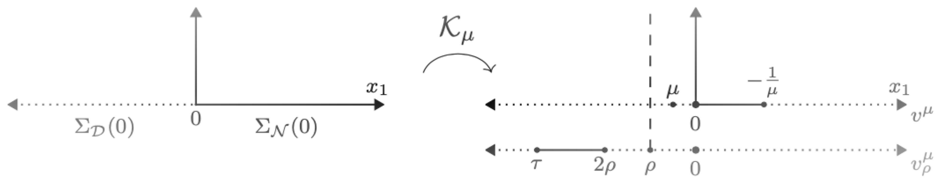

Fig. 3: The Kelvin transform centered at P_μ, $\mu \leq 0$ acting on $\Sigma_{\mathcal{D}}(0)$ (doted line) and $\Sigma_{\mathcal{N}}(0)$ for the functions v^μ and v_ρ^μ. The set $\Sigma_{\mathcal{N}}(0)$ is transformed into those $x \in \mathbb{R}^N_+$ such that $0 < x_1 < -\frac{1}{\mu}$, so v_ρ^μ satisfies a Neumann condition on $\tau < x_1 < 2\rho$ with $\tau = 2\rho + \frac{1}{\mu}$.

Now we extend progressively the region in which the monotonicity holds reaching Υ_μ for $\mu > 0$. First, observe that we cannot continue as before due to the singularity of the Kelvin transform at the origin: we cannot take a moving plane starting at $\rho = -\infty$ since for ρ large there are points where the Neumann boundary condition holds (and the solution is positive) which are reflected to the Dirichlet part of the boundary. In terms of the test functions, for ρ large enough the function $(V - V_\rho)^+$ is not allowed to be chosen as test function for the problem satisfied by the reflected function V_ρ , since it does not vanish at those points of the boundary where the Dirichlet condition for V_ρ holds.

Nevertheless, an inequality similar to (3.3) holds for $(v^\mu - v_\rho^\mu)^+$ if ρ is close to 0 so that we extend the inequality $v^\mu(x) < v_\rho^\mu(x) = v^\mu(x_\rho)$ for every $\rho < 0$ fixed, moving μ from $\mu = 0$ where the strict inequality is true up to $\mu = \frac{-1}{2\rho}$.

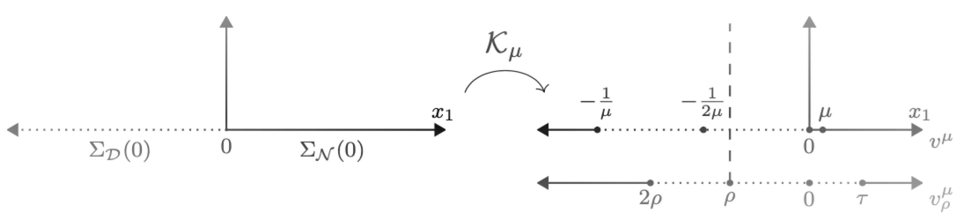

Fig. 4: The Kelvin transform centered at P_μ, $\mu \geq 0$ acting on $\Sigma_{\mathcal{D}}(0)$ (doted line) and $\Sigma_{\mathcal{N}}(0)$ for the functions v^μ and v_ρ^μ. The set $\Sigma_{\mathcal{D}}(0)$ is transformed into the $x \in \mathbb{R}^N_+$ such that $x_N = 0$ and $-\frac{1}{\mu} < x_1 < 0$, so the reflected function v_ρ^μ satisfies a Dirichlet condition on $2\rho < x_1 < \tau$ with $\tau = 2\rho + \frac{1}{\mu}$. It follows that for $x \in \Upsilon_\rho$ the function v^μ vanish where the Dirichlet condition holds for v_ρ^μ.

If $\mu \geq 0$, the fractional Kelvin transform centered at the point P_μ (denoted by $v^\mu(x)$) satisfies a Dirichlet boundary condition at points $x \in \mathbb{R}^N_+$ with $x_N = 0$ and $\frac{-1}{\mu} < x_1 < 0$ ($x_1 < 0$ if $\mu = 0$ as in the previous step) and a Neumann condition on the remaining part of the boundary. Then, if $-\frac{1}{2\mu} < \rho < 0$ it follows that V^μ, and hence $(V^\mu - V^\mu_\rho)^+$, vanishes where the Dirichlet condition holds for V^μ and also where the Dirichlet condition holds for the reflected function V^μ_ρ (therefore φ_ε is an allowed test function).

Thus, proceeding exactly as in the case $\mu = 0$, we obtain

$$\int_{\Upsilon^*_\rho} y^{1-2s}|\nabla(V^\mu - V^\mu_\rho)^+|^2 dxdy$$
$$\leq C_\rho \left(\int_{\mathcal{A}^\mu_\rho} \frac{1}{|x|^{2N}} dx \right)^{\frac{2s}{N}} \int_{\Upsilon^*_\rho} y^{1-2s}|\nabla(V^\mu - V^\mu_\rho)^+|^2 dxdy,$$

where C_ρ is increasing with respect to ρ and $\mathcal{A}^\mu_\rho = \{x \in \Upsilon_\rho \backslash O_\rho : v^\mu \geq v^\mu_\rho\}$.

If we now fix $\rho < 0$ the previous estimate holds for any $\mu \in (0, -\frac{1}{2\rho})$ and, since $\frac{1}{|x|^{2N}} \in L^1(\Upsilon_\rho)$, applying the Dominated Convergence Theorem we conclude $\chi_{\mathcal{A}^\mu_\rho} \cdot \frac{1}{|x|^{2N}} \to 0$ as $\mu \to 0$ in $\mathbb{R}^N \backslash \{T_\rho \cup P_\rho\}$, we recall that $P_\rho = (2\rho, 0, \ldots, 0)$ is the reflected point of the origin, which is the singular point of every transform V^μ. As a consequence

$$C_\rho \left(\int_{\mathcal{A}^\mu_\rho} \frac{1}{|x|^{2N}} dx \right)^{\frac{2s}{N}} < 1,$$

for some $\rho_0 \in (\frac{-1}{2\mu}, 0)$ and the monotonicity follows. Finally, suppose that $\mu_0 < -\frac{1}{2\rho_0}$ is maximal such that $v^\mu \leq v^\mu_\rho$ in Υ_ρ for all $0 < \mu < \mu_0$. Then, by the maximum principle, $v^\mu < v^\mu_\rho$ and hence $\mathcal{A}^\mu_\rho \to \emptyset$ as $\mu \to \mu_0$. Thus, there exists $\epsilon > 0$ such that

$$C_\rho \left(\int_{\mathcal{A}^\mu_\rho} \frac{1}{|x|^{2N}} dx \right)^{\frac{2s}{N}} < 1 \quad \text{for} \quad \mu \in (\mu_0, \mu_0 + \epsilon).$$

We conclude that $v^\mu \leq v^\mu_\rho$ for $\mu > \mu_0$ and close to μ_0 in contradiction with the maximality of μ_0.

