A sub-supersolution method for nonlinear elliptic singular systems with natural growth and some applications

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Abstract

In this paper we give a sub-supersolution method for nonlinear elliptic singular systems with quadratic gradient whose model system is the following

$$\begin{cases} -\Delta u + v^\beta \dfrac{|\nabla u|^2}{u^\alpha} = f_1(x,u,v) & \text{in } \Omega, \\[2mm] -\Delta v + u^\mu \dfrac{|\nabla v|^2}{v^\gamma} = f_2(x,u,v) & \text{in } \Omega, \\[1mm] u = v = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a smooth bounded domain of $\mathbb{R}^N$ ($N \geq 3$), $\beta, \mu \geq 0$, $0 < \alpha, \gamma < 1$ and regular f_1, f_2 functions. Moreover, we apply it to prove existence of solution for some systems, including the classical Lotka-Volterra models with gradient terms. Specifically, we study the competition and the symbiotic Lokta-Volterra systems.

Keywords: Sub-supersolution method, Natural growth, Singular gradient systems, Lotka-Volterra.
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1. Introduction

The aim of this paper is to provide a sub-supersolution method for the following nonlinear elliptic singular system with natural growth

$$
\begin{cases}
-\Delta u + g_1(v)\dfrac{|\nabla u|^2}{u^\alpha} = f_1(x, u, v) & \text{in } \Omega, \\[3mm]
-\Delta v + g_2(u)\dfrac{|\nabla v|^2}{v^\gamma} = f_2(x, u, v) & \text{in } \Omega, \\[1mm]
u = v = 0 & \text{on } \partial\Omega,
\end{cases}
\tag{1.1}
$$

where Ω is a smooth bounded domain of $\mathbb{R}^N$ ($N \geq 3$), $0 < \alpha, \gamma < 1$, the functions $g_1, g_2 \in C([0, +\infty))$ and $f_1, f_2 \in C(\overline{\Omega} \times [0, +\infty) \times [0, +\infty))$ verifying some general conditions detailed below.

Regarding the literature there are several papers about equations with quadratic gradient terms. The existence of solutions of the equation

$$
\begin{cases}
-\Delta u + g(u)|\nabla u|^2 = a(x) & \text{in } \Omega, \\
u = 0 & \text{on } \partial\Omega,
\end{cases}
\tag{1.2}
$$

for every function $a(x)$ in a given Lebesgue space has been systematically studied in [5, 8, 9] and references therein (in fact, for a more general nonlinear term $H(x, u, \nabla u)$ instead of $g(u)|\nabla u|^2$). They consider in the lower order term a continuous g in $\mathbb{R}$ which does not satisfy any growth restriction and the sign condition $g(s)s \geq 0$ for every $s \in \mathbb{R}$ is assumed. Thanks to the presence of the lower order term the Dirichlet problem associated to the equation is allowed to have finite energy weak solutions.

In [15] and [4] some of the above results were extended to the case of systems. Specifically, in [4] the authors study systems of elliptic equations with quadratic gradient. They consider a general system

$$
\begin{cases}
-\Delta u_i + H_i(x, u, \nabla u) = a_i(x) & \text{in } \Omega, \\
u = 0 & \text{on } \partial\Omega, \quad i = 1, \ldots, n
\end{cases}
$$

where $u = (u_1, \ldots, u_n)$, $a_i \in H^{-1}(\Omega)$ and the quadratic terms $H_i(x, u, \nabla u)$ satisfy a more general one-side condition than the sign condition, but in the case $H_i(x, u, \nabla u) = g_i(u)|\nabla u|^2$ this one-side hypothesis is equivalent to

the sign condition. In their case g_i is continuous in $\mathbb{R}^n$ and they prove the existence of solution in the Sobolev space.

In the last years, equation (1.2) has attracted much attention by the presence of singular terms in front of the gradient, see [1, 2, 6] and references therein.

In [11] we proved that a sub-supersolution method works for equations of the form

$$\begin{cases} -\Delta u + \dfrac{|\nabla u|^2}{u^\alpha} = f(\lambda, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

and we apply it to different models.

In this paper we focus our attention to systems with quadratic gradient and singular terms as (1.1).

Let us mention that the sub-supersolution method is valid for semilinear systems, see for instance [13] and [20]. In this case, when $g_1 \equiv g_2 \equiv 0$, the natural extension of the scalar definition of sub-supersolution depends on the monotonicity of the functions f_1 and f_2 with respect to v and u, respectively. A general definition was given in [13] and [20] where a pair of functions $(\underline{u}, \underline{v})$, $(\overline{u}, \overline{v})$, $\underline{u}, \overline{u}, \underline{v}, \overline{v} \in H^1(\Omega) \cap L^\infty(\Omega)$ is called a sub-supersolution if

$$\begin{array}{cc} \underline{u} \le \overline{u}, \qquad \underline{v} \le \overline{v} & \text{in } \Omega, \\ \underline{u} \le 0 \le \overline{u}, \quad \underline{v} \le 0 \le \overline{v} & \text{on } \partial\Omega, \end{array}$$

and

$$\begin{array}{cc} -\Delta\underline{u} \le f_1(x, \underline{u}, v), & -\Delta\overline{u} \ge f_1(x, \overline{u}, v), \quad \forall v \in [\underline{v}, \overline{v}], \\ -\Delta\underline{v} \le f_2(x, u, \underline{v}), & -\Delta\overline{v} \ge f_2(x, u, \overline{v}), \quad \forall u \in [\underline{u}, \overline{u}], \end{array}$$

where, given two ordered functions $z \le w$, we have denoted

$$[z, w] := \{q \in L^\infty(\Omega) : z(x) \le q(x) \le w(x)\}$$

(see also [19] where it is proved the validity of the method for singular semilinear systems). Assuming the existence of a sub-supersolution, $(\underline{u}, \underline{v})$, $(\overline{u}, \overline{v})$, there exists a solution $(u, v) \in I \equiv [\underline{u}, \overline{u}] \times [\underline{v}, \overline{v}]$ of the semilinear system (i.e. (1.1) with $g_1 \equiv g_2 \equiv 0$).

When the reaction terms depend on the gradient, i.e.,

$$-\Delta u = f_1(x, u, v, \nabla u, \nabla v), \quad -\Delta v = f_2(x, u, v, \nabla u, \nabla v)$$

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and the functions f_1 and f_2 are regular verifying some hypotheses, the definition is (see [21])

$$-\Delta \underline{u} \leq f_1(x, \underline{u}, v, \nabla \underline{u}, \nabla v), \quad -\Delta \overline{u} \geq f_1(x, \overline{u}, v, \nabla \overline{u}, \nabla v), \quad \forall v \in [\underline{v}, \overline{v}],$$
$$-\Delta \underline{v} \leq f_2(x, u, \underline{v}, \nabla u, \nabla \underline{v}), \quad -\Delta \overline{v} \geq f_2(x, u, \overline{v}, \nabla u, \nabla \overline{v}), \quad \forall u \in [\underline{u}, \overline{u}].$$

Assuming again the existence of a sub-supersolution, the existence of a solution $(u, v) \in I$ follows.

In this paper, we use the above definition of sub-supersolution, and taking advantage of the form of equation, we overcome the singularities difficulty of the system (1.1) with respect to u and v. Hence, for our system (1.1) we define a couple of sub-super solution as follows

$$-\Delta \underline{u} + g_1(v)\frac{|\nabla \underline{u}|^2}{\underline{u}^\alpha} \leq f_1(x, \underline{u}, v), \quad -\Delta \overline{u} + g_1(v)\frac{|\nabla \overline{u}|^2}{\overline{u}^\alpha} \geq f_1(x, \overline{u}, v), \quad \forall v \in [\underline{v}, \overline{v}],$$

$$-\Delta \underline{v} + g_2(u)\frac{|\nabla \underline{v}|^2}{\underline{v}^\gamma} \leq f_2(x, u, \underline{v}), \quad -\Delta \overline{v} + g_2(u)\frac{|\nabla \overline{v}|^2}{\overline{v}^\gamma} \geq f_2(x, u, \overline{v}), \quad \forall u \in [\underline{u}, \overline{u}].$$

Moreover, we apply this method to prove existence of positive solution for some systems, including the classical Lotka-Volterra models confronted with the Laplacian operator perturbed by a singular gradient term, that is, the following systems

$$
\begin{cases}
-\Delta u + g_1(v)\dfrac{|\nabla u|^2}{u^\alpha} = u(\lambda - u - bv) & \text{in } \Omega, \\[2mm]
-\Delta v + g_2(u)\dfrac{|\nabla v|^2}{v^\gamma} = v(\mu - v - cu) & \text{in } \Omega, \\[2mm]
u = v = 0 & \text{on } \partial\Omega,
\end{cases}
\tag{1.3}
$$

where $\lambda, \mu \in \mathbb{R}$ and $b \cdot c > 0$. Here, $u(x)$ and $v(x)$ denote the densities of two species, λ and μ represent the growth rates of the species, b and c measure the interaction rates between both species; if $b, c > 0$ they are competing and if $b, c < 0$ cooperating. Moreover, in (1.3) a nonlinear convective term is included with a singular term. This term is accompanied by a nonlinear function depending on the other species. We give conditions on λ and μ that assure the existence of a coexistence state of (1.3), that is, a solution with both components positive.

The structure of the article is: in Section 2 we study an auxiliary scalar equation that we use in Section 3 to prove the validity of the sub-supersolution method for (1.1). Section 4 is devoted to applications of the method.

