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HOMOGENIZATION OF SINGULAR QUASILINEAR ELLIPTIC PROBLEMS WITH NATURAL GROWTH IN A DOMAIN WITH MANY SMALL HOLES

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Abstract. In this paper we consider the homogenization problem for quasilinear elliptic equations with singularities in the gradient, whose model is the following

$$\begin{cases} -\Delta u^\varepsilon + \dfrac{|\nabla u^\varepsilon|^2}{(u^\varepsilon)^\theta} = f(x) & \text{in } \Omega^\varepsilon, \\ u^\varepsilon = 0 & \text{on } \partial\Omega^\varepsilon, \end{cases}$$

where Ω is an open bounded set of $\mathbb{R}^N$, $\theta \in (0,1)$ and f is positive function that belongs to a certain Lebesgue's space. The homogenization of these equations is posed in a sequence of domains Ω^ε obtained by removing many small holes from a fixed domain Ω. We also give a corrector result.

1. Introduction.

We study a homogenization problem for a singular quasilinear elliptic problem with quadratic gradient, specifically

$$\begin{cases} -\Delta u^\varepsilon + g(u^\varepsilon)|\nabla u^\varepsilon|^2 = f(x) & \text{in } \Omega^\varepsilon \\ u^\varepsilon = 0 & \text{on } \partial\Omega^\varepsilon, \end{cases} \tag{1}$$

where Ω^ε is a sequence of open sets which are included in a fixed bounded open set Ω of $\mathbb{R}^N$, $N \geq 3$, $g \in C(0,+\infty) \cap L^1(0,1)$ is a positive function and $f \in L^{\frac{2N}{N+2}}(\Omega)$, $f \geq 0$, $f \not\equiv 0$.

We study the asymptotic behaviour, as ε goes to zero, of a sequence of problems posed in domains Ω^ε obtained by removing many small holes from a fixed domain

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Ω, in the framework of [11] for the linear case. In such paper it has been shown (see also [21] or [12] for a more general framework) that for every $f \in L^2(\Omega)$, the (unique) solution u^ε of

$$\begin{cases} -\Delta u^\varepsilon = f(x) & \text{in } \Omega^\varepsilon, \\ u^\varepsilon = 0 & \text{on } \partial\Omega^\varepsilon, \end{cases} \tag{2}$$

satisfies that, with $\tilde{u}^\varepsilon$ denoting the extension of u^ε by zero in $\Omega \setminus \Omega^\varepsilon$, $\tilde{u}^\varepsilon \rightharpoonup u$ in $H_0^1(\Omega)$, where u is the (unique) solution of

$$\begin{cases} u \in H_0^1(\Omega) \cap L^2(\Omega; d\mu), \\ \displaystyle\int_\Omega \nabla u \nabla \varphi + \int_\Omega u\varphi \, d\mu = \int_\Omega f\varphi, \ \ \forall \varphi \in H_0^1(\Omega) \cap L^2(\Omega; d\mu), \ \varphi \geq 0, \end{cases} \tag{3}$$

with μ is a nonnegative finite Radon measure depending only on the holes. In [11] there is an example of holes for which μ is a positive constant and u satisfies

$$\begin{cases} -\Delta u + \mu u = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

It is widely remarked the presence of the "strange term" μu (which is the asymptotic memory of the fact that $\tilde{u^\varepsilon}$ was zero on the holes) appearing in the limit equation (3).

In [12] the authors deal with the homogenization problem in a general framework. In that paper it is not essential to have an a priori control of the set where the weak limit of $\tilde{u}^\varepsilon$ vanishes. However, since we are dealing with a singularity at $u = 0$ we adopt the framework of [11] as in references [16, 17, 18] in which is studied the existence of solution and homogenization of the problem

$$\begin{cases} -\operatorname{div} A(x) Du^\varepsilon = \dfrac{f(x)}{(u^\varepsilon)^\gamma} & \text{in } \Omega^\varepsilon, \\ u^\varepsilon = 0 & \text{on } \partial\Omega^\varepsilon, \end{cases}$$

where $A \in L^\infty(\Omega)^{N \times N}$ is a coercive matrix and $\gamma > 0$, and [7] which deals with homogenization of this problem in varying matrices.

In [10] was studied problem (1) in the case $g(s) = -\gamma$ where γ is a real constant. The author used a suitable change of unknown function, $z^\varepsilon = e^{\gamma u^\varepsilon} - 1$, and he obtains a new problem

$$\begin{cases} -\Delta z^\varepsilon = f\gamma e^{\gamma u^\varepsilon} & \text{in } \Omega^\varepsilon, \\ z^\varepsilon = 0 & \text{on } \partial\Omega^\varepsilon. \end{cases}$$

A careful analysis of this semilinear homogenization problem allows the author to pass to the limit as in the linear case. Undoing the change of variable he proved that, as in the linear case, a new term appears in the equation that satisfies u, but in this case the new term $(e^{\gamma u} - 1)\mu/(\gamma e^{\gamma u})$ is nonlinear (μ is the same measure). Specifically the homogenized problem is

$$\begin{cases} -\Delta u + \dfrac{e^{\gamma u} - 1}{\gamma e^{\gamma u}} \mu + \gamma |\nabla u|^2 = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

or equivalently, in the case where μ is a Radon measure, the solution u satisfies

$$\begin{cases} u \in H_0^1(\Omega) \cap L^2(\Omega; d\mu), \\ \displaystyle\int_\Omega \nabla u \nabla \varphi + \int_\Omega \frac{e^{\gamma u} - 1}{\gamma e^{\gamma u}} \varphi \, d\mu + \gamma \int_\Omega |\nabla u|^2 \varphi = \int_\Omega f\varphi, \end{cases}$$

for every $\varphi \in H_0^1(\Omega) \cap L^2(\Omega; d\mu)$, $\varphi \geq 0$.

The difference is that the "new equation" is no more linear. As the author remarked, this means that the perturbation of the linear problem (2) by a nonlinear term, namely $\gamma|\nabla u^\varepsilon|^2$, changes the structure of the new term in the limit equation. Moreover, a corrector result was proved, is that to say, a representation of ∇u^ε in the strong topology of $L^2(\Omega)^N$. Similar results were proved in [9] in which the nonlinear perturbation of (2) is a general function of the form $H(x, u, \nabla u)$, where H has a (at most) natural growth in the gradient.

We remark that in all the cases the lower order term is bounded respect to u and, up to our knowledge, the singular problem (1) has not been considered yet. In [15] there a is first homogenization result for a singular quasilinear equation (but with the nonlinearity on the right-hand side) for a fixed domain with oscillating coefficients.

In the present paper, inspired by [10], we consider singular functions g, using the results in [1, 4, 6] we have the existence of solution $u^\varepsilon \in H_0^1(\Omega^\varepsilon)$, in the sense of distributions, to problem (1) and in [5] it has been proved the uniqueness. In [19] the authors prove the existence of solution for f bounded and a general lower order term.

Observe that $g(s) \geq 0$ for every $s > 0$, thus it is easy to prove that u^ε is bounded in $H_0^1(\Omega^\varepsilon)$ and in $L^\infty(\Omega^\varepsilon)$ following [20] and [22] respectively. Moreover we can prove that $\tilde{u^\varepsilon}$ is a bounded sequence in $H_0^1(\Omega) \cap L^\infty(\Omega)$. Therefore, up to a subsequence, we get that $\tilde{u^\varepsilon}$ converges to some u weakly in $H_0^1(\Omega)$. The general questions we are concerned with are the following. Do the solutions u^ε converge to a limit u when the parameter ε tends to zero? If this limit exists, can it be characterized? Will the result be the same result as in the non singular case? In principle the answer is not obvious at all since, as ε tends to zero, the number of holes becomes greater and greater and the singular set for the right-hand side (which includes at least the holes' boundary) tends to "invade" the entire Ω.