To summarize, for every $\rho < 0$ and $\mu \leq -\frac{1}{2\rho}$ we have $v^\mu \leq v^\mu_\rho$ in Υ_ρ or, equivalently, fixed $\mu > 0$ the inequality holds for every $-\frac{1}{2\mu} < \rho < 0$. Letting $\rho \to 0$ we get $v^\mu \leq v^\mu_0$ in Υ_0, i.e., $v^\mu(x_1, x') \leq v^\mu(-x_1, x')$ for all x with $x_1 < 0$, so that $u \leq u_\mu$ in Υ_μ with $\mu > 0$. Since $\mu > 0$ is arbitrary we get that u is nondecreasing in the x_1-direction in whole $\mathbb{R}^N_+$. $\square$

REMARK 3.2. Let us observe that the method described in the above theorem in the x_1-direction may be applied to any other direction $x_2, \dots, x_{N-1}$, centered at any point P of the form $P = (0, P_2, \dots, P_{N-1}, 0)$, with a hyperplane orthogonal to both to the e_1 and e_n directions. Thus, due to the arbitrary of the point P, we can deduce that u does not depend to the $x_2, \dots, x_{N-1}$ variables.

4. A priori bounds in $L^\infty(\Omega)$.

In this section we prove Theorem 1.2 exploiting the blow-up method by Guidas-Spruck (see [23]). To this aim we will make use of the estimates proved in [12, Theorem 1.2] that guarantee the compactness needed in order to accomplish this limit step. Then, with the same ideas, we prove Theorem 1.3 using the uniform estimates proved in [12, Theorem 1.3] for the moving boundary conditions (as in hypotheses (B_1)-(B_3)).

Proof of Theorem 1.2. We argue by contradiction: set $\Lambda > 0$ given by Theorem 1.1 and assume that there exists sequences $\{\lambda_k\} \subset [0, \Lambda]$, $\{u_k\}$ of solutions to problems (P_{λ_k}) and $\{p_k\} \subset \overline{\Omega}$ of points verifying

$$M_k = \sup_{x\in\overline{\Omega}} u_k(x) = u_k(p_k) \to +\infty, \quad \text{as } k \to \infty.$$

Let us set $\mu_k = M_k^{-\frac{r-1}{2s}}$ and define the functions $v_k(y) = \frac{1}{M_k} u(p_k + \mu_k y)$. Note that $v_k(y)$ is defined in $\Omega_k = \frac{1}{\mu_k}(\Omega - p_k)$ as well as $v_k(0) = 1$ and $\|v_k\|_{L^\infty(\Omega_k)} \leq 1$ for all $k \geq 0$. Moreover, the scaled function v_k satisfies the problem

$$\begin{cases} (-\Delta)^s v_k = \lambda_k M_k^{q-r} v_k^q + v_k^r \quad v_k > 0, & \text{in } \Omega_k = \frac{1}{\mu_k}(\Omega - p_k), \\ v_k = 0 & \text{on } \Sigma_{\mathcal{D}}^k, \\ \dfrac{\partial v_k}{\partial \nu} = 0 & \text{on } \Sigma_{\mathcal{N}}^k, \end{cases}$$

where $\Sigma_{\mathcal{D}}^k$ and $\Sigma_{\mathcal{N}}^k$ are the transformed boundary manifolds.

Now we study the limit problem obtained as $k \to \infty$. To carry out this step we need some compactness properties for the sequence $\{v_k\}$ in order to guarantee the convergence in some sense. By [12, Theorem 1.2] the sequence $\{v_k\}$ is uniformly bounded in $C^\gamma(\overline{\Omega_k})$ for some $\gamma \in \left(0, \frac{1}{2}\right)$. Then, by the Ascoli-Arzelá Theorem, there exists a subsequence $\overline{\{v_k\}}$ uniformly convergent over compact sets in $\overline{\mathbb{R}_+^N}$ to a function $v \in C^\eta(\overline{\mathbb{R}_+^N})$ for some $0 < \eta < \gamma < \frac{1}{2}$. Moreover $\|v\|_{L^\infty(\mathbb{R}^N)} \leq 1$ and $v(0) = 1$.

On the other hand, the problem satisfied by the limit function v depends on the position of the point $p = \lim_{k\to\infty} p_k$. Let us set

$$d_k^{\mathcal{D}} = dist(p_k, \Sigma_{\mathcal{D}}^k) \quad \text{and} \quad d_k^{\mathcal{N}} = dist(p_k, \Sigma_{\mathcal{N}}^k).$$

and define $d_k^\Omega = \min\{d_k^{\mathcal{D}}, d_k^{\mathcal{N}}\}$. We distinguish several cases according to the behavior, up to subsequences, of the sequences $\frac{d_k^i}{\mu_k}$ with $i = \Omega, \mathcal{D}, \mathcal{N}$.

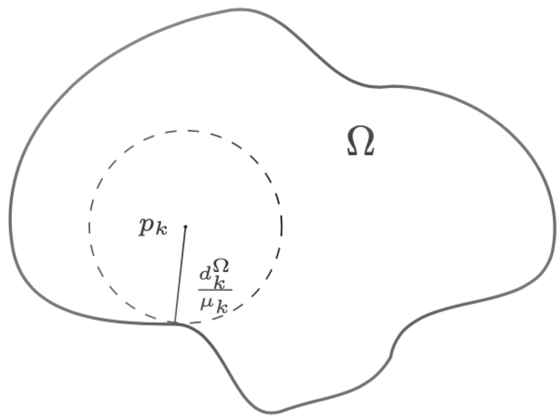

Fig. 5: The relevant geometry after dilation of variables lies in a neighbourhood of p_k such as the one of the picture.

1. **Interior case:** $\left\{\frac{d_k^\Omega}{\mu_k}\right\} \to +\infty$.

Since $B_{d_k^\Omega/\mu_k}(0) \subset \Omega_k$ (see Figure 5) we have that $\Omega_k \to \mathbb{R}^N$ and the limit function v is a positive bounded solution to

$$(-\Delta)^s v = v^r \quad \text{in } \mathbb{R}^N,$$

Then, by [14, Theorem 1] (see also [9, Theorem 3.1])we conclude $v \equiv 0$, in contradiction with $v(0) = 1$.

2. **Boundary Cases:** $\left\{\frac{d_k^\Omega}{\mu_k}\right\} \to d^\Omega \in \overline{\mathbb{R}}_+$.

In this situation we have several possibilities:

2.1 Dirichlet Case: $\left\{\frac{d_k^{\mathcal{D}}}{\mu_k}\right\} \to d^{\mathcal{D}} \in \overline{\mathbb{R}}_+$ and $\left\{\frac{d_k^{\mathcal{N}}}{\mu_k}\right\} \to +\infty$.