2. An auxiliary scalar equation

Freezing one of the unknown in each equation of (1.1) we are led to consider the scalar boundary value problem

$$\begin{cases} -\Delta w + m(x)\dfrac{|\nabla w|^2}{w^\theta} = f(x,w) & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.4}$$

for convenient functions m and f and a parameter $0 < \theta < 1$. In order to use monotone methods for (2.4), as was pointed out in [11] for $m(x)$ constant, it is useful to consider a positive function $g \in C(0, +\infty)$ such that, denoting $G(u) = \int_1^u g(s)ds$, the function $e^{-G(s)}$ belongs to $L^1(0,1)$ and we define also Ψ by

$$\Psi(s) := \int_0^s e^{-G(t)}dt, \quad s > 0.$$

Observe that if there exists $M \geq 0$ such that $f(x,s) + Ms$ is nondecreasing for a.e. $x \in \Omega$ then $f(x,s) + M\Psi(s)e^{G(s)}$ is also nondecreasing for a.e. $x \in \Omega$. Thus, adding the term $M\Psi(w)e^{G(w)}$ in (2.4) it becomes

$$\begin{cases} -\Delta w + m(x)\dfrac{|\nabla w|^2}{w^\theta} + M\Psi(w)e^{G(w)} = f(x,w) + M\Psi(w)e^{G(w)} & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases}$$

Therefore, in order to study (2.4) using sub-supersolution, we need to establish a comparison principle for the problem

$$\begin{cases} -\Delta w + m(x)\dfrac{|\nabla w|^2}{w^\theta} + M\Psi(w)e^{G(w)} = f_0(x) & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.5}$$

where $0 < m(x) \in L^\infty(\Omega)$, $0 < \theta < 1$ and $0 \leq f_0(x) \in L^{2N/(N+2)}(\Omega)$, $f_0 \not\equiv 0$. Observe that this problem is similar to that studied in [11] but here the function g, from which are defined G and Ψ, is arbitrary and non necessarily related with the gradient lower order term. When $M = 0$, that is,

$$\begin{cases} -\Delta w + m(x)\dfrac{|\nabla w|^2}{w^\theta} = f_0(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

this problem has solution (see [6]) and it is unique (see [3]).

The concept of sub and super-solution for the problem (2.5) is the following:

Definition 2.1. *A sub-solution of (2.5) is a function $\underline{u} \in H_0^1(\Omega)$ such that $0 < \underline{u}$ a.e. in Ω, $\dfrac{|\nabla \underline{u}|^2}{\underline{u}^\theta}$, $\Psi(\underline{u})e^{G(\underline{u})} \in L^1(\Omega)$ and for every $\phi \in H_0^1(\Omega) \cap L^\infty(\Omega)$, $\phi \geq 0$,*

$$\int_\Omega \nabla \underline{u} \cdot \nabla \phi + \int_\Omega m(x)\frac{|\nabla \underline{u}|^2}{\underline{u}^\theta}\phi + M\int_\Omega \Psi(\underline{u})e^{G(\underline{u})}\phi \leq \int_\Omega f_0(x)\phi.$$

Similarly $\overline{u} \in H^1(\Omega)$ such that $0 < \overline{u}$ a.e. in Ω, $\dfrac{|\nabla \overline{u}|^2}{\overline{u}^\theta}$, $\Psi(\overline{u})e^{G(\overline{u})} \in L^1(\Omega)$ and for every $\phi \in H_0^1(\Omega) \cap L^\infty(\Omega)$, $\phi \geq 0$,

$$\int_\Omega \nabla \overline{u} \cdot \nabla \phi + \int_\Omega m(x)\frac{|\nabla \overline{u}|^2}{\overline{u}^\theta}\phi + M\int_\Omega \Psi(\overline{u})e^{G(\overline{u})}\phi \geq \int_\Omega f_0(x)\phi,$$

is called a super-solution of (2.5). We say that $u \in H_0^1(\Omega)$ is a solution of (2.5) if it is a sub and super-solution of (2.5).

We recall some classical results about the regularity of the equation (2.5) with $f_0(x) = f(x, w(x))$, that is the non-linear equation

$$\begin{cases} -\Delta w + m(x)\dfrac{|\nabla w|^2}{w^\theta} + M\Psi(w)e^{G(w)} = f(x, w) & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.6}$$

Concretely, the following two lemmas can be deduced from [22], summarizing some known $L^\infty(\Omega)$-estimates for sub-solutions of (2.6). The first one deals with a subcritical function f, here the $L^\infty(\Omega)$-estimate follows from a standard bootstrap argument.

Lemma 2.2. *Assume that there exists $C > 0$ such that $|f(x, s)| \leq C(1 + |s|^q)$ $(q < (N+2)/(N-2))$ for every $s \geq 0$, a.e. $x \in \Omega$, and that u is a sub-solution of (2.6), then $u \in L^\infty(\Omega)$.*

Remark 2.3. *Once we have proved that it is bounded, under the conditions of the previous lemma, we have that any solution u is continuous in Ω arguing as in [14] (see Remark 2.6 in [1] for a detailed proof). Moreover, since $\partial\Omega$ is smooth, $u \in C^{0,\alpha}(\overline{\Omega})$ for some $\alpha \in (0, 1)$.*

Lemma 2.4. *Assume that there exists s_0 such that $f(x, s) \leq 0$ a.e. $x \in \Omega$ and for every $s > s_0$. Assume also that u is a sub-solution of (2.6), then $u \in L^\infty(\Omega)$ and $\|u\|_\infty \leq s_0$.*

We can prove that problem (2.5) has a solution arguing as in Lemma 3.3 of [11]. We include here a sketch of the proof in order to show how to deal with the term $m(x) \in L^{\infty}(\Omega)$.

Lemma 2.5. *Assume that $g \in L^1(0,1)$. Then there exists a solution for (2.5).*

Proof. We use an approximative scheme, namely

$$\begin{cases} -\Delta u_n + B_n(x, u_n, \nabla u_n) = \min\{f_0(x), n\} & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.7}$$

where the function $B_n(x, s, p)$ is given, for every $(x, s, p) \in \Omega \times \mathbb{R} \times \mathbb{R}^N$ and $n \in \mathbb{N}$, by

$$B_n(x, s, p) = \frac{m(x)s^+|p|^2}{(\frac{1}{n} + s^+)^{\theta+1}(1 + \frac{1}{n}|p|^2)} + \frac{Me^{\int_1^{s^+} g(t+\frac{1}{n})dt} \int_0^{s^+} e^{-\int_1^t g(\sigma+\frac{1}{n})d\sigma} dt}{1 + \frac{1}{n}e^{\int_1^{s^+} g(t+\frac{1}{n})dt} \int_0^{s^+} e^{-\int_1^t g(\sigma+\frac{1}{n})d\sigma} dt}.$$

Since $B_n(x, s, p)s \geq 0$ and $B_n(x, s, p) \leq \|m\|_{L^{\infty}(\Omega)} n(n^{\theta} + M)$, the existence of solution $u_n \in H_0^1(\Omega)$ of (2.7) is deduced from [16]. Moreover, since $B_n(x, s, p) \geq 0$ then $-\Delta u_n \leq f_0(x)$ and, using [22], the sequence u_n is bounded in $L^{\infty}(\Omega)$, that is, there exists $R > 0$ such that

$$\|u_n\|_{L^{\infty}(\Omega)} \leq R.$$

Moreover, taking u_n^- as test function we obtain that $u_n \geq 0$. Similarly, taking u_n as test function and using the positivity of the lower order term we get that u_n is bounded in $H_0^1(\Omega)$. Even more, taking $\min\{u_n, \varepsilon\}/\varepsilon$ as test function and using Fatou Lemma, see [6], as $\varepsilon \to 0$ yields that

$$\int_{\Omega} B_n(x, u_n, \nabla u_n) \leq \|f_0\|_1.$$

Therefore u_n weakly converges to $u \in H_0^1(\Omega)$, $\nabla u_n \to \nabla u$ a.e. (see [7, Theorem 2.1]) and using Fatou Lemma as $n \to \infty$,

$$m(x)\frac{|\nabla u|^2}{u^{\theta}}\chi_{\{u>0\}} \in L^1(\Omega) \quad \text{and} \quad \Psi(u)e^{G(u)}\chi_{\{u>0\}} \in L^1(\Omega).$$

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In particular, since g is integrable at zero, we have that $\Psi(u)e^{G(u)}$ is bounded at zero and thus, $\Psi(u)e^{G(u)} \in L^1(\Omega)$.