In our case, the function g may presents a singularity at $u = 0$. Our main result is to prove that for every $f \in L^{\frac{2N}{N+2}}(\Omega)$, $f \geq 0$, $f \not\equiv 0$, the unique solution u^ε to problem (1) satisfies $\tilde{u^\varepsilon} \rightharpoonup u$ in $H_0^1(\Omega)$, where u is the (unique) solution of problem

$$\begin{cases} -\Delta u + g(u)|\nabla u|^2 + \mu\Psi(u)e^{G(u)} = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

in the case μ constant, where $G(s) = \int_1^s g(t)dt$ and $\Psi(s) = \int_0^s e^{-G(t)}dt$ for every $s > 0$, or of the problem, in the case where μ is a Radon measure,

$$\begin{cases} u \in H_0^1(\Omega) \cap L^2(\Omega; d\mu), \\ \displaystyle\int_\Omega \nabla u \nabla\varphi + \int_\Omega \Psi(u)e^{G(u)}\varphi\, d\mu + \int_{\Omega\cap\{u>0\}} g(u)|\nabla u|^2\varphi = \int_\Omega f\varphi, \end{cases}$$

for every $\varphi \in H_0^1(\Omega) \cap L^2(\Omega; d\mu)$, $\varphi \geq 0$.

In [8] the authors prove the existence and uniqueness of solution of the previous problem in the case where μ is a constant and the proof of the uniqueness can be adapted in the case where μ is a Radon measure.

Observe that we can not guarantee that the solution of the previous problem satisfies $u > 0$ in Ω. The proof of this fact is usually based on a change of variables to obtain a semilinear problem where the strong maximum principle is satisfied. In our case, this procedure leads to a problem of the form

$$\begin{cases} -\Delta z + \mu z = f_1(x) \geq 0 & \text{in } \Omega, \\ z = 0 & \text{on } \partial\Omega, \end{cases}$$

and in [18] the authors give an explicit counterexample which shows that in general the strong maximum principle fails in the case where the operator involves a zeroth-order term μz when $\mu \in \mathcal{M}_b(\Omega)$, $\mu \geq 0$. Therefore, we do not impose $0 < u^\varepsilon$ in the notion of solution of the singular problem (1) and we proceed here with the notion of solution in Definition 2.1. Although we prove that both concepts are equivalents for g integrable at zero we think that Definition 2.1 is more appropriate if $g \notin L^1(0,1)$. In addition, the techniques used to prove existence of solution in this sense are important in order to deal with the homogenization of the singular problem (1).

The plan of the paper is the following. We firstly prove that the problem

$$\begin{cases} -\Delta u + g(u)|\nabla u|^2 = f(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{4}$$

has solution in a suitable sense that we will detail in Section 2. In Section 3 we give the precise assumptions of the perforated domains, following the framework of [11], and we prove our homogenization result for the singular quasilinear problem (1). The last part of Section 3 is devoted to prove the corrector result.

Let us explicitly state that we have chosen to present the results and to perform the proofs in the case $N \geq 3$. However, all the results hold true also in the case $N = 2$ provided we replace the assumption $f \in L^{\frac{2N}{N+2}}(\Omega)$ with $f \in L^p(\Omega)$ for some $p > 1$.

Notation: As usual, we consider the positive and negative part functions defined on $\mathbb{R}$ by $s^+ = \max\{s, 0\}$ and $s^- = \min\{s, 0\}$, respectively.

For any $k > 0$ and $s \geq 0$ we set $T_k(s) = \min\{k, s\}$, $G_k(s) = s - T_k(s)$ and $S_k(s) = \min\{1/k, \max\{k, s\}\}$.

For any $1 < p < N$, $p^* = \frac{Np}{N-p}$ is the Sobolev conjugate exponent of p. As usual, $\mathcal{S}$ denotes the best Sobolev constant, i.e.,

$$\mathcal{S} = \sup_{\|u\|_{H_0^1(\Omega)}=1} \|u\|_{L^{2^*}(\Omega)}.$$

We denote by $\mathcal{D}(\Omega)$ the space of the functions $C^\infty(\Omega)$ whose support is compact and included on Ω, by $\mathcal{D}'(\Omega)$ the space of distributions on Ω and $\mathcal{M}_b(\Omega)$ denotes the space of the finite Radon measures.

For $l : \Omega \longrightarrow [0, +\infty]$ a measurable function we denote

$$\{l = 0\} = \{x \in \Omega : l(x) = 0\},\ \{l > 0\} = \{x \in \Omega : l(x) > 0\}$$

and for some $a < b$, $\{a < l < b\} = \{x \in \Omega : a < l(x) < b\}$.

2. **Framework of the quasilinear problem.** Let Ω be an open and bounded subset of $\mathbb{R}^N$ $(N \geq 3)$ and $f \in L^{\frac{2N}{N+2}}(\Omega)$, $f \geq 0$, $f \not\equiv 0$. We consider the boundary value problem (4) with g a continuous function in $(0, +\infty)$.

Definition 2.1. We say that $u \in H_0^1(\Omega)$ is a positive solution for (4) if $u(x) \geq 0$ for a.e. $x \in \Omega$, $g(u)|\nabla u|^2 \in L^1(\{x \in \Omega : u(x) > 0\})$ and

$$\int_\Omega \nabla u \nabla \varphi + \int_{\{u>0\}} g(u)|\nabla u|^2 \varphi = \int_\Omega f(x)\varphi,$$

for every $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$.

Remark 1. Arguing as in [2, Lemma 2.1] it is equivalent to consider test functions not necessarily bounded. More precisely, if $u \in H_0^1(\Omega)$ with $u(x) \geq 0$ for a.e. $x \in \Omega$ and $g(u)|\nabla u|^2 \in L^1(\{x \in \Omega : u(x) > 0\})$ satisfies

$$\int_\Omega \nabla u \nabla \varphi + \int_{\{u>0\}} g(u)|\nabla u|^2 \varphi \leq \int_\Omega f(x)\varphi, \ \forall 0 \leq \varphi \in H_0^1(\Omega) \cap L^\infty(\Omega),$$

(i.e. u is a sub-solution of (4)) then u satisfies this inequality for every $0 \leq \varphi \in H_0^1(\Omega)$. $\square$

Remark 2. Observe that in [6] it is proved the existence of $u \in H_0^1(\Omega)$ such that $u(x) > 0$ for a.e. $x \in \Omega$, $g(u)|\nabla u|^2 \in L^1(\Omega)$ and

$$\int_\Omega \nabla u \nabla \varphi + \int_\Omega g(u)|\nabla u|^2 \varphi = \int_\Omega f(x)\varphi,$$

for every $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$. In particular, u is solution in the sense of Definition 2.1. (See also [1] where a stronger hypothesis on f allows to consider more general functions g). $\square$

The next result is a direct consequence of the Stampacchia method [22], we include here only a sketch of the proof.

Lemma 2.2. *There exists a positive constant $C_{f,\Omega}$ such that for every $g \geq 0$ and every sub-solution $u \in H_0^1(\Omega)$ of (4) we have that*

$$\|u\|_{H_0^1(\Omega)} \leq C_{f,\Omega}.$$

If, in addition, $f \in L^q(\Omega)$ for some $q > N/2$ then $u \in L^\infty(\Omega)$ and

$$\|u\|_{L^\infty(\Omega)} \leq C_{f,\Omega}.$$

Proof. Taking u as test function (see Remark (1)) and neglecting the positive lower order term we have that

$$\|u\|_{H_0^1(\Omega)} \leq \mathcal{S}\|f\|_{L^{2N/(N+2)}(\Omega)},$$

where $\|w\|_{L^{2^*}(\Omega)} \leq \mathcal{S}\|w\|_{H_0^1(\Omega)}$ for every $w \in H_0^1(\Omega)$. Moreover, if $f \in L^q(\Omega)$ for some $q > N/2$, neglecting again the lower order term, the standard Stampaccchia method gives us the existence of $C > 0$, depending only on f and Ω, such that

$$\|u\|_{L^\infty(\Omega)} \leq C.$$

This concludes the proof. $\square$

Remark 3. Observe that if $\Omega_1 \subset \Omega_2$ are open and bounded and $f \in L^q(\Omega_2)$ for some $q > N/2$ then we can take $C_{f,\Omega_1} \leq C_{f,\Omega_2}$. $\square$

We recall that, for every $s > 0$, the function $S_\delta(s) = \min\{\delta^{-1}, \max\{\delta, s\}\}$, $G(s) = \int_1^s g(t)dt$ and $\Psi(s) = \int_0^s e^{-G(t)}dt$.

