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Álgebra homológica Gorenstein relativa a un módulo de tipo tilting

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Gorenstein homological algebra relative to a tilting-like module

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Dedication

To my beloved parents
To my beloved wife

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Rachid El Maaouy
December, 2023

Contents

Resumen

Recientemente, en la rama de investigación del Álgebra Homológica Gorenstein, se ha introducido una nueva perspectiva que le ha dado un nuevo e importante impulso. Esta nueva perspectiva consiste en estudiar los módulos Gorenstein (inyectivos, proyectivos o planos) en base a un módulo de referencia con determinadas propiedades (un módulo semidualizante), en lugar de tomar como referencia el anillo sobre el que se construyen los módulos. Sin embargo, las condiciones exigidas a un módulo para incluirlo en la clase de los semidualizantes son bastante restrictivas, y no todas parecen ser del todo necesarias para desarrollar una teoría de (co)homología satisfactoria. Así, en los últimos años se han publicado varios artículos con resultados muy relevantes que dan respuesta al problema de hasta qué punto se puede alejar de ser semidualizante el módulo sobre el que se relativizan las clases de módulos Gorenstein, sin que esta pérdida de propiedades se traslade a los resultados esperables. De esta manera, aparecen los módulos w-tilting y w-cotilting.

Los objetivos fundamentales de esta tesis siguen tres direcciones.

Primero investigamos la teoría de los módulos G_C-proyectivos y de la dimensión asociada a la clase formada por estos módulos, en la categoría de módulos sobre un anillo de matrices triangulares. En concreto, se estudiarán, sobre este tipo de anillos, los conceptos fundamentales del álgebra homológica Gorenstein relativa: hablamos de los módulos w-tilting, de los G_C-proyectivos, de las dimensiones G_C-proyectivas de los módulos y de la dimensión G_C-proyectiva global del anillo.

En la segunda parte de la tesis hacemos un estudio exhaustivo del ambiente Gorenstein plano relativo, en términos tanto estructurales como de las propiedades de las dimensiones asociadas. Todo esto, por supuesto, respecto a un módulo que no será necesariamente semidualizante.

Finalmente, investigamos la clase de los módulos G_C-planos desde una nueva y prometedora perspectiva: desde el punto de vista homotópico. Más concretamente, desde el punto de vista de las estructuras de modelos abelianas. Así, la última parte de la tesis se dedica al estudio de la existencia de una estructura de modelos abeliana a partir de la clase de los módulos G_C-planos, y a la investigación de sus propiedades desde este punto de vista.

Palabras claves. Anillo de matricies triangulares; dimensión Gorenstein proyectiva relativa; módulo

w-tilting; dimensión Gorenstein plana relativa; módulo w^{+}-tilting; estructura de modelos abeliana.

Abstract

In recent years, a variant of Gorenstein homological algebra has been successfully introduced. It consists of replacing, in certain situations, the base ring by a semidualizing module C. Recently, and since (semidualizing defining properties are quite restrictive, relevant works have been published with the aim to know to what extent the conditions imposed on the module C can be reduced): the concepts of w-tilting and w-cotilting modules appear and a satisfactory theory has been developed.

The goal of this thesis goes in three directions.

First, we investigate the theory of the G_C-projective modules and dimensions in the category of modules over triangular matrix rings. Namely, several fundamental concepts of relative Gorenstein homological algebra (w-tilting, G_C-projective modules, G_C-projective dimensions and the global G_C-projective dimension) are characterized over such rings.

In the second part, we extensively study the relative Gorenstein flat behavior, in terms of both structural and dimension properties, with respect to a non-necessarily semidualizing module.

Finally, we investigate the class of the G_C-flat modules from a fresh and different perspective: that of homotopical aspect. In particular, from the abelian model structures perspective. Therefore, the rest of this thesis is devoted to investigating the existence of an abelian model structure involving the class of G_C-flat modules and then to further study it from this perspective.

Key Words. Triangular matrix ring; relative Gorenstein projective dimension; w-tilting module; relative Gorenstein flat dimension; w^+-tilting module; abelian model structure

Résumé

Ces dernières années, une variante de l'algèbre homologique de Gorenstein a été introduite avec succès. Elle consiste à remplacer, dans certaines situations, l'anneau de base par un module semidualisant C. Récemment, et puisque les conditions requises pour qu'un module soit semidualisant sont assez restrictives, et toutes ne semblent pas entièrement nécessaires pour développer une théorie de (co)homologie satisfaisante, des travaux pertinents ont été publiés dans le but de savoir dans quelle mesure les conditions imposées au module C peuvent être réduites: les concepts de modules w-tilting et w-cotilting apparaissent et une théorie satisfaisante a été développée.

Les objectifs fondamentaux de cette thèse suivent trois directions.

Tout d'abord, nous étudions la théorie des dimensions $\mathrm{G_C}$-projectives dans la catégorie des modules sur l'anneau matriciel triangulaire. Plus précisément, plusieurs concepts fondamentaux de l'algèbre homologique de Gorenstein relative sont caractérisés sur ce type d'anneaux: on parle des modules w-tilting, des modules $\mathrm{G_C}$-projectifs, des dimensions $\mathrm{G_C}$-projectives et de la dimension $\mathrm{G_C}$-projective globale de l'anneau.

Dans la deuxième partie, nous étudions en détail le comportement de Gorenstein plat relatif, à la fois en termes structurels et en termes de propriétés des dimensions associées, par rapport à un module qui n'est pas nécessairement semidualisant.

Enfin, nous étudions la classe des modules $\mathrm{G_C}$-plats d'un point de vue (nouveau) et prometteur: celui de l'aspect homotopique. Plus précisément, du point de vue des structures de modèles abéliens. Par conséquent, le reste de cette thèse est consacré à l'étude de l'existence d'une structure de modèle abélien impliquant la classe des modules $\mathrm{G_C}$-plats et ensuite à l'étudier plus en détail de ce point de vue.

Mots Clés. Anneau matriciel triangulaire; dimension projective de Gorenstein relative; module w-tilting; dimension plat de Gorenstein relative; module w^+-tilting; structure du modèle abélien

<u>Author's papers involved in this thesis</u> :

1. D. Bennis, R. El Maaouy, J. R. García Rozas and L. Oyonarte, *Relative Gorenstein dimensions over triangular matrix rings*, Mathematics. **9** (2021), 2676.

2. D. Bennis, R. El Maaouy, J. R. García Rozas and L. Oyonarte, *Relative Gorenstein flat modules and dimension*, Comm. Alg. **50** (2022), 3853-3882.

3. D. Bennis, R. El Maaouy, J. R. García Rozas and L. Oyonarte, *Relative Gorenstein flat modules and Foxby classes and their model structures*, submitted.

4. D. Bennis, R. El Maaouy, J. R. García Rozas and L. Oyonarte, *Relative weak global Gorenstein dimension, AB-contexts and model structures*, submitted.

Introduction

Throughout this thesis, R and S will be associative (non-necessarily commutative) rings with identity, and all modules will be, unless otherwise specified, unital left R-modules or right S-modules.

In modern homological algebra, one of the main reasons for introducing numerical invariants is to measure 'how far' a module or a ring is from possessing some special properties. To make this statement precise, let us introduce some terminology. Given a class $\mathscr{X}$ of R-modules, we say that an R-module M is said to have an $\mathscr{X}$-resolution dimension less than or equal to an integer $n \geq 0$ if M has an $\mathscr{X}$-resolution of length n, that is, there exists an exact sequence $0 \to X_n \to \cdots \to X_1 \to X_0 \to M \to 0$ with each $X_i \in \mathscr{X}$.

One of the main examples in classical homological algebra is the projective dimension of modules. This dimension can be obtained by taking $\mathscr{X}$ as the class of projective R-modules and measures, for instance, how far a module is from being projective, while the global dimension of a ring R, the supremum of the projective dimension of every R-module, measures how far R is from being semisimple.

This trend of "showing that finiteness of homological dimensions of modules characterizes modules and rings with certain properties" began with Auslander, Buchsbaum and Serre in 1956 when they showed that a commutative noetherian local ring R is regular (the maximal ideal $\mathfrak{m}$ can be generated by d elements where d is the Krull dimension) if and only if every R-module M has finite projective dimension.

In line with this idea, the notion of G-dimension, a refinement of the projective dimension, was introduced by Auslander in [3] and developed by Auslander and Bridger in [4]. This is a homological dimension of finitely generated modules over a commutative noetherian ring R and is defined as the $\mathscr{X}$-resolution dimension with $\mathscr{X}$ the class of the so-called totally reflexive modules. By definition, these are the finitely generated R–modules M satisfying the following conditions:

- $\mathrm{Ext}^i_R(M,R) = 0 = \mathrm{Ext}^i_R(\mathrm{Hom}_R(M,R),R)$ for all $i \geq 1$.
- The natural biduality morphism $M \to \mathrm{Hom}_R(\mathrm{Hom}_R(M,R),R)$.

Auslander and Bridger used this homological dimension to characterize local Gorenstein rings (rings of finite self-injective dimension), parallel to the Auslander-Buchsbaum-Serre characterization of regular rings. That is, they showed that a commutative noetherian local ring R is Gorenstein if and only if every finitely generated R-module M has finite G-dimension.

This theory of G-dimension has been extended in several directions, for different purposes and motivated by many reasons.

One of the most successful approaches is the one taken in the 1990s by Enochs et al. in [36, 39]. Motivated by the classical homological algebra, Enochs and Jenda in [36] and Enochs, Jenda and Torrecillas in [39] introduced Gorenstein projective and Gorenstein flat modules, respectively, that generalized the notion of modules with G-dimension 0 to the case of arbitrary modules, in the sense that a finitely generated module over a noetherian ring is Gorenstein projective if and only if it is Gorenstein flat if and only if it has G-dimension zero. Dually, and to make the theory complete, Enochs and Jenda also defined in [36] the notion of Gorenstein injective modules. These three types of modules, together with their related dimensions, form the basis of what is known as "Gorenstein homological algebra".

Golod, on the other hand, considered another interesting extension. Instead of working with respect to the regular module R in the definition of totally reflexive modules, one can build a homological dimension with respect to certain modules with nice homological properties. These are the semidualizing modules.

Recall that a semidualizing module over a commutative noetherian R is a finitely generated module C satisfying the following assertions:

• $\mathrm{Hom}_R(C,C)$ is (canonically) isomorphic to R.

• $\mathrm{Ext}^i_R(C,C) = 0$ for $i \geq 1$.

With respect to a semidualizing module C, Golod introduced the $\mathrm{G_C}$-dimension of finitely generated modules over a commutative noetherian ring R, which is also a refinement of the projective dimension. He showed that this relative homological dimension shares many nice and fundamental homological properties of Auslander's G–dimension.

Holm and Jørgensen, motivated in part by the generalizations of the G-dimension made by Enochs, Jenda, and Torrecillas, extended Golod's study of $\mathrm{G_C}$-dimension of finitely generated modules to the case of arbitrary modules over commutative noetherian rings with respect to a semidualizing module C. They introduced C-Gorenstein projective and C-Gorenstein flat R-modules as the analogues of modules of $\mathrm{G_C}$-dimension 0 in one hand, and as relative versions of Gorenstein projective and Gorenstein flat modules on the other hand. And they also introduced C-Gorenstein injective modules.

In particular, when $C = R$, these relative Gorenstein modules coincide with the absolute ones. Therefore, all the previous approaches have been generalized and unified with C-Gorenstein projective, flat and injective modules and their associated dimensions. This gave rise to the theory of relative Gorenstein homological algebra, where "relative" here refers to the module C.

Back to semidualizing modules, these modules play a central role in relative (Gorenstein) homological algebra. They were introduced by Foxby in 1972 over commutative noetherian rings under the name PG-modules of rank one ([45]), while Golod ([54]) and Vasconcelos ([81]) rediscovered them and continued their study under different names, suitable modules and spherical modules, respectively. However, it was Christensen who used the name "semidualizing" to refer to these modules ([27]).

The necessity of making the theory of relative homological algebra more flexible and less restrictive has motivated many researchers to extend the definition of a semidualizing module to a more general setting. First, Araya-Takahashi-Yoshino ([1]) extended the definition to a pair of non-commutative noetherian rings, while White in 2010 ([83]) extended it to the non-noetherian, but commutative rings. This allowed her to further study C-Gorenstein projective modules and dimensions, where she used the name "G_C-projective" instead of "C-Gorenstein projective". Holm and White ([62]), on the other hand, made the most general extension to a semidualizing module C over two associative rings R and S such that C becomes an (R,S)-bimodule with excellent duality properties. This general setting allowed many researchers (see for instance [70, 86, 48]) to further study the theory of relative Gorenstein homological algebra.

But still, requiring C to be a semidualizing module is by no means quite restrictive. As explained in [17], in order to study the theory of G_C-projective and G_C-injective dimensions with C being a semidualizing (R,S)-bimodule, the condition $\mathrm{End}_S(C) \cong R$ seems to be too restrictive and in some cases it approaches C to be projective, and C being projective would mean that this relative theory would turn out to be the absolute classical one. Therefore, one may ask the following question:

Question. Is the condition on C to be semidualizing necessary so that the relative (Gorenstein) homological algebra preserve its main properties?

Bennis, Garcia and Oyonarte answered this question in [17] in the case of G_C-projectivity, and G_C-injectivity. They found the minimum conditions for C to still have a nice theory to develop. Modules satisfying these conditions were called w-tilting, and dually, w-cotilting. Consequently, this led to a series of papers ([17, 18, 19, 11]) in which the authors developed a satisfactory theory of many of the homological aspects of G_C-projective and G_C-injective dimensions of modules and rings with respect to w-tilting and w-cotilting modules, respectively.

The goal of this thesis goes in three directions.

First, we further investigate the properties of the G_C-projective modules and dimensions. This will be achieved via the category of modules over triangular matrix rings. Such rings appear naturally in many areas of algebra and play an important role in many fields of mathematics. In particular, in representation theory of algebras (see for instance [6]). But more importantly, they are a very useful tool for constructing examples and counter-examples.

The second aim is to answer the above question in the case of G_C-flatness. That is, we introduce and investigate modules C such that the theory of G_C-flat modules and dimensions behaves homologically best and show that is preserves many of the homological properties, which are known to hold over noetherian or coherent rings with respect to a semidualizing module. In addition, the theory developed throughout the thesis will also include results that are new even in the case where C is semidualizing.

The third and final objective of this thesis is the study of the class of the G_C-flat modules from a fresh and different perspective: the homotopical aspect. In particular, from the abelian model structures perspective. Recall that, roughly speaking, an abelian model structure on an abelian category $\mathscr{A}$ is given by three classes of morphisms of $\mathscr{A}$, called cofibrations, fibrations and weak equivalences, that are compatible with the abelian structure, and from which it is possible to introduce a homotopy theory in $\mathscr{A}$. (Abelian) model structures are interesting because they establish the theoretical framework for formally inverting the weak equivalences. Therefore, the approach we will take in this thesis is to investigate the existence of an abelian model structure involving the class of G_C-flat modules, and to study it further from this perspective.

This thesis is divided into five chapters. Let us discuss the contents of each chapter.

Chapter I.

This chapter is devoted to the preliminaries. Our focus will be on the basic terminology and results we'll be using throughout the thesis.

Chapter II.

Let A and B be rings and U be a (B,A)-bimodule. The ring $T = \left(\begin{smallmatrix} A & 0 \\ U & B \end{smallmatrix}\right)$ is known as the (formal) triangular matrix ring with usual matrix addition and multiplication. Modules over such rings can be described in a very concrete and useful way. Recall that a left module over T is a triple $M = \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}_{\varphi^M}$, where $M_1 \in A$-Mod, $M_2 \in B$-Mod and $\varphi^M : U \otimes_A M_1 \to M_2$ is a B-morphism. With this approach, triangular matrix rings and modules over them have proved to be, among other things, a rich source of examples and counterexamples.

The main objective of this chapter is to study the fundamental concepts of relative Gorenstein homological algebra (w-tilting, G_C-projective modules, G_C-projective dimensions and the global G_C-projective dimension) over T.

This chapter is organized as follows:

In Section 2.1, we study how to construct w-tilting (tilting, semidualizing) modules over T using w-tilting (tilting, semidualizing) modules over A and B under some assumptions on the bimodule U. In Definition 2.1.2, we introduce (weakly) C-compatible (B,A)-bimodules with respect to a T-module of the form $C := \mathbf{p}(C_1,C_2) = \begin{pmatrix} C_1 \\ (U\otimes_A C_1)\oplus C_2 \end{pmatrix}$. Then, given two w-tilting modules ${}_AC_1$ and ${}_BC_2$, we prove in Proposition 2.1.5 that $C := \mathbf{p}(C_1,C_2)$ is a w-tilting T-module when U is weakly C-compatible.

In Section 2.2, we first describe relative Gorenstein projective modules over T. Let $C = \mathbf{p}(C_1, C_2)$ be a T-module. We prove in Theorem 2.2.3 that if U is C-compatible then the following assertions are equivalent for any T-module $M = \binom{M_1}{M_2}_{\varphi^M}$:

(1) M is $\mathrm{G_C}$-projective.

(2) M_1 is a $\mathrm{G_{C_1}}$-projective A-module, $\mathrm{Coker}\varphi^M$ is a $\mathrm{G_{C_2}}$-projective B-module and $\varphi^M : U \otimes_A M_1 \to M_2$ is a monomorphism.

As an application, we prove the converse of Proposition 2.1.5. Also, when C is w-tilting, we characterize when a T-morphism is a special precover (see Proposition 2.2.7). Then, in Theorem 2.2.8, we prove that the class of $\mathrm{G_C}$-projective T-modules is a special precovering if and only if so are the classes of $\mathrm{G_{C_1}}$-projective A-modules and $\mathrm{G_{C_2}}$-projective B-modules, respectively.

Finally, in Section 2.3, we give an estimate of the $\mathrm{G_C}$-projective dimension of a left T-module and the left global $\mathrm{G_C}$-projective dimension of T.

Recall that the global $\mathrm{G_C}$-projective dimension of a ring R, denoted as $\mathrm{G_C}-\mathrm{PD}(R)$, is the supremum of the $\mathrm{G_C}$-projective dimensions of all R-modules.

First, it is proven that, given a T-module $M = \binom{M_1}{M_2}_{\varphi^M}$, if $C = \mathbf{p}(C_1, C_2)$ is w-tilting, U is C-compatible and

$$\mathrm{SG_{C_2}} - \mathrm{PD}(B) := \sup\{\mathrm{G_{C_2}}{-}\mathrm{pd}(\mathrm{U} \otimes_{\mathrm{A}} \mathrm{G}) \mid \mathrm{G} \in \mathrm{G_{C_1}P(A)}\} < \infty,$$

then

$$\begin{gathered}\max\{\mathrm{G_{C_1}}{-}\mathrm{pd_A}(\mathrm{M_1}), (\mathrm{G_{C_2}}{-}\mathrm{pd_B}(\mathrm{M_2})) - (\mathrm{SG_{C_2}} - \mathrm{PD(B)})\}\\ \leq \mathrm{G_C}{-}\mathrm{pd}_T(M) \leq\\ \max\{(\mathrm{G_{C_1}}{-}\mathrm{pd_A}(\mathrm{M_1})) + (\mathrm{SG_{C_2}} - \mathrm{PD(B)}) + 1, \mathrm{G_{C_2}}{-}\mathrm{pd_B}(\mathrm{M_2})\}.\end{gathered}$$

As an application, we prove that, if $C = \mathbf{p}(C_1, C_2)$ is w-tilting and U is C-compatible then

$$\begin{gathered}\max\{\mathrm{G_{C_1}} - \mathrm{PD}(A), \mathrm{G_{C_2}} - \mathrm{PD}(B)\}\\ \leq \mathrm{G_C} - \mathrm{PD}(T) \leq\\ \max\{\mathrm{G_{C_1}} - \mathrm{PD}(A) + \mathrm{SG_{C_2}} - \mathrm{PD}(B) + 1, \mathrm{G_{C_2}} - \mathrm{PD}(B)\}.\end{gathered}$$

We conclude this section with some cases in which this estimation becomes an exact formula (Corollaries 2.3.9 and 2.3.10).

Chapter III.

In this chapter we develop the theory of $\mathrm{G_C}$-flatness with respect to a non-necessarily semidualizing module C. We are mainly interested in modules C such that the $\mathrm{G_C}$-flat modules and dimensions have good homological properties.

This chapter is organized as follows:

In section 3.1, we give a new concept of relative flat modules on which we will construct relative Gorenstein flat modules. Namely, given an R-module C, we say that an R-module M is $\mathscr{F}_C$-flat provided that $M^+ \in \mathrm{Prod}_R(C^+)$, where $\mathrm{Prod}(C^+)$ stands for the class of all modules which are isomorphic to direct summands of direct products of the right R-module $C^+ := \mathrm{Hom}_R(C, \mathbb{Q}/\mathbb{Z})$. We call the class of all $\mathscr{F}_C$-flat modules $\mathscr{F}_C(R)$. We will find the link between this class $\mathscr{F}_C(R)$ and the class of all flat left S-modules, $S := \mathrm{End}_R(C)$ being the endomorphisms ring of C, and we prove that $\mathscr{F}_C(R) = \mathrm{Add}_R(\mathscr{F}_C(R))$. As $\mathscr{F}_C(R)$ is an Add-type class, this suggests the existence of (pre)covers and (pre)envelope by modules $\mathscr{F}_C(R)$, which we investigate in Proposition 3.1.6(1) and Theorem 3.1.7. In Proposition 3.1.6, we also prove some nice and fundamental homological properties of $\mathscr{F}_C(R)$. Namely: $\mathscr{F}_C(R)$ is always closed under pure extensions, pure submodules and pure quotients, and when C satisfies some orthogonality conditions then $\mathscr{F}_C(R)$ is also closed under arbitrary extensions and direct limits.

In section 3.2, we use $\mathscr{F}_C$-flat modules to define the concept of $\mathrm{G_C}$-flat modules with no restrictions on either the ring R or the module C. $\mathrm{G_C}$-flat modules will be constructed over $\mathscr{F}_C(R)$ as $\mathrm{G_C}$-projective (resp., $\mathrm{G_C}$-injective) modules were constructed over $\mathrm{Add}_R(C)$ (resp., $\mathrm{Prod}_R(C)$) (see sections 2.2 and 3.2).

We will study what are the homological properties of $\mathrm{G_CF}(R)$, the class of $\mathrm{G_C}$-flat R-modules, and check how it is related to other important classes of modules as those of (relative) flat, (relative) projective, (relative) Gorenstein projective, (relative) Gorenstein flat, etc.

As mentioned above, the development of this study will be carried out with respect to a module C satisfying significantly less restrictive conditions than semidualizing modules: we will call these new modules w^+-tilting, and we shall see that they properly generalize w-tilting (and so semidualizing) modules (Proposition 3.2.3 and Example 3.2.2).

In Section 3.3, we investigate a natural problem originally raised by Sather-Wagstaff, Sharif and White in [79]. Namely, they wonder what is the result of the iteration of constructing Gorenstein objects, that is, if we compute Gorenstein modules by taking as the base class a given one, $\mathscr{X}$, and call this new class $\mathscr{G}(\mathscr{X})$, then we compute Gorenstein modules taking as base class $\mathscr{G}(\mathscr{X})$ and call this new class $\mathscr{G}^2(\mathscr{X})$, and so on, when is a class $\mathscr{X}$ such that $\mathscr{G}^n(\mathscr{X}) = \mathscr{G}(\mathscr{X})$ for some n?

It is another of our purposes in this section to study what is the situation when working with the class of $\mathrm{G_C}$-flat modules. We will prove in Theorem 3.3.1 that indeed $\mathrm{G_CF}^2(R) = \mathrm{G_CF}(R)$.

As a consequence, we show that all exact complexes having as components modules in any class in between $\mathscr{F}(R) \cup \mathscr{F}_C(R)$ and $\mathrm{G_CF}(R)$, regardless of the positions they hold in the complex, will give $\mathrm{G_C}$-flat modules as syzygies (Corollary 3.3.2).

Another interesting consequence of the stability is the answer to the following natural question: when is any Gorenstein flat module a G_C-flat module? We will give a set of equivalent conditions for this to hold (Corollary 3.3.3).

Section 3.4 is devoted to the treatment of the dimension relative to the class of G_C-flat modules: the G_C-flat dimension. We will carry out the traditional homological study of dimensions, which includes the link of the dimensions of the modules in a short exact sequence (Proposition 3.4.9), the characterization of modules of finite dimension in terms of the vanishing of the right homological functor, in our case the Tor functor (Theorem 3.4.7 and Corollary 3.4.8), and of course the comparison of the G_C-flat dimension with the dimensions relative to the classes involved in the definition of $G_CF(R)$, that is, $\mathscr{F}_C(R)$ and the class of flat modules (Theorems 3.4.12 and 3.4.14).

Finally, in Section 3.5, we are mainly interested in the global G_C-flat dimension of R, i.e., the supremum of the G_C-flat dimensions of all R-modules. Our first main result is to provide a simple way to compute this global dimension. We prove that finiteness of the global G_C-flat dimension of R depends only on the finiteness of the flat dimension of the $\mathscr{I}_{C^+}$-injective right R-modules (modules in $\mathrm{Prod}_R(C^+)$) and the $\mathscr{F}_C$-flat dimension of the injective left R-modules (Theorem 3.5.5).

It is well known that the weak global dimension of any ring R can be computed by means of the flat dimension of either the left or right R-modules. More precisely, we have the following equality:

$$\sup\{\mathrm{fd}_R(M)|M \text{ is a left } R\text{-module}\} = \sup\{\mathrm{fd}_R(M)|M \text{ is a right } R\text{-module}\}.$$

The theory of Gorenstein flat dimension relative to a semidualizing module is usually studied over commutative rings. But, once we remove the commutativity of the ring, taking into account that the definition of a semidualizing (R,S)-bimodule C is left-right symmetric, a question similar to that of the weak global dimension arises:

Question: Does the following equality hold true?

$$\sup\{G_C\text{-}\mathrm{fd}_R(M)|M \in R\text{-Mod}\} = \sup\{G_C\text{-}\mathrm{fd}_S(M)|M \in \text{Mod-}S\}$$

As an application of Theorem 3.5.5, we give a positive answer to this question when the class of G_C-flat left R-modules and the class of G_C-flat right S-modules are both closed under extensions (see Theorem 3.5.9 and Corollary 3.5.10).

Consequently, we obtain a positive answer (Corollary 3.5.11) to Bennis' conjecture ([10]) for any ring R. However, this conjecture was recently solved independently by S. Bouchiba ([22]) and later by Christensen, Estrada, and Thompson ([29]). We note that our approach is different from theirs. Theorem 3.5.9, in particular, provides a new and simpler proof that sheds more light on such symmetries.

Chapter IV:

The purpose of this chapter is to construct a new hereditary abelian model structure on the category of left R-modules that involves the class of $\mathrm{G_C}$-flat modules and the well-known Bass class $\mathscr{B}_C(R)$, and use the homotopy category of this model structure in order to further study these classes.

Following Hovey [64, Theorem 2.2], there is a one-to-one correspondence between the class of abelian model structures and the class of Hovey triples, i.e., three classes of objects $(\mathscr{Q},\mathscr{W},\mathscr{R})$ satisfying that $\mathscr{W}$ is thick and both $(\mathscr{Q},\mathscr{W}\cap\mathscr{R})$ and $(\mathscr{Q}\cap\mathscr{W},\mathscr{R})$ are complete cotorsion pairs (see Sections 1.5 and 1.7). This establishes a relation between model category theory and representation theory via cotorsion pairs. If the abelian model structure is hereditary (i.e., both of the complete cotorsion pairs induced by the Hovey triple are hereditary), then its homotopy category $\mathrm{Ho}(\mathscr{A})$ is the stable category of a Frobenius category, thus it is triangulated. This is an important situation in which one obtains a triangulated category from the point of view of model category theory. We refer the reader to Section 1.7 for more details.

This chapter is organized as follows.

In Section 4.1, we introduce and study new concepts of relative cotorsion modules: strongly $\mathscr{C}_{\mathrm{C}}$-cotorsion and n-$\mathscr{C}_{\mathrm{C}}$-cotorsion modules for a given integer $n\geq 1$. We are mainly interested in their links with other known classes of modules such as cotorsion modules (Proposition 4.1.5 and Corollary 4.1.6), as well as in their homological properties. It is investigated when these new classes are the right half of a (perfect, complete, hereditary) cotorsion pair (Theorem 4.1.7).

In Section 4.2, we introduce and investigate a new concept of Gorenstein cotorsion modules: $\mathrm{G_C}$-cotorsion. We characterize when the pair ($\mathrm{G_C}$-flat,$\mathrm{G_C}$-cotorsion) is a hereditary and perfect cotorsion pair (Theorem 4.2.6).

In Section 4.3, we use the results from the previous sections to construct, under certain conditions (Theorem 4.3.6), a new hereditary abelian model structure on the category of R-modules

$$\mathscr{M} := (\mathrm{G_CF}(R),\mathscr{W},\mathscr{H}_C(R)),$$

called the **$\mathrm{G_C}$-flat model structure**, and defined as follows:

(1) A morphism f is a cofibration (trivial cofibration) if and only if it is a monomorphism with $\mathrm{G_C}$-flat ($\mathscr{V}_C$-flat) cokernel.

(2) A morphism g is a fibration (trivial fibration) if and only if it is an epimorphism with $\mathscr{H}_C$-cotorsion ($\mathrm{G_C}$-cotorsion) kernel.

Here, $\mathscr{H}_C(R) := \mathscr{B}_C(R)\cap\mathscr{C}_{\mathrm{C}}(R)$ and $\mathscr{V}_C(R) := {}^{\perp}(\mathscr{B}_C(R)\cap\mathscr{C}_{\mathrm{C}}(R))$ is the left Ext-orthogonal class of $\mathscr{H}_C(R)$. Modules in $\mathscr{V}_C(R)$ and $\mathscr{H}_C(R)$ are called $\mathscr{V}_C$-flat and $\mathscr{H}_C$-cotorsion modules, respectively. We call them this way as they satisfy most of the nice properties of flat and cotorsion modules.

Two main consequences of this theorem are obtained (Corollary 4.3.8):

(a) The full subcategory $\mathrm{G_CF}(R)\cap\mathscr{H}_C(R)$ is a Frobenius category whose projective-injective objects are exactly the objects in $\mathscr{F}_C(R)\cap\mathscr{C}_C(R)$.

(b) The homotopy category $\mathrm{Ho}(\mathscr{M})$ is triangulated equivalent to the stable category

$$\underline{\mathrm{G_CF}(R)\cap\mathscr{H}_C(R)}.$$

In Section 4.4, we investigate a question naturally motivated by the previous section:

Question. What are the trivial objects in the $\mathrm{G_C}$-flat model structure?

We answer this question under the assumption that the global $\mathrm{G_C}$-flat dimension of R is finite (see Theorem 4.4.7). In this case, trivial objects coincide with modules having finite $\mathscr{V}_C$-flat dimensions. i.e, finite $\mathscr{V}_C(R)$-resolution dimension.

This problem of lack of information about trivial objects raises even in the general setting of abelian categories. The main tool used to construct the $\mathrm{G_C}$-flat model structure is a well-known result by Gillespie ([51, Theorem 1.1]). This result is a very useful tool for building model structures and has been used by many researchers for this purpose.

Researchers interested in model structures would like to know more about trivial objects than what this result provides. The importance of these objects comes mainly from the fact that they determine the associated homotopy category, as explained in the fundamental theorem of model categories ([63, Theorem 1.2.10]).

A new approach is therefore appreciated. For instance, Šaroch and Šťovíček ([78, Sections 4 and 5]), and later Estrada, Iacob and Pérez ([42]) and the author of this thesis ([32]), have recently used new techniques to construct new model structures with an explicit description of the trivial objects.

Our last main result, which is used to prove Theorem 4.4.7, gives one more step towards a better understanding of this class of trivial objects: assume that $(\mathscr{Q},\widetilde{\mathscr{R}})$ and $(\widetilde{\mathscr{Q}},\mathscr{R})$ are complete hereditary cotorsion pairs in an abelian category $\mathscr{A}$ such that

(a) $\widetilde{\mathscr{Q}}\subseteq\mathscr{Q}$ (or equivalently, $\mathscr{R}\subseteq\widetilde{\mathscr{R}}$).

(b) $\mathscr{Q}\cap\widetilde{\mathscr{R}}=\widetilde{\mathscr{Q}}\cap\mathscr{R}$.

Then, the following assertions hold:

(1) If $\sup\{\mathscr{Q}-\mathrm{resdim}_{\mathscr{A}}(A)|A\in\mathscr{A}\}<\infty$, then $(\mathscr{Q},\mathscr{W},\mathscr{R})$ is a Hovey triple where

$$\mathscr{W}=\{A\in\mathscr{A},\ \widetilde{\mathscr{Q}}-\mathrm{resdim}_{\mathscr{A}}(A)<\infty\}.$$

(2) If $\sup\{\mathscr{R}-\mathrm{coresdim}_{\mathscr{A}}(A)|A\in\mathscr{A}\}<\infty$, then $(\mathscr{Q},\mathscr{W},\mathscr{R})$ is a Hovey triple where

$$\mathscr{W}=\{A\in\mathscr{A},\ \widetilde{\mathscr{R}}-\mathrm{coresdim}_{\mathscr{A}}(A)<\infty\}.$$

The proof of this result is based on an interesting connection between Hovey triples and (weak) AB-contexts. The later have been studied by Auslander and Buchweitz in [5] and used by Hashimoto in his book [59] (see Section 1.7). (Weak) AB-contexts are known to be useful for generating special (pre-)covers and (pre-)envelopes. It turns out that they are also useful for constructing Hovey triples as we will show in Theorem 4.4.5.

Chapter V:

This is the last chapter in which we discuss some of the open questions raised in this thesis or related to the subject of it.

CHAPTER 1

PRELIMINARIES

In this chapter we introduce the basic terminology and notation that we will use, as well as the fundamental definitions and results needed for this thesis.

1.1 Notation and terminology

Throughout this thesis, R and S will be associative (non-necessarily commutative) rings with identity, and all modules will be, unless otherwise specified, unital left R-modules or right S-modules. When right R-modules need to be used, they will be denoted as M_R, while in these cases left R-modules will be denoted by $_RM$. In some cases, right R-modules (resp, left S-modules) will be identified with left (resp., right) modules over the opposite ring R^{op} (resp, S^{op}).

We also use $\mathscr{A}$ to denote an abelian category. By a subcategory of $\mathscr{A}$, we will always mean a full subcategory which is closed under isomorphisms. Any class of objects of $\mathscr{A}$ will be thought of as a (full) subcategory. Conversely, any subcategory can be identified with its class of objects.

Given an integer $n \geq 1$, to any given class of objects $\mathscr{X}$, we associate its right and left n-th Ext-orthogonal classes

$$\mathscr{X}^{\perp_n} = \{M \in \mathscr{A} \mid \operatorname{Ext}^i_{\mathscr{A}}(X,M) = 0, \forall X \in \mathscr{X}, \forall i = 1,..,n\},$$

$$^{\perp_n}\mathscr{X} = \{M \in \mathscr{A} \mid \operatorname{Ext}^i_{\mathscr{A}}(M,X) = 0, \forall X \in \mathscr{X}, \forall i = 1,..,n\}.$$

In particular, we set

$$\mathscr{X}^{\perp} = \mathscr{X}^{\perp_1}, \qquad \mathscr{X}^{\perp_\infty} = \cap_{n\geq 1}\mathscr{X}^{\perp_n}, \qquad ^{\perp}\mathscr{X} = {}^{\perp_1}\mathscr{X}, \qquad ^{\perp_\infty}\mathscr{X} = \cap_{n\geq 1}{}^{\perp_n}\mathscr{X}.$$

and if $\mathscr{X} = \{X\}$, we simply write $\mathscr{X}^{\perp_n} = X^{\perp_n}$ and $^{\perp_n}\mathscr{X} = {}^{\perp_n}X$.

Here are some other standard notations that we will also be using throughout:

***R*-Mod**	The category of all left R-modules.
Mod-*R*	The category of all right R-modules.
$\mathscr{P}$	The class of projective modules.
$\mathscr{I}$	The class of injective modules.
$\mathscr{F}$	The class of flat modules.
M^+	The character module $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$.
$\mathrm{Add}(M)$	The class of all modules which are isomorphic to direct summands of direct sums of copies of a module M.
$\mathrm{add}(M)$	The class of all modules which are isomorphic to direct summands of finite direct sums of copies of a module M.
$\mathrm{Prod}(M)$	The class of all modules which are isomorphic to direct summands of direct products of copies of a module M

1.2 Resolution and coresolution dimensions

Definition 1.2.1. *Let $\mathscr{X}$ be a class of objects of $\mathscr{A}$ and consider a sequence in $\mathscr{A}$*

$$\boldsymbol{X} : \cdots \to X_1 \to X_0 \to X_{-1} \to \cdots$$

(1) ***X** is called* $\mathrm{Hom}_{\mathscr{A}}(\mathscr{X}, -)$***-exact** if the induced sequence of abelian groups*

$$\cdots \to \mathrm{Hom}_{\mathscr{A}}(X, X_1) \to \mathrm{Hom}_{\mathscr{A}}(X, X_0) \to \mathrm{Hom}_{\mathscr{A}}(X, X_{-1}) \to \cdots$$

is exact for every $X \in \mathscr{X}$.

(2) ***X** is called* $\mathrm{Hom}_{\mathscr{A}}(-, \mathscr{X})$***-exact** if the induced sequence of abelian groups*

$$\cdots \to \mathrm{Hom}_{\mathscr{A}}(X_{-1}, X) \to \mathrm{Hom}_{\mathscr{A}}(X_0, X) \to \mathrm{Hom}_{\mathscr{A}}(X_1, X) \to \cdots$$

is exact for every $X \in \mathscr{X}$.

(3) *In case $\mathscr{A} = R$-Mod and $\mathscr{Y}$ is a class of right R-modules, **X** is called $(\mathscr{Y} \otimes_R -)$-**exact** if the induced sequence of abelian groups*

$$\cdots \to Y \otimes_R X_1 \to Y \otimes_R X_0 \to Y \otimes_R X_{-1} \to \cdots$$

is exact for every $Y \in \mathscr{Y}$.

Definition 1.2.2. *An $\mathscr{X}$**-resolution** of an object $A \in \mathscr{A}$ is an exact complex*

$$\cdots \to X_1 \to X_0 \to A \to 0$$

*where each $X_i \in \mathscr{X}$. Dually, an $\mathscr{X}$**-coresolution** of A is an exact complex*

$$0 \to A \to X^0 \to X^1 \to X^2 \to \cdots$$

with $X^i \in \mathscr{X}$.

In particular, $\mathscr{P}$-resolution, $\mathscr{F}$-resolution and $\mathscr{I}$-coresolution will mean the usual projective, flat and injective resolutions, respectively.

Definition 1.2.3 ([5])**.** *An object $A \in \mathscr{A}$ is said to have $\mathscr{X}$**-resolution dimension** less than or equal to an integer $n \geq 0$, $\mathscr{X} - \text{resdim}_{\mathscr{A}}(A) \leq n$, if A has a finite $\mathscr{X}$-resolution:*

$$0 \to X_n \to \cdots \to X_1 \to X_0 \to M \to 0.$$

If n is the least non negative integer for which such a sequence exists, then its $\mathscr{X}$-resolution dimension is precisely n, and if there is no such n, then we define its $\mathscr{X}$-resolution dimension as ∞. We denote by $\widehat{\mathscr{X}}$ (or, $\text{res}\widehat{\mathscr{X}}$) the (full) subcategory of objects in $\mathscr{A}$ having a finite $\mathscr{X}$-resolution.

One could also define $\mathscr{X}$-resolution dimension of classes in a natural way. The $\mathscr{X}$-resolution dimension of a class $\mathscr{Y} \subseteq \mathscr{A}$, is defined as

$$\mathscr{X} - \text{resdim}_{\mathscr{A}}(\mathscr{Y}) = \sup\{\mathscr{X} - \text{resdim}_{\mathscr{A}}(Y) | Y \in \mathscr{Y}\}.$$

In particular, we define the **global $\mathscr{X}$- resolution dimension** of $\mathscr{A}$ as

$$\text{glresdim}_{\mathscr{X}}(\mathscr{A}) := \mathscr{X} - \text{resdim}_{\mathscr{A}}(\mathscr{A}) = \sup\{\mathscr{X} - \text{resdim}_{\mathscr{A}}(A) | \, A \in \mathscr{A}\}.$$

Example 1.2.4. *Let $\mathscr{A} = R$-Mod and $\mathscr{X} = \mathscr{P}(R)$. Then,*

(a) $\mathscr{X} - \text{resdim}_{\mathscr{A}}(-) = \text{pd}_R(-)$*, that is, the $\mathscr{X}$-resolution dimension coincides with the projective dimension.*

(b) $\text{glresdim}_{\mathscr{X}}(\mathscr{A}) = \text{gldim}(R)$*, that is, the global $\mathscr{X}$- resolution dimension of R-Mod coincides with the global dimension of R.*

Example 1.2.5. *Let $\mathscr{A} = R$-Mod and $\mathscr{X} = \mathscr{F}(R)$. Then,*

(a) $\mathscr{X} - \text{resdim}_{\mathscr{A}}(-) = \text{fd}_R(-)$*, that is, the $\mathscr{X}$-resolution dimension coincides with the flat dimension.*

(b) $\text{glresdim}_{\mathscr{X}}(\mathscr{A}) = \text{wdim}(R)$*, that is, the global $\mathscr{X}$-resolution dimension of R-Mod coincides with the weak global dimension of R.*

The question of the closure under certain properties of a given class of objects is a typical problem for those classes having interest in homological and homotopical algebra. Here we recall all such closure properties needed in this thesis.

A class $\mathscr{X}$ of objects of $\mathscr{A}$ is said to be:

1. Closed under **direct summands** if for every two objects X_1 and X_2,
$$\mathscr{X}_1 \oplus X_2 \in \mathscr{X} \Rightarrow X_1, X_2 \in \mathscr{X}.$$
2. Closed under **direct sums** if whenever $(X_i)_i$ is a family of objects,
$$\forall i, X_i \in \mathscr{X} \Rightarrow \oplus_i X_i \in \mathscr{X}.$$
3. Closed under **kernels of epimorphisms** if for any exact sequence $0 \to X_1 \to X_2 \to X_3 \to 0$,
$$X_2, X_3 \in \mathscr{X} \Rightarrow X_1 \in \mathscr{X}.$$
4. Closed under **extensions** if for any short exact sequence $0 \to X_1 \to X_2 \to X_3 \to 0$,
$$X_1, X_3 \in \mathscr{X} \Rightarrow X_2 \in \mathscr{X}.$$
5. Closed under **cokernels of monomorphisms** if for any exact sequence $0 \to X_1 \to X_2 \to X_3 \to 0$,
$$X_1, X_2 \in \mathscr{X} \Rightarrow X_3 \in \mathscr{X}.$$
6. Closed under **pure submodules** if for any pure exact sequence $0 \to X_1 \to X_2 \to X_3 \to 0$,
$$X_2 \in \mathscr{X} \Rightarrow X_1 \in \mathscr{X}.$$
7. Closed under **pure extensions** if for any pure exact sequence $0 \to X_1 \to X_2 \to X_3 \to 0$,
$$X_1, X_3 \in \mathscr{X} \Rightarrow X_3 \in \mathscr{X}.$$
8. Closed under **pure quotients** if for any pure exact sequence $0 \to X_1 \to X_2 \to X_3 \to 0$,
$$X_2 \in \mathscr{X} \Rightarrow X_3 \in \mathscr{X}.$$
9. Closed under **direct limits** if whenever $\{(M_i, \varphi_{ji}), i, j \in I\}$ is a direct system, where I is a directed set,
$$\forall i, X_i \in \mathscr{X} \Rightarrow \varinjlim_i X_i \in \mathscr{X}.$$

10. **Projectively resolving** if $\mathscr{X}$ contains all projective objects of $\mathscr{A}$ and $\mathscr{X}$ is closed under extensions and kernels of epimorphisms.

11. **Injectively coresolving** if $\mathscr{X}$ contains all injective objects of $\mathscr{A}$ and $\mathscr{X}$ is closed under extensions and cokernels of monomorphisms.

12. **Left** (resp., **right**) **thick** if it is closed under extensions, kernels of epimorphisms (resp., cokernels of monomorphisms), and direct summands.

13. **Thick** if it is left and right thick.

1.3 Relative Gorenstein homological algebra

In this section we recall the basic notions from the theory of (Gorenstein) homological algebra relative to a module C.

Given an R-module C, a **complete $\mathscr{P}_C$-projective** complex is a $\mathrm{Hom}_R(-,\mathrm{Add}_R(C))$-exact exact complex of R-modules

$$\mathbf{X}:\ \cdots \to P_1 \to P_0 \to C_{-1} \to C_{-2} \to \cdots,$$

with all $P_i \in \mathscr{P}(R)$ and $C_j \in \mathrm{Add}_R(C)$.

Dually, a **complete $\mathscr{I}_C$-injective** complex is a $\mathrm{Hom}_R(\mathrm{Prod}_R(C),-)$-exact exact complex of R-modules

$$\mathbf{Y}:\ \cdots \to C_1 \to C_0 \to E_{-1} \to E_{-2} \to \cdots,$$

where $C_i \in \mathrm{Prod}_R(C)$, $E_j \in \mathscr{I}(R)$, and such that $M \cong \mathrm{Im}(C_0 \to E_{-1})$.

Definition 1.3.1 ([17])**.** *An R-module M is said to be* $\mathrm{G_C}$***-projective*** *if there exists a complete $\mathscr{P}_C$-projective complex $\mathbf{X}$ as above such that $M \cong \mathrm{Im}(P_0 \to C_{-1})$.*

Dually, an R-module M is said to be $\mathrm{G_C}$***-injective*** *if there exists a complete $\mathscr{I}_C$-injective complex $\mathscr{Y}$ as above such that $M \cong \mathrm{Im}(C_0 \to E_{-1})$.*

We use $\mathrm{G_CP}(R)$ and $\mathrm{G_CI}(R)$ to denote the class of all $\mathrm{G_C}$-projective and $\mathrm{G_C}$-injective R-modules, respectively.

The notion of semidualizing modules is one of the principal notions in relative (Gorenstein) homological algebra. The following general version of semidualizing is due to Holm and White [62].

Definition 1.3.2 ([62], Definition 2.1)**.** *An (R,S)-bimodule C is* ***semidualizing*** *if:*

1. *${}_RC$ and C_S both admit a degreewise finite projective resolution.*

2. $\mathrm{Ext}_R^{\geq 1}(C,C) = \mathrm{Ext}_S^{\geq 1}(C,C) = 0$.

3. *The natural homothety maps* $R \to \mathrm{Hom}_S(C,C)$ *and* $S \to \mathrm{Hom}_R(C,C)$ *are both ring isomorphisms.*

By a **degreewise finite projective resolution** we mean a projective resolution in which every projective module is finitely generated.

Definition 1.3.3 ([82], Section 3.)**.** *An R-module C is called **(Wakamatsu) tilting** if it has the following properties:*

1. *${}_RC$ admits a degreewise finite projective resolution.*

2. $\mathrm{Ext}_R^{\geq 1}(C,C) = 0$.

3. *There exists a* $\mathrm{Hom}_R(-,C)$*-exact* $\mathrm{add}_R(C)$*-coresolution*

$$0 \to R \to C_{-1} \to C_{-2} \to \cdots$$

Given an (R,S)-bimodule C, it was proven by Wakamatsu ([82, Corollary 3.2]) that C is semidualizing if and only if ${}_RC$ is tilting with $S = \mathrm{End}_R(C)$ if and only if C_S is tilting with $R = \mathrm{End}_S(C)$. The following notion, due to Bennis, García Rozas and Oyonarte ([17]), generalizes these two concepts and it will play a crucial role in this thesis.

Definition 1.3.4 ([17])**.** *An R-module C is said to be **weakly Wakamatsu tilting (w-tilting** for short) if it satisfies the following two properties:*

(T1) *C is **Σ-self-orthogonal**, that is,* $\mathrm{Ext}_R^{i\geq 1}(C,C^{(I)}) = 0$ *for every set I.*

(T2) *There exists a* $\mathrm{Hom}_R(-,\mathrm{Add}_R(C))$*-exact* $\mathrm{Add}_R(C)$*-coresolution*

$$0 \to R \to C_{-1} \to C_{-2} \to \cdots$$

*Dually, an R-module U is said to be **w-cotilting** if it satisfies the following two properties:*

(C1) *C is **∏-self-orthogonal**, that is,* $\mathrm{Ext}_R^{i\geq 1}(U^I,U) = 0$ *for every set I.*

(C2) *There is an injective cogenerator D in R-Mod which admits a* $\mathrm{Hom}_R(\mathrm{Prod}_R(U),-)$*-exact* $\mathrm{Prod}_R(U)$*-resolution*

$$\cdots \to U_1 \to U_0 \to D \to 0$$

It is immediate from the definition that w-(co)tilting modules can be characterized as follows.