In order to pass to the limit and to prove that u is the solution of (2.5) it is essential to prove that $u > 0$. In order to do that we follow the ideas in [6]. Given $\tilde{m} \geq \|m\|_{L^\infty(\Omega)}$ we take $e^{-\tilde{m}\int_1^{u_n} \frac{1}{t^\theta}dt}\phi$, with $0 \leq \phi \in C_0^\infty(\Omega)$, as test function in (2.7) and we obtain

$$\int_\Omega e^{-\tilde{m}\int_1^{u_n} \frac{1}{t^\theta}dt}\nabla u_n \cdot \nabla\phi + \int_\Omega \left(B_n(x, u_n, \nabla u_n) - \frac{\tilde{m}}{u_n^\theta}\right)e^{-\tilde{m}\int_1^{u_n} \frac{1}{t^\theta}dt}\phi =$$
$$= \int_\Omega \min\{f_0(x), n\}e^{-\tilde{m}\int_1^{u_n} \frac{1}{t^\theta}dt}\phi \geq \int_\Omega \min\{f_0(x), 1\}e^{-\tilde{m}\int_1^{u_n} \frac{1}{t^\theta}dt}\phi.$$

We can use now that for $0 < s < R$

$$\left(B_n(x, s, p) - \frac{\tilde{m}}{s^\theta}\right)e^{-\int_1^s \frac{1}{t^\theta}dt} \leq \frac{m(x) - \tilde{m}}{s^\theta}|p|^2 e^{-\int_1^s \frac{1}{t^\theta}dt} +$$
$$+Me^{\int_1^s \left(g(t+\frac{1}{n})-\frac{\tilde{m}}{t^\theta}\right)dt}\int_0^s e^{-\int_1^t g(\sigma+\frac{1}{n})d\sigma}dt$$
$$\leq Me^{\int_1^s \left(g(t+\frac{1}{n})-\frac{\tilde{m}}{t^\theta}\right)dt}\int_0^s e^{-\int_1^t g(\sigma+\frac{1}{n})d\sigma}dt$$
$$\leq C\int_0^s e^{-\tilde{m}\int_1^t \frac{1}{\sigma^\theta}d\sigma}dt.$$

(2.8)

The last inequality is due to the fact that

$$e^{\int_1^s \left(g(t+\frac{1}{n})-\frac{\tilde{m}}{t^\theta}\right)dt} = e^{\int_{1+\frac{1}{n}}^{s+\frac{1}{n}} g(\sigma)d\sigma - \int_1^s \frac{\tilde{m}}{t^\theta}dt} \leq e^{\int_1^{R+1} g(\sigma)d\sigma + \int_0^1 \frac{\tilde{m}}{t^\theta}dt}$$

and

$$\int_0^s e^{-\int_1^t g(\sigma+\frac{1}{n})d\sigma}dt = \int_0^s e^{\int_1^t \left(\frac{\tilde{m}}{\sigma^\theta}-g(\sigma+\frac{1}{n})\right)d\sigma}e^{-\tilde{m}\int_1^t \frac{1}{\sigma^\theta}d\sigma}dt$$
$$\leq e^{\int_1^{R+1} \frac{\tilde{m}}{\sigma^\theta}d\sigma + \int_0^1 g(\sigma+\frac{1}{n})d\sigma}\int_0^s e^{-\tilde{m}\int_1^t \frac{1}{\sigma^\theta}d\sigma}dt$$
$$\leq e^{\int_1^{R+1} \frac{\tilde{m}}{\sigma^\theta}d\sigma + \int_0^2 g(\sigma)d\sigma}\int_0^s e^{-\tilde{m}\int_1^t \frac{1}{\sigma^\theta}d\sigma}dt.$$

Thus, we can take $C = Me^{\int_1^{R+1} g(\sigma)d\sigma + \int_0^1 \frac{\tilde{m}}{t^\theta}dt}e^{\int_1^{R+1} \frac{\tilde{m}}{\sigma^\theta}d\sigma + \int_0^2 g(\sigma)d\sigma}$, using (2.8) in (2.8) and denoting $\tilde{\Psi}(s) = \int_0^s e^{-\tilde{m}\int_1^t \frac{1}{\sigma^\theta}d\sigma}dt$ we get

$$\int_\Omega \nabla\tilde{\Psi}(u_n) \cdot \nabla\phi + \tilde{C}\int_\Omega \tilde{\Psi}(u_n)\phi \geq \int_\Omega \min\{f_0(x), 1\}e^{-\tilde{m}\int_1^{u_n} \frac{1}{t^\theta}dt}\phi.$$

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From now on the proof continues exactly as in [6]. Passing to the limit in the previous inequality it follows that

$$\int_\Omega \nabla\tilde\Psi(u) \cdot \nabla\phi + \tilde C \int_\Omega \tilde\Psi(u)\phi \geq \int_\Omega \min\{f_0(x), 1\} e^{-\tilde m \int_1^u \frac{1}{t^\theta} dt} \phi.$$

Thus, the strong maximum principle allows us to assure that $0 < \tilde\Psi(u) \leq e^{\int_0^1 \frac{\tilde m}{\sigma^\theta} d\sigma} u$, in particular $u > 0$, and we can to pass to the limit in the approximated problem to deduce that $u \in H_0^1(\Omega)$ is a solution of (2.5) arguing as in [6]. □

With respect to uniqueness we prove below a comparison principle for this equation that assures that this solution is unique. This comparison principle is one of the keystones of the proof of our method. In the case $M = 0$ it correspond to the comparison principle in [3, Corollary 3.5]. We include here, for the convenience of the reader, the proof of that result with the new term $M\Psi(u)e^{G(u)}$ at the left-hand side of the equation, that is to say, we prove a comparison principle for the problem (2.5) although the proof follows with no significant change that of Theorem 1.1 in [3].

Proposition 2.6. *Assume that $0 < \theta < 1$ and $0 < m(x) \in L^\infty(\Omega)$. Let $\underline u, \overline u \in C(\overline\Omega)$ be, respectively, a sub and a super-solution of (2.5). Suppose also that $g \in C^1(0, +\infty)$, $e^{-G(t)} \in L^1(0, 1)$, $g \geq 0$ and there exists $\tau \geq 0$ such that for a.e. $x \in \Omega$ and for every $0 < s < \max\{\|\underline u\|_{L^\infty(\Omega)}, \|\overline u\|_{L^\infty(\Omega)}\}$ we have*

$$\tau\left[\left(-g'(s) - \frac{m(x)\theta}{s^{\theta+1}}\right) + \left(-g(s) + \frac{m(x)}{s^\theta}\right)g(s)\right] \geq \left(-g(s) + \frac{m(x)}{s^\theta}\right)^2.$$
(2.9)

Then $\underline u \leq \overline u$.

Proof. We will use the usual function $G_\varepsilon(s) = (s - \varepsilon)^+$ for every $s \in \mathbb{R}$. We also define $w = \Psi(\underline u) - \Psi(\overline u)$ and observe that $G_\varepsilon(w)$ is bounded and has compact support in Ω. In particular, $e^{-G(\underline u)}, e^{-G(\overline u)}, g(\underline u), g(\overline u)$ are bounded on the support of $G_\varepsilon(w)$. Thus, for n equal to the integer part of $\tau + 1$, we can take $e^{-G(\underline u)} G_\varepsilon(w)^n$ as test function in the inequality satisfied by $\underline u$ and

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$e^{-G(\overline{u})}G_\varepsilon(w)^n$ in the inequality satisfied by $\overline{u}$. Subtracting we have

$$0 \geq \int_\Omega \left(-g(\underline{u}) + \frac{m(x)}{\underline{u}^\theta}\right) e^{-G(\underline{u})}|\nabla\underline{u}|^2 G_\varepsilon(w)^n -$$
$$- \int_\Omega \left(-g(\overline{u}) + \frac{m(x)}{\overline{u}^\theta}\right) e^{-G(\overline{u})}|\nabla\overline{u}|^2 G_\varepsilon(w)^n +$$
$$+ n\int_\Omega G_\varepsilon(w)^{n-1}(e^{-G(\underline{u})}\nabla\underline{u} - e^{-G(\overline{u})}\nabla\overline{u}) \cdot \nabla w,$$

where we have used that

$$\int_\Omega (\Psi(\underline{u}) - \Psi(\overline{u}))G_\varepsilon(w)^n \geq 0.$$

We denote $s = \Psi^{-1}(t\Psi(\underline{u}) + (1-t)\Psi(\overline{u}))$ and $\xi = t\nabla\Psi(\underline{u}) + (1-t)\nabla\Psi(\overline{u})$, this means that

$$0 \geq \int_{\{w>\varepsilon\}} G_\varepsilon(w)^n \int_0^1 \frac{d}{dt}\left(\left(-g(s) + \frac{m(x)}{s^\theta}\right)e^{G(s)}|\xi|^2\right) dt +$$
$$+ n\int_{\{w>\varepsilon\}} G_\varepsilon(w)^{n-1}|\nabla w|^2.$$

Now we perform the derivative and we get

$$0 \geq \int_{\{w>\varepsilon\}} wG_\varepsilon(w)^n \int_0^1 \left(\left(-g'(s) - \frac{m(x)\theta}{s^{\theta+1}}\right)e^{2G(s)}|\xi|^2\right) dt +$$
$$+ \int_{\{w>\varepsilon\}} wG_\varepsilon(w)^n \int_0^1 \left(-g(s) + \frac{m(x)}{s^\theta}\right)g(s)e^{2G(s)}|\xi|^2 \, dt +$$
$$+ \int_{\{w>\varepsilon\}} G_\varepsilon(w)^n \int_0^1 \left(-g(s) + \frac{m(x)}{s^\theta}\right)e^{G(s)}2\xi \cdot \nabla w \, dt +$$
$$+ n\int_{\{w>\varepsilon\}} G_\varepsilon(w)^{n-1}|\nabla w|^2.$$

Multiplying by $\frac{\tau}{n}$ and taking into account that, by Young's inequality,

$$\frac{\tau}{n}\left|G_\varepsilon(w)^n\left(-g(s) + \frac{m(x)}{s^\theta}\right)e^{G(s)}2\xi \cdot \nabla w\right| \leq$$
$$\leq \frac{\tau^2}{n}G_\varepsilon(w)^{n-1}|\nabla w|^2 + \frac{G_\varepsilon(w)^{n+1}}{n}\left(-g(s) + \frac{m(x)}{s^\theta}\right)^2 e^{2G(s)}|\xi|^2,$$

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it follows that

$$0 \geq \tau\left(1 - \frac{\tau}{n}\right)\int_{\{w>\varepsilon\}} G_\varepsilon(w)^{n-1}|\nabla w|^2 +$$

$$+ \int_{\{w>\varepsilon\}} \int_0^1 \frac{wG_\varepsilon(w)^n \tau e^{2G(s)}}{n}\left[\left(-g'(s) - \frac{m(x)\theta}{s^{\theta+1}}\right)|\xi|^2 +\right.$$

$$\left. + \left(-g(s) + \frac{m(x)}{s^\theta}\right)g(s)|\xi|^2 - \frac{G_\varepsilon(w)}{\tau w}\left(-g(s) + \frac{m(x)}{s^\theta}\right)^2 |\xi|^2\right]dt \geq 0.$$

The last inequality due to the fact that $G_\varepsilon(w)/w \leq 1$, $M \geq 0$ and (2.9). We deduce that the integrands are zero, which implies that $G_\varepsilon(w) = 0$ for every $\varepsilon > 0$, i.e., $w^+ \equiv 0$, concluding the proof. $\qquad\square$

The following technical result plays an essential role in the further work, and it was proved in [3, Corollary 3.5] (see condition (3.6) of that paper).

