Existence of a continuum of solutions for a quasilinear elliptic singular problem

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Abstract

In this paper we study the existence of positive solution $u \in H_0^1(\Omega)$ for some quasilinear elliptic equations, having lower order terms with quadratic growth in the gradient and singularities, whose model is

$$-\Delta u + \mu(x)\frac{|\nabla u|^2}{u^\gamma + u^\beta} = \lambda u^p + f_0(x),\ x \in \Omega,\ 0 < \gamma \le \beta, 0 < p.$$

Using topological methods we obtain the existence of an unbounded continuum of solutions. In the case $\mu(x)$ constant we derive the existence of solution for every $\lambda > 0$ if $1 < \gamma < 2$ for any β and $p < 1$. Even more for $\mu \in L^\infty(\Omega)$ we prove this result if $\beta \le 1$ and $p < 2 - \beta$.

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1. Introduction

We consider the following boundary value problem

$$\begin{cases} -\Delta u + \mu(x) g(u)\,|\nabla u|^2 = \lambda u^p + f_0(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \qquad (P_\lambda)$$

where Ω is a smooth bounded and open subset of $\mathbb{R}^N$, $N \ge 3$, $p \ge 0$. The functions $\mu \in L^\infty(\Omega)$ and $g \in C^1((0, +\infty))$ are nonnegative; notice that g can become singular at zero. We are assume $0 \lneqq f_0 \in L^q(\Omega)$ for some $q > N/2$.

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By a subsolution (respectively, supersolution) of problem (P_λ) we mean a function $u \in H_0^1(\Omega) \cap C(\overline{\Omega})$ with $u > 0$ a.e. $x \in \Omega$, $g(u)\,|\nabla u|^2 \in L^1(\Omega)$ and which satisfies the following inequality:

$$\int_\Omega \nabla u \nabla \varphi + \int_\Omega \mu(x) g(u)\,|\nabla u|^2 \varphi \overset{(\geq)}{\leq} \int_\Omega (\lambda u^p + f_0)\varphi\,,$$

for every $0 \leq \varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$. A solution is a function which is both a subsolution and a supersolution.

Problem (P_λ) involves a quasilinear elliptic differential operator with quadratic gradient terms. This kind of differential operators with natural growth were considered in [1, 2] and since then different associated boundary value problems have been studied. A well known case is the existence of the solution of (P_0) when g is continuous at $u = 0$ (see for instance [3], [4] and [2]).

Alternatively, problem (P_0) for functions g with a singularity at zero, has also been extensively studied in [5, 6, 7, 8]. Existence of solutions was discussed in [9] in the case $\sqrt{g} \in L^1(0,1)$ by imposing the following condition

$$\operatorname{ess\,inf}\{f_0(x) : x \in \omega\} > 0\,, \ \forall \omega \subset\subset \Omega\,. \tag{1}$$

Results concerning (P_λ) for $\lambda \neq 0$ were obtained in [10, 11] in the case $g(s) = 1/s^\gamma$ where the model problem is

$$\begin{cases} -\Delta u + \mu(x)\dfrac{|\nabla u|^2}{u^\gamma} = \lambda u^p + f_0(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{R_λ}$$

with $\mu(x)$ as a constant function. More precisely, with $\gamma < 1$ and $\gamma + p < 2$ (region I in Figure 1 below), the existence of a solution for each $\lambda \geq 0$ was proved in [10] by means of topological methods and in [11] by using an approximative scheme.

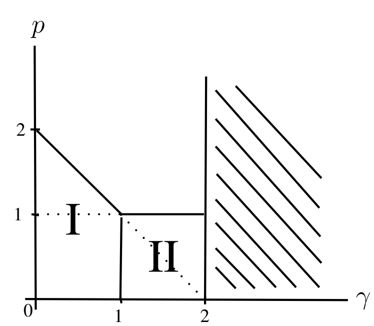

Figure 1:

Notice that if $\gamma \geq 2$ it makes no sense to search solutions of (R_λ). Indeed, as it is proved in [12], $\frac{|\nabla u|^2}{u^\gamma} \notin L^1(\Omega)$.

However, the techniques employed in [10, 11] can not be applied in the case $\mu(x)$ not constant or where $p < 1 \leq \gamma < 2$ (region II in Figure 1 above). In this paper, we complete the previous results and we extend them for a more general function g in order to show the following: *"the values of λ for which there exists a solution of (P_λ) depends on the behavior of g at infinity"*. In fact, in contrast with the results when $g \equiv 0$, in some cases we obtain solutions for every positive λ, that is, the gradient term produces a regularizing effect. We deal with (P_λ) for a function g exhibiting a different behavior at zero and at infinity. In particular, we are mainly interested in the case of functions $g(s) = 1/(s^\gamma + s^\beta)$ with $\gamma \leq \beta$. In this way, we consider the model problem

$$\begin{cases} -\Delta u + \mu(x)\dfrac{|\nabla u|^2}{u^\gamma + u^\beta} = \lambda u^p + f_0(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{Q_λ}$$

as a natural extension of the problem (R_λ). Observe that for $\lambda = 0$, as was mentioned above, problem (Q_0) has been extensively studied. Our main goal is to exploit this known case to obtain an unbounded continuum Σ of solutions of (Q_λ), namely, a connected and closed subset of

$$\{(\lambda, u) \in [0, +\infty) \times C(\overline{\Omega}) : \, u \text{ is a solution of } (Q_\lambda)\},$$

for suitable values of p, γ and β, which extend the previous existence results. In particular, beginning with the case $\mu(x)$ constant and $\gamma < 2$, we prove in Theorem 1.1 the existence of an unbounded continuum Σ. In Theorem 1.2 we deal with non-constant $\mu(x)$ in the case $\beta \leq 1$.

Theorem 1.1. *Assume $\mu(x) = \mu$ is constant and that $f_0 \in L^q(\Omega)$ with $q > \frac{N}{2}$ satisfies* (1). *Then:*

i) If $1 \leq \gamma < 2$ and $0 < p < 1$ then problem (Q_λ) admits at least one solution for every $\lambda \geq 0$.

ii) If $\gamma < 1 < \beta$ and $1 \leq p$, then there exists $\lambda_*, \lambda^* > 0$ such that (Q_λ) admits no solution for $\lambda > \lambda^*$ and at least one solution for $0 \leq \lambda < \lambda_*$.