Lemma 1.3.5.

(1) *An R-module C is w-tilting if and only if both C and R are* $\mathrm{G_C}$*-projective modules.*

(2) *An R-module U is w-cotilting if and only if U and D are* $\mathrm{G_U}$*-injective modules for some injective cogenerator D for R-Mod.*

Examples 1.3.6. *(Gorenstein projective and injective modules). If C is a projective generator, that is,* $\mathrm{Add}_R(C) = \mathscr{P}(R)$*, then C is a w-tilting R-module and* $\mathrm{G_C}$*-projective R-modules are exactly* ***Gorenstein projective*** *modules.*

Dually, Given an injective cogenerator U, that is, $\mathrm{Prod}_R(U) = \mathscr{I}(R)$*, then C is a w-cotilting R-module and* $\mathrm{G_C}$*-injective R-modules are exactly* ***Gorenstein injective*** *modules.*

We use $\mathscr{GP}(R)$ and $\mathscr{GI}(R)$ to denote the class of all Gorenstein projective and Gorenstein injective R-modules, respectively.

The following example is a non trivial example of a w-tilting module.

Example 1.3.7 ([17], Section 2)**.** *Consider a left noetherian ring R and a Gorenstein injective R-module M which is not injective. If*

$$\cdots \to E_1 \to E_0 \to E_{-1} \to E_{-2} \cdots$$

is the complete injective resolution associated to M, that is, $M = \mathrm{Im}(E_0 \to E_{-1})$*, take* $C = (\oplus_{i\leq 1} E_i) \oplus (\oplus E_{i\geq 1} E^i)$ *where*

$$0 \to R \to E^1 \to E^2 \to \cdots$$

is an injective resolution of R. Then, C is a w-tilting and M is $\mathrm{G_C}$*-projective.*

We will now recall and discuss two types of classes of modules that are of interest to us in this thesis. These classes are known as Foxby classes and can be traced back to Foxby ([45]). As in [17], we recall their definitions without any assumptions on the ring R or the module C.

Definition 1.3.8. *Associated to an* (R,S)*-bimodule C, we have the* ***Auslander*** *and* ***Bass*** *classes,* $\mathscr{A}_C(S)$ *and* $\mathscr{B}_C(R)$*, respectively, defined as follows:*

(A) $\mathscr{A}_C(S)$ *is the class of all left S-modules M satisfying:*

(A1) $\mathrm{Tor}^S_{\geq 1}(C,M) = 0$.

(A2) $\mathrm{Ext}_R^{\geq 1}(C, C\otimes_S M) = 0$.

(A3) *The canonical map* $\mu_M : M \to \mathrm{Hom}_R(C, C\otimes_S M)$ *is an isomorphism of left S-modules.*

(B) *$\mathscr{B}_C(R)$ consists of all left R-modules N satisfying:*

(B1) $\mathrm{Ext}_R^{\geq 1}(C,N)=0$.

(B2) $\mathrm{Tor}_{\geq 1}^S(C,\mathrm{Hom}_R(C,N))=0$.

(B3) *The canonical map $\nu_N : C\otimes_S \mathrm{Hom}_R(C,N)\to N$ is an isomorphism of R-modules.*

On the other hand, one can define the classes $\mathscr{A}_C(R)$ and $\mathscr{B}_C(S)$ of right R-modules and right S-modules, respectively.

We may refer to modules in Foxby classes $\mathscr{A}_C$ and $\mathscr{B}_C$ as $\mathscr{A}_C$**-Auslander** and $\mathscr{B}_C$**-Bass** modules, respectively. It is an important property of Auslander and Bass classes that they are equivalent under the pair of functors [47, Proposition 2.1]:

$$\mathscr{B}_C(R) \underset{C\otimes_S -}{\overset{\mathrm{Hom}_R(C,-)}{\rightleftarrows}} \mathscr{A}_C(S) \quad \text{and} \quad \mathscr{A}_C(R) \underset{\mathrm{Hom}_S(C,-)}{\overset{-\otimes_R C}{\rightleftarrows}} \mathscr{B}_C(S)\ .$$

Consequently, Bass classes can be defined via Auslander classes and vice-versa:

$$\mathscr{B}_C(R)=C\otimes_S \mathscr{A}_C(S) \text{ and } \mathscr{A}_C(R)=\mathrm{Hom}_S(C,\mathscr{B}_C(S)).$$

Remark 1.3.9. *In the case ${}_RC_S = {}_RR_R$, the classes $\mathscr{A}_C(S)$ and $\mathscr{B}_C(R)$ coincide with the class of left R-modules.*

Throughout the rest of this section, S will be, unless otherwise stated, the endomorphism ring of C, $S=\mathrm{End}_R(C)$.

Recall that an R-module M is called **self-small** if the canonical morphism

$$\mathrm{Hom}_R(M,M^{(I)})\to \mathrm{Hom}_R(M,M)^{(I)}$$

is an isomorphism, for every set I. Examples of self-small modules are finitely generated modules. Keeping in mind that $S=\mathrm{End}_R(C)$, the module ${}_RC$ is self-small if and only if, for every set I, the canonical map $\mu_{S^{(I)}} : S^{(I)}\to \mathrm{Hom}_S(C,C\otimes_S S^{(I)})$ is an isomorphism.

Inspired by the duality between Foxby classes, we propose the dual notion to that of self-small.

Definition 1.3.10. *An R-module ${}_RM$ is said to be* ***self-cosmall****, if the canonical morphisms*

$$(M^+)^I\otimes_R M\to (M^+\otimes_R M)^I \text{ and } M^+\otimes_R M\to \mathrm{Hom}_R(M,M)^+$$

are isomorphisms for every set I.

Remark 1.3.11.

(1) *The module ${}_RC$ is self-cosmall if and only if the canonical morphism*

$$\nu_{(S^+)^I} : \mathrm{Hom}_S(C,(S^+)^I) \otimes_R C \to (S^+)^I$$

is an isomorphism for every set I.

(2) *Following [37, Theorem 3.2.11 and 3.2.22], any finitely presented R-module is self-cosmall.*

It is straightforward to prove the following:

Lemma 1.3.12.

1. *${}_RC$ is self-small if and only if there is an equivalence of categories:*

$$\mathrm{Add}_R(C) \underset{C\otimes_S -}{\overset{\mathrm{Hom}_R(C,-)}{\rightleftarrows}} \mathscr{P}(S)$$

2. *C is self-cosmall if and only if there exists an equivalence of categories*

$$\mathrm{Prod}_R(C^+) \underset{\mathrm{Hom}_S(C,-)}{\overset{-\otimes_R C}{\rightleftarrows}} \mathscr{I}(S)$$

Corollary 1.3.13. *The following assertions hold:*

1. *([11, Proposition 3.1]) If C is self-small, then* $\mathrm{Add}_R(C) = C \otimes_S \mathscr{P}(S)$.

2. *If ${}_RC$ is self-cosmall, then* $\mathrm{Prod}_R(C^+) = \mathrm{Hom}_S(C, \mathscr{I}(S))$.

Lemma 1.3.14. *The following assertions hold:*

1. *$\mathscr{P}(S) \subseteq \mathscr{A}_C(S)$ if and only if ${}_RC$ is Σ-self-orthogonal and self-small.*

   In this case, $\mathrm{Add}_R(C) \subseteq \mathscr{B}_C(R)$.

2. *$\mathscr{I}(S) \subseteq \mathscr{B}_C(S)$ if and only if ${}_RC$ is $\prod$-*Tor*-orthogonal and self-cosmall.*

   In this case, $\mathrm{Prod}_R(C^+) \subseteq \mathscr{A}_C(R)$.

Proof. 1. Follows by [17, Proposition 5.4(1)] and the equality $\mathrm{Add}_R(C) = C \otimes_S \mathscr{P}(S)$ when C is self-small.

2. By the dual argument to that of [17, Proposition 5.4(1)] and (1). ■

With respect to the terminology used in this thesis, modules in the class $\mathrm{Add}_R(C)$ (resp., $\mathrm{Prod}_R(C)$) will be called $\mathscr{P}_{\mathrm{C}}$**-projective** (res., $\mathscr{I}_C$**-injective**).

Remark 1.3.15.

1. *By Lemma 1.3.14, when C is self-small (resp., self-cosmall), the class of $\mathscr{P}_{\mathrm{C}}$-projective (resp., $\mathscr{I}_{\mathrm{C}^+}$-injective) modules coincides with the class of **C-projective** (resp., **C-injective**) modules in the sense of Holm and White [62], that is, modules in the class $C \otimes_S \mathscr{P}(S)$ (resp., $\mathrm{Hom}_S(C, \mathscr{I}(S))$).*

2. *By Lemma 1.3.14, the adjoint pair $(C \otimes_S -, \mathrm{Hom}_R(C,-))$ is left semidualizing (in the sense of [47, Definition 2.1]) if and only if ${}_RC$ is Σ-self-orthogonal and self-small.*

3. *When C is considered as a right module over an arbitrary ring S, there is a version of each definition and result presented in this thesis. For example, If $R = \mathrm{End}_S(C)$, then we have the following equalities*

$$\mathrm{Add}_S(C) = \mathscr{P}(R) \otimes_R C \text{ and } \mathrm{Prod}_S(C^+) = \mathrm{Hom}_R(C, \mathscr{I}(R))$$

when C_S is self-small and self-cosmall, respectively.

1.4 The Category of modules over triangular matrix rings

In this section, we present some ways of working with the category of modules over triangular matrix rings and recall some basic facts and results.

Let A and B be two rings and U be a (B,A)-bimodule. We use

$$T := \begin{pmatrix} A & 0 \\ U & B \end{pmatrix}$$

to denote the set of all matrices of the form $\left(\begin{smallmatrix} a & 0 \\ u & b \end{smallmatrix}\right)$, where $a \in A, b \in B, u \in U$. With respect to matrix addition and multiplication:

$$\left(\begin{smallmatrix} a & 0 \\ u & b \end{smallmatrix}\right) + \left(\begin{smallmatrix} a' & 0 \\ u' & b' \end{smallmatrix}\right) = \left(\begin{smallmatrix} a+a' & 0 \\ u+u' & b+b' \end{smallmatrix}\right) \text{ and } \left(\begin{smallmatrix} a & 0 \\ u & b \end{smallmatrix}\right)\left(\begin{smallmatrix} a' & 0 \\ u' & b' \end{smallmatrix}\right) = \left(\begin{smallmatrix} aa' & 0 \\ ua'+bu' & bb' \end{smallmatrix}\right)$$

T is a ring called the **(generalized) triangular matrix ring.**

Remark 1.4.1. *If $U = 0$, then the triangular matrix ring $T = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ can be identified with the direct product of rings $A \times B$.*

Throughout this thesis, $T = \begin{pmatrix} A & 0 \\ U & B \end{pmatrix}$ will be a triangular matrix ring.

Let ${}_T\Omega$ be (the category) defined as follows:

• **Objects:** are triples $M = \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}_{\varphi^M}$, where $M_1 \in A$-Mod, $M_2 \in B$-Mod and $\varphi^M : U \otimes_A M_1 \to M_2$ is a morphism of B-modules.

• **Morphisms**: are pairs $\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$ such that $f_1 \in \mathrm{Hom}_A(M_1, N_1)$, $f_2 \in \mathrm{Hom}_B(M_2, N_2)$, satisfying that the following diagram is commutative

$$\begin{array}{ccc} U \otimes_A M_1 & \xrightarrow{\varphi^M} & M_2 \\ {\scriptstyle 1_U \otimes f_1}\downarrow & & \downarrow{\scriptstyle f_2} \\ U \otimes_A N_1 & \xrightarrow{\varphi^N} & N_2 \end{array}$$

• **Composition:** is defined componentwise. That is, if $f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} : M \to N$ and $g = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} : N \to L$ are two morphisms in ${}_T\Omega$, then the composition gf is defined to be the pair of morphisms $gf := \begin{pmatrix} g_1 f_1 \\ g_2 f_2 \end{pmatrix} : M \to L$ (which is a morphism in ${}_T\Omega$).

The relationship between T-Mod and ${}_T\Omega$ is given via the functor $F : {}_T\Omega \to T$-Mod which is defined on objects and morphisms as follows:

• For an object $M = \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}_{\varphi^M}$ in ${}_T\Omega$, $F(M)$ is defined as the abelian group $F(M) = M_1 \oplus M_2$ with T-module structure given by

$$\begin{pmatrix} a & 0 \\ u & b \end{pmatrix} (m_1, m_2) = (am_1, \varphi^M(u \otimes m_1) \oplus bm_2).$$

• If $f : \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} : \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}_{\varphi^M} \to \begin{pmatrix} N_1 \\ N_2 \end{pmatrix}_{\varphi^N}$ is a morphism in ${}_T\Omega$, we let

$$F(f) := f_1 \oplus f_2 : M_1 \oplus M_2 \to N_1 \oplus N_2.$$

The following result allows us to identify the categories T-Mod and ${}_T\Omega$ by means of the functor F.

Theorem 1.4.2 ([56], Theorem 1.5, see also [44])**.** *T-Mod is equivalent to ${}_T\Omega$.*

In order to give a description of injective T-modules, we use the natural isomorphism

$$\mathrm{Hom}_B(U \otimes_A M_1, M_2) \cong \mathrm{Hom}_A(M_1, \mathrm{Hom}_B(U, M_2)),$$

to obtain an alternative description T-modules and T-homomorphisms in terms of maps $\widetilde{\varphi^M} : M_1 \to \mathrm{Hom}_B(U, M_2)$ given by $\widetilde{\varphi^M}(x)(u) = \varphi^M(u \otimes x)$ for each $u \in U$ and $x \in M_1$.

Let $\widetilde{{}_T\Omega}$ be the category whose:

• **Objects**: are triples $M = \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}_{\widetilde{\varphi^M}}$, where $M_1 \in A$-Mod, $M_2 \in B$-Mod and $\widetilde{\varphi^M} : M_1 \to \mathrm{Hom}_B(U, M_2)$ is an A-morphism.

• **Morphisms**: are pairs $\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$ such that $f_1 \in \mathrm{Hom}_A(M_1, N_1)$, $f_2 \in \mathrm{Hom}_B(M_2, N_2)$ satisfying that the following diagram is commutative

$$\begin{array}{ccc} M_1 & \xrightarrow{\widetilde{\varphi^M}} & \mathrm{Hom}_B(U, M_2) \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle \mathrm{Hom}_B(U, f_2)} \\ N_1 & \xrightarrow{\widetilde{\varphi^N}} & \mathrm{Hom}_B(U, N_2). \end{array}$$

Proposition 1.4.3. *$\widetilde{{}_T\Omega}$ is isomorphic to ${}_T\Omega$.*

Analogously, the category Mod-T of right T-modules is equivalent to the category Ω_T whose objects are triples $M = \big(M_1, M_2\big)_{\varphi_M}$, where $M_1 \in$ Mod-A, $M_2 \in$ Mod-B and $\varphi_M : M_2 \otimes_B U \to M_1$ is an A-morphism, and whose morphisms from $\big(M_1, M_2\big)_{\varphi_M}$ to $\big(N_1, N_2\big)_{\varphi_N}$ are pairs $\big(f_1, f_2\big)$ such that $f_1 \in \mathrm{Hom}_A(M_1, N_1)$, $f_2 \in \mathrm{Hom}_B(M_2, N_2)$ satisfying that the following diagram

$$\begin{array}{ccc} M_2 \otimes_B U & \xrightarrow{\varphi_M} & M_1 \\ {\scriptstyle f_2 \otimes 1_U}\downarrow & & \downarrow{\scriptstyle f_1} \\ N_2 \otimes_B U & \xrightarrow{\varphi_N} & N_1 \end{array}$$

is commutative.

Alternatively, the category Mod-T of right T-modules is also equivalent to the category $\widetilde{\Omega_T}$ whose objects are triples $M = \big(M_1, M_2\big)_{\widetilde{\varphi_M}}$, where $M_1 \in$ Mod-A, $M_2 \in$ Mod-B and $\widetilde{\varphi_M} : M_2 \to \mathrm{Hom}_A(U, M_1)$ is an A-morphism, and whose morphisms from $\big(M_1, M_2\big)_{\widetilde{\varphi_M}}$ to $\big(N_1, N_2\big)_{\widetilde{\varphi_N}}$ are pairs $\big(f_1, f_2\big)$ such that $f_1 \in \mathrm{Hom}_A(M_1, N_1)$, $f_2 \in \mathrm{Hom}_B(M_2, N_2)$ satisfying that the following diagram

$$\begin{array}{ccc} M_2 & \xrightarrow{\widetilde{\varphi_M}} & \operatorname{Hom}_A(U,M_1) \\ {\scriptstyle f_2}\downarrow & & \downarrow{\scriptstyle \operatorname{Hom}_A(U,f_1)} \\ N_2 & \xrightarrow{\widetilde{\varphi_N}} & \operatorname{Hom}_A(U,N_1) \end{array}$$

is commutative.

From now on, we will identify both categories ${}_T\Omega$ and $\widetilde{{}_T\Omega}$ (resp., Ω_T and $\widetilde{\Omega_T}$) with T-Mod (resp., Mod-T).

In this case, a sequence of left (resp., right) T-modules

$$\begin{pmatrix} \mathscr{E}_1 \\ \mathscr{E}_2 \end{pmatrix} : 0 \to \begin{pmatrix} M_1 \\ M_2 \end{pmatrix} \to \begin{pmatrix} M_1' \\ M_2' \end{pmatrix} \to \begin{pmatrix} M_1'' \\ M_2'' \end{pmatrix} \to 0$$

$$(\textit{resp.}, \ (\mathscr{E}_1, \mathscr{E}_2) : 0 \to (M_1, M_2) \to (M_1', M_2') \to (M_1'', M_2'') \to 0)$$

is exact if and only if both sequences

$$\mathscr{E}_1 : 0 \to M_1 \to M_1' \to M_1'' \to 0 \text{ and } \mathscr{E}_2 : 0 \to M_2 \to M_2' \to M_2'' \to 0$$

are exact in the corresponding categories.

Example 1.4.4. *${}_TT$ corresponds to $\begin{pmatrix} A \\ U \oplus B \end{pmatrix}_{\varphi^T}$, where $\varphi^T : U \otimes A \to U \oplus B$ is given by $\varphi^T(u \otimes a) = (ua, 0)$ while T_T corresponds to $(A \oplus U, B)_{\varphi_T}$, where $\varphi_T : B \otimes_B U \to A \oplus U$ is given by $\varphi_T(b \otimes u) = (0, bu)$.*

There are some pairs of adjoint functors between the category T-Mod and the product category A-Mod $\times B$-Mod:

$$A\text{-Mod} \times B\text{-Mod} \underset{\mathbf{q}}{\overset{\mathbf{p}}{\rightleftarrows}} T\text{-Mod},$$

$$A\text{-Mod} \times B\text{-Mod} \underset{\mathbf{q}}{\overset{\mathbf{h}}{\rightleftarrows}} T\text{-Mod},$$

$$A\text{-Mod} \times B\text{-Mod} \underset{\mathbf{s}}{\overset{\mathbf{r}}{\rightleftarrows}} T\text{-Mod}$$

which are defined as follows:

1. $\mathbf{p}: A\text{-Mod} \times B\text{-Mod} \to T\text{-Mod}$: for each object (M_1, M_2) of $A\text{-Mod}\times B\text{-Mod}$, $\mathbf{p}(M_1, M_2) = \begin{pmatrix} M_1 \\ (U \otimes_A M_1) \oplus M_2 \end{pmatrix}$ with the obvious map and for any morphism (f_1, f_2) in $A\text{-Mod}\times B\text{-Mod}$, $\mathbf{p}(f_1, f_2) = \begin{pmatrix} f_1 \\ (1_U \otimes_A f_1) \oplus f_2 \end{pmatrix}$.

2. $\mathbf{q}: T\text{-Mod} \to A\text{-Mod} \times B\text{-Mod}$:

$$\mathbf{q}(M) = (M_1, M_2) \text{ and } \mathbf{q}(f) = (f_1, f_2)$$

   for each left T-module $M = \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}$ and for each morphism $f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$ in T-Mod.

3. $\mathbf{h}: A\text{-Mod} \times B\text{-Mod} \to T\text{-Mod}$: for each object (M_1, M_2) of $A\text{-Mod}\times B\text{-Mod}$, $\mathbf{h}(M_1, M_2) = \begin{pmatrix} M_1 \oplus \mathrm{Hom}_B(U, M_2) \\ M_2 \end{pmatrix}$ with the obvious map and for any morphism (f_1, f_2) in $A\text{-Mod}\times B\text{-Mod}$, $\mathbf{h}(f_1, f_2) = \begin{pmatrix} f_1 \oplus \mathrm{Hom}_B(U, f_2) \\ f_2 \end{pmatrix}$.

4. $\mathbf{r}: A\text{-Mod} \times B\text{-Mod} \to T\text{-Mod}$: for each object (M_1, M_2) of $A\text{-Mod}\times B\text{-Mod}$, $\mathbf{r}(M_1, M_2) = \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}$ with the zero map and for any morphism (f_1, f_2) in $A\text{-Mod}\times B$-Mod, $\mathbf{r}(f_1, f_2) = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$.

5. $\mathbf{s}: T\text{-Mod} \to A\text{-Mod} \times B\text{-Mod}$:

$$\mathbf{s}(M) = (M_1, \mathrm{Coker}\varphi^M) \text{ and } \mathbf{s}(f) = (f_1, \overline{f}_2)$$

   for each T-module $M = \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}$ and for each morphism $f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$ in T-Mod, where $\overline{f}_2 : \mathrm{Coker}\varphi^M \to \mathrm{Coker}\varphi^M$ is the induced map.

The functors $\mathbf{p}, \mathbf{q}$ and $\mathbf{h}$ were introduced by Enochs, Cortés-Izurdiaga and Torrecillas in [34] and have played an important role in the characterization of Gorenstein projective and injective modules over triangular matrix rings.

We also note that all the above functors can be induced from adjoint functors defined on a more general construction called trivial extension [44].

Here are some facts about these functors, which are used frequently in this thesis.

(1) The pairs $(\mathbf{p},\mathbf{q})$, $(\mathbf{q},\mathbf{h})$ and $(\mathbf{s},\mathbf{r})$ are adjoint pairs. For a future reference, we list these adjointness isomorphisms:

$$\mathrm{Hom}_T(\begin{pmatrix} M_1 \\ (U\otimes_A M_1)\oplus M_2 \end{pmatrix}, N) \cong \mathrm{Hom}_A(M_1,N_1)\oplus \mathrm{Hom}_B(M_2,N_2).$$

$$\mathrm{Hom}_T(M, \begin{pmatrix} N_1 \\ N_2 \end{pmatrix}_0) \cong \mathrm{Hom}_A(M_1,N_1)\oplus \mathrm{Hom}_B(\mathrm{Coker}\varphi^M,N_2).$$

$$\mathrm{Hom}_T(M, \begin{pmatrix} N_1\oplus \mathrm{Hom}_B(U,N_2) \\ N_2 \end{pmatrix}) \cong \mathrm{Hom}_A(M_1,N_1)\oplus \mathrm{Hom}_B(M_2,N_2).$$

(2) The functor $\mathbf{q}$ is exact. Consequently, $\mathbf{p}$ preserves projective objects and $\mathbf{h}$ preserves injective objects.

(3) The functor $\mathbf{p}$ preserves direct limits (in particular direct sums), while the functor $\mathbf{h}$ preserves inverse limits (in particular direct products).

Lemma 1.4.5. *Let* $X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}_{\varphi^X}$ *be a* T*-module and* $(C_1,C_2)\in A$*-Mod* $\times$ B*-Mod.*

(1) $X\in \mathrm{Add}_T(\boldsymbol{p}(C_1,C_2))$ *if and only if*

  (i) $X\cong \boldsymbol{p}(X_1,\mathrm{Coker}\varphi^X)$,

  (ii) $X_1\in \mathrm{Add}_A(C_1)$ *and* $\mathrm{Coker}\varphi^X \in \mathrm{Add}_B(C_2)$.

  In this case, φ^X *is a monomorphism.*

(2) $X\in \mathrm{Prod}_T(\boldsymbol{h}(C_1,C_2))$ *if and only if*

  (i) $X\cong \boldsymbol{h}(\mathrm{Ker}\widetilde{\varphi^X},X_2)$,

  (ii) $\mathrm{Ker}\widetilde{\varphi^X}\in \mathrm{Prod}_A(C_1)$ *and* $X_2\in \mathrm{Prod}_B(C_2)$.

  In this case, $\widetilde{\varphi^X}$ *is an epimorphism.*

Proof. We only need to prove (1), since (2) is dual.

For the "if"part. If $X_1\in \mathrm{Add}_A(C_1)$ and $\mathrm{Coker}\varphi^X\in \mathrm{Add}_B(C_2)$, then $X_1\oplus Y_1 = C_1^{(I_1)}$ and $\mathrm{Coker}\varphi^X\oplus Y_2 = C_2^{(I_2)}$ for some $(Y_1,Y_2)\in A\text{-Mod}\times B\text{-Mod}$ and some sets I_1 and I_2. Without loss of generality, we may assume that $I = I_1 = I_2$. Then:

$$\begin{aligned} X\oplus \mathbf{p}(Y_1,Y_2) &\cong \mathbf{p}(X_1,\mathrm{Coker}\varphi^X)\oplus \mathbf{p}(Y_1,Y_2) \\ &\cong \mathbf{p}(X_1\oplus Y_1,\mathrm{Coker}\varphi^X\oplus Y_2) \\ &\cong \mathbf{p}(C_1^{(I)},C_2^{(I)}) \\ &\cong \mathbf{p}(C_1,C_2)^{(I)}. \end{aligned}$$

Hence, $X \in \mathrm{Add}_T(\mathbf{p}(C_1,C_2))$.

Conversely, let $X \in \mathrm{Add}_T(\mathbf{p}(C_1,C_2))$ and $Y = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}_{\varphi^Y}$ be a T-module such that $X \oplus Y = \mathbf{p}(C_1,C_2)^{(I)}$ for some set I. Then, φ^X is a monomorphism, as X is a direct summand of $C := \mathbf{p}(C_1,C_2)^{(I)}$ and φ^C is a monomorphism. Consider now the split exact sequence

$$0 \to Y \overset{\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}}{\longrightarrow} C \overset{\begin{pmatrix} p_1 \\ p_2 \end{pmatrix}}{\longrightarrow} X \to 0$$

which induces the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & U\otimes_A Y_1 & \xrightarrow{1_U\otimes\lambda_1} & U\otimes_A C_1^{(I)} & \xrightarrow{1_U\otimes p_1} & U\otimes_A X_1 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \varphi^Y} & & \downarrow{\scriptstyle \varphi^C} & & \downarrow{\scriptstyle \varphi^X} & & \\
0 & \longrightarrow & Y_2 & \xrightarrow{\lambda_2} & U\otimes_A C_1^{(I)} \oplus C_2^{(I)} & \xrightarrow{p_2} & X_2 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \overline{\varphi^Y}} & & \downarrow{\scriptstyle \overline{\varphi^C}} & & \downarrow{\scriptstyle \overline{\varphi^X}} & & \\
0 & \longrightarrow & \mathrm{Coker}\varphi^Y & \xrightarrow{\overline{\lambda_2}} & C_2^{(I)} & \xrightarrow{\overline{p_2}} & \mathrm{Coker}\varphi^X & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where $\overline{\varphi^X}$, $\overline{\varphi^C}$ and $\overline{\varphi^X}$ are the cokernel morphisms. Clearly, $p_1 : C_1^{(I)} \to X_1$ and $\overline{p_2} : C_2^{(I)} \to \mathrm{Coker}\varphi^X$ are split epimorphisms. Then, $X_1 \in \mathrm{Add}_A(C_1)$ and $\mathrm{Coker}\varphi^X \in \mathrm{Add}_B(C_2)$.

It remains to prove that $X \cong \mathbf{p}(X_1, \mathrm{Coker}\varphi^X)$. For this, it suffices to prove that the short exact sequence

$$0 \to U\otimes_A X_1 \overset{\varphi^X}{\to} X_2 \overset{\overline{\varphi^X}}{\to} \mathrm{Coker}\varphi^X \to 0$$

splits.

Let r_2 be the retraction of $\overline{p_2}$. If $i : C_2^{(I)} \to (U\otimes_A C_1^{(I)}) \oplus C_2^{(I)}$ denotes the canonical injection, then $\overline{\varphi^X} p_2 i r_2 = \overline{p_2}\,\overline{\varphi^C} i r_2 = \overline{p_2} r_2 = 1_{\mathrm{Coker}\varphi^X}$ and the proof is finished. ∎

The previous lemma can be used to characterize projective and injective modules over T. Since the class of projective modules over T is nothing but the class $\mathrm{Add}_T(T)$, when we take $C_1 = A$ and $C_2 = B$ in Lemma 1.4.5(1), we recover the characterization of projective T-modules given in [58, Theorem 3.1]. On the other hand, the class of injective T-modules coincides with the class $\mathrm{Prod}_T(T^+)$ where $T^+ = \mathrm{Hom}_{\mathbb{Z}}(T, \mathbb{Q}/\mathbb{Z})$ is the character right T-module of ${}_TT$.

Let us recall the structure of character modules over T.

Let $N = \big(N_1, N_2\big)_{\varphi_N}$ be a right T-module and G an arbitrary abelian group. The abelian group $\mathrm{Hom}_{\mathbb{Z}}(N,G)$ is a left T-module with T-module structure given by:

$$\begin{aligned} T \times \mathrm{Hom}_{\mathbb{Z}}(N,G) &\longrightarrow \mathrm{Hom}_{\mathbb{Z}}(N,G) \\ (t,\alpha) &\longmapsto t\cdot\alpha : n \mapsto t\cdot\alpha(n) = \alpha(nt). \end{aligned}$$

Similarly, the groups $\mathrm{Hom}_A(N_1,G)$ and $\mathrm{Hom}_B(N_1,G)$ are a left A-module and B-module, respectively. This defines a left T-module $H := \begin{pmatrix} \mathrm{Hom}_{\mathbb{Z}}(N_1,G) \\ \mathrm{Hom}_{\mathbb{Z}}(N_2,G) \end{pmatrix}_{\varphi^H}$ where

$$\varphi^H : U \otimes_A \mathrm{Hom}_{\mathbb{Z}}(N_1,G) \to \mathrm{Hom}_{\mathbb{Z}}(N_2,G)$$

is defined by $\varphi^H(u \otimes f)(n_2) = f\varphi_N(n_2 \otimes u)$ for any $f \in \mathrm{Hom}_{\mathbb{Z}}(N_1,G)$, $u \in U$ and $n_2 \in N_2$.

There exists a canonical isomorphism of left T-modules

$$\begin{aligned} \mathrm{Hom}_{\mathbb{Z}}(N,G) &\longrightarrow \begin{pmatrix} \mathrm{Hom}_{\mathbb{Z}}(N_1,G) \\ \mathrm{Hom}_{\mathbb{Z}}(N_2,G) \end{pmatrix}_{\varphi^H} \\ f &\longmapsto \begin{pmatrix} f_{|N_1} \\ f_{|N_2} \end{pmatrix} \end{aligned}$$

With this isomorphism, we identify the T-modules

$$\mathrm{Hom}_{\mathbb{Z}}(N,G) \text{ and } \begin{pmatrix} \mathrm{Hom}_{\mathbb{Z}}(N_1,G) \\ \mathrm{Hom}_{\mathbb{Z}}(N_2,G) \end{pmatrix}_{\varphi^{\mathrm{Hom}_{\mathbb{Z}}(N,G)}}$$

In particular, we identify the character module $N^+ = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Q}/\mathbb{Z})$ with the T-module $\begin{pmatrix} N_1^+ \\ N_2^+ \end{pmatrix}_{\varphi^{N^+}}$ where $\varphi^{N^+} : U \otimes_A N_1^+ \to N_2^+$ is defined by $\varphi^{N^+}(u \otimes f)(n_2) = f\varphi_N(n_2 \otimes u)$ for any $f \in N_1^+$, $u \in U$ and $n_2 \in N_2$.

Theorem 1.4.6. *Let* $M = \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}_{\varphi^M}$ *be a* T*-module.*

(1) *([58, Theorem 3.1])* M *is projective if and only if* M_1 *is projective in* A*-Mod,* $\mathrm{Coker}\varphi^M$ *is projective in* B*-Mod and* φ^M *is injective.*

(2) *([57, Proposition 5.1])* M *is injective if and only if* $\mathrm{Ker}\widetilde{\varphi^M}$ *is injective in* A*-Mod,* M_2 *is injective in* B*-Mod and* $\widetilde{\varphi^M}$ *is surjective.*

Proof. (1) By taking $C_1 = A$ and $C_2 = B$ in Lemma 1.4.5(1).

(2) If we take $T_T = \big(A \oplus U, B\big)$, then the injective cogenerator T-module $T^+ = \mathrm{Hom}_{\mathbb{Z}}(T, \mathbb{Q}/\mathbb{Z})$ can be identified with $\begin{pmatrix} A^+ \oplus U^+ \\ B^+ \end{pmatrix} \cong \mathbf{q}(A^+, B^+)$. So by taking $C_1 = A^+$

and $C_2 = B^+$ in Lemma 1.4.5(2), we recover the characterization of injective T-modules.
■

Lemma 1.4.7. *Let* $M = \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}_{\varphi^M}$ *and* $N = \begin{pmatrix} N_1 \\ N_2 \end{pmatrix}_{\varphi^N}$ *be two T-modules and $n \geq 1$ be an integer number. Then, we have the following natural isomorphisms:*

(1) *If* $\mathrm{Tor}^A_{1\leq i\leq n}(U,M_1) = 0$*, then* $\mathrm{Ext}^n_T(\begin{pmatrix} M_1 \\ U\otimes_A M_1 \end{pmatrix},N) \cong \mathrm{Ext}^n_A(M_1,N_1)$.

(2) $\mathrm{Ext}^n_T(\begin{pmatrix} 0 \\ M_2 \end{pmatrix},N) \cong \mathrm{Ext}^n_B(M_2,N_2)$.

(3) $\mathrm{Ext}^n_T(M,\begin{pmatrix} N_1 \\ 0 \end{pmatrix}) \cong \mathrm{Ext}^n_A(M_1,N_1)$.

(4) *If* $\mathrm{Ext}^{1\leq i\leq n}_B(U,N_2) = 0$*, then* $\mathrm{Ext}^n_T(M,\begin{pmatrix} \mathrm{Hom}_B(U,N_2) \\ N_2 \end{pmatrix}) \cong \mathrm{Ext}^n_B(M_2,N_2)$.

Proof. . We prove only (1) since (2) is similar and (3) and (4) are dual.

Assume that $\mathrm{Tor}^A_{1\leq i\leq n}(U,M_1) = 0$ and consider an exact sequence of A-modules

$$0 \to K_1 \to P_1 \to M_1 \to 0$$

where P_1 is projective. Then, there exists an exact sequence of T-modules

$$0 \to \mathbf{p}(K_1,0) \to \mathbf{p}(P_1,0) \to \mathbf{p}(M_1,0) = \begin{pmatrix} M_1 \\ U\otimes_A M_1 \end{pmatrix} \to 0$$

where $\mathbf{p}(P_1,0)$ is projective as $\mathbf{p}$ preserves projective objects.

Let $n = 1$. By applying the functor $\mathrm{Hom}_T(-,N)$ to the above short exact sequence and since $\mathbf{p}(P_1,0)$ and P_1 are projectives, we get a commutative diagram with exact rows:

$$\begin{array}{ccccccc} \mathrm{Hom}_T(\mathbf{p}(P_1,0),N) & \longrightarrow & \mathrm{Hom}_T(\mathbf{p}(K_1,0),N) & \twoheadrightarrow & \mathrm{Ext}^1_T(\mathbf{p}(M_1,0),N) & \longrightarrow & 0 \\ \downarrow\cong & & \downarrow\cong & & \downarrow & & \\ \mathrm{Hom}_A(P_1,N_1) & \longrightarrow & \mathrm{Hom}_A(K_1,N_1) & \twoheadrightarrow & \mathrm{Ext}^1_A(M_1,N_1) & \longrightarrow & 0 \end{array}$$

where the first two columns are just the natural isomorphisms given by adjointeness and the last two horizontal morphisms are epimorphisms. Thus, the induced map

$$\mathrm{Ext}^1_T(\mathbf{p}(M_1,0),N) \to \mathrm{Ext}^1_A(M_1,N_1)$$

is an isomorphism such that the above diagram is commutative.

Assume that $n > 1$. Using the long exact sequences, we get a commutative diagram with exact rows:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{Ext}_T^{n-1}(\mathbf{p}(K_1,0),N) & \xrightarrow[\cong]{f} & \mathrm{Ext}_T^{n}(\mathbf{p}(M_1,0),N) & \longrightarrow & 0 \\ & & \sigma \downarrow \cong & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{Ext}_A^{n-1}(K_1,N_1) & \xrightarrow[\cong]{g} & \mathrm{Ext}_A^{n}(M_1,N_1) & \longrightarrow & 0 \end{array}$$

where σ is a natural isomorphism by induction, since $\mathrm{Tor}_k^A(U,K_1) = 0$ for every $k \in \{1,\cdots,n-1\}$ because of the exactness of the sequence

$$0 = \mathrm{Tor}_{k+1}^A(U,M_1) \to \mathrm{Tor}_k^A(U,K_1) \to \mathrm{Tor}_k^A(U,P_1) = 0.$$

Thus, the composite map

$$g\sigma f^{-1} : \mathrm{Ext}_T^n(\mathbf{p}(M_1,0),N) \to \mathrm{Ext}_A^n(M_1,N_1)$$

is a natural isomorphism, as desired. ■

1.5 Covers, envelopes and cotorsion pairs

In this section we recall the basic definitions and results of approximation theory (precovers, preenvelopes, cotorsion pairs, etc.).

The general references for the content of this section are Enochs & Jenda's book [37] and Göbel & Trlifaj's book [53].

Definition 1.5.1 ([37], Chapter 8)**.** *An $\mathscr{X}$-**precover** of an object A is a morphism $f : X \to A$ with $X \in \mathscr{X}$, such that $f_* : \mathrm{Hom}_{\mathscr{A}}(X',X) \to \mathrm{Hom}_{\mathscr{A}}(X',M)$ is surjective for every $X' \in \mathscr{X}$. That is, for each morphism $f' : X' \to A$ there is a morphism $g : X' \to X$ such that $f' = gf$:*

$$\begin{array}{ccc} & X' & \\ {\scriptstyle g}\swarrow & & \searrow{\scriptstyle f'} \\ X & \xrightarrow{\quad f \quad} & A \end{array}$$

*If, moreover, any endomorphism $g : X \to X$ such that $fg = f$ is an automorphism of X, then $f : X \to A$ is called an $\mathscr{X}$-**cover** of A.*

*The class $\mathscr{X}$ is called **(pre)covering** if every object of $\mathscr{A}$ has an $\mathscr{X}$-(pre)cover.*

*$\mathscr{X}$-**(pre)envelope** morphisms and **(pre)enveloping** classes are defined dually.*

Remark 1.5.2. *We note that an $\mathscr{X}$-precover $f : X \to A$ is not necessarily an epimorphism. However, if $\mathscr{A}$ has enough projective objects and all these objects belong to $\mathscr{X}$, then f is necessarily an epimorphism. Dually, if $\mathscr{A}$ has enough injective objects and all these objects belong to $\mathscr{X}$, then any $\mathscr{X}$-preenvelope is a monomorphism.*

Definition 1.5.3 ([37], Definition 7.1.6)**.** *An $\mathscr{X}$-precover $f : X \to A$ is called **special** if it is an epimorphism and* $\mathrm{Ker}f \in \mathscr{X}^{\perp}$*. Dually, an $\mathscr{X}$-preenvelope $g : A \to Y$ is called **special** if it is a monomorphism and* $\mathrm{Coker}g \in^{\perp} \mathscr{Y}$*.*

The following is known as Wakamatsu Lemma.

Proposition 1.5.4 ([84], Lemmas 2.1.1 and 2.1.2)**.** *Assume that $\mathscr{X} \subseteq R$-Mod is closed under extensions.*

1. *If $\varphi : X \to M$ is an $\mathscr{X}$-cover of M, then* $\mathrm{Ker}\varphi \in \mathscr{X}^{\perp}$*.*
2. *If $\varphi : M \to X$ is an $\mathscr{X}$-envelope of M, then* $\mathrm{Coker}\varphi \in {}^{\perp}\mathscr{X}$*.*

Consequently, any covering class $\mathscr{X} \subseteq R$-Mod that is closed under extensions and such that $\mathscr{P}(R) \subseteq \mathscr{X}$ (resp., $\mathscr{I}(R) \subseteq \mathscr{X}$) is a special precovering (resp., preenveloping).

Definition 1.5.5 ([37], definition 7.1.2)**.** *A pair of classes $(\mathscr{X},\mathscr{Y})$ of objects in $\mathscr{A}$ is said to be a **cotorsion pair** (or **cotorsion theory**) if $\mathscr{X}^{\perp} = \mathscr{Y}$ and $\mathscr{X} =^{\perp} \mathscr{Y}$.*

*In this case, the class $\mathscr{X} \cap \mathscr{Y}$ is called the **core** of the cotorsion pair $(\mathscr{X},\mathscr{Y})$.*

Clearly, for any class $\mathscr{X}$ of objects of $\mathscr{A}$, $\mathscr{X} \subseteq {}^{\perp}(\mathscr{X}^{\perp})$ and $\mathscr{X} \subseteq ({}^{\perp}\mathscr{X})^{\perp}$. Moreover, it is easy to see that $({}^{\perp}\mathscr{X},({}^{\perp}\mathscr{X})^{\perp})$ and $({}^{\perp}(\mathscr{X}^{\perp}),\mathscr{X}^{\perp})$ are cotorsion pairs, called the cotorsion pairs **generated** and **cogenerated**, respectively, by the class $\mathscr{X}$.

Examples 1.5.6. *In $\mathscr{A} = R$-Mod, the pairs of classes $(\mathscr{P}(R), R\text{-Mod})$ and $(R\text{Mod}, \mathscr{I}(R))$ are easily seen to be cotorsion pairs.*

Another interesting and non-trivial example of cotorsion pairs is given in Theorem 1.5.14.

Definition 1.5.7. *([37, Definition 7.1.5] and [53]). A cotorsion pair $(\mathscr{X},\mathscr{Y})$ is called **complete** if it satisfies the following two assertions:*

1. *$(\mathscr{X},\mathscr{Y})$ **has enough injectives**, that is, for any object A of $\mathscr{A}$, there exists an exact sequence $0 \to Y \to X \to A \to 0$ with $X \in \mathscr{X}$ and $Y \in \mathscr{Y}$.*
2. *$(\mathscr{X},\mathscr{Y})$ **has enough projectives**, that is, for any object A of $\mathscr{A}$, there exists an exact sequence $0 \to A \to Y \to X \to 0$ with $X \in \mathscr{X}$ and $Y \in \mathscr{Y}$.*

In other words, a cotorsion pair $(\mathscr{X},\mathscr{Y})$ is complete if $\mathscr{X}$ is special precovering and $\mathscr{Y}$ is special precovering.

Lemma 1.5.8 ([53], Lemma 2.2.6)**.** *Let $(\mathscr{X},\mathscr{Y})$ be a cotorsion pair in R-Mod. Then, the following are equivalent:*

1. *$(\mathscr{X},\mathscr{Y})$ is complete.*
2. *$\mathscr{X}$ is a special precovering.*
3. *$\mathscr{Y}$ is a special preenveloping.*

Definition 1.5.9. *A cotorsion pair $(\mathscr{X},\mathscr{Y})$ is called* ***perfect*** *if $\mathscr{X}$ is covering and $\mathscr{Y}$ is enveloping.*

Clearly, perfect cotorsion pairs are complete. The converse is not true in general. For instance, the cotorsion pair $(\mathscr{P}(R), R\text{-Mod})$ is perfect if and only if R is left perfect (i.e., every projective R-module has a projective cover).

The following theorem shows that when a class of modules $\mathscr{X}$ is closed under direct limits, then a cotorsion pair $(\mathscr{X},\mathscr{Y})$ is perfect whenever it is complete.

Theorem 1.5.10 ([37], Corollary 5.2.7 and Theorem 7.2.6)**.** *Let $\mathscr{X}$ be a class of R-modules closed under direct limits. If an R-module has an $\mathscr{X}$-precover, then it has an $\mathscr{X}$-cover.*

Consequently, if $(\mathscr{X},\mathscr{Y})$ is a complete cotorsion pair, then it is perfect.

A powerful method for constructing complete cotorsion pairs, and then approximations, is to cogenerate one from a set in the following sense.

Definition 1.5.11. *A cotorsion pair $(\mathscr{X},\mathscr{Y})$ is said to be* ***cogenerated*** *by a set if there exists a set $\mathscr{S}$ such that $\mathscr{S}^{\perp} = \mathscr{Y}$.*

Example 1.5.12. *In $\mathscr{A} = R\text{-Mod}$, the cotorsion pair $(R\text{-}Mod, \mathscr{I}(R))$ is cogenerated by the set of modules R/I where I is a left ideal of R.*

Dually, the cotorsion pair $(\mathscr{P}(R), R\text{-}Mod)$ is cogenerated by ${}_RR$.

Theorem 1.5.13 ([53], Theorem 3.2.1)**.** *If $(\mathscr{X},\mathscr{Y})$ is a cotorsion pair of modules cogenerated by a set, then it is complete.*

This theorem was proved by Eklof and Trlifaj and received a lot of attention after being used by Enochs [21, Theorem 3] to prove his flat cover conjecture:

Flat Cover Conjecture. Every module over any ring has a flat cover.

Recall that an R-module M is called **cotorsion** if $\mathrm{Ext}^1_R(F,M) = 0$ for all flat R-modules F. We let $\mathscr{C}(R) = \mathscr{F}(R)^{\perp}$ denote the class of all cotorsion R-modules.

Theorem 1.5.14. *(**Enochs cotorsion pair**.) The pair $(\mathscr{F}(R),\mathscr{C}(R))$ is a perfect cotorsion pair.*

Consequently, every R-module has a flat cover and a cotorsion envelope.

Proof. It is well-known that the class of flat modules is closed under direct limits and by [37, Proposition] and Theorem 1.5.13, $(\mathscr{F}(R),\mathscr{C}(R))$ is a complete cotorsion pair. So, this result follows by Theorem 1.5.10. ■

Definition 1.5.15 ([46], Definition 1.2.10)**.** *A cotorsion pair $(\mathscr{X},\mathscr{Y})$ in $\mathscr{A}$ is called **hereditary** if $\mathrm{Ext}^i_{\mathscr{A}}(\mathscr{X},\mathscr{Y})=0$ for every $i\geq 1$.*

It is clear that $(\mathscr{X},\mathscr{Y})$ being hereditary implies that $\mathscr{X}$ is closed under kernels of epimorphisms and $\mathscr{Y}$ is closed under cokernels of monomorphisms.

Conversely, in R-Mod (or more generally, in any abelian category with enough projective and injective objects), we have the following lemma:

Lemma 1.5.16 ([46], Theorem 1.2.10)**.** *Let $(\mathscr{X},\mathscr{Y})$ be a cotorsion pair in R-Mod. Then, the following are equivalent:*

1. *$(\mathscr{X},\mathscr{Y})$ is hereditary.*
2. *$\mathscr{X}$ is closed under kernels of epimorphisms.*
3. *$\mathscr{Y}$ is closed under cokernels of monomorphisms.*

The following lemma will be needed later.

Lemma 1.5.17. *Let $\mathscr{X}$ be a set of R-modules. the following assertions hold.*

1. *$\mathscr{X}^{\perp_\infty}=M^{\perp}$ for some R-module M.*
2. *${}^{\perp_\infty}\mathscr{X}={}^{\perp}M$ for some R-module M.*

Proof. The proof of (2) is dual to that of (1), so we only prove (1).

Let X be the direct sum of all the modules in $\mathscr{X}$. Consider any projective resolution of X

$$\cdots\to P_1\overset{f_1}{\to}P_0\overset{f_0}{\to}X\to 0$$

and let $K_{i+1}=\mathrm{Ker}(f_i)$. Clearly, for any $i\geq 1$ and any R-module A, we have $\mathrm{Ext}^1_R(K_i,A)\cong\mathrm{Ext}^{i+1}_R(X,A)$. If we let $M=X\oplus(\oplus_{i\geq 1}K_i)$, then

$$\mathrm{Ext}^1_R(M,A)\cong\mathrm{Ext}^1_R(X,A)\oplus(\prod_{i\geq 1}\mathrm{Ext}^1_R(K_i,A))\cong\prod_{i\geq 1}\mathrm{Ext}^i_R(X,A).$$ ■

We end this section with one last useful way of constructing approximations.