Lemma 2.7. *Fix $m \in L^\infty(\Omega)$, $m > 0$ in Ω and $\nu > 0$. Then, there exist $g \in C^1(0,\infty)\cap L^1(0,1)$, in particular $e^{-G(t)} \in L^1(0,1)$, and τ such that if $\underline{u}$ and $\overline{u}$ are a sub and a supersolution of (2.5) such that $\max\{\|\underline{u}\|_{L^\infty(\Omega)}, \|\overline{u}\|_{L^\infty(\Omega)}\} \leq \nu$, g satisfies condition (2.9), and as consequence*

$$\underline{u} \leq \overline{u}.$$

In fact, g depends on $\|m\|_{L^\infty(\Omega)}$ and θ, but neither ν nor τ. Specifically, fixed d, C, m_1 with $0 < \theta < d < 1$, $C > 0$ and

$$m_1 \leq \min\left\{dC, C\left(\frac{d-\theta}{1-\theta}\right)^{1-\theta}\right\},$$

for any $m \in L^\infty(\Omega)$ with $\|m\|_\infty \leq m_1$ we can choose $g(s) \equiv g_{\theta,d,C}(s)$ given by

$$g_{\theta,d,C}(s) = \begin{cases} \dfrac{dC}{s^\theta}, & s < \left(\frac{\theta}{C}\right)^{\frac{1}{1-\theta}}, \\[2mm] \dfrac{d\theta}{\theta s + \left(\frac{\theta}{C}\right)^{\frac{1}{1-\theta}}(1-\theta)}, & s \geq \left(\frac{\theta}{C}\right)^{\frac{1}{1-\theta}}, \end{cases}$$

for every $s > 0$. Moreover, τ is such that

$$\tau > \max\left\{\frac{dC + m_1}{C(1-d)}, \frac{2d\left(m_1^2\nu^{2(1-\theta)} + \theta^2\right)}{(1-d)\theta^2}, \frac{2m_1^2\left(\frac{1-\theta}{d-\theta}\right)^{2(1-\theta)} + 2d^2C^2}{d(1-d)C^2\left(1 - \frac{m_1}{C}\left(\frac{1-\theta}{d-\theta}\right)^{1-\theta}\right)}\right\}.$$

3. The sub-supersolution method

Now, we are ready to state the method of sub and super-solutions in order to get existence of solutions of (1.1). In view of the results of the previous section the concept of sub and super-solution for (1.1) is the following.

Definition 3.1. *A pair* $(\underline{u}, \overline{u}), (\underline{v}, \overline{v})$ *is a sub-supersolution of (1.1) if* $\underline{u}, \overline{u}, \underline{v}, \overline{v} \in H^1(\Omega) \cap C(\overline{\Omega})$, $\underline{u}, \underline{v} \in H_0^1(\Omega)$ *such that*

1. $0 < \underline{u} \leq \overline{u}$, $0 < \underline{v} \leq \overline{v}$ *almost everywhere in* Ω,
2. $\dfrac{|\nabla \underline{u}|^2}{\underline{u}^\alpha}, \dfrac{|\nabla \overline{u}|^2}{\overline{u}^\alpha}, \dfrac{|\nabla \underline{v}|^2}{\underline{v}^\gamma}, \dfrac{|\nabla \overline{v}|^2}{\overline{v}^\gamma} \in L^1(\Omega)$,
3. *for every* $\phi \in H_0^1(\Omega) \cap L^\infty(\Omega)$, $\phi > 0$,

$$\int_\Omega \nabla \underline{u} \cdot \nabla \phi + \int_\Omega g_1(v) \frac{|\nabla \underline{u}|^2}{\underline{u}^\alpha} \phi - \int_\Omega f_1(x, \underline{u}, v)\phi \leq 0 \leq$$
$$\leq \int_\Omega \nabla \overline{u} \cdot \nabla \phi + \int_\Omega g_1(v) \frac{|\nabla \overline{u}|^2}{\overline{u}^\alpha} \phi - \int_\Omega f_1(x, \overline{u}, v)\phi \quad \forall v \in [\underline{v}, \overline{v}],$$

$$(3.10)$$

4. *for every* $\phi \in H_0^1(\Omega) \cap L^\infty(\Omega)$, $\phi > 0$,

$$\int_\Omega \nabla \underline{v} \cdot \nabla \phi + \int_\Omega g_2(u) \frac{|\nabla \underline{v}|^2}{\underline{v}^\gamma} \phi - \int_\Omega f_2(x, u, \underline{v})\phi \leq 0 \leq$$
$$\leq \int_\Omega \nabla \overline{v} \cdot \nabla \phi + \int_\Omega g_2(u) \frac{|\nabla \overline{v}|^2}{\overline{v}^\gamma} \phi - \int_\Omega f_2(x, u, \overline{v})\phi \quad \forall u \in [\underline{u}, \overline{u}].$$

$$(3.11)$$

Remark 3.2. *Observe that any of the four inequalities in (3.10) and (3.11) is verified if it is satisfied in the classical sense, for instance, the first inequality in (3.10) is satisfied if* $\underline{u}$ *is twice differentiable and*

$$-\Delta \underline{u} + g_1(v)\frac{|\nabla \underline{u}|^2}{\underline{u}^\alpha} - f_1(x, \underline{u}, v) \leq 0 \ \text{a.e.} \ x \in \Omega, \forall v \in [\underline{v}, \overline{v}].$$

Theorem 3.3. *Assume that* $(\underline{u}, \overline{u})$, $(\underline{v}, \overline{v})$ *is pair of sub-supersolution of (1.1) and denote* $I := [\underline{u}, \overline{u}] \times [\underline{v}, \overline{v}] \subset C(\overline{\Omega}) \times C(\overline{\Omega})$. *Assume also the following conditions on* f_1, f_2, g_1 *and* g_2:

(F) *There exists a constant* $M \geq 0$ *such that the maps* $s \mapsto f_1(x, s, v) + Ms$ *and* $r \mapsto f_2(x, u, r) + Mr$ *are positive and increasing for* $(s, r) \in [0, \sup_{\overline{\Omega}} \overline{u}] \times [0, \sup_{\overline{\Omega}} \overline{v}]$ *and for all* $(u, v) \in I$.

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(G) g_1, g_2 *are non-negative functions and* $g_i(s) = 0$ *if and only if* $s = 0$ *for* $i = 1, 2$.

Then, there exists a solution (u, v) *of (1.1) such that* $(u, v) \in I$.

Proof. Taking into account (G), there exits a positive number $0 < m_1$ such that

$$0 < g_1(z(x)), g_2(w(x)) \leq m_1, \ \forall (w, z) \in I.$$

Then, taking $\nu = \max\{\|\overline{v}\|_{L^\infty(\Omega)}, \|\overline{u}\|_{L^\infty(\Omega)}\}$, by Lemma 2.7 there exist $h_1, h_2 \in C^1(0, +\infty) \cap L^1(0, 1)$ and $\tau_1, \tau_2 \geq 0$ such that for a.e. $x \in \Omega$, for every $0 < s < \nu$ and for every $(w, z) \in I$ we have

$$\tau_1 \left[\left(-h_1'(s) - \frac{g_1(z(x))\alpha}{s^{\alpha+1}} \right) + \left(-h_1(s) + \frac{g_1(z(x))}{s^\alpha} \right) h_1(s) \right] \geq$$
$$\geq \left(-h_1(s) + \frac{g_1(z(x))}{s^\alpha} \right)^2$$

and

$$\tau_2 \left[\left(-h_2'(s) - \frac{g_2(w(x))\gamma}{s^{\gamma+1}} \right) + \left(-h_2(s) + \frac{g_2(w(x))}{s^\gamma} \right) h_2(s) \right] \geq$$
$$\geq \left(-h_2(s) + \frac{g_2(w(x))}{s^\gamma} \right)^2.$$

We would like to remark again that neither h_i nor τ_i depend on (w, z), see Lemma 2.7.