Moreover, there exists an unbounded continuum Σ of solutions of (Q_λ), such that there exists u_λ solution of (Q_λ) with $(\lambda, u_\lambda) \in \Sigma$ for every $\lambda \geq 0$ (item i)*) or every $0 \leq \lambda < \lambda_*$ (item* ii)*).*

We would like to stress that in the case of item i), it is not required assumptions on the parameter β. This is because in order to $\frac{|\nabla u|^2}{u^\gamma + u^\beta}$ be an integrable function we only need the natural hypothesis $\gamma < 2$ which is a condition at zero. In other words, the behavior of g at infinity has not a role in the solutions set. Conversely, item ii) shows that no regularizing effect take place since there is no solution for all positive λ.

Moreover, observe that this theorem improve the results of [10, 11] since item *i*) with $\gamma = \beta$ gives us existence results of the problem (R_λ) in the case that (γ, p) belongs to Region II of Figure 1 above.

Furthermore, our techniques also allow us to work with non-constant function $\mu(x)$ when the parameter (γ, p) belongs to the corresponding Region I of the Figure 1 above. In fact, if we suppose that there exist positive constants m, M such that

$$m \leq \mu(x) \leq M, \quad \text{a.e. } x \in \Omega, \tag{2}$$

we prove the following theorem.

Theorem 1.2. *Assume that* $0 < \gamma \leq \beta \leq 1$, $0 < p < 2 - \beta$, $f_0 \in L^q(\Omega)$ *with* $q > \frac{N}{2}$ *and* (2) *where* $M < 2$ *in the case* $\gamma = \beta = 1$ *and* $M > 0$ *otherwise. Then there exists an unbounded continuum* Σ *of solutions of* (Q_λ)*, such that there exists* u_λ *solution of* (Q_λ) *with* $(\lambda, u_\lambda) \in \Sigma$ *for every* $\lambda \geq 0$.

Note that this theorem with $\gamma = \beta < 1$ improves again the results of [10] since we can consider non-constant function $\mu(x)$. Furthermore, it improves also [11] except regularity of f_0; in this work the authors consider data f_0 belonging to $L^{\frac{2N}{2N-\gamma(N-2)}}(\Omega)$.

In addition, since we deal with $\gamma < \beta$ and the function $g(s) = 1/(s^\gamma + s^\beta)$ behaves at infinity as $1/s^\beta$ do, we also show that the hypothesis $p < 2 - \beta$ is a restriction in the behavior of g at infinity, rather than in the singularity at zero.

We obtain the existence of the continuum in the above two theorems by using a double approach. Initially, for a convenient sequence of approximated problems, we can derive the existence of Σ_n by means of Leray-Schauder degree techniques and Rabinowitz continuation theorem as in [10]. This requires the uniqueness of the solution for the problem (P_0), in order to set the problem as a fixed point problem for a compact operator. This uniqueness result can not be deduced from [6] if μ is not a constant. Conditions to have uniqueness results for (P_0) were obtained in [13]. Secondly, we use a topological lemma to obtain a continuum of solutions as the limit of this approximative scheme Σ_n. It is also important to note that condition (1) becomes crucial when applying this approach in Theorem 1.1.

The rest of the paper is structured as follows, Section 2 presents the main a priori estimates (this is essentially contained in [14] and [11]). Section 3 provides, for sequences of solutions of (P_λ), compactness properties and continua of solutions. Section 4 provides proofs of the main theorems. Finally the Appendix contains the proof of some a priori estimates and results related to the uniqueness of solution of the problem (P_0).

2. Preliminaries

In this section, according the values for p, we obtain L^∞ estimates for solutions of problem (P_λ).

As usual, for every $s \in \mathbb{R}$, we denote by $s^+ = \max\{s, 0\}$, $s^- = s - s^+$, $T_\varepsilon(s) = s \min\{1, \varepsilon/|s|\}$ and $G_\varepsilon(s) = s - T_\varepsilon(s)$.

Next lemma is consequence of the classical Stampacchia method [14]. We include the proof in the Appendix, by convenience of the reader, using the Hartman-Stampacchia variant [15] (see also [16]).

Lemma 2.1. *Let Λ be a positive number. Assume that $0 < p < 1$ and $f_0 \in L^q(\Omega)$ with $q > \frac{N}{2}$, then there exists a positive constant $C > 0$ such that, for every $g \geq 0$ and every solution u of (P_λ) with $0 < \lambda < \Lambda$, one has $\|u\|_{L^\infty(\Omega)} \leq C$.*

The next lemma shows that, for a convenient decay of g at infinity, the previous result is true even for some cases where $p \geq 1$.

Lemma 2.2. *Let Λ be a positive number. Assume (2) and that $f_0 \in L^q(\Omega)$ with $q > \frac{N}{2}$. Let g_0 also be a nonnegative function in $C((0, +\infty))$ sastifying*

$$\liminf_{t\to\infty} t^\beta g_0(t) > 0, \tag{3}$$

where $1 \leq p < 2 - \beta$. Then there exists a positive constant $C > 0$ such that, for every $g \geq g_0$ and every solution u of (P_λ) with $0 < \lambda < \Lambda$, one has $\|u\|_{L^\infty(\Omega)} \leq C$.

Proof. We follow the arguments of [11, Theorem 2.1] and we prove that the right hand side of (P_λ) is (uniformly) bounded in $L^r(\Omega)$, for some $r > \frac{N}{2}$. Thus the conclusion follows by the classical Stampacchia boundedness theorem and by the positive sign on the quadratic gradient lower order term.

We claim that *there exists a positive constant $C > 0$ and $\sigma \geq pN/2$ such that, for every $g \geq g_0$ and every solution u of (P_λ) with $0 < \lambda < \Lambda$, one has $\|u\|_{L^\sigma(\Omega)} \leq C$.* Thus we can take $r = \min\{q, \sigma/p\}$ to complete the proof.

In order to prove the claim we take $\sigma = (2-\beta)s^{**}$ for some s with

$$\max\left\{\frac{Np}{2(2-\beta+p)}, \frac{2N}{2N-\beta(N-2)}\right\} < s < \frac{N}{2}. \tag{4}$$

We observe that since $\frac{Np}{2(2-\beta+p)} < s$ we have that $(2-\beta)s^{**} > pN/2$. In addition, (4) assures that $\theta = \frac{(2-\beta)s^{**}}{2^*} > 1$ and, for $0 < \delta < 1$, we use $(u+\delta)^{2\theta+\beta-2} - \delta^{2\theta+\beta-2}$ as test function taking into account [10, Lemma 2.1].