Definition 1.5.18 ([60], Definition 2.1)**.** *A left (right) **duality pair** of R-modules is a pair of classes $(\mathscr{X},\mathscr{Y})$, $\mathscr{X}$ being a class of left (right) R-modules and $\mathscr{Y}$ being a class of right (left) R-modules, subject to the following conditions:*

(1) *$M\in\mathscr{X}$ if and only if $M^+\in\mathscr{Y}$.*

(2) *$\mathscr{Y}$ is closed under direct summands and finite direct sums.*

A duality pair $(\mathscr{X}, \mathscr{Y})$ is called ***coproduct-closed*** *(resp.,* ***product-closed****) if the class $\mathscr{X}$ is closed under direct sums (resp., direct products), and it is called* ***perfect*** *if it is coproduct-closed, the class $\mathscr{X}$ is closed under extensions and $R \in \mathscr{X}$.*

Duality pairs were introduced by Holm and Jørgensen and have been proved to be very useful in showing the existence of (pre)covers and (pre)envelopes in the category of modules.

Theorem 1.5.19 ([60], Theorem 3.1)**.** *Let $(\mathscr{X}, \mathscr{Y})$ be a (left) duality pair. Then, $\mathscr{X}$ is closed under pure submodules, pure quotients and pure extensions. Furthermore, the following hold:*

(a) *If $(\mathscr{X}, \mathscr{Y})$ is product-closed, then $\mathscr{X}$ is preenveloping.*

(b) *If $(\mathscr{X}, \mathscr{Y})$ is coproduct-closed, then $\mathscr{X}$ is covering.*

(c) *If $(\mathscr{X}, \mathscr{Y})$ is perfect, then $(\mathscr{X}, \mathscr{X}^{\perp})$ is a perfect cotorsion pair.*

1.6 Abelian model structures

In this section we recall the definition of abelian model structures and their main properties needed for this thesis.

Consider a class of morphisms $\mathscr{W}$ in a category $\mathscr{A}$. One can formally, as in [63, Definition 1.2.1], invert the morphisms in $\mathscr{W}$ to get a 'category' $\mathscr{A}[\mathscr{W}^{-1}]$ in which the morphisms in $\mathscr{W}$ have been forced to become isomorphisms. But this rises a foundational problem: this 'category' may not be a category in the sense that the class of morphisms between two objects may not be a set.

Model categories were introduced in 1967 by Quillen [76]. One of the standard results about model categories is that if $\mathscr{W}$ is the class of weak equivalences in a model structure on $\mathscr{A}$, then $\mathscr{A}[\mathscr{W}^{-1}]$ is a category in the usual sense and can be represented via the model structure as we will explain later.

Recall from [63, Definition 1.1.3] that a **model structure** on a category $\mathscr{A}$ is a triple $(\mathrm{Cof}, \mathrm{Weak}, \mathrm{Fib})$ of subclasses of morphisms, called **cofibrations**, **weak equivalences** and **fibrations**, satisfying some axioms. The purpose of these axioms is to provide a general frame work for homotopy theory. A **model category** is a bicomplete category, i.e., a category with (small) limits and colimits, equipped with a model structure on it.

Standard references for model structures are the books of Hovey ([63]) and Dwyer and Spalinski ([31]). For an alternative approach, Beligiannis and I. Reiten [8] is also useful.

In this thesis, we are interested in model structures on abelian categories. Throughout this section, $\mathscr{A}$ denotes an abelian category not necessarily bicomplete. As explained by Gillespie in [50, Section 4], this requirement is not needed when working over abelian categories.

Recall ([63]) that in a model category $\mathscr{A}$, a **trivial cofibration** (resp., **fibration**) is a morphism which is both a weak equivalence and a cofibration (resp., fibration). An object $X \in \mathscr{A}$ is said to be

• (**trivially**) **cofibrant** if $0 \to X$ is a (trivial) cofibration.

• (**trivially**) **fibrant** if $X \to 0$ is a (trivial) fibration.

• **trivial** if $0 \to X$ is a weak equivalence. Or equivalently, if $X \to 0$ is a weak equivalence.

Abelian model structures with respect to some proper classes of short exact sequences were introduced by Hovey in [64]. These are model structures on abelian categories in the sense of Quillen ([76]) with some compatibility with the abelian structure. With respect to the class of short exact sequences of an abelian category $\mathscr{A}$, we have the following definition.

Definition 1.6.1 ([64], Definition 2.1 and Proposition 4.2)**.** *Let $\mathscr{A}$ be an abelian category. A model structure* $(\mathrm{Cof}, \mathrm{Weak}, \mathrm{Fib})$ *on $\mathscr{A}$ is said to be an* ***abelian model structure*** *if each of the following holds.*

1. *A morphism f is a (trivial) cofibration if and only if it is a monomorphism with a (trivially) cofibrant cokernel.*

2. *A morphism g is a (trivial) fibration if and only if it is an admissible epimorphism with a (trivially) fibrant kernel.*

This definition suggests that we can study abelian model structures by focusing our attention on objects (cofibrant, trivial, and fibrant) instead of morphisms (cofibrations, weak equivalences, and fibrations). This brings us to Hovey's correspondence: a correspondence between abelian model structures and complete cotorsion pairs.

Theorem 1.6.2. *(**Hovey's Correspondence** [64, Theorem 2.2]) . Let $\mathscr{A}$ be an abelian category. Assume that $\mathscr{A}$ has an abelian model structure and let $\mathscr{Q}$, $\mathscr{W}$ and $\mathscr{R}$ denote the classes of cofibrant, trivial objects and fibrant, respectively. Then, $\mathscr{W}$ is thick in $\mathscr{A}$ and both $(\mathscr{Q}, \mathscr{W} \cap \mathscr{R})$ and $(\mathscr{Q} \cap \mathscr{W}, \mathscr{R})$ are complete cotorsion pairs in $\mathscr{A}$.*

Conversely, given three classes $\mathscr{Q}$, $\mathscr{R}$ and $\mathscr{W}$ in $\mathscr{A}$ such that $\mathscr{W}$ is thick in $\mathscr{A}$ and both $(\mathscr{Q}, \mathscr{W} \cap \mathscr{R})$ and $(\mathscr{Q} \cap \mathscr{W}, \mathscr{R})$ are complete cotorsion pairs in $\mathscr{A}$, there is an abelian model structure on $\mathscr{A}$, where $\mathscr{Q}$ is the class of cofibrant objects, $\mathscr{W}$ is the class of trivial objects and $\mathscr{R}$ is the class of fibrant objects.

It follows from the Hovey's correspondence that an abelian model structure can be identified with a triple $\mathscr{M} = (\mathscr{Q}, \mathscr{W}, \mathscr{R})$ as in Theorem 1.6.2. By an abuse of language, we often refer to such a triple as an abelian model structure. Alternatively, we often call $\mathscr{M}$ a **Hovey triple**.

A Hovey triple $(\mathscr{Q}, \mathscr{W}, \mathscr{R})$ is called **hereditary** if both of the associated cotorsion pairs are hereditary. The class $\mathscr{Q} \cap \mathscr{W} \cap \mathscr{R}$ is called the **core** of the Hovey triple.

In practice, it is usually quite challenging to prove that a category has an abelian model structure (or equivalently, a Hovey triple). We now present a result due to Gillespie ([51, Theorem 1.1]) that provides a convenient way to construct Hovey triples.

Given a Hovey triple $(\mathscr{Q}, \mathscr{W}, \mathscr{R})$, Gillespie noticed that the cotorsion pairs

$$(\mathscr{Q}, \widetilde{\mathscr{R}}) := (\mathscr{Q}, \mathscr{W} \cap \mathscr{R}) \qquad \text{and} \qquad (\widetilde{\mathscr{Q}}, \mathscr{R}) := (\mathscr{Q} \cap \mathscr{W}, \mathscr{R})$$

satisfy the following properties:

(a) $\widetilde{\mathscr{R}} \subseteq \mathscr{R}$ and $\widetilde{\mathscr{Q}} \subseteq \mathscr{Q}$,

(b) $\widetilde{\mathscr{Q}} \cap \mathscr{R} = \mathscr{Q} \cap \widetilde{\mathscr{R}}$,

Interestingly, under the hereditary property, there is a converse to this as the following result shows.

Theorem 1.6.3 ([51], Theorem 1.1)**.** *Given two complete and hereditary cotorsion pairs*

$$(\widetilde{\mathscr{Q}}, \mathscr{R}) \qquad \textit{and} \qquad (\mathscr{Q}, \widetilde{\mathscr{R}})$$

in $\mathscr{A}$ satisfying the two conditions:

(a) *$\widetilde{\mathscr{R}} \subseteq \mathscr{R}$ (or equivalently, $\widetilde{\mathscr{Q}} \subseteq \mathscr{Q}$).*

(b) *$\widetilde{\mathscr{Q}} \cap \mathscr{R} = \mathscr{Q} \cap \widetilde{\mathscr{R}}$.*

there is a unique thick class $\mathscr{W}$ such that $(\mathscr{Q}, \mathscr{W}, \mathscr{R})$ is a Hovey triple. Moreover, the class $\mathscr{W}$ is characterized by:

$$\begin{aligned} \mathscr{W} &= \{X \in \mathscr{A} \mid \exists \textit{ an exact sequence } 0 \to X \to R' \to Q' \to 0 \textit{ with } R' \in \widetilde{\mathscr{R}}, Q' \in \widetilde{\mathscr{Q}}\} \\ &= \{X \in \mathscr{A} \mid \exists \textit{ an exact sequence } 0 \to R' \to Q' \to X \to 0 \textit{ with } R' \in \widetilde{\mathscr{R}}, Q' \in \widetilde{\mathscr{Q}}\}. \end{aligned}$$

This powerful result allows us to construct hereditary Hovey triples even before we have any ideas about trivial objects.

We end this section by a construction called the homotopy category of an abelian model category and some of its main properties.

Let X be an object in a model category $\mathscr{A}$. Then, the unique morphism $0 \to X$ can be factorized as a cofibration $i_X : 0 \to QX$ followed by a trivial fibration $p_X ; QX \to X$.

Dually, the unique morphism $X \to 0$ can be factorized as a trivial cofibration $j_X : X \to RX$ followed by a fibration $q_X : RX \to 0$.

$$\begin{array}{ccc} 0 & \xrightarrow{\qquad\qquad} & X \\ & {\scriptstyle i_X}\searrow \qquad \nearrow{\scriptstyle p_X} & \\ & QX & \end{array} \qquad \text{and} \qquad \begin{array}{ccc} X & \xrightarrow{\qquad\qquad} & 0 \\ & {\scriptstyle j_Y}\searrow \qquad \nearrow{\scriptstyle q_X} & \\ & RX & \end{array}$$

The objects QX and RX are called **cofibrant** and **fibrant replacements** of X, respectively, and by a **bifibrant replacement** of X we will mean the object RQX.

In the following result, we summarize the essentials about the homotopy category of an abelian model category. It follows from [63, Theorem 1.2.10] and [31, Sections 5 and 6], except that the homotopy relation in this way is due to Gillespie [50, Proposition 4.4].

Theorem 1.6.4. *(The fundamental theorem of abelian model categories) Let $\mathscr{M} = (\mathscr{Q}, \mathscr{W}, \mathscr{R})$ be an abelian model structure on $\mathscr{A}$. Then, there is a category, denoted as $\mathrm{Ho}_{\mathscr{A}}(\mathscr{M})$ and called the **homotopy category** of $\mathscr{M}$, with the same objects as $\mathscr{A}$ and*

$$\mathrm{Hom}_{\mathrm{Ho}(\mathscr{A})}(X,Y) = \mathrm{Hom}_{\mathscr{A}}(RQX, RQY)/\sim$$

where $f \sim g$ if and only if $g - f$ factors through an object of the core $\mathscr{Q} \cap \mathscr{W} \cap \mathscr{R}$.

Moreover, the following conditions hold:

(a) *The inclusion $\mathscr{A}_{cf} := \mathscr{Q} \cap \mathscr{R} \hookrightarrow \mathscr{A}$ induces an equivalence of categories*

$$(\mathscr{Q} \cap \mathscr{R})/\sim \hookrightarrow \mathrm{Ho}(\mathscr{M})$$

where $f \sim g$ if and only if $g - f$ factors through an object of the core $\mathscr{Q} \cap \mathscr{W} \cap \mathscr{R}$.

(b) *There is a functor $\gamma_{\mathscr{A}} : \mathscr{A} \to \mathrm{Ho}(\mathscr{M})$ that is the identity on objects and that sends a morphism $f : A \to B$ to the homotopy class $[f']$, where f' and $\tilde{f}$ any morphisms making the diagrams commute:*

$$\begin{array}{ccccc} RQA & \xleftarrow[j_{QA}]{} & QA & \xrightarrow{p_B} & A \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle \tilde{f}} & & \downarrow{\scriptstyle f} \\ RQB & \xleftarrow[j_{QB}]{} & QB & \xrightarrow{p_B} & B \end{array}$$

(c) *The functor $\gamma_{\mathscr{A}}$ is a localization of $\mathscr{A}$ with respect to $\mathscr{W}$ in the sense of [31, Definition 6.1], and hence there is a canonical equivalence of categories*

$$\mathscr{A}[\mathscr{W}^{-1}] \cong \mathrm{Ho}(\mathscr{M}).$$

One more important feature of hereditary abelian model structures is that the associated homotopy category is always triangulated, in the sense of Verdier.

Recall that a **triangulated category** is an additive category $\mathscr{T}$ with an invertible additive endofunctor $\Sigma : \mathscr{T} \to \mathscr{T}$ and a class of "distinguished triangles" satisfying some properties (see [75] for the precise definition and more details).

Recall also that an **exact category** is an additive category, which may not have all kernels and cokernels, but with an exact structure, that is, a distinguished class $\mathscr{E}$ of ker-coker sequences which are called conflations, subject to certain axioms (see Bühler [24] for instance). For example, given an additive category $\mathscr{B}$, the pair $(\mathscr{B},\mathscr{E})$ is an exact category in the following cases:

- $\mathscr{B}$ is an abelian category and $\mathscr{E}$ is the class of all short exact sequences.
- $\mathscr{B}$ is a subcategory of an abelian category $\mathscr{A}$ that is closed under extensions and $\mathscr{E}$ is the class short exact sequences with terms in $\mathscr{B}$. We will call this exact structure the **induced exact structure**.

Definition 1.6.5. *A* ***Frobenius category*** *is an exact category with enough injectives and projectives and such that the projective objects coincide with the injective objects.*

Given a Frobenius category $\mathscr{F}$, we can form the **stable category** $\underline{\mathscr{F}} := \mathscr{F}/\sim$, which has the same objects as $\mathscr{A}$ and $\mathrm{Hom}_{\mathscr{F}/\sim}(X,Y) = \mathrm{Hom}_{\mathscr{F}}(X,Y)/\sim$, where $f \sim g$ if and only if $f-g$ factors through a projective-injective object. The main fact about a Frobenius category $\mathscr{F}$ is that the stable category is canonically triangulated and it encodes the corresponding relative homological algebra on $\mathscr{F}$.

The following result is the key reason why the homotopy category of a hereditary abelian model structure $\mathscr{M}$ turns out to be triangulated.

Theorem 1.6.6. *([50, Sections 4 and 5] & [49, Proposition 4.2 and Theorem 4.3])*
Let $\mathscr{M} = (\mathscr{Q},\mathscr{W},\mathscr{R})$ be a hereditary abelian model structure. Then, the subcategory $\mathscr{Q}\cap\mathscr{R}$, along with the induced exact structure, is a Frobenius category. The projective-injective objects are exactly the objects in $\mathscr{Q}\cap\mathscr{W}\cap\mathscr{R}$. Moreover, the inclusion $\mathscr{Q}\cap\mathscr{R} \hookrightarrow \mathscr{A}$ induces a triangle equivalence

$$(\mathscr{Q}\cap\mathscr{R})/\sim \hookrightarrow \mathrm{Ho}_{\mathscr{A}}(\mathscr{M}).$$

1.7 Auslander-Buchweitz theory

In this section we recall some notions and results from Auslander-Buchweitz theory and their relations with the Frobenius pairs recently introduced by Becerril, Mendoza, Pérez and Santiago in [7]

Let $\mathscr{X}$ and ω be two subcategories of $\mathscr{A}$. The subcategory ω is a **cogenerator** in $\mathscr{X}$ if $\omega \subseteq \mathscr{X}$ and for any $X \in \mathscr{X}$ there exists an exact sequence $0 \to X \to W \to X' \to 0$

with $X \in \mathscr{X}'$ and $W \in \omega$. If, in addition, $\mathrm{Ext}^{i\geq 1}_{\mathscr{A}}(\mathscr{X},\omega)=0$, it is called **injective cogenerator** for $\mathscr{X}$.

The following definition is taken from Hashimoto ([59, Theorem 1.1.2.10]). As this definition can be dualized, we add "left" to distinguish it from its dual.

Definition 1.7.1 ([59])**.** *A triple $(\mathscr{X},\mathscr{Y},\omega)$ of subcategories of $\mathscr{A}$ is called **left weak Auslander-Buchweitz context** (or **weak AB-context** for short), if the following three conditions are satisfied:*

(AB1) *$\mathscr{X}$ is left thick.*

(AB2) *$\mathscr{Y}$ is right thick and $\mathscr{Y} \subseteq \mathrm{res}\widehat{\mathscr{X}}$.*

(AB3) *$\omega = \mathscr{X} \cap \mathscr{Y}$ and ω is an injective cogenerator for $\mathscr{X}$.*

*If moreover $\mathrm{res}\widehat{\mathscr{X}} = \mathscr{A}$, then it is called a left **Auslander-Buchweitz context** (or **AB-context** for short).*

We refer to the dual concept as right (weak) AB-context, for which we use the notation $(\nu,\mathscr{X},\mathscr{Y})$.

It follows from [59, Theorem 1.1.2.10(1)] that the middle term $\mathscr{Y}$ in a left weak AB-context $(\mathscr{X},\mathscr{Y},\omega)$ is determined by the term ω, as one has $Y = \mathrm{res}\widehat{\omega}$. Based on this fact, Becerril, Mendoza, Pérez and Santiago ([7]) have recently introduced the notion of (left) Frobenius pairs and shown that there is a one-to-one correspondence between these two concepts.

Definition 1.7.2 ([7], Definition 2.5)**.** *A pair $(\mathscr{X},\omega)$ of subcategories of $\mathscr{A}$ is said to be a **left Frobenius pair** if the following three conditions are satisfied:*

(F1) *$\mathscr{X}$ is left thick.*

(F2) *ω is closed under direct summands.*

(F3) *ω is an injective cogenerator for $\mathscr{X}$.*

A right Frobenius pair is defined dually and we use $(\mu,\mathscr{Y})$ to denote it.

Theorem 1.7.3 ([7], Theorem 5.4(1))**.** *Consider the following classes of objects*

$$\mathscr{F} := \{(\mathscr{X},\omega) \subseteq \mathscr{A}\times\mathscr{A} \mid (\mathscr{X},\omega) \text{ is a left Frobenius pair}\}$$

$$\mathscr{C} := \{(\mathscr{X},\mathscr{Y},\omega) \subseteq \mathscr{A}\times\mathscr{A}\times\mathscr{A} \mid (\mathscr{X},\mathscr{Y},\omega) \text{ is a left weak AB-context }\}$$

There exists a one-to-one correspondence

$$\begin{aligned} \Phi : \mathscr{F} &\longrightarrow \mathscr{C} \\ (\mathscr{X},\omega) &\longmapsto (\mathscr{X},\mathrm{res}\widehat{\omega},\mathscr{X}\cap\mathrm{res}\widehat{\omega}) \end{aligned}$$

with inverse

$$\begin{aligned} \Psi : \mathscr{C} &\longrightarrow \mathscr{F} \\ (\mathscr{X}, \mathscr{Y}, \omega) &\longmapsto (\mathscr{X}, \omega) \end{aligned}$$

We conclude this section with the following result, which links Frobenius pairs and cotorsion pairs.

Proposition 1.7.4 ([69], Propositions 2.5 and 2.10)**.** *Let $\mathscr{X}$ and $\mathscr{Y}$ be two classes of objects of $\mathscr{A}$. If $(\mathscr{X}, \mathscr{Y})$ is a complete hereditary cotorsion pair in $\mathscr{A}$, Then, $(\mathscr{X}, \mathscr{X} \cap \mathscr{Y})$ is a left Frobenius pair.*

Conversely, assume that $\mathrm{res}\,\widehat{\mathscr{X}} = \mathscr{A}$ *and $\mathscr{Y}$ is closed under extensions and cokernels of monomorphisms. If $(\mathscr{X}, \mathscr{X} \cap \mathscr{Y})$ is a left Frobenius pair in $\mathscr{A}$, then $(\mathscr{X}, \mathscr{Y})$ is a complete hereditary cotorsion pair.*

RELATIVE GORENSTEIN HOMOLOGICAL DIMENSIONS OVER TRIANGULAR MATRIX RINGS

In this chapter, several notions of relative Gorenstein homological algebra over a triangular matrix ring are investigated. We first study how to construct w-tilting (tilting, semidualizing) modules over T using the corresponding ones over A and B. We show that when U is relative (weakly) compatible, we are able to describe the structure of $\mathrm{G_C}$-projective modules over T. As an application, we study when a morphism in T-Mod is a special $\mathrm{G_C}$-projective precover and when the class $\mathrm{G_CP}(T)$ is a special precovering class. In addition, we study the relative global Gorenstein dimension of T. In some cases, we show that it can be computed from the relative global Gorenstein dimensions of A and B.

Throughout this chapter, $T = \begin{pmatrix} A & 0 \\ U & B \end{pmatrix}$ will always be a triangular matrix ring.

2.1 Weakly Wakamatsu tilting modules

In this section, we study when the functor $\mathbf{p}$ preserves w-tilting modules.

It is well known that the functor $\mathbf{p}$ preserves projective modules. However, the functor $\mathbf{p}$ does not preserve w-tilting modules in general, as the following example shows.

Example 2.1.1. *Let Q be the quiver*

$$v_1 \longrightarrow v_2 \longrightarrow \cdots \longrightarrow v_n$$

with $n \geq 1$ vertices and let $R = kQ$ be the path algebra over an algebraic closed field k. For each $i = 1, \cdots, n$ set $P_i = Rv_i$ and $I_i = \mathrm{Hom}_k(v_iR, k) = (v_iR)^*$. It follows by [12, Example 3.5] that

$$C_1 := \bigoplus_{i=1}^{n} P_i = R \text{ and } C_2 := \bigoplus_{i=1}^{n} I_i = R^*$$

are semidualizing (R,R)-bimodules and then w-tilting R-modules. Now, consider the triangular matrix ring

$$T(R) = \begin{pmatrix} R & 0 \\ R & R \end{pmatrix}.$$

We claim that $\boldsymbol{p}(C_1,C_2)$ is not a w-tilting $T(R)$-module.

Since R is left hereditary by [6, Proposition 1.4], but not semisimple as Q is not discrete, we get that I_i is not projective for some i. Therefore, $\mathrm{pd}_R(I_i) = 1$ and hence $\mathrm{Ext}^1_R(I_1,R) \neq 0$. Using now Lemma 1.4.7, we get that $\mathrm{Ext}^1_{T(R)}(\boldsymbol{p}(C_1,C_2),\boldsymbol{p}(C_1,C_2)) \cong \mathrm{Ext}^1_R(C_1,C_1) \oplus \mathrm{Ext}^1_R(C_2,C_1) \oplus \mathrm{Ext}^1_R(C_2,C_2) \cong \mathrm{Ext}^1_R(I_1,R) \neq 0$. Thus, $\boldsymbol{p}(C_1,C_2)$ is not a w-tilting $T(R)$-module.

Motivated by the definition of compatible bimodules in [88, Definition 1.1], we introduce the following definition, which will be crucial throughout the rest of this chapter.

Definition 2.1.2. *Let $(C_1,C_2) \in A\text{-Mod} \times B\text{-Mod}$ and $C = \boldsymbol{p}(C_1,C_2)$. The bimodule $_BU_A$ is said to be **C-compatible** if the following two conditions hold:*

(P1) *The complex $U \otimes_A \boldsymbol{X}_1$ is exact for every exact sequence in A-Mod*

$$\boldsymbol{X}_1 : \cdots \to P_1^1 \to P_1^0 \to C_1^0 \to C_1^1 \to \cdots$$

where the P_1^i's are all projective and $C_1^i \in \mathrm{Add}_A(C_1)$ $\forall i$.

(P2) $\mathrm{Hom}(\boldsymbol{X}_2, U \otimes_A \mathrm{Add}_A(C_1))$ *is exact for any complete $\mathscr{P}_{C_2}$-projective sequence $\boldsymbol{X}_2$.*

*Moreover, U is called **weakly C-compatible** if it satisfies* (P2) *and the following condition*

(WP1) *$U \otimes_A \boldsymbol{X}_1$ is exact for any complete $\mathscr{P}_{C_1}$-projective sequence $\boldsymbol{X}_1$.*

*When $C = {}_TT = \boldsymbol{p}(A,B)$, the bimodule U will be called simply **(weakly) compatible**.*

Remark 2.1.3.

1. *It is clear by the definition that every C-compatible bimodule is weakly C-compatible.*

2. *The (B,A)-bimodule U is weakly compatible if and only if the functor $U \otimes_A - : A\text{-Mod} \to B\text{-Mod}$ is weak compatible in the sense of [66].*

3. *If A and B are Artin algebras and since ${}_TT = \boldsymbol{p}(A,B)$, it is easy to see that ${}_TT$-compatible bimodules are nothing but compatible (B,A)-bimodules as defined in [88].*

Given a T-module $C = \mathbf{p}(C_1, C_2)$, we have simple characterizations of conditions (WP1) and (P2) if C_1 and C_2 are w-tilting.

Proposition 2.1.4. *Let $C = \boldsymbol{p}(C_1, C_2)$ be a T-module.*

1. *If C_1 is w-tilting, then the following assertions are equivalent:*

   (i) *U satisfies* (WP1).

   (ii) $\mathrm{Tor}_1^A(U, G_1) = 0$, $\forall G_1 \in \mathrm{G_{C_1}P}(A)$.

   (iii) $\mathrm{Tor}_{i\geq 1}^A(U, G_1) = 0$, $\forall G_1 \in \mathrm{G_{C_1}P}(A)$.

   In this case, $\mathrm{Tor}_{i\geq 1}^A(U, C_1) = 0$.

2. *If C_2 is w-tilting, then the following assertions are equivalent:*

   (i) U satisfies (P2).

   (ii) $\mathrm{Ext}_B^1(G_2, U \otimes_A X_1) = 0$, $\forall G_2 \in \mathrm{G_{C_2}P}(B)$, $\forall X_1 \in \mathrm{Add}_A(C_1)$.

   (iii) $\mathrm{Ext}_B^{i\geq 1}(G_2, U \otimes_A X_1) = 0$, $\forall G_2 \in \mathrm{G_{C_2}P}(B)$, $\forall X_1 \in \mathrm{Add}_A(C_1)$.

   In this case, $\mathrm{Ext}_B^{i\geq 1}(C_2, U \otimes_A X_1) = 0$, $\forall X_1 \in \mathrm{Add}_A(C_1)$.

Proof. We only prove (1), since (2) is similar.

(i)$\Rightarrow$ (iii) Let $G_1 \in \mathrm{G_{C_1}P}(R)$. Then, there exists a complete $\mathscr{P}_{C_1}$-projective complex of G_1

$$\mathbf{X}_1 : \cdots \to P_1^1 \to P_1^0 \to C_1^0 \to C_1^1 \to \cdots$$

By hypothesis, $U \otimes_A \mathbf{X}_1$ is exact, which means in particular that $\mathrm{Tor}_{i\geq 1}^A(U, G_1) = 0$.

(iii) $\Rightarrow$ (ii) Clear.

(ii) $\Rightarrow$ (i) Follows by [17, Corollary 2.13].

Finally, to prove that $\mathrm{Tor}_{i\geq 1}^A(U, C_1) = 0$, note that $C_1 \in \mathrm{G_{C_1}P}(A)$ by [17, Theorem 2.12]. ■

In the following proposition, we study when $\mathbf{p}$ preserves w-tilting (tilting) modules.

Proposition 2.1.5. *Let $C = \boldsymbol{p}(C_1, C_2)$ be a T-module and assume that U is weakly C-compatible. If C_1 and C_2 are w-tilting (tilting), then $\boldsymbol{p}(C_1, C_2)$ is w-tilting (tilting).*

Proof. Since the functor $\mathbf{p}$ preserves finitely generated modules (see for instance [67, Proposition 3.2.1]), we only need prove the statement for w-tilting.

Assume that C_1 and C_2 are w-tilting and let I be a set. Then, $\mathrm{Ext}_A^{i\geq 1}(C_1, C_1^{(I)}) = 0$ and $\mathrm{Ext}_B^{i\geq 1}(C_2, C_2^{(I)}) = 0$. Thus, by Proposition 2.1.4, we have $\mathrm{Ext}_B^{i\geq 1}(C_2, U\otimes_A C_1^{(I)}) = 0$ and $\mathrm{Tor}_{i\geq 1}^A(U, C_1) = 0$, and using Lemma 1.4.7, for every $n \geq 1$ we get that

$$\begin{aligned}\mathrm{Ext}_T^n(C, C^{(I)}) &= \mathrm{Ext}_T^n(\mathbf{p}(C_1,C_2), \mathbf{p}(C_1,C_2)^{(I)})\\ &\cong \mathrm{Ext}_A^n(C_1, C_1^{(I)}) \oplus \mathrm{Ext}_B^n(C_2, U\otimes_A C_1^{(I)}) \oplus \mathrm{Ext}_B^n(C_2, C_2^{(I)})\\ &= 0.\end{aligned}$$

Moreover, there exist exact sequences

$$\mathbf{X}_1 :\ 0 \to A \to C_1^0 \to C_1^1 \to \cdots \qquad \text{and} \qquad \mathbf{X}_2 :\ 0 \to B \to C_2^0 \to C_2^1 \to \cdots$$

which are $\mathrm{Hom}_A(-, \mathrm{Add}_A(C_1))$-exact and $\mathrm{Hom}_B(-, \mathrm{Add}_B(C_2))$-exact, respectively, and such that $C_1^i \in \mathrm{Add}_A(C_1)$ and $C_2^i \in \mathrm{Add}_B(C_2)$ for every $i \in \mathbb{N}$. Since U is weakly C-compatible, the complex $U\otimes_A \mathbf{X}_1$ is exact. So we construct in T-Mod the exact sequence

$$\mathbf{p}(\mathbf{X}_1,\mathbf{X}_2) :\ 0 \to T \to \mathbf{p}(C_1^0, C_2^0) \to \mathbf{p}(C_1^1, C_2^1) \to \cdots$$

where $\mathbf{p}(C_1^i, C_2^i) \in \mathrm{Add}_T(\mathbf{p}(C_1,C_2))$, $\forall i \in \mathbb{N}$, by Lemma 1.4.5(1).

Let $X \in \mathrm{Add}_T(\mathbf{p}(C_1,C_2))$. As a consequence of Lemma 1.4.5(1), $X = \mathbf{p}(X_1,X_2)$ where $X_1 \in \mathrm{Add}_A(C_1)$ and $X_2 \in \mathrm{Add}_B(C_2)$. Using the adjoitness $(\mathbf{p},\mathbf{q})$, we get an isomorphism of complexes

$$\mathrm{Hom}_T(\mathbf{p}(\mathbf{X}_1,\mathbf{X}_2), X) \cong \mathrm{Hom}_A(\mathbf{X}_1, X_1) \oplus \mathrm{Hom}_B(\mathbf{X}_2, U\otimes X_1) \oplus \mathrm{Hom}_B(\mathbf{X}_2, X_2).$$

However, the complexes $\mathrm{Hom}_A(\mathbf{X}_1, X_1)$ and $\mathrm{Hom}_B(\mathbf{X}_2, X_2)$ are exact and the complex $\mathrm{Hom}_B(\mathbf{X}_2, U\otimes X_1)$ is also exact since U is weakly C-compatible. Then, the complex $\mathrm{Hom}_T(\mathbf{p}(\mathbf{X}_1,\mathbf{X}_2), X)$ is exact, as well, and the proof is finished. ■

Now, we illustrate Proposition 2.1.5 with two applications.

Corollary 2.1.6. *Let $C = \mathbf{p}(C_1,C_2)$ be a T-module, A' and B' be two rings such that C_1 and C_2 are (A,A')- and (B,B')-bimodules and assume that U is weakly C-compatible. If C_1 and C_2 are semidualizing (A,A')- and (B,B')-bimodules, then $\mathbf{p}(C_1,C_2)$ is a semidualizing $(T, \mathrm{End}_T(C))$-bimodule.*

Proof. Follows by Proposition 2.1.5 and [82, Corollary 3.2]. ■

Corollary 2.1.7. *Let R and S be rings with S_R flat, $\theta : R \to S$ be a ring homomorphism, and $T(\theta) =: \begin{pmatrix} R & 0 \\ S & S \end{pmatrix}$. Let C_1 be an R-module such that $S\otimes_R C_1 \in \mathrm{Add}_R(C_1)$ (for instance, if $S = R$ or S_R is projective with R commutative). If ${}_RC_1$ is w-tilting, then*

1. $S \otimes_R C_1$ is a w-tilting S-module;

2. $C = \begin{pmatrix} C_1 \\ (S \otimes_R C_1) \oplus (S \otimes_R C_1) \end{pmatrix}$ is a w-tilting $T(\theta)$-module.

Proof. Let $C_2 = S \otimes_R C_1$ and note that $C = \mathbf{p}(C_1, C_2)$ and that ${}_S S_R$ is weakly C-compatible as S_R is flat and $S \otimes_R C_1 \in \mathrm{Add}_R(C_2)$. So, by Proposition 2.1.5, we only need to prove that C_2 is a w-tilting S-module.

Since ${}_R C_1$ is w-tilting, there exist $\mathrm{Hom}_R(-, \mathrm{Add}_R(C_1))$-exact exact sequences

$$\mathbf{P} : \cdots \to P_1^1 \to P_1^0 \to C_1 \to 0 \qquad \text{and} \qquad \mathbf{X} : 0 \to R \to C_1^0 \to C_1^1 \to \cdots$$

with each ${}_R P_1^i$ projective and ${}_R C_1^i \in \mathrm{Add}_R(C_1)$. Since S_R is flat, we get the exact sequences of S-modules

$$S \otimes_R \mathbf{P} : \cdots \to S \otimes_R P_1^1 \to S \otimes_R P_1^0 \to C_2 \to 0$$

and

$$S \otimes_R \mathbf{X} : 0 \to S \to S \otimes_R C_1^0 \to S \otimes_R C_1^1 \to \cdots$$

with each $S \otimes_R P_i$ a projective S-module and $S \otimes_R C_i \in \mathrm{Add}_R(C_2)$.

We prove now that $S \otimes_R \mathbf{P}$ and $S \otimes_R \mathbf{X}$ are $\mathrm{Hom}_S(-, \mathrm{Add}_S(C_2))$-exact.

Let I be a set. Then, $\mathrm{Hom}_S(S \otimes_R \mathbf{P}, S \otimes_R C_1^{(I)}) \cong \mathrm{Hom}_R(\mathbf{P}, \mathrm{Hom}_S(S, S \otimes_R C_1^{(I)})) \cong \mathrm{Hom}_R(\mathbf{P}, S \otimes_R C_1^{(I)})$ is exact since $S \otimes_R C_1^{(I)} \in \mathrm{Add}_R(C_1)$. Similarly, the complex $S \otimes_R \mathbf{X}$ is $\mathrm{Hom}_S(-, \mathrm{Add}_S(C_2))$-exact. ■

We end this section with an example of a w-tilting module that is neither projective nor injective.

Example 2.1.8. *Take R and C_2 as in example 2.1.1. By Corollary 2.1.7, $C = \begin{pmatrix} C_2 \\ C_2 \oplus C_2 \end{pmatrix} = \boldsymbol{p}(C_2, C_2)$ is a w-tilting $T(R)$-module. By Lemma 1.4.6, C is not projective since C_2 is not and it is not injective since the map $\widetilde{\varphi^C} : C_2 \to C_2 \oplus C_2$ is not an epimorphism.*

Moreover, by [6, Proposition 2.6], $gl.dim(T(R)) = gl.dim(R) + 1 \leq 2$. So, if

$$0 \to T(R) \to E^0 \to E^1 \to E^2 \to 0$$

is an injective resolution of $T(R)$, then $C^1 = E^0 \oplus E^1 \oplus E^2$ is a w-tilting $T(R)$-module. Note that $T(R)$ has at least three w-tilting modules:

$$C^1,\ C^2 = T(R) \text{ and } C^3 = C.$$

2.2 Relative Gorenstein projective modules

In this section, we describe $\mathrm{G_C}$-projective modules over T. Then, we use this description to study when the class of $\mathrm{G_C}$-projective T-modules is a special precovering class.

Clearly, the functor $\mathbf{p}$ preserves projective modules. Therefore, we start by studying when the functor $\mathbf{p}$ also preserves relative Gorenstein projective modules.

Lemma 2.2.1. *Let $C = \boldsymbol{p}(C_1, C_2)$ be a T-module and U be weakly C-compatible.*

1. *If $M_1 \in \mathrm{G_{C_2}P}(A)$, then $\boldsymbol{p}(M_A, 0) = \begin{pmatrix} M_1 \\ U \otimes_A M_1 \end{pmatrix} \in \mathrm{G_C P}(T)$.*

2. *If $M_2 \in \mathrm{G_{C_2}P}(B)$, then $\boldsymbol{p}(0, M_2) = \begin{pmatrix} 0 \\ M_2 \end{pmatrix} \in \mathrm{G_C P}(T)$.*

Proof. 1. Suppose that $M_1 \in \mathrm{G_{C_1}P}(A)$. There exists a complete $\mathscr{P}_{C_1}$-projective complex of M_1

$$\mathbf{X}_1 : \cdots \to P_1^1 \to P_1^0 \to C_1^0 \to C_1^1 \to \cdots .$$

Using the fact that U is weakly C-compatible, we get that the complex $U \otimes_A \mathbf{X}_1$ is exact in B-Mod, which implies that the complex

$$\mathbf{p}(\mathbf{X}_1, 0) : \cdots \to \mathbf{p}(P_1^1, 0) \to \mathbf{p}(P_1^0, 0) \to \mathbf{p}(C_1^0, 0) \to \mathbf{p}(C_1^1, 0) \to \cdots$$

is exact with $\mathbf{p}(M_1, 0) \cong \mathrm{Im}(\mathbf{p}(P_1^0, 0) \to \mathbf{p}(C_1^0, 0))$.

Clearly, $\mathbf{p}(P_1^i, 0) \in \mathscr{P}(T)$ and $\mathbf{p}(C_1^i, 0) \in \mathrm{Add}_T(C)$ $\forall i \in \mathbb{N}$ by Lemmas 1.4.6(1) and 1.4.5(1). If $X \in \mathrm{Add}_T(C)$, then $X_1 \in \mathrm{Add}_A(C_1)$ by Lemma 1.4.5(1) and using the adjoint pair $(\mathbf{p}, \mathbf{q})$, we obtain that the complex $\mathrm{Hom}_T(\mathbf{p}(\mathbf{X}_1, 0), X) \cong \mathrm{Hom}_A(\mathbf{X}_1, X_1)$ is exact. Hence, $\mathbf{p}(M_1, 0)$ is $\mathrm{G_C}$-projective.

2. Suppose that M_2 is $\mathrm{G_{C_2}}$-projective. There exists a complete $\mathscr{P}_{C_2}$-projective complex of M_2

$$\mathbf{X}_2 : \cdots \to P_2^1 \to P_2^0 \to C_2^0 \to C_2^1 \to \cdots .$$

Clearly, the complex

$$\mathbf{p}(0, \mathbf{X}_2) : \cdots \to \mathbf{p}(0, P_2^1) \to \mathbf{p}(0, P_2^0) \to \mathbf{p}(0, C_2^0) \to \mathbf{p}(0, C_2^1) \to \cdots$$

is exact with $\mathbf{p}(0, M_2) \cong \mathrm{Im}(\mathbf{p}(0, P_2^0) \to \mathbf{p}(0, C_2^0))$, $\mathbf{p}(0, P_2^i) \in \mathscr{P}(T)$ and $\mathbf{p}(0, C_2^i) \in \mathrm{Add}_T(C)$ $\forall i$, by Lemmas 1.4.6(1) and 1.4.5(1). Let $X \in \mathrm{Add}_T(C)$. Then, by Lemma 1.4.5(1), $X = \mathbf{p}(X_1, X_2)$ where $X_1 \in \mathrm{Add}_A(C_1)$ and $X_2 \in \mathrm{Add}_B(C_2)$. Using the adjoint pair $(\mathbf{p}, \mathbf{q})$, we obtain the isomorphism of complexes:

$$\mathrm{Hom}_T(\mathbf{p}(0, \mathbf{X}_2), X) \cong \mathrm{Hom}_B(\mathbf{X}_2, U \otimes_A X_1) \oplus \mathrm{Hom}_B(\mathbf{X}_2, X_2).$$

The complex $\mathrm{Hom}_B(\mathbf{X}_2, X_2)$ is exact and since U is weakly C-compatible, the complex $\mathrm{Hom}_B(\mathbf{X}_2, U \otimes_A X_1)$ is also exact. This means that $\mathrm{Hom}_T(\mathbf{p}(0, \mathbf{X}_2), X)$ is exact as well and $\mathbf{p}(0, M_2)$ is $\mathrm{G_C}$-projective. ■

Proposition 2.2.2. *Let $C = \mathbf{p}(C_1, C_2)$ be a T-module. If ${}_BU_A$ is weakly C-compatible, then the functor $\mathbf{p}$ sends $G_{(C_1,C_2)}$-projectives to $\mathrm{G_C}$-projectives. The converse holds provided that C_1 and C_2 are w-tilting.*

In particular, $\mathbf{p}$ preserves Gorenstein projective modules if and only if U is weakly compatible.

Proof. Note that
$$\mathbf{p}(M_1, M_2) = \begin{pmatrix} M_1 \\ U \otimes_A M_1 \end{pmatrix} \oplus \begin{pmatrix} 0 \\ M_2 \end{pmatrix}.$$
So this direction follows from Lemma 2.2.1 and [17, Proposition 2.5].

Conversely, assume that C_1 and C_2 are w-tilting. By Proposition 2.1.4, it suffices to prove that $\mathrm{Tor}_1^A(U, \mathrm{G_{C_1}P}(A)) = 0 = \mathrm{Ext}_B^1(\mathrm{G_{C_2}P}(B), U \otimes_A \mathrm{Add}_A(C_1))$.

Let $G_1 \in \mathrm{G_{C_1}P}(A)$. By [17, Corollary 2.13], there exits an exact sequence $0 \to L_1 \xrightarrow{\iota} P_1 \to G_1 \to 0$, where ${}_AP_1$ is projective and L_1 is $\mathrm{G_{C_1}}$-projective. Note that $A, C_1 \in \mathrm{G_{C_1}P}(A)$ and $B, C_2 \in \mathrm{G_{C_2}P}(B)$ by Lemma 1.3.5. Then, ${}_TT = \mathbf{p}(A,B)$ and $C = \mathbf{p}(C_1, C_2)$ are $\mathrm{G_C}$-projective, which imply by Lemma 1.3.5, that C is w-tilting. Moreover, $\mathbf{p}(L_1, 0)$ is also $\mathrm{G_C}$-projective and by [17, Corollary 2.13], there exists a short exact sequence
$$0 \to \mathbf{p}(L_1, 0) \to X \to H \to 0$$
where $X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}_{\varphi^X} \in \mathrm{Add}_T(C)$ and $H = \begin{pmatrix} H_1 \\ H_2 \end{pmatrix}_{\varphi^H}$ is $\mathrm{G_C}$-projective.

Since $X_1 \in \mathrm{Add}_A(C_1)$, we have the following commutative diagram with exact rows:
$$\begin{array}{ccccccccc} 0 & \longrightarrow & L_1 & \xrightarrow{\iota} & P_1 & \longrightarrow & G_1 & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & L_1 & \longrightarrow & X_1 & \longrightarrow & H_1 & \longrightarrow & 0 \end{array}$$
So if we apply the functor $U \otimes_A -$ to the above diagram, we get the following commutative diagram with exact rows:
$$\begin{array}{ccccccccc} & & U \otimes_A L_1 & \xrightarrow{1_U \otimes \iota} & U \otimes_A P_1 & \longrightarrow & U \otimes_A G_1 & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ & & U \otimes_A L_1 & \longrightarrow & U \otimes_A X_1 & \longrightarrow & U \otimes_A H_1 & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & U \otimes_A L_1 & \longrightarrow & X_2 & \longrightarrow & H_2 & \longrightarrow & 0 \end{array}$$
The commutativity of this diagram implies that the map $1_U \otimes \iota$ injective, and since P_1 is projective, $\mathrm{Tor}_1^A(U, G_1) = 0$.

Now, let $G_2 \in \mathrm{G}_{\mathrm{C}_2}\mathrm{P}(B)$ and $Y_2 \in \mathrm{Add}_A(C_1)$. By hypothesis, $\mathbf{p}(0,G_2)$ is $\mathrm{G_C}$-projective and by Lemma 1.4.5, $\mathbf{p}(Y_1,0) \in \mathrm{Add}_T(C)$. Hence,

$$\mathrm{Ext}^1_B(G_2, U \otimes_A Y_1) = \mathrm{Ext}^1_T(\mathbf{p}(0,G_2), \mathbf{p}(Y_1,0)) = 0$$

by Lemma 1.4.7 and [17, Proposition 2.4]. ■

Theorem 2.2.3. *Let* $M = \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}_{\varphi^M}$ *and* $C = \boldsymbol{p}(C_1,C_2)$ *be two* T*-modules. If* U *is* C*-compatible, then the following assertions are equivalent:*

(1) M *is* $\mathrm{G_C}$*-projective.*

(2) *The following two assertions hold:*

  (i) φ^M *is a monomorphism.*

  (ii) M_1 *is* $\mathrm{G}_{\mathrm{C}_1}$*-projective and* Coker φ^M *is* $\mathrm{G}_{\mathrm{C}_2}$*-projective.*

In this case, if C_2 *is* Σ*-self-orthogonal, then* $U \otimes_A M_1$ *is* $\mathrm{G}_{\mathrm{C}_2}$*-projective if and only if* M_2 *is* $\mathrm{G}_{\mathrm{C}_2}$*-projective.*

Proof. 2. $\Rightarrow$ 1. Since φ^M is a monomorphism, there exists an exact sequence in T-Mod

$$0 \to \begin{pmatrix} M_1 \\ U \otimes_A M_1 \end{pmatrix} \to M \to \begin{pmatrix} 0 \\ \mathrm{Coker}\varphi^M \end{pmatrix} \to 0$$

Note that $\begin{pmatrix} M_1 \\ U \otimes_A M_1 \end{pmatrix}$ *and* $\begin{pmatrix} 0 \\ \mathrm{Coker}\varphi^M \end{pmatrix}$ are $\mathrm{G_C}$-projective T-modules by Lemma 2.2.1. Therefore, M is $\mathrm{G_C}$-projective by [17, Proposition 2.5].