Now, as $\Sigma_{\mathcal{D}}$ is a $(N-1)$-dimensional smooth manifold, we have that, up to a rotation

$$\Omega_k \to \Omega_{d^{\mathcal{D}}} \equiv \{x \in \mathbb{R}^N : x_N > -d^{\mathcal{D}}\},$$

and the limit function v is a positive solution to

$$\begin{cases} (-\Delta)^s v = v^r & \text{in } \Omega_{d^{\mathcal{D}}} , \\ \qquad v = 0 & \text{in } \{x_N = -d^{\mathcal{D}}\}, \end{cases}$$

with $\|v\|_{L^\infty(\Omega_{d^{\mathcal{D}}})} \leq 1$ and $v(0) = 1$. Thus, if $d^{\mathcal{D}} = 0$ we have a contradiction with the continuity since $v(0) = 1$ while if $d^{\mathcal{D}} > 0$ we have a contradiction with [9, Theorem 3.4]

2.2 Neumann case: $\left\{\frac{d_k^{\mathcal{D}}}{\mu_k}\right\} \to +\infty$ and $\left\{\frac{d_k^{\mathcal{N}}}{\mu_k}\right\} \to d^{\mathcal{N}} \in \overline{\mathbb{R}}_+$.
As before, since $\Sigma_{\mathcal{N}}$ is a $(N-1)$-dimensional smooth manifold, we have that, up to rotation,

$$\Omega_k \to \Omega_{d^{\mathcal{N}}} \equiv \{x \in \mathbb{R}^N : x_N > -d^{\mathcal{N}}\},$$

and the limit function v is a positive solution to

$$\begin{cases} (-\Delta)^s v = v^r & \text{in } \Omega_{d^{\mathcal{N}}}, \\ \frac{\partial v}{\partial x_N} = 0 & \text{in } \{x_N = -d^{\mathcal{N}}\}, \end{cases}$$

with $\|v\|_{L^\infty(\Omega_{d^{\mathcal{N}}})} \le 1$ and $v(0) = 1$. Then, if we define the translated function $w(x) = v(x_1, x_2, \dots, x_N + d^{\mathcal{N}})$ it follows that

$$\begin{cases} (-\Delta)^s w = w^r & \text{in } \mathbb{R}^N_+, \\ \frac{\partial w}{\partial x_N} = 0 & \text{in } \{x_N = 0\}, \end{cases}$$

with $\|w\|_{L^\infty(\mathbb{R}^N_+)} \le 1$ and $w(0, 0, \dots, d^{\mathcal{N}}) = 1$. Extending to the whole space by reflection through the hyperplane $\{x_N = 0\}$, thanks to [9, Theorem 3.1], it follows that $w \equiv 0$ and we get a contradiction with $w(0, 0, \dots, d^{\mathcal{N}}) = 1$.

2.3 Interphase Case: $\left\{\frac{d_k^{\mathcal{D}}}{\mu_k}\right\} \to d^{\mathcal{D}} \in \overline{\mathbb{R}}_+$ and $\left\{\frac{d_k^{\mathcal{N}}}{\mu_k}\right\} \to d^{\mathcal{N}} \in \overline{\mathbb{R}}_+$.
Let us set $d^\Omega = \min\{d^{\mathcal{D}}, d^{\mathcal{N}}\} \ge 0$ and note that $\Sigma_{\mathcal{D}}^k$, $\Sigma_{\mathcal{N}}^k$ and $\Gamma_k = \Sigma_{\mathcal{D}}^k \cap \overline{\Sigma_{\mathcal{N}}^k}$ are smooth manifolds by hypotheses ($\mathfrak{B}$). Hence, we can assume that, up to a rotation,

$$\Omega_k \to \Omega_{d^\Omega} \equiv \{x \in \mathbb{R}^N : x_N > -d^\Omega\},$$

and the interphase $\Gamma_k \to \{x_1 = \tau\}$ for some finite $\tau \in \mathbb{R}$. Then the limit function v is a positive solution to

$$\begin{cases} (-\Delta)^s v = v^r & \text{in } \Omega_{d^\Omega}, \\ v = 0 & \text{in } \{x_N = -d^\Omega\} \cap \{x_1 \le \tau\}, \\ \frac{\partial v}{\partial x_N} = 0 & \text{in } \{x_N = -d^\Omega\} \cap \{x_1 > \tau\}, \end{cases}$$

with $\|v\|_{L^\infty(\Omega_{d^\Omega})} \le 1$ and $v(0) = 1$.

1) If $d^\Omega = 0$ and $\tau \ge 0$ we get a contradiction with the continuity of v, since the maximum is achieved at a point on the Dirichlet boundary where $v \equiv 0$.
2) If $d^\Omega > 0$ and $\tau \ge 0$ we get a contradiction with the monotonicity (Theorem 3.1) and the Hopf Lemma at the maximum point. Indeed it is sufficient to have the monotonicity of the solution v with respect to the x_1-direction up to $x_1 = \tau$.
3) If $\tau < 0$, we reach, once again, a contradiction with the monotonicity and the Hopf Lemma at the point of maximum. In this

step it is necessary to use the monotonicity of v with respect to the x_1-direction in the whole space.

□

With the same ideas, we can prove the next result concerning the moving boundary conditions.

Proof of Theorem 1.3. As we did in Theorem 1.2, we argue by contradiction. Assume that there exists a sequence $\{u_\alpha\}_{\alpha\in I_\varepsilon}$ of solutions to problems $(P_{\alpha,\lambda})$, a sequence of points $\{p_\alpha\}\subset\overline{\Omega}$, $\overline{\alpha}\in I_\varepsilon$ and a sequence of numbers $\mu_\alpha = M_\alpha^{\frac{1-r}{2s}}$ verifying

$$M_\alpha = \sup_{x\in\overline{\Omega}} u_\alpha(x) = u_\alpha(p_\alpha) \to +\infty, \text{ as } \alpha\to\overline{\alpha}\,.$$

We have to distinguish several cases. The interior, Dirichlet and Neumann cases can be proved following the corresponding cases in Theorem 1.2.

As far as the interface case is concerned, we need some compactness for the sequence $\{u_\alpha\}$ as $\alpha\to\overline{\alpha}$. Since we are considering sets $\Sigma_{\mathcal{D}}(\alpha)$ with $\alpha\in I_\varepsilon=[\varepsilon,|\partial\Omega|]$ for some $\varepsilon>0$ and satisfying hypotheses $(\mathfrak{B}_\alpha)$ and (B_1)-(B_3), by [12, Theorem 1.3] the sequence $\{u_\alpha\}$ is uniformly bounded in $C^\gamma(\overline{\Omega})$ for some $\gamma\in\left(0,\frac{1}{2}\right)$ and so the conclusion follows as in the corresponding case in Theorem 1.2. □

5. Minimal and mountain-pass solutions

We devote this section to the proof of Theorem 1.1, exploiting the extension technique. We recall that in terms of the s-extension, problem (P_λ) can be reformulated as

$$\begin{cases} -\mathrm{div}(y^{1-2s}\nabla U)=0 & \text{in } \mathcal{C}_\Omega,\\ B(U)=0 & \text{on } \partial_L\mathcal{C}_\Omega,\\ U>0 & \text{on } \Omega\times\{y=0\},\\ \dfrac{\partial U}{\partial\nu^s}=f_\lambda(U) & \text{on } \Omega\times\{y=0\},\end{cases}\qquad (P_\lambda^*)$$

where $f_\lambda(s)=\lambda|s|^{q-1}s+|s|^{r-1}s$. Associated to the problem (P_λ^*) we consider the Euler-Lagrange functional $J_\lambda : H^1_{\Sigma^*_{\mathcal{D}}}(\mathcal{C}_\Omega, y^{1-2s}dxdy)\to\mathbb{R}$ given by

$$J_\lambda(U)=\frac{\kappa_s}{2}\int_{\mathcal{C}_\Omega} y^{1-2s}|\nabla U|^2dxdy-\int_\Omega F_\lambda(U(x,0))dx,$$

where $F_\lambda(s)\equiv\int_0^s f_\lambda(\tau)d\tau$. Since J_λ does not satisfies the Palais-Smale (PS for short) condition, due to the unboundedness of the cylinder $\mathcal{C}_\Omega$, we show the PS condition for the functional I_λ.