We obtain, dropping negative terms,

$$\begin{aligned} &\int_\Omega |\nabla u|^2 (u+\delta)^{2\theta+\beta-3}\Big[(2\theta+\beta-2) + m(u+\delta)g(u)\Big] \\ &\quad \leq M\delta^{2\theta+\beta-2}\int_\Omega g(u)|\nabla u|^2 + \int_\Omega [\Lambda u^p + f_0](u+\delta)^{2\theta+\beta-2}. \end{aligned} \tag{5}$$

Using (3) we deduce the existence of a positive constant $C > 0$ such that

$$\frac{1+tg_0(t)}{(t+1)^{1-\beta}} \geq C, \forall t \geq 0.$$

Hence, since $g \geq g_0$ and $\delta < 1$, we have the inequality

$$1 + tg(t) \geq C(t+\delta)^{1-\beta}, \forall t \geq 0.$$

Therefore, from (5) we obtain, using also Sobolev inequality,

$$\begin{aligned} C\mathcal{S}\Big(\int_\Omega \big[(u+\delta)^\theta-\delta^\theta\big]^{2^*}\Big)^{\frac{2}{2^*}} &\le C\int_\Omega |\nabla u|^2(u+\delta)^{2\theta-2} \\ &\le M\delta^{2\theta+\beta-2}\int_\Omega g(u)|\nabla u|^2+\int_\Omega [\Lambda u^p+f_0](u+\delta)^{2\theta+\beta-2}\,, \end{aligned} \tag{6}$$

where $\mathcal{S}$ is the Sobolev embedding constant. Letting δ tend to zero, we get

$$C\mathcal{S}\left(\int_\Omega u^{2^*\theta}\right)^{\frac{2}{2^*}} \le C\int_\Omega |\nabla u|^2 u^{2\theta-2} \le \Lambda\int_\Omega u^{2\theta+\beta+p-2}+\int_\Omega f_0 u^{2\theta+\beta-2}\,. \tag{7}$$

Thanks to the choice of θ, we have $2^*\theta=(2\theta+\beta-2)s'=(2-\beta)s^{**}$. Thus, using Hölder inequality, and recalling that $s^{**}(2-\beta)\ge 2^*>2>2-\beta>p$, we deduce

$$\begin{aligned} \left(\int_\Omega u^{(2-\beta)s^{**}}\right)^{\frac{2}{2^*}} &\le C\left(\int_\Omega u^{(2-\beta)s^{**}}\right)^{\frac{2\theta+\beta+p-2}{(2-\beta)s^{**}}} \\ &+C\,\|f_0\|_{L^s(\Omega)}\left(\int_\Omega u^{(2-\theta)s^{**}}\right)^{\frac{1}{s'}}\,. \end{aligned} \tag{8}$$

Now we point out that $\frac{2}{2^*}>\frac{1}{s'}$, since $s<\frac{N}{2}$, and that $\frac{2}{2^*}>\frac{2\theta+\beta+p-2}{(2-\beta)s^{**}}$, since $2-\beta>p$. Therefore, from (8) it follows the claim which allows to finish the proof. □

3. Global continua of solutions

Let $\mathcal{M}$ be the solution set for (P_λ), namely

$$\mathcal{M}=\{(\lambda,u)\in[0,+\infty)\times C(\overline{\Omega}) :\ u \text{ is a solution of } (P_\lambda)\}.$$

Continua of solutions in $\mathcal{M}$ are obtained in this section by using degree computations and Rabinowitz continuation theorem. In this way, we set (P_λ) as a fixed point problem for a compact operator.

Next result gives sufficient conditions to assure that solutions of (P_λ) are uniformly bounded from below by a positive constant in compact subsets. In fact, we can consider lower order terms of the form $h(u)|\nabla u|^2$ with

$$\begin{aligned} &h\in C((0,+\infty)) \text{ is a nonnegative function, nonincreasing} \\ &\qquad\qquad \text{in a neighborhood of zero with } \sqrt{h}\in L^1(0,1), \end{aligned} \tag{9}$$

and data f_0 satisfying

(F) If $e^{-\int_1^s h(t)dt}\in L^1(0,1)$ then f_0 is nonnegative and nontrivial. In other case f_0 satisfies (1).

Lemma 3.1. *Assume that h verifies* (9) *and $f_0 \in L^1(\Omega)$ satisfies* (F). Then for each $\omega \subset\subset \Omega$ there exists a positive constant c_ω such that $z(x) \geq c_\omega > 0$ a.e. $x \in \omega$, for every $0 < z \in H_0^1(\Omega) \cap C(\Omega)$ supersolution of

$$-\Delta z + h(z)|\nabla z|^2 = f_0 \quad \text{in } \Omega.$$

Proof. On the one hand, if f_0 satisfies (1) the proof can be found in [9, Proposition 2.3]. On the other hand, if $e^{-\int_1^s h(t)dt} \in L^1(0,1)$, then f_0 is a general nonnegative and nontrivial function and we split the proof into two cases: when $h \in L^1(0,1)$ we conclude by [10, Proposition 2.4], while if $h \notin L^1(0,1)$ we follow the arguments of [8, Theorem 3.1]. □

Remark 3.2. We notice that if we assume $h(s) = \frac{C}{s^\gamma}$ then $e^{-\int_1^s h(t)dt} \in L^1(0,1)$ if and only if $\gamma < 1$ or if $\gamma = 1$ and $C < 1$.

The following lemma ensures the compactness properties required later to deal with our topological approach.

Lemma 3.3. *Assume that $0 \lneqq f_0 \in L^q(\Omega)$ with $q > \frac{N}{2}$ and $\mu \in L^\infty(\Omega)$. Let assume that $0 < u_n \in H_0^1(\Omega) \cap C(\overline{\Omega})$ satisfies*

$$\begin{cases} -\Delta u_n + \mu(x) g_n(u_n)|\nabla u_n|^2 = \lambda_n w_n^p + f_0 & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega, \end{cases} \tag{10}$$

with $0 \leq \lambda_n$ bounded in $\mathbb{R}$, $0 \leq w_n$ bounded in $C(\overline{\Omega})$ and $0 \leq g_n$ a sequence of functions in $C((0,+\infty))$. Then, up to a subsequence, u_n is strongly convergent in $C(\overline{\Omega})$ to $u \in H_0^1(\Omega) \cap C(\overline{\Omega})$. If, in addition, $\lambda_n \to \lambda$, $w_n \to w$ in $C(\overline{\Omega})$, $g_n(s) \to g(s)$ uniformly in $C([a,b])$ for every $0 < a < b < \infty$, $g_n(s) \leq h(s)$ for some h verifying (9) *and f_0 satisfies (F), then u is a solution of problem*

$$\begin{cases} -\Delta u + \mu(x) g(u)|\nabla u|^2 = \lambda w^p + f_0(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega\,. \end{cases} \tag{11}$$