Now we define

$$G_i(s) := \int_0^s h_i(t)dt, \quad \Psi_i(s) := \int_0^s e^{-G_i(t)}dt, s > 0, \ i = 1, 2.$$

On the other hand, observe that, using *(F)*, we get that the maps

$$s \mapsto f_1(x, s, v) + M\Psi_1(s)e^{G_1(s)}, r \mapsto f_2(x, u, r) + M\Psi_2(r)e^{G_2(r)}$$

are increasing for $(s, r) \in [0, \sup_{\overline{\Omega}} \overline{u}] \times [0, \sup_{\overline{\Omega}} \overline{v}]$ for all $(u, v) \in I$.

We define the operator

$$T : I \mapsto C(\overline{\Omega}) \times C(\overline{\Omega}), \ (w, z) \mapsto (u, v) := T(w, z)$$

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where $(u, v) \in H_0^1(\Omega) \times H_0^1(\Omega)$ are solutions of

$$
\begin{cases}
-\Delta u + g_1(z(x))\dfrac{|\nabla u|^2}{u^\alpha} + M\Psi_1(u)e^{G_1(u)} = F_1(x), \\
-\Delta v + g_2(w(x))\dfrac{|\nabla v|^2}{v^\gamma} + M\Psi_2(v)e^{G_2(v)} = F_2(x),
\end{cases}
\tag{3.12}
$$

with

$$
F_1(x) := f_1(x, w, z) + M\Psi_1(w)e^{G_1(w)}
$$

and

$$
F_2(x) := f_2(x, w, z) + M\Psi_2(z)e^{G_2(z)}.
$$

Taking into account *(F)*, we have that $F_1, F_2 > 0$. Moreover, $F_1, F_2 \in L^{\frac{2N}{N+2}}(\Omega)$, in fact, $F_1, F_2 \in L^\infty(\Omega)$. Hence, the existence of u and v can be obtained from Lemma 2.5 while the uniqueness from Proposition 2.6. Therefore T is well defined.

Now, we prove that $T(I) \subset I$. Indeed, take $(w, z) \in I$ and consider $(u, v) = T(w, z)$, i.e. the unique solution to the problem (3.12). We are going to show that $(u, v) \in I$.

Indeed, using the definition of sub-super solution, it follows that

$$
-\Delta u + g_1(z)\frac{|\nabla u|^2}{u^\alpha} + M\Psi_1(u)e^{G_1(u)} = f_1(x, w, z) + M\Psi_1(w)e^{G_1(w)} \geq
$$

$$
\geq f_1(x, \underline{u}, z) + M\Psi_1(\underline{u})e^{G_1(\underline{u})} \geq -\Delta\underline{u} + g_1(z)\frac{|\nabla \underline{u}|^2}{\underline{u}^\alpha} + M\Psi_1(\underline{u})e^{G_1(\underline{u})}.
$$

Proposition 2.6 allows to assure that $u \geq \underline{u}$. Analogously $u \leq \overline{u}$ and $\underline{v} \leq v \leq \overline{v}$.

Next we show that T maps bounded sets into relatively compact sets. Indeed, given a bounded sequence (w_n, z_n) in I and denoting by $(u_n, v_n) = T(w_n, z_n)$ we have that u_n and v_n are bounded in $H_0^1(\Omega)$. Even more, regularity arguments (see [22] and [14]) show that sequences u_n, v_n are also bounded in $C^{0,\alpha}(\overline{\Omega})$ and the compact embedding of $C^{0,\alpha}(\overline{\Omega})$ in $C(\overline{\Omega})$ implies that $u_n \to u$ and $v_n \to v$ strongly in $C(\overline{\Omega})$.

Finally, in order to have that T is compact, we prove that T is continuous. In order to do that, we claim that if $(w_n, z_n) \to (w, z)$ strongly in I then $(u, v) = T(w, z)$ (observe that this in particular implies that not only a subsequence but the whole sequence (u_n, v_n) strongly converges to $T(w, z)$ in $C(\overline{\Omega}) \times C(\overline{\Omega})$).

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In order to prove the claim we observe that

$$\int_\Omega \nabla u_n \cdot \nabla \phi + \int_\Omega g_1(z_n) \frac{|\nabla u_n|^2}{u_n^\alpha} \phi + M \int_\Omega \Psi_1(u_n) e^{G_1(u_n)} \phi =$$
$$= \int_\Omega f_1(x, w_n, z_n) \phi + M \int_\Omega \Psi_1(w_n) e^{G_1(w_n)} \phi$$

for every $\phi \in H_0^1(\Omega) \cap L^\infty(\Omega)$. We recall that, as in the proof of Lemma 2.5, the lower order term is bounded in $L^1(\Omega)$ and we may assume that $u_n \to u$ weakly in $H_0^1(\Omega)$, strongly in $L^p(\Omega)$ ($p < 2^*$), $u_n(x) \to u(x)$ and $\nabla u_n(x) \to \nabla u(x)$ a.e. $x \in \Omega$. Even more, since $u_n \geq \underline{u}$ we have that $u \geq \underline{u} > 0$. In order to pass to the limit, arguing as in [6], we first consider $\phi \geq 0$ and using Lebesgue theorem as well as the weak convergence

$$\int_\Omega g_1(z_n) \frac{|\nabla u_n|^2}{u_n^\alpha} \phi \to \int_\Omega f_1(x, w, z) \phi + M \int_\Omega \Psi_1(w) e^{G_1(w)} \phi -$$
$$- \int_\Omega \nabla u \cdot \nabla \phi - M \int_\Omega \Psi_1(u) e^{G_1(u)} \phi.$$

Thus, using Fatou lemma

$$\int_\Omega g_1(z) \frac{|\nabla u|^2}{u^\alpha} \phi \leq \int_\Omega f_1(x, w, z) \phi + M \int_\Omega \Psi_1(w) e^{G_1(w)} \phi -$$
$$- \int_\Omega \nabla u \cdot \nabla \phi - M \int_\Omega \Psi_1(u) e^{G_1(u)} \phi.$$

Now we prove the reverse inequality. We take $\tilde{m} > 0$ such that $\tilde{m} \geq g_1(z)$ for every $z \in [\underline{v}, \overline{v}]$. We consider ϕ with compact support, therefore we can take $e^{\int_{u_n}^u \frac{\tilde{m}}{s^\alpha} ds} \phi$ as test function in the equation satisfied by u_n and we get

$$\int_\Omega e^{\int_{u_n}^u \frac{\tilde{m}}{s^\alpha} ds} \nabla u_n \cdot \nabla \phi + \tilde{m} \int_\Omega e^{\int_{u_n}^u \frac{\tilde{m}}{s^\alpha} ds} \phi \frac{\nabla u}{u^\alpha} \cdot \nabla u_n +$$
$$+ \int_\Omega (M \Psi_1(u_n) e^{G_1(u_n)} - f_1(x, w_n, z_n) - M \Psi_1(w_n) e^{G_1(w_n)}) e^{\int_{u_n}^u \frac{\tilde{m}}{s^\alpha} ds} \phi =$$
$$= \int_\Omega \frac{\tilde{m} - g_1(z_n)}{u_n^\alpha} |\nabla u_n|^2 e^{\int_{u_n}^u \frac{\tilde{m}}{s^\alpha} ds} \phi.$$

We can pass to the limit in the left hand side of the previous inequality (using Lebesgue theorem and the weak convergence) and then Fatou Lemma in the

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right hand side assures that

$$\int_\Omega \nabla u \cdot \nabla \phi + \tilde{m} \int_\Omega \phi \frac{|\nabla u|^2}{u^\alpha} +$$
$$+ \int_\Omega (M\Psi_1(u)e^{G_1(u)} - f_1(x,w,z) - M\Psi_1(w)e^{G_1(w)})\phi \geq$$
$$\geq \int_\Omega \frac{\tilde{m} - g_1(z)}{u^\alpha}|\nabla u|^2\phi$$

or equivalently

$$\int_\Omega \nabla u \cdot \nabla \phi + \int_\Omega g_1(z)\frac{|\nabla u|^2}{u^\alpha}\phi + M \int \Psi_1(u)e^{G_1(u)}\phi \geq$$
$$\geq \int_\Omega (f_1(x,w,z) + M\Psi_1(w)e^{G_1(w)})\phi.$$

From both inequalities it yields that

$$\int_\Omega \nabla u \cdot \nabla \phi + \int_\Omega g_1(z)\frac{|\nabla u|^2}{u^\alpha}\phi + M \int \Psi_1(u)e^{G_1(u)}\phi =$$
$$= \int_\Omega (f_1(x,w,z) + M\Psi_1(w)e^{G_1(w)})\phi,$$

for every $\phi \in H_0^1(\Omega) \cap C_c(\Omega)$ and by density for every $\phi \in H_0^1(\Omega) \cap L^\infty(\Omega)$. Analogously $v_n \to v$ with $(u,v) = T(w,z)$.

Therefore, applying the Schauder Fixed Point Theorem we conclude the proof. $\square$

4. Applications

In this section we apply the sub-supersolution method to different systems. For that, we need some previous results.

Given $q \in L^\infty(\Omega)$, we denote by $\lambda_1(q)$ the principal eigenvalue of

$$\begin{cases} -\Delta u + q(x)u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Recall that the map $q \mapsto \lambda_1(q)$ is increasing. For simplicity, we denote by $\lambda_1 := \lambda_1(0)$.

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First consider the classical logistic equation

$$\begin{cases} -\Delta u = u(\lambda + m(x) - u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.13}$$

It is well known that there exists a unique positive solution of (4.13) if and only if $\lambda > \lambda_1(-m)$. If we denote the unique positive solution by $\theta_{\lambda+m}$ we have that

$$\theta_{\lambda+m} \le \lambda + m_M, \tag{4.14}$$

where we have denoted $m_M = \sup_\Omega m$.