LEMMA 5.1. *Let $\{u_n\} \subset H^s_{\Sigma_{\mathcal{D}}}(\Omega)$ be a PS sequence, i.e., $I_\lambda(u_n) \to c$ and $I'_\lambda(u_n) \to 0$. Then, there exist a subsequence (again denoted by) u_n strongly convergent in $H^s_{\Sigma_{\mathcal{D}}}(\Omega)$.*

P r o o f. Since $I_\lambda(u_n) \to c$ we have that $\|u_n\|_{H^s_{\Sigma_{\mathcal{D}}}(\Omega)} \le C$ uniformly for some positive constant. By the Sobolev embeddings, there exists a subsequence still denoted by $\{u_n\}$ such that

$$u_n \to u \quad \text{in } L^r(\Omega), \text{ for any } 1 \le r < 2^*_s, \tag{5.1}$$

and

$$u_n \rightharpoonup u \quad \text{in } H^s_{\Sigma_{\mathcal{D}}}(\Omega). \tag{5.2}$$

Using that $I'_\lambda(u_n) \to 0$ together with (5.1)-(5.2), we have the strong convergence proving the PS condition. □

P r o o f o f T h e o r e m 1.1-(1). Consider the eigenvalue problem associated to the first eigenvalue λ_1^s, and let φ_1 be the positive normalized in $L^2(\Omega)$ associated eigenfunction. Using φ_1 as a test function in problem (P_λ), we have

$$(\lambda_1^s - \lambda)\int_\Omega u\varphi_1 dx = \int_\Omega u^r \varphi_1 dx,$$

and hence necessarily $\lambda < \lambda_1^s$. On the other hand, using the fractional Sobolev inequality together with Poincaré inequality we find

$$\begin{aligned} I_\lambda(v) &= \frac{1}{2}\int_\Omega |(-\Delta)^{s/2} v|^2 dx - \frac{\lambda}{2}\int_\Omega |v|^2 dx - \frac{1}{r+1}\int_\Omega |v|^{r+1} dx \\ &\ge c_1\left(1 - \frac{\lambda}{\lambda_1^s}\right)\int_\Omega |(-\Delta)^{s/2} v|^2 dx - c_2\left(\int_\Omega |(-\Delta)^{s/2} v|^2 dx\right)^{(r+1)/2}, \end{aligned}$$

for positive constants c_1, c_2. Therefore, $v = 0$ is a local minimum for I_λ and, since $I_\lambda(tv) \to -\infty$ as $t \to \infty$, the functional I_λ satisfies the hypotheses of the Mountain Pass Theorem by Ambrosetti-Rabinowitz [4]. Hence, by Lemma 5.1, we obtain the existence of at least one solution for $0 < \lambda < \lambda_1^s$. Even more, the bifurcation result is a consequence of the classical Rabinowitz Theorem [26]. □

Next, in order to continue with the proof of Theorem 1.1, we establish some preliminary results. Some of these results can be proved for more general nonlinearities $f(u)$, with f at least continuous, satisfying the growth condition $0 \le f(s) \le c(1+|s|^p)$ for some $p > 0$. In such cases we will denote the associated extension problem as (P^*_f).

The first result deals with the sub and supersolutions method, the proof is rather standard and so we omit it.

Lemma 5.2. *Suppose that there exist a subsolution U_1 and a supersolution U_2 to (P_f^*), i.e., $U_1, U_2 \in H^1_{\Sigma^*_{\mathcal{D}}}(\mathcal{C}_\Omega, y^{1-2s}dxdy)$ such that $B(U_1) \leq 0$, $B(U_2) \geq 0$ on $\partial_L \mathcal{C}_\Omega$ and for every nonnegative $\phi \in H^1_{\Sigma^*_{\mathcal{D}}}(\mathcal{C}_\Omega, y^{1-2s}dxdy)$ the following inequalities are satisfied:*

$$\kappa_s \int_{\mathcal{C}_\Omega} y^{1-2s} \nabla U_1 \nabla \phi dxdy \leq \int_\Omega f(U_1(x,0))\phi(x,0)dx\,,$$
$$\kappa_s \int_{\mathcal{C}_\Omega} y^{1-2s} \nabla U_2 \nabla \phi dxdy \geq \int_\Omega f(U_2(x,0))\phi(x,0)dx\,,$$

respectively. Assume moreover that $U_1 \leq U_2$ in $\mathcal{C}_\Omega$. Then, there exists a solution U verifying $U_1 \leq U \leq U_2$ in $\mathcal{C}_\Omega$.

Next we deal with a comparison result.

Lemma 5.3. *Let $U_1, U_2 \in H^1_{\Sigma^*_{\mathcal{D}}}(\mathcal{C}_\Omega, y^{1-2s}dxdy)$ be respectively a positive subsolution and a positive supersolution to (P_f^*) and assume that $f(t)/t$ is decreasing for $t > 0$. Then $U_1 \leq U_2$ in $\mathcal{C}_\Omega$.*

P r o o f. The proof is similar to the proof of [3, Lemma 3.3]. By definition we have, for any positive test functions ϕ_1, $\phi_2 \in H^1_{\Sigma^*_{\mathcal{D}}}(\mathcal{C}_\Omega)$ that

$$\kappa_s \int_{\mathcal{C}_\Omega} y^{1-2s} \nabla U_1 \nabla \phi_1 dxdy \leq \int_\Omega f(u_1)\phi_1(x,0)dx,$$
$$\kappa_s \int_{\mathcal{C}_\Omega} y^{1-2s} \nabla U_2 \nabla \phi_2 dxdy \geq \int_\Omega f(u_2)\phi_2(x,0)dx,$$

where $u_1 = U_1(x,0)$ and $u_2 = U_2(x,0)$. Let $\theta(t)$ be a smooth non-decreasing function such that $\theta(t) = 0$ for $t \leq 0$, $\theta(t) = 1$ for $t \geq 1$, set $\theta_\varepsilon(t) = \theta(t/\varepsilon)$, and define the test functions φ_1 and φ_2 as

$$\varphi_1 = U_2 \theta_\varepsilon\left(U_1 - U_2\right), \quad \varphi_2 = U_1 \theta_\varepsilon\left(U_1 - U_2\right).$$

From the above inequalities we obtain

$$j_\varepsilon := \kappa_s \int_{\mathcal{C}_\Omega} y^{1-2s} \left\langle U_1 \nabla U_2 - U_2 \nabla U_1, \nabla(U_1 - U_2)\right\rangle \theta'_\varepsilon\left(U_1 - U_2\right) dxdy$$
$$\geq \int_\Omega u_1 u_2 \left(\frac{f(u_2)}{u_2} - \frac{f(u_1)}{u_1}\right) \theta_\varepsilon\left(u_1 - u_2\right) dx.$$

On the other hand,

$$\begin{aligned} j_\varepsilon &\le \kappa_s \int_{\mathcal{C}_\Omega} y^{1-2s} \langle \nabla U_1, (U_1 - U_2) \nabla (U_1 - U_2) \rangle \, \theta_\varepsilon' (U_1 - U_2) \, dxdy \\ &= \kappa_s \int_{\mathcal{C}_\Omega} y^{1-2s} \langle \nabla U_1, \nabla \eta_\varepsilon (U_1 - U_2) \rangle \, dxdy \\ &= \int_\Omega f(u_1) \eta_\varepsilon (u_1 - u_2) dx, \end{aligned}$$

where $\eta_\varepsilon'(t) = t\theta_\varepsilon'(t)$. Since $0 \le \eta_\varepsilon \le \varepsilon$, we find $j_\varepsilon \le c\varepsilon$. Then, letting $\varepsilon \to 0^+$ we conclude

$$\int\limits_{\Omega \cap \{u_1 > u_2\}} u_1 u_2 \left(\frac{f(u_2)}{u_2} - \frac{f(u_1)}{u_1} \right) dx \le 0.$$

Taking in mind the hypotheses on f, it follows $u_1 \le u_2$ in Ω. The result for the whole cylinder $\mathcal{C}_\Omega$ follows by the maximum principle. □

Next we focus on the remaining assertions in Theorem 1.1-(2). Thus, from now on we assume that $0 < q < 1$.