Moreover, if the problem (11) *admits a unique solution then the whole sequence u_n converges strongly to u in $C(\overline{\Omega})$.*

Proof. Since the sequence $f_n := \lambda_n w_n^p + f_0$ is bounded in $L^q(\Omega)$ for some $q > N/2$, we can deduce, as in the proof of Lemma 2.1, or by using the Stampacchia technique in [14] that $\|u_n\|_{L^\infty(\Omega)} \leq c_\infty$ for some positive constant c_∞. In addition, applying [16, Theorem 6.1] we deduce that the sequence u_n is bounded in $C^{0,\alpha}(\overline{\Omega})$. Consequently, Ascolí-Arzelá Theorem assures that u_n possesses a subsequence converging in $C(\overline{\Omega})$. This conclude the first part of the lemma.

In order to prove the second part we observe that, since the sequence u_n is bounded in $H_0^1(\Omega)$ (arguing again as in the proof of Lemma 2.1, Step I) we can assume that u_n converges weakly to u in $H_0^1(\Omega)$. Now we prove that u is solution of problem (11), i.e. $u > 0$, $g(u)|\nabla u|^2 \in L^1(\Omega)$ and satisfies,

$$\int_\Omega \nabla u \nabla \varphi + \int_\Omega \mu(x) g(u)\,|\nabla u|^2 \varphi = \int_\Omega (\lambda w^p + f_0)\varphi, \tag{12}$$

for every $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$.

By Lemma 3.1 given $\omega \subset\subset \Omega$ there exists $c_\omega > 0$ such that $u_n(x) \geq c_\omega$ a.e. $x \in \omega$ for every $n \in \mathbb{N}$. In particular, using that u_n converges strongly to u in $C(\overline{\Omega})$, we deduce $u > 0$ in Ω. Even more, the strong convergence of g_n to g in $C([c_\omega, c_\infty])$ assures that $g_n(u_n) \to g(u)$ a.e. in Ω.

Next, by the first part of the proof of Theorem 3.1 in [7] we have that $\mu(x)g(u)|\nabla u|^2 \in L^1(\Omega)$. We include the proof by convenience of the reader. Indeed, taking $\varphi = \frac{T_\epsilon(u_n)}{\epsilon}$ as test function in (10) and dropping the positive term coming from the principal part we get

$$\int_\Omega \mu(x)g_n(u_n)|\nabla u_n|^2 \frac{T_\epsilon(u_n)}{\epsilon} \leq \int_\Omega (\lambda_n w_n^p + f_0)\frac{T_\epsilon(u_n)}{\epsilon}.$$

Since $\displaystyle\int_\Omega (\lambda_n w_n^p + f_0) \leq C$, we obtain

$$\int_\Omega \mu(x)g_n(u_n)|\nabla u_n|^2 \frac{T_\epsilon(u_n)}{\epsilon} \leq C.$$

The limit as $\epsilon \to 0$ implies, using that $\lim\limits_{\epsilon\to 0} \frac{T_\epsilon(u_n)}{\epsilon} = 1$,

$$\int_\Omega \mu(x)g_n(u_n)|\nabla u_n|^2 \leq C\,.$$

Furthermore, the results of [17, Theorem 2.1] imply that (up to a subsequence) $\nabla u_n \to \nabla u$ strongly in $(L^q(\Omega))^N$ $(1 < q < 2)$, particularly, it converges almost everywhere in Ω. Then, the last inequality gives us after applying the Fatou lemma that

$$\int_\Omega \mu(x)g(u)|\nabla u|^2 \leq C\,,$$

which proves our claim.

To finish, following closely [7], we prove that u satisfies the equation (12). Since $\varphi = \varphi^+ + \varphi^-$, it is enough to prove (12) for every nonegative function $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$. Furthermore, by density, it is sufficient to prove it when $0 \leq \varphi \in H_0^1(\Omega) \cap C_c(\Omega)$.

We divide the proof into two steps.

Step I. The function u satisfies

$$\int_\Omega \nabla u \nabla \phi + \int_\Omega \mu(x)g(u)|\nabla u|^2\phi \leq \int_\Omega \lambda w^p\,\phi + \int_\Omega f_0\phi,$$

for all $0 \leq \phi \in H_0^1(\Omega) \cap C_c(\Omega)$. Indeed, since $\mu(x)g_n(u_n)|\nabla u_n|^2 \geq 0$, $g_n(u_n) \to g(u)$ a.e. $x \in \Omega$, ∇u_n converges weakly in $(L^2(\Omega))^N$ and a.e. $x \in \Omega$ to ∇u and w_n^p converges to w^p strongly in $L^2(\Omega)$, then we obtain the result taking a function $0 \leq \phi \in H_0^1(\Omega) \cap C_c(\Omega)$ as a test function in (10) and applying Fatou lemma.

Step II. The function u satisfies

$$\int_\Omega \nabla u \nabla \phi + \int_\Omega \mu(x) g(u) |\nabla u|^2 \phi \geq \int_\Omega \lambda w^p \phi + \int_\Omega f_0 \phi,$$

for all $0 \leq \phi \in H_0^1(\Omega) \cap C_c(\Omega)$. We fix $0 \leq \phi \in H_0^1(\Omega) \cap C_c(\Omega)$ and we define the function

$$H(t) = \int_1^t M h\,(s)\, ds,$$

where $M = \|\mu\|_{L^\infty(\Omega)}$. Let call $\omega = \text{supp}\ \phi$ and observe, thanks to Lemma 3.1 there exists a positive constants c_ω such that $c_\omega \leq u_n$ in ω for every $n \in \mathbb{N}$. Moreover, the boundedness in $L^\infty(\Omega)$ of the sequence $\{u_n\}$ implies $u_n \leq c_\infty$. Therefore, for n big enough

$$|H(u) - H(u_n)| \leq M \int_{c_w}^{c_\infty} h(s) ds \leq M(c_\infty - c_\omega) \max_{s \in [c_\omega, c_\infty]} h(s) < \infty,$$

a.e. $x \in \omega$. In addition, one can similarly deduce, that

$$|H(u) - H(u_n)| \leq M |u - u_n| \max_{s \in [c_\omega, c_\infty]} h(s), \text{ a.e. } x \in \omega.$$