1. $\Rightarrow$ 2. There exists a $\mathrm{Hom}_T(-, \mathrm{Add}_T(C))$-exact sequence in T-Mod

$$\mathbf{X} = \cdots \to P^1 \to P^0 \to C^0 \to C^1 \to \cdots$$

where $C^i = \begin{pmatrix} C^i_1 \\ C^i_2 \end{pmatrix}_{\varphi^{C^i}} \in \mathrm{Add}_T(C)$, $P^i = \begin{pmatrix} P^i_1 \\ P^i_2 \end{pmatrix}_{\varphi^{P^i}} \in \mathscr{P}(T)$ $\forall i \in \mathbb{N}$, and such that $M \cong \mathrm{Im}(P^0 \to C^0)$. Then, we obtain the exact sequence

$$\mathbf{X}_1 = \cdots \to P^1_1 \to P^0_1 \to C^0_1 \to C^1_1 \to \cdots$$

where $C^i_1 \in \mathrm{Add}_A(C_1)$, $P^i_1 \in \mathscr{P}(A)$ $\forall i \in \mathbb{N}$ by Lemmas 1.4.5(1) and 1.4.6(1), and such that $M_1 \cong \mathrm{Im}(P^0_1 \to C^0_1)$. Since U is C-compatible, the complex $U \otimes_A \mathbf{X}_1$ is exact with

$U \otimes_A M_1 \cong \mathrm{Im}(U \otimes_A P_1^0 \to U \otimes_A C_1^0)$. If $\iota_1 : M_1 \to C_1^0$ and $\iota_2 : M_2 \to C_2^0$ are the inclusions, then $1_U \otimes \iota_1$ is a monomorphism and the following diagram commutes:

$$\begin{array}{ccc}
U \otimes_A M_1 & \xrightarrow{1_U \otimes \iota_1} & U \otimes_A C_1^0 \\
\varphi^M \downarrow & & \downarrow \varphi^{C^0} \\
M_2 & \xrightarrow{\iota_2} & C_2^0
\end{array}$$

By Lemma 1.4.5(1), φ^{C^0} is a monomorphism, then φ^M is also monomorphism. Moreover, for every $i \in \mathbb{N}$, φ^{P^i} and φ^{C^i} are monomorphism by Lemmas 1.4.6 and 1.4.5(1). Then, the following diagram with exact columns

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & U \otimes_A P_1^1 & \longrightarrow & U \otimes_A P_1^0 & \longrightarrow & U \otimes_A C_1^0 & \longrightarrow & U \otimes_A C_1^1 & \longrightarrow & \cdots \\
 & & \varphi^{P^1} \downarrow & & \varphi^{P^0} \downarrow & & \varphi^{C^0} \downarrow & & \varphi^{C^1} \downarrow & & \\
\cdots & \longrightarrow & P_2^1 & \longrightarrow & P_2^0 & \longrightarrow & C_2^0 & \longrightarrow & C_2^1 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & \mathrm{Coker}\varphi^{P^1} & \longrightarrow & \mathrm{Coker}\varphi^{P^O} & \longrightarrow & \mathrm{Coker}\varphi^{C^O} & \longrightarrow & \mathrm{Coker}\varphi^{C^1} & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & & 0 & &
\end{array}$$

is commutative. Since the first row and the second row are exact, we get the exact sequence of B-modules

$$\overline{\mathbf{X}}_2 : \cdots \to \mathrm{Coker}\varphi^{P^1} \to \mathrm{Coker}\varphi^{P^O} \to \mathrm{Coker}\varphi^{C^O} \to \mathrm{Coker}\varphi^{C^1} \to \cdots$$

where $\mathrm{Coker}\varphi^{P^i} \in \mathscr{P}(B)$, $\mathrm{Coker}\varphi^{C_1} \in \mathrm{Add}_B(C_2)$ by Lemmas 1.4.6 and 1.4.5(1), and such that $\mathrm{Coker}\varphi^M = \mathrm{Im}(\mathrm{Coker}\varphi^{P^O} \to \mathrm{Coker}\varphi^{C^O})$.

It remains to see that the two sequences $\mathbf{X}_1$ and $\overline{\mathbf{X}}_2$ are $\mathrm{Hom}_A(-,\mathrm{Add}(C_1))$-exact and $\mathrm{Hom}_B(-,\mathrm{Add}_B(C_2))$-exact, respectively.

Let $X_1 \in \mathrm{Add}_A(C_1)$ and $X_2 \in \mathrm{Add}_B(C_2)$. Then $\mathbf{p}(X_1,0) \in \mathrm{Add}_T(C)$ and $\mathbf{p}(0,X_2) \in \mathrm{Add}_T(C)$ by Lemma 1.4.5(1). Therefore, by using the adjoint pair $(\mathbf{s},\mathbf{r})$, we obtain that

$$\mathrm{Hom}_B(\overline{\mathbf{X}}_2, X_2) \cong \mathrm{Hom}_{A\times B}(\mathbf{s}(\mathbf{X}),(0,X_2)) \cong \mathrm{Hom}_T(\mathbf{X},\mathbf{r}(0,X_2)) \cong \mathrm{Hom}_T(\mathbf{X},\begin{pmatrix} 0 \\ X_2 \end{pmatrix})$$

is exact. Using now the adjointness $(\mathbf{s},\mathbf{r})$ and $(\mathbf{q},\mathbf{h})$, we get that

$$\mathrm{Hom}_T(\mathbf{X},\begin{pmatrix}0\\ U\otimes_A X_1\end{pmatrix})\cong \mathrm{Hom}_B(\overline{\mathbf{X}}_2,U\otimes_A X_1) \text{ and } \mathrm{Hom}_T(\mathbf{X},\begin{pmatrix}X_1\\ 0\end{pmatrix})\cong \mathrm{Hom}_A(\mathbf{X}_1,X_1).$$

Note that $C^i\cong \mathbf{p}(C_1^i,\mathrm{Coker}\varphi^{C^i})\cong \mathbf{p}(C_1^i,0)\oplus\mathbf{p}(0,\mathrm{Coker}\varphi^{C^i})$ by Lemma 1.4.5(1). Hence, $\mathrm{Ext}^1_T(C^i,\begin{pmatrix}0\\ U\otimes_A X_1\end{pmatrix})\cong \mathrm{Ext}^1_B(\mathrm{Coker}\varphi^{C^i},U\otimes_A X_1)=0$ by Lemma 1.4.7. Therefore, if we apply the functor $\mathrm{Hom}_T(\mathbf{X},-)$ to the sequence

$$0\to\begin{pmatrix}0\\ U\otimes_A X_1\end{pmatrix}\to\begin{pmatrix}X_1\\ U\otimes_A X_1\end{pmatrix}\to\begin{pmatrix}X_1\\ 0\end{pmatrix}\to 0,$$

we get the following exact sequence of complexes

$$0\to \mathrm{Hom}_B(\overline{\mathbf{X}}_2,U\otimes_A X_1)\to \mathrm{Hom}_T(\mathbf{X},\begin{pmatrix}X_1\\ U\otimes_A X_1\end{pmatrix})\to \mathrm{Hom}_A(\mathbf{X}_1,X_1)\to 0.$$

Since U is C-compatible, it follows that $\mathrm{Hom}_B(\overline{\mathbf{X}}_2,U\otimes_A X_1)$ is exact and by hypothesis, $\mathrm{Hom}_T(\mathbf{X},\begin{pmatrix}X_1\\ U\otimes_A X_1\end{pmatrix})$ is also exact. Thus, $\mathrm{Hom}_A(\mathbf{X}_1,X_1)$ is exact and the proof is finished. ∎

The following consequence of the above theorem gives the converse of Proposition 2.1.5.

Corollary 2.2.4. *Let $C=\mathbf{p}(C_1,C_2)$ and assume that U is C-compatible. Then C is w-tilting if and only if C_1 and C_2 are w-tilting.*

Proof. It follows by an easy application of Proposition 1.3.5 and Theorem 2.2.3 on the T-modules $C=\mathbf{p}(C_1,C_2)$ and ${}_TT=\mathbf{p}(A,B)$. ∎

One would like to know if every w-tilting T-module has the form $\mathbf{p}(C_1,C_2)$ where C_1 and C_2 are w-tilting. The following example gives a negative answer to this question.

Recall that a ring R is quasi-Frobenius if projective and injective R-modules coincide.

Example 2.2.5. *Let R be a quasi-Frobenius ring and $T(R)=\begin{pmatrix}R & 0\\ R & R\end{pmatrix}$. Consider the exact sequence of T-modules*

$$0\to {}_TT=\begin{pmatrix}R\\ R\oplus R\end{pmatrix}\to\begin{pmatrix}R\oplus R\\ R\oplus R\end{pmatrix}\to\begin{pmatrix}R\\ 0\end{pmatrix}\to 0.$$

By Lemma 1.4.6, $I^0 = \begin{pmatrix} R\oplus R \\ R\oplus R \end{pmatrix}$ *and* $I^1 = \begin{pmatrix} R \\ 0 \end{pmatrix}$ *both are injective* $T(R)$*-modules. Note that* $T(R)$ *is noetherian ([55, Proposition 1.7]) and then we can see that* $C := I^0 \oplus I^1$ *is a w-tilting* $T(R)$*-module but does not have the form* $\boldsymbol{p}(C_1,C_2)$ *where* C_1 *and* C_2 *are w-tilting by Lemma 1.4.5(1) since* $I^1 \in \mathrm{Add}_{T(R)}(C)$ *and* φ^{I^1} *is not a monomorphism.*

As an immediate consequence of Theorem 2.2.3, we have the following.

Corollary 2.2.6. *Let* R *be a ring and* $T(R) = \begin{pmatrix} R & 0 \\ R & R \end{pmatrix}$*. If* $M = \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}_{\varphi^M}$ *and* $C = \boldsymbol{p}(C_1,C_1)$ *are two* $T(R)$*-modules with* C_1 Σ*-self-orthogonal, then the following assertions are equivalent:*

1. *M is* $\mathrm{G_C}$*-projective a* $T(R)$*-module.*
2. M_1 *and* $\mathrm{Coker}\varphi^M$ *are* $\mathrm{G_{C_1}}$*-projective R-modules and* φ^M *is a monomorphism.*
3. M_2 *and* $\mathrm{Coker}\varphi^M$ *are* $\mathrm{G_{C_1}}$*-projective R-modules and* φ^M *is a monomorphism.*

Our aim now is to study special $\mathrm{G_C P}(T)$-precovers in T-Mod. We start with the following result.

Proposition 2.2.7. *Let* $C = \boldsymbol{p}(C_1,C_2)$ *be w-tilting,* U *be* C*-compatible,* $M = \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}_{\varphi^M}$ *and* $G = \begin{pmatrix} G_1 \\ G_2 \end{pmatrix}_{\varphi^G}$ *two* T*-modules with* G $\mathrm{G_C}$*-projective. Then,*

$$f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} : G \longrightarrow M$$

is a special $\mathrm{G_C P}(T)$*-precover if and only if the two assertions hold:*

(i) $G_1 \stackrel{f_1}{\to} M_1$ *is a special* $\mathrm{G_{C_1} P}(A)$*-precover.*

(ii) $G_2 \stackrel{f_2}{\to} M_2$ *is surjective with kernel in* $\mathrm{G_{C_2} P}(B)^{\perp}$*.*

In this case, if $G_2 \in \mathrm{G_{C_2} P}(B)$*, then* $G_2 \stackrel{f_2}{\to} M_2$ *is a special* $\mathrm{G_{C_2} P}(B)$*-precover.*

Proof. First, let $K = \mathrm{Ker} f = \begin{pmatrix} K_1 \\ K_2 \end{pmatrix}_{\varphi^K}$ and note that, since C_1 is w-tilting, $\mathrm{Tor}_1^A(U,H_1) = 0$ for every $H_1 \in \mathrm{G_{C_1} P}(A)$ by Proposition 2.1.4(1).

($\Rightarrow$) Since f is an epimorphism, so are f_1 and f_2.

Let $H_1 \in \mathrm{G_{C_1}P}(A)$ and $H_2 \in \mathrm{G_{C_2}P}(B)$. Then, $\begin{pmatrix} H_1 \\ U\otimes_A H_1 \end{pmatrix}, \begin{pmatrix} 0 \\ H_2 \end{pmatrix} \in \mathrm{G_CP}(T)$ by Theorem 2.2.3.

Now, using Lemma 1.4.7 and the fact that K lies in $\mathrm{G_CP}(R)^{\perp}$, we obtain that

$$\mathrm{Ext}^1_A(H_1,K_1) \cong \mathrm{Ext}^1_T(\begin{pmatrix} H_1 \\ U\otimes_A H_1 \end{pmatrix},K) = 0 \text{ and } \mathrm{Ext}^1_B(H_2,K_2) \cong \mathrm{Ext}^1_T(\begin{pmatrix} 0 \\ H_2 \end{pmatrix},K) = 0.$$

It remains to see that $G_1 \in \mathrm{G_{C_1}P}(A)$, which is true by Theorem 2.2.3 since G is $\mathrm{G_C}$-projective.

($\Leftarrow$) The morphism f is an epimorphism since f_1 and f_2 are. Therefore, we only need to prove that K lies in $\mathrm{G_CP}(R)^{\perp}$.

Let $H \in \mathrm{G_CP}(R)$. By Theorem 2.2.3, we have the short exact sequence of T-modules

$$0 \to \begin{pmatrix} H_1 \\ U\otimes_A H_1 \end{pmatrix} \to H \to \begin{pmatrix} 0 \\ \mathrm{Coker}\varphi^H \end{pmatrix} \to 0$$

where H_1 is $\mathrm{G_{C_1}}$-projective and $\mathrm{Coker}\varphi^H$ is $\mathrm{G_{C_2}}$-projective. Thus, by hypothesis and Lemma 1.4.7 we obtain that

$$\mathrm{Ext}^1_T(\begin{pmatrix} H_1 \\ U\otimes_A H_1 \end{pmatrix},K) \cong \mathrm{Ext}^1_A(H_1,K_1) = 0$$

and

$$\mathrm{Ext}^1_T(\begin{pmatrix} 0 \\ \mathrm{Coker}\varphi^H \end{pmatrix},K) \cong \mathrm{Ext}^1_B(\mathrm{Coker}\varphi^H,K_2) = 0.$$

Then, the exactness of the following sequence

$$\mathrm{Ext}^1_T(\begin{pmatrix} 0 \\ \mathrm{Coker}\varphi^H \end{pmatrix},K) \to \mathrm{Ext}^1_T(H,K) \to \mathrm{Ext}^1_T(\begin{pmatrix} H_1 \\ U\otimes_A H_1 \end{pmatrix},K)$$

implies that $\mathrm{Ext}^1_T(H,K) = 0$. ∎

Theorem 2.2.8. *Let $C = \boldsymbol{p}(C_1,C_2)$ be w-tilting and U be C-compatible. Then, the class $\mathrm{G_CP}(T)$ is special precovering in T-Mod if and only if the classes $\mathrm{G_{C_1}P}(A)$ and $\mathrm{G_{C_2}P}(B)$ are special precovering in A-Mod and B-Mod, respectively.*

Proof. ($\Rightarrow$) Let M_1 be an A-module and $\begin{pmatrix} G_1 \\ G_2 \end{pmatrix}_{\varphi^G} \to \begin{pmatrix} M_1 \\ 0 \end{pmatrix}$ be a special $\mathrm{G_C}$-projective precover in T-Mod. Then, by Proposition 2.2.7, $G_1 \to M_1$ is a special $\mathrm{G_{C_1}P}(A)$-precover in A-Mod.

Let M_2 be a B-module and $\begin{pmatrix} 0 \\ f_2 \end{pmatrix} : \begin{pmatrix} G_1 \\ G_2 \end{pmatrix}_{\varphi^G} \to \begin{pmatrix} 0 \\ M_2 \end{pmatrix}$ be a special $\mathrm{G_CP}(T)$-precover in T-Mod. Following Proposition 2.2.7, $G_1 \to 0$ is a special $\mathrm{G_{C_1}P}(A)$-precover. Then, $\mathrm{Ext}^1_A(\mathrm{G_{C_1}P}(A),G_1) = 0$.

On the other hand, by [17, Proposition 2.8], there exists an exact sequence of A-modules

$$0 \to G_1 \to X_1 \to H_1 \to 0$$

where $X_1 \in \mathrm{Add}_A(C_1)$ and H_1 is G_{C_1}-projective. However, this sequence splits since $\mathrm{Ext}^1_A(H_1, G_1) = 0$, which implies that $G_1 \in \mathrm{Add}_A(C_1)$.

Let $K = \begin{pmatrix} K_1 \\ K_2 \end{pmatrix}_{\varphi^K}$ be the kernel of $\begin{pmatrix} 0 \\ f_2 \end{pmatrix}$. Note that $K_1 = G_1$. Therefore, there exists a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & U \otimes_A G_1 & = & U \otimes_A G_1 & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \varphi^K} & & \downarrow{\scriptstyle \varphi^G} & & \downarrow & & \\
0 & \longrightarrow & K_2 & \longrightarrow & G_2 & \longrightarrow & M_2 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
 & & \mathrm{Coker}\varphi^K & \longrightarrow & \mathrm{Coker}\varphi^G & \longrightarrow & M_2 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Using the Snake Lemma, there exists an exact sequence of B-modules

$$0 \to \mathrm{Coker}\varphi^K \to \mathrm{Coker}\varphi^G \to M_2 \to 0$$

where $\mathrm{Coker}\varphi^G$ is G_{C_2}-projective by Theorem 2.2.3. It remains to see that $\mathrm{Coker}\varphi^K$ lies in $\mathrm{G}_{C_2}\mathrm{P}(B)^{\perp}$.

Let $H_2 \in \mathrm{G}_{C_2}\mathrm{P}(B)$. Then, $\mathrm{Ext}^1_B(H_2, K_2) = 0$ by Proposition 2.2.7 and $\mathrm{Ext}^{i\geq 1}_B(H_2, U \otimes_A G_1) = 0$ by Proposition 2.1.4(2). From the above diagram, φ^K is a monomorphism. So, if we apply the functor $\mathrm{Hom}_B(H_2, -)$ to the short exact sequence

$$0 \to U \otimes_A G_1 \to K_2 \to \mathrm{Coker}\varphi^K \to 0,$$

we get an exact sequence

$$0 = \mathrm{Ext}^1_B(H_2, K_2) \to \mathrm{Ext}^1_B(H_2, \overline{K}_2) \to \mathrm{Ext}^2_B(H_2, U \otimes_A G_1) = 0,$$

which implies that $\mathrm{Ext}^1_B(H_2, \mathrm{Coker}\varphi^K) = 0$.

($\Leftarrow$) Note that the functor $U \otimes_A - : A\text{-Mod} \to B\text{-Mod}$ is $\mathrm{G}_{C_1}\mathrm{P}(A)$-exact in the sense of [66] since $\mathrm{Tor}^A_1(U, \mathrm{G}_{C_1}\mathrm{P}(A)) = 0$ by Proposition 2.1.4. Thus, this direction follows by [66, Theorem 1.1] since $\mathrm{G}_C\mathrm{P}(T) = \{M = \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}_{\varphi^M} \in T\text{-Mod} \mid M_1 \in \mathrm{G}_{C_1}\mathrm{P}(A)$, $\mathrm{Coker}\varphi^M \in \mathrm{G}_{C_2}\mathrm{P}(B)$ and φ^M is a monomorphism$\}$ by Theorem 2.2.3. ■

Corollary 2.2.9. *Let R be a ring, $T(R) = \begin{pmatrix} R & 0 \\ R & R \end{pmatrix}$ and $C = \boldsymbol{p}(C_1, C_1)$ be a w-tilting $T(R)$-module. Then,* $\mathrm{G_C P}(T(R))$ *is a special precovering class if and only if* $\mathrm{G_{C_1} P}(R)$ *is a special precovering class.*

2.3 Relative global Gorenstein dimension

In this section we investigate the $\mathrm{G_C}$-projective dimension of T-modules and the left $\mathrm{G_C}$-projective global dimension of T.

Recall [18] that the $\mathrm{G_C}$**-projective dimension** of an R-module M is defined as

$$\mathrm{G_C}-\mathrm{pd}_R(M) := \mathrm{G_C P}(R) - \mathrm{resdim}_R(M)$$

and the (left) **global** $\mathrm{G_C}$**-projective** of R is defined as

$$\mathrm{G_C} - \mathrm{PD}(R) := \mathrm{G_C P}(R) - \mathrm{glresdim}(R) = \sup\{\mathrm{G_C}-\mathrm{pd}_R(M) |\ M \text{ is an } R\text{-module }\}.$$

Lemma 2.3.1. *Let $C = \boldsymbol{p}(C_1, C_2)$ be w-tilting and U be C-compatible.*

1. $\mathrm{G_{C_2}}-\mathrm{pd_B}(\mathrm{M_2}) = \mathrm{G_C}-\mathrm{pd_T}(\begin{pmatrix} 0 \\ M_2 \end{pmatrix})$.

2. $\mathrm{G_{C_1}}-\mathrm{pd_A}(\mathrm{M_1}) \leq \mathrm{G_C}-\mathrm{pd_T}(\begin{pmatrix} M_1 \\ U \otimes_A M_1 \end{pmatrix})$, *and the equality holds if* $\mathrm{Tor}^A_{i\geq 1}(U, M_1) = 0$.

Proof. 1. Let $n \in \mathbb{N}$ and consider an exact sequence of B-modules

$$0 \to K_2^n \to G_2^{n-1} \to \cdots \to G_2^0 \to M_2 \to 0$$

where each G_2^i is $\mathrm{G_{C_2}}$-projective. Thus, there exists an exact sequence of T-modules

$$0 \to \begin{pmatrix} 0 \\ K_2^n \end{pmatrix} \to \begin{pmatrix} 0 \\ G_2^{n-1} \end{pmatrix} \to \cdots \to \begin{pmatrix} 0 \\ G_2^0 \end{pmatrix} \to \begin{pmatrix} 0 \\ M_2 \end{pmatrix} \to 0$$

where each $\begin{pmatrix} 0 \\ G_2^i \end{pmatrix}$ is $\mathrm{G_C}$-projective by Theorem 2.2.3. Again, by Theorem 2.2.3, $\begin{pmatrix} 0 \\ K_2^n \end{pmatrix}$ is $\mathrm{G_C}$-projective if and only if K_2^n is $\mathrm{G_{C_2}}$-projective which means that $\mathrm{G_C}-\mathrm{pd}_T(\begin{pmatrix} 0 \\ M_2 \end{pmatrix}) \leq n$ if and only if $\mathrm{G_{C_2}}-\mathrm{pd_B}(\mathrm{M_2}) \leq \mathrm{n}$ by [17, Theorem 3.8]. Hence, $\mathrm{G_C}-\mathrm{pd}_T(\begin{pmatrix} 0 \\ M_2 \end{pmatrix}) = \mathrm{G_{C_2}}-\mathrm{pd_B}(\mathrm{M_2})$.

2. We may assume that $n = \mathrm{G_C}-\mathrm{pd}_T(\begin{pmatrix} M_1 \\ U\otimes_A M_1 \end{pmatrix}) < \infty$. By Definition, there exists an exact sequence of T-modules

$$0 \to G^n \to G^{n-1} \to \cdots \to G^0 \to \begin{pmatrix} M_1 \\ U\otimes_A M_1 \end{pmatrix} \to 0$$

where each $G^i = \begin{pmatrix} G_1^i \\ G_2^i \end{pmatrix}_{\varphi^{G^i}}$ is $\mathrm{G_C}$-projective. Thus, there exists an exact sequence of A-modules

$$0 \to G_1^n \to G_1^{n-1} \to \cdots \to G_1^0 \to M_1 \to 0$$

where each G_1^i is $\mathrm{G_{C_1}}$-projective by Theorem 2.2.3. So, $\mathrm{G_{C_1}-pd_A(M_1) \leq n}$.

Conversely, we prove that $\mathrm{G_C}-\mathrm{pd}_T(\begin{pmatrix} M_1 \\ U\otimes_A M_1 \end{pmatrix}) \leq \mathrm{G_{C_1}-pd_A(M_1)}$. We may assume that $m := \mathrm{G_{C_1}-pd_A(M_1)} < \infty$.

The hypothesis means that if

$$\mathbf{X}_1 \ : \ 0 \to K_1^m \to P_1^{m-1} \to \cdots \to P_1^0 \to M_1 \to 0$$

is an exact sequence of A-modules where each P_1^i is projective, then the complex $U\otimes_A \mathbf{X}_1$ is exact. Since C_1 is w-tilting, each P_i is $\mathrm{G_{C_1}}$-projective by [17, Proposition 2.11] and then K^m is $\mathrm{G_{C_1}}$-projective by [17, Theorem 3.8]. Thus, there exists an exact sequence of T-modules

$$0 \to \begin{pmatrix} K_1^m \\ U\otimes_A K_1^m \end{pmatrix} \to \begin{pmatrix} P_1^{m-1} \\ U\otimes_A P_1^{m-1} \end{pmatrix} \to \cdots \to \begin{pmatrix} P_1^0 \\ U\otimes_A P_1^0 \end{pmatrix} \to \begin{pmatrix} M_1 \\ U\otimes_A M_1 \end{pmatrix} \to 0$$

where $\begin{pmatrix} K_1^m \\ U\otimes_A K_1^m \end{pmatrix}$ and all $\begin{pmatrix} P_1^i \\ U\otimes_A P_1^i \end{pmatrix}$ are $\mathrm{G_C}$-projectives by Theorem 2.2.3. Therefore, $\mathrm{G_C}-\mathrm{pd}_T(\begin{pmatrix} M_1 \\ U\otimes_A M_1 \end{pmatrix}) \leq m = \mathrm{G_{C_1}-pd_A(M_1)}$. ■

Given a T-module $C = \mathbf{p}(C_1, C_2)$, we introduce a strong notion of the global $\mathrm{G_{C_2}}$-projective dimension of B as

$$\mathrm{SG_{C_2}} - \mathrm{PD}(B) = \sup\{\mathrm{G_{C_2}-pd_B(U\otimes_A G) \mid G \in G_{C_1}P(A)}\}.$$

When $C = R$, we simply write $\mathrm{G_{C_2}} - \mathrm{PD}(B) = \mathrm{SGPD}(B)$.

This homological invariant is crucial when we estimate the $\mathrm{G_C}$-projective of T-modules and the global $\mathrm{G_C}$-projective dimension of T.

Remark 2.3.2.

1. *Clearly,* $\mathrm{SG_{C_2}} - \mathrm{PD}(B) \leq \mathrm{G_{C_2}} - \mathrm{PD}(B)$.

2. *Note that* $\mathrm{pd}_B(U) = \sup\{\mathrm{pd}_B(U \otimes_A P) \mid {}_AP$ *is projective* $\}$. *Therefore, in the classical case, the strong left global dimension of B is nothing but the projective dimension of* ${}_BU$.

Theorem 2.3.3. *Let $C = \boldsymbol{p}(C_1, C_2)$ be w-tilting, U be C-compatible,* $M = \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}_{\varphi^M}$ *a T-module and* $k := \mathrm{SG_{C_2}-PD}(B) < \infty$ *. Then,*

$$max\{\mathrm{G_{C_1}\!-\!pd_A(M_1), G_{C_2}\!-\!pd_B(M_2) - k)}\}$$
$$\leq \mathrm{G_C}\!-\!\mathrm{pd}_T(M) \leq$$
$$max\{\mathrm{G_{C_1}\!-\!pd_A(M_1) + k + 1, G_{C_2}\!-\!pd_B(M_2)}\}$$

Proof. First of all, note that C_1 and C_2 are w-tilting by Proposition 2.2.4.

Let us first prove that $max\{\mathrm{G_{C_1}\!-\!pd_A(M_1), G_{C_2}\!-\!pd_B(M_2) - k}\} \leq \mathrm{G_C\!-\!pd_T(M)}$. We may assume that $n := \mathrm{G_C}\!-\!\mathrm{pd}_T(M) < \infty$. Then, there exists an exact sequence of T-modules

$$0 \to G^n \to G^{n-1} \to \cdots \to G^0 \to M \to 0$$

where each $G^i = \begin{pmatrix} G_1^i \\ G_2^i \end{pmatrix}_{\varphi^{G^i}}$ is $\mathrm{G_C}$-projective. Thus, there exists an exact sequence of A-modules

$$0 \to G_1^n \to G_1^{n-1} \to \cdots \to G_1^0 \to M_1 \to 0$$

where each G_1^i is $\mathrm{G_{C_1}}$-projective by Theorem 2.2.3. So, $\mathrm{G_{C_1}\!-\!pd_A(M_1) \leq n}$. By Theorem 2.2.3, for each i, there exists an exact sequence of B-modules

$$0 \to U \otimes_A G_1^i \to G_2^i \to \mathrm{Coker}\varphi^{G^i} \to 0$$

where $\mathrm{Coker}\varphi^{G^i}$ is $\mathrm{G_{C_2}}$-projective. Then, $\mathrm{G_{C_2}\!-\!pd_B(G_2^i) = G_{C_2}\!-\!pd_B(U \otimes_A G_1^i) \leq k}$ by [17, Proposition 3.11(1)]. So, using the exact sequence of B-modules

$$0 \to G_2^n \to G_2^{n-1} \to \cdots \to G_2^0 \to M_2 \to 0$$

and [17, Proposition 3.11(4)], we get that $\mathrm{G_{C_2}\!-\!pd(M_2) \leq n + k}$.

Next we prove that $\mathrm{G_C}\!-\!\mathrm{pd}_T(M) \leq \max\{\mathrm{G_{C_1}\!-\!pd_A(M_1) + k + 1, G_{C_2}\!-\!pd_B(M_2)}\}$. We may assume that

$$m := \max\{\mathrm{G_{C_1}\!-\!pd_A(M_1) + k + 1, G_{C_2}\!-\!pd_B(M_2)}\} < \infty.$$

Then, $n_1 := \mathrm{G_{C_1}\!-\!pd_A(M_1)} < \infty$ and $n_2 := \mathrm{G_{C_2}\!-\!pd_B(M_2)} < \infty$. Since $\mathrm{G_{C_1}\!-\!pd_A(M_1) = n_1 \leq m - k - 1}$, there exists an exact sequence of A-modules

$$0 \to G_1^{m-k-1} \to \cdots \to G_1^{n_2-k} \to \cdots \xrightarrow{f_1^1} G_1^0 \xrightarrow{f_1^0} M_1 \to 0$$

where each G_1^i is $\mathrm{G_{C_1}}$-projective. Since C_2 is w-tilting, there exists an exact sequence of B-modules $G_2^0 \overset{g_2^0}{\to} M_2 \to 0$ where G_2^0 is $\mathrm{G_{C_2}}$-projective by [17, Corollary 2.14]. Let $K_1^i = \mathrm{Ker} f_1^{i-1}$ and define the map $f_2^0 : U \otimes_A G_1^0 \oplus G_2^0 \to M_2$ to be $(\varphi^M(1_U \otimes f_1^0)) \oplus g_2^0$. Then, we get an exact sequence of T-modules

$$0 \to \begin{pmatrix} K_1^1 \\ K_2^1 \end{pmatrix}_{\varphi^{K^1}} \to \begin{pmatrix} G_1^0 \\ (U \otimes_A G_1^0) \oplus G_2^0 \end{pmatrix} \overset{\begin{pmatrix} f_1^0 \\ f_2^0 \end{pmatrix}}{\to} M \to 0.$$

Similarly, there exists an exact sequence of B-modules $G_2^1 \overset{g_2^1}{\to} K_2^1 \to 0$ where G_2^1 is $\mathrm{G_{C_2}}$-projective and then we get an exact sequence of T-modules

$$0 \to \begin{pmatrix} K_1^2 \\ K_2^2 \end{pmatrix}_{\varphi^{K^2}} \to \begin{pmatrix} G_1^1 \\ (U \otimes_A G_1^1) \oplus G_2^1 \end{pmatrix} \to \begin{pmatrix} K_1^1 \\ K_2^1 \end{pmatrix}_{\varphi^{K^1}} \to 0.$$

Repeating this process, we get the exact sequence of T-modules

$$0 \to \begin{pmatrix} 0 \\ K_2^{m-k} \end{pmatrix} \to \begin{pmatrix} G_1^{m-k-1} \\ (U \otimes_A G_1^{m-k-1}) \oplus G_2^{m-k-1} \end{pmatrix} \overset{\begin{pmatrix} f_1^{m-k-1} \\ f_2^{m-k-1} \end{pmatrix}}{\longrightarrow}$$

$$\cdots \to \begin{pmatrix} G_1^1 \\ (U \otimes_A G_1^1) \oplus G_2^1 \end{pmatrix} \overset{\begin{pmatrix} f_1^1 \\ f_2^1 \end{pmatrix}}{\longrightarrow} \begin{pmatrix} G_1^0 \\ (U \otimes_A G_1^0) \oplus G_2^0 \end{pmatrix} \overset{\begin{pmatrix} f_1^0 \\ f_2^0 \end{pmatrix}}{\longrightarrow} M \to 0$$

Note that $\mathrm{G_{C_2}}-\mathrm{pd_B}((\mathrm{U} \otimes_\mathrm{A} \mathrm{G_1^i}) \oplus \mathrm{G_2^i}) = \mathrm{G_{C_2}}-\mathrm{pd_B}(\mathrm{U} \otimes_\mathrm{A} \mathrm{G_1^i}) \le \mathrm{k}$, for every $i \in \{0, \cdots, m-k-1\}$. So, by [17, Proposition 3.11(2)] and using the exact sequence

$$0 \to K_2^{m-k} \to (U \otimes_A G_1^{m-k-1}) \oplus G_2^{m-k-1} \overset{f_2^{m-k-1}}{\longrightarrow} \cdots \to (U \otimes_A G_1^0) \oplus G_2^0 \overset{f_2^0}{\to} M_2 \to 0,$$

we get that $\mathrm{G_{C_2}}-\mathrm{pd_B}(\mathrm{K_2^{m-k}}) \le \mathrm{k}$. This means that there exists an exact sequence of B-modules

$$0 \to G_2^m \to \cdots \to G_2^{m-k+1} \to G_2^{m-k} \to K_2^{m-k} \to 0.$$

Thus, there exists an exact sequence of T-modules

$$0 \to \begin{pmatrix} 0 \\ G_2^m \end{pmatrix} \to \cdots \to \begin{pmatrix} 0 \\ G_2^{m-k+1} \end{pmatrix} \to \begin{pmatrix} 0 \\ G_2^{m-k} \end{pmatrix} \to \begin{pmatrix} G_1^{m-k-1} \\ (U \otimes_A G_1^{m-k-1}) \oplus G_2^{m-k-1} \end{pmatrix} \overset{f^{m-k-1}}{\longrightarrow}$$

$$\cdots \to \begin{pmatrix} G_1^1 \\ (U \otimes_A G_1^1) \oplus G_2^1 \end{pmatrix} \overset{f^1}{\to} \begin{pmatrix} G_1^0 \\ (U \otimes_A G_1^0) \oplus G_2^0 \end{pmatrix} \overset{f^0}{\to} M \to 0.$$

By Theorem 2.2.3, all $\begin{pmatrix} G_1^i \\ (U \otimes_A G_1^i) \oplus G_2^i \end{pmatrix}$ and all $\begin{pmatrix} 0 \\ G_2^j \end{pmatrix}$ are $\mathrm{G_C}$-projective. Thus, $\mathrm{G_C}-\mathrm{pd}_T(M) \le m$. ■

The following consequence of Theorem 2.3.3 extends [34, Proposition 2.8(1)] and [92, Theorem 2.7(1)] to the relative setting.

Corollary 2.3.4. *Let $C = \boldsymbol{p}(C_1, C_2)$ be w-tilting, U be C-compatible and $M = \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}_{\varphi^M}$ be a T-module. If $\mathrm{SG_{C_2}-PD}(B) < \infty$, then $\mathrm{G_C{-}pd}_T(M) < \infty$ if and only if $\mathrm{G_{C_1}-pd_A(M_1)} < \infty$ and $\mathrm{G_{C_2}{-}pd_B(M_2)} < \infty$.*

Theorem 2.3.5. *Let $C = \boldsymbol{p}(C_1, C_2)$ be w-tilting and U be C-compatible. Then*

$$\max\{\mathrm{G_{C_1}-PD}(A), \mathrm{G_{C_2}-PD}(B)\}$$

$$\leq \mathrm{G_C - PD}(T) \leq$$

$$\max\{\mathrm{G_{C_1}-PD}(A) + \mathrm{SG_{C_2}-PD}(B) + 1, \mathrm{G_{C_2}-PD}(B)\}.$$

Proof. We prove first that $\max\{\mathrm{G_{C_1}-PD}(A), \mathrm{G_{C_2}-PD}(B)\} \leq \mathrm{G_C - PD}(T)$. We may assume that $n := \mathrm{G_C - PD}(T) < \infty$.

Let M_1 be an A-module and M_2 be a B-module. Since $\mathrm{G_C{-}pd}_T(\begin{pmatrix} M_1 \\ U \otimes_A M_2 \end{pmatrix}) \leq n$ and $\mathrm{G_C{-}pd}_T(\begin{pmatrix} 0 \\ M_2 \end{pmatrix}) \leq n$, $\mathrm{G_{C_1}-pd_A(M_1) \leq n}$ and $\mathrm{G_{C_2}{-}pd_Bf(M_2) \leq n}$ by Lemma 2.3.1. Thus, $\mathrm{G_{C_1}-PD}(A) \leq n$ and $\mathrm{G_{C_2}-PD}(B) \leq n$.

Next, we prove that

$$\mathrm{G_C - PD}(T) \leq \max\{\mathrm{G_{C_1}-PD}(A) + 1 + \mathrm{SG_{C_2}-PD}(B), \mathrm{G_{C_2}-PD}(B)\}.$$

We may assume that

$$m := \max\{\mathrm{G_{C_1}-PD}(A) + 1 + \mathrm{SG_{C_2}-PD}(B), \mathrm{G_{C_2}-PD}(B)\} < \infty.$$

Then, $n_1 := \mathrm{G_{C_1}-PD}(A) < \infty$ and $k := \mathrm{SG_{C_2}-PD}(B) \leq n_2 := \mathrm{G_{C_2}-PD}(B) < \infty$.

Finally, if $M = \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}_{\varphi^M}$ is a T-module, then $\mathrm{G_C{-}pd}_T(M) \leq \max\{n_1 + k + 1, n_2\} \leq m$ by Theorem 2.3.3 ■

Corollary 2.3.6. *Let $C = \boldsymbol{p}(C_1, C_2)$ be w-tilting and U be C-compatible. Then,*
$\mathrm{G_C - PD}(T) < \infty$ if and only if $\mathrm{G_{C_1}-PD}(A) < \infty$ and $\mathrm{G_{C_2}-PD}(B) < \infty$.

Recall that a ring R is called left Gorenstein regular if the category R-Mod is Gorenstein ([34, Definition 2.1] and [35, Definition 2.18]).

On the other hand, we know by [20, Theorem 1.1], that the following equality holds:

$$\sup\{\mathrm{Gpd}_R(M) \mid M \in R\text{-Mod}\} = \sup\{\mathrm{Gid}_R(M) \mid M \in R\text{-Mod}\}.$$

and this common value is call the (left) global Gorenstein dimension of R, denoted by $\text{l.Ggldim}(R)$. As a consequence of [35, Theorem 2.28], a ring R is left Gorenstein regular if and only if the global Gorenstein dimension of R is finite.

Enochs, Izurdiaga and Torrecillas, characterized in [34, Theorem 3.1] when T is left Gorenstein regular under the conditions that ${}_BU$ has finite projective dimension and U_A has finite flat dimension. As a direct consequence of Corollary 2.3.6, we refine this result.

Corollary 2.3.7. *Assume that U is compatible. Then, T is left Gorenstein regular if and only if so are A and B.*

There are some cases when the estimate in Theorem 2.3.5 becomes an exact formula, which computes the G_C-projective global dimension of T.

Corollary 2.3.8. *Let $C = \boldsymbol{p}(C_1, C_2)$ be w-tilting. Assume that $U = 0$. Then,*

$$\text{G}_\text{C} - \text{PD}(T) = \max\{\text{G}_{\text{C}_1} - \text{PD}(A), \text{G}_{\text{C}_2} - \text{PD}(B)\}.$$

In particular,

$$\text{GPD}(T) = \max\{\text{GPD}(A), \text{GPD}(B)\}.$$

Proof. Using a similar argument as the one in the proof of Theorems 2.3.3 and 2.3.5, we can prove this statement. We only need to notice that if $U = 0$, then a T-module $M = \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}_{\varphi^M}$ is G_C-projective if and only if M_1 is G_{C_1}-projective and M_2 is G_{C_2}-projective (since φ^M is always a monomorphism and $M_2 = \text{Coker}\varphi^M$) by Theorem 2.2.3. ■

Recall that an injective cogenerator E in R-Mod is said to be strong if any R-module embeds in a direct sum of copies of E. When C is w-tilting and R is left noetherian, it follows by [18, Corollary 2.3] that C is a strong injective cogenerator if and only if $\text{G}_\text{C} - \text{PD}(R) = 0$.

Corollary 2.3.9. *Let $C = \boldsymbol{p}(C_1, C_2)$ be w-tilting and $U \neq 0$ be C-compatible. If A is left noetherian and ${}_AC_1$ is a strong injective cogenerator, then*

$$\text{G}_\text{C} - \text{PD}(T) = \max\{\text{SG}_{\text{C}_2} - \text{PD}(B) + 1, \text{G}_{\text{C}_2} - \text{PD}(B)\}.$$

In particular, if A is quasi-Frobenius and U is compatible, then

$$\text{GPD}(T) = \max\{\text{SGPD}(B) + 1, \text{GPD}(B)\}.$$

Proof. Note first that $\text{G}_{\text{C}_1} - \text{PD}(A) = 0$ by [18, Corollary 2.3], and by Theorem 2.2.3 $\begin{pmatrix} A \\ 0 \end{pmatrix}$ is not G_C-projective since $U \neq 0$. Hence, $\text{G}_{\text{C}_2} - \text{PD}(B) \geq \text{G}_\text{C} - \text{pd}_T(\begin{pmatrix} A \\ 0 \end{pmatrix}) \geq 1$.

Using now Theorem 2.3.5, we have the inequality

$$\mathrm{G_{C_2}} - \mathrm{PD}(B) \leq \mathrm{G_C} - \mathrm{PD}(T) \leq \max\{\mathrm{SG_{C_2}} - \mathrm{PD}(B) + 1, \mathrm{G_{C_2}} - \mathrm{PD}(B)\}.$$

Therefore, the case $\mathrm{SG_{C_2}} - \mathrm{PD}(B) + 1 \leq \mathrm{G_{C_2}} - \mathrm{PD}(B)$ is clear and we only need to prove the result when $\mathrm{SG_{C_2}} - \mathrm{PD}(B) + 1 > n := \mathrm{G_{C_2}} - \mathrm{PD}(B)$.

Since $\mathrm{G_{C_2}} - \mathrm{pd_B}(\mathrm{U} \otimes_\mathrm{A} \mathrm{G}) \leq \mathrm{G_{C_2}} - \mathrm{PD(B)} = \mathrm{n}$ for any $G \in \mathrm{G_{C_1}P}(A)$, $\mathrm{SG_{C_2}} - \mathrm{PD}(B) = n$. Let G_1 be a $\mathrm{G_{C_1}}$-projective A-module with $\mathrm{G_{C_2}} - \mathrm{pd}(\mathrm{U} \otimes_\mathrm{A} \mathrm{G_1}) = \mathrm{n}$ and consider the following short exact sequence

$$0 \to \begin{pmatrix} 0 \\ U \otimes_A G_1 \end{pmatrix} \to \begin{pmatrix} G_1 \\ U \otimes_A G_1 \end{pmatrix} \to \begin{pmatrix} G_1 \\ 0 \end{pmatrix} \to 0.$$

By Theorem 2.2.3 $\begin{pmatrix} G_1 \\ U \otimes_A G_1 \end{pmatrix}$, is $\mathrm{G_C}$-projective and by Lemma 2.3.1

$$\mathrm{G_C} - \mathrm{pd}_T(\begin{pmatrix} 0 \\ U \otimes_A G_1 \end{pmatrix}) = \mathrm{G_{C_2}} - \mathrm{pd_B}(\mathrm{U} \otimes_\mathrm{A} \mathrm{G}) = \mathrm{n}.$$

Thus, by [17, Proposition 3.11(4)]

$$\mathrm{G_C} - \mathrm{pd}_T(\begin{pmatrix} G_1 \\ 0 \end{pmatrix}) = \mathrm{G_C} - \mathrm{pd}_T(\begin{pmatrix} 0 \\ U \otimes_A G_1 \end{pmatrix}) + 1 = n + 1.$$

This shows that $\mathrm{G_C} - \mathrm{PD}(T) = \mathrm{SG_{C_2}} - \mathrm{PD}(B) + 1$.

The last equality follows by the first equality and [20, Proposition 2.6]. ∎

We shall say that a ring R is left n-**Gorenstein regular** if $n = \mathrm{Ggldim}(R) < \infty$.

Corollary 2.3.10. *Let R be a ring, $T(R) = \begin{pmatrix} R & 0 \\ R & R \end{pmatrix}$ and $C = \boldsymbol{p}(C_1, C_1)$ where C_1 is w-tilting. Then,*

$$\mathrm{G_C} - \mathrm{PD}(T(R)) = \mathrm{G_{C_1}} - \mathrm{PD}(R) + 1.$$

In particular,

$$\mathrm{GPD}(T(R)) = \mathrm{GPD}(R) + 1.$$

Consequently, given an integer $n \geq 0$, $T(R)$ is left $(n+1)$-Gorenstein regular if and only if R left n-Gorenstein regular.

Proof. Note first that C is a w-tilting $T(R)$-module, R is a C-compatible (R,R)-bimodule and $S\mathrm{G_{C_1}} - \mathrm{PD}(R) = 0$. Therefore, by Theorem 2.3.5,

$$\mathrm{G_{C_1}} - \mathrm{PD}(R) \leq \mathrm{G_C} - \mathrm{PD}(T(R)) \leq \mathrm{G_{C_1}} - \mathrm{PD}(R) + 1.$$

The case $\mathrm{G_{C_1}} - \mathrm{PD}(R) = \infty$ is clear.

Assume that $n := \mathrm{G_{C_1}} - \mathrm{PD}(R) < \infty$. Then, there exists an R-module M with $\mathrm{G_{C_1}}-\mathrm{pd_R(M)} = \mathrm{n}$ and $\mathrm{Ext}^n_R(M,X) \neq 0$ for some $X \in \mathrm{Add}_R(C_1)$ by [17, Theorem 3.8]. If we apply the functor $\mathrm{Hom}_{T(R)}(-,\begin{pmatrix}0\\X\end{pmatrix})$ to the exact sequence of $T(R)$-modules

$$0 \to \begin{pmatrix}0\\M\end{pmatrix} \to \begin{pmatrix}M\\M\end{pmatrix}_{1_M} \to \begin{pmatrix}M\\0\end{pmatrix} \to 0,$$

we get an exact sequence

$$\mathrm{Ext}^n_{T(R)}((\begin{smallmatrix}M\\M\end{smallmatrix}),(\begin{smallmatrix}0\\X\end{smallmatrix})) \to \mathrm{Ext}^n_{T(R)}((\begin{smallmatrix}0\\M\end{smallmatrix}),(\begin{smallmatrix}0\\X\end{smallmatrix})) \to \mathrm{Ext}^{n+1}_{T(R)}((\begin{smallmatrix}M\\0\end{smallmatrix}),(\begin{smallmatrix}0\\X\end{smallmatrix})) \to \mathrm{Ext}^{n+1}_{T(R)}((\begin{smallmatrix}M\\M\end{smallmatrix}),(\begin{smallmatrix}0\\X\end{smallmatrix})).$$

By Lemma 1.4.7, $\mathrm{Ext}^{i\geq 1}_{T(R)}(\begin{pmatrix}M\\M\end{pmatrix},\begin{pmatrix}0\\X\end{pmatrix}) \cong \mathrm{Ext}^{i\geq 1}_R(M,0) = 0$. Using Lemma 1.4.7 again and the above long exact sequence,

$$\mathrm{Ext}^{n+1}_{T(R)}(\begin{pmatrix}M\\0\end{pmatrix},\begin{pmatrix}0\\X\end{pmatrix}) \cong \mathrm{Ext}^n_{T(R)}(\begin{pmatrix}0\\M\end{pmatrix},\begin{pmatrix}0\\X\end{pmatrix}) \cong \mathrm{Ext}^n_R(M,X) \neq 0.$$

Since $\begin{pmatrix}0\\X\end{pmatrix} \in \mathrm{Add}_{T(R)}(C)$ by Lemma 1.4.5(1), it follows that $n < \mathrm{G_C}-\mathrm{pd}_{T(R)}(\begin{pmatrix}M\\0\end{pmatrix})$ by [17, Theorem 3.8]. But, $\mathrm{G_C}-\mathrm{pd}_{T(R)}(\begin{pmatrix}M\\0\end{pmatrix}) \leq \mathrm{G_C} - \mathrm{PD}(T(R)) \leq n+1$. Therefore, $\mathrm{G_C}-\mathrm{pd}_{T(R)}(\begin{pmatrix}M\\0\end{pmatrix}) = n+1$, which means that $\mathrm{G_C} - \mathrm{PD}(T(R)) = n+1$. ■

CHAPTER 3

RELATIVE GORENSTEIN FLAT MODULES AND DIMENSIONS

In this chapter, a new concept of relative Gorenstein flat modules is introduced. We give a survey of their behavior in terms of structural and dimension properties. We also tackle the classical problem of the stability of the Gorenstein condition under iterations of its construction. We conclude this chapter by introducing and studying the weak Gorenstein global dimension of a ring R with respect to an R-module C. We provide several characterizations of when this homological invariant is bounded. As an application, we prove that the weak Gorenstein global dimension of R relative to a semidualizing (R,S)-bimodule C can be computed either by the $\mathrm{G_C}$-flat dimension of the left R-modules or right S-modules, just like the (absolute) weak global dimension. As a consequence, a new argument for solving Bennis' conjecture is obtained.