LEMMA 5.4. *Let Λ be defined by*

$$\Lambda = \sup\{\lambda > 0 : (P_\lambda) \text{ has solution}\},$$

then, $0 < \Lambda < \infty$.

P r o o f. As for the linear case, consider the eigenvalue problem associated to the first eigenvalue λ_1^s, and let φ_1 the associated eigenfunction. Using φ_1 as a test function in problem (P_λ), we have

$$\int_\Omega (\lambda u^q + u^r) \varphi_1 dx = \lambda_1^s \int_\Omega u \varphi_1 dx. \tag{5.3}$$

Since there exists a constant $c = c(r,q) > 1$ such that $\lambda t^q + t^r > c\lambda^\delta t$ with $\delta = \frac{r}{r-q}$, for any $t > 0$, from (5.3) we deduce $c\lambda^\delta < \lambda_1^s$ and hence $\Lambda < \infty$. In particular, this also proves that there is no solution to (P_λ) for $\lambda > \Lambda$.

In order to prove that $\Lambda > 0$, we prove, by means of the sub and supersolution technique, the existence of solution to (P_λ^*) for any small positive λ. Indeed, for $\varepsilon > 0$ small enough, $\underline{U} = \varepsilon E_s[\varphi_1]$ is a subsolution to (P_λ^*). A supersolution can be constructed as an appropiate multiple of the function G, the solution to

$$\begin{cases} -\text{div}(y^{1-2s} \nabla G) = 0 & \text{in } \mathcal{C}_\Omega, \\ B(G) = 0 & \text{on } \partial_L \mathcal{C}_\Omega, \\ \dfrac{\partial G}{\partial \nu^s} = 1 & \text{on } \Omega \times \{y = 0\}. \end{cases}$$

Since the trace function $g(x) = G(x,0)$ is a solution to

$$\begin{cases} (-\Delta)^s g = 1 & \text{in } \Omega, \\ B(g) = 0 & \text{on } \partial\Omega, \end{cases}$$

by [12, Theorem 3.7] we have $\|g\|_{L^\infty(\Omega)} < +\infty$. Next, since $0 < q < 1 < r$ we can find $\lambda_0 > 0$ such that for all $0 < \lambda \leq \lambda_0$ there exists $M = M(\lambda)$ such that

$$M \geq \lambda M^q \|g\|^q_{L^\infty(\Omega)} + M^r \|g\|^r_{L^\infty(\Omega)}. \tag{5.4}$$

As a consequence, the function $h = Mg$ satisfies $M = (-\Delta)^s h \geq \lambda h^q + h^r$ and, by the maximum principle, the extension function $\overline{U} = E_s[h]$ is a supersolution and $\underline{U} \leq \overline{U}$. Applying Lemma 5.2 we conclude the existence of a solution U to problem (P^*_λ). Therefore, its trace $u(x) = U(x,0)$ is a solution to problem (P_λ), $\lambda < \lambda_0$. □

REMARK 5.1. In the proof of Lemma 5.4, precisely in (5.4), we can choose $M = M(\lambda)$ verifying $M(\lambda) \to 0$ as $\lambda \to 0$, proving that $\|u_\lambda\|_{L^\infty(\Omega)} \to 0$ as $\lambda \to 0$. Indeed, it is enough to choose $M(\lambda) = \lambda^\eta$ with $0 < \eta < \frac{1}{1-q}$.

REMARK 5.2. Although Lemma 5.4 provides the existence of a solution for small $\lambda > 0$, we can also prove this result studying the associated functional I_λ. Indeed,

$$\begin{aligned} I_\lambda(v) &= \frac{1}{2}\int_\Omega |(-\Delta)^{s/2} v|^2 dx - \frac{\lambda}{q+1}\int_\Omega |v|^{q+1} dx - \frac{1}{r+1}\int_\Omega |v|^{r+1} dx \\ &\geq \frac{1}{2}\int_\Omega |(-\Delta)^{s/2} v|^2 dx - \lambda c_1 \left(\int_\Omega |(-\Delta)^{s/2} v|^2 dx\right)^{(q+1)/2} \\ &\quad - c_2 \left(\int_\Omega |(-\Delta)^{s/2} v|^2 dx\right)^{(r+1)/2}, \end{aligned}$$

for some positive constants c_1 and c_2. Then, for sufficiently small λ, there exist (at least) two solutions to problem (P_λ), one given by minimization and another given by the Mountain-Pass Theorem. The proof is rather common, based on the geometry of the function $g(t) = \frac{1}{2}t^2 - \lambda c_1 t^{q+1} - c_2 t^{r+1}$ (see for instance [4]).

Next we show that there exists a solution for every $\lambda \in (0, \Lambda)$.

LEMMA 5.5. *Problem (P_λ) has at least a positive minimal solution for every $0 < \lambda < \Lambda$. Moreover, the family $\{u_\lambda\}$ of minimal solutions is increasing with respect to λ.*

P r o o f. By definition of Λ, for any $0 < \lambda < \Lambda$ there exists $\mu \in (\lambda, \Lambda]$ such that (P^*_μ) admits a solution U_μ. It is easy to see that U_μ is a supersolution for (P^*_λ).

On the other hand, let V_λ be the unique solution to problem (P^*_f) with $f(t) = \lambda t^q$ (the existence can be deduced by minimization, while uniqueness follows from Lemma 5.3). It is clear that V_λ is a subsolution to problem (P^*_λ) and, by Lemma 5.3, we have $V_\lambda \le U_\mu$. Therefore, thanks to Lemma 5.2, we conclude that there is a solution to (P^*_λ) and, as a consequence, for the whole open interval $(0, \Lambda)$.

Finally, we prove the existence of a minimal solution for all $0 < \lambda < \Lambda$. Indeed, given a solution u to (P_λ) we take $U = E_s(u)$ and, by Lemma 5.3 being U solution to problem (P^*_λ), it satisfies $V_\lambda \le U$ with V_λ solution to problem (P^*_f) with $f(t) = \lambda t^q$. Then, the function $v_\lambda(x) = V_\lambda(x, 0)$ is a subsolution of problem (P_λ) and the monotone iteration procedure described by

$$(-\Delta)^s u_{n+1} = \lambda u_n^q + u_n^r, \quad u_n \in H^s_{\Sigma_{\mathcal{D}}}(\Omega) \quad \text{with} \quad u_0 = v_\lambda,$$

verifies $u_n \le U(x, 0) = u$ and $u_n \nearrow u_\lambda$ with u_λ solution to problem (P_λ). In particular $u_\lambda \le u$ and we conclude that u_λ is a minimal solution. The monotonicity follows directly from the first part of the proof, taking $U_\mu = E_s(u_\mu)$ which leads to $u_\lambda \le u_\mu$ whenever $0 < \lambda < \mu \le \Lambda$. □

Lemma 5.6. *Problem (P^*_λ) has at least one solution if $\lambda = \Lambda$.*

To prove Lemma 5.6 we extend [3, Lemma 3.5] to the fractional framework. This result guarantees that the linearized equation corresponding to (P_λ) has non-negative eigenvalues at the minimal solution.