Lemma 2.3. *Assume that u is a solution of (4) in the sense of Definition 2.1. Then*

1. *If $g \notin L^1(0,1)$ then $f(x) > 0$ for a.e. $x \in \Omega$ implies that $u(x) > 0$ for a.e. $x \in \Omega$.*
2. *If $g \in L^1(0,1)$ then $f \geq 0$ and $f \not\equiv 0$ implies that $u(x) > 0$ for a.e. $x \in \Omega$. Moreover,*
$$\int_\Omega e^{-G(u)}\nabla u \nabla \phi = \int_\Omega f e^{-G(u)}\phi, \quad \forall \phi \in H_0^1(\Omega) \cap L^\infty(\Omega),\ \phi \geq 0. \tag{5}$$
Even more, for every $\omega \subset\subset \Omega$ there exists $c_\omega > 0$ such that $u > c_\omega$ a.e. in ω. The constant c_ω does not depend on u if, in addition, $g \in L^1(1,+\infty)$ or $f \in L^q(\Omega)$ for some $q > N/2$.

Remark 4. In the present paper we do not consider the case in which g can be not integrable at zero. However, we prove in (i) of the previous Lemma a result that improves a hypothesis used in [1], that is that we do not impose here f to be greater than a positive constant in compact subsets of Ω. $\square$

Proof. Let $\phi \in H_0^1(\Omega) \cap L^\infty(\Omega)$ with $\phi \geq 0$. First we observe that, for $0 < \delta < 1$, we can take $\varphi = e^{-G(S_\delta(u))}\phi$ as test function in (4) and we obtain that
$$\int_\Omega e^{-G(S_\delta(u))}\nabla u \nabla \phi + \int_{\{0<u\leq\delta\}} g(u)|\nabla u|^2 e^{-G(\delta)}\phi + \int_{\{u>\delta^{-1}\}} g(u)|\nabla u|^2 e^{-G(\delta^{-1})}\phi = \int_\Omega f e^{-G(S_\delta(u))}\phi.$$

Multiplying by $e^{G(\delta)}$ we deduce
$$\int_\Omega e^{G(\delta)-G(S_\delta(u))}\nabla u \nabla \phi + \int_{\{0<u\leq\delta\}} g(u)|\nabla u|^2 \phi + \int_{\{u>\delta^{-1}\}} g(u)|\nabla u|^2 e^{G(\delta)-G(\delta^{-1})}\phi = \int_\Omega f e^{G(\delta)-G(S_\delta(u))}\phi.$$

Now we pass to the limit as δ tends to zero. Since G is increasing and $\delta \leq S_\delta(u)$ we have that $e^{G(\delta)-G(S_\delta(u))} \leq 1$, in particular, using Lebesgue theorem
$$\lim_{\delta\to 0}\int_\Omega e^{G(\delta)-G(S_\delta(u))}\nabla u \nabla \phi = \int_\Omega e^{G(0)-G(u)}\nabla u \nabla \phi = \int_{\{u>0\}} e^{G(0)-G(u)}\nabla u \nabla \phi.$$

Analogously, passing to the limit as δ tends to zero
$$\lim_{\delta\to 0}\int_\Omega f e^{G(\delta)-G(S_\delta(u))}\phi = \int_\Omega f e^{G(0)-G(u)}\phi = \int_{\{u=0\}} f\phi + \int_{\{u>0\}} f e^{G(0)-G(u)}\phi.$$

Moreover, since $g(u)|\nabla u|^2 \in L^1(\{x \in \Omega : u(x) > 0\})$ we have that
$$\lim_{\delta\to 0}\int_{\{0<u\leq\delta\}} g(u)|\nabla u|^2\phi = 0$$
and
$$\lim_{\delta\to 0}\int_{\{u>\delta^{-1}\}} g(u)|\nabla u|^2 e^{G(\delta)-G(\delta^{-1})}\phi = 0.$$
Summarizing we obtain that
$$\int_{\{u>0\}} e^{G(0)-G(u)}\nabla u \nabla \phi = \int_{\{u=0\}} f\phi + \int_{\{u>0\}} f e^{G(0)-G(u)}\phi. \tag{6}$$

In the case of item (1) we have that $G(0) = -\infty$ since $g \notin L^1(0,1)$. In this case, from (6) we deduce that

$$\int_{\{u=0\}} f\phi = 0,$$

which implies that either $u(x) > 0$ for a.e. $x \in \Omega$ or $f(x) = 0$ for a.e. $x \in \{u = 0\}$, which conclude the proof in this case.

In the case of item (2), i.e. $g \in L^1(0,1)$ then $-\infty < G(0) < 0$ and from (6) we obtain directly (5). Moreover $\Psi(u) \in H_0^1(\Omega)$ and fixing $L > 0$ it follows that

$$\int_\Omega \nabla\Psi(u)\nabla\phi = \int_\Omega fe^{-G(u)}\phi \geq \int_{\{u<L\}} fe^{-G(u)}\phi \geq \int_\Omega f\chi_{\{u<L\}}e^{-G(L)}\phi,$$

for every $0 \leq \phi \in H_0^1(\Omega)$ (see Remark (1)). Therefore, the comparison principle assures that $\Psi(u) \geq z \in H_0^1(\Omega) \cap C(\Omega)$ the unique solution of

$$-\Delta z = f\chi_{\{u<L\}}e^{-G(L)},\ z \in H_0^1(\Omega).$$

Since $f\chi_{\{u<L\}} \not\equiv 0$, the strong maximum principle guarantee that $z > 0$ for every $x \in \Omega$ and we can take $c_\omega = \inf_{x\in\omega} z(x) > 0$ for every $\omega \subset\subset \Omega$. Moreover, the function z (and then the constant c_ω) does not depend on u if either $L = +\infty$ with $g \in L^1(1,+\infty)$ or $L = C_{f,\Omega}$ in the case $f \in L^q(\Omega)$ for some $q > N/2$ and $C_{f,\Omega}$ given by Lemma (2.2). □

Remark 5. Taking into account Lemma 2.3 we have sufficient conditions to have (5) satisfied. This is the key point to prove the uniqueness result in [5] which, under these conditions, it is also true for solutions in the sense of Definition 2.1. Using then Remark 2 we have that both concepts of solution are equivalent in the cases where uniqueness of solution holds. This is the case when g is integrable at zero (in [3] it is proved a uniqueness result, in the case $g(s) = c/s$ with $c < 1$ if $\partial\Omega$ is smooth, whose proof cannot be adapted for solutions in the sense of Definition 2.1). □

We include now the proof of the existence of solution in the sense of Definition 2.1.

Theorem 2.4. *Assume that $g \in C((0,\infty)) \cap L^1(0,1)$ such that $\limsup_{s\to 0} g(s)s < +\infty$ and $f \in L^{\frac{2N}{N+2}}(\Omega)$, $f \geq 0$, $f \not\equiv 0$. Then there exists a solution of (4) in the sense of Definition 2.1.*

Proof. For every $n \in \mathbb{N}$, there exists $u_n \in H_0^1(\Omega)\cap L^\infty(\Omega)$, $u_n(x) > 0$ for a.e. $x \in \Omega$, solution of

$$\begin{cases} -\Delta u_n + g(u_n + 1/n)|\nabla u_n|^2 = T_n(f(x)) & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{7}$$

Since $g(s + 1/n) \geq 0$ for every $s \geq 0$ we can use Lemma 2.2 and

$$\|u_n\|_{H_0^1(\Omega)} \leq C_{f,\Omega}.$$

Therefore, there exists $u \in H_0^1(\Omega)$ such that, up to a subsequence, u_n converges to u, weakly in $H_0^1(\Omega)$, strongly in $L^p(\Omega)$ with $p < 2^*$ and $u_n(x) \to u(x)$ for a.e. $x \in \Omega$.