Throughout this chapter, S will be, unless otherwise stated, the endomorphism ring of C, $S = \mathrm{End}_R(C)$.

3.1 Relative flat modules

In this section we introduce a class of relatively flat modules, $\mathscr{F}_\mathrm{C}$-flat modules, and check, besides its links with the class of flat modules, what homological properties it satisfies.

Establishing the basic properties of $\mathscr{F}_\mathrm{C}(R)$ will be fundamental for the development of this article since $\mathscr{F}_\mathrm{C}(R)$ will be the class on which $\mathrm{G_C}$-flat modules will be built in later sections.

Definition 3.1.1. *An R-module M is said to be **$\mathscr{F}_\mathrm{C}$-flat** if M^+ belongs to the class* $\mathrm{Prod}_R(C^+)$*, and we will denote the class of all $\mathscr{F}_\mathrm{C}$-flat modules as $\mathscr{F}_\mathrm{C}(R)$.*

It is clear that $\mathscr{F}_C(R) = \mathscr{F}(R)$ when $C = R$. So, flat modules are particular cases of $\mathscr{F}_C$-flat modules.

Under the hypothesis that ${}_RC$ is finitely presented, the classes $\mathscr{F}_C(R)$ and $\mathrm{Prod}_R(C^+)$ can be nicely described from the classes of flat and injective modules, respectively.

Proposition 3.1.2. *Suppose that ${}_RC$ is finitely presented. The following assertions hold:*

1. $\mathrm{Prod}_R(C^+) = \mathrm{Hom}_S(C, \mathscr{I}(S))$.
2. $\mathscr{F}_C(R) = C \otimes_S \mathscr{F}(S)$.

Proof. (1) This equality follows from Lemma 1.3.14(2), since any finitely presented module is self-cosmall.

(2) Given any $C \otimes_S F \in C \otimes_S \mathscr{F}(S)$, we have $(C \otimes_S F)^+ \cong \mathrm{Hom}_S(C, F^+) \in \mathrm{Prod}_R(C^+)$ by (1) since F^+ is an injective right S-module, so $C \otimes_S F \in \mathscr{F}_C(R)$.

Conversely, for any R-module M we have a natural homomorphism

$$\begin{array}{rrcl} \nu_M : & C \otimes_S \mathrm{Hom}_R(C,M) & \to & M \\ & c \otimes f & \mapsto & f(c) \end{array}$$

and the natural map $\tau : \mathrm{Hom}_R(C,M)^+ \to M^+ \otimes_R C$ is an isomorphism of right S-modules by [37, Theorem 3.2.11] since ${}_RC$ is finitely presented.

If $M \in \mathscr{F}_C(R)$, then $M^+ \in \mathrm{Prod}_R(C^+)$, so by (1) there exists an injective right S-module I such that $M^+ \cong \mathrm{Hom}_S(C,I)$. But then $\mathrm{Hom}_R(C,M)^+ \cong M^+ \otimes_R C \cong I_S$ is an injective right S-module so $\mathrm{Hom}_R(C,M)$ is a flat left S-module. Therefore, if ν_M were an isomorphism of R-modules for every $M \in \mathscr{F}_C(R)$, we would have $M \in C \otimes_S \mathscr{F}(S)$ for every $M \in \mathscr{F}_C(R)$, as desired.

Let us then assume that $M \in \mathscr{F}_C(R)$.

It follows by Lemma 1.3.12 that the natural homomorphism

$$\begin{array}{l} \mu_{M^+} : M^+ \longrightarrow \mathrm{Hom}_S(C, M^+ \otimes_R C) \\ f \mapsto \mu_{M^+}(f) : c \mapsto \mu_{M^+}(f)(c) = f \otimes c \end{array}$$

is an isomorphism of right R-modules for every $M \in \mathscr{F}_C(R)$. Then, if we call $\gamma : (C \otimes_S \mathrm{Hom}_R(C,M))^+ \to \mathrm{Hom}_S(C, \mathrm{Hom}_R(C,M)^+)$ the isomorphism of right R-modules given by the adjunction, we have a commutative diagram

$$\begin{array}{ccc} M^+ & \xrightarrow[\cong]{\mu_{M^+}} & \mathrm{Hom}_S(C, M^+ \otimes_R C) \\ {\scriptstyle (\nu_M)^+} \downarrow & & \cong \downarrow {\scriptstyle \mathrm{Hom}_R(C,\tau)} \\ (C \otimes_S \mathrm{Hom}_R(C,M))^+ & \xrightarrow[\gamma]{\cong} & \mathrm{Hom}_S(C, \mathrm{Hom}_R(C,M)^+) \end{array}$$

which shows that $(\nu_M)^+$ is an isomorphism of right R-modules and so that ν_M is an isomorphism of R-modules. ■

Now we want to know the general behavior of $\mathscr{F}_C(R)$. Let us start with the following definition.

Definition 3.1.3. *A left R-module M is said to be* $\prod$-Tor-***orthogonal*** *provided that* $\operatorname{Tor}^R_{i\geq 1}\left((M^+)^I, M\right) = 0$ *for every set I.*

From the canonical isomorphism of abelian groups

$$\operatorname{Ext}_R^{i\geq 1}((M^+)^I, M^+) \cong \operatorname{Tor}^R_{i\geq 1}((M^+)^I, M)^+$$

(see [37, Theorem 3.2.1]), it immediately follows that the module M is $\prod$-Tor-orthogonal if and only if M^+ is $\prod$-self-orthogonal.

We state that as a proposition.

Proposition 3.1.4. *A left R-module M is* $\prod$-Tor-*orthogonal if and only if the right R-module* M^+ *is* $\prod$-*self-orthogonal.*

The following is a $\prod$-Tor-orthogonality test. It will be needed later.

Proposition 3.1.5. *The module C is* $\prod$-Tor-*orthogonal if and only if* $\operatorname{Tor}^R_{i\geq 1}(X,Y) = 0$ *for every* $X \in \operatorname{Prod}_R(C^+)$ *and every* $Y \in \mathscr{F}_C(R)$.

Proof. Of course, if C is $\prod$-Tor-orthogonal then $\operatorname{Tor}^R_{\geq 1}(X,C) = 0$ for any $X \in \operatorname{Prod}_R(C^+)$, and if $Y \in \mathscr{F}_C(R)$, then there are some $Z \in$ Mod-R and some set I such that $Y^+ \oplus Z = (C^+)^I$. Thus,

$$\operatorname{Tor}^R_i(X,Y)^+ \oplus \operatorname{Ext}^i_R(X,Z) \cong \operatorname{Ext}^i_R(X,Y^+) \oplus \operatorname{Ext}^i_R(X,Z) \cong \operatorname{Ext}^i_R\left(X,(C^+)^I\right)$$

$$\cong \operatorname{Ext}^i_R\left(X,C^+\right)^I \cong \left(\operatorname{Tor}^R_i(X,C)^+\right)^I = 0.$$

The converse is clear. ∎

The next result is inspired in the ideas of [21].

Proposition 3.1.6. *The following assertions hold:*

1. $\mathscr{F}_C(R) = \operatorname{Add}_R(\mathscr{F}_C(R))$. *Consequently,* $(\mathscr{F}_C(R), \mathscr{I}_{C^+}(R))$ *is a coproduct-closed left duality pair. In particular* $\mathscr{F}_C(R)$ *is covering.*

2. $\mathscr{F}_C(R)$ *is closed under pure submodules, pure quotients, and pure extensions.*

3. *If C is* $\prod$-Tor-*orthogonal then* $\mathscr{F}_C(R)$ *is closed under extensions.*

4. $\mathscr{F}_C(R) = \varinjlim \mathscr{F}_C(R)$, *that is,* $\mathscr{F}_C(R)$ *is closed under direct limits.*

Proof. That $(\mathscr{F}_{\mathrm{C}}(R), \mathrm{Prod}_R(C^+))$ is a coproduct-closed left duality pair is clear. The fact that $\mathscr{F}_{\mathrm{C}}(R)$ is covering and assertion *2* follow from Theorem 1.5.19.

3. Let $M,\ L \in \mathscr{F}_{\mathrm{C}}(R)$. If $0 \to M \to N \to L \to 0$ is exact then

$$0 \to L^+ \to N^+ \to M^+ \to 0$$

is also exact, and it splits since $\mathrm{Ext}^1_R(M^+, L^+) \cong \mathrm{Tor}^R_1(M^+, L)^+$ (see [37, Thorem 3.2.1]) and C is $\prod$-Tor-orthogonal. But, $L^+, M^+ \in \mathrm{Prod}_R(C^+)$, so finally $N^+ \in \mathrm{Prod}_R(C^+)$ and then $N \in \mathscr{F}_{\mathrm{C}}(R)$.

4. Let $\{M_i;\ i \in I\}$ be a directed system of R-modules in $\mathscr{F}_{\mathrm{C}}(R)$. By *1*, $\oplus_i M_i \in \mathscr{F}_{\mathrm{C}}(R)$, and the canonical map $\oplus_i M_i \to \varinjlim_i M_i$ is a pure epimorphism, so by *2*, it follows that $\varinjlim M_i \in \mathscr{F}_{\mathrm{C}}(R)$. ■

We finish the section by finding conditions for $\mathscr{F}_{\mathrm{C}}(R)$ to be preenveloping.

Theorem 3.1.7. *The following assertions are equivalent:*

1. *$\mathscr{F}_{\mathrm{C}}(R)$ is preenveloping.*
2. *$\mathscr{F}_{\mathrm{C}}(R)$ is closed under direct products.*
3. *$\forall_R N$, $N \in \mathscr{F}_{\mathrm{C}}(R)$ if and only if $N^{++} \in \mathscr{F}_{\mathrm{C}}(R)$.*
4. *$\forall M_R$, $M \in \mathrm{Prod}_R(C^+) \Rightarrow M^+ \in \mathscr{F}_{\mathrm{C}}(R)$.*

If $_RC$ is finitely presented, the above assertions are equivalent to

5. *S is right coherent and C_S is finitely presented.*

Proof. *1.*$\Leftrightarrow$*2.* It is well known that every preenveloping class of modules is closed under arbitrary direct products, so we only need to check *2* $\Rightarrow$ *1*. But if $\mathscr{F}_{\mathrm{C}}(R)$ is closed under direct products we can apply [77, Corollary 3.5(c)] since $\mathscr{F}_{\mathrm{C}}(R)$ is closed under pure submodules by Proposition 3.1.6, so we are done.

2. $\Rightarrow$ 3. $\mathscr{F}_{\mathrm{C}}(R)$ is closed under direct limits and pure submodules, so *2* says that it is a definable subcategory of R-Mod, (see [74, pg. 1390]), and so, if we call $\overline{\mathrm{Prod}}(\mathscr{F}_{\mathrm{C}}(R)^+)$ the closure of $\mathrm{Prod}(\mathscr{F}_{\mathrm{C}}(R)^+)$ under pure submodules, we get by [74, Corollary 4.6] that $\big(\mathscr{F}_{\mathrm{C}}(R), \overline{\mathrm{Prod}}(\mathscr{F}_{\mathrm{C}}(R)^+)\big)$ is a left duality pair (given by the Pontryagin duality) and that $\big(\overline{\mathrm{Prod}}(\mathscr{F}_{\mathrm{C}}(R)^+), \mathscr{F}_{\mathrm{C}}(R)\big)$ is a right duality pair. This gives the result.

3. $\Rightarrow$ 2. Let $(V_i)_{i\in I}$ be a family of $\mathscr{F}_{\mathrm{C}}$-flat R-modules. Note that the class $\mathscr{F}_{\mathrm{C}}(R)$ is closed under direct sums and pure submodules. Then, $\oplus_{i\in I} V_i \in \mathscr{F}_{\mathrm{C}}(R)$ and by hypothesis, $(\prod_{i\in I} V_i^+)^+ = (\oplus_{i\in I} V_i)^{++}$ is $\mathscr{F}_{\mathrm{C}}$-flat. But $\oplus_{i\in I} V_i^+$ is a pure submodule of $\prod_{i\in I} V_i^+$ by [25, Lemma 1(1)], and hence $(\prod_{i\in I} V_i^+)^+ \to (\oplus_{i\in I} V_i^+)^+ \to 0$ splits. This implies

that $\prod_{i\in I} V_i^{++} \cong (\oplus_{i\in I} V_i^{+})^{+}$ is $\mathscr{F}_{\mathrm{C}}$-flat. But $\prod_{i\in I} V_i$ is a pure submodule of $\prod_{i\in I} V_i^{++}$ by [25, Lemma 1(2)]. Hence, $\prod_{i\in I} V_i$ is $\mathscr{F}_{\mathrm{C}}$-flat.

3. $\Rightarrow$ 4. Let $M \in \mathrm{Prod}_R(C^+)$ and $M \oplus L = (C^+)^I = (C^{(I)})^+$ for some R-module L and some set I. Then, $M^+ \oplus L^+ = (C^{(I)})^{++}$.

Since the class $\mathscr{F}_{\mathrm{C}}(R)$ is closed under direct sums and summands, and by hypothesis, $M^+ \in \mathscr{F}_{\mathrm{C}}(R)$.

4. $\Rightarrow$ 3. If $N \in \mathscr{F}_{\mathrm{C}}(R)$ then $N^+ \in \mathrm{Prod}_R(C^+)$ by definition, and then $N^{++} \in \mathscr{F}_{\mathrm{C}}(R)$ by the hypothesis.

Conversely, N is a pure submodule of N^{++} and $\mathscr{F}_{\mathrm{C}}(R)$ is closed under pure submodules by Proposition 3.1.6(2), so $N \in \mathscr{F}_C(R)$.

For the last equivalence, let us assume that ${}_RC$ is finitely presented.

2. $\Rightarrow$ 5. By [37, Theorem 3.2.24], to prove that S is right coherent, it suffices to prove that S^I is a flat left S-module for any set I. Since ${}_RC$ is finitely presented, it follows by [37, Theorem 3.2.7] that the canonical morphism $\tau_X : C \otimes_S \mathrm{Hom}_R(C,X) \to X$ is an isomorphism for every $\mathscr{F}_{\mathrm{C}}$-flat R-module $X = C \otimes_S F$. In particular, the map τ_{C^I} is an isomorphism, and so is the canonical morphism $\tau : C \otimes_S S^I \to C^I$. Hence, C_S is finitely presented by [37, Theorem 3.2.22]. Moreover, there are natural isomorphisms $S^I \cong \mathrm{Hom}_R(C,C)^I \cong \mathrm{Hom}_R(C,C^I)$. But, by hypothesis $C^I = C \otimes_S G$ is $\mathscr{F}_{\mathrm{C}}$-flat. Therefore, $S^I \cong {}_SG$ is a flat S-module.

5. $\Leftarrow$ 2. Let $V_i = C \otimes_S F_i$, where each F_i is a flat S-module. Since S is right coherent and C_S is finitely presented, $\prod_{i\in I} F_i$ is a flat left S-module by [37, Theorem 3.2.24] and $\prod_{i\in I} V_i = \prod_{i\in I}(C \otimes_S F_i) \cong C \otimes_S \prod_{i\in I} F_i$ is $\mathscr{F}_{\mathrm{C}}$-flat. ■

3.2 Relative Gorenstein flat modules

In this section we introduce and study relative Gorenstein flat modules. We are interested in discovering the main homological properties of this new class of modules, and check its links with other known classes of relative Gorenstein modules ($\mathrm{G_C}$-projective modules, $\mathrm{G_{C^+}}$-injective modules, etc.).

As mentioned above, the development of this study will be based on modules C with less restrictive conditions than semidualizing modules, which we will call w^+-tilting modules. We will see that indeed this class properly generalizes w-tilting (and so semidualizing) modules (Proposition 3.2.3 and Example 3.2.2).

We start by introducing the type of modules over which the main properties of $\mathrm{G_C}$-flat modules will be proved: w^+-tilting modules.

Definition 3.2.1. *The left R-module C is said to be w^+**-tilting** if it satisfies the following two properties:*

1. *C is* $\prod$-Tor-*orthogonal.*

2. *There exists an exact and* $(\mathrm{Prod}_R(C^+)\otimes_R -)$*-exact sequence of R-modules*

$$0\to R\to C^0\to C^1\to\cdots$$

with $C^i\in\mathscr{F}_\mathrm{C}(R)$ *for every* $i\in\mathbb{N}$.

Now we provide a way to build w^+-tilting modules which, as a consequence, will lead to find an example of a w^+-tilting module which is not w-tilting (and so not semidualizing), showing this way, as promised before, that the concept of a w^+-tilting module properly generalizes those of a w-tilting and a semidualizing module.

Example 3.2.2. *Let R be a left coherent and non-noetherian ring, and find an FP-injective coresolution*

$$\boldsymbol{X}:\ 0\to R\overset{\alpha^0}{\to}F^0\overset{\alpha^1}{\to}F^1\overset{\alpha^2}{\to}\cdots$$

of ${}_RR$*. Then, the module* $C=\oplus_{i\geq 0}F^i$ *is FP-injective, so since R is left coherent we have that* C^+ *is flat (see for instance [25, Theorem 1]), and so that* $(C^+)^I$ *is flat for every index set I. This means that every module in* $\mathrm{Prod}_R(C^+)$ *is flat and so our complex* $\boldsymbol{X}$ *is* $(\mathrm{Prod}_R(C^+)\otimes_R -)$*-exact. Moreover, by construction, every* F^i *belongs to* $\mathscr{F}_\mathrm{C}(R)$*. But* $(C^+)^I$ *being flat also implies* $\mathrm{Tor}_i^R\left((C^+)^I,C\right)=0$ *for every* $i\geq 1$*, that is, C is* $\prod$-Tor*-orthogonal. Therefore, C is* w^+*-tilting and we have just given a way to produce* w^+*-tilting modules.*

Moreover, R being left coherent and non-noetherian guarantees the existence of an FP-injective and non-injective R-module N, so there is a non-split exact sequence

$$0\to N\to E(N)\to L\to 0$$

(so $\mathrm{Ext}_R^1(L,N)\neq 0$*) with L FP-injective. Therefore, the module* $N\oplus L$ *is FP-injective and satisfies* $\mathrm{Ext}_R^1(N\oplus L,N\oplus L)\neq 0$*, and the sequence*

$$0\to R\xrightarrow{\beta^0}F^0\oplus N\oplus L\xrightarrow{\beta^1}F^1\oplus N\oplus L\oplus N\oplus L\xrightarrow{\beta^2}F^2\oplus N\oplus L\oplus N\oplus L\xrightarrow{\beta^3}\cdots$$

given by $\beta^0(r)=(\alpha^0(r),0,0)$, $\beta^1(x,n,l)=(\alpha^1(x),n,l,0,0)$*, and, for every* $i\geq 2$,

$$\beta^i(x,n,l,n',l')=\begin{cases}(\alpha^i(x),0,0,n',l') & \text{if } i \text{ is even}\\ (\alpha^i(x),n,l,0,0) & \text{if } i \text{ is odd}\end{cases}$$

is indeed an FP-injective coresolution of R.

Hence, by the comments above, the module

$$C=(F^0\oplus N\oplus L)\oplus\left(\oplus_{i\geq 1}\left(F^i\oplus N\oplus L\oplus N\oplus L\right)\right)$$

is w^+*-tilting. However, C cannot be w-tilting, and so it cannot be semidualizing, since* $\mathrm{Ext}_R^1(C,C)\neq 0$.

Let us now show the relation between w-tilting and w^+-tilting modules.

Proposition 3.2.3. *Assume that ${}_RC$ has a degreewise finite projective resolution. Then, ${}_RC$ is $\prod$-*Tor*-orthogonal if and if it is Σ-self-orthogonal.*

Moreover, if ${}_RC$ w-tilting, then it is w^+-tilting.

In particular, every semidualizing (R,S)-bimodule is w^+-tilting both as a left R-module and as a right S-module.

Proof. Following [53, Lemma 1.2.11(d)], we have the natural isomorphism

$$\text{(I)} \quad \mathrm{Tor}_i^R((C^{(I)})^+, C) \cong \mathrm{Ext}_R^i(C, C^{(I)})^+$$

for any index set I. Thus, ${}_RC$ is $\prod$-Tor-orthogonal if and if it is Σ-self-orthogonal.

Assume now that ${}_RC$ is w-tilting. By definition, there exists a complete $\mathscr{P}_C$-projective complex

$$\mathbf{X}_1 = 0 \to R \to C_{-1} \to C_{-2} \to C_{-3} \to \cdots .$$

Moreover, there exist modules $C_i' \in \mathrm{Add}_R(C)$ such that

$$C_{-1} \oplus C_{-1}' \cong C^{(I_{-1})} \text{ and } C_i \oplus C_{i-1}' \oplus C_i' \cong C^{(I_i)}$$

for some sets I_i. So, by adding the exact sequence $0 \to C_i' \to C_i' \to 0$ to $\mathbf{X}_1$ in degrees i and $i+1$, we get another complete $\mathscr{P}_C$-projective complex

$$\mathbf{X}_2 = 0 \to R \overset{f_0}{\to} C^{(I_{-1})} \overset{f_{-1}}{\to} C^{(I_{-2})} \overset{f_{-2}}{\to} \cdots .$$

Since ${}_RR$ is finitely generated, there exist an integer $n_{-1} \geq 1$ and a set I_{-1}' such that $C^{(I_{-1})} = C^{n_{-1}} \oplus C^{(I_{-1}')}$ with $\mathrm{Im} f_0 \subseteq C^{n_{-1}}$. Let $K_{-2} = C^{n_{-1}}/\mathrm{Im} f_0$. and consider the exact sequence $\mathscr{E} := 0 \to R \to C^{n_{-1}} \to K_{-2} \to 0$. Since the class $\mathscr{F}\mathscr{P}_\infty(R)$ of all R-modules having a degreewise finite projective resolution is known to be thick (see for instance [23, Theorem 1.8]), $C^{n_{-1}}$ and K_{-2} have degreewise finite projective resolutions. This implies (using a similar computation to that of (I)) that

$$\mathrm{Tor}_i^R\left((C^+)^I, K_{-2}\right) \cong \mathrm{Tor}_i^R\left((C^{(I)})^+, K_{-2}\right) \cong \mathrm{Ext}_R^i\left(K_{-2}, C^{(I)}\right)^+ = 0 \ \forall i \geq 1.$$

Hence, the sequence $E \otimes_R \mathscr{E}$ is exact for every $E \in \mathrm{Prod}_R(C^+)$.

So, if we show that K_{-2} has a $\mathrm{Hom}_R(-, \mathrm{Add}_R(C))$-exact $\mathrm{Add}_R(C)$-coresolution, we can inductively construct a $(\mathrm{Prod}_R(C^+) \otimes_R -)$-exact exact resolution

$$\begin{array}{ccccccccccc} 0 \to R \to & C^{n_{-1}} & & \longrightarrow & & C^{n_{-2}} & & \longrightarrow & & C^{n_{-3}} & \longrightarrow \cdots \\ & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow \\ & & & K_{-2} & & & & K_{-3} & & & & K_{-4} \end{array}$$

which implies that C is w^+-tilting as desired.

However, we immediately get such a $\mathrm{Hom}_R(-, \mathrm{Add}_R(C))$-exact $\mathrm{Add}_R(C)$-coresolution once we show that $K_{-2} \in \mathrm{G_CP}(R)$.

Let $K = \mathrm{Im} f_0$ and consider the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & R & \longrightarrow & C^{n-1} & \longrightarrow & K_{-2} & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & R & \longrightarrow & C^{(I_{-1})} & \longrightarrow & K & \longrightarrow & 0.
\end{array}$$

Since R and C^{n-1} are $\mathrm{G_C}$-projective, to see that K_{-2} is $\mathrm{G_C}$-projective it suffices to show that $\mathrm{Ext}^1_R(K_{-2}, \mathrm{Add}_R(C)) = 0$ by [17, Lemma 3.13].

Let $X \in \mathrm{Add}_R(C)$. By applying the functor $\mathrm{Hom}_R(-,X)$ to the above diagram, we get the following commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 \longrightarrow & \mathrm{Hom}_R(K,X) & \longrightarrow & \mathrm{Hom}_R(C^{(I_{-1})},X) & \longrightarrow & \mathrm{Hom}_R(R,X) & \longrightarrow & 0 & \\
 & \downarrow & & \downarrow & & \| & & & \\
0 \longrightarrow & \mathrm{Hom}_R(K_{-2},X) & \longrightarrow & \mathrm{Hom}_R(C^{n-1},X) & \longrightarrow & \mathrm{Hom}_R(R,X) & \longrightarrow & \mathrm{Ext}^1_R(K_{-1},X) & \longrightarrow 0.
\end{array}$$

The commutativity of the right square shows that $\mathrm{Hom}_R(C^{n-1},X) \to \mathrm{Hom}_R(M,X)$ is a surjective map. Hence, $\mathrm{Ext}^1_R(K_{-2},X) = 0$. ■

We now give the concept of what we will understand by a $\mathrm{G_C}$-flat module. This type of modules were already studied by Holm and Jørgensen in [61] in the more restrictive setting of commutative noetherian rings, and when the module C is semidualizing. It is clear from Proposition 3.1.2 that both coincide when ${}_RC$ is finitely presented.

Definition 3.2.4. *An R-module M is said to be* $\mathrm{G_C}$***-flat*** *if there exists an exact and* $(\mathrm{Prod}_R(C^+) \otimes_R -)$*-exact sequence*

$$\boldsymbol{X}: \; \cdots \to F_1 \to F_0 \to C^0 \to C^1 \to \cdots$$

with $C^i \in \mathscr{F}_\mathrm{C}(R)$ *and* $F_i \in \mathscr{F}(R)$ *for every* $i \in \mathbb{N}$*, such that* $M \cong \mathrm{Im}(F_0 \to C^0)$.

We call the above $(\mathrm{Prod}_R(C^+) \otimes_R -)$*-exact exact sequence a* ***complete*** $\mathscr{F}_\mathrm{C}$***-flat resolution*** *of M and denote the class of all* $\mathrm{G_C}$*-flat R-modules by* $\mathrm{G_CF}(R)$.

Remarks 3.2.5.

1. *If* $C = R$*, then* $\mathrm{G_CF}(R)$ *is exactly the class of all* ***Gorenstein flat*** *modules, but in general these two classes are different as can be seen in Example 3.2.6 below or in Proposition 3.2.23.*

2. *C is $\prod$-Tor-orthogonal if and only if it is $\mathrm{G_C}$-flat, since C being $\prod$-Tor-orthogonal implies that any flat resolution of C is a sequence as that of Definition 3.2.4.*

3. *C is w^+-tilting if and only if both R and C are $\mathrm{G_C}$-flat.*

Next we show that the definition of $\mathrm{G_C}$-flat modules differs, in general, from the one of Gorenstein flat modules. The next example follows the argument of Example 3.2.2.

Example 3.2.6. *Let R be a coherent ring, choose an injective module E and call $C = E^{(I)}$. Then, E^+ is a flat module and, since R is coherent, $C^+ \cong (E^+)^I$ is a flat module. This means, as in Example 3.2.2, that the $(\mathrm{Prod}_R(C^+) \otimes_R -)$-exactness of any exact sequence of modules is guaranteed and so, if*

$$\cdots \to F_1 \to F_0 \to C \to 0$$

is a flat resolution of C, the sequence

$$\begin{array}{ccccccccc} \cdots \longrightarrow F_1 \longrightarrow & F_0 & & \longrightarrow & & C & \longrightarrow 0 \longrightarrow \cdots \\ & & \searrow & & \nearrow\!\!\!\!\!\!= & & \\ & & & C & & & \end{array}$$

shows that C is $\mathrm{G_C}$-flat, and of course it is not Gorenstein flat in general, nor it is a w^+-tilting module (R needs not be $\mathrm{G_C}$-flat).

The same argument shows that if M is any module,

$$0 \to M \to E^0 \to \cdots$$

is an injective coresolution, and we let $C = \oplus_{i \geq 0} E^i$, then M is $\mathrm{G_C}$-flat.

Proposition 3.2.7. *The class $\mathrm{G_CF}(R)$ is always closed under direct sums.*

Proof. It follows from the fact that the classes $\mathscr{F}_{\mathrm{C}}(R)$ and $\mathscr{F}(R)$ are closed under direct sums and the tensor product commutes with direct sums. ∎

The following is a standard characterization of Gorenstein objects, which immediately follows from the definition.

Proposition 3.2.8. *An R-module M is $\mathrm{G_C}$-flat if and only if the following statements hold:*

1. $\mathrm{Tor}^R_{i \geq 1}(\mathrm{Prod}_R(C^+), M) = 0$.

2. *There exists an exact and $(\mathrm{Prod}_R(C^+) \otimes_R -)$-exact sequence of R-modules*

$$\boldsymbol{X} : 0 \to M \to C^0 \to C^1 \to \cdots$$

with $C^i \in \mathscr{F}_{\mathrm{C}}(R)$ for every $i \in \mathbb{N}$.

Corollary 3.2.9. *If C is $\prod$-*Tor*-orthogonal and*

$$\cdots \to F_1 \to F_0 \to C^0 \to C^1 \to \cdots$$

*is a complete $\mathscr{F}_{\mathrm{C}}$-flat resolution, then $K_i = \mathrm{Ker}(C^i \to C^{i+1})$ is $\mathrm{G_C}$-flat for every $i \geq 0$. Consequently, the inclusion $\mathscr{F}_{\mathrm{C}}(R) \subseteq \mathrm{G_CF}(R)$ holds when C is $\prod$-*Tor*-orthogonal.*

Proof. K_0 is $\mathrm{G_C}$-flat by definition, so $\mathrm{Tor}^R_{i\geq 0}(\mathrm{Prod}_R(C^+), K_0) = 0$, and then, using Proposition 3.1.5 we get

$$\mathrm{Tor}^R_i\left(\mathrm{Prod}_R(C^+), K_1\right) \cong \mathrm{Tor}^R_{i-1}\left(\mathrm{Prod}_R(C^+), K_0\right) = 0 \ \forall i \geq 2.$$

Now, the sequence $0 \to K_0 \to C^0 \to K_1 \to 0$ is $(\mathrm{Prod}_R(C^+) \otimes_R -)$-exact, so also $\mathrm{Tor}^R_1(\mathrm{Prod}_R(C^+), K_1) = 0$ and then K_1 is $\mathrm{G_C}$-flat.

By induction we get that all K_i are $\mathrm{G_C}$-flat. ■

Making use of this, we can prove the following nice characterization of $\mathrm{G_C}$-flat modules.

Proposition 3.2.10. *Suppose that C is $\prod$-*Tor*-orthogonal. An R-module M is $\mathrm{G_C}$-flat if and only if there exists a short exact sequence*

$$0 \to M \to V \to G \to 0$$

with $V \in \mathscr{F}_{\mathrm{C}}(R)$ and $G \in \mathrm{G_CF}(R)$.

Proof. If M is $\mathrm{G_C}$-flat, then it is the kernel $M = \mathrm{Ker}(F_0 \to C^0)$ of a complete $\mathscr{F}_{\mathrm{C}}$-resolution

$$\cdots \to F_1 \to F_0 \to C^0 \to C^1 \to \cdots$$

and so we have the short exact sequence $0 \to M \to C^0 \to K^1 \to 0$ with $C^0 \in \mathscr{F}_{\mathrm{C}}(R)$ and $K^1 \in \mathrm{G_CF}(R)$ by Corollary 3.2.9.

Conversely, by Proposition 3.2.8 we have $\mathrm{Tor}^R_{i\geq 1}(\mathrm{Prod}_R(C^+), G) = 0$, which implies by dimension shifting and Proposition 3.1.5 that $\mathrm{Tor}^R_{i\geq 1}(\mathrm{Prod}_R(C^+), M) = 0$. Moreover, there exists a $(\mathrm{Prod}_R(C^+) \otimes_R -)$-exact $\mathscr{F}_{\mathrm{C}}(R)$-coresolution

$$\mathbf{X} : 0 \to G \to C^0 \to C^1 \to \cdots,$$

so combining this with the short exact sequence $0 \to M \to V \to G \to 0$, we get that M has a $(\mathrm{Prod}_R(C^+) \otimes_R -)$-exact $\mathscr{F}_{\mathrm{C}}(R)$-coresolution and then that M is $\mathrm{G_C}$-flat, again by Proposition 3.2.8. ■

Now, it is very likely that w^+-tilting and w-cotilting modules on one side, and $\mathrm{G_C}$-flat and G_{C^+}-injective modules on the other, be connected somehow. It is our purpose to check what are these connections. The proof of the next result is modeled on that of [28, Theorem 6.4.2].

Theorem 3.2.11. *If an R-module M is $\mathrm{G_C}$-flat, then the right R-module M^+ is $\mathrm{G_{C^+}}$-injective. If, in addition, ${}_RC$ is $\prod$-Tor-orthogonal and $\mathscr{F}_\mathrm{C}(R)$ is closed under direct products, then the converse holds too.*

Proof. The canonical isomorphism of abelian groups $\mathrm{Tor}_R^i(N,M)^+ \cong \mathrm{Ext}_R^i(N,M^+)$ $\forall i \geq 1$ gives that

$$\mathrm{Tor}_R^i\left(\mathrm{Prod}_R(C^+),M\right) = 0 \;\forall i \geq 1 \Leftrightarrow M^+ \in \mathrm{Prod}_R(C^+)^{\perp_\infty},$$

so we only have to check the equivalence between 2. in Proposition 3.2.8 and 2. in [17, Proposition 4.5]. That is, M has a $(\mathrm{Prod}_R(C^+)\otimes_R -)$-exact $\mathscr{F}_\mathrm{C}(R)$-coresolution if and only if M^+ has a $\mathrm{Hom}_R(\mathrm{Prod}_R(C^+),-)$-exact $\mathrm{Prod}_R(C^+)$-resolution

Thus, if M is $\mathrm{G_C}$-flat then there is a $(\mathrm{Prod}_R(C^+)\otimes_R -)$-exact $\mathscr{F}_\mathrm{C}(R)$-coresolution

$$\mathbf{X}:\; 0 \to M \to C_0 \to C_1 \to \cdots$$

which gives rise to a $\mathrm{Prod}_R(C^+)$-resolution

$$\mathbf{X}^+:\; \cdots \to C_1^+ \to C_0^+ \to M^+ \to 0.$$

This resolution is $\mathrm{Hom}_R(\mathrm{Prod}_R(C^+),-)$-exact by the natural isomorphism of complexes $\mathrm{Hom}_R(X,\mathbf{X}^+) \cong (X\otimes_R \mathbf{X})^+$, so M^+ is G_{W^+}-injective.

Conversely, assume that $\mathscr{F}_\mathrm{C}(R)$ is closed under direct products. By Theorem 3.1.7, M has an $\mathscr{F}_\mathrm{C}(R)$-preenvelope $M \to C_0$, and by [17, Proposition 4.6], M^+ is $\mathrm{G_{C^+}}$-injective, so there is an epimorphism $L \twoheadrightarrow M^+$ for some $L \in \mathrm{Prod}_R(C^+)$. Therefore, we have a monomorphism $M \hookrightarrow M^{++} \hookrightarrow L^+$, which proves that our original preenvelope $M \to C_0$ is indeed a monomorphism. We then get an exact and $\mathrm{Hom}_R(-,\mathscr{F}_\mathrm{C}(R))$-exact sequence of R-modules

$$0 \to M \to C_0 \to L_0 \to 0,$$

and so an exact sequence of right R-modules

$$0 \to L_0^+ \to C_0^+ \to M^+ \to 0$$

with $C_0^+ \in \mathrm{Prod}_R(C^+) \subseteq \mathrm{G_{C^+}I}(R)$.

If we prove that L_0^+ is $\mathrm{G_{C^+}}$-injective, we can repeat the process and find another exact and $\mathrm{Hom}_R(-,\mathscr{F}_\mathrm{C}(R))$-exact sequence

$$0 \to L_0 \to C_1 \to L_1 \to 0.$$

Gluing the two sequences together we will have the exact and $\mathrm{Hom}_R(-,\mathscr{F}_\mathrm{C}(R))$-exact sequence

$$0 \to M \to C_0 \to C_1$$

with $C_i \in \mathscr{F}_C(R)$, and so, by induction, we will get a $\mathrm{Hom}_R(-,\mathscr{F}_C(R))$-exact $\mathscr{F}_C(R)$-coresolution of R-modules

$$\mathbf{X}: 0 \to M \to C_0 \to C_1 \to C_2 \to \cdots$$

of M. But then, for any $L \in \mathrm{Prod}_R(C^+)$, we will have

$$(L \otimes_R \mathbf{X})^+ \cong \mathrm{Hom}_R\left(\mathbf{X}, L^+\right),$$

which is exact by construction since $L^+ \in \mathscr{F}_C(R)$ by Theorem 3.1.7.

Therefore, it only remains to prove that L_0^+ is G_{C^+}-injective, and this will follow if we prove that $L_0^+ \in \mathrm{Prod}_R(C^+)^{\perp_\infty}$ by the dual of [17, Lemma 3.13].

Choose then any $L \in \mathrm{Prod}_R(C^+)$. The commutative diagram

$$\begin{array}{ccccccc}
\mathrm{Hom}_R(C_0,L^+) & \longrightarrow & \mathrm{Hom}_R(M,L^+) & \longrightarrow & 0 & & \\
\cong\downarrow & & \cong\downarrow & & & & \\
\mathrm{Hom}_R(L,C_0^+) & \longrightarrow & \mathrm{Hom}_R(L,M^+) & \longrightarrow & \mathrm{Ext}_R^1(L,L_0^+) & \longrightarrow & 0
\end{array}$$

shows that $\mathrm{Ext}_R^1(L,L_0^+) = 0$, and if $i \geq 2$, the long exact sequence associated to the bottom row shows that $\mathrm{Ext}_R^i(L,L_0^+) \cong \mathrm{Ext}^i(L,C_0^+)$ since M^+ is G_{C^+}-injective and $L \in \mathrm{Prod}_R(C^+)$. But C^+ is $\prod$-self-orthogonal so $\mathrm{Ext}_R^i\left((C^+)^I,(C^+)^J\right) = 0$ for every $i \geq 1$ and every couple of sets I and J, and $\mathrm{Ext}_R^i(L,C_0^+)$ is a direct summand of some $\mathrm{Ext}_R^i\left((C^+)^I,(C^+)^J\right)$, so we get that $\mathrm{Ext}_R^i(L,L_0^+) = 0$ for every $i \geq 2$. ■

Corollary 3.2.12. *If C is a w^+-tilting R-module, then C^+ is a w-cotilting right R-module. If $\mathscr{F}_C(R)$ is closed under direct products, then the converse holds too.*

Proof. Follows by Theorem 3.2.11, Remark 3.2.5(3) and Lemma 1.3.5(2) ■

The question of the closure under extensions of a given class of modules is a typical problem for those classes having interest in homological algebra. We can now give a necessary condition in our context of $\mathrm{G_CF}(R)$.

Corollary 3.2.13. *Assume that ${}_RC$ is $\prod$-*Tor*-orthogonal. If $\mathscr{F}_C(R)$ is closed under direct products, then $\mathrm{G_CF}(R)$ is closed under extensions.*

In particular, this the case when S is right coherent and both ${}_RC$ and C_S are finitely presented.

Proof. The first part follows by the fact that the class of G_{C^+}-injective right R-modules is closed under extensions ([17, Proposition 4.7]) together with Theorem 3.2.11.

The last claim follows by Theorem 3.1.7. ■

As mentioned above, many of the nice homological properties of a given class rely on the fact that it is closed under extensions (when it is so). Of course this applies to our class $\mathrm{G_CF}(R)$, so it is worth establishing the following terminology, adopted for the first time in [9]. Many of the results to be given from now on will be proved under this condition on the ring.

Definition 3.2.14. *A ring R is said to be* $\mathrm{G_CF}$***-closed*** *provided that the class* $\mathrm{G_CF}(R)$ *is closed under extensions.*

It has recently been shown by Šaroch and Šťovičeck ([78]) that any ring R is $\mathrm{G_CF}$-closed in case that $C = R$. In fact, this holds for any flat generator R-module C (see Corollary 3.2.24). Therefore, the following question becomes natural at this point:

Question: For what modules C is any ring $\mathrm{G_CF}$-closed?

All the examples given in this thesis provide positive answers to this question. However, the authors do not have a general answer to it.

We now find conditions for the class $\mathrm{G_CF}(R)$ to have the properties that a class is expected to have to develop a nice theory of homology. As one could expect, by similarity with the classical case of Gorenstein flat modules, the class $\mathrm{G_CF}(R)$ is indeed closed under kernels of monomorphisms (at least under certain hypotheses on the ring) and direct summands. But also, as we will now see, it is closed under direct limits.

The following is then a version of what was proved in [17] for $\mathrm{G_C}$-projective and for $\mathrm{G_C}$-injective modules.

Proposition 3.2.15. *If R is* $\mathrm{G_CF}$*-closed and C is* $\prod$-Tor*-orthogonal, then the class* $\mathrm{G_CF}(R)$ *is:*

1. *Closed under kernels of epimorphisms.*

2. *Closed under direct summands*

3. *Closed under direct limits.*

Proof. 1. Let $0 \to M \to N \to L \to 0$ be a short exact sequence of R-modules with $N,\ L \in \mathrm{G_CF}(R)$. By Proposition 3.2.10, there exists a short exact sequence $0 \to N \to$

$V \to G \to 0$ with $V \in \mathscr{F}_C(R)$ and $G \in \mathrm{G_CF}(R)$. Consider the pushout diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M & \longrightarrow & N & \longrightarrow & L & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M & \longrightarrow & V & \dashrightarrow & K & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & G & = & G & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Since $\mathrm{G_CF}(R)$ is closed under extensions, K is $\mathrm{G_C}$-flat and hence, again by Proposition 3.2.10 (using the middle row), M is $\mathrm{G_C}$-flat.

2. Now, we know that $\mathrm{G_CF}(R)$ is closed under direct sums, so using Eilenberg's swindle we get that it is also closed under direct summands.

3. The closure of $\mathrm{G_CF}(R)$ under direct limits can be reasoned following the arguments of [87, Lemma 3.1] since $\mathscr{F}_C(R)$ is closed under direct limits by Proposition 3.1.6. ■

Though we do not know when the class $\mathrm{G_C}$-flat R-modules is closed under cokernels of monomorphisms, we can give conditions to ensure when the cokernel of a monomorphism in $\mathrm{G_CF}(R)$ belongs to the class. The following is a simple adaptation of [9, Theorem 2.3(3)] for $\mathrm{G_C}$-flat modules, and both proofs follow the same argument, using in this adaptation Proposition 3.2.10.

Proposition 3.2.16. *Assume that R is $\mathrm{G_CF}$-closed and that C is $\prod$-Tor-orthogonal, and let*

$$0 \to G_1 \to G_0 \to M \to 0$$

be exact with $G_0, G_1 \in \mathrm{G_CF}(R)$. If $\mathrm{Tor}_1^R(\mathrm{Prod}_R(C^+), M) = 0$, then $M \in \mathrm{G_CF}(R)$.

We now want to check the behavior of $\mathrm{G_CF}(R)$ with respect to flat and projective modules. As one could expect, over w^+-tilting modules, $\mathrm{G_C}$-flat modules generalize both projective and flat modules.

Proposition 3.2.17. *Let R be $\mathrm{G_CF}$-closed and C be $\prod$-Tor-orthogonal. The following assertions are equivalent:*

1. *C is w^+-tilting.*

2. *$R \in \mathrm{G_CF}(R)$.*

3. $\mathscr{P}(R) \subseteq \mathrm{G_CF}(R)$.

4. $\mathscr{F}(R) \subseteq \mathrm{G_CF}(R)$.

Proof. (1) $\Leftrightarrow$ (2) and (4) $\Rightarrow$ (3) $\Rightarrow$ (2) are clear.

(2) $\Rightarrow$ (3) holds since $\mathrm{G_CF}(R)$ is closed under direct sums and summands.

(3) $\Rightarrow$ (4) follows since $\mathrm{G_CF}(R)$ is closed under direct limits. ∎

Corollary 3.2.18. *If R is $\mathrm{G_CF}$-closed and C is w^+-tilting, then for every R-module M there exists an exact sequence*

$$\cdots \to G_1 \to G_0 \to M \to 0$$

with each G_i $\mathrm{G_C}$-flat.

In Corollary 3.2.9 we proved that in a complete $\mathscr{F}_\mathrm{C}$-flat resolution

$$\cdots \to F_1 \to F_0 \to C^0 \to C^1 \to \cdots,$$

every kernel $\mathrm{Ker}(C^i \to C^{i+1})$ is $\mathrm{G_C}$-flat. However, we still did not know what happens in the left part of the sequence. Now we state the conditions we need to ensure the $\mathrm{G_C}$-flatness of kernels in the left part of that sequence.

Corollary 3.2.19. *Suppose R is $\mathrm{G_CF}$-closed. If C is w^+-tilting and*

$$\cdots \to F_1 \to F_0 \to C^0 \to C^1 \to \cdots$$

is a complete $\mathscr{F}_\mathrm{C}$-flat resolution, then every image $I_i = \mathrm{Im}\,(F_{i+1} \to F_i)$ is $\mathrm{G_C}$-flat.

Proof. By Proposition 3.2.17, we know $F_0 \in \mathrm{G_CF}(R)$, and $\mathrm{Im}\left(F_0 \to C^0\right)$ is also $\mathrm{G_C}$-flat by definition. Then, applying Proposition 3.2.15 we get that I_0 is $\mathrm{G_C}$-flat.

But then we get, using induction, that all I_i are $\mathrm{G_C}$-flat. ∎

Remark 3.2.20. *As a consequence of the last corollary, and having in mind Corollary 3.2.9, we see that when C is w^+-tilting and $\mathrm{G_CF}(R)$ is closed under extensions, all kernels in the sequences providing the class of $\mathrm{G_C}$-flat modules are indeed $\mathrm{G_C}$-flat.*

Now, by Corollary 3.2.13 and Propositions 3.2.15 and 3.2.17 we know when $\mathrm{G_CF}(R)$ is projectively resolving.

Theorem 3.2.21. *Assuming that R is $\mathrm{G_CF}$-closed, the module C is w^+-tilting if and only if $\mathrm{G_CF}(R)$ is projectively resolving and $\mathscr{F}_\mathrm{C}(R) \subseteq \mathrm{G_CF}(R)$.*

When a class is projectively resolving, the Comparison Lemma can be proved. The following is a version of the Comparison Lemma adapted to our class $\mathrm{G_CF}(R)$. It follows immediately from [4, Lemma 3.12].

Corollary 3.2.22. *Let R be* $\mathrm{G_CF}$*-closed and C be w^+-tilting. Given two exact sequences*

$$0 \to K_n \to G_{n-1} \cdots \to G_1 \to G_0 \to M \to 0$$

and

$$0 \to L_n \to H_{n-1} \cdots \to H_1 \to H_0 \to M \to 0$$

with all G_i and H_i $\mathrm{G_C}$*-flat, the kernel K_n is* $\mathrm{G_C}$*-flat if and only if L_n is* $\mathrm{G_C}$*-flat.*

We finish this section by proving that, with respect to w^+-tilting modules, the classes of classical flat, Gorenstein flat and Gorenstein injective modules, either all coincide with those of relative flat, relative Gorenstein flat and relative Gorenstein injective modules respectively, or none of them coincide.

By a **generator** (of R-Mod) we mean a module G such that for any R-module M there is an epimorphism $G^{(I)} \to M$ for some set I. Dually, G is a **cogenerator** if for any R-module M there is an a monomorphism $M \to G^I$ for some set I.

It is easy to verify the following three assertions:
(a) C is a projective generator if and only if $\mathrm{Add}_R(C) = \mathscr{P}(R)$.
(b) C is a flat generator if and only if $\mathscr{F}_\mathrm{C}(R) = \mathscr{F}(R)$.
(c) C is an injective cogenerator if and only if $\mathrm{Prod}_R(C) = \mathscr{I}(R)$.

Proposition 3.2.23. *If C is w^+-tilting, the following assertions are equivalent:*

(1) $\mathrm{G_CF}(R) = \mathscr{G}\mathscr{F}(R)$.

(2) *C is a flat generator R-module.*

(3) *C^+ is an injective cogenerator right R-module.*

(4) $\mathrm{G_{C^+}I}(R) = \mathscr{G}\mathscr{I}(R)$.