In particular, there exists a positive constant C_ϕ (depending only on ϕ) such that

$$e^{H(u) - H(u_n)} \phi \leq C_\phi.$$

Even more,

$$\nabla \left(e^{H(u) - H(u_n)} \phi \right) =$$

$$e^{H(u) - H(u_n)} \left(M \phi h(u) \nabla u - M \phi h(u_n) \nabla u_n + \nabla \phi \right) \in L^2(\Omega).$$

Thus, taking $\varphi = e^{H(u) - H(u_n)} \phi$ as a test function in (10), we get

$$\begin{aligned} \int_\Omega \nabla u_n \nabla \phi\, e^{H(u) - H(u_n)} &+ M \int_\Omega h(u) \nabla u \nabla u_n e^{H(u) - H(u_n)} \phi \\ &- \int_\Omega \left(\lambda_n w_n^p + f_0 \right) e^{H(u) - H(u_n)} \phi \\ &= \int_\Omega \left(M h(u_n) - \mu(x) g_n(u_n) \right) |\nabla u_n|^2 e^{H(u) - H(u_n)}\, \phi\,. \end{aligned} \quad (13)$$

Next, we want to pass to the limit in the above expression. Observe that, since ∇u_n converges weakly in $(L^2(\Omega))^N$, we have

$$\int_\Omega \nabla u_n \nabla \phi\, e^{H(u) - H(u_n)} \longrightarrow \int_\Omega \nabla u \nabla \phi.$$

In addition, since the function $\phi h(u)$ and the sequence $\lambda_n w_n^p$ are bounded, we obtain using the Lebesgue Theorem

$$\int_\Omega h(u) \nabla u \nabla u_n e^{H(u) - H(u_n)}\, \phi \longrightarrow \int_\Omega h(u) |\nabla u|^2 \phi,$$

and

$$\int_\Omega (\lambda_n w_n^p + f_0)\, e^{H(u)-H(u_n)}\, \phi \longrightarrow \int_\Omega (\lambda w^p + f_0)\, \phi\,.$$

To finish, since $Mh(u_n) - \mu(x)g_n(u_n) \geq 0$, we deduce the inequality desired applying the Fatou Lemma in the right hand side of (13).

Summarizing Step I and Step II we conclude the proof. $\square$

As can be observed, uniqueness of solution for (P_0) plays a fundamental role. In order to use the uniqueness result in [13, Theorem 1.1] we have to assume that the function g satisfies in addition that for every $\nu > 0$ there exists $\theta_\nu \geq 0$ and a nonnegative function $\tilde{g} \in C^1((0,+\infty))$ with $e^{-\int_1^s \tilde{g}(t)dt} \in L^1(0,1)$ such that for every $0 < s < \nu$ and for a.e. $x \in \Omega$

$$\begin{aligned} \theta_\nu \left[(\mu(x)g'(s) - \tilde{g}'(s)) + \tilde{g}(s)(\mu(x)g(s) - \tilde{g}(s))\right] \\ \geq (\mu(x)g(s) - \tilde{g}(s))^2\,. \end{aligned} \tag{14}$$

Remark 3.4. In the case $\mu(x) = \mu$ for some positive constant μ, we can use the uniqueness result for problem (P_0) in [6] for functions $g \in L^1(0,1)$. Observe that, in that case, condition (14) is also trivially satisfied with $\tilde{g}(s) = \mu g(s)$. On the other hand, for a non-constant function $\mu(x)$, it is proved in [13] that condition (14) is also satisfied in the case $g(s) = 1/s^\gamma$ with $\gamma < 1$. Moreover, in the case $g(s) = 1/s$, assuming in addition that $M < 1$, we can choose $\tilde{g}(s) = c/s$ for some $M < c < 1$ and we have that (14) is satisfied with $\theta_\nu \geq \frac{c}{1-c}$. Others particular cases that it will be used in the proof of Theorem 1.2 can be found in the Appendix.

Finally, next result ensures existence of an unbounded, connected and closed subset of $\mathcal{M}$.

Theorem 3.5. *Assume* (2), *g satisfies* (14), *$g(s) \leq h(s)$ for some function h verifying* (9) *and $f_0 \in L^q(\Omega)$ with $q > N/2$ satisfies (F). Then there exists an unbounded continuum $\Sigma \subset \mathcal{M}$ such that $(0, u_0) \in \Sigma$, where u_0 is the unique solution of* (P_0).

Proof. Firstly, we observe the problem (P_0) admits a unique solution $0 < u \in H_0^1(\Omega) \cap C(\overline{\Omega})$. Indeed, the existence is due to [7] and [8] if $0 \lneqq f_0$ and due to [9, Theorem 1.1] if f_0 satisfies (1). Alternatively, the uniqueness is deduced using [13, Theorem 1.1].

Hence, we can define $K : [0,1] \times \mathbb{R} \times C(\overline{\Omega}) \to C(\overline{\Omega})$ by setting $K(t,\lambda,w)$ as the unique solution $0 < u \in C(\overline{\Omega})$ of the problem

$$\begin{cases} -\Delta u + t\mu(x)g(u)\,|\nabla u|^2 = \lambda^+(w^+)^p + f_0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{15}$$

for every $\lambda \in \mathbb{R}$, $t \in [0,1]$ and $w \in C(\overline{\Omega})$. With this notation problem (P_λ) can be rewritten as a fixed point problem, namely,

$$u = K_\lambda^1(u),$$

with $K^t_\lambda(u) = K(t,\lambda,u)$. Moreover, since g satisfies (14) Lemma 3.3 assures that K is compact and we can use Leray-Schauder degree to study (P_λ).