Proposition 5.1. *Let u_λ be the minimal solution to (P_λ) and define $a_\lambda = a_\lambda(x) = \lambda q u_\lambda^{q-1} + r u_\lambda^{r-1}$. Then, the operator $[(-\Delta)^s - a_\lambda(x)]$ with mixed boundary conditions has a first eigenvalue $\nu_1 \ge 0$. In particular it follows that*

$$\int_\Omega \left(|(-\Delta)^{s/2} v|^2 - a_\lambda v^2 \right) dx \ge 0, \quad \text{for any} \quad v \in H^s_{\Sigma_{\mathcal{D}}}(\Omega). \tag{5.5}$$

P r o o f. By contradiction, assume that $\nu_1 < 0$ and let $\phi_1 > 0$ be the first eigenfunction. Let $\alpha > 0$ and observe that since $0 < q < 1$,

$$
\begin{aligned}
&(-\Delta)^s(u_\lambda-\alpha\phi_1)-\left(\lambda(u_\lambda-\alpha\phi_1)^q+(u_\lambda-\alpha\phi_1)^r\right)\\
&=\lambda u_\lambda^q+u_\lambda^r-\alpha\nu_1\phi_1-\alpha\left(\lambda q u_\lambda^{q-1}+ru_\lambda^{r-1}\right)\phi_1-\lambda(u_\lambda-\alpha\phi_1)^q-(u_\lambda-\alpha\phi_1)^r\\
&\geq u_\lambda^r-\alpha\nu_1\phi_1-\alpha r u_\lambda^{r-1}\phi_1-(u_\lambda-\alpha\phi_1)^r\\
&=-\alpha\nu_1\phi_1+o(\alpha\phi_1).
\end{aligned}
$$

Using that $\nu_1<0$, $\phi_1>0$, for $\alpha>0$ sufficiently small we have that

$$(-\Delta)^s(u_\lambda-\alpha\phi_1)-\left(\lambda(u_\lambda-\alpha\phi_1)^q+(u_\lambda-\alpha\phi_1)^r\right)\geq 0,$$

proving that $u_\lambda-\alpha\phi_1$ is a supersolution of (P_λ).

Now, let $\psi=\lambda^{\frac{1}{q-1}}v$, with v a solution to

$$\left\{\begin{array}{ll}(-\Delta)^s v=v^q & \text{in } \Omega,\\ B(v)=0 & \text{on } \partial\Omega,\end{array}\right. \tag{5.6}$$

that turns out to be a subsolution of (P_λ).

Then $\psi\leq u_\lambda-\alpha\phi_1$ and problem (P_λ) has a solution $\widetilde{u}$ such that $\psi\leq\widetilde{u}\leq u_\lambda-\alpha\phi_1$ in contradiction with the minimality of u_λ. □

Proof of Lemma 5.6. Let $\{\lambda_n\}$ be a sequence such that $\lambda_n\nearrow\Lambda$ and denote by $u_n=u_{\lambda_n}$ the minimal solution to problem (P_{λ_n}). Let $U_n=E_s[u_n]$, then

$$I_{\lambda_n}(u_n)=\frac{1}{2}\int_\Omega|(-\Delta)^{\frac{s}{2}}u_n|^2dx-\frac{\lambda_n}{q+1}\int_\Omega u_n^{q+1}dx-\frac{1}{r+1}\int_\Omega u_n^{r+1}dx.$$

Moreover, since u_n is a solution to (P_{λ_n}), it also satisfies

$$\int_\Omega|(-\Delta)^{\frac{s}{2}}u_n|^2dx=\lambda_n\int_\Omega u_n^{q+1}dx+\int_\Omega u_n^{r+1}dx.$$

On the other hand, using (5.5) with $v=u_n$,

$$\int_\Omega|(-\Delta)^{\frac{s}{2}}u_n|^2dx-\lambda_n q\int_\Omega u_n^{q+1}dx-r\int_\Omega u_n^{r+1}dx\geq 0.$$

As in [3, Lemma 3.5], we conclude $I_{\lambda_n}(u_n)<0$. Since $I'_{\lambda_n}(u_n)=0$, we obtain that $\|u_n\|_{H^s_{\Sigma_{\mathcal{D}}}(\Omega)}\leq C$. Hence, there exists a weakly convergent subsequence $u_n\to u\in H^s_{\Sigma_{\mathcal{D}}}(\Omega)$ and, as a consequence, u is a weak solution of (P_λ) for $\lambda=\Lambda$. □

Next we assure the existence of a second solution to (P_λ) for every $0<\lambda<\Lambda$ following the ideas of [5], developed to concave-convex problems in [2, 9] for the classical Laplacian and the fractional Laplacian respectively. In order to find a second solution by means of variational methods it is essential to have a first solution which is also a local minimum of the associated functional J_λ.

LEMMA 5.7. *Problem* (P_λ) *has at least two solutions for each* $\lambda \in (0, \Lambda)$.

P r o o f. The proof follows exactly as in [9, Lemma 5.11]. □

Now we can conclude the proof of Theorem 1.1.

P r o o f o f T h e o r e m 1.1-(2). Part a) follows by Lemma 5.5. Moreover part b) is a consequence of Lemma 5.6, part c) of Lemma 5.4 while part d) holds true thanks to Lemma 5.7. □

5.1. **Moving the boundary conditions.** Now we prove Theorem 1.4, i.e., the assertions on the behavior of the minimal and mountain pass solutions when we move the boundary conditions (see hypotheses (B_1)-(B_3)). To this aim, we need the following result.

LEMMA 5.8. *Let* v *be the solution to problem* (5.6). *There exists a constant* $\beta > 0$ *such that*

$$\|\phi\|^2_{H^s_{\Sigma_{\mathcal{D}}}(\Omega)} - q\int_\Omega v^{q-1}\phi^2 dx \geq \beta\|\phi\|^2_{L^2(\Omega)}, \quad \text{for all } \phi \in H^s_{\Sigma_{\mathcal{D}}}(\Omega). \tag{5.7}$$

P r o o f. Since we always consider boundary conditions such that $|\Sigma_{\mathcal{D}}| = \alpha > 0$, the function v can be obtained as

$$\min\left\{\|\phi\|^2_{H^s_{\Sigma_{\mathcal{D}}}(\Omega)} - \frac{1}{q+1}\|\phi\|^{q+1}_{L^{q+1}(\Omega)} : \phi \in H^s_{\Sigma_{\mathcal{D}}}(\Omega)\right\},$$

and thus, by (5.5)

$$\|\phi\|^2_{H^s_{\Sigma_{\mathcal{D}}}(\Omega)} - q\int_\Omega v^{q-1}\phi^2 dx \geq 0, \quad \text{for all } \phi \in H^s_{\Sigma_{\mathcal{D}}}(\Omega).$$

As a consequence, the linearized problem

$$\left\{\begin{array}{rl} (-\Delta)^s\varphi - qv^{q-1}\varphi = \mu\varphi & \text{in } \Omega, \\ B(\varphi) = 0 & \text{on } \partial\Omega, \end{array}\right. \tag{5.8}$$

has a non-negative first eigenvalue μ_1. Let φ_1 be the first eigenfunction and assume $\mu_1 = 0$. Since v is a solution to (5.6), then

$$q\int_\Omega v^q\varphi_1 dx = \int_\Omega v^q\varphi_1 dx.$$

which is a contradiction. Hence $\mu_1 > 0$. □

LEMMA 5.9. *There exists* $A > 0$ *such that for all* $\lambda \in (0, \Lambda)$ *the problem* (P_λ) *has at most one solution satisfying* $\|u\|_{L^\infty(\Omega)} < A$.