Proof. (1) $\Rightarrow$ (2) If we prove that

$$\mathrm{G_CF}(R)^+ \cap {}^\perp\big(\mathrm{G_CF}(R)^+\big) = \mathscr{F}_\mathrm{C}(R)^+ \text{ and } \mathscr{G}\mathscr{F}(R)^+ \cap {}^\perp\big(\mathscr{G}\mathscr{F}(R)^+\big) = \mathscr{F}(R)^+$$

we will have

$$\mathscr{F}_\mathrm{C}(R)^+ = \mathrm{G_CF}(R)^+ \cap {}^\perp\big(\mathrm{G_CF}(R)^+\big) = \mathscr{G}\mathscr{F}(R)^+ \cap {}^\perp\big(\mathscr{G}\mathscr{F}(R)^+\big) = \mathscr{F}(R)^+$$

and we will be done.

On one side, it is clear that $\mathscr{F}_\mathrm{C}(R) \subseteq \mathrm{G_CF}(R)$ since C is $\prod$-Tor-orthogonal and so that $\mathscr{F}_\mathrm{C}(R)^+ \subseteq \mathrm{G_CF}(R)^+$, and on the other we know that $\mathrm{Ext}_R^{i\geq 1}(\mathscr{F}_\mathrm{C}(R)^+, X^+) \cong \mathrm{Tor}^R_{i\geq 1}(\mathscr{F}_\mathrm{C}(R)^+, X)^+ = 0$ for every $X \in \mathrm{G_CF}(R)$, so $\mathscr{F}_\mathrm{C}(R)^+ \subseteq {}^\perp(\mathrm{G_CF}(R)^+)$.

Conversely, let $X \in \mathrm{G_CF}(R)$ be such that $X^+ \in \mathrm{G_CF}(R)^+ \cap {}^\perp(\mathrm{G_CF}(R)^+)$. By Proposition 3.2.10, there is an exact sequence $0 \to X \to V \to L \to 0$ with $V \in \mathscr{F}_\mathrm{C}(R)$ and

$L \in \mathrm{G_C F}(R)$, so being $X^+ \in {}^{\perp}(\mathrm{G_C F}(R)^+)$ implies that the exact sequence $0 \to L^+ \to V^+ \to X^+ \to 0$ splits and so that $X^+ \in \mathscr{F}_{\mathrm{C}}(R)^+$.

The second equality follows similarly.

$(2) \Rightarrow (3)$ $C \in \mathscr{F}_{\mathrm{C}}(R) = \mathscr{F}(R) \Rightarrow C^+ \in \mathscr{I}(R) \Rightarrow \mathrm{Prod}_R(C^+) \subseteq \mathscr{I}(R)$.

Conversely, $R \in \mathscr{F}(R) = \mathscr{F}_{\mathrm{C}}(R) \Rightarrow R^+ \in \mathrm{Prod}_R(C^+)$, but R^+ is an injective cogenerator in Mod-R, so $\mathscr{I}(R) \subseteq \mathrm{Prod}_R(C^+)$.

$(3) \Rightarrow (2)$ $A \in \mathscr{F}_{\mathrm{C}}(R) \Leftrightarrow A^+ \in \mathrm{Prod}_R(C^+) = \mathscr{I}(R) \Leftrightarrow A \in \mathscr{F}(R)$.

$(2) \Rightarrow (1)$ Let

$$\cdots \to F_1 \to F_0 \to C^0 \to C^1 \to \cdots$$

be an exact sequence and call $M \cong \mathrm{Im}(F_0 \to C^0)$. By hypothesis and $(2) \Leftrightarrow (3)$ we have $\mathscr{F}_{\mathrm{C}}(R) = \mathscr{F}(R)$ and $\mathrm{Prod}_R(C^+) = \mathscr{I}(R)$, so M is $\mathrm{G_C}$-flat if and only if it is Gorenstein flat. ■

As a consequence of Proposition 3.2.23 and [78, Theorem 4.11], we get the following case in which the class of $\mathrm{G_C}$-flat modules is closed under extensions.

Corollary 3.2.24. *If C is a flat generator R-module, then R is* $\mathrm{G_C F}$*-closed.*

3.3 Stability of relative Gorenstein flat modules

This section is a one-theorem and two consequences section devoted to give an answer to the very interesting problem of the stability of a Gorenstein class of modules. In our specific case, the class $\mathrm{G_C F}(R)$. In other words, given a $(\mathrm{Prod}_R(C^+) \otimes_R -)$-exact exact sequence of $\mathrm{G_C}$-flat R-modules

$$\cdots \to G_1 \to G_0 \to G^0 \to G^1 \to \cdots,$$

is the module $M \cong \mathrm{Im}(G_0 \to G^1)$ $\mathrm{G_C}$-flat?

We call $\mathrm{G^2_C F}(R)$ the class of all modules M defined as above, that is, modules with $(\mathrm{Prod}_R(C^+) \otimes_R -)$-exact $\mathrm{G_C F}(R)$-resolutions and coresolutions, and we will prove that indeed, $\mathrm{G^2_C F}(R) = \mathrm{G_C F}(R)$.

The first consequence deals with the question of the symmetry of $\mathrm{G_C}$-flat modules with respect to classes in between $\mathscr{F}(R) \cup \mathscr{F}_{\mathrm{C}}(R)$ and $\mathrm{G_C F}(R)$, that is: all exact and $(\mathrm{Prod}_R(C^+) \otimes_R -)$-exact sequences having as components modules in any class between $\mathscr{F}(R) \cup \mathscr{F}_{\mathrm{C}}(R)$ and $\mathrm{G_C F}(R)$, regardless the positions they hold, will give $\mathrm{G_C}$-flat modules as 0-syzygies.

And the second consequence is a very natural question: when is the class of Gorenstein flat modules included in that of $\mathrm{G_C}$-flat modules?

Theorem 3.3.1. *The equality* $\mathrm{G^2_C F}(R) = \mathrm{G_C F}(R)$ *holds for any* $\mathrm{G_C F}$*-closed ring and any w^+-tilting module C.*

Proof. The proof will be done by showing that $(\mathrm{Prod}_R(C^+)\otimes_R\text{-})$-exact $\mathscr{F}(R)$-resolutions ($\mathscr{F}_\mathrm{C}(R)$-coresolutions) of a module exist if and only if $(\mathrm{Prod}_R(C^+)\otimes_R -)$-exact $\mathrm{G_CF}(R)$-resolutions ($\mathrm{G_CF}(R)$-coresolutions) exist.

We know by Corollary 3.2.9 that $\mathscr{F}_\mathrm{C}(R)\subseteq \mathrm{G_CF}(R)$, so every $\mathscr{F}_\mathrm{C}(R)$-coresolution is indeed a $\mathrm{G_CF}(R)$-coresolution, and similarly, $\mathscr{F}(R)\subseteq \mathrm{G_CF}(R)$ by Proposition 3.2.17, so every $\mathscr{F}(R)$-resolution is a $\mathrm{G_CF}(R)$-resolution.

Conversely, suppose a module M has a $(\mathrm{Prod}_R(C^+)\otimes_R\text{-})$-exact $\mathrm{G_CF}(R)$-resolution

$$\cdots\to G_1\to G_0\to M\to 0.$$

Choose an exact sequence

$$0\to L_0\to F_0\to G_0\to 0$$

with F_0 flat. As G_0 and F_0 are $\mathrm{G_C}$-flat (see Proposition 3.2.17), Proposition 3.2.15 gives that L_0 is also $\mathrm{G_C}$-flat. Call $K_i=\mathrm{Im}\,(G_{i+1}\to G_i)$ and construct the pullback

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & L_0 & = & L_0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & D_0 & \dashrightarrow & F_0 & \to & M & \to & 0\\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & K_0 & \to & G_0 & \to & M & \to & 0\\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Since G_0 is $\mathrm{G_C}$-flat, the middle column is $(\mathrm{Prod}_R(C^+)\otimes_R -)$-exact and then the whole diagram is $(\mathrm{Prod}_R(C^+)\otimes_R -)$-exact since the bottom row is so as well.

Now compute the pullback of $D_0\to K_0$ and $G_1\to K_0$:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & K_1 & = & K_1 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & L_0 & \to & G_1' & \dashrightarrow & G_1 & \to & 0\\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & L_0 & \to & D_0 & \to & K_0 & \to & 0\\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

By the assumption on the ring, $\mathrm{G_C F}(R)$ is closed under extensions so the middle row shows that $G'_1 \in \mathrm{G_C F}(R)$.

But the bottom row and the right column are $(\mathrm{Prod}_R(C^+) \otimes_R -)$-exact, so the middle row is $(\mathrm{Prod}_R(C^+) \otimes_R -)$-exact too, and then we have a $(\mathrm{Prod}_R(C^+) \otimes_R -)$-exact exact sequence

$$\begin{array}{ccccccccc} \cdots \longrightarrow G_1 & & \longrightarrow & & G'_1 & & \longrightarrow & & F_0 \longrightarrow M \longrightarrow 0 \\ & \searrow & & \nearrow & & \searrow & & \nearrow & \\ & & K_1 & & & & D_0 & & \end{array}$$

with $F_0 \in \mathscr{F}(R)$, and G_i, $G'_1 \in \mathrm{G_C F}(R)$ for every $i \geq 1$. In particular, the module D_0 has a $(\mathrm{Prod}_R(C^+) \otimes_R -)$-exact $\mathrm{G_C F}(R)$-resolution, so we can repeat the argument inductively and get a $(\mathrm{Prod}_R(C^+) \otimes_R -)$-exact $\mathscr{F}(R)$-resolution of M.

It only remains to be shown that every module having a $(\mathrm{Prod}_R(C^+) \otimes_R -)$-exact $\mathrm{G_C F}(R)$-coresolution also has a $(\mathrm{Prod}_R(C^+) \otimes_R -)$-exact $\mathscr{F}_\mathrm{C}(R)$-coresolution, and this will be done in a dual manner to that of $\mathscr{F}(R)$-resolutions.

Suppose then that M has a $(\mathrm{Prod}_R(C^+) \otimes_R -)$-exact $\mathrm{G_C F}(R)$-coresolution

$$0 \to M \to G_0 \to G_1 \to G_2 \to \cdots$$

and call $K_i = \mathrm{Ker}\,(G_i \to G_{i+1})$. By Proposition 3.2.10, there is an exact sequence

$$0 \to G_0 \to A_0 \to G'_0 \to 0$$

for some $A_0 \in \mathscr{F}_\mathrm{C}(R)$ and some $\mathrm{G_C}$-flat module G'_0. Compute the pushout of $G_0 \to A_0$ and $G_0 \to K_1$

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M & \longrightarrow & G_0 & \longrightarrow & K_1 & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M & \longrightarrow & A_0 & \longrightarrow & P & \longrightarrow & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & G'_0 & = & G'_0 & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0 & & \end{array}$$

The top row is $(\mathrm{Prod}_R(C^+) \otimes_R -)$-exact by assumption, and the middle and right columns are $(\mathrm{Prod}_R(C^+) \otimes_R -)$-exact since G'_0 is $\mathrm{G_C}$-flat. Therefore, the middle row is also $(\mathrm{Prod}_R(C^+) \otimes_R -)$-exact.

Next, compute the pushout of $K_1 \to G_1$ and $K_1 \to P$

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & K_1 & \longrightarrow & G_1 & \longrightarrow & K_2 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & P & \longrightarrow & P' & \longrightarrow & K_2 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & G_0' & = & G_0' & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Again, G_1 and G_0' are $\mathrm{G_C}$-flat so is P', and the middle row must be $(\mathrm{Prod}_R(C^+)\otimes_R \text{-})$-exact since the top row and the two columns are. Thus, we have a $(\mathrm{Prod}_R(C^+)\otimes_R -)$-exact sequence

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & M & \longrightarrow & A_0 & \longrightarrow & P' & \longrightarrow & G_2 & \longrightarrow & G_3 & \longrightarrow & \cdots \\
 & & & & & \searrow \quad \nearrow & & \searrow \quad \nearrow & & & & & \\
 & & & & & P & & K_2 & & & & &
\end{array}$$

with $A_0 \in \mathscr{F}_{\mathrm{C}}(R)$ and $P',\ G_i \in \mathrm{G_CF}(R)$. In particular, P has a $(\mathrm{Prod}_R(C^+)\otimes_R -)$-exact $\mathrm{G_CF}(R)$-coresolution, so we can repeat the argument and by induction construct a $(\mathrm{Prod}_R(C^+)\otimes_R -)$-exact $\mathscr{F}_{\mathrm{C}}(R)$-coresolution of M. ■

We now prove the symmetry of $\mathrm{G_C}$-flat modules with respect to classes in between $\mathscr{F}(R)\cup\mathscr{F}_{\mathrm{C}}(R)$ and $\mathrm{G_CF}(R)$.

Corollary 3.3.2. *Let R be $\mathrm{G_CF}$-closed. Then, the following assertions are equivalent.*

(1) C is w^+-tilting.

(2) $M \in \mathrm{G_CF}(R)$ if and only if there exists a $(\mathrm{Prod}_R(C^+)\otimes_R -)$-exact exact sequence of R-modules

$$\cdots \to V_1 \to V_0 \to V^0 \to V^1 \to \cdots$$

with $M \cong \mathrm{Im}(V_0 \to V_1)$ and V_i, V^i belonging to some class $\mathscr{V}$ such that $\mathscr{F}(R)\cup\mathscr{F}_{\mathrm{C}}(R) \subseteq \mathscr{V} \subseteq \mathrm{G_CF}(R)$.

(3) $M \in \mathrm{G_CF}(R)$ if and only if there exists a $(\mathrm{Prod}_R(C^+)\otimes_R -)$-exact exact sequence of R-modules

$$\cdots \to V_1 \to V_0 \to V^0 \to V^1 \to \cdots$$

with $M \cong \mathrm{Im}(V_0 \to V_1)$ and $V_i, V^i \in \mathscr{F}(R)\cup\mathscr{F}_{\mathrm{C}}(R)$.

Proof. (1) $\Rightarrow$ (2) is a direct consequence of Theorem 3.3.1.
(2) $\Rightarrow$ (3) Clear.
(3) $\Rightarrow$ (1) Clearly the sequences

$$\cdots \to 0 \to R \xrightarrow{1_R} R \to 0 \cdots \text{ and } \cdots \to 0 \to C \xrightarrow{1_C} C \to 0 \cdots$$

are complete $\mathscr{F}_{\mathrm{C}}$-flat resolutions, so $R, C \in \mathrm{G_CF}(R)$ by hypothesis. Hence, C is w^+-tilting. ■

Corollary 3.3.3. *Let R be* $\mathrm{G_CF}$*-closed and C be* w^+*-tilting. The following conditions are equivalent.*

1. $\mathscr{GF}(R) \subseteq \mathrm{G_CF}(R)$.
2. $\mathrm{Tor}^R_{i\geq 1}(\mathrm{Prod}_R(C^+), \mathscr{GF}(R)) = 0$.
3. $\mathrm{Tor}^R_1(\mathrm{Prod}_R(C^+), \mathscr{GF}(R)) = 0$.

Thus, if $\mathrm{id}_R(C^+) < \infty$ *or* $\mathrm{fd}_R\left((C^+)^I\right) < \infty$ *for every set I, then* $\mathscr{GF}(R) \subseteq \mathrm{G_CF}(R)$.

Proof. The case $\mathrm{fd}_R\left((C^+)^I\right) < \infty$ for every set I is clear. If $\mathrm{id}_R(C^+)$ is finite, then the injective dimension of every module in $\mathrm{Prod}_R(C^+)$ is also finite. But, $\mathrm{Tor}^R_{\geq 1}(E, G) = 0$ for any $E \in \mathscr{I}(R)$ and any $G \in \mathscr{GF}(R)$, so taking a finite injective coresolution of any module $X \in \mathrm{Prod}_R(C^+)$, we see by dimension shifting that $\mathrm{Tor}^R_{i\geq 1}(X, \mathscr{GF}(R)) = 0$. ■

3.4 Relative Gorenstein flat dimensions

In this section, we define and study $\mathscr{F}_{\mathrm{C}}$-flat and $\mathrm{G_C}$-flat dimensions. In addition to finding the connections between them, we study in depth the properties of the $\mathrm{G_C}$-flat dimension, the relations between the dimensions of the modules in a short exact sequence, the link with the vanishing of Tor, etc.

Definition 3.4.1. *The* $\mathscr{F}_{\mathrm{C}}(R)$*-resolution dimension of an R-module M is called* $\mathscr{F}_{\mathrm{C}}$***-flat dimension*** *of M, and it is denoted as* $\mathscr{F}_C\!-\!\mathrm{fd}_R(M)$.

Remark 3.4.2. *Over a commutative ring and when C is semidualizing, the classes* $\mathscr{F}_{\mathrm{C}}(R)$ *and* $C \otimes_R \mathscr{F}(R)$ *coincide (see Proposition 3.1.2 for a wider version of this), so the* $\mathscr{F}_{\mathrm{C}}$*-flat dimension of a module defined above generalizes the one given in [80].*

The question that first comes to our mind is whether or not the $\mathscr{F}_{\mathrm{C}}$-flat dimension and the $\mathscr{P}_{\mathrm{C}}$-projective or the $\mathscr{I}_{\mathrm{C}}$-injective dimensions are related somehow. We start by recalling these two concepts.

Definition 3.4.3 ([19]). *The* $\mathscr{P}_{\mathrm{C}}$***-projective dimension*** *of a module* M, $\mathscr{P}_C-\mathrm{pd}_R(M)$, *is defined as its* $\mathrm{Add}_R(C)$*-resolution dimension, and the* $\mathscr{I}_{\mathrm{C}}$***-injective dimension*** *of* M *is defined as its* $\mathrm{Prod}_R(C)$*-coresolution dimension, and it is denoted by* $\mathscr{I}_C-\mathrm{id}_R(M)$.

Proposition 3.4.4. *Let* C *and* M *be two arbitrary* R*-modules. The following statements hold:*

(1) $\mathscr{F}_C-\mathrm{fd}_R(M) \leq \mathscr{P}_C-\mathrm{pd}_R(M)$.

(2) $\mathscr{I}_{\mathrm{C}^+}-\mathrm{id}_R(M^+) \leq \mathscr{F}_C-\mathrm{fd}_R(M)$.

(3) If M *is* $\mathrm{G_C}$*-flat and* $\mathscr{F}_C-\mathrm{fd}_R(M) < \infty$*, then* $M \in \mathscr{F}_{\mathrm{C}}(R)$.

Proof. (1) Clear since $\mathrm{Add}_R(C) \subseteq \mathscr{F}_{\mathrm{C}}(R)$.

(2) Applying the functor $\mathrm{Hom}_R(-,\mathbb{Q}/\mathbb{Z})$ to any $\mathscr{F}_{\mathrm{C}}(R)$-resolution of M, we get a $\mathrm{Prod}_R(C^+)$-coresolution of M^+, so the inequality holds.

(3) By *(2)*, we have $\mathscr{I}_{\mathrm{C}^+}-\mathrm{id}_R(M^+) < \infty$, and by Theorem 3.2.11 that M^+ is $\mathrm{G}_{\mathrm{C}^+}$-injective, so $M^+ \in \mathrm{Prod}_R(C^+)$ by [19, Proposition 3.4(2)]. Hence, $M \in \mathscr{F}_{\mathrm{C}}(R)$. ■

It is a fact already proven at this point how important would be to know when a class with which one works in any homological aspect is closed under extensions. In Corollary 3.2.13, we gave a necessary condition for the class $\mathrm{G_CF}(R)$ to be closed under extensions, and now, with the use of the dimensions just introduced, we can give another and interesting condition.

Corollary 3.4.5. *Let* R *be a ring such that every* R*-module has finite* $\mathscr{F}_{\mathrm{C}}$*-flat dimension (for instance, a ring of finite weak dimension when* $C = R$*). If* C *is* $\prod$-Tor*-orthogonal, then* $\mathrm{G_CF}(R)$ *is closed under extensions.*

Proof. Proposition 3.4.4 gives the inclusion $\mathrm{G_CF}(R) \subseteq \mathscr{F}_{\mathrm{C}}(R)$, Proposition 3.2.9 completes the equality $\mathrm{G_CF}(R) = \mathscr{F}_{\mathrm{C}}(R)$, and Proposition 3.1.6(3) guarantees that $\mathscr{F}_{\mathrm{C}}(R)$ (and so $\mathrm{G_CF}(R)$) is closed under extensions. ■

We now give the concept of relative Gorenstein flat dimension.

Definition 3.4.6. *The* $\mathrm{G_C}$***-flat dimension*** *of an* R*-module* M, $\mathrm{G_C}\text{-}\mathrm{fd}_R(M)$*, is defined as the* $\mathrm{G_CF}(R)$*-resolution dimension of* M.

As a first consequence of the definition we clearly see that, when C is $\prod$-Tor-orthogonal, the $\mathrm{G_C}$-flat dimension of any module is always less than or equal to its $\mathscr{F}_{\mathrm{C}}$-flat dimension, since in this case $\mathscr{F}_{\mathrm{C}}(R) \subseteq \mathrm{G_CF}(R)$.

It is a very natural problem to relate a given dimension (relative to a class $\mathscr{A}$) with the vanishing properties of the functors which, one way or another, have influence in the computation of $\mathscr{A}$. This is our next goal for the class of $\mathrm{G_C}$-flat modules. And to do so, we first give a typical characterization of the size of all G_C-flat resolutions of a module, in terms of a given such resolution. Its proof is standard, but we think it is worth giving it for completeness.

Theorem 3.4.7. *Assume R is $G_C F$-closed and C is w^+-tilting. If M is an R-module and $n \geq 0$ is an integer, the following assertions are equivalent:*

(1) $G_C\text{-fd}_R(M) \leq n$.

(2) $G_C\text{-fd}_R(M) < \infty$ *and* $\mathrm{Tor}^R_{i>n}(\mathrm{Prod}_R(C^+), M) = 0$.

(3) For every exact sequence of $0 \to K_n \to G_{n-1} \to \cdots \to G_0 \to M \to 0$*, if each G_i is G_C-flat, then so is K_n.*

Proof. (1)⇒(2) We have an exact sequence

$$0 \to G_n \to \cdots \to G_0 \to M \to 0$$

with all G_i G_C-flat. If we call $K_i = \mathrm{Ker}\,(G_{i-1} \to G_{i-2})\ \forall i \geq 1$ (with $G_{-1} = M$), we have

$$\mathrm{Tor}^R_{n+i}(X, M) \cong \mathrm{Tor}^R_i(X, G_n)\ \forall i \geq 1,\ \forall X \in \mathrm{Prod}_R(C^+).$$

(2)⇒(3) There is an exact sequence

$$0 \to G_m \to \cdots \to G_0 \to M \to 0$$

with all G_i G_C-flat. If $m \leq n$ then we are done, so suppose $m > n$ and call $K_i = \mathrm{Ker}\,(G_{i-1} \to G_{i-2})\ \forall i \geq 1$ (again $G_{-1} = M$). We want to show that K_n is G_C-flat.

For every $i \geq 1$ and every $j \geq 0$ we have $\mathrm{Tor}^R_i\left(\mathrm{Prod}_R(C^+), G_j\right) = 0$ so there is an isomorphism of abelian groups

$$\mathrm{Tor}^R_i\left(X, K_j\right) \cong \mathrm{Tor}^R_{i+j}(X, M)\ \forall i \geq 1,\ \forall j \geq 0, \forall X \in \mathrm{Prod}_R(C^+).$$

Then, using the hypotheses, we get $\mathrm{Tor}^R_i\left(\mathrm{Prod}_R(C^+), K_j\right) = 0\ \forall i \geq 1,\ \forall j \geq n$, so applying Proposition 3.2.16 to each sequence

$$0 \to K_{j+1} \to G_j \to K_j \to 0,\ j \geq n$$

($K_m = G_m$) we see by induction that $K_j \in G_C F(R)\ \forall j \geq n$.

(3)⇒(1) We know by Corollary 3.2.18 that M has a G_C-flat resolution, so the result is clear. ■

As an immediate consequence of the above, we can see that the standard functorial characterization of the dimension of a module, when it is finite, can also be proved in the case of the G_C-flat dimension. Moreover, the G_C-flat dimension of a direct sum can also be computed.

Corollary 3.4.8. *Assume R is $\mathrm{G_CF}$-closed and C is w^+-tilting. The following assertions hold:*

1. If M is of finite $\mathrm{G_C}$-flat dimension, then

$$\mathrm{G_C}\text{-}\mathrm{fd}_R(M) = \sup\left\{i \in \mathbb{N};\ \mathrm{Tor}_i^R(X,M) \neq 0 \textit{ for some } X \in \mathrm{Prod}_R(C^+)\right\}.$$

2. For any family of left R-modules $\{M_i;\ i \in I\}$, one has

$$\mathrm{G_C}\text{-}\mathrm{fd}_R(\oplus_{i\in I} M_i) = \sup\{\mathrm{G_C}\text{-}\mathrm{fd}_R(M_i);\ i \in I\}.$$

The next result provides a generalization of classical (in)equalities. It gives the connections between the $\mathrm{G_C}$-flat dimension of the modules in a short exact sequence.

Proposition 3.4.9. *Let R be $\mathrm{G_CF}$-closed and C be w^+-tiling. Given a short exact sequence of R-modules $0 \to K \to M \to N \to 0$, we have:*

(1) If any two of K, M or N have finite $\mathrm{G_C}$-flat dimension, then so has the third.

(2) $\mathrm{G_C}\text{-}\mathrm{fd}_R(K) \leq \sup\{\mathrm{G_C}\text{-}\mathrm{fd}_R(M), \mathrm{G_C}\text{-}\mathrm{fd}_R(N) - 1\}$, *and the equality holds whenever* $\mathrm{G_C}\text{-}\mathrm{fd}_R(M) \neq \mathrm{G_C}\text{-}\mathrm{fd}_R(N)$.

(3) $\mathrm{G_C}\text{-}\mathrm{fd}_R(M) \leq \sup\{\mathrm{G_C}\text{-}\mathrm{fd}_R(K), \mathrm{G_C}\text{-}\mathrm{fd}_R(N)\}$, *and the equality holds whenever* $\mathrm{G_C}\text{-}\mathrm{fd}_R(N) \neq \mathrm{G_C}\text{-}\mathrm{fd}_R(K) + 1$.

(4) $\mathrm{G_C}\text{-}\mathrm{fd}_R(N) \leq \sup\{\mathrm{G_C}\text{-}\mathrm{fd}_R(M), \mathrm{G_C}\text{-}\mathrm{fd}_R(K) + 1\}$, *and the equality holds whenever* $\mathrm{G_C}\text{-}\mathrm{fd}_R(K) \neq \mathrm{G_C}\text{-}\mathrm{fd}_R(M)$.

Proof. (1) Let

$$\cdots \to P'_m \to \cdots \to P'_0 \to K \to 0 \ \text{ and } \ \cdots \to P''_m \to \cdots \to P''_0 \to N \to 0$$

be two projective resolutions (and so $\mathrm{G_C}$-flat resolutions by Proposition 3.2.17) of K and N, respectively, and apply the Horseshoe Lemma to get the commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K_n & \longrightarrow & M_n & \longrightarrow & N_n & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P_0' & \longrightarrow & P_0' \oplus P_0'' & \longrightarrow & P_0'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

If K and N have $\mathrm{G_C}$-flat dimension $\leq n$, then both K_n and N_n are $\mathrm{G_C}$-flat modules by Theorem 3.4.7 and then $M_n \in \mathrm{G_CF}(R)$ since R is $\mathrm{G_CF}$-closed.

If $\mathrm{G_C}\text{-fd}_R(K), \mathrm{G_C}\text{-fd}_R(M) \leq n$, then $K_n, M_n \in \mathrm{G_CF}(R)$ and then $N_n \in \mathrm{G_CF}(R)$ by Proposition 3.2.16.

And if $\mathrm{G_C}\text{-fd}_R(M), \mathrm{G_C}\text{-fd}_R(N) \leq n$, then $M_n, N_n \in \mathrm{G_CF}(R)$ and then $K_n \in \mathrm{G_CF}(R)$ by Proposition 3.2.15.

We now prove *(2)* and avoid *(3)* and *(4)* since their proofs all follow the same argument.

And to prove *(2)*, we can suppose $n = \sup\{\mathrm{G_C}\text{-fd}_R(M), \mathrm{G_C}\text{-fd}_R(N) - 1\} < \infty$ since the infinite case is clear.

In this case we have $\mathrm{Tor}^R_{i>n}(X,M) = 0$ and $\mathrm{Tor}^R_{i>n+1}(X,N) = 0$ for any $X \in \mathrm{Prod}_R(C^+)$ by Theorem 3.4.7, so from the long exact sequence

$$\cdots \to \mathrm{Tor}^R_{i+1}(X,N) \to \mathrm{Tor}^R_i(X,K) \to \mathrm{Tor}^R_i(X,M) \to \mathrm{Tor}^R_i(X,N) \to \cdots,$$

we immediately see that $\mathrm{Tor}^R_i(X,K) = 0$ for every $i > n$ and every $X \in \mathrm{Prod}_R(C^+)$. But of course $\mathrm{G_C}\text{-fd}_R(K) < \infty$ by *(1)*. So, $\mathrm{G_C}\text{-fd}_R(K) \leq n$ again by Theorem 3.4.7.

Assume now that $\mathrm{G_C}\text{-fd}_R(M) \neq \mathrm{G_C}\text{-fd}_R(N)$ and let us prove that $\mathrm{Tor}^R_n(X_0,K) \neq 0$ for some $X_0 \in \mathrm{Prod}_R(C^+)$.

If $\mathrm{G_C}\text{-fd}_R(M) > \mathrm{G_C}\text{-fd}_R(N)$ then $\mathrm{G_C}\text{-fd}_R(M) = n > \mathrm{G_C}\text{-fd}_R(N)$, which implies that $\mathrm{Tor}^R_{i\geq n}(X,N) = 0$ for every $X \in \mathrm{Prod}_R(C^+)$ and there exists some $X_0 \in \mathrm{Prod}_R(C^+)$ for which $\mathrm{Tor}^R_n(X_0,M) \neq 0$. Then, from the exact sequence of Tor above, we get that $\mathrm{Tor}^R_n(X_0,K) \cong \mathrm{Tor}^R_n(X_0,M) \neq 0$.

On the other hand, if $\mathrm{G_C}$-$\mathrm{fd}_R(M) < \mathrm{G_C}$-$\mathrm{fd}_R(N)$, then $\mathrm{G_C}$-$\mathrm{fd}_R(N) = n+1$, which implies that $\mathrm{Tor}^R_{i\geq n+1}(X,M) = 0$ for every $X \in \mathrm{Prod}_R(C^+)$ and $\mathrm{Tor}^R_{n+1}(X_0,N) \neq 0$ for some $X_0 \in \mathrm{Prod}_R(C^+)$. Again, from the exact sequence above we get that $\mathrm{Tor}^R_n(X_0,K) \neq 0$. ■

Corollary 3.4.10. *When R is* $\mathrm{G_CF}$*-closed and C is w^+-tilting, the class of modules of finite* $\mathrm{G_C}$*-flat dimension is closed under finite direct sums and direct summands. Moreover, it is also projectively resolving.*

Proof. That the class is closed under finite direct sums and direct summands follows by Corollary 3.4.8, and that it is projectively resolving by Propositions 3.4.9 and 3.2.17. ■

Proposition 3.4.11. *Let R be* $\mathrm{G_CF}$*-closed, $n \geq 0$ be an integer, M be an R-module, and consider the following assertions:*

(1) $\mathrm{G_C}$-$\mathrm{fd}_R(M) \leq n$.

(2) There exists an exact sequence of R-modules $0 \to M \to V \to G \to 0$*, where G is* $\mathrm{G_C}$*-flat and* $\mathscr{F}_C - \mathrm{fd}_R(V) \leq n$.

If C is $\prod$-Tor*-orthogonal then (1)⇒(2) holds. If, in addition, C is w^+-tilting then (2)⇒(1) holds too.*

Proof. (1)⇒(2) We use induction on the $\mathrm{G_C}$-flat dimension of M.

The case $n = 0$ follows directly from Proposition 3.2.10.

Assume then that $n \geq 1$. There exists an exact sequence

$$0 \to G_n \to \cdots \to G_0 \to M \to 0$$

where $G_i \in \mathrm{G_CF}(R)$ for every $i \in \{0,...,n\}$. Let $K = \mathrm{Ker}(G_0 \to M)$. Clearly, $\mathrm{G_C}$-$\mathrm{fd}_R(K) \leq n-1$ so by induction there exists an exact sequence

$$0 \to K \to V' \to G' \to 0,$$

where G' is $\mathrm{G_C}$-flat and V' admits a $\mathscr{F}_\mathrm{C}$-flat resolution of the type:

$$0 \to F_n \to \cdots \to F_1 \to V' \to 0.$$

Consider the pushout diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & K & \longrightarrow & G_0 & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & V' & \dashrightarrow & D & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & G' & = & G' & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

By the middle column D is $\mathrm{G_C}$-flat. Then, there exists a short exact sequence of R-modules $0 \to D \to F_0 \to G \to 0$ where $F_0 \in \mathscr{F}_{\mathrm{C}}(R)$ and $G \in \mathrm{G_C F}(R)$. Consider then another pushout diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & V' & \longrightarrow & D & \longrightarrow & M & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & V' & \longrightarrow & F_0 & \dashrightarrow & V & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & G & = & G & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

We now see that $\mathscr{F}_C\text{-}\mathrm{fd}_R(V) \leq n$ since we have the $\mathscr{F}_{\mathrm{C}}$-flat resolution of V

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & F_n & \longrightarrow & \cdots & \longrightarrow & F_1 & & \longrightarrow & & F_0 & \longrightarrow & V \longrightarrow 0\ , \\
 & & & & & & & \searrow & & \nearrow & & & \\
 & & & & & & & & V' & & & &
\end{array}$$

so the exact sequence $0 \to M \to V \to G \to 0$ completes de proof.

(2)⇒(1) Follows from Proposition 3.4.9(2). ■

We mentioned above that over $\prod$-Tor-orthogonal modules C, the $\mathrm{G_C}$-flat dimension of any module is less than or equal to its $\mathscr{F}_{\mathrm{C}}$-flat dimension. But, indeed we can go a little further and see that, under the typical conditions on R, if the $\mathscr{F}_C$-flat dimension is finite then it coincides with the G_C-flat dimension.

Theorem 3.4.12. *If C is $\prod$-Tor-orthogonal, then $G_C\text{-fd}_R(M) \leq \mathscr{F}_C-\text{fd}(M)$ for every module M. If R is $G_C F$-closed and C is w^+-tilting, then*

$$\mathscr{F}_C-\text{fd}_R(M) < \infty \Rightarrow G_C\text{-fd}_R(M) = \mathscr{F}_C-\text{fd}_R(M).$$

Proof. Assume $\mathscr{F}_C-\text{fd}_R(M) < \infty$ and call $n = G_C\text{-fd}_R(M) < \infty$. We use induction on n, and the case $n = 0$ follows from Proposition 3.4.4(3), so let $n \geq 1$.

Since $\mathscr{F}_C-\text{fd}_R(M) < \infty$, there exists an exact sequence of R-modules

$$0 \to K \to F \to M \to 0$$

with $F \in \mathscr{F}_C(R)$ and $\mathscr{F}_C-\text{fd}(K) < \infty$. But, $F \in \mathscr{F}_C(R) \subseteq G_C F(R)$ so $G_C\text{-fd}_R(K) = n-1$ by Proposition 3.4.9 since R is $G_C F$-closed, and then, by induction, $\mathscr{F}_C-\text{fd}_R(K) = G_C\text{-fd}_R(K) = n-1$. Using again the above short exact sequence, we get that $\mathscr{F}_C-\text{fd}_R(M) \leq n = G_C\text{-fd}_R(M)$. Therefore, $\mathscr{F}_C-\text{fd}_R(M) = G_C\text{-fd}_R(M)$. ■

In Theorem 3.4.12, we gave a sufficient condition to have the equality between the given dimensions. Though we did not find a necessary condition, we can prove that the equality does not always hold, by giving an example in which the inequality is strict.

Example 3.4.13. *Let R be coherent and choose a non-FP-injective module M. Find an injective coresolution and a flat resolution of M and glue them together:*

$$\begin{array}{ccccccccc} \cdots \longrightarrow & F_1 & \longrightarrow & F_0 & \longrightarrow & E^0 & \longrightarrow & E^1 & \longrightarrow \cdots \\ & & & & \searrow \; M \; \nearrow & & & & \end{array}$$

Then, set $C = \oplus_{i\geq 0} E^i$. Since $(E^i)^+$ is flat for every i and R is coherent, C^+ is flat and so the exact sequence above is $(\text{Prod}_R(C^+) \otimes_R -)$-exact. This means that $G_C\text{-fd}_R(M) = 0$. However, M cannot belong to $\mathscr{F}_C(R)$ since $\text{Prod}_R(C^+) \subseteq \mathscr{F}(R)$ and M^+ is not flat, so $\mathscr{F}_C-\text{fd}_R(M) \neq 0$ (and therefore $\mathscr{F}_C-\text{fd}_R(M) = \infty$ by Theorem 3.4.12).

When C is w^+-tilting we have $\mathscr{F}(R) \subseteq G_C F(R)$ and so the G_C-flat dimension can be seen as a refinement of, not only the $\mathscr{F}_C$-flat dimension, but also the flat dimension. Therefore, a natural question comes up immediately: does Theorem 3.4.12 hold if we replace the class $\mathscr{F}_C(R)$ by $\mathscr{F}(R)$? In other words, if the flat dimension of a module is finite, does it coincide with the G_C-flat dimension of the module?

Theorem 3.4.14. *Let R be $G_C F$-closed and C be w^+-tilting. For any R-module M we have*

$$G_C\text{-fd}_R(M) \leq \text{fd}_R(M),$$

and if $R \in \mathscr{F}_C(R)$, then we have $\text{fd}_R(M) < \infty \Rightarrow G_C\text{-fd}_R(M) = \text{fd}_R(M)$.

Proof. As mentioned above we have $\mathrm{G_C}\text{-}\mathrm{fd}_R(M) \leq \mathrm{fd}_R(M)$ so we only need to prove the second assertion.

We then assume that $n = \mathrm{fd}_R(M)$ is finite and use induction on n.

If $n = 0$, there is nothing to prove.

If $n = 1$, there exists an exact sequence $0 \to F_1 \to F_0 \to M \to 0$ with F_1 and F_0 flat. We know that $\mathrm{G_C}\text{-}\mathrm{fd}_R(M) \leq \mathrm{fd}_R(M) = 1$ and if $\mathrm{G_C}\text{-}\mathrm{fd}_R(M) = 0$, then the sequence $0 \to M^+ \to F_0^+ \to F_1^+ \to 0$ would split since

$$\mathrm{Ext}^1_R(F_1^+, M^+) \cong \mathrm{Tor}^R_1(F_1^+, M)^+ = 0$$

($F_1 \in \mathscr{F}(R) \subseteq \mathscr{F}_{\mathrm{C}}(R)$). But, this means that M is flat, contradicting the hypothesis $\mathrm{fd}_R(M) = 1$. Hence, $\mathrm{G_C}\text{-}\mathrm{fd}_R(M) = 1 = \mathrm{fd}_R(M)$.

Finally, if $n > 1$, consider an exact sequence $0 \to K \to F \to M \to 0$ with F flat and $\mathrm{fd}_R(K) = n-1$. By induction $\mathrm{G_C}\text{-}\mathrm{fd}(K) = n-1$, so using 3.4.9(4) we get $\mathrm{G_C}\text{-}\mathrm{fd}(M) = n$. ■

It is also natural to compare the $\mathrm{G_C}$-flat dimension of a module and the $\mathrm{G_{C^+}}$-injective dimension of its character module.

Corollary 3.4.15. *For any R-module M the inequality*

$$\mathrm{G_{C^+}}\text{-}\mathrm{id}_R(M^+) \leq \mathrm{G_C}\text{-}\mathrm{fd}_R(M)$$

holds. If moreover C is $\prod$-Tor-*orthogonal and* $\mathscr{F}_{\mathrm{C}}(R)$ *is closed under direct products, then this inequality is indeed an equality.*

Proof. Apply Theorem 3.2.11. ■

And with the link given in the last result we can realize that the connection between the two classes is indeed deep. We state such a connection to finish this section.

Corollary 3.4.16. *Let R be* $\mathrm{G_C F}$*-closed, C be* w^+*-tilting and* $\mathscr{F}_{\mathrm{C}}(R)$ *be closed under direct products. For any integer number* $n \geq 0$ *define the classes*

$$\overline{\mathrm{G_C F}(R)}^{\,n} = \{{}_RM \mid \mathrm{G_C}\text{-}\mathrm{fd}_R(M) \leq n\} \text{ and } \overline{\mathrm{G_{C^+}I}(R)}^{\,n} = \{N_R \mid \mathrm{G_{C^+}}\text{-}\mathrm{id}_R(N) \leq n\}.$$

The pair

$$(\overline{\mathrm{G_C F}(R)}^{\,n}, \overline{\mathrm{G_{C^+}I}(R)}^{\,n})$$

is a perfect duality pair.

Proof. The given pair is a duality pair by Corollary 3.4.15 and [17, Proposition 4.7].

Moreover, $\overline{\mathrm{G_C F}(R)}^{\,n}$ is closed under direct sums by Proposition 3.4.8, it is closed under extensions by Proposition 3.4.9(3), and of course $R \in \mathrm{G_C F}(R) \subseteq \overline{\mathrm{G_C F}(R)}^{\,n}$. ■

3.5 Relative weak Gorenstein global dimension

In this section we define and study the global $\mathrm{G_C}$-flat dimension of R.

Definition 3.5.1. *The* ***global*** $\mathrm{G_C}$***-flat dimension*** *of R is defined as the supremum, if it exists, of the* $\mathrm{G_C}$*-flat dimension of every R-module:*

$$\mathrm{G_C}-\mathrm{FD}(R) := \sup\{\mathrm{G_C}\text{-}\mathrm{fd}_R(M)|\ M \textit{ is an } R\textit{-module }\}.$$

We set $\mathrm{G_C}-\mathrm{FD}(R) = \infty$ *if such a supremum does not exist.*

Remark 3.5.2.

1. *The global* $\mathrm{G_C}$*-flat dimension of rings was briefly studied in [90, 89] over commutative rings with C being a semidualizing R-module.*

2. *When ${}_RC$ is a flat generator, we recover* $\mathrm{GFD}(R) := \mathrm{G_C}-\mathrm{FD}(R)$*, the weak global Gorenstein dimension of R.*

The first main theorem of this section is Theorem 3.5.5. In order to prove it, we will need the following two lemmas.

Lemma 3.5.3. *Let R be* $\mathrm{G_CF}$*-closed and ${}_RC$ be* $\prod$-Tor*-orthogonal. For an R-module M and an integer $n \geq 1$, if* $\mathrm{G_C}\text{-}\mathrm{fd}_R(M) \leq n$*, then there exists an exact sequence of R-modules $0 \to F \to G \to M \to 0$, where G is* $\mathrm{G_C}$*-flat and* $\mathscr{F}_C-\mathrm{fd}_R(F) \leq n-1$.

Proof. By definition, there is a short exact sequence of R-modules $0 \to K \to G_0 \to M \to 0$, where G_0 is $\mathrm{G_C}$-flat and $\mathrm{G_C}\text{-}\mathrm{fd}_R(K) \leq n-1$. By Proposition 3.4.11, there is an exact sequence of R-modules $0 \to K \to F \to L \to 0$, where L is $\mathrm{G_C}$-flat and $\mathscr{F}_C-\mathrm{fd}_R(F) \leq n-1$. Then, from the pushout diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & K & \longrightarrow & G_0 & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & F & \dashrightarrow & G & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & L & = & L & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}.$$

wee see that G is $\mathrm{G_C}$-flat as G_0 and L are $\mathrm{G_C}$-flat and R is $\mathrm{G_CF}$-closed. ∎

In the absolute case (see [10, Corollary 2.3]), the following key lemma is based on the fact that the class of modules of finite flat dimension is closed under direct summands, which is not the case in our relative setting. Here, we adopt a different proof.

Lemma 3.5.4. *Let R be* $\mathrm{G_C F}$*-closed and C be a* w^+*-tilting R-module. If M is an injective R-module, then* $\mathscr{F}_C\!-\!\mathrm{fd}_R(M) = \mathrm{G_C}\text{-}\mathrm{fd}_R(M)$.

Proof. The inequality $\mathrm{G_C}\text{-}\mathrm{fd}_R(M) \leq \mathscr{F}_C\!-\!\mathrm{fd}_R(M)$ holds by Theorem 3.4.12. For the other inequality, we may assume that $n = \mathrm{G_C}\text{-}\mathrm{fd}_R(M) < \infty$.

If $n = 0$, then M is $\mathrm{G_C}$-flat and since M is injective, there exists by Proposition 3.2.10 a split exact sequence $0 \to M \to V \to L \to 0$ with $V \in \mathscr{F}_\mathrm{C}(R)$. Thus, $M \in \mathscr{F}_\mathrm{C}(R)$ and hence $\mathscr{F}_C\!-\!\mathrm{fd}_R(M) = 0$.

Let us now assume that $n \geq 1$. By Proposition 3.4.11 and Lemma 3.5.3, there exist two exact sequences of R-modules

$$0 \to M \to F_1 \to G_1 \to 0 \text{ and } 0 \to F_2 \to G_2 \to M \to 0$$

where G_1 and G_2 are $\mathrm{G_C}$-flat and $\mathscr{F}_C\!-\!\mathrm{fd}_R(F_1) \leq n$ and $\mathscr{F}_C\!-\!\mathrm{fd}_R(F_2) \leq n-1$. Since M is injective, the first sequence splits and so $M \oplus G_1 \cong F_1$. Then, adding the second sequence to $0 \to 0 \to G_1 \to G_1 \to 0$ we get a short exact sequence of the form $0 \to F_2 \to G_2 \oplus G_1 \to F_1 \to 0$.

Since $\mathscr{F}_C\!-\!\mathrm{fd}_R(F_1) \leq n$ and $\mathscr{F}_C\!-\!\mathrm{fd}_R(F_2) \leq n-1$, there exist exact sequences

$$0 \to X_{n-1} \to \cdots \to X_0 \to F_1 \to 0 \text{ and } 0 \to Y_n \to Y_{n-1} \to \cdots \to Y_0 \to F_2 \to 0$$

where $X_i, Y_j \in \mathscr{F}_\mathrm{C}(R)$. It follows by Proposition 3.1.4 that C^+ is $\prod$-self-orthogonal. This implies that $\mathrm{Ext}_R^{k\geq 1}(Y_j^+, E) = 0$ for all $j = 0, \cdots, n$ and all $E \in \mathrm{Prod}_R(C^+)$), so by dimension shifting we get that $\mathrm{Ext}_R^1(F_2^+, E) = 0$. Hence, the exact sequence

$$0 \to F_1^+ \to G_1^+ \oplus G_2^+ \to F_2^+ \to 0$$

is $\mathrm{Hom}_R(-, \mathrm{Prod}_R(C^+))$-exact and by Horseshoe Lemma we get the commutative diagram with exact rows and columns:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F_1^+ & \longrightarrow & G_1^+ \oplus G_2^+ & \longrightarrow & F_2^+ & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & X_0^+ & \longrightarrow & X_0^+ \oplus Y_0^+ & \longrightarrow & Y_0^+ & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & X_{n-1}^+ & \longrightarrow & X_{n-1}^+ \oplus Y_{n-1}^+ & \longrightarrow & Y_{n-1}^+ & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & 0 & \longrightarrow & K & \longrightarrow & Y_n^+ & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
.
$$

From the diagram, one can see that $K \cong Y_n^+ \in \mathrm{Prod}_R(C^+)$. Then, we get $\mathscr{I}_{C^+}-\mathrm{id}_R(G_1^+ \oplus G_2^+) \leq n$. By Proposition 3.2.7 and Theorem 3.2.11, $(G_1 \oplus G_2)^+$ is $\mathrm{G_{C^+}}$-injective, since G_1 and G_2 are $\mathrm{G_C}$-flat. This implies by [18, Proposition 3.4(2)] that $(G_1 \oplus G_2)^+ \cong G_1^+ \oplus G_2^+ \in \mathrm{Prod}_R(C^+)$, which gives that $G_2 \in \mathscr{F}_C(R)$. Thus, $\mathscr{F}_C-\mathrm{fd}_R(M) \leq n = \mathrm{G_C}\text{-}\mathrm{fd}_R(M)$. ■

Theorem 3.5.5. *Assume that R is* $\mathrm{G_CF}$*-closed and* $_RC$ *is* w^+*-tilting. Then, for a positive integer n, the following assertions are equivalent:*

1. $\mathrm{G_C}-\mathrm{FD}(R) \leq n$.
2. *The following two assertions hold:*
   (a) $\mathrm{fd}_R(M) \leq n$ *for every* $\mathscr{I}_{C^+}$*-injective right R-module M.*
   (b) $\mathscr{F}_C-\mathrm{fd}_R(M) \leq n$ *for every injective left R-module M.*
3. *The following two assertions hold:*
   (a) $\mathrm{fd}_R(M) \leq n$ *for every* $\mathscr{I}_{C^+}$*-injective right R-module M.*
   (b) $\mathrm{G_C}\text{-}\mathrm{fd}_R(M) \leq n$ *for every injective left R-module M.*

Consequently, the $\mathrm{G_C}$*-flat global dimension of R can be computed via the following formula:*

$$\mathrm{G_C}-\mathrm{FD}(R) = \max\{\mathrm{fd}_R(\mathrm{Prod}_R(C^+)), \mathscr{F}_C\text{−}\mathrm{fd}_R(\mathscr{I}(R))\}.$$

Proof. 2. $\Leftrightarrow$ 3. Follows by Lemma 3.5.4.