We define $Z_\delta(s) = T_1((2 - s/\delta)^+)$ for every $\delta > 0$ and

$$\mathcal{G}_n(s) = \int_{1+\frac{1}{n}}^{s+\frac{1}{n}} g(t)dt$$

for every $n \in \mathbb{N}$. Observe that for $\phi \in H_0^1(\Omega) \cap L^\infty(\Omega)$ with $\phi \geq 0$, we can take $\varphi = e^{-\mathcal{G}_n(u_n)} Z_\delta(u_n)\phi$ as test function in (7) and using that $f \geq 0$ we obtain

$$\int_\Omega e^{-\mathcal{G}_n(u_n)} Z_\delta(u_n) \nabla u_n \nabla \phi \geq \frac{1}{\delta} \int_{\{\delta \leq u_n \leq 2\delta\}} |\nabla u_n|^2 e^{-\mathcal{G}_n(u_n)} \phi \geq$$
$$(u_n < 2\delta < 1 \text{ is used }) \geq \frac{e^{-\mathcal{G}_n(1)}}{\delta} \int_{\{\delta \leq u_n \leq 2\delta\}} |\nabla u_n|^2 \phi = \frac{1}{\delta} \int_{\{\delta \leq u_n \leq 2\delta\}} |\nabla u_n|^2 \phi.$$

Now we pass to the limit as $n \to \infty$. Observe that, since $g \in L^1(0,1)$ then $\mathcal{G}_n(u_n) \to G(u)$ and, since $e^{-\mathcal{G}_n(u_n)} Z_\delta(u_n)$ is bounded, Lebesgue's Theorem allows to deduce that $e^{-\mathcal{G}_n(u_n)} Z_\delta(u_n) \nabla \phi$ strongly converges in $L^2(\Omega)$ to $e^{-G(u)} Z_\delta(u) \nabla \phi$. Therefore, using the weak convergence of u_n in $H_0^1(\Omega)$ we can pass to the limit in the left hand side

$$\int_\Omega e^{-G(u)} Z_\delta(u) \nabla u \nabla \phi \geq \limsup_{n \to \infty} \frac{1}{\delta} \int_{\{\delta \leq u_n \leq 2\delta\}} |\nabla u_n|^2 \phi.$$

Using that $Z_\delta(s) \to 0$ as $\delta \to 0$ we obtain, passing to the limit as $\delta \to 0$, that

$$\limsup_{\delta \to 0} \limsup_{n \to \infty} \frac{1}{\delta} \int_{\{\delta \leq u_n \leq 2\delta\}} |\nabla u_n|^2 \phi = 0. \tag{8}$$

Now we take $\varphi = Z_\delta(u_n)\phi$ with $\phi \in H_0^1(\Omega) \cap L^\infty(\Omega)$, $\phi \geq 0$, as test function in (7) and we obtain that

$$\int_\Omega Z_\delta(u_n) \nabla u_n \nabla \phi + \int_{\{u_n \leq \delta\}} g(u_n + 1/n) |\nabla u_n|^2 \phi \leq$$
$$\leq \frac{1}{\delta} \int_{\{\delta \leq u_n \leq 2\delta\}} (1 + g(u_n + 1/n) u_n) |\nabla u_n|^2 \phi + \int_\Omega f Z_\delta(u_n) \phi \leq$$
$$\leq \frac{1 + c}{\delta} \int_{\{\delta \leq u_n \leq 2\delta\}} |\nabla u_n|^2 \phi + \int_\Omega f Z_\delta(u_n) \phi,$$

where c is a positive constant such that $\limsup_{s \to 0} g(s)s \leq c$.

Taking limits as $n \to \infty$ and then as $\delta \to 0$ we obtain, using (8) that

$$\limsup_{\delta \to 0} \limsup_{n \to \infty} \int_{\{u_n \leq \delta\}} g(u_n + 1/n) |\nabla u_n|^2 \phi = 0.$$

Finally taking $\phi \in H_0^1(\Omega) \cap L^\infty(\Omega)$, with $\phi \geq 0$, as test function in (7) we have

$$\int_\Omega \nabla u_n \nabla \phi + \int_{\{u_n \leq \delta\}} g(u_n + 1/n) |\nabla u_n|^2 \phi$$
$$+ \int_{\{\delta < u_n\}} g(u_n + 1/n) |\nabla u_n|^2 \phi = \int_\Omega T_n(f) \phi.$$

Now we claim that

$$\lim_{n \to \infty} \int_{\{\delta < u_n\}} g(u_n + 1/n) |\nabla u_n|^2 \phi = \int_{\{\delta < u\}} g(u) |\nabla u|^2 \phi.$$

This can be proved as in [1] writing, for every $k > 0$, $G_\delta(u_n) = T_k(G_\delta(u_n)) + G_k(G_\delta(u_n))$ and taking into account that, fixed k, δ, $T_k(G_\delta(u_n))$ strongly converges to $T_k(G_\delta(u))$ in $H_0^1(\Omega)$ and $\|\nabla G_k(G_\delta(u_n))\|_{L^2(\Omega)^N}$ tends to 0 uniformly in n as $k \to \infty$.

Therefore, as $n \to \infty$,

$$\int_\Omega \nabla u \nabla \phi + \limsup_{n\to+\infty} \int_{\{u_n \le \delta\}} g(u_n + 1/n)|\nabla u_n|^2 \phi + \int_{\{\delta < u\}} g(u)|\nabla u|^2 \phi = \int_\Omega f\phi,$$

and taking limit as $\delta \to 0$

$$\int_\Omega \nabla u \nabla \phi + \int_{\{0<u\}} g(u)|\nabla u|^2 \phi = \int_\Omega f\phi,$$

and u is a solution of (4). □

3. Homogenization for the problem (1).

3.1. The perforated domains.

In this Section, we describe the geometry of the domains, following [11], in which we study our homogenization result.

Let Ω be an open and bounded set of $\mathbb{R}^N$ ($N \ge 2$). Consider for every ε, where ε takes its values in a sequence of positive numbers which tends to zero, some closed subsets T_i^ε of $\mathbb{R}^N$, $1 \le i \le n(\varepsilon)$, which are the holes. The domain Ω^ε is defined by removing the holes T_i^ε from Ω, that is

$$\Omega^\varepsilon = \Omega - \bigcup_{i=1}^{n(\varepsilon)} T_i^\varepsilon.$$

Hypotheses on the holes. We suppose that the sequence of domains Ω^ε is such that there exist a sequence of functions w^ε, a distribution $\mu \in \mathcal{D}'(\Omega)$ and two sequences of distributions $\mu^\varepsilon \in \mathcal{D}'(\Omega)$ and $\lambda^\varepsilon \in \mathcal{D}'(\Omega)$ such that

$$w^\varepsilon \in H^1(\Omega) \cap L^\infty(\Omega), \tag{9}$$

$$0 \le w^\varepsilon \le 1 \text{ a.e. } x \in \Omega, \tag{10}$$

$$\forall \psi \in H_0^1(\Omega) \cap L^\infty(\Omega),\ w^\varepsilon \psi \in H_0^1(\Omega^\varepsilon), \tag{11}$$

$$w^\varepsilon \rightharpoonup 1 \text{ in } H^1(\Omega) \text{ weakly, in } L^\infty(\Omega) \text{ weakly-star and a.e. in } \Omega, \tag{12}$$

$$\mu \in H^{-1}(\Omega), \tag{13}$$

$$\begin{cases} -\Delta w^\varepsilon = \mu^\varepsilon - \lambda^\varepsilon \text{ in } \mathcal{D}'(\Omega), \\ \text{with } \mu^\varepsilon \in H^{-1}(\Omega),\ \lambda^\varepsilon \in H^{-1}(\Omega), \\ \mu^\varepsilon \ge 0 \text{ in } \mathcal{D}'(\Omega), \\ \mu^\varepsilon \to \mu \text{ in } H^{-1}(\Omega) \text{ strongly}, \\ \langle \lambda^\varepsilon, \tilde{z^\varepsilon} \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} = 0\ \ \forall z^\varepsilon \in H_0^1(\Omega^\varepsilon). \end{cases} \tag{14}$$