P r o o f. Let $A > 0$ such that $rA^{r-1} < \beta$, with β given by (5.7). Assumme by contradiction that there exists a second solution $u = u_\lambda + w$ of (P_λ) such that $\|u\|_{L^\infty(\Omega)} \le A$. Since u_λ is the minimal solution, $w \ge 0$. Let $\zeta(x) = \lambda^{\frac{1}{1-q}} v(x)$ with v the solution to (5.6), so that $(-\Delta)^s \zeta = \lambda \zeta^q$. Moreover, u_λ is also a supersolution of (5.6), and hence, by Lemma 5.3, $u_\lambda \ge \lambda^{\frac{1}{1-q}} v$. On the other hand, since $u = u_\lambda + w$ is a solution to (P_λ) we have

$$(-\Delta)^s (u_\lambda + w) = \lambda (u_\lambda + w)^q + (u_\lambda + w)^r.$$

By concavity, $\lambda(u_\lambda + w)^q \le \lambda u_\lambda^q + \lambda q u_\lambda^{q-1} w$ and hence

$$(-\Delta)^s w \le \lambda q u_\lambda^{q-1} w + (u_\lambda + w)^r - u_\lambda^r.$$

Furthermore, since $u_\lambda \ge \lambda^{\frac{1}{1-q}} v$, one also has $u_\lambda^{q-1} \le \lambda^{-1} v^{q-1}$ and as we are assuming $\|u_\lambda\|_{L^\infty(\Omega)} \le A$, we find

$$\begin{aligned}(-\Delta)^s w &\le q v^{q-1} + (u_\lambda + w)^r - u_\lambda^r \\ &\le q v^{q-1} + r A^{r-1} w.\end{aligned}$$

Multiplying the above inequality by w and using (5.7) we conclude

$$\beta \int_\Omega w^2 dx \le r A^{r-1} \int_\Omega w^2 dx.$$

Since $\beta < rA^{r-1}$, it follows $w = 0$. □

Now we can perform the proof of Theorem 1.4.

P r o o f o f T h e o r e m 1.4. First we claim that if $A = A(\alpha)$ is the associated constant to $(P_{\alpha,\lambda})$ obtained in Lemma 5.9, then $A(\alpha) \to 0$ as $\alpha \to 0$.

Indeed, it is enough to observe that

$$0 < \mu_1 \le \lambda_1^s(\alpha) = \inf_{\substack{u \in H^s_{\Sigma_{\mathcal{D}(\alpha)}}(\Omega) \\ u \not\equiv 0}} \frac{\|u\|^2_{H^s_{\Sigma_{\mathcal{D}(\alpha)}}(\Omega)}}{\|u\|^2_{L^2(\Omega)}},$$

where μ_1 is the first eigenvalue of the linearized eigenvalue problem (5.8).

Since by Remark 2.1 $\lambda_1^s(\alpha)$ as $\alpha \searrow 0$, the result follows.

In particular we deduce:

(1) From the proof of Lemma 5.4, we have $c\Lambda^\delta(\alpha) < \lambda_1^s(\alpha)$ and arguing as above $\Lambda(\alpha) \to 0$ as $\alpha \to 0$.
(2) There exist atmost one solution u to (P_λ) with $(\lambda, \|u\|_\infty) \in (0, \Lambda(\alpha)) \times (0, A(\alpha))$, that is the minimal solution and, since $A(\alpha) \searrow 0$ as $\alpha \to 0$, the minimal solution converges to zero as $\alpha \searrow 0$.

Now we prove that for $0 < \lambda < \Lambda(\alpha)$ small enough, the solution to problem $(P_{\alpha,\lambda})$ obtained by the Mountain Pass Theorem, u_α, satisfies

$$\|u_\alpha\|_{H^s(\Omega)} \to 0, \quad \text{as} \quad \alpha \searrow 0.$$

The proof follows the lines of [16, Lemma 5.12]. Let us consider the funcional at $\lambda = 0$

$$\begin{aligned} I_0(u_\alpha) &= \frac{1}{2}\int_\Omega |(-\Delta)^{\frac{s}{2}} u_\alpha|^2 dx - \frac{1}{r+1}\int_\Omega u_\alpha^{r+1} dx \\ &= \frac{1}{2}\|u_\alpha\|^2_{H^s_{\Sigma_{\mathcal{D}(\alpha)}}(\Omega)} - \frac{1}{r+1}\|u_\alpha\|^{r+1}_{L^{r+1}(\Omega)} \\ &\geq \frac{1}{2}\|u_\alpha\|^2_{H^s_{\Sigma_{\mathcal{D}(\alpha)}}(\Omega)} - \frac{1}{r+1}|\Omega|^{1-\frac{r+1}{2_s^*}}\left(1 + \frac{1}{\lambda_1^s(\alpha)}\right)^{\frac{r+1}{2}} \|u_\alpha\|^{r+1}_{H^s_{\Sigma_{\mathcal{D}(\alpha)}}(\Omega)}. \end{aligned}$$

Let us define $g(t) = \frac{1}{2}t^2 - c_2(r, |\Omega|)\lambda_1^s(\alpha)^{-s\frac{r+1}{2}} t^{r+1}$. It is easy to see that if t_α is such that $g'(t_\alpha) = 0$ then $t_\alpha \leq c(r, |\Omega|)\lambda_1^{s\mu}(\alpha)$ with $\mu = \frac{r+1}{2(r-1)}$, so that $t_\alpha \to 0$ as $\alpha \searrow 0$. Hence, the Mountain Pass solution converges to zero as $\alpha \searrow 0$. □

Remark 5.3. As a conclusion of the above arguments:

(1) Both the minimal solution u_λ and the mountain pass solution u_{mp}, converge to zero as $\alpha \searrow 0$.
(2) If we set $\alpha \in I_\varepsilon = [\varepsilon, |\partial\Omega|]$ with $\varepsilon > 0$, under hypotheses $(\mathfrak{B}_\alpha)$ and (B_1)-(B_3), there exist M_ε, Λ_ε such that the family $\mathcal{S}_\varepsilon \subset [0, \Lambda_\varepsilon] \times [0, M_\varepsilon]$ (see Theorem 1.3 for the definition of $\mathcal{S}_\varepsilon$).
(3) To conclude, it is interesting to point out Theorem 8 by Denzler in [18], where the author proved that

$$\sup_{0<\alpha<|\partial\Omega|} \{\lambda_1(\alpha) : \alpha = |\Sigma_{\mathcal{D}}|\} = \lambda_1(|\partial\Omega|),$$

which in particular proves that there are configurations of the distribution of the manifolds $\Sigma_{\mathcal{D}}$ and $\Sigma_{\mathcal{N}}$ on $\partial\Omega$ such that [16, Lemma 4.1] does not apply and hence $\lambda_1^s(\alpha) \not\to 0$ as $\alpha \searrow 0$. But this is not our case under hypotheses $(\mathfrak{B}_\alpha)$ and (B_1)-(B_3), in which [16, Lemma 4.1] applies proving that $\lambda_1^s(\alpha) \to 0$ as $\alpha \searrow 0$.