1. $\Rightarrow$ 3. The assertion (b) holds by definition. Let us prove (a).

Note that $\mathrm{fd}_R(M) \leq n$ if and only if $\mathrm{Tor}^R_{n+1}(M,A) = 0$ for every R-module A. But $\mathrm{G_C}-\mathrm{FD}(R) \leq n$ means that for every such A there is an exact sequence

$$0 \to G_n \to \cdots \to G_0 \to A \to 0$$

with $G_i \in \mathrm{G_C F}(R)$ for all $i \geq 0$. Now, $M \in \mathrm{Prod}_R(C^+)$ implies that $\mathrm{Tor}^R_{i\geq 1}(M,G_i) = 0$ for all i so $\mathrm{Tor}^R_{n+1}(M,A) \cong \mathrm{Tor}^R_1(M,G_n) = 0$.

3. $\Rightarrow$ 1. Let M be an R-module. Consider a projective resolution and an injective coresolution of M:

$$\cdots \to P_1 \to P_0 \to P_{-1} = M \to 0 \quad \text{and} \quad 0 \to M = I_{-1} \to I_0 \to I_1 \to \cdots,$$

respectively. Decomposing these exact sequences into short exact ones, we get, for every integer $i \in \mathbb{N}$,

$$0 \to N_{i+1} \to P_i \to N_i \to 0 \quad \text{and} \quad 0 \to K_i \to I_i \to K_{i+1} \to 0$$

where $N_i = \mathrm{Im}(P_i \to P_{i-1})$ and $K_i = \mathrm{Ker}(I_i \to I_{i+1})$. Adding the direct sum of the first sequences,

$$0 \to \oplus_{i\in\mathbb{N}} N_{i+1} \to \oplus_{i\in\mathbb{N}} P_i \to M \oplus (\oplus_{i\in\mathbb{N}} N_{i+1}) \to 0,$$

to the direct product of the second ones,

$$0 \to M \oplus (\prod_{i\in\mathbb{N}} K_{i+1}) \to \prod_{i\in\mathbb{N}} I_i \to \prod_{i\in\mathbb{N}} K_{i+1} \to 0,$$

we get the exact sequence

$$0 \to M \oplus (N \oplus K) \to P \oplus I \to M \oplus (N \oplus K) \to 0$$

where $N = \oplus_{i\in\mathbb{N}} N_{i+1}$, $K = \prod_{i\in\mathbb{N}} K_{i+1}$, $P = \oplus_{i\in\mathbb{N}} P_i$ and $I = \prod_{i\in\mathbb{N}} I_i$.

Now, consider a projective resolution of $M \oplus (N \oplus K)$:

$$\cdots \to H_1 \to H_0 \to M \oplus (N \oplus K) \to 0.$$

Thus, by Horseshoe Lemma, we get a commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & J_n & \longrightarrow & H & \longrightarrow & J_n & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H_{n-1} & \longrightarrow & H_{n-1}\oplus H_{n-1} & \longrightarrow & H_{n-1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H_0 & \longrightarrow & H_0\oplus H_0 & \longrightarrow & H_0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M\oplus(N\oplus K) & \longrightarrow & P\oplus I & \longrightarrow & M\oplus(N\oplus K) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Note that P is $\mathrm{G_C}$-flat by Proposition 3.2.17 and since I is injective, $\mathrm{G_C}\text{-fd}_R(P\oplus I) = \mathrm{G_C}\text{-fd}_R(I) \leq n$ by Corollary 3.4.8(2) and the hypotheses. But, since each H_i is $\mathrm{G_C}$-flat, H is $\mathrm{G_C}$-flat as well by Theorem 3.4.7.

Now, let X be any $\mathscr{I}_{\mathrm{C}^+}$-injective right R-module. By hypothesis, $\mathrm{fd}_R(X) \leq n$. Then, using the projective resolution of $M\oplus(N\oplus K)$, we get

$$\mathrm{Tor}_1^R(X,J_n) \cong \mathrm{Tor}_{n+1}^R(X,M\oplus(N\oplus K)) = 0.$$

So, the sequence $0\to J_n\to H\to J_n\to 0$ is $(\mathrm{Prod}_R(C^+)\otimes_R -)$-exact. Then, assembling these sequences, we get a $(\mathrm{Prod}_R(C^+)\otimes_R -)$-exact exact sequence

$$\cdots\to H\xrightarrow{f} H\xrightarrow{f} H\to\cdots$$

with $J_n = \mathrm{Ker} f$. Therefore, J_n is $\mathrm{G_C}$-flat by Theorem 3.3.1 and then $\mathrm{G_C}\text{-fd}_R(M) \leq \mathrm{G_C}\text{-fd}_R(M\oplus(N\oplus K)) \leq n$ by Corollary 3.4.8(2) as desired. ■

The following special case of Theorem 3.5.5 was proved by Emmanouil ([33, Theorem 5.3]) when the Gorenstein weak dimension of R is finite. Here we drop this finiteness condition.

Corollary 3.5.6. *The weak global Gorenstein dimension of R can be computed via the following simple formulas:*

$$\begin{array}{rcl}\mathrm{GFD}(R) & = & \max\{\mathrm{fd}_{R^{op}}(\mathscr{I}(R^{op})),\mathrm{Gfd}_R(\mathscr{I}(R))\} \\ & = & \max\{\mathrm{fd}_{R^{op}}(\mathscr{I}(R^{op})),\mathrm{fd}_R(\mathscr{I}(R))\}.\end{array}$$

For completeness, we state the right version of Theorem 3.5.5.

Theorem 3.5.7. *Assume that S is any* $\mathrm{G_CF}$*-closed ring and C is a w^+-tilting S-module. The global* $\mathrm{G_C}$*-flat dimension of S can be computed as follows:*

$$\begin{aligned}\mathrm{G_C}-\mathrm{FD}(S) &= \max\{\mathrm{fd}_S(\mathrm{Prod}_S(C^+)), \mathscr{F}_C\text{−}\mathrm{fd}_S(\mathscr{I}(S))\}\\ &= \max\{\mathrm{fd}_S(\mathrm{Prod}_S(C^+)), \mathrm{G_C}\text{-}\mathrm{fd}_S(\mathscr{I}(S))\}.\end{aligned}$$

Lemma 3.5.8. *Let C be a semidualizing (R,S)-bimodule. The following equalities hold:*

1. $\mathrm{fd}_S(\mathrm{Hom}_R(C,E)) = \mathscr{F}_C\text{−}\mathrm{fd}_R(E)$ *for every injective left R-module.*
   Consequently, $\mathrm{fd}_S(\mathrm{Prod}_S(C^+)) = \mathscr{F}_C\text{−}\mathrm{fd}_R(\mathscr{I}(R))$.
2. $\mathrm{fd}_R(\mathrm{Hom}_S(C,E)) = \mathscr{F}_C\text{−}\mathrm{fd}_S(E)$ *for every injective right S-module.*
   Consequently, $\mathrm{fd}_R(\mathrm{Prod}_R(C^+)) = \mathscr{F}_C\text{−}\mathrm{fd}_S(\mathscr{I}(S))$.

Proof. Assertion (2) can be proved in a similar way to (1); so we only prove (1).

First, we prove that $\mathrm{fd}_S(\mathrm{Hom}_R(C,E)) \leq \mathscr{F}_C\text{−}\mathrm{fd}_R(E)$.

We may assume that $n = \mathscr{F}_C\text{−}\mathrm{fd}_R(E) < \infty$ since the infinite case is clear. Recall (Proposition 3.1.2(2)) that with our hypothesis we have $\mathscr{F}_\mathrm{C}(R) = C \otimes_S \mathscr{F}(S)$, so there exists an exact sequence of R-modules:

$$0 \to C \otimes_S F_n \to \cdots \to C \otimes_S F_0 \to E \to 0$$

where each ${}_SF_i$ is flat. By [37, Theorem 3.2.15 and Remark 3.2.27], for every $k \geq 1$ we get

$$\begin{aligned}\mathrm{Ext}_R^k(C, C \otimes_S F_i)^+ &\cong (\mathrm{Ext}_R^k(C,C) \otimes_S F_i)^+\\ &\cong \mathrm{Hom}_S(F, \mathrm{Ext}_R^k(C,C)^+) = 0.\end{aligned}$$

Then, $\mathrm{Ext}_R^k(C, C \otimes_S F_i) = 0$ for every $i \geq 1$. Hence, the sequence of left S-modules

$$0 \to \mathrm{Hom}_R(C, C \otimes_S F_n) \to \cdots \to \mathrm{Hom}_R(C, C \otimes_S F_0) \to \mathrm{Hom}_R(C,E) \to 0$$

is exact. But, for each $i = 0, \cdots, n$

$$\mathrm{Hom}_R(C, C \otimes_S F_i) \cong \mathrm{Hom}_R(C,C) \otimes_S F_i \cong {}_SF_i$$

by [37, Theorem 3.2.14], which implies that $\mathrm{fd}_S(\mathrm{Hom}_R(C,E)) \leq n$.

Now we prove the other inequality. Suppose that $n = \mathrm{fd}_S(\mathrm{Hom}_R(C,E)) < \infty$. Then, there exists a finite flat resolution of ${}_S\mathrm{Hom}_R(C,E)$:

$$0 \to F_n \to \cdots \to F_0 \to \mathrm{Hom}_R(C,E) \to 0.$$

Using [53, Lemma 1.2.11], we get, for every $k \geq 0$,

$$\mathrm{Tor}_k^S(C, \mathrm{Hom}_R(C,E)) \cong \mathrm{Hom}_R(\mathrm{Ext}_S^k(C,C),E) \cong \begin{cases} E \text{ if } k=0 \\ \\ 0 \text{ if } k \geq 1 \end{cases}.$$

Then, we get an exact sequence

$$0 \to C \otimes_S F_n \to \cdots \to C \otimes_S F_0 \to E \to 0$$

where each $C \otimes_S F_i$ is $\mathscr{F}_{\mathrm{C}}$-flat. Hence, $\mathscr{F}_C\mathrm{-fd}_R(E) \leq n = \mathrm{fd}_S(\mathrm{Hom}_R(C,E))$.

Finally, keeping in mind that $\mathrm{Prod}_S(C^+) = \mathrm{Hom}_R(C, \mathscr{I}(R))$, we get the last equality. ■

The following result is the second main result of this section.

Theorem 3.5.9. *Let C be a semidualizing (R,S)-bimodule such that R and S are* $\mathrm{G_C}$F-*closed rings. Then*

$$\mathrm{G_C} - \mathrm{FD}(R) = \mathrm{G_C} - \mathrm{FD}(S).$$

In this case, we define the common value of these two numbers to be the $\mathrm{G_C}$*-flat dimension of the pair of rings (R,S) and we denote it by* $\mathrm{G_C} - \mathrm{FD}(R,S)$.

Proof. By Theorems 3.5.5 and 3.5.7, we have the following two formulas:

$$\mathrm{G_C} - \mathrm{FD}(R) = \max\{\mathrm{fd}_R(\mathrm{Prod}_R(C^+)), \mathscr{F}_C\mathrm{-fd}_R(\mathscr{I}(R))\},$$

$$\mathrm{G_C} - \mathrm{FD}(S) = \max\{\mathrm{fd}_S(\mathrm{Prod}_S(C^+)), \mathscr{F}_C\mathrm{-fd}_S(\mathscr{I}(S))\}.$$

On the other hand, by Lemma 3.5.8, we know that

$$\mathrm{fd}_R(\mathrm{Prod}_R(C^+)) = \mathscr{F}_C\mathrm{-fd}_S(\mathscr{I}(S)) \text{ and } \mathrm{fd}_R(\mathrm{Prod}_S(C^+)) = \mathscr{F}_C\mathrm{-fd}_R(\mathscr{I}(R)).$$

Therefore, $\mathrm{G_C} - \mathrm{FD}(R) = \mathrm{G_C} - \mathrm{FD}(S)$. ■

We end this section with some consequences of Theorem 3.5.9.

Substituting the $\mathrm{G_C}$F-closeness condition in Theorem 3.5.9 by the coherence condition, keeping in mind Corollary 3.2.13, we get the following special case.

Corollary 3.5.10. *Let C be a semidualizing (R,S)-bimodule such that R is left coherent and S is right coherent. Then*

$$\mathrm{G_C} - \mathrm{FD}(R) = \mathrm{G_C} - \mathrm{FD}(S).$$

Bennis conjectured in [10] that the weak global Gorenstein dimension of R is symmetric. That is, the equality $\mathrm{GFD}(R) = \mathrm{GFD}(R^{op})$ holds.

This conjecture has been investigated by many authors. Mahdou and Tamekkante [71, Corollary 2.8] solved it in the case where R is (two-sided) coherent and Emmanouil [33, Theorem 5.3] solved it in the case where both R and R^{op} have finite Gorenstein weak global dimension. Bouchiba [22, Corollary 3.5], on the other hand, showed that this conjecture is true when the classes of Gorenstein flat left and right R-modules are closed under extensions. But, this is always true by Corollary 3.2.24. Thus, this conjecture is solved. Recently, Christensen, Estrada, and Thompson have re-established this fact using a different approach [29, Corollary 1.4].

As a direct consequence of Theorem 3.5.9, we obtain a third proof. Note that our proof is simple and completely different from the ones given in [22] and [29].

Corollary 3.5.11. *([22, Corollary 3.5.] and [29, Corollary 1.4]). Over any ring R, the weak global Gorenstein dimension of R is symmetric, that is, we have the equality:*

$$\mathrm{GFD}(R) = \mathrm{GFD}(R^{op}).$$

Recall from [10] that a ring R is n-IF if every left and right injective R-module has flat dimension at most n. A two-sided noetherian ring is n-IF if and only if it is n-Gorenstein by [37, Theorem 9.1.11]. Bennis characterized in [10, Theorem 2.8] n-IF rings provided that they are (two-sided) coherent. As another consequence of Theorem 3.5.5, we drop the coherence assumption.

Corollary 3.5.12. *A ring R is n-IF if and only if* $\mathrm{GFD}(R) \leq n$.

Assume that R is left coherent and S is right coherent. Zhu and Ding ([91, Theorem 2.6]) proved that if ${}_RC$ and C_S have finite FP-injective dimension, then $\text{FP-id}_R(C) = \text{FP-id}_S(C)$. We show next that this happens exactly when R (or S) has finite global $\mathrm{G_C}$-flat dimension. But first, we need the following lemma.

Lemma 3.5.13.

1. *If R is left coherent, then* $\mathrm{fd}_R(\mathrm{Prod}_R(C^+)) = \text{FP-id}_R(C)$.
2. *If S is right coherent, then* $\mathrm{fd}_S(\mathrm{Prod}_S(C^+)) = \text{FP-id}_S(C)$.

Proof. We only prove (1) since (2) has a similar proof.

Following [43, Theorem 2.2], we get that $\text{FP-id}_R(C) = \mathrm{fd}_R(C^+) \leq \mathrm{fd}_R(\mathrm{Prod}_R(C^+))$. Conversely, if $\text{FP-id}_R(C) = \infty$, then the equality holds true. So, we may assume that $n = \text{FP-id}_R(C) < \infty$.

Let $X \in \mathrm{Prod}_R(C^+)$. Then, X is a direct summand of some $(C^+)^I$. Using again [43, Theorem 2.2], we get that $\mathrm{fd}_R(C^+) = \text{FP-id}_R(C) = n$. But, the direct product of any family flat resolutions is a flat resolution by [37, Theorem 3.2.24]. This means that $\mathrm{fd}_R((C^+)^I) \leq n$. Hence, $\mathrm{fd}_R(X) \leq \mathrm{fd}_R((C^+)^I) \leq n$. Consequently, $\mathrm{fd}_R(\mathrm{Prod}_R(C^+)) \leq n$.
■

Assume that R and S are left and right noetherian rings, respectively. Recall ([38, Definition 3.1]) that a semidualizing (R,S)-bimodule is called dualizing if C_S and $_RC$ both have finite injective dimension. Replacing noetherian by coherent and injective by FP-injective, we get the following weaker notion.

Definition 3.5.14. *Let R and S be left and right coherent rings, respectively. A semidualizing (R,S)-bimodule C is said to be **weak dualizing** if $_RC$ and C_S both have finite FP-injective dimension. In this case, C is said to be **weak n-dualizing** where* $n = \text{FP-id}_R(C) = \text{FP-id}_S(C)$.

Corollary 3.5.15. *Assume that C is a semidualizing (R,S)-bimodule such that R is left coherent and S is right coherent. Then,* $\mathrm{G_C}-\mathrm{FD}(R,S) \leq n$ *if and only if C is weak n-dualizing.*

Consequently, $\mathrm{G_C}-\mathrm{FD}(R,S) = \max\{\text{FP-id}_R(C), \text{FP-id}_S(C)\}$.

Proof. Follows from Corollary 3.5.10, Theorem 3.5.5, Lemmas 3.5.8 and 3.5.13 and the comment just before Lemma 3.5.13. ■

Corollary 3.5.16. *Assume that C is a semidualizing (R,S)-bimodule such that R is left coherent and S is right coherent. Then,* $\mathrm{G_C}-\mathrm{FD}(R,S) = 0$ *if and only if both $_RC$ and C_S are FP-injective modules.*

RELATIVE GORENSTEIN FLAT MODULES AND THEIR MODEL STRUCTURES

Given a ring R and an R-module C, we introduce and study new concepts of relative Gorenstein cotorsion and cotorsion modules: $\mathrm{G_C}$-cotorsion and (strongly) $\mathscr{C}_\mathrm{C}$-cotorsion modules. As an application, we prove that there is a unique hereditary abelian model structure on the category of R-modules, in which the cofibrations are the monomorphisms with $\mathrm{G_C}$-flat cokernel and the fibrations are the epimorphisms with $\mathscr{C}_\mathrm{C}$-cotorsion kernel belonging to the Bass class $\mathscr{B}_C(R)$. Moreover, we also give a concrete description of the weak equivalences under the assumption that R has finite global $\mathrm{G_C}$-flat dimension. To prove this point, an interesting connection between abelian model structures and AB-weak contexts is proved. This connection leads to a result that can be applied to obtain abelian model structures with a simpler description of trivial objects.

Throughout this chapter, unless otherwise stated, S will be the endomorphism ring of C, $S = \mathrm{End}_R(C)$, and $\mathscr{A}$ an abelian category.

4.1 Relative cotorsion modules

In this section we introduce some classes of relatively cotorsion modules. Besides their links with other known classes of modules (cotorsion, flat, $\mathscr{F}_\mathrm{C}$-flat, etc.), we are interested in discovering the main homological properties of these new classes.

Recall that a module M is **cotorsion** if $\mathrm{Ext}^1(F,M) = 0$ for every flat module F, equivalently, if $\mathrm{Ext}^i(F,M) = 0$ for every flat module F and $i \geq 1$ ([37, Definiton 5.3.22]). In the following definition, we extend the concept of cotorsion modules to our relative setting.

Definition 4.1.1. *Given an integer $n \geq 1$, an R-module M is said to be **n-$\mathscr{C}_\mathrm{C}$-cotorsion** if $\mathrm{Ext}^i_R(N,M) = 0$ for all $\mathscr{F}_\mathrm{C}$-flat modules N and $1 \leq i \leq n$.*

• *M is called* ***$\mathscr{C}_{\mathrm{C}}$-cotorsion*** *if it is 1-$\mathscr{C}_{\mathrm{C}}$-cotorsion.*
• *M is called* ***strongly $\mathscr{C}_{\mathrm{C}}$-cotorsion*** *if it is n-$\mathscr{C}_{\mathrm{C}}$-cotorsion for every $n \geq 1$.*

We use $\mathscr{C}_{\mathrm{C}}(R)$ (resp., $\mathscr{C}^n_{\mathrm{C}}(R)$, $\mathscr{S}\mathscr{C}_C(R)$) to denote the class of all $\mathscr{C}_{\mathrm{C}}$-cotorsion (resp., n-$\mathscr{C}_{\mathrm{C}}$-cotorsion, strongly $\mathscr{C}_{\mathrm{C}}$-cotorsion) R-modules.

Remarks 4.1.2.

1. *When R is a commutative noetherian ring and $_RC$ is a semidualizing R-module, strongly $\mathscr{C}_{\mathrm{C}}$-cotorsion modules coincide with the C-cotorsion modules defined in [79] and the strongly C-cotorsion modules defined in [26].*

2. *Given an integer $n \geq 1$, every $(n+1)$-$\mathscr{C}_{\mathrm{C}}$-cotorsion R-module is n-$\mathscr{C}_{\mathrm{C}}$-cotorsion. Moreover, we have the following ascending sequence:*

$$\mathscr{S}\mathscr{C}_C(R) \subseteq \cdots \subseteq \mathscr{C}^{n+1}_{\mathrm{C}}(R) \subseteq \mathscr{C}^n_{\mathrm{C}}(R) \subseteq \cdots \subseteq \mathscr{C}_{\mathrm{C}}(R),$$

where $\mathscr{S}\mathscr{C}_C(R)$ can be written as $\mathscr{S}\mathscr{C}_C(R) = \cap_{n\geq 1}\mathscr{C}^n_{\mathrm{C}}(R)$.

Examples 4.1.3.

1. *When $_RC$ is a flat generator, $\mathscr{C}(R) = \mathscr{C}_{\mathrm{C}}(R) = \mathscr{S}\mathscr{C}_C(R)$.*

2. *Every injective module is (strongly) $\mathscr{C}_{\mathrm{C}}$-cotorsion.*

3. *Assume that $_RC$ is $\prod$-Tor-orthogonal. Given any $\mathscr{F}_{\mathrm{C}}$-cover $\varphi : F \to M$ (which exists by Proposition 3.1.6(1)), $\mathrm{Ker}\varphi$ is a $\mathscr{C}_{\mathrm{C}}$-cotorsion module (Lemma 1.5.4(1)).*

4. *Recall that a module M is called* ***Gorenstein cotorsion*** *if $\mathrm{Ext}^1_R(G,M) = 0$ for every Gorenstein flat module G. If $_RC$ is flat, then $\mathscr{F}_{\mathrm{C}}(R) \subseteq \mathscr{F}(R) \subseteq \mathscr{G}\mathscr{F}(R)$. Hence, both cotorsion and Gorenstein cotorsion modules are $\mathscr{C}_{\mathrm{C}}$-cotorsion.*

5. *Assume that every $\mathscr{F}_{\mathrm{C}}$-flat R-module has finite injective dimension. Then, $\mathscr{I}(R)^{\perp_\infty} \subseteq \mathscr{S}\mathscr{C}_C(R)$. In particular, every Gorenstein injective R-module is (strongly) $\mathscr{C}_{\mathrm{C}}$-cotorsion.*

It is unknown whether or not the class of $\mathscr{C}_{\mathrm{C}}$-cotorsion R-modules is closed under cokernels of monomorphisms. However, when this happens, all the introduced relative cotorsion R-modules coincide.

Proposition 4.1.4. *Let $n \geq 1$ be an integer.*

(1) *The class $\mathscr{S}\mathscr{C}_C(R)$ is closed under cokernels of monomorphisms.*

(2) *The following assertions are equivalent:*

(a) The class of n-$\mathscr{C}_{\mathrm{C}}$-cotorsion modules is closed under cokernels of monomorphisms.

(b) Every n-$\mathscr{C}_{\mathrm{C}}$-cotorsion module is $(n+1)$-$\mathscr{C}_{\mathrm{C}}$-cotorsion.

(c) Every n-$\mathscr{C}_{\mathrm{C}}$-cotorsion module is strongly $\mathscr{C}_{\mathrm{C}}$-cotorsion.

In this case, $\mathscr{S}\mathscr{C}_C(R) = \cdots = \mathscr{C}_{\mathrm{C}}^{n+1}(R) = \mathscr{C}_{\mathrm{C}}^{n}(R)$.

Proof. (1) Straightforward.

(2) We only prove $(a) \Rightarrow (b)$, since the other implications are clear. Let X be n-$\mathscr{C}_{\mathrm{C}}$-cotorsion and consider a short exact sequence of R-modules

$$0 \to X \to I \to L \to 0$$

with I injective. By hypothesis, L is n-$\mathscr{C}_{\mathrm{C}}$-cotorsion. So, the exact sequence

$$0 = \mathrm{Ext}_R^n(F,L) \to \mathrm{Ext}_R^{n+1}(F,X) \to \mathrm{Ext}_R^{n+1}(F,I) = 0$$

shows that $\mathrm{Ext}_R^{n+1}(F,X) = 0$ for every $F \in \mathscr{F}_{\mathrm{C}}(R)$. ∎

In light of Proposition 3.1.2, it is natural to wonder whether there is a relation between $\mathscr{C}_{\mathrm{C}}$-cotorsion R-modules and cotorsion S-modules. The following result gives such a useful relation.

Proposition 4.1.5. *Assume that ${}_RC$ is finitely presented and let $n \geq 1$, be an integer. An R-module M is n-$\mathscr{C}_{\mathrm{C}}$-cotorsion if and only if $M \in C^{\perp_n}$ and* $\mathrm{Hom}_R(C,M)$ *is a cotorsion left S-module.*

Consequently, $\mathscr{B}_C(R) \cap \mathscr{C}_{\mathrm{C}}^n(R) = C \otimes_S (\mathscr{A}_C(S) \cap \mathscr{C}(S))$.

Proof. $(\Rightarrow)$ Assume that M is n-$\mathscr{C}_{\mathrm{C}}$-cotorsion. Clearly $M \in C^{\perp_n}$, since ${}_RC$ is $\mathscr{F}_{\mathrm{C}}$-flat. We prove now that $\mathrm{Hom}_R(C,M)$ is a cotorsion S-module.

Let F be a flat S-module and consider an exact sequence of left S-modules

$$0 \to K \to P \to F \to 0$$

with P projective. Note that this sequence is pure, which implies that ${}_SK$ is flat and the induced sequence $0 \to C \otimes_S K \to C \otimes_S P \to C \otimes_S F \to 0$ is exact. Since $C \otimes_S F$ is $\mathscr{F}_{\mathrm{C}}$-flat, $\mathrm{Ext}_R^1(C \otimes_S F, M) = 0$. Then, we can construct the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
\mathrm{Hom}_R(C\otimes_S P, M) & \longrightarrow & \mathrm{Hom}_R(C\otimes_S K, M) & \longrightarrow & 0 & & \\
\downarrow \cong & & \downarrow \cong & & & & \\
\mathrm{Hom}_S(P, \mathrm{Hom}_R(C,M)) & \longrightarrow & \mathrm{Hom}_S(K, \mathrm{Hom}_R(C,M)) & \longrightarrow & \mathrm{Ext}_S^1(F, \mathrm{Hom}_R(C,M)) & \longrightarrow & 0
\end{array}$$

where the first two vertical morphisms are the adjoint isomorphisms. We then see that $\mathrm{Ext}^1_S(F,\mathrm{Hom}_R(C,M))=0$ and hence that $\mathrm{Hom}_R(C,M)$ is cotorsion.

($\Leftarrow$) Conversely, we proceed by induction on n.

Consider an exact sequence of R-modules $0\to M\to I\to L\to 0$, where I is injective. Since $\mathrm{Ext}^1_R(C,M)=0$, the induced sequence

$$0\to \mathrm{Hom}_R(C,M)\to \mathrm{Hom}_R(C,I)\to \mathrm{Hom}_R(C,L)\to 0$$

is exact. Let $C\otimes_S F$ be an $\mathscr{F}_C$-flat R-module and $n=1$.

By the the implication ($\Rightarrow$), $\mathrm{Hom}_R(C,I)$ is cotorsion since I is $\mathscr{C}_C$-cotorsion. Then, $\mathrm{Ext}^1_R(F,\mathrm{Hom}_R(C,I))=0$. Also, we have by hypothesis that $\mathrm{Ext}^1_R(F,\mathrm{Hom}_R(C,M))=0$. Consider now the commutative diagram with exact rows

$$\begin{array}{ccccccc}
\mathrm{Hom}_R(C\otimes_S F,I) & \longrightarrow & \mathrm{Hom}_R(C\otimes_S F,L) & \longrightarrow & \mathrm{Ext}^1_R(C\otimes_S F,M) & \to & 0\\
\downarrow\cong & & \downarrow\cong & & & & \\
\mathrm{Hom}_S(F,\mathrm{Hom}_R(C,I)) & \to & \mathrm{Hom}_S(F,\mathrm{Hom}_R(C,L)) & & \longrightarrow & & 0
\end{array}$$

where the first two vertical maps are the adjoint isomorphisms. Hence, $\mathrm{Ext}^1_R(C\otimes_S F,M)=0$ and then M is $\mathscr{C}_C$-cotorsion.

Assume now that $n\geq 2$. By induction, M is n-$\mathscr{C}_C$-cotorsion, so we only need to prove that $\mathrm{Ext}^{n+1}_R(C\otimes_S F,M)=0$.

Since ${}_RI$ is injective (and then $(n+1)$-$\mathscr{C}_C$-cotorsion), $\mathrm{Hom}_R(C,I)$ is a cotorsion S-module and then the exact sequences

$$0=\mathrm{Ext}^k_R(C,I)\to \mathrm{Ext}^k_R(C,L)\to \mathrm{Ext}^{k+1}_R(C,M)=0$$

and

$$0=\mathrm{Ext}^k_S(F,\mathrm{Hom}_R(C,I))\to \mathrm{Ext}^k_S(F,\mathrm{Hom}_R(C,L))\to \mathrm{Ext}^k_S(F,\mathrm{Hom}_R(C,M))=0$$

show that $\mathrm{Ext}^k_S(F,\mathrm{Hom}_R(C,L))=0=\mathrm{Ext}^k_R(C,L)$ for every $k=1,\ldots,n$. Using induction again, we get that L is n-$\mathscr{C}_C$-cotorsion. Now using the exact sequence

$$0=\mathrm{Ext}^n_R(C\otimes_S F,L)\to \mathrm{Ext}^{n+1}_R(C\otimes_S F,M)\to \mathrm{Ext}^{n+1}_R(C\otimes_S F,I)=0,$$

we get that $\mathrm{Ext}^{n+1}_R(C\otimes_S F,M)=0$ as desired, and the proof of the equivalence is finished.

We prove now the equality $\mathscr{B}_C(R)\cap\mathscr{C}^n_C(R)=C\otimes_S(\mathscr{A}_C(S)\cap\mathscr{C}(S))$.

Let M be an R-module. If $M\in\mathscr{B}_C(R)\cap\mathscr{C}^n_C(R)$, then $M=C\otimes_S F$ for some ${}_SF\in\mathscr{A}_C(S)$. Moreover, since $M\in\mathscr{C}^n_C(R)$, $F\cong\mathrm{Hom}_S(C,C\otimes_S F)=\mathrm{Hom}_S(C,M)$ is a cotorsion S-module by the above equivalence. Hence, $M=C\otimes_S F\in C\otimes_S(\mathscr{A}_C(S)\cap\mathscr{C}(S))$.

For the other inclusion, assume that $M=C\otimes_S F$ with ${}_SF\in\mathscr{A}_C(S)\cap\mathscr{C}(S)$. Clearly, M is in $\mathscr{B}_C(R)$. On the other hand, we have that $\mathrm{Hom}_R(C,M)\cong F$ is a cotorsion S-module and $\mathrm{Ext}^i_R(C,M)=0$ for every $i\geq 1$, since $M\in\mathscr{B}_C(R)$. Using again the equivalence proved above, we get that $M\in\mathscr{C}^n_C(R)$. Thus, the equality holds. ∎

Corollary 4.1.6. *Assume that ${}_RC$ is finitely presented. An R-module M is strongly $\mathscr{C}_C$-cotorsion if and only if $M \in C^{\perp_\infty}$ and* $\operatorname{Hom}_R(C,M)$ *is a cotorsion left S-module.*

Consequently, $\mathscr{B}_C(R) \cap \mathscr{S}\mathscr{C}_C(R) = C \otimes_S (\mathscr{A}_C(S) \cap \mathscr{C}(S)) = \mathscr{B}_C(R) \cap \mathscr{C}_C(R)$.

Given a regular cardinal number κ, following [40, Definition 2.1] and [68, Definition 3.6], a class $\mathscr{X}$ of R-modules or complexes of R-modules is a **κ-Kaplansky** class if for every object $M \in \mathscr{X}$ and every $x \in M$ there exists a subobject N of M that contains x, with the property that $|N| \leq \kappa$ and both N and M/N are in $\mathscr{X}$. We say that $\mathscr{X}$ is a **Kaplansky** class if it is a κ-Kaplansky class for some regular cardinal κ.

Two proofs were given by Bican, El Bashir and Enochs in [21], showing that the class of flat modules forms the left side of a complete cotorsion pair $(\mathscr{F}(R),\mathscr{C}(R))$. In that paper, the authors solved the Flat cover Conjecture. In the following result, whose proof is inspired by that given by Enochs, we prove a relative version of this conjecture.

Theorem 4.1.7. *Let C be* $\prod$-Tor-*orthogonal. The following assertions hold:*

1. $\left({}^{\perp}\mathscr{C}_C(R),\mathscr{C}_C(R)\right)$ *is a complete cotorsion pair cogenerated by a set. Moreover, every R-module has a $\mathscr{C}_C$-cotorsion envelope.*

2. $\left({}^{\perp}\mathscr{S}\mathscr{C}_C(R),\mathscr{S}\mathscr{C}_C(R)\right)$ *is a complete hereditary cotorsion pair cogenerated by a set.*

3. *The following assertions are equivalent:*

   (a) $\left(\mathscr{F}_C(R),\mathscr{F}_C(R)^{\perp}\right)$ *is a perfect cotorsion pair cogenerated by a set.*

   (b) *Every R-module has a special $\mathscr{F}_C$-flat precover.*

   (c) *Every flat R-module is $\mathscr{F}_C$-flat.*

   (d) *R is a $\mathscr{F}_C$-flat R-module.*

Proof. 1. Assume that $\kappa \geq |R| + \aleph_0$ is a regular cardinal number and let $F \in \mathscr{F}_C(R)$. Proceeding by transfinite induction we construct a continuous chain

$$0 \neq F_0 \subseteq F_1 \subseteq \cdots \subseteq F_\alpha \subseteq F_{\alpha+1} \subseteq \cdots \subseteq F$$

of pure submodules of F with each $F_0, F_{\alpha+1}/F_\alpha \in \mathscr{F}_C(R)$ and $|F_0|$, $|F_{\alpha+1}/F_\alpha| \leq \kappa$.

• Let $x \in F$. By [53, Lemma 1.2.17(a)], there is a pure submodule $F_0 \subseteq F$ with $x \in F_0$ and such that $|F_0| \leq \kappa$. Since the class $\mathscr{F}_C(R)$ is closed under pure submodules and pure quotients by Proposition 3.1.6(2), $F_0, F/F_0 \in \mathscr{F}_C(R)$. This means, in particular, that $\mathscr{F}_C(R)$ is a κ-Kaplansky class.

• For any ordinal number α, consider F/F_α and choose any element $x_{\alpha+1} + F_\alpha \in F/F_\alpha$. Using again [53, Lemma 1.2.17(a)], there exists a pure submodule $F_{\alpha+1}/F_\alpha \subseteq F/F_\alpha$ with $x_{\alpha+1} + F_\alpha \in F_{\alpha+1}/F_\alpha$ and such that $|F_{\alpha+1}/F_\alpha| \leq \kappa$. Since $F_\alpha \subseteq F$ and

$F_{\alpha+1}/F_\alpha \subseteq F/F_\alpha$ are pure submodules, $F_{\alpha+1} \subseteq F$ is a pure submodule by [53, Lemma 1.2.17(b)]. Moreover, $F_{\alpha+1}/F_\alpha$, $F/F_{\alpha+1} \cong (F/F_\alpha)/(F_{\alpha+1}/F_\alpha) \in \mathscr{F}_C(R)$.

• For a limit ordinal β, we define $F_\beta = \cup_{\alpha<\beta} F_\alpha$. As each F_α is a pure submodule of F, then so is F_β by [53, Lemma 1.2.17(d)].

Therefore, we see we can find an ordinal number λ such that $F = \cup_{\alpha<\lambda} F_\alpha$ with $(F_\alpha)_{\alpha<\lambda}$ being a continuous chain of submodules of F with desired properties.

Choose a representative of each isomorphism class of $\mathscr{F}_C(R)$ with cardinality at most κ and let X be their direct sum. Clearly, $\mathscr{C}_C(R) \subseteq X^\perp$ and Eklof's Lemma [37, Theorem 7.3.4] gives that $X^\perp \subseteq \mathscr{F}_C(R)^\perp = \mathscr{C}_C(R)$. Thus, $\mathscr{C}_C(R) = X^\perp$ and hence $\big({}^\perp\mathscr{C}_C(R), \mathscr{C}_C(R)\big)$ is a complete cotorsion pair by [37, Theorem 7.4.1].

The last statement follows by [40, Theorem 2.8].

2. If we prove that $\mathscr{S}\mathscr{C}_C(R) = M^\perp$ for some R-module M, this assertion will follow by Proposition 4.1.4 and Theorem 1.5.13.

By the proof of item 1, we have $\mathscr{C}_C(R) = X^\perp$. We claim that $\mathscr{S}\mathscr{C}_C(R) = X^{\perp_\infty}$. Clearly, $\mathscr{S}\mathscr{C}_C(R) \subseteq X^{\perp_\infty}$. Conversely, take $N \in X^{\perp_\infty}$ and let $0 \to N \to I \to L \to 0$ be a short exact sequence of R-modules where I is injective. Note that $L \in X^{\perp_\infty} \subseteq X^\perp = \mathscr{C}_C(R)$, so by the long exact sequence, we get that

$$0 = \mathrm{Ext}^1_R(F,L) \to \mathrm{Ext}^2_R(F,N) \to \mathrm{Ext}^2_R(F,I) = 0$$

for every $F \in \mathscr{F}_C(R)$. Hence, $\mathrm{Ext}^2_R(F,N) = 0$.

Repeating this process, we get that $\mathrm{Ext}^i_R(F,N) = 0$ for every $i \geq 1$. Therefore, $X^{\perp_\infty} \subseteq \mathscr{S}\mathscr{C}_C(R)$ and then $\mathscr{S}\mathscr{C}_C(R) = X^{\perp_\infty}$. This means by Lemma 1.5.17 that $\mathscr{S}\mathscr{C}_C(R) = X^{\perp_\infty} = M^\perp$ for some R-module M.

3. The implications $(a) \Rightarrow (b)$ and $(c) \Rightarrow (d)$ are obvious.

$(b) \Rightarrow (c)$ Let F be a flat R-module and consider a special $\mathscr{F}_C$-flat precover of F: $0 \to K \to X \to F \to 0$. Since F is flat, this sequence is pure. But, $\mathscr{F}_C(R)$ is closed under pure quotient, so F is $\mathscr{F}_C$-flat.

$(d) \Rightarrow (a)$ By Proposition 3.1.6(4), $\mathscr{F}_C(R)$ is closed under direct limits. So, by (1) and Theorem 1.5.10, we only need to show that $\mathscr{F}_C(R) = {}^\perp\mathscr{C}_C(R)$.

Clearly, $\mathscr{F}_C(R) \subseteq {}^\perp\mathscr{C}_C(R)$. Conversely, take $X \in {}^\perp\mathscr{C}_C(R)$ and consider an $\mathscr{F}_C$-flat cover $f : F \to X$, which exists by Proposition 3.1.6(1). Since the class $\mathscr{F}_C(R)$ is closed under direct sums and summands and ${}_RR$ is $\mathscr{F}_C$-flat, we get that $\mathscr{P}(R) \subseteq \mathscr{F}_C(R)$. Hence, the morphism f is surjective and $K = \mathrm{Ker} f$ is $\mathscr{C}_C$-cotorsion by Lemma 1.5.4. But since $X \in {}^\perp\mathscr{C}_C(R)$, X will be a direct summand of $F \in \mathscr{F}_C(R)$. Hence, $X \in \mathscr{F}_C(R)$ and thus $\mathscr{F}_C(R) = {}^\perp\mathscr{C}_C(R)$. ■

4.2 Relative Gorenstein cotorsion modules

In this section, we prove that the class of G_C-flat modules is the left hand class of a perfect hereditary cotorsion pair. Consequently, every module has a G_C-flat cover.

Definition 4.2.1. *An R-module M is said to be* $\mathrm{G_C}$***-cotorsion*** *if* $\mathrm{Ext}^1_R(N,M)=0$ *for all* $\mathrm{G_C}$*-flat modules N.*

We use $\mathrm{G_CC}(R)$ *to denote the class of all* $\mathrm{G_C}$*-cotorsion R-modules.*

Remark 4.2.2. *When* $C=R$*,* $\mathrm{G_CC}(R)$ *coincides with* $\mathscr{GC}(R)=\mathscr{GF}(R)^{\perp}$*, the class of* ***Gorenstein cotorsion*** *modules.*

$\mathrm{G_C}$-cotorsion modules are at the same time cotorsion and (strongly) $\mathscr{C}_{\mathrm{C}}$-cotorsion, as the following result shows.

Proposition 4.2.3. *Let C be* $\prod$-Tor*-orthogonal. Then, every* $\mathrm{G_C}$*-cotorsion module is* $\mathscr{C}_{\mathrm{C}}$*-cotorsion. Moreover, if R is* $\mathrm{G_CF}$*-closed and C is* w^+*-tilting then*

$$\mathrm{G_CC}(R)=\mathscr{C}(R)\cap\mathscr{SC}_C(R)\cap\mathscr{L}_C(R),$$

where $\mathscr{L}_C(R)$ *is the class of R-modules M such that the complex* $\mathrm{Hom}_R(\boldsymbol{X},M)$ *is exact for any complete* $\mathscr{F}_{\mathrm{C}}$*-flat complex* $\boldsymbol{X}$*.*

Proof. The first statement follows from the inclusion $\mathscr{F}_{\mathrm{C}}(R)\subseteq\mathrm{G_CF}(R)$ by Corollary 3.2.9 . Now we prove the equality.

($\subseteq$) Assume that M is $\mathrm{G_C}$-cotorsion. Then, M is cotorsion since $\mathscr{F}(R)\subseteq\mathrm{G_CF}(R)$ by Proposition 3.2.17.

Since $\mathscr{F}_{\mathrm{C}}(R)\subseteq\mathrm{G_CF}(R)$, $\mathrm{G_CF}(R)^{\perp_\infty}\subseteq\mathscr{SC}_C(R)$. So, to show that $M\in\mathscr{SC}_C(R)$ it suffices to show that $\mathrm{Ext}^i_R(G,M)=0$ for every $i\geq 2$ and every $G\in\mathrm{G_CF}(R)$.

Take a projective resolution of G

$$\cdots\to P_1\to P_0\to G\to 0$$

and let $K_i=\mathrm{Ker}(P_{i-1}\to P_{i-2})$. Since $P_i,G\in\mathrm{G_CF}(R)$, each $K_i\in\mathrm{G_CF}(R)$ by Proposition 3.2.15(1). Hence, $\mathrm{Ext}^i_R(G,M)\cong\mathrm{Ext}^{i-1}_R(K_1,M)\cong\cdots\cong\mathrm{Ext}^1_R(K_{i-1},M)=0$.

Consider now a complete $\mathscr{F}_{\mathrm{C}}$-flat complex

$$\mathbf{X}:\cdots\to F_1\to F_0\to C_{-1}\to C_{-2}\to\cdots.$$

By Corollaries 3.2.9 and 3.2.19, every image $I_i=\mathrm{Im}(F_{i+1}\to F_i)$ and kernel $K_j=\mathrm{Ker}(C_j\to C_{j-1})$ is $\mathrm{G_C}$-flat. Then, $\mathrm{Ext}^1_R(I_i,M)=0=\mathrm{Ext}^1_R(K_j,M)$ for all $i\geq 0$ and $j\leq -1$, which implies that $\mathbf{X}$ is $\mathrm{Hom}_R(-,M)$-exact.

($\supseteq$) Let $M\in\mathscr{C}(R)\cap\mathscr{SC}_C(R)\cap\mathscr{L}_C(R)$ and N be $\mathrm{G_C}$-flat. Then, there exists a complete $\mathscr{F}_{\mathrm{C}}$-flat resolution $\mathbf{X}$ as above such that $N=\mathrm{Ker}(C_{-1}\to C_{-2})$. Consider the short exact sequence $0\to I_0\to F_0\to N\to 0$. Since this sequence is $\mathrm{Hom}_R(-,M)$-exact and M is cotorsion, the exactness of sequence

$$0\to\mathrm{Hom}_R(N,M)\to\mathrm{Hom}_R(F_0,M)\to\mathrm{Hom}_R(I_0,M)\to\mathrm{Ext}^1_R(N,M)\to 0$$

shows that $\mathrm{Ext}^1_R(N,M)=0$. Thus, M is $\mathrm{G_C}$-cotorsion. ∎

Lemma 4.2.4. *Assume that R is* $\mathrm{G_C F}$*-closed and C is w^+-tilting. Then, the class of* $\mathrm{G_C}$*-flat R-modules is a Kaplansky class.*

Proof. Let $\widetilde{\mathscr{A}}$ be the class of all $(\mathrm{Prod}_R(C^+)\otimes_R -)$-exact exact complexes of R-modules with components in $\mathscr{A} := \mathscr{F}(R)\cup\mathscr{F}_{\mathrm{C}}(R)$. Since $\mathscr{F}_{\mathrm{C}}(R)$ and $\mathscr{F}(R)$ are closed under direct summands, pure submodules and pure quotients, so is the class $\mathscr{A}$. On the other hand, similarly to the proof of [41, Theorem 3.7], we get that the class $\tilde{\mathscr{A}}$ is closed under pure subcomplexes and pure quotients (see [41, Definition 3.1]). Using now [41, Proposition 3.4], we get that the class $\tilde{\mathscr{A}}$ is a Kaplansky class. But since $\mathrm{G_C F}(R)$ is the class of 0-syzygies of exact sequences in $\widetilde{\mathscr{A}}$ by [14, Corollary 5.2], we get that it is also a Kaplansky class, as desired. ■

Corollary 4.2.5. *Assume that R is* $\mathrm{G_C F}$*-closed and C is w^+-tilting. Then,* $\mathrm{G_C F}(R)$ *is preenveloping if and only if it is closed under direct products.*

Proof. Follows from [40, Theorem 2.5] since the class $\mathrm{G_C F}$ is closed under direct limits by Proposition 3.2.15(3). ■

Theorem 4.2.6. *Let C be* $\prod$-Tor*-orthogonal. The following assertions are equivalent:*

1. $(\mathrm{G_C F}(R), \mathrm{G_C C}(R))$ *is a perfect hereditary cotorsion pair cogenerated by a set.*

2. *R is* $\mathrm{G_C F}$*-closed and C is w^+-tilting.*

In this case, $\mathrm{G_C F}(R)$ *is covering and* $\mathrm{G_C C}(R)$ *is enveloping.*

Proof. 2. ⇒ 1. Following Lemma 4.2.4, $\mathrm{G_C F}(R)$ is a Kaplansky class and by Propositions 3.2.15 and Theorem 3.2.21 it is closed under direct limits and projectively resolving. Therefore, $(\mathrm{G_C F}(R), \mathrm{G_C C}(R))$ is a hereditary perfect cotorsion pair by [40, Theorem 2.9].

Finally, as in the proof of Theorem 4.1.7(1), our pair is cogenerated by a set.