The meaning of assumption (11) is that

$$w^\varepsilon = 0 \text{ on } \bigcup_{i=1}^{n(\varepsilon)} T_i^\varepsilon, \tag{15}$$

while the meaning of the last statement of (14) is that the distribution λ^ε only acts on the holes T_i^ε, $i = 1, \dots, n(\varepsilon)$, since taking $z^\varepsilon \in \mathcal{D}(\Omega^\varepsilon)$ implies that

$$-\Delta w^\varepsilon = \mu^\varepsilon \text{ in } \mathcal{D}'(\Omega^\varepsilon).$$

Taking $z^\varepsilon = w^\varepsilon \phi$, with $\phi \in \mathcal{D}(\Omega)$, $\phi \ge 0$, as test function in (14) we have

$$\int_\Omega \phi |\nabla w^\varepsilon|^2 + \int_\Omega w^\varepsilon \nabla w^\varepsilon \nabla \phi = \langle \mu^\varepsilon, w^\varepsilon \phi \rangle_{H^{-1}(\Omega), H_0^1(\Omega)},$$

from which we easily deduce that

$$\int_\Omega \phi|\nabla w^\varepsilon|^2 \to \langle \mu,\phi\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}\ \forall\phi\in\mathcal{D}(\Omega),\ \phi\geq 0,$$

and therefore that $\mu\geq 0$. The distribution $\mu\in H^{-1}(\Omega)$ is therefore also a nonnegative measure. Moreover, since

$$\begin{cases}\forall\phi\in\mathcal{D}(\Omega),\phi\geq 0,\\ \displaystyle\int_\Omega \phi d\mu=\langle\mu,\phi\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}\leq\limsup_\varepsilon\int_\Omega\phi|\nabla w^\varepsilon|^2\leq\\ \displaystyle\leq\|\phi\|_{L^\infty(\Omega)}\limsup_\varepsilon\int_\Omega|\nabla w^\varepsilon|^2\leq C\|\phi\|_{L^\infty(\Omega)},\end{cases}$$

the measure μ is a finite Radon measure, or in other terms $\mu\in\mathcal{M}_b(\Omega)$.

It is then (well) known (see e.g. [13] Section 1 and [14] Section 2.2 for more details) that if $z\in H_0^1(\Omega)$, then z (or more exactly its quasi-continuous representative for the $H_0^1(\Omega)$ capacity) satisfies

$$z\in L^1(\Omega;d\mu)\ \text{ with }\ \langle\mu,z\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}=\int_\Omega z\,d\mu\,;$$

moreover if $z\in H_0^1(\Omega)\cap L^\infty(\Omega)$, then z satisfies

$$z\in L^\infty(\Omega;d\mu)\ \text{ with }\ \|z\|_{L^\infty(\Omega;d\mu)}=\|z\|_{L^\infty(\Omega)};$$

therefore when $z\in H_0^1(\Omega)\cap L^\infty(\Omega)$, then z belongs to $L^1(\Omega;d\mu)\cap L^\infty(\Omega;d\mu)$ and therefore to $L^p(\Omega;d\mu)$ for every p, $1\leq p\leq+\infty$.

3.2. **The homogenization result for the singular quasilinear problem (1).**

In the case $g\in C(0,+\infty)\cap L^1(0,1)$, for every $f\in L^{\frac{2N}{N+2}}(\Omega)$, $f\geq 0$, $f\not\equiv 0$, there exists a unique solution $u^\varepsilon\in H_0^1(\Omega^\varepsilon)$ in the sense of Definition 2.1 of the problem (1). This is consequence of Theorem 2.4 and Remark 5 (see also [1], [4] and [6] for the existence and [5] for the uniqueness in the usual sense).

Moreover, using Lemma 2.3, $u^\varepsilon(x)>0$ for a.e. $x\in\Omega^\varepsilon$, $g(u^\varepsilon)|\nabla u^\varepsilon|^2\in L^1(\Omega^\varepsilon)$ and for every $\varphi^\varepsilon\in H_0^1(\Omega^\varepsilon)\cap L^\infty(\Omega^\varepsilon)$, $\varphi^\varepsilon\geq 0$, we have

$$\int_{\Omega^\varepsilon}\nabla u^\varepsilon\nabla\varphi^\varepsilon+\int_{\Omega^\varepsilon}g(u^\varepsilon)|\nabla u^\varepsilon|^2\varphi^\varepsilon=\int_{\Omega^\varepsilon}f\varphi^\varepsilon. \tag{16}$$

Remark 6. From item (ii) of Lemma 2.3 we also deduce that, for every $\omega^\varepsilon\subset\subset\Omega^\varepsilon$, there exists $c_{\omega^\varepsilon}>0$ such that $u^\varepsilon>c_{\omega^\varepsilon}$ a.e. in ω^ε. This fact will be used in the proof of Theorem 3.2. □

In order to deal with the main result in the case where no estimate in $L^\infty(\Omega^\varepsilon)$ is known we need to impose a convenient behavior of g at infinity in the sense of the following definition.

Definition 3.1. We say that condition (FG) is satisfied if one of the following assumptions is satisfied

1. $f\in L^q(\Omega)$ for some $q>N/2$ or,
2. $f\in L^{\frac{2N}{N+2}}(\Omega)$, $g\in L^1(1,+\infty)$ and $\limsup_{s\to+\infty}g(s)<+\infty$.

Theorem 3.2. *Assume that $g \in C((0,\infty)) \cap L^1(0,1)$ is a positive function such that $\limsup_{s\to 0} g(s)s < +\infty$ and condition (FG), in Definition 3.1, is satisfied. Assume also that the sequence of perforated domains Ω^ε satisfies (9), (10), (11), (12), (13) and (14). Then there exists a subsequence, still labelled by ε, such that for this subsequence the solution u^ε to problem (1) in the sense of Definition 2.1, satisfies $\tilde{u}^\varepsilon$ weakly converges to $u \in H_0^1(\Omega)$ the unique solution of the problem*

$$\begin{cases} -\Delta u + g(u)|\nabla u|^2 + \mu\Psi(u)e^{G(u)} = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{17}$$

in the sense that $u \in H_0^1(\Omega)\cap L^2(\Omega; d\mu)$, $u \geq 0$, $g(u)|\nabla u|^2 \in L^1(\{x \in \Omega : u(x) > 0\})$ and for every $\varphi \in H_0^1(\Omega) \cap L^2(\Omega; d\mu)$, $\varphi \geq 0$ we have

$$\int_\Omega \nabla u \nabla\varphi + \int_\Omega \Psi(u)e^{G(u)}\varphi d\mu + \int_{\{u>0\}} g(u)|\nabla u|^2\varphi = \int_\Omega f(x)\varphi, \tag{18}$$

where $G(s) = \int_1^s g(t)dt$ and $\Psi(s) = \int_0^s e^{-G(t)}dt$.

Proof. We deal with the proof in the case of item (2) of condition (FG) in Definition 3.1. Observe that in the case of item (1) we can argue even easier since, by Lemma 2.2 and Remark 3 we have that $\|u^\varepsilon\|_{L^\infty(\Omega^\varepsilon)} \leq C_{f,\Omega}$. In fact, we can consider $\tilde{g} \in C((0,\infty))$ such that $\tilde{g}(s) = g(s)$ for $s \leq C_{f,\Omega}$ and $\tilde{g}(s) = 0$ for $s > C_{f,\Omega} + 1$ and argue with $\tilde{g}$ instead of g.

Step 1. Observe that $\|\tilde{u}^\varepsilon\|_{H_0^1(\Omega)} = \|u^\varepsilon\|_{H_0^1(\Omega^\varepsilon)}$. Thus, since $g \geq 0$, we can use Lemma 2.2 and Remark 3 to deduce a uniform estimate of $\|\tilde{u}^\varepsilon\|_{H_0^1(\Omega)}$. As a consequence there exists $u \in H_0^1(\Omega)$ such that $\tilde{u}^\varepsilon \to u$ weakly in $H_0^1(\Omega)$, strongly in $L^q(\Omega)$ $(q < 2^*)$ and a.e. in Ω.