The result follows, as in [10], from the Rabinowitz's Theorem [18, Theorem 3.2]. We only have to compute the index of the solution u_0 and show that it is different from zero. Let us denote $u_t = K(1-t,0,0)$ i.e., u_t is the unique positive solution in $H^1_0(\Omega)\cap C(\overline{\Omega})$ of the problem

$$\begin{cases} -\Delta u + (1-t)\mu(x)g(u)\,|\nabla u|^2 = f_0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega\,. \end{cases}$$

Define the homotopy $J : [0,1]\times C(\overline{\Omega}) \to C(\overline{\Omega})$ by $J(t,w) = u_t$ for every $t \in [0,1]$ and $w \in C(\overline{\Omega})$. Observe that $J(t,w) = K(1-t,0,0)$ and thus, using Lemma 3.3, we have that J is compact. Moreover, $J(0,w) = u_0$ and $J(1,w) = (-\Delta)^{-1}(f_0(x))$. Therefore

$$i(K^1_0,u_0) = i(J(0,\cdot),u_0) = i(J(1,\cdot),u_1) = i((-\Delta)^{-1}(f_0(x)),u_1) = 1.$$

Consequently, since $i(K^1_0,u_0) = 1$, we conclude the proof by using Rabinowitz's theorem. □

4. Proof of the main results

In order to prove Theorem 1.1 and Theorem 1.2 we recall, for the convenience of the reader, the following definition and topological result (see [19]):

Definition 4.1. Let $\{S_n\} \subset X$ be any infinite collection of point sets, not necessarily different. The set of all points x of our space X such that every neighborhood of x contains points of infinitely many sets of $\{S_n\}$ is called the *superior limit*. The set of all points y such that every neighborhood of y contains points of all but a finite number of the sets of $\{S_n\}$ is called the *inferior limit*.

From the definiton, we have at once for any system $\{S_n\}$

$$\liminf S_n \subset \limsup S_n$$

Lemma 4.2 ([19] Whyburn)**.** *Let X be a metric space. If $\{S_n\}$ is a sequence of connected subsets of X such that $\bigcup S_n$ is relatively compact and $\liminf S_n$ is not empty, then the $\limsup S_n$ is connected.*

The trick, in the proof of Theorem 1.1 and Theorem 1.2 is to use Lemma 4.2 where S_n is a continuum of solutions of the following approximated problems

$$\begin{cases} -\Delta u + \mu(x)\,\dfrac{|\nabla u|^2}{(u+\frac{1}{n})^\gamma + (u+\frac{1}{n})^\beta} = \lambda u^p + f_0(x)\,, & \text{in } \Omega, \\ u = 0\,, & \text{on } \partial\Omega, \end{cases} \qquad (Q_{n,\lambda})$$

for $n \in \mathbb{N}$ and $\gamma \le \beta$.

Proof of Theorem 1.1. First we deal with item i). We consider, for $n \in \mathbb{N}$, the approximated problems $(Q_{n,\lambda})$ and the idea is to use Theorem 3.5 with $h(s) = \frac{1}{s^\gamma + s^\beta}$ and f_0 satisfying (1). We observe that, under the assumption μ constant, the function $g_n(s) = \frac{1}{(s+1/n)^\gamma + (s+1/n)^\beta}$ satisfies (14) without restrictions in γ and β (recall Remark 3.4). Now by Theorem 3.5, there exists a continuum Σ_n in $[0, +\infty) \times C(\overline{\Omega})$ of positive solutions of $(Q_{n,\lambda})$ such that $(0, u_n) \in \Sigma_n$ with u_n solution of $(Q_{n,0})$. One can observe that by Lemma 2.1, one has $\mathrm{Proj}_{[0,\infty)}\Sigma_n = [0, \infty)$.

For obtaining the existence of an unbounded continuum Σ of solutions of (Q_λ) we apply the result of Lemma 4.2. Indeed, for every $\Lambda > 0$ we take $S_{n,\Lambda}$ the connected component of $\Sigma_n \cap \big([0, \Lambda] \times C(\overline{\Omega})\big)$ such that $(0, u_n) \in S_{n,\Lambda}$. Since Σ_n is unbounded and $\mathrm{Proj}_{[0,\infty)}\Sigma_n = [0, \infty)$, we deduce that $\mathrm{Proj}_{[0,\Lambda]}S_{n,\Lambda} = [0, \Lambda]$. Moreover, Lemma 3.3 with $\lambda_n = 0$ assures that, up to (not relabeled) subsequences, u_n converges strongly to u solution of (Q_0), which implies $(0, u) \in \liminf S_{n,\Lambda}$. Even more, given a sequence $(\lambda_m, u_m) \in \bigcup_{k\in\mathbb{N}} S_{k,\Lambda}$ we have that, for some $k_m \in \mathbb{N}$

$$\begin{cases} -\Delta u_m + \mu(x) g_{k_m}(u_m)|\nabla u_m|^2 = \lambda_m u_m^p + f_0(x) & \text{in } \Omega, \\ u_m = 0 & \text{on } \partial\Omega, \end{cases}$$

with $0 \leq \lambda_m < \Lambda$ and $\|u_m\|_{L^\infty(\Omega)} \leq c_\Lambda$. As we can suppose that $k_m \to \infty$, then the first part of Lemma 3.3, with $w_n = u_m$, assures that (λ_m, u_m) admits a strongly convergent subsequence. In particular we deduce that $\bigcup_{k\in\mathbb{N}} S_{k,\Lambda}$ is relatively compact. We notice that if the sequence k_m is bounded then, up to a sequence, (λ_m, u_m) converges in $\bigcup_{k\in\mathbb{N}} S_{k,\Lambda}$. Now we can use Lemma 4.2 to deduce that $\Gamma_\Lambda = \limsup S_{n,\Lambda}$ is a continuum which, using the second part of Lemma 3.3, is contained in $\mathcal{M}$. In fact, since for every $n \in \mathbb{N}$ there exists $(\Lambda, u_n) \in S_{n,\Lambda}$, then we have that $\mathrm{Proj}_{[0,\Lambda]}\Gamma_\Lambda = [0, \Lambda]$. Furthermore, by construction, Γ_Λ is increasing in Λ and we can take $\Sigma = \bigcup_{n\in\mathbb{N}} \Gamma_n$. Observe that since $(0, u) \in \Gamma_n$ for every $n \in \mathbb{N}$ then $\Sigma \subset \mathcal{M}$ is a connected set in $[0, +\infty) \times C(\overline{\Omega})$. Moreover, $\mathrm{Proj}_{[0,\infty)}\Sigma = \bigcup_{n\in\mathbb{N}}[0, n] = [0, \infty)$.