Acknowledgements

E. Colorado and A. Ortega are partially supported by the Ministry of Economy and Competitiveness of Spain and FEDER under Grant number MTM2016-80618-P, and J. Carmona is partially supported by Ministerio de Ciencia, Innovación y Universidades (MCIU), Agencia Estatal de Investigación (AEI) and (FEDER) Fondo Europeo de Desarrollo Regional under Research Project PGC2018-096422-B-I00 and Junta de Andalucía FQM-194.

References

[1] B. Abdellaoui, E. Colorado, I. Peral, Some remarks on elliptic equations with singular potentials and mixed boundary conditions. *Adv. Nonlinear Stud.* **4**, No 4 (2004), 503–533.

[2] B. Abdellaoui, E. Colorado, I. Peral, Effect of the boundary conditions in the behavior of the optimal constant of some Caffarelli-Kohn-Nirenberg inequalities. Application to some doubly critical nonlinear elliptic problems. *Adv. Differential Equations* **11**, No 6 (2006), 667–672.

[3] A. Ambrosetti, H. Brezis, G. Cerami, Combined effects of concave and convex nonlinearities in some elliptic problems. *J. Funct. Anal.* **122**, No 2 (1994), 519–543.

[4] A. Ambrosetti, P.H. Rabinowitz, Dual variational methods in critical point theory and applications. *J. Funct. Anal.* **14**, (1973), 349–381.

[5] S. Alama, Semilinear elliptic equations with sublinear indefinite nonlinearities. *Adv. Differential Equations* **4** (1999), 813–842.

[6] B. Barrios, E. Colorado, A. De Pablo, U. Sánchez, On some critical problems for the fractional Laplacian operator. *J. Differential Equations* **252**, No 11 (2012), 6133–6162.

[7] B. Barrios, E. Colorado, R. Servadei, F. Soria, A critical fractional equation with concave-convex power nonlinearities. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **32**, No 4 (2015), 875–900.

[8] L. Boccardo, M. Escobedo, I. Peral, A Dirichlet problem involving critical exponents. *Nonlinear Anal.* **24**, No 11 (1995), 1639–1648.

[9] C. Brändle. E. Colorado, A. de Pablo, U. Sanchez, A concave-convex elliptic problem involving the fractional Laplacian. *Proc. Roy. Soc. Edinburgh Sect. A* **143**, (2013), 39–71.

[10] X. Cabré, J. Tan, Positive solutions of nonlinear problems involving the square root of the Laplacian. *Adv. Math.* **224**, No 5 (2010), 2052–2093.

[11] L. Caffarelli, L. Silvestre, An extension problem related to the fractional Laplacian. *Comm. Partial Differential Equations* **32**, No 7-9 (2007), 1245–1260.

[12] J. Carmona, E. Colorado, T. Leonori, A. Ortega, Regularity of solutions to a fractional elliptic problem with mixed Dirichlet-Neumann boundary data. *Adv. in Calculus of Variations* **2020** (2020); doi: 10.1515/acv-2019-0029.

[13] F. Charro, E. Colorado, I. Peral, Multiplicity of solutions to uniformly elliptic fully nonlinear equations with concave-convex right-hand side. *J. Differential Equations* **246** (2009), 4221–4248.

[14] W. Chen, C. Li, B. Ou, Qualitative Properties of Solutions for an Integral Equation. *Disc. & Cont. Dynamics Sys.* **12** (2005), 347–354.

[15] E. Colorado, A. Ortega, The Brezis-Nirenberg problem for the fractional Laplacian with mixed Dirichlet-Neumann boundary conditions. *J. Math. Anal. Appl.* **473**, No 2 (2019), 1002–1025.

[16] E. Colorado, I. Peral, Semilinear elliptic problems with mixed Dirichlet-Neumann boundary conditions. *J. Funct. Anal.* **199**, No 2 (2003), 468–507.

[17] L. Damascelli, F. Gladiali, Some nonexistence results for positive solutions of elliptic equations in unbounded domains. *Rev. Mat. Iberoamericana* **20**, No 1 (2004), 67–86.

[18] J. Denzler, Bounds for the heat diffusion through windows of given area. *J. Math. Anal. Appl.* **217**, No 2 (1998), 405–422.

[19] E.B. Fabes, C.E. Kenig, R.P. Serapioni, The local regularity of solutions of degenerate elliptic equations. *Comm. Partial Differential Equations* **7**, No 1 (1982), 77–116.

[20] J. García Azorero, I. Peral, Multiplicity of solutions for elliptic problems with critical exponent or with a nonsymmetric term. *Trans. Amer. Math. Soc.* **323** (1991), 877–895.

[21] J. García Azorero, J. Manfredi, I. Peral, Sobolev versus Hölder local minimizers and global multiplicity for some quasilinear elliptic equations. *Commun. Contemp. Math.* **2**, No 3 (2000), 385–404.

[22] B. Gidas, W.M. Ni, L. Nirenberg, Symmetry and related properties via the maximum principle. *Comm. Math. Phys.* **68**, No 3 (1979), 209–243.

[23] B. Gidas, J. Spruck, A priori bounds for positive solutions of nonlinear elliptic equations. *Comm. Partial Differential Equations* **6** No 8 (1981), 883–901.

[24] P.-L. Lions, The concentration-compactness principle in the calculus of variations. The limit case. II. *Rev. Mat. Iberoamericana* **1**, No 2 (1985), 45–121.

[25] J.-L. Lions, E. Magenes, *Non-homogeneous Boundary Value Problems and Applications*, Vol. I. Springer-Verlag, New York-Heidelberg (1972).

[26] P.H. Rabinowitz, Some global results for nonlinear eigenvalue problems. *J. Funct. Analysis* **7**, (1971), 487–513.

[27] J. Serrin, A symmetry problem in potential theory. *Arch. Rational Mech. Anal.* **43** (1971), 304–318.

[1] *Departamento de Matemáticas*
Universidad de Almería
Ctra. Sacramento s/n, La Cañada de San Urbano
04120 Almería, SPAIN

e-mail: jcarmona@ual.es

[2] *Departamento de Matemáticas*
Universidad Carlos III de Madrid
Avenida de la Universidad 30
28911 Leganés (Madrid), SPAIN
e-mail: ecolorad@math.uc3m.es (Corresponding author)
Received: March 29, 2019 , Revised: July 30, 2020

[3] *Dipartimento di Scienze di Base e Applicate per l'Ingegneria*
Università di Roma "Sapienza"
Via Antonio Scarpa 10
00161 Roma, ITALY
e-mail: tommaso.leonori@sbai.uniroma1.it

[4] *Departamento de Matemáticas*
Universidad Carlos III de Madrid
Avenida de la Universidad 30
28911 Leganés (Madrid), SPAIN
e-mail: alortega@math.uc3m.es