Now consider the logistic equation with a singular term and natural growth

$$\begin{cases} -\Delta u + k\dfrac{|\nabla u|^2}{u^\alpha} = u(\lambda + m(x) - u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.15}$$

where $k > 0$, $0 < \alpha < 1$, $\lambda \in \mathbb{R}$ and $m \in L^\infty(\Omega)$.

Equation (4.15) was studied in [11] when $m \equiv 0$. We prove now the following result.

Theorem 4.1. *Any weak solution u of (4.15) is bounded, in fact, $u \in C^{0,\alpha}(\overline{\Omega})$ and*

$$u \le \theta_{\lambda+m} \quad \text{in } \Omega.$$

Moreover, there exists a positive solution of (4.15) if and only if

$$\lambda > \lambda_1(-m).$$

Furthermore, in this case there exists a maximal positive solution of (4.15), denoted by $\Theta_{[\lambda+m,\alpha,k]}$.

Finally, $\lambda + m \mapsto \Theta_{[\lambda+m,\alpha,k]}$ is increasing and $k \to \Theta_{[\lambda+m,\alpha,k]}$ is decreasing.

Proof. It is clear that if u is a solution of (4.15), then u is bounded. So, it is a bounded subsolution of (4.13) and hence

$$u \le \theta_{\lambda+m}.$$

Assume that $\lambda \le \lambda_1(-m)$ and that there exists a positive solution of (4.15). Then multiplying (4.15) by φ_1, a positive eigenfunction associated to $\lambda_1(-m)$, we get a contradiction.

Suppose from now on that $\lambda > \lambda_1(-m)$. We apply the sub-supersolution method from [11] to prove the existence of a positive solution. Let us remark that although Theorem 3.5 (the sub-supersolution method) in [11] was proved for functions $f(\lambda, u)$, it is easily extended for functions $f(x, u)$.

Take
$$(\underline{u}, \overline{u}) = (\varepsilon \varphi_1^a, \theta_{\lambda+m})$$
where $\varepsilon > 0$, $a > 0$ and φ_1 is a positive eigenfunction associated to $\lambda_1(-m)$, that is, it satisfies
$$\begin{cases} -\Delta\varphi_1 - m(x)\varphi_1 = \lambda_1(-m)\varphi_1 & \text{in } \Omega, \\ \varphi_1 = 0 & \text{on } \partial\Omega. \end{cases}$$

First, we recall that $|\nabla\overline{u}|^2/\overline{u}^\alpha \in L^1(\Omega)$, see for instance Lemma 2.5 in [11].

On the other hand, observe that
$$\nabla(\varphi_1^a) = a\varphi_1^{a-1}\nabla\varphi_1,$$
$$\Delta(\varphi_1^a) = a((a-1)\varphi_1^{a-2}|\nabla\varphi_1|^2 + \varphi_1^a(-m(x) - \lambda_1(-m))).$$

Hence, $|\nabla\underline{u}|^2/\underline{u}^\alpha \in L^1(\Omega)$ if $a > 1/(2-\alpha)$.

Since $\lambda > \lambda_1(-m)$, we can choose $a > 1$ and $0 < \eta < \lambda - \lambda_1(-m)$ such that
$$\lambda > \lambda_1(-m)a + (a-1)m(x) + \eta.$$

Then, $\underline{u}$ is subsolution of (4.15) if
$$a\varphi_1^{-2}|\nabla\varphi_1|^2\left((1-a) + ka\varepsilon^{1-\alpha}\varphi_1^{a(1-\alpha)}\right) + \varepsilon\varphi_1^a \le \lambda - \lambda_1(-m)a - (a-1)m(x).$$

Observe that the above inequality is true if
$$ka\varepsilon^{1-\alpha}\varphi_1^{a(1-\alpha)} \le a - 1 \text{ and } \varepsilon\varphi_1^a \le \eta,$$

i.e., taking ε small enough. Finally, by the strong maximum principle we have that $\partial\theta_{\lambda+m}/\partial n < 0$ on $\partial\Omega$, where n is the outer unit normal. Thus, taking ε small $\varepsilon\varphi_1^a \le \theta_{\lambda+m}$ and so there exists a positive solution of (4.15) for $\lambda > \lambda_1(-m)$.

We go a little further now. We want to prove the existence of maximal solution of (4.15). Define the sequence $u^0 = \overline{u}$ and for $n \ge 1$
$$\begin{cases} -\Delta u^n + k\dfrac{|\nabla u^n|^2}{(u^n)^\alpha} + M\Psi(u^n)e^{G(u^n)} = f(x, u^{n-1}) + M\Psi(u^{n-1})e^{G(u^{n-1})} & \text{in } \Omega, \\ u^n = 0 & \text{on } \partial\Omega, \end{cases}$$
$$(4.16)$$

where $G(u) = \int_1^u g(s)ds$, $\Psi(s) := \int_0^s e^{-G(t)}dt$, with $g(u) = ku^{-\alpha}$, and $M \geq 0$ is such that $f(x, s) + Ms$ is nondecreasing for a.e. $x \in \Omega$, where

$$f(x, s) = s(\lambda + m(x) - s), \qquad s \in [0, \sup_{\overline{\Omega}} \overline{u}].$$

Then, $\{u^n\}$ is well-defined, and by Comparison Principle (see Lemma 3.2 in [11]) we have that

$$\underline{u} \leq \ldots \leq u^{n+1} \leq u^n \leq \ldots \leq u^1 \leq u^0 = \overline{u}. \tag{4.17}$$

Indeed, it is easy to show that $\overline{u}$ is supersolution of the equation of u^1, and then $u^1 \leq \overline{u}$. Assume now that $\underline{u} \leq u^n \leq u^{n-1}$. Then, u^n is supersolution of the equation that verifies u^{n+1}, and $\underline{u}$ is subsolution, and so $\underline{u} \leq u^{n+1} \leq u^n$.

Moreover, we can show that

$$u^n \to u^* \quad \text{in } H_0^1(\Omega),$$

with u^* a positive solution of (4.15). We claim that u^* is the maximal solution. Indeed, take any positive solution u of (4.15). We know that $u \leq \theta_{\lambda+m} = \overline{u}$, and then we can take as $\underline{u} = u$ in the above reasoning. Then, by (4.17) we obtain that

$$u \leq u^*.$$

This shows the claim.

Assume that $\mu_1 + m_1 \leq \mu_2 + m_2$ and $\mu_1 > \lambda_1(-m_1)$. Then, it is clear that $\mu_2 > \lambda_1(-m_2)$. Moreover, $\Theta_{[\mu_1+m_1,\alpha,k]}$ is sub-solution of (4.15) for $\lambda + m = \mu_2 + m_2$ and $\overline{u} = K$, $K > 0$ is a supersolution for large K. Hence, there exists a solution u of (4.15) for $\lambda + m = \mu_2 + m_2$ such that

$$\Theta_{[\mu_1+m_1,\alpha,k]} \leq u \leq K,$$

and so, since $\Theta_{[\mu_2+m_2,\alpha,k]}$ is the maximal solution, it follows that

$$\Theta_{[\mu_1+m_1,\alpha,k]} \leq \Theta_{[\mu_2+m_2,\alpha,k]}.$$

Using the same ideas it is easy to prove that the map $k \to \Theta_{[\lambda+m,\alpha,k]}$ is decreasing. $\qquad \square$

Remark 4.2.   1. *Observe that when $k = 0$, $\Theta_{[\lambda+m,\alpha,0]} = \theta_{\lambda+m}$.*

    2. *Observe that the existence result of (4.15) does not depend on the value of $k > 0$.*

4.1. Example 1

We consider the system

$$\begin{cases} -\Delta u + g_1(v)\dfrac{|\nabla u|^2}{u^\alpha} = \lambda u^q v^p & \text{in } \Omega, \\ -\Delta v + g_2(u)\dfrac{|\nabla v|^2}{v^\gamma} = \mu u^r v^s & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \qquad (4.18)$$

where $\lambda, \mu \in \mathbb{R}$, $p, q, r, s \geq 0$, and g_1 and g_2 verify (G). The existence results depend on the size of $p + q$ and $r + s$ and the increase and decrease of the functions g_1 and g_2.

First, it is clear that if $\lambda \leq 0$ or $\mu \leq 0$, by the maximum principle (4.18) does not have positive solutions. So, we assume that $\lambda, \mu > 0$. It is clear that in this case $f_1(x, u, v) = \lambda u^q v^p$ and $f_2(x, u, v) = \mu u^r v^s$ verify (F) for any pair of sub-supersolution of (4.18).