Step 2. For every $\varphi^\varepsilon \in H_0^1(\Omega^\varepsilon) \cap L^\infty(\Omega^\varepsilon)$ we have that

$$\begin{aligned} \int_{\Omega^\varepsilon\cap\{u>\delta\}} g(u)e^{G(u)-G(u^\varepsilon)}\varphi_\varepsilon \nabla u\nabla u^\varepsilon + \int_{\Omega^\varepsilon} e^{G(H_\delta(u))-G(u^\varepsilon)}\nabla u^\varepsilon\nabla\varphi^\varepsilon \\ = \int_{\Omega^\varepsilon} fe^{G(H_\delta(u))-G(u^\varepsilon)}\varphi^\varepsilon, \end{aligned} \tag{19}$$

where $H_\delta(s) = G_\delta(s) + \delta = \max\{s, \delta\}$. Indeed, there exists $\varphi_{\varepsilon,n} \in C_c^\infty(\Omega^\varepsilon)$ with $\varphi_{\varepsilon,n} \to \varphi^\varepsilon$ in $H_0^1(\Omega^\varepsilon)$ as $n \to +\infty$ and, using condition (FG) (see Definition 3.1) and taking into account Remark 6, we have that

$$e^{G(H_\delta(u))-G(u^\varepsilon)}\varphi_{\varepsilon,n} \in L^\infty(\Omega^\varepsilon) \text{ and } \left(g(u)\chi_{\{u>\delta\}}\nabla u - g(u^\varepsilon)\nabla u^\varepsilon\right)\varphi_{\varepsilon,n} \in L^2(\Omega^\varepsilon)^N.$$

Thus we can take $e^{G(H_\delta(u))-G(u^\varepsilon)}\varphi_{\varepsilon,n}$ as test function in (16) and we get

$$\begin{aligned} \int_{\Omega^\varepsilon\cap\{u>\delta\}} g(u)e^{G(u)-G(u^\varepsilon)}\varphi_{\varepsilon,n}\nabla u\nabla u^\varepsilon + \int_{\Omega^\varepsilon} e^{G(H_\delta(u))-G(u^\varepsilon)}\nabla u^\varepsilon\nabla\varphi_{\varepsilon,n} = \\ = \int_{\Omega^\varepsilon} fe^{G(H_\delta(u))-G(u^\varepsilon)}\varphi_{\varepsilon,n}. \end{aligned}$$

Now, using that $g \in L^1(0,+\infty)$ and it is bounded at infinity, we can pass to the limit in n obtaining the desired result.

Step 3. $u \in H_0^1(\Omega)$ is solution of (17). As usual the idea of the proof is to take, for $\phi \in H_0^1(\Omega) \cap L^\infty(\Omega)$ with $\phi \geq 0$, $\varphi_\varepsilon = w^\varepsilon\phi$ as test function in (19) and then pass to the limit as $\varepsilon, \delta \to 0$ (observe that (9), (10) and (11) imply that $w^\varepsilon\phi \in H_0^1(\Omega^\varepsilon) \cap L^\infty(\Omega^\varepsilon)$ for every $\phi \in H_0^1(\Omega) \cap L^\infty(\Omega)$).

Therefore, denoting $\varphi_{\varepsilon,\delta}=e^{G(H_\delta(u))-G(\tilde u^\varepsilon)}w^\varepsilon\phi$

$$\int_{\Omega^\varepsilon\cap\{u>\delta\}} g(u)\varphi_{\varepsilon,\delta}\nabla u\nabla u^\varepsilon+\int_{\Omega^\varepsilon} e^{G(H_\delta(u))-G(u^\varepsilon)}\phi\nabla u^\varepsilon\nabla w^\varepsilon \\ +\int_{\Omega^\varepsilon} e^{G(H_\delta(u))-G(u^\varepsilon)}w^\varepsilon\nabla u^\varepsilon\nabla\phi=\int_{\Omega^\varepsilon} f\varphi_{\varepsilon,\delta}.$$

Observe that, since $w^\varepsilon=0$ in $\Omega\setminus\overline{\Omega}^\varepsilon$ (see (15)) this is equivalent to

$$\int_\Omega e^{G(H_\delta(u))-G(\tilde u^\varepsilon)}w^\varepsilon\nabla\tilde u^\varepsilon\nabla\phi+\int_\Omega e^{G(H_\delta(u))-G(\tilde u^\varepsilon)}\phi\nabla\tilde u^\varepsilon\nabla w^\varepsilon+ \\ +\int_{\Omega\cap\{u>\delta\}} g(u)\varphi_{\varepsilon,\delta}\nabla u\nabla\tilde u^\varepsilon=\int_\Omega f e^{G(H_\delta(u))-G(\tilde u^\varepsilon)}w^\varepsilon\phi. \tag{20}$$

Observe that in view of (14) one has

$$\int_\Omega e^{G(H_\delta(u))-G(\tilde u^\varepsilon)}\phi\nabla\tilde u^\varepsilon\nabla w^\varepsilon=\int_\Omega\nabla w^\varepsilon\nabla\left(e^{G(H_\delta(u))}\phi\int_0^{\tilde u^\varepsilon}e^{-G(s)}ds\right) \\ -\int_\Omega\left(\int_0^{\tilde u^\varepsilon}e^{-G(s)}ds\right)\nabla w^\varepsilon\nabla\left(e^{G(H_\delta(u))}\phi\right) \\ =\left\langle\mu^\varepsilon,e^{G(H_\delta(u))}\phi\int_0^{\tilde u^\varepsilon}e^{-G(s)}ds\right\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} \\ -\int_\Omega\left(\int_0^{\tilde u^\varepsilon}e^{-G(s)}ds\right)\nabla w^\varepsilon\nabla\left(e^{G(H_\delta(u))}\phi\right),$$

therefore we can write (20) in the following sense

$$\int_\Omega e^{G(H_\delta(u))-G(\tilde u^\varepsilon)}w^\varepsilon\nabla\tilde u^\varepsilon\nabla\phi+\left\langle\mu^\varepsilon,e^{G(H_\delta(u))}\phi\int_0^{\tilde u^\varepsilon}e^{-G(s)}ds\right\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} \\ -\int_\Omega\left(\int_0^{\tilde u^\varepsilon}e^{-G(s)}ds\right)\nabla w^\varepsilon\nabla\left(e^{G(H_\delta(u))}\phi\right)+\int_{\Omega\cap\{u>\delta\}}g(u)\varphi_{\varepsilon,\delta}\nabla u\nabla\tilde u^\varepsilon \\ =\int_\Omega f e^{G(H_\delta(u))-G(\tilde u^\varepsilon)}w^\varepsilon\phi.$$

Using Lebesgue theorem ($e^{G(H_\delta(u))-G(\tilde u^\varepsilon)}w^\varepsilon\to\chi_{\{u>\delta\}}+e^{G(\delta)-G(u)}\chi_{\{u\le\delta\}}$ a.e and it is dominated in $L^1(\Omega)$, for every fixed δ) we have that

$$\lim_{\delta\to0}\lim_{\varepsilon\to0}\int_\Omega f e^{G(H_\delta(u))-G(\tilde u^\varepsilon)}w^\varepsilon\phi=\int_\Omega f\phi.$$

Moreover, since $\tilde u^\varepsilon\to u$ weakly in $H_0^1(\Omega)$, (11) and (12) we have that

$$\lim_{\delta\to0}\lim_{\varepsilon\to0}\int_\Omega e^{G(H_\delta(u))-G(\tilde u^\varepsilon)}w^\varepsilon\nabla\tilde u^\varepsilon\nabla\phi=\int_\Omega\nabla u\nabla\phi,$$

$$\lim_{\varepsilon\to0}\int_{\Omega\cap\{u>\delta\}}g(u)\varphi_{\varepsilon,\delta}\nabla u\nabla\tilde u^\varepsilon=\int_{\Omega\cap\{u>\delta\}}g(u)|\nabla u|^2\phi$$

and

$$\lim_{\varepsilon\to0}\int_\Omega\left(\int_0^{\tilde u^\varepsilon}e^{-G(s)}ds\right)\nabla w^\varepsilon\nabla\left(e^{G(H_\delta(u))}\phi\right)=0.$$