Now we deal with the proof in the case of item ii). In this case, since $\mu(x)$ is constant and $\gamma < 1$, we have that $g(s) = \frac{1}{s^\gamma + s^\beta}$ verifies (14) and (9). Thus, the unbounded continuum Σ of solutions of (Q_λ) is obtained from Theorem 3.5. In addition, the projection of Σ to the λ-axis has to be bounded, since we can use [10, Theorem 5.1] to deduce the existence of λ^*. Observe that $g \in L^1(0, +\infty)$ and

$$\lim_{s\to\infty} \frac{s^p}{\int_0^s e^{\int_r^s g(t)dt} dr} = \lim_{s\to\infty} \frac{e^{\int_1^s \left(\frac{p}{t} - g(t)\right)dt}}{\int_0^s e^{-\int_1^r g(t)dt} dr} = \lim_{s\to\infty} \left(\frac{p}{s} - g(s)\right) s^p$$

$$= \lim_{s\to\infty} s^{p-1}\left(p - \frac{s^{1-\beta}}{s^{\gamma-\beta} + 1}\right) = \begin{cases} 1, & p = 1, \\ \infty, & p > 1. \end{cases}$$

Therefore g verifies condition (1.6) in [10]. □

Proof of Theorem 1.2. We observe that, for every $n \in \mathbb{N}$ fixed, the function $g_n(s) = \frac{1}{(s+1/n)^\gamma + (s+1/n)^\beta}$ satisfies (14) for $\beta \leq 1$ and general $\mu(x)$ (see *Cases 1-*

3 of Appendix). Thus by Theorem 3.5 there exists a continuum Σ_n in $[0,+\infty)\times C(\overline{\Omega})$ of positive solutions of $(Q_{n,\lambda})$ such that $(0,u_n)\in\Sigma_n$ with u_n solution of $(Q_{n,0})$. We claim that $\mathrm{Proj}_{[0,\infty)}\Sigma_n=[0,\infty)$. Indeed, this is a consequence of the bound on the norm, for λ in bounded sets, of the solutions of $(Q_{n,\lambda})$. More precisely, this bound is obtained by means of Lemma 2.1, for $p<1$ and Lemma 2.2 with $g_0(s)=\frac{1}{(s+1)^\gamma+(s+1)^\beta}$ for $p\geq 1$.

The existence of the unbounded continuum Σ with $\mathrm{Proj}_{[0,\infty)}\Sigma=[0,\infty)$ is deduced now arguing as in the proof of Theorem 1.1, observe that Lemma 3.3 with $\lambda_n=0$ assures that, passing to subsequence, u_n converges strongly to u solution of (Q_0). To conclude, we note by Remark 3.4 the need to consider $M<2$ in the case $\gamma=\beta=1$. □

Remark 4.3. Thanks to *Case 4* of Appendix it is worth stressing that the previous theorem could be extended to $\gamma=1<\beta$ if $M\leq 1$.

Remark 4.4. A simplest proof of Theorem 1.2 can be obtained in the particular case $\gamma=\beta\leq 1$. Indeed, the function $g(s)=1/s^\gamma$ with $\gamma<1$ satisfies condition (14) and this condition is also satisfied in the case $\gamma=1$ if, in addition, we assume that $M<1$ (see Remark 3.4). Consequently applying directly Theorem 3.5 for $\gamma<1$ and Remark 3.2 for $\gamma=1$ we can deduce the existence of an unbounded continuum Σ of solutions of (R_λ). Moreover, using Lemma 2.1 in the case $p<1$ or Lemma 2.2, with $\beta=\gamma$ and $g_0(s)=1/s^\gamma$, in the case $p>1$, we can assure that $\mathrm{Proj}_{[0,\infty)}\Sigma=[0,\infty)$, concluding the claim.

Appendix

We devote this appendix to include the proof of Lemma 2.1 as well as the proof of (14) in some particular cases.

Proof of Lemma 2.1. We choose suitable test functions taking into account [10, Lemma 2.1]. We divide the proof into two steps:
STEP I. There exists a positive constant C such that, for every $g\geq 0$ and every solution u of (P_λ) with $0<\lambda<\Lambda$, one has $\|u\|_{H_0^1(\Omega)}\leq C$.

Indeed, take $\varphi=u$ as a test function to obtain, dropping the positive term given by the lower order term, that

$$\int_\Omega |\nabla u|^2 \leq \int_\Omega \lambda u^{p+1} + \int_\Omega f_0\, u\,.$$

Since $p+1<2$, we can use Hölder and Sobolev inequalities in the right hand side to conclude

$$\int_\Omega |\nabla u|^2 \leq c\left(\left(\int_\Omega |\nabla u|^2\right)^{\frac{p+1}{2}} + \left(\int_\Omega |\nabla u|^2\right)^{\frac{1}{2}}\right),$$

for some positive constant c depending only on Λ,Ω,f_0 and p. This inequality give us Step I with C the unique positive solution of the equation $s^2=c(s^{p+1}+s)$.

STEP II. There exists $C > 0$ such that, for every $g \geq 0$ and every solution u of (P_λ) with $0 < \lambda < \Lambda$, one has $\|u\|_{L^\infty(\Omega)} \leq C\|u\|_{L^1(\Omega)}$.

Given $k > 1$, we take $\varphi = G_k(u)$ as a test function in (P_λ). Hence, dropping the positive lower order term and using Hölder's inequality in the right hand side, we have

$$\int_\Omega |\nabla G_k(u)|^2 \leq \int_{A_k} (\lambda + f_0)u^2 \leq \|\lambda + f_0\|_{L^q(\Omega)} \left(\int_{A_k} u^{2q'}\right)^{\frac{1}{q'}}, \tag{16}$$

where $A_k = \{x \in \Omega : u(x) > k\}$. Throughout the proof, C denotes different positive constants depending only on Λ, f_0, p and Ω.

Firstly, we estimate the right hand side of (16) using Hölder and Sobolev inequalities and the fact that $u = T_k(u) + G_k(u)$. Thus,

$$\begin{aligned}
\left(\int_{A_k} u^{2q'}\right)^{\frac{1}{q'}} &\leq C\left(k^{2q'}|A_k| + \int_{A_k} G_k(u)^{2q'}\right)^{\frac{1}{q'}} \\
&\leq C\left(\int_{A_k} G_k(u)^{2q'}\right)^{\frac{1}{q'}} + C\,k^2|A_k|^{\frac{1}{q'}} \\
&\leq C\left(\int_\Omega G_k(u)^{2^*}\right)^{\frac{2}{2^*}} |A_k|^{\frac{1}{q'}-\frac{2}{2^*}} + C\,k^2|A_k|^{\frac{1}{q'}} \\
&\leq C\,|A_k|^{\frac{1}{q'}-\frac{2}{2^*}} \int_\Omega |\nabla G_k(u)|^2 + C\,k^2|A_k|^{\frac{1}{q'}}.
\end{aligned}$$