Theorem 4.3. *Assume that $p + q < 1$ and $r + s < 1$. Then, there exists a positive solution if and only if $\lambda, \mu > 0$.*

Proof. Take
$$(\overline{u}, \overline{v}) = (Ke, Ke), \quad (\underline{u}, \underline{v}) = (\varepsilon\varphi_1^a, \varepsilon\varphi_1^a),$$

where e is the unique positive solution of

$$\begin{cases} -\Delta e = 1 & \text{in } \Omega, \\ e = 0 & \text{on } \partial\Omega, \end{cases}$$

φ_1 is a positive principal eigenfunction associated to λ_1, K and ε are positive constants chosen large and small enough, respectively, and $a > 1$. Indeed, item (1) of Definition 3.1 is trivially satisfied. Moreover, item (2) is a direct consequence of Lemma 2.5 in [11] taking $a > \frac{1}{2-\alpha}$. With respect to items (3) and (4) we observe that for every $v \in [\underline{v}, \overline{v}]$

$$-\Delta\overline{u} + g_1(v)\frac{|\nabla\overline{u}|^2}{\overline{u}^\alpha} - f_1(x, \overline{u}, v) = K + g_1(v)K^{2-\alpha}\frac{|\nabla e|^2}{e^\alpha} - \lambda K^q e^q v^p \geq$$

$$\geq K + g_1(v)K^{2-\alpha}\frac{|\nabla e|^2}{e^\alpha} - \lambda K^{p+q}e^{p+q} \geq 0.$$

The last inequality is due to $g_1 \geq 0$, $p + q < 1$ and to the fact that we can choose

$$K^{1-p-q} \geq \lambda\|e\|_\infty^{p+q}.$$

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Similarly, in order to get

$$0 \leq -\Delta \overline{v} + g_2(u)\frac{|\nabla \overline{v}|^2}{\overline{v}^\gamma} - f_2(x, u, \overline{v})\phi \quad \forall u \in [\underline{u}, \overline{u}],$$

it is sufficient that

$$K^{1-r-s} \geq \mu \|e\|_\infty^{r+s}.$$

On the other hand,

$$-\Delta \underline{u} + g_1(v)\frac{|\nabla \underline{u}|^2}{\underline{u}^\alpha} - f_1(x, \underline{u}, v)\phi =$$

$$= -\mathrm{div}(a\varepsilon\varphi_1^{a-1}\nabla\varphi_1) + g_1(v)a^2\varepsilon^{2-\alpha}\varphi_1^{2(a-1)-a\alpha}|\nabla\varphi_1|^2 - \lambda\varepsilon^q\varphi_1^{aq}v^p \leq$$

$$\leq \lambda_1 a\varepsilon\varphi_1^a + a(1-a)\varepsilon\varphi_1^{a-2}|\nabla\varphi_1|^2 + g_1(v)a^2\varepsilon^{2-\alpha}\varphi_1^{2(a-1)-a\alpha}|\nabla\varphi_1|^2 - \lambda\varepsilon^{q+p}\varphi_1^{aq+ap}.$$

Extracting $\varepsilon^{q+p}\varphi_1^{a(p+q)}$ we obtain that this quantity is negative if

$$\lambda_1 a\varepsilon^{1-(p+q)}\varphi_1^{a(1-(p+q))} +$$
$$+ a\varepsilon^{1-(p+q)}\varphi_1^{a(1-(p+q))-2}|\nabla\varphi_1|^2(1 - a + g_1(v)a\varepsilon^{1-\alpha}\varphi_1^{a(1-\alpha)}) - \lambda \leq 0$$

for all $v \in [\underline{v}, \overline{v}]$. But, since g_1 in continuous in $[0, \sup_{\overline{\Omega}} \overline{v}]$, it follows the existence of a positive constant $R(\mu)$ such that $g_1(v) \leq R(\mu)$ for all $v \in [\underline{v}, \overline{v}]$. Thus, we can choose ε such that

$$1 - a + g_1(v)a\varepsilon^{1-\alpha}\varphi_1^{a(1-\alpha)} < 0 \quad \text{and} \quad \lambda_1 a\varepsilon^{1-(p+q)}\varphi_1^{a(1-(p+q))} - \lambda < 0.$$

Moreover, we can take ε small and K large such that $\underline{u} \leq \overline{u}$. Similarly, we can argue with $\underline{v}, \overline{v}$. $\qquad \square$

Theorem 4.4. *Assume that $p + q < 1$ and $1 \leq r + s < 2 - \gamma$.*

1. *Assume that g_2 is increasing, $g_2(0) > 0$. Then, for each $\lambda > 0$ there exists $K(\lambda) > 0$ such that if $\lambda > 0$ and $\mu > K(\lambda)$, system (4.18) possesses at least a positive solution.*

2. *Assume that g_2 is decreasing. Then, there exists $K > 0$ such that if $\lambda > 0$ and $\mu > K$ system (4.18) possesses at least a positive solution.*

Proof. With the same notation of the previous proof we take on both cases

$$(\overline{u}, \overline{v}) = (K_1 e, K_2 E), \quad (\underline{u}, \underline{v}) = (\varepsilon\varphi_1^a, \varphi_1^a),$$

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where $E(x) = \sigma \cdot x + b$ for some $\sigma \in \mathbb{R}^N$ with $|\sigma| > 0$ and $b > 0$ such that $E(x) > \eta > 0$ for every $x \in \Omega$. Thus, items (1) and (2) of Definition 3.1 are satisfied. In order to verify the second inequality in (3.10) and in (3.11) it is enough to take $K_1^{1-p-q} \geq \lambda \|e\|_\infty^{p+q}$ and

$$K_2(-\Delta E) + g_2(u)K_2^{2-\gamma}\frac{|\nabla E|^2}{E^\gamma} \geq \mu K_2^{r+s}e^r E^s, \quad \forall u \in [\underline{u}, \overline{u}],$$

for which it suffices that

$$g_2(u)K_2^{2-\gamma-r-s}|\sigma|^2 \geq \mu \|e\|_\infty^r \|E\|_\infty^{s+\gamma}, \quad \forall u \in [\underline{u}, \overline{u}].$$

If g_2 is increasing, then $g_2(u) \geq g_2(0)$ and so we need that $2 - \gamma > r + s$ and $g_2(0) > 0$. Thus, in this case, K_1 and K_2 depend only on λ and μ, respectively.

However, if g_2 is decreasing then $g_2(u) \geq g_2(K_1\|e\|_\infty) > 0$ and, using that $2 - \gamma > r + s$ we can choose K_2 depending on K_1 (which depends on λ) and μ.

On the other hand, the first inequality in (3.10) is satisfied if ε is small enough and (3.11) is satisfied if

$$a\varphi_1^{a(1-(r+s))-2}|\nabla\varphi_1|^2\left[(1-a) + g_2(u)a\varphi_1^{a(1-\alpha)}\right] + \varphi_1^{1-(r+s)} \leq \mu, \quad \forall u \in [\underline{u}, \overline{u}].$$

If g_2 is increasing, then $g_2(u) \leq g_2(K_1(\lambda)e) \leq K_3(\lambda)$, and so the above inequality is true for $\mu > K(\lambda)$ for some constant $K(\lambda) > 0$.

If g_2 is decreasing, then $g_2(u) \leq g_2(0)$ and so the above inequality is true for μ large and independent of λ. This completes the proof. $\qquad\square$

Remark 4.5. 1. *Similar results hold for the case $2 - \alpha > p + q \geq 1$ and $r + s < 1$.*

2. *We could obtain results for any positive function g_2 imposing more restrictive conditions on λ and μ.*

Using similar arguments of the proofs of the above results, we can show the following result.

Theorem 4.6. *Assume that $1 \leq p + q < 2 - \alpha$ and $1 \leq r + s < 2 - \gamma$.*

1. *Assume that g_1 and g_2 are increasing and $g_1(0), g_2(0) > 0$. Then, there exist $K_1(\mu)$ and $K_2(\lambda)$ such that if $\lambda > K_1(\mu)$ and $\mu > K_2(\lambda)$ system (4.18) possesses at least a positive solution.*

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2. *Assume that g_1 is increasing, g_2 decreasing and $g_1(0) > 0$. Then, there exist $K_1(\mu)$ and K_2 such that if $\lambda > K_1(\mu)$ and $\mu > K_2$, system (4.18) possesses at least a positive solution.*

3. *Assume that g_1 and g_2 are decreasing. Then, there exist $K_1, K_2 > 0$ such that if $\lambda > K_1$ and $\mu > K_2$, system (4.18) possesses at least a positive solution.*

4.2. Example 2: competition Lotka-Volterra system

We consider the system

$$
\begin{cases}
-\Delta u + g_1(v)\dfrac{|\nabla u|^2}{u^\alpha} = u(\lambda - u - bv) & \text{in } \Omega, \\[2mm]
-\Delta v + g_2(u)\dfrac{|\nabla v|^2}{v^\gamma} = v(\mu - v - cu) & \text{in } \Omega, \\[2mm]
u = v = 0 & \text{on } \partial\Omega,
\end{cases}
\tag{4.19}
$$

where $\lambda, \mu \in \mathbb{R}$, $b, c \geq 0$, and g_1 and g_2 verify (G). When $g_1 \equiv g_2 \equiv 0$ system (4.19) is the classical competition Lotka-Volterra model, studied extensively in the last years, see for instance [10].

First, it is clear that if $\lambda \leq \lambda_1$ or $\mu \leq \lambda_1$ then (4.19) does not have positive solution. So, assume that $\lambda, \mu > \lambda_1$. Again, it is clear that $f_1(x, u, v) = u(\lambda - u - bv)$ and $f_2(x, u, v) = v(\mu - v - cu)$ verify (F) for any pair of sub-supersolution of (4.19).

Theorem 4.7. *Assume that one of the following conditions holds:*

1. *g_1 and g_2 are increasing and (λ, μ) satisfies*

$$
\lambda > \lambda_1(b\Theta_{[\mu,\gamma,g_2(0)]}) \quad and \quad \mu > \lambda_1(c\Theta_{[\lambda,\alpha,g_1(0)]});
\tag{4.20}
$$

2. *g_1 and g_2 are decreasing and (λ, μ) satisfies*

$$
\lambda > \lambda_1(b\Theta_{[\mu,\gamma,g_2(\lambda)]}) \quad and \quad \mu > \lambda_1(c\Theta_{[\lambda,\alpha,g_1(\mu)]});
\tag{4.21}
$$

3. *g_1 is increasing, g_2 is decreasing and (λ, μ) satisfies*

$$
\lambda > \lambda_1(b\Theta_{[\mu,\gamma,g_2(0)]}) \quad and \quad \mu > \lambda_1(c\Theta_{[\lambda,\alpha,g_1(\mu)]}).
\tag{4.22}
$$

Then (4.19) possesses a least a positive solution.

Remark 4.8. *Observe that conditions (4.20), (4.21) and (4.22) define regions in the plane (λ, μ) which could possibly be empty. For the semilinear case, that is $g_1 \equiv g_2 \equiv 0$, it can be shown, see for example [18] and [17], that these regions are not empty, imposing some conditions (b or c small). The study of these regions are out of the scope of this paper, but let us remark some aspects. Observe that the map*

$$\lambda \in [\lambda_1, \infty) \mapsto \lambda_1(c\Theta_{[\lambda,\alpha,g_1(0)]})$$

is increasing. Hence, for example, the region defined by (4.20) in not empty if b or c is small.

Proof. (1) Assume that g_1 and g_2 are increasing. Then, take

$$(\overline{u}, \overline{v}) = (\Theta_{[\lambda,\alpha,g_1(0)]}, \Theta_{[\mu,\gamma,g_2(0)]}),$$
$$(\underline{u}, \underline{v}) = (\Theta_{[\lambda-b\Theta_{[\mu,\gamma,g_2(0)]},\alpha,R]}, \Theta_{[\mu-c\Theta_{[\lambda,\alpha,g_1(0)]},\gamma,S]}),$$

for some positive constants R and S to be chosen. Since $\underline{u}, \underline{v}, \overline{u}$ and $\overline{v}$ are solutions of logistic equations as (4.15), then items (1) and (2) of Definition 3.1 are satisfied.

Using the equation of $\overline{u}$, it can be shown that $\overline{u}$ satisfies the second inequality in (3.10) if

$$\overline{u}(\lambda - \overline{u}) - g_1(0)\frac{|\nabla \overline{u}|^2}{|\overline{u}|^\alpha} \geq \overline{u}(\lambda - \overline{u} - b\underline{v}) - g_1(\underline{v})\frac{|\nabla \overline{u}|^2}{|\overline{u}|^\alpha},$$

or equivalently,

$$b\overline{u}\underline{v} + (g_1(\underline{v}) - g_1(0))\frac{|\nabla \overline{u}|^2}{|\overline{u}|^\alpha} \geq 0,$$

which is true because g_1 in increasing and $\underline{v} > 0$.

For $\underline{u}$, we need that

$$\frac{|\nabla \underline{u}|^2}{|\underline{u}|^\alpha}(g_1(\overline{v}) - R) \leq 0, \quad \forall v \in [\underline{v}, \overline{v}].$$

Take $R \geq g_1(\overline{v})$.

Observe that by the increase of the map $\lambda + m \mapsto \Theta_{[\lambda+m,\alpha,k]}$, it follows that

$$\underline{u} \leq \Theta_{[\lambda,\alpha,R]} \leq \Theta_{[\lambda,\alpha,g_1(0)]} = \overline{u},$$

this last inequality because $R \geq g_1(0)$.

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(2) Assume that g_1 and g_2 are decreasing. Then, take

$$(\overline{u}, \overline{v}) = (\Theta_{[\lambda, \alpha, g_1(\mu)]}, \Theta_{[\mu, \gamma, g_2(\lambda)]}),$$
$$(\underline{u}, \underline{v}) = (\Theta_{[\lambda - b\Theta_{[\mu, \gamma, g_2(\lambda)]}, \alpha, g_1(0)]}, \Theta_{[\mu - c\Theta_{[\lambda, \alpha, g_1(\mu)]}, \gamma, g_2(0)]}).$$

Indeed, observe that, with a similar argument to the used in the first paragraph, $\overline{u}$ satisfies the second inequality in (3.10) if

$$\overline{u}(\lambda - \overline{u}) - g_1(\mu)\frac{|\nabla \overline{u}|^2}{|\overline{u}|^\alpha} \geq \overline{u}(\lambda - \overline{u} - b\underline{v}) - g_1(\overline{v})\frac{|\nabla \overline{u}|^2}{|\overline{u}|^\alpha},$$

for what it is sufficient that

$$g_1(\overline{v}) \geq g_1(\mu).$$

But, from (4.14) we have that $\overline{v} \leq \theta_\mu \leq \mu$, and since g_1 is decreasing, it follows that

$$g_1(\overline{v}) \geq g_1(\mu).$$

With respect to $\underline{u}$, it can be proved that $\underline{u}$ satisfies the first inequality in (3.10) because $g_1(\underline{v}) \leq g_1(0)$.

Again, it can be shown that $\underline{u} \leq \overline{u}$.

(3) Assume that g_1 is increasing and g_2 is decreasing. Then, take in this case

$$(\overline{u}, \overline{v}) = (\Theta_{[\lambda, \alpha, g_1(0)]}, \Theta_{[\mu, \gamma, g_2(\lambda)]}),$$
$$(\underline{u}, \underline{v}) = (\Theta_{[\lambda - b\Theta_{[\mu, \gamma, g_2(\lambda)]}, \alpha, R]}, \Theta_{[\mu - c\Theta_{[\lambda, \alpha, g_1(0)]}, \gamma, g_2(0)]}).$$

$\square$

4.3. Example 3: symbiotic Lotka-Volterra system

We consider the system

$$\begin{cases} -\Delta u + g_1(v)\dfrac{|\nabla u|^2}{u^\alpha} = u(\lambda - u + bv) & \text{in } \Omega, \\[2mm] -\Delta v + g_2(u)\dfrac{|\nabla v|^2}{v^\gamma} = v(\mu - v + cu) & \text{in } \Omega, \\[2mm] u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.23}$$

where $\lambda, \mu \in \mathbb{R}$, $b, c > 0$, g_1 and g_2 verify (G).

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Theorem 4.9. *Assume that $bc < 1$ and (λ, μ) satisfies*

$$\lambda > \lambda_1(-b\Theta_{[\mu,\gamma,\bar{g}_2]}) \quad and \quad \mu > \lambda_1(-c\Theta_{[\lambda,\alpha,\bar{g}_1]}), \qquad (4.24)$$

where $\bar{g}_i = g_i(0)$ when g_i is decreasing and

$$\bar{g}_1 = g_1\left(\frac{\mu + c\lambda}{1 - bc}\right) \quad when\ g_1\ is\ increasing,$$

and

$$\bar{g}_2 = g_2\left(\frac{\lambda + b\mu}{1 - bc}\right) \quad when\ g_2\ is\ increasing.$$

Then (4.23) possesses a least a positive solution.

Proof. First, recall that $\Theta_{[\mu,\gamma,\bar{g}_2]} \leq \mu$, and then if λ and μ verify (4.24), we have that

$$\lambda > \lambda_1(-b\Theta_{[\mu,\gamma,\bar{g}_2]}) \geq \lambda_1(-b\mu) = \lambda_1 - b\mu,$$

and so $\lambda + b\mu > 0$. Analogously, $\mu + c\lambda > 0$.

Now, take

$$(\bar{u}, \bar{v}) = (R, S)$$

where R and S are large positive constants and

$$(\underline{u}, \underline{v}) = (\Theta_{[\lambda + b\Theta_{[\mu,\gamma,\bar{g}_2]},\alpha,\bar{g}_1]}, \Theta_{[\mu + c\Theta_{[\lambda,\alpha,\bar{g}_1]},\gamma,\bar{g}_2]}).$$

Indeed, R and S must verify

$$\lambda - R + bS \leq 0 \quad and \quad \mu - S + cR \leq 0.$$

Since $bc < 1$, we can take

$$R = \frac{\lambda + b\mu}{1 - bc}, \qquad S = \frac{\mu + c\lambda}{1 - bc}.$$

On the other hand, $\underline{u}$ is subsolution provided of

$$g_1(v) \leq \bar{g}_1, \quad \forall v \in [\underline{v}, \bar{v}].$$

Then, if g_1 is decreasing (respectively increasing) we can take $\bar{g}_1 = g_1(0)$ (respectively $\bar{g}_1 = g_1(\bar{v}) = g_1(\frac{\mu + c\lambda}{1 - bc})$).

Finally, observe that

$$\underline{u} = \Theta_{[\lambda + b\Theta_{[\mu,\gamma,\bar{g}_2]},\alpha,\bar{g}_1]} \leq \lambda + b(\Theta_{[\mu,\gamma,\bar{g}_2]})_M \leq \lambda + b\mu \leq \frac{\lambda + b\mu}{1 - bc} = \bar{u}.$$

□

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Remark 4.10. *Observe again that condition (4.24) could define an empty region in the plane (λ, μ). As in Remark 4.8 we point out that the maps*

$$\lambda \in [\lambda_1, \infty) \mapsto \lambda_1(-c\Theta_{[\lambda,\alpha,g_1(0)]})$$

and

$$\mu \in [\lambda_1, \infty) \mapsto \lambda_1(-b\Theta_{[\mu,\gamma,g_2(0)]})$$

are decreasing, and so the region defined by (4.24) is non empty when g_1 and g_2 are decreasing, see also [12] for the semilinear case $g_1 \equiv g_2 \equiv 0$