On the other hand, using again that $\tilde{u}^\varepsilon \to u$ weakly in $H_0^1(\Omega)$, (11), (12) and (14), it follows that

$$\lim_{\varepsilon\to 0}\left\langle \mu^\varepsilon, e^{G(H_\delta(u))}\phi\int_0^{\tilde{u}^\varepsilon} e^{-G(s)}ds\right\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} = \left\langle \mu, e^{G(H_\delta(u))}\phi\Psi(u)\right\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}.$$

Observe that $e^{G(H_\delta(u))}\phi\psi(u) \le Cu \in L^1(\Omega; d\mu)$ and Lebesgue theorem allows to assure that

$$\lim_{\delta\to 0}\int_\Omega e^{G(H_\delta(u))}\phi\Psi(u)d\mu = \int_\Omega e^{G(u)}\phi\Psi(u)d\mu.$$

Thus, taking now limits in all the terms as $\delta \to 0$ we have that $g(u)|\nabla u|^2\phi \in L^1(\{u>0\})$ and

$$\int_\Omega \nabla u\nabla\phi + \int_\Omega e^{G(u)}\Psi(u)\phi\, d\mu + \int_{\Omega\cap\{u>0\}} g(u)|\nabla u|^2\phi = \int_\Omega f\phi, \tag{21}$$

for every $\phi \in H_0^1(\Omega)\cap L^\infty(\Omega)$, $\phi \ge 0$.

Step 4. Uniqueness of the solution of (17) follows using the ideas contained in Theorem 2.7 in [5] (see also [8]). Assume that problem (17) admits two solutions $u_1, u_2 \in H_0^1(\Omega)\cap L^2(\Omega; d\mu)$. For $\varepsilon < 1$ we take $e^{-G(S_\varepsilon(u_1))}T_k\left((\Psi(u_1)-\Psi(u_2))^+\right) \in H_0^1(\Omega)\cap L^\infty(\Omega)$ as test function in (21) for $u = u_1$ and we take the test function $e^{-G(S_\varepsilon(u_2))}T_k\left((\Psi(u_1)-\Psi(u_2))^+\right) \in H_0^1(\Omega)\cap L^\infty(\Omega)$ for $u = u_2$, where we recall that $S_\varepsilon(s) = \min\{\max\{\varepsilon, s\}, 1/\varepsilon\}$, for every $s > 0$. Subtracting and taking into account that $\Psi(s)$ is strictly increasing and $e^{-G(s)}$ is strictly decreasing we have

$$\begin{aligned}
&\int_\Omega e^{-G(S_\varepsilon(u_1))}\nabla u_1\cdot\nabla T_k\left((\Psi(u_1)-\Psi(u_2))^+\right)+\\
&+\int_{\{\varepsilon>u_1\}\cup\{u_1>1/\varepsilon\}} e^{-G(S_\varepsilon(u_1))}g(u_1)|\nabla u_1|^2 T_k\left((\Psi(u_1)-\Psi(u_2))^+\right)+\\
&+\int_\Omega e^{-G(S_\varepsilon(u_1))}e^{G(u_1)}\Psi(u_1)T_k\left((\Psi(u_1)-\Psi(u_2))^+\right)d\mu-\\
&-\int_\Omega e^{-G(S_\varepsilon(u_2))}\nabla u_2\cdot\nabla T_k\left((\Psi(u_1)-\Psi(u_2))^+\right)-\\
&-\int_{\{\varepsilon>u_2\}\cup\{u_2>1/\varepsilon\}} e^{-G(S_\varepsilon(u_2))}g(u_2)|\nabla u_2|^2 T_k\left((\Psi(u_1)-\Psi(u_2))^+\right)-\\
&-\int_\Omega e^{-G(S_\varepsilon(u_2))}e^{G(u_2)}\Psi(u_2)T_k\left((\Psi(u_1)-\Psi(u_2))^+\right)d\mu=\\
&=\int_\Omega f(x)(e^{-G(S_\varepsilon(u_1))}-e^{-G(S_\varepsilon(u_2))})T_k\left((\Psi(u_1)-\Psi(u_2))^+\right)\le 0.
\end{aligned}$$

Observe that functions $e^{-G(S_\varepsilon(u_1))}$ and $e^{-G(S_\varepsilon(u_2))}$ are bounded and thus we can pass to the limit as ε goes to zero and we obtain that

$$\begin{aligned}
0 \le &\int_\Omega \left|\nabla T_k\left((\Psi(u_1)-\Psi(u_2))^+\right)\right|^2\\
&+\int_\Omega (\Psi(u_1)-\Psi(u_2))T_k\left((\Psi(u_1)-\Psi(u_2))^+\right)d\mu \le 0.
\end{aligned}$$

Thus $(\Psi(u_1)-\Psi(u_2))^+ = 0$ and consequently $u_1 \le u_2$ (since ψ is strictly increasing). Interchanging u_1 and u_2 we get the reverse inequality.

Step 5. Let us finally prove that $u \in L^2(\Omega; d\mu)$, $g(u)|\nabla u|^2\chi_{\{u>0\}} \in L^1(\Omega)$ and that (18) holds true.

Taking $\phi = T_k(u) \in H_0^1(\Omega) \cap L^\infty(\Omega)$ in (21) we obtain

$$\int_\Omega e^{G(u)}\Psi(u)T_k(u)\, d\mu \leq \int_\Omega fu,$$

and using Fatou Lemma we infer that $e^{G(u)}\Psi(u)u \in L^1(\Omega; d\mu)$. Moreover, taking into account that $e^{G(s)}\psi(s) \geq s$, for every $s > 0$ it follows that $u \in L^2(\Omega; d\mu)$.

Taking $\frac{T_\varepsilon(u)}{\varepsilon}$ as test function in (21) and using Fatou Lemma as $\varepsilon \to 0$ yields that

$$g(u)|\nabla u|^2\chi_{\{u>0\}} \in L^1(\Omega).$$

Finally, given $\varphi \in H_0^1(\Omega) \cap L^2(\Omega; d\mu)$, with $\varphi \geq 0$, we take $T_k(\varphi)$ as test function in (21) and we get that (18) is satisfied taking limit as k goes to infinity. □

3.3. **Corrector result.**

In order to prove that the solution given by Theorem 3.2 is strictly positive we assume that the measure μ is such that

$$-\Delta w + \mu w \text{ verifies the strong maximum principle.} \tag{22}$$

Theorem 3.3. *Assume that hypotheses of Theorem 3.2 are satisfied. Suppose also that μ satisfies (22). Then*

$$\tilde{u}^\varepsilon = \Psi^{-1}(w^\varepsilon\Psi(u) + r^\varepsilon)$$

with $r^\varepsilon \to 0$ strongly in $H_0^1(\Omega)$.