1. ⇒ 2. By hypothesis, ${}^{\perp}\mathrm{G_C C}(R) = \mathrm{G_C F}(R)$ so, the class $\mathrm{G_C F}(R)$ is closed under extensions, and since $R \in \mathrm{G_C F}(R)$ we get that C is w^+-tilting by Proposition 3.2.17. ■

Corollary 4.2.7. *Let C be a semidualizing (R,S)-bimodule such that S is right coherent. The following assertions hold:*

1. $(\mathrm{G_C F}(R), \mathrm{G_C C}(R))$ *is a perfect and hereditary cotorsion pair.*

2. $\mathrm{G_C F}(R)$ *is special precovering.*

3. $\mathrm{G_C C}(R)$ *is special preenveloping.*

Proof. Follows from Theorem 4.2.6, Lemma 1.5.4 and Lemma 3.2.13. ■

4.3 Relative Gorenstein flat model structure

In this section we construct a hereditary Hovey triple on the category of R-modules, in which the cofibrant objects coincide with $\mathrm{G_C}$-flat modules and the fibrant objects coincide with $\mathscr{C}_\mathrm{C}$-cotorsion modules belonging to the Bass class $\mathscr{B}_C(R)$.

We start with the following proposition. Its proof is inspired by an argument due to Estrada, Iacob and Pérez in [42, Proposition 4.1].

Proposition 4.3.1. *Assume that R is $\mathrm{G_CF}$-closed and C is w^+-tilting. Then,*

$$\mathrm{G_CF}(R)\cap \mathrm{G_CC}(R) = \mathscr{F}_\mathrm{C}(R)\cap \mathscr{C}_\mathrm{C}(R).$$

Proof. ($\subseteq$) Assume that $M \in \mathrm{G_CF}(R)\cap \mathrm{G_CC}(R)$. Then, $M \in \mathscr{C}_\mathrm{C}(R)$ by Proposition 4.2.3. Moreover, since M is $\mathrm{G_C}$-flat, there exists by Proposition 3.2.10 an exact sequence of R-modules $0 \to M \to V \to G \to 0$ where V is $\mathscr{F}_\mathrm{C}$-flat and G is $\mathrm{G_C}$-flat. This short exact sequence splits since G is $\mathrm{G_C}$-flat and M is $\mathrm{G_C}$-cotorsion. Hence, $M \in \mathscr{F}_\mathrm{C}(R)$.

($\supseteq$) Assume that $M \in \mathscr{F}_\mathrm{C}(R)\cap \mathscr{C}_\mathrm{C}(R)$. Clearly $M \in \mathrm{G_CF}(R)$. Now we prove that $M \in \mathrm{G_CC}(R)$.

By Theorem 4.2.6, M has a special $\mathrm{G_CC}(R)$-preenvelope

$$0 \to M \to X \to G \to 0.$$

Since M and G are $\mathrm{G_C}$-flat modules and R is $\mathrm{G_CF}$-closed, X is $\mathrm{G_C}$-flat as well and then $X \in \mathrm{G_CF}(R)\cap \mathrm{G_CC}(R) \subseteq \mathscr{F}_\mathrm{C}(R)\cap \mathscr{C}_\mathrm{C}(R)$ by the first inclusion. Now, since M and X are $\mathscr{F}_\mathrm{C}$-flat and G is $\mathrm{G_C}$-flat, $G \in \mathscr{F}_\mathrm{C}(R)$ by Theorem 3.4.12. Then, the short exact sequence splits. Hence, M is $\mathrm{G_C}$-cotorsion. ■

The pair $(\mathscr{F}_\mathrm{C}(R), \mathscr{C}_\mathrm{C}(R))$ satisfies all the conditions necessary to construct our desired Hovey triple, except that it is not a cotorsion pair in general (see Theorem 4.1.7).

This motivates us to introduce new concepts that will serve our purpose perfectly:

Definition 4.3.2. *An R-module H is said to be $\mathscr{H}_C$-**cotorsion** if it belongs to $\mathscr{H}_C(R) := \mathscr{B}_C(R)\cap \mathscr{C}_\mathrm{C}(R)$. An R-module V is said to be $\mathscr{V}_C$-**flat** if $\mathrm{Ext}^1_R(V,H) = 0$ for all $\mathscr{H}_C$-cotorsion R-modules.*

Set $\mathscr{V}_C(R) := {}^{\perp}\mathscr{H}_C(R)$ the class of all $\mathscr{V}_C$-flat R-modules.

If $C = R$, then $\mathscr{F}_\mathrm{C}(R) = \mathscr{V}_C(R) = \mathscr{F}(R)$ and $\mathscr{C}_\mathrm{C}(R) = \mathscr{H}_C(R) = \mathscr{C}(R)$.

However, this not true in general as one can see from the following proposition.

Proposition 4.3.3. *Assume that ${}_RC$ is w-tilting and has a degreewise finite projective resolution. Then, $(\mathscr{V}_C(R), \mathscr{H}_C(R))$ is a hereditary complete cotorsion pair cogenerated by a set.*

Proof. First of all, note that C is Tor-$\prod$-orthogonal by Proposition 3.2.3 and $\mathscr{B}_C(R) \cap \mathscr{C}_C(R) = \mathscr{B}_C(R) \cap \mathscr{S}\mathscr{C}_C(R)$ by Corollary 4.1.6. Now, using [11, Theorem 3.10], we see that $\mathscr{B}_C(R) = \mathscr{X}_1^{\perp_\infty}$ for some set $\mathscr{X}_1$. Similarly, $\mathscr{S}\mathscr{C}_C(R) = \mathscr{X}_2^{\perp_\infty}$ for some set $\mathscr{X}_2$ by Theorem 4.1.7(2). Then, $\mathscr{B}_C(R) \cap \mathscr{C}_C(R) = \mathscr{X}_1^{\perp_\infty} \cap \mathscr{X}_2^{\perp_\infty} = (\mathscr{X}_1 \cup \mathscr{X}_2)^{\perp_\infty}$. Thus, $\mathscr{H}_C(R) = \mathscr{B}_C(R) \cap \mathscr{C}_C(R) = M^{\perp}$ for some R-module M by Lemma 1.5.17. Hence, our pair is hereditary and cogenerated by a set and then complete by Theorem 1.5.13. ■

Under strong conditions, we get a different description of the core of the cotorsion pair $(\mathrm{G_CF}(R), \mathrm{G_CC}(R))$ which is the last ingredient to get our desired model structure.

But first we need the following two lemma.

Lemma 4.3.4. *Assume that $_RC$ has a degreewise projective resolution*

1. *$\mathscr{F}(S) \subseteq \mathscr{A}_C(S)$ if and only if $_RC$ is $\prod$-Tor-orthogonal.*

   In this case, $\mathscr{F}_C(R) \subseteq \mathscr{B}_C(R)$.

2. *If $_RC$ is $\prod$-Tor-orthogonal, then any R-module in $\mathscr{B}_C(R)$ has an epic $\mathscr{F}_C$-flat cover with kernel in $\mathscr{B}_C(R)$.*

Proof. 1. Using Proposition 3.2.3, the "if" part follows by [11, Proposition 5.2] and the "only if" part follows by Lemma 1.3.14(1) since $\mathscr{P}(S) \subseteq \mathscr{F}(S)$.

2. Let $M \in \mathscr{B}_C(R)$. Then, there exists by Proposition 3.1.6(1) an $\mathscr{F}_C$-flat cover $\gamma : L \to M$ which is epic by [11, Proposition 3.8]. It remains to show that $\mathrm{Ker}\gamma$ is in $\mathscr{B}_C(R)$.

Since $F \in \mathscr{F}_C(R) \subseteq \mathscr{B}_C(R)$ and $M \in \mathscr{B}_C(R)$, we deduce that $\mathrm{Ext}_R^{k\geq 1}(C,F) = 0$ and $\mathrm{Ext}_R^{k\geq 1}(C,M) = 0$, so applying $\mathrm{Hom}_R(C,-)$ to the exact sequence

$$0 \to \mathrm{Ker}\gamma \to F \xrightarrow{\gamma} M \to 0$$

we immediately get that $\mathrm{Ext}_R^{k\geq 1}(C,\mathrm{Ker}\gamma)$. Similarly, the fact that $F, M \in \mathscr{B}_C(R)$ implies $\mathrm{Tor}^S_{k\geq 1}(C, \mathrm{Hom}_R(C,M)) = 0$ and $\mathrm{Tor}^S_{k\geq 1}(C, \mathrm{Hom}_R(C,F)) = 0$, so applying the functor $C \otimes_S -$ to the exact sequence

$$0 \to \mathrm{Hom}_R(C,\mathrm{Ker}\gamma) \to \mathrm{Hom}_R(C,F) \to \mathrm{Hom}_R(C,M) \to 0$$

we get that $\mathrm{Tor}^S_{k\geq 1}(C, \mathrm{Hom}_R(C,\mathrm{Ker}\gamma)) \cong \mathrm{Tor}^S_{k\geq 1}(C, \mathrm{Hom}_R(C,F)) = 0$.

Finally, from the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \to & C\otimes_S \mathrm{Hom}_R(C,\mathrm{Ker}\gamma) & \to & C\otimes_S \mathrm{Hom}_R(C,F) & \to & C\otimes_S \mathrm{Hom}_R(C,M) & \to & 0 \\ & & & & \downarrow \cong & & \downarrow \cong & & \\ 0 & \to & \mathrm{Ker}\gamma & \to & F & \to & M & \to & 0 \end{array}$$

we get that $\mathrm{Ker}\gamma$ is naturally isomorphic to $C \otimes_S \mathrm{Hom}_R(C,\mathrm{Ker}\gamma)$. ■

Proposition 4.3.5. *Assume that R is* $\mathrm{G_CF}$*-closed and C is w^+-tilting admitting a degreewise finite projective resolution. Then,*

$$\mathrm{G_CF}(R)\cap\mathrm{G_CC}(R)=\mathscr{V}_C(R)\cap\mathscr{H}_C(R).$$

Proof. ($\subseteq$) By Proposition 4.3.1, $\mathrm{G_CF}(R)\cap\mathrm{G_CC}(R)=\mathscr{F}_\mathrm{C}(R)\cap\mathscr{C}_\mathrm{C}(R)\subseteq\mathscr{C}_\mathrm{C}(R)$. But $\mathscr{F}_\mathrm{C}(R)\subseteq\mathscr{B}_C(R)$ by Lemma 4.3.4(1). Then, $\mathrm{G_CF}(R)\cap\mathrm{G_CC}(R)\subseteq\mathscr{B}_C(R)\cap\mathscr{C}_\mathrm{C}(R)=\mathscr{H}_C(R)$.

On the other hand, we have $\mathscr{B}_C(R)\cap\mathscr{C}_\mathrm{C}(R)\subseteq\mathscr{C}_\mathrm{C}(R)$, which implies that ${}^\perp\mathscr{C}_\mathrm{C}(R)\subseteq{}^\perp\mathscr{H}_C(R)=\mathscr{V}_C(R)$. But, $\mathscr{F}_\mathrm{C}(R)\subseteq{}^\perp\mathscr{C}_\mathrm{C}(R)$. Hence, using again Proposition 4.3.1, $\mathrm{G_CF}(R)\cap\mathrm{G_CC}(R)=\mathscr{F}_\mathrm{C}(R)\cap\mathscr{C}_\mathrm{C}(R)\subseteq\mathscr{F}_\mathrm{C}(R)\subseteq\mathscr{V}_C(R)$.

($\supseteq$) Conversely, let $M\in\mathscr{V}_C(R)\cap\mathscr{H}_C(R)$. By Proposition 4.3.1, we only need show that $M\in\mathscr{F}_\mathrm{C}(R)\cap\mathscr{C}_\mathrm{C}(R)$.

Clearly, $M\in\mathscr{C}_\mathrm{C}(R)$. Since $M\in\mathscr{B}_C(R)$, Lemma 4.3.4(2) says that there exists an epic $\mathscr{F}_\mathrm{C}$-flat cover $\gamma:F\twoheadrightarrow M$ with $K:=\mathrm{Ker}\gamma\in\mathscr{B}_C(R)$. Wakamatsu Lemma (see Lemma 1.5.4) implies that $K\in\mathscr{C}_\mathrm{C}(R)$, that is, $K\in\mathscr{B}(R)\cap\mathscr{C}_\mathrm{C}(R)=\mathscr{H}_C(R)$. Since $M\in\mathscr{V}_C(R)$, the short exact sequence $0\to K\to F\to M\to 0$ splits and hence $M\in\mathscr{F}_\mathrm{C}(R)$, as desired. ■

Theorem 4.3.6. *Assume that R is* $\mathrm{G_CF}$*-closed and C is w-tilting admitting a degreewise finite projective resolution. Then, there exists a unique hereditary abelian model structure on R-Mod, called the* $\mathrm{G_C}$***-flat model structure**,*

$$(\mathrm{G_CF}(R),\mathscr{W},\mathscr{H}_C(R)),$$

as follows:

- *The cofibrant objects coincide with* $\mathrm{G_C}$*-flat modules.*
- *The fibrant objects coincide with $\mathscr{H}_C$-cotorsion modules.*
- *The class of trivially cofibrant objects coincide with $\mathscr{V}_C$-flat modules.*
- *The trivially fibrant objects coincide with* $\mathrm{G_C}$*-cotorsion modules.*

Proof. It follows from Theorem 4.1.7 and Theorem 4.2.6 that the pairs

$$(\mathrm{G_CF}(R),\mathrm{G_CC}(R))\text{ and }(\mathscr{V}_C(R),\mathscr{H}_C(R))$$

are complete and hereditary cotorsion pairs. By Proposition 4.3.3, these cotorsion pairs have the same core. Let us now show that $\mathrm{G_CC}(R)\subseteq\mathscr{H}_C(R)=\mathscr{B}_C(R)\cap\mathscr{C}_\mathrm{C}(R)$.

The inclusion $\mathrm{G_CC}(R)\subseteq\mathscr{C}_\mathrm{C}(R)$ holds by Proposition 4.2.3. On the other hand, by the proof of Proposition 3.2.3, we have a complete $\mathscr{P}_\mathrm{C}$-projective complex

$$0\to R\overset{t_0}{\to}C_0\overset{t_1}{\to}C_1\overset{t_2}{\to}\cdots$$

which is also $(\mathrm{Prod}_R(C^+)\otimes_R-)$-exact. Then, $\mathscr{B}_C(R)=(C\oplus(\oplus_{i\geq 0}\mathrm{Coker}t_i))^\perp$ by [11, Theorem 3.10], and each $\mathrm{Coker}t_i$ is $\mathrm{G_C}$-flat by Corollary 3.2.9. It follows that $C\oplus$

$(\oplus_{i\geq 0}\mathrm{Coker}t_i) \in \mathrm{G_CF}(R)$, which implies that $\mathrm{G_CC}(R) \subseteq (C \oplus (\oplus_{i\geq 0}\mathrm{Coker}t_i))^{\perp} = \mathscr{B}_C(R)$ and hence $\mathrm{G_CC}(R) \subseteq \mathscr{H}_C(R)$. Thus, Theorem 1.6.3 gives the desired Hovey triple. ■

Remark 4.3.7. *Under Hovey's correspondence between abelian model structures and Hovey triples, the* $\mathrm{G_C}$*-flat model structure is described as follows:*
• *A morphism f is a cofibration (trivial cofibration) if and only if it is a monomorphism with* $\mathrm{G_C}$*-flat (*$\mathscr{V}_C$*-flat) cokernel.*
• *A morphism g is a fibration (trivial fibration) if and only if it is an epimorphism with* $\mathscr{H}_C$*-cotorsion (*$\mathrm{G_CC}$*-cotorsion) kernel.*

The first part of the following result was proved by Hu, Geng, Wu and Li in [65, Theorem 4.3] when R is a commutative noetherian ring and C is a semidualizing R-module. Here we obtain it with a different approach and with less assumptions.

Corollary 4.3.8. *Assume that R is* $\mathrm{G_CF}$*-closed and C is w-tilting admitting a degreewise finite projective resolution. Then, the category* $\mathrm{G_CF}(R) \cap \mathscr{H}_C(R)$*, along with the induced exact structure, is a Frobenius category. The projective-injective objects are exactly the objects in* $\mathscr{F}_C(R) \cap \mathscr{C}_C(R)$*. Moreover, the homotopy category of the* $\mathrm{G_C}$*-flat model structure is triangle equivalent to the stable category*

$$\underline{\mathrm{G_CF}(R) \cap \mathscr{H}_C(R)} := (\mathrm{G_CF}(R) \cap \mathscr{H}_C(R))/\sim$$

where $f \sim g$ *if and only if* $f - g$ *factors through an object in* $\mathscr{F}_C(R) \cap \mathscr{C}_C(R)$*.*

Proof. Follows using Theorem 4.3.6 and Proposition 4.3.1 together with Theorem 1.6.6. ■

The Gorenstein flat model structure goes back to Gillespie and Hovey [52, Theorem 3.12] when the ring is Iwanaga-Gorenstein. Recently, Šaroch and Šťovíček proved in [78] the existence of this model structure over any ring.

Corollary 4.3.9. *(**The Gorenstein flat model structure**) For any finitely generated projective generator* $_RC$*, there exists a unique hereditary abelian model structure on R-Mod where* $\mathscr{GF}(R) = \mathrm{G_CF}(R)$ *is the class of cofibrant objects and* $\mathscr{C}(R) = \mathscr{H}_C(R)$ *is the class of fibrant objects.*

In this case, the category $\mathscr{GF}(R) \cap \mathscr{C}(R)$ *is a Frobenius category where the projective-injective objects are exactly the flat-cotorsion R-modules. Moreover, the homotopy category of the Gorenstein flat model structure is triangle equivalent to the stable category* $\underline{\mathscr{GF}(R) \cap \mathscr{C}(R)}$*.*

Proof. Since $_RC$ is a projective generator, $\mathscr{F}_C(R) = \mathscr{F}(R)$ and $\mathrm{G_CF}(R) = \mathscr{GF}(R)$ by Proposition 3.2.23, and hence $\mathscr{H}_C(R) = \mathscr{C}(R)$. Moreover, R is $\mathrm{G_CF}$-closed by Corollary 3.2.24. Thus, by Theorem 4.3.6 we have the desired model structure. ■

4.4 Relative Gorenstein flat model structure under some finiteness conditions

In this section we describe the class of weak equivalences in the G_C-flat model structure found in Theorem 4.3.6, under the finiteness of the global G_C-flat dimension of R. This result is a consequence of a more general result (Theorem 4.4.6) that we will also prove in this section. First, we need to develop some preliminary results that we will use throughout this section.

By a **special proper $\mathscr{X}$-resolution** of an object A, we will mean an $\mathscr{X}$-resolution of A

$$\cdots \to X_1 \to X_0 \to A \to 0$$

in which each $\mathrm{Ker}(X_i \to X_{i-1}) \in \mathscr{X}^{\perp}$ with $X_{-1} := A$. Note that any object A of $\mathscr{A}$ has a special proper $\mathscr{X}$-resolution if and only if $\mathscr{X}$ is a special precovering class.

Proposition 4.4.1. *Assume that $(\mathscr{X}, \mathscr{Y})$ is a complete hereditary cotorsion pair in $\mathscr{A}$. The following assertions are equivalent for any object A in $\mathscr{A}$ and any integer $n \geq 1$.*

(a) $\mathscr{X} - \mathrm{resdim}_{\mathscr{A}}(A) \leq n$

(b) For any special proper $\mathscr{X}$-resolution of A

$$\cdots \to X_n \to X_{n-1} \to \cdots \to X_1 \to X_0 \to A \to 0$$

we have $K_{n-1} := \mathrm{Ker}(X_{n-1} \to X_{n-2}) \in \mathscr{X}$ with $X_{-1} = A$.

(c) There exists a special proper $\mathscr{X}$-resolution of A

$$0 \to X_n \to \cdots \to X_0 \to A \to 0.$$

(d) $\mathrm{Ext}^k_{\mathscr{A}}(A,Y) = 0$ *for all objects $Y \in \mathscr{Y}$ and all $k \geq n+1$.*

(e) $\mathrm{Ext}^{n+1}_{\mathscr{A}}(A,Y) = 0$ *for all objects $Y \in \mathscr{Y}$.*

Consequently,

$$\mathscr{X} - \mathrm{resdim}_{\mathscr{A}}(A) = \sup\{i \in \mathbb{N} : \mathrm{Ext}^i_{\mathscr{A}}(A,Y) \neq 0 \text{ for some } Y \in \mathscr{Y}\}.$$

Proof. $(a) \Rightarrow (b)$ We proceed by induction.

If $n = 1$, that is, $\mathscr{X} - \mathrm{resdim}_{\mathscr{A}}(A) \leq 1$, there exists an exact sequence

$$0 \to Z_1 \to Z_0 \to A \to 0,$$

with $Z_1, Z_0 \in \mathscr{X}$. Let us show that $K_0 := \mathrm{Ker}(X_0 \to A) \in \mathscr{X}$. We have the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Z_1 & \longrightarrow & Z_0 & \longrightarrow & A & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & K_0 & \longrightarrow & X_0 & \longrightarrow & A & \longrightarrow & 0 \end{array}$$

which induces the mapping cone that leads to the short exact sequence

$$0 \to Z_1 \to Z_0 \oplus K_0 \to X_0 \to 0.$$

Therefore, $K_0 \in \mathscr{X}$ as $Z_1, X_0 \in \mathscr{X}$ and $\mathscr{X}$ is closed under extensions and direct summands.

For the case $n \geq 2$, there exists an exact sequence $0 \to Z \to Z_0 \to A \to 0$, with $Z_0 \in \mathscr{X}$ and $\mathscr{X} - \mathrm{resdim}_{\mathscr{A}}(Z) \leq n-1$. Since $K_0 \in \mathscr{X}^{\perp}$, we can construct the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & K_1 & \longrightarrow & L & \longrightarrow & Z & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & X_1 & \longrightarrow & X_1 \oplus Z_0 & \longrightarrow & Z_0 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & K_0 & \longrightarrow & X_0 & \longrightarrow & A & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

Note that $L \in \mathscr{X}$ as $(\mathscr{X}, \mathscr{Y})$ is a hereditary cotorsion pair and $X_0, X_1, Z_0 \in \mathscr{X}$. Now, $\mathscr{X} - \mathrm{resdim}_{\mathscr{A}}(Z) \leq n-1$ and Z has the following special proper $\mathscr{X}$-resolution:

$$\cdots \to X'_{n-2} = X_{n-1} \to X'_{n-3} = X_{n-2} \to \cdots \to X'_1 = X_2 \to X'_0 = L \to Z \to 0$$

By induction, $K_{n-1} = \mathrm{Ker}(X_{n-1} \to X_{n-2}) = \mathrm{Ker}(X'_{n-2} \to X'_{n-3}) \in \mathscr{X}$ as desired.

$(b) \Rightarrow (c)$ Follows by the by the hypotheses and the fact that $\mathscr{X}$ is a special precovering class.

$(c) \Rightarrow (d)$ Since $(\mathscr{X}, \mathscr{Y})$ is a hereditary cotorsion pair, for any object $Y \in \mathscr{Y}$ and any integer $k \geq n+1$, we get the following isomorphisms:

$$\mathrm{Ext}^k_{\mathscr{A}}(A, Y) \cong \mathrm{Ext}^{k-1}_{\mathscr{A}}(K_0, Y) \cong \cdots \cong \mathrm{Ext}^{k-n}_{\mathscr{A}}(K_{n-1}, Y) = \mathrm{Ext}^{k-n}_{\mathscr{A}}(X_n, Y) = 0.$$

$(d) \Rightarrow (e)$ Clear.

$(e) \Rightarrow (a)$ Consider a special proper $\mathscr{X}$-resolution of A

$$\cdots \to X_n \to \cdots \to X_0 \to A \to 0.$$

Since $K_n \in \mathscr{Y}$ and each $X_i \in \mathscr{X}$, $\mathrm{Ext}^1_{\mathscr{A}}(K_{n-1}, K_n) \cong \cdots \cong \mathrm{Ext}^{n+1}_{\mathscr{A}}(A, K_n) = 0$. Hence, the short exact sequence $0 \to K_n \to X_n \to K_{n-1} \to 0$ splits. Therefore, $K_{n-1} \in \mathscr{X}$ and $\mathscr{X} - \mathrm{resdim}_{\mathscr{A}}(A) \leq n$. ■

Using the description of the $\mathscr{X}$-resolution dimension given in Proposition 4.4.1, the following result is standard and its proof is straightforward.

Proposition 4.4.2. *Let $(\mathscr{X}, \mathscr{Y})$ be a complete hereditary cotorsion pair in $\mathscr{A}$. The following assertions hold.*

1. *Given a short exact sequence $\mathscr{E} : 0 \to M \to N \to L \to 0$ in $\mathscr{A}$, we have:*
   (a) $\mathscr{X} - \mathrm{resdim}_{\mathscr{A}}(M) \leq \max\{\mathscr{X} - \mathrm{resdim}_{\mathscr{A}}(N), \mathscr{X} - \mathrm{resdim}_{\mathscr{A}}(L) - 1\}$.
   The equality holds when $\mathscr{X} - \mathrm{resdim}_{\mathscr{A}}(N) \neq \mathscr{X} - \mathrm{resdim}_{\mathscr{A}}(L)$.
   (b) $\mathscr{X} - \mathrm{resdim}_{\mathscr{A}}(N) \leq \max\{\mathscr{X} - \mathrm{resdim}_{\mathscr{A}}(M), \mathscr{X} - \mathrm{resdim}_{\mathscr{A}}(L)\}$.
   The equality holds when $\mathscr{X} - \mathrm{resdim}_{\mathscr{A}}(L) \neq \mathscr{X} - \mathrm{resdim}_{\mathscr{A}}(M) + 1$
   (c) $\mathscr{X} - \mathrm{resdim}_{\mathscr{A}}(L) \leq \max\{\mathscr{X} - \mathrm{resdim}_{\mathscr{A}}(N), \mathscr{X} - \mathrm{resdim}_{\mathscr{A}}(M) + 1\}$.
   The equality holds when $\mathscr{X} - \mathrm{resdim}_{\mathscr{A}}(M) \neq \mathscr{X} - \mathrm{resdim}_{\mathscr{A}}(N)$.
2. *For any family $(M_i)_{i=1,\cdots,n}$ of objects in $\mathscr{A}$, we have*

$$\mathscr{X} - \mathrm{resdim}_{\mathscr{A}}(\oplus_{i=1}^n M_i) = \sup\{\mathscr{X} - \mathrm{resdim}_{\mathscr{A}}(M_i) | i = 1, \cdots, n\}.$$

Consequently, $\widehat{\mathscr{X}}$ is thick.

Lemma 4.4.3. *Let $\mathscr{X}, \mathscr{G} \subseteq \mathscr{A}$ be two classes such that $\mathscr{X}$ is closed under direct summands. Set $\mathscr{H} = \mathscr{G}^{\perp}$ and $\mathscr{Y} = \mathscr{X}^{\perp}$. If $\mathscr{X}$ is a cogenerator for $\mathscr{G}$, then*

$$\mathscr{G} \cap \mathscr{H} \subseteq \mathscr{X} \cap \mathscr{Y}.$$

If, in addition, $(\mathscr{G}, \mathscr{H})$ is a complete cotorsion pair and $\mathscr{G} \cap \widehat{\mathscr{X}} = \mathscr{X}$, then

$$\mathscr{G} \cap \mathscr{H} = \mathscr{X} \cap \mathscr{Y}.$$

Proof. Let A be an object in $\mathscr{A}$. For the first inclusion, assume that $A \in \mathscr{G} \cap \mathscr{H}$. Since $\mathscr{X}$ is a cogenerator for $\mathscr{G}$, $\mathscr{X} \subseteq \mathscr{G}$ and then $A \in \mathscr{H} \subseteq \mathscr{Y}$. Moreover, there exists an exact sequence $0 \to A \to X \to G \to 0$ with $X \in \mathscr{X}$ and $G \in \mathscr{G}$. This sequence splits as $A \in \mathscr{H}$ and $G \in \mathscr{G}$. Hence, $A \in \mathscr{X}$.

Conversely, assume that that $A \in \mathscr{X} \cap \mathscr{Y}$. Then, $A \in \mathscr{X} \subseteq \mathscr{G}$. Since the cotorsion pair $(\mathscr{G}, \mathscr{H})$ is complete, there exists an exact sequence

$$0 \to A \to W \to G \to 0$$

with $W \in \mathscr{H}$ and $G \in \mathscr{G}$. Note that $W \in \mathscr{G} \cap \mathscr{H}$ and $\mathscr{G} \cap \mathscr{H} \subseteq \mathscr{X} \cap \mathscr{Y}$ by the previous inclusion. Now, we have $G \in \mathscr{G}$ with $\mathscr{X} - \text{resdim}_{\mathscr{A}}(G) \leq 1 < \infty$, that is, $G \in \mathscr{G} \cap \widehat{\mathscr{X}}$. Hence, $G \in \mathscr{X}$ and this sequence splits. Therefore, $A \in \mathscr{H}$. ■

The following two results relate cotorsion pairs and Hovey triples with (weak) Ab-contexts in a convenient way. The first one is inspired by two results due to Liang and Yang ([69, Proposition 2.5 and Proposition 2.10]).

Proposition 4.4.4. *Let $\mathscr{X}$ and $\mathscr{G}$ be two classes of objects of $\mathscr{A}$ such that $\mathscr{X}$ is closed under direct summands. Set $\mathscr{Y} = \mathscr{X}^{\perp}$ and $\mathscr{H} = \mathscr{G}^{\perp}$.*

1. *Assume that $\mathscr{X}$ is a cogenerator for $\mathscr{G}$ and $\mathscr{G} \cap \widehat{\mathscr{X}} = \mathscr{X}$. If $(\mathscr{G}, \mathscr{H})$ is a hereditary complete cotorsion pair, then $(\mathscr{G}, \widehat{\mathscr{X} \cap \mathscr{Y}}, \mathscr{X} \cap \mathscr{Y})$ is a left weak AB-context.*

2. *Assume that $(\mathscr{X}, \mathscr{Y})$ is a complete hereditary cotorsion pair. If $(\mathscr{G}, \widehat{\mathscr{X} \cap \mathscr{Y}}, \mathscr{X} \cap \mathscr{Y})$ is a left AB-context, then $(\mathscr{G}, \mathscr{H})$ is a hereditary complete cotorsion pair with $\mathscr{H} = \widehat{\mathscr{X} \cap \mathscr{Y}} = \widehat{\mathscr{X}} \cap \mathscr{Y}$.*

Proof. 1. By Proposition 1.7.4, the pair $(\mathscr{G}, \mathscr{G} \cap \mathscr{H})$ is a left Frobenius pair. Moreover, using the one-to-one correspondence between left AB-contexts and left Frobenius pairs from Theorem 1.7.3, we get that the triple

$$(\mathscr{G}, (\widehat{\mathscr{G} \cap \mathscr{H}}), \mathscr{G} \cap (\widehat{\mathscr{G} \cap \mathscr{H}})) = (\mathscr{G}, (\widehat{\mathscr{G} \cap \mathscr{H}}), \mathscr{G} \cap \mathscr{H})$$

is a left weak AB-context. Finally, applying Lemma 4.4.3, we get our desired left weak AB-context $(\mathscr{G}, \widehat{\mathscr{X} \cap \mathscr{Y}}, \mathscr{X} \cap \mathscr{Y})$.

2. By [59, Theorem 1.12.10], we get that $(\mathscr{G}, \mathscr{H})$ is a complete hereditary cotorsion pair with $\mathscr{H} = \widehat{\mathscr{X} \cap \mathscr{Y}}$. Now let us show that $\mathscr{H} = \widehat{\mathscr{X}} \cap \mathscr{Y}$.

If $A \in \mathscr{H}$, there exists a finite $(\mathscr{X} \cap \mathscr{Y})$-resolution of A

$$0 \to X_n \to \cdots \to X_0 \to A \to 0.$$

Since each $X_i \in \mathscr{X} \cap \mathscr{Y} \subseteq \mathscr{X}$, $A \in \widehat{\mathscr{X}}$. We also know that $(\mathscr{X}, \mathscr{Y})$ is hereditary so $A \in \mathscr{Y}$. Hence, $A \in \widehat{\mathscr{X}} \cap \mathscr{Y}$.

Conversely, if $A \in \widehat{\mathscr{X}} \cap \mathscr{Y}$ then $A \in \widehat{\mathscr{X}}$. Moreover, by Proposition 4.4.1 there exists a finite special proper $\mathscr{X}$-resolution of A

$$0 \to X_n \to \cdots \to X_0 \to A \to 0$$

Since $\mathscr{Y}$ is closed under extensions, each $X_i \in \mathscr{Y}$ and then $X_i \in \mathscr{X} \cap \mathscr{Y}$. Hence, $A \in \widehat{\mathscr{X} \cap \mathscr{Y}} = \mathscr{H}$. ■

Assume that $(\mathscr{Q},\mathscr{W},\mathscr{R})$ is a hereditary Hovey triple and let

$$(\widetilde{\mathscr{Q}},\mathscr{R}) := (\mathscr{Q}\cap\mathscr{W},\mathscr{R}) \text{ and } (\mathscr{Q},\widetilde{\mathscr{R}}) := (\mathscr{Q},\mathscr{W}\cap\mathscr{R})$$

be the associated cotorsion pairs. Then, the following two assertions always hold:

• $(\widetilde{\mathscr{Q}},\mathscr{R})$ is a complete hereditary cotorsion pair.

• $\left(\mathscr{Q},\widehat{(\mathscr{Q}\cap\widetilde{\mathscr{R}})},\widetilde{\mathscr{Q}}\cap\mathscr{R}\right)$ is a left weak AB-context: indeed, since $(\mathscr{Q},\widetilde{\mathscr{R}})$ is a complete hereditary cotorsion pair and $\mathscr{Q}\cap\widetilde{\mathscr{R}}=\widetilde{\mathscr{Q}}\cap\mathscr{R}$, we get this left weak AB-context by letting $\mathscr{G}=\mathscr{X}=\mathscr{Q}$ in Proposition 4.4.4(1).

Under the finiteness assumption $\mathscr{A}=\widehat{\mathscr{G}}$, the following result shows that there is a converse to this, giving a new characterization of hereditary Hovey triples in terms of AB-contexts. We note that such a relation between Hovey triples and (weak) AB-contexts was also noted by A. Xu in [85, Theorem 4.2] in a particular setting.

Theorem 4.4.5. *Let $\mathscr{X}\subseteq\mathscr{G}\subseteq\mathscr{A}$ be two classes such that:*

1. *$(\mathscr{X},\mathscr{Y})$ is a complete hereditary cotorsion pair.*
2. *$(\mathscr{G},\mathscr{H},\mathscr{X}\cap\mathscr{Y})$ is a left AB-context.*

Then, $(\mathscr{G},\widehat{\mathscr{X}},\mathscr{Y})$ is a hereditary Hovey triple.

Proof. We divide the proof into three parts:

(a) The class $\widehat{\mathscr{X}}$ is thick by Proposition 4.4.2.

(b) By assumption and [59, Theorem 1.12.10(1)], $\mathscr{H}=\widehat{\mathscr{X}\cap\mathscr{Y}}$. Then, $\mathscr{H}=\widehat{\mathscr{X}}\cap\mathscr{Y}$ and the pair $(\mathscr{G},\widehat{\mathscr{X}}\cap\mathscr{Y})$ is a hereditary complete cotorsion pair by Proposition 4.4.4(2).

(c) It remains to show that $(\mathscr{G}\cap\widehat{\mathscr{X}},\mathscr{Y})$ is a hereditary complete cotorsion pair. By the assumptions, we only need to show $\mathscr{X}=\mathscr{G}\cap\widehat{\mathscr{X}}$.

The inclusion $\mathscr{X}\subseteq\mathscr{G}\cap\widehat{\mathscr{X}}$ is clear.

Conversely, assume $A\in\mathscr{G}\cap\widehat{\mathscr{X}}$. Then, $n=\mathscr{X}-\text{resdim}(A)<\infty$. If $n=0$, that is, $X\in\mathscr{X}$, then we are done. Assume now that $n\geq 1$. Let us proceed by induction on n.

There exists an exact sequence $0\to K\to X\to A\to 0$ with $\mathscr{X}-\text{resdim}_{\mathscr{A}}(K)=n-1$ and $X\in\mathscr{X}$. Since $(\mathscr{X},\mathscr{Y})$ is complete, there exists an exact sequence $0\to K\to Y\to$

$L \to 0$ with $\mathscr{Y} \in \mathscr{Y}$ and $L \in \mathscr{X}$. Consider now the following pushout:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & K & \longrightarrow & X & \longrightarrow & A & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & Y & \dashrightarrow & H & \longrightarrow & A & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & L & = & L & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Since $A \in \mathscr{G}$, $X \in \mathscr{X} \subseteq \mathscr{G}$ and $\mathscr{G}$ is closed under kernels of epimorphisms, we deduce that $K \in \mathscr{G}$. So, by induction, $K \in \mathscr{X}$. Hence, $Y \in \mathscr{X} \cap \mathscr{Y}$.

On the other hand, $\mathscr{X} \cap \mathscr{Y} = \mathscr{G} \cap \mathscr{H}$ by hypothesis. Then, the middle short exact sequence splits as $A \in \mathscr{G}$ and $Y \in \mathscr{H} = \mathscr{G}^{\perp}$. But $H \in \mathscr{X}$ by the middle column, so $A \in \mathscr{X}$ as desired. ■

We are now ready to prove our first main result of this section.

Theorem 4.4.6. *Assume that $(\mathscr{Q},\widetilde{\mathscr{R}})$ and $(\widetilde{\mathscr{Q}},\mathscr{R})$ are two complete hereditary cotorsion pairs such that:*

(a) $\widetilde{\mathscr{Q}} \subseteq \mathscr{Q}$ *(or equivalently, $\widetilde{\mathscr{R}} \subseteq \mathscr{R}$).*

(b) $\mathscr{Q} \cap \widetilde{\mathscr{R}} = \widetilde{\mathscr{Q}} \cap \mathscr{R}$.

If $\mathscr{Q} - \operatorname{resdim}_{\mathscr{A}}(\mathscr{A}) < \infty$, then $(\mathscr{Q},\widehat{(\widetilde{\mathscr{Q}})},\mathscr{R})$ is a hereditary Hovey triple.

Proof. Taking advantage of our previous notations, let us set

$$(\mathscr{G},\mathscr{H}) = (\mathscr{Q},\widetilde{\mathscr{R}}) \text{ and } (\mathscr{X},\mathscr{Y}) = (\widetilde{\mathscr{Q}},\mathscr{R}).$$

By Theorem 4.4.5 and the hypotheses, it suffices to show that

$$(\mathscr{G},\mathscr{H},\mathscr{X} \cap \mathscr{Y}) = (\mathscr{Q},\widetilde{\mathscr{R}},\widetilde{\mathscr{Q}} \cap \mathscr{R})$$

is a left weak AB-context.

Notice that $\mathscr{G}$ is a cogenerator for $\mathscr{G}$ with $\mathscr{G} \cap \widehat{\mathscr{G}} = \mathscr{G}$, then $(\mathscr{G},\widehat{\mathscr{G} \cap \mathscr{H}},\mathscr{G} \cap \mathscr{H})$ is a left AB-context by Proposition 4.4.4(1) and so a left AB-context as $\mathscr{A} = \widehat{\mathscr{Q}}$. Then, since $\mathscr{G} \cap \mathscr{H} = \mathscr{X} \cap \mathscr{Y}$, we can apply Proposition 4.4.4(2) to get that $\mathscr{H} = \widehat{\mathscr{X} \cap \mathscr{Y}}$ and so that $(\mathscr{G},\mathscr{H},\mathscr{X} \cap \mathscr{Y})$ is a left AB-context as desired. ■

In the following result is (the second main result of this section), we describe the trivial objects of the $\mathrm{G_C}$-flat model structure constructed in Theorem 4.3.6.

Theorem 4.4.7. *Assume that R is* $\mathrm{G_CF}$*-closed and C is w-tilting admitting a degreewise finite projective resolution. If R has finite global* $\mathrm{G_C}$*-flat dimension, then there exists an abelian model structure on R-Mod,*

$$\left(\mathrm{G_CF}(R),\widehat{\mathscr{V}_C(R)},\mathscr{H}_C(R)\right),$$

as follows:

- *The cofibrant objects coincide with* $\mathrm{G_C}$*-flat R-modules.*
- *The trivial objects coincide with modules having finite* $\mathscr{V}_C$*-flat dimension.*
- *The fibrant objects coincide with* $\mathscr{H}_C$*-cotorsion R-modules.*

Proof. We know by Theorem 4.3.6 that $(\mathrm{G_CF}(R),\mathrm{G_C}(R))$ and $(\mathscr{V}_C(R),\mathscr{H}_C(R))$ are two complete hereditary cotorsion pairs with the same core, that is,

$$\mathrm{G_CF}(R)\cap\mathrm{G_CC}(R)=\mathscr{V}_C(R)\cap\mathscr{H}_C(R),$$

and such that $\mathrm{G_CC}(R)\subseteq\mathscr{H}_C(R)$. Thus, this result follows by Theorem 4.4.6. ■

The following consequence has been proven by A. Xu ([85, Corollary 4.6(3)]) over right coherent rings.

Corollary 4.4.8. *For any ring R with finite weak global Gorenstein dimension we have a hereditary abelian model structure on R-Mod*

$$(\mathscr{G}\mathscr{F}(R),\widehat{\mathscr{F}(R)},\mathscr{C}(R)).$$

PERSPECTIVES: OPEN ROBLEMS

In this brief chapter, we discuss some of the open questions raised in this thesis or related to the subject of it.

Question A: When is the class of G_C-flat modules closed under extensions?

This question is of great importance since almost all the homological properties of G_C-flat modules are based on this property, i.e., closure under extensions.

When C is Tor-$\prod$-orthogonal, we have seen three cases in which this happens:

(a) C is a flat generator.

(b) $\mathscr{F}_C(R)$ is closed under products. In particular, this the case when $S := \mathrm{End}_R(C)$ is right coherent and both ${}_RC$ and C_S are finitely presented.

(c) R has finite global $\mathscr{F}_C$-flat dimension, i.e.,

$$\sup\{\mathscr{F}_C\text{−}\mathrm{fd}_R(M) | M \text{ is an } R\text{-module}\} < \infty.$$

Question B. When is the pair $(\mathrm{G_CP}(R), \mathrm{G_CP}(R)^{\perp})$ a complete cotorsion pair?

This question is still open even in the absolute case. In this setting, some partial affirmative answers have been given recently in [32, Corollary 4.13(2)] and [30, Corollary 5.10(3)]

Another positive answer can be obtained under the assumption that R has finite global G_C-projective dimension with respect to a w-tilting module C. To see this, it suffices by [17, Corollary 3.6] to show that $\mathrm{G_CP}(R) = {}^{\perp}(\mathrm{G_CP}(R)^{\perp})$.

Given an R-module $M \in {}^{\perp}(\mathrm{G_CP}(R)^{\perp})$, take a special $\mathrm{G_C}$-projective (which exists by [17, Corollary 3.6])

$$0 \to K \to G \to M \to 0.$$

This short exact sequence splits as $K \in \mathrm{G_CP}(R)^{\perp}$. Hence, $M \in \mathrm{G_CP}(R)$.

Question B brings us to the next question:

Question C. When is the class $\mathrm{G_CP}(R)$ the first part of a (non-trivial) Hovey triple?

$$(\mathrm{G_CP}(R), \mathscr{W}, \mathscr{R})$$

By non-trivial here we mean that $\mathscr{W}$ is not the class of all modules.

Assume that C is self-small and w-tilting. A natural candidate class for the class of fibrant objects $\mathscr{R}$ in this model structure (when it exists) is the Bass class $\mathscr{B}_C(R)$. This can be seen by the following assertions (required in Theorem 1.6.3):

(a) $({}^{\perp}\mathscr{B}_C(R), \mathscr{B}_C(R))$ is a complete hereditary cotorsion pair.

(b) $\mathrm{G_CP}(R) \cap \mathrm{G_CP}(R)^{\perp} = {}^{\perp}\mathscr{B}_C(R) \cap \mathscr{B}_C(R)$.

(c) $\mathrm{G_CP}(R)^{\perp} \subseteq \mathscr{B}_C(R)$.

Property (a) follows by [11, Corollary 3.12] while (b) follows by [15, Proposition 5.2] and the proof of [17, Proposition 2.18]. In order to show (c), let us consider a $\mathrm{Hom}_R(-, \mathrm{Add}_R(C))$-exact $\mathrm{Add}_R(C)$-coresolution of R:

$$0 \to R \to C_0 \to C_1 \to \cdots$$

Then $C \oplus (\oplus_i \mathrm{Coker} f_i) \in \mathrm{G_CP}(R)$ by [17, Propositions 2.5 and 2.6 and Remark 2.7] and hence $\mathrm{G_CP}(R)^{\perp} \subseteq (C \oplus (\oplus_i \mathrm{Coker} f_i))^{\perp} = \mathscr{B}_C(R)$ by [11, Theorem 3.10].

From this discussion, keeping in mind Theorem 4.4.6, we obtain under the assumption that R has finite global $\mathrm{G_C}$-projective dimension, a hereditary Hovey triple

$$(\mathrm{G_CP}(R), \widehat{{}^{\perp}\mathscr{B}_C(R)}, \mathscr{B}_C(R))$$

In this case, $(\mathrm{G_CP}(R), \mathrm{G_CP}(R))$ is a hereditary complete cotorsion pair

Of course one could ask dual the questions to **Questions B and C**.

Question $\mathrm{B^{op}}$**.** When is the pair $(\mathrm{G_UI}(R), \mathrm{G_UI}(R)^{\perp})$ a complete cotorsion pair?

Question $\mathrm{C^{op}}$**.** When is the class $\mathrm{G_UI}(R)$ the third part of a (non-trivial) Hovey triple?

$$(\mathscr{Q},\mathscr{W},\mathrm{G_UI}(R))$$

Question D. When does the inequality $\mathrm{G_CP}(R) \subseteq \mathrm{G_CF}(R)$ hold true?

Note that an affirmative answer to this question shows that if C is a w-tilting module so is also w^+-tilting.

Question E. When do we have the following equivalence

$$\forall_R M, \quad M \in \mathrm{G_CF}(R) \Leftrightarrow M^+ \in \mathrm{G_{C^+}I}(R)$$

Again, these last two questions are still open even in the absolute case.

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Resumen

En los últimos años se ha introducido con éxito una variante del álgebra homológica Gorenstein, consistente, a grandes rasgos, en sustituir, en ciertas situaciones, el anillo base por un módulo semidualizante C. Recientemente, y dado que las propiedades que deben satisfacer los módulos semidualizantes son bastante restrictivas, se han publicado trabajos relevantes en los que se estudia hasta qué punto se pueden rebajar las condiciones impuestas al módulo C: aparece el concepto de módulo w-tilting (ver por ejemplo [D. Bennis, J.R. García Rozas and L. Oyonarte, Relative Gorenstein dimensions, Mediterr. J. Math. 13 (2016) 65-91]). En esta tesis doctoral pretendemos descubrir las propiedades de los módulos C-Gorenstein proyectivos (inyectivos y planos) en categorías de módulos sobre determinados tipos de anillos que han despertado el interés de la comunidad matemática desde hace algunos años, y que incluyen construcciones como las álgebras de matrices de matrices triangulares y extensiones triviales de álgebras.

Abstract

In recent years, a variant of Gorenstein homological algebra has been successfully introduced. It consists of replacing, in certain situations, the base ring by a semidualizing module C. Recently, and since semidualizing defining properties are quite restrictive, relevant works have been published with the aim to know to what extent the conditions imposed on the module C can be reduced: the concept of w-tilting module appears (see for example [D. Bennis, JR García Rozas and L Oyonarte, Relative Gorenstein dimensions, Mediterr. J. Math. 13 (2016) 65-91]). In this PhD we intend to discover the properties of the C-Gorenstein projective modules (injective and flat respectively) in categories of modules on certain types of rings that have attracted the interest of the mathematical community for some years, and which include constructions such as triangular matrix algebras and trivial extensions of algebras.

Résumé

Ces dernières années, une variante de l'algèbre homologique de Gorenstein a été introduite avec succès. Elle consiste à remplacer, dans certaines situations, l'anneau de base par un module semidualisant C. Récemment, et puisque les conditions requises pour qu'un module soit semidualisant sont assez restrictives, des travaux pertinents ont été publiés dans le but de savoir dans quelle mesure les conditions imposées au module C peuvent être réduites : le concept de module w-tilting apparaît (voir par exemple [D. Bennis, JR García Rozas et L Oyonarte, Relative Gorenstein dimensions, Mediterr. J. Math. 13 (2016) 65-91]). Dans cette thèse, nous avons l'intention de découvrir les propriétés des modules projectifs de C-Gorenstein (injectifs et plats respectivement) dans les catégories de modules sur certains types d'anneaux qui ont attiré l'intérêt de la communauté mathématique depuis quelques années, et qui incluent des constructions telles que les anneaux matriciels triangulaires.

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