Consequently, from (16) we have,

$$\int_\Omega |\nabla G_k(u)|^2 \leq C\,\|\lambda + f_0\|_{L^q(\Omega)} \left(|A_k|^{\frac{1}{q'}-\frac{2}{2^*}} \int_\Omega |\nabla G_k(u)|^2 + k^2|A_k|^{\frac{1}{q'}}\right).$$

Using Step I we have that $k|A_k| \leq \|u\|_{L^1(\Omega)} \leq C$ and, since $\frac{1}{q'} - \frac{2}{2^*} > 0$, we can choose k big enough such that

$$\int_\Omega |\nabla G_k(u)|^2 \leq C\,\|\lambda + f_0\|_{L^q(\Omega)}\, k^2|A_k|^{\frac{1}{q'}}.$$

Using Hölder and Sobolev inequalities and the above inequality we conclude

$$\int_{A_k} G_k(u) \leq C\,k\,|A_k|^{1+\frac{1}{2q'}-\frac{1}{2^*}},$$

which gives us the result applying [15, Lemma 7.2] (see also [16, Lemma 5.1, pag 71]).

Summarizing Step I and Step II, we conclude the proof. □

Now we prove (14) for $g(s) = \frac{1}{\left(s+\frac{1}{n}\right)^\gamma + \left(s+\frac{1}{n}\right)^\beta}$ and $0 < \gamma \leq \beta \leq 1$ or $M < 1 = \gamma < \beta$.

Proof of (14). For every $\nu > 0$, we take $\tilde{g}(s) = h(s)g(s)$ for a convenient function $h \in C^1([0, +\infty))$, such that, for some $\theta_\nu \geq 0$

$$\theta_\nu \left(\left(\mu(x)\frac{g'(s)}{g^2(s)} - h(s)\frac{g'(s)}{g^2(s)} - \frac{h'(s)}{g(s)} \right) + h(s)(\mu(x) - h(s)) \right) \\ \geq (\mu(x) - h(s))^2, \quad \forall s < \nu.$$

Observe that this inequality is trivially satisfied if $h(s) = \mu(x)$ and $h'(s) \leq 0$ while, in other case, it is equivalent to prove that the function

$$\sigma(x, s) \equiv \frac{(\mu(x) - h(s)) \left(h(s) + \frac{g'(s)}{g^2(s)} \right) - \frac{h'(s)}{g(s)}}{(\mu(x) - h(s))^2}$$

is bounded from below by a positive constant. We point out that

$$\frac{g'(s)}{g^2(s)} = -\gamma \left(s + \frac{1}{n} \right)^{\gamma - 1} - \beta \left(s + \frac{1}{n} \right)^{\beta - 1}.$$

Now we choose the function $h(s)$ based on the different values of γ and β.
Case 1. $\gamma \leq \beta < 1$.

In this case we take $h(s) = -g'(s)/g^2(s) = \gamma(s + \frac{1}{n})^{\gamma-1} + \beta(s + \frac{1}{n})^{\beta-1}$. Thus

$$h'(s) = \gamma(\gamma - 1) \left(s + \frac{1}{n} \right)^{\gamma - 2} + \beta(\beta - 1) \left(s + \frac{1}{n} \right)^{\beta - 2} < 0.$$

In particular, we have that $\sigma(x, s)$ is given by

$$\frac{\left(\gamma(1-\gamma) \left(s + \frac{1}{n} \right)^{\gamma-2} + \beta(1-\beta) \left(s + \frac{1}{n} \right)^{\beta-2} \right) \left(\left(s + \frac{1}{n} \right)^{\gamma} + \left(s + \frac{1}{n} \right)^{\beta} \right)}{\left(\mu(x) - \gamma \left(s + \frac{1}{n} \right)^{\gamma-1} - \beta \left(s + \frac{1}{n} \right)^{\beta-1} \right)^2} \\ = \frac{\left(\gamma(1-\gamma) \left(s + \frac{1}{n} \right)^{\gamma-\beta} + \beta(1-\beta) \right) \left(\left(s + \frac{1}{n} \right)^{\gamma-\beta} + 1 \right)}{\left(\mu(x) \left(s + \frac{1}{n} \right)^{1-\beta} - \gamma \left(s + \frac{1}{n} \right)^{\gamma-\beta} - \beta \right)^2}$$

We conclude by taking into account that this function (which may take infinite values) only vanishes for $s \to +\infty$.
Case 2. $\gamma < \beta = 1$.

In this case we take again $h(s) = -g'(s)/g^2(s) = \gamma(s + \frac{1}{n})^{\gamma-1} + 1$. Thus

$$h'(s) = \gamma(\gamma - 1) \left(s + \frac{1}{n} \right)^{\gamma - 2} < 0.$$

In particular, we have

$$\sigma(x, s) = \frac{\left(\gamma(1-\gamma) \left(s + \frac{1}{n} \right)^{\gamma-2} \right) \left(\left(s + \frac{1}{n} \right)^{\gamma} + \left(s + \frac{1}{n} \right) \right)}{\left(\mu(x) - \gamma \left(s + \frac{1}{n} \right)^{\gamma-1} - 1 \right)^2}.$$

We conclude, as before, by taking into account that this function only vanishes for $s \to +\infty$.
Case 3. $\gamma = \beta = 1$.

In this case we can choose $h(s) = 2 + \frac{1}{1+3ns}$ and we have

$$\sigma(x,s) = \frac{(\mu(x) - 2 - \frac{1}{1+3ns})\frac{1}{1+3ns} + \frac{6n(s+1/n)}{(1+3ns)^2}}{\left(\mu(x) - 2 - \frac{1}{1+3ns}\right)^2}$$
$$> \frac{\frac{-3-6ns}{(1+3ns)^2} + \frac{6n(s+1/n)}{(1+3ns)^2}}{\left(\mu(x) - 2 - \frac{1}{1+3ns}\right)^2} = \frac{3}{((\mu(x)-2)(1+3ns)-1)^2}$$

We conclude again using that this function only vanishes for $s \to +\infty$.
Case 4. $M \leq 1 = \gamma < \beta$.

In this case we can choose $h(s) = 1$ and, since $\frac{g'(s)}{g^2(s)} = -1 - \beta(s + \frac{1}{n})^{\beta-1}$, we have

$$\sigma(x,s) = \frac{1 - 1 - \beta(s+\frac{1}{n})^{\beta-1}}{\mu(x) - 1} = \frac{\beta(s+\frac{1}{n})^{\beta-1}}{1-\mu(x)} \geq \frac{\beta}{n^{\beta-1}(1-\mu(x))}.$$

□

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