Remark 7. Observe that, due to the presence of Ψ^{-1} and Ψ, this is not a standard corrector result. The change arises from the nonlinear nature of the lower order term. □

Proof. We take $e^{-G(u^\varepsilon)}\varphi$ with $\varphi \in C_c^\infty(\Omega^\varepsilon)$, $\varphi \geq 0$. Then we have that

$$\int_{\Omega^\varepsilon} e^{-G(u^\varepsilon)}\nabla u^\varepsilon \nabla\varphi = \int_{\Omega^\varepsilon} fe^{-G(u^\varepsilon)}\varphi.$$

This equality is true, by density, for every $0 \leq \varphi \in H_0^1(\Omega^\varepsilon)$. We choose $\varphi = \Psi(u^\varepsilon)$ and we have that

$$\int_\Omega e^{-2G(\tilde{u}^\varepsilon)}|\nabla\tilde{u}^\varepsilon|^2 = \int_\Omega fe^{-G(\tilde{u}^\varepsilon)}\Psi(\tilde{u}^\varepsilon).$$

Using Lebesgue theorem we obtain that

$$\lim_{\varepsilon\to 0}\int_\Omega e^{-2G(\tilde{u}^\varepsilon)}|\nabla\tilde{u}^\varepsilon|^2 = \int_\Omega fe^{-G(u)}\Psi(u).$$

Observe that, for $H_\delta(s)$ defined as before by $H_\delta(s) = \max\{\delta, s\}$, we can take $e^{-G(H_\delta(u))}\phi$ with $\phi \in H_0^1(\Omega) \cap L^\infty(\Omega)$, $\phi \geq 0$, as test function in the equation (18) and we obtain

$$\int_\Omega e^{-G(H_\delta(u))}\nabla u\nabla\phi + \int_\Omega \Psi(u)e^{G(u)-G(H_\delta(u))}\phi d\mu + \int_{\{0<u\leq\delta\}} g(u)|\nabla u|^2 e^{-G(H_\delta(u))}\phi$$
$$= \int_\Omega f(x)e^{-G(H_\delta(u))}\phi \geq \int_\Omega fe^{-\int_1^\infty g(t)dt}\phi.$$

Now, since $H_\delta(u) \geq u$, we have that

$$\int_\Omega \Psi(u)\phi d\mu \geq \int_\Omega \Psi(u) e^{G(u)-G(H_\delta(u))}\phi d\mu.$$

Moreover, we can pass to the limit in δ to obtain that

$$\lim_{\delta\to 0}\int_\Omega e^{-G(H_\delta(u))}\nabla u \nabla \phi = \int_\Omega \nabla\Psi(u)\nabla\phi$$

and, since $g(u)|\nabla u|^2 \in L^1(\{x\in\Omega : u(x)>0\})$, that

$$\lim_{\delta\to 0}\int_{\{0<u\leq\delta\}} g(u)|\nabla u|^2 e^{-G(\delta)}\phi = 0.$$

Hence, we deduce that

$$\int_\Omega \nabla\Psi(u)\nabla\phi + \int_\Omega \Psi(u)\phi d\mu \geq \int_\Omega f e^{-\int_1^\infty g(t)dt}\phi.$$

Thus, using (22) for $w = \Psi(u)$, the strong maximum principle allow us to assure that $0 < \Psi(u)$. In particular, since Ψ is increasing, for every $\omega \subset\subset \Omega$ there exists $c_\omega > 0$ such that $u > c_\omega$ a.e. in ω. Thus, we can take $e^{-G(u)}\varphi$ with $\varphi \in \mathcal{D}(\Omega)$, $\varphi \geq 0$, as test function in the equation satisfied by u and we obtain that

$$\int_\Omega e^{-G(u)}\nabla u\nabla\varphi + \langle \mu, \Psi(u)\varphi\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} = \int_\Omega f e^{-G(u)}\varphi.$$

In particular, by density, for $\varphi = \Psi(u)$ we have

$$\lim_{\varepsilon\to 0}\int_\Omega e^{-2G(\tilde{u}^\varepsilon)}|\nabla\tilde{u}^\varepsilon|^2 = \int_\Omega e^{-2G(u)}|\nabla u|^2 + \langle \mu, \Psi(u)^2\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}. \tag{23}$$

On the other hand, using (14) we know that

$$\begin{aligned}
&\int_\Omega e^{-G(\tilde{u}^\varepsilon)}\nabla\tilde{u}^\varepsilon\nabla(w^\varepsilon\Psi(u)) \\
=&\int_\Omega e^{-G(\tilde{u}^\varepsilon)}\Psi(u)\nabla\tilde{u}^\varepsilon\nabla w^\varepsilon + \int_\Omega e^{-G(\tilde{u}^\varepsilon)}e^{-G(u)}w^\varepsilon\nabla\tilde{u}^\varepsilon\nabla u \\
=&\langle \mu^\varepsilon, \Psi(\tilde{u}^\varepsilon)\Psi(u)\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} - \int_\Omega e^{-G(u)}\Psi(\tilde{u}^\varepsilon)\nabla u\nabla w^\varepsilon \\
&+ \int_\Omega e^{-G(\tilde{u}^\varepsilon)}e^{-G(u)}w^\varepsilon\nabla\tilde{u}^\varepsilon\nabla u.
\end{aligned}$$

Taking into account that, up to a subsequence, $\Psi(\tilde{u}^\varepsilon)\Psi(u) \to \Psi(u)^2$ weakly in $H_0^1(\Omega)$

$$\lim_{\varepsilon\to 0}\langle \mu^\varepsilon, \Psi(\tilde{u^\varepsilon})\Psi(u)\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} = \langle \mu, \Psi(u)^2\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}.$$

Moreover, the weak convergence in $L^2(\Omega)^N$ of ∇w_ε to $\nabla 1$ (see (11)) and the strong convergence in $L^2(\Omega)$ of $e^{-G(u)}\Psi(\tilde{u}^\varepsilon)\nabla u$ to $e^{-G(u)}\Psi(u)\nabla u$ implies that

$$\lim_{\varepsilon\to 0}\int_\Omega e^{-G(u)}\Psi(\tilde{u}^\varepsilon)\nabla u\nabla w^\varepsilon = 0.$$

Even more, the weak convergence in $L^2(\Omega)^N$ of $\nabla\tilde{u}_\varepsilon$ to ∇u and the strong convergence of $e^{-G(\tilde{u}^\varepsilon)}e^{-G(u)}w^\varepsilon\nabla u$ to $e^{-2G(u)}\nabla u$ in $L^2(\Omega)$ implies that

$$\lim_{\varepsilon\to 0}\int_\Omega e^{-G(\tilde{u}^\varepsilon)}e^{-G(u)}w^\varepsilon\nabla\tilde{u}^\varepsilon\nabla u = \int_\Omega e^{-2G(u)}|\nabla u|^2.$$

Consequently

$$\lim_{\varepsilon\to 0}\int_\Omega e^{-G(\tilde u^\varepsilon)}\nabla\tilde u^\varepsilon\nabla(w^\varepsilon\Psi(u)) = \int_\Omega e^{-2G(u)}|\nabla u|^2 + \langle \mu, \Psi(u)^2\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}. \quad (24)$$

Finally we have that, taking into account (14)

$$\int_\Omega \nabla(w^\varepsilon\Psi(u))\nabla(w^\varepsilon\Psi(u)) =$$
$$= \int_\Omega \Psi(u)\nabla(w^\varepsilon\Psi(u))\nabla w^\varepsilon + \int_\Omega w^\varepsilon e^{-G(u)}\nabla(w^\varepsilon\Psi(u))\nabla u =$$
$$= \langle \mu^\varepsilon, w^\varepsilon\Psi(u)^2\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} - \int_\Omega w^\varepsilon\Psi(u)\nabla\Psi(u)\nabla w^\varepsilon + \int_\Omega w^\varepsilon e^{-G(u)}\Psi(u)\nabla w^\varepsilon\nabla u +$$
$$+ \int_\Omega (w^\varepsilon)^2 e^{-2G(u)}|\nabla u|^2.$$

Arguing as above, using that $w^\varepsilon\Psi(u)^2$ weakly converges to $\Psi(u)^2$ in $H_0^1(\Omega)$, that $w^\varepsilon e^{-G(u)}\Psi(u)\nabla u$ is strongly convergent in $L^2(\Omega)^N$ and Lebesgue theorem, we obtain

$$\lim_{\varepsilon\to 0}\int_\Omega \nabla(w^\varepsilon\Psi(u))\nabla(w^\varepsilon\Psi(u)) = \int_\Omega e^{-2G(u)}|\nabla u|^2 + \langle \mu, \Psi(u)^2\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}. \quad (25)$$

Using (23), (24) and (25) we deduce that r^ε strongly converges to zero in $H_0^1(\Omega)$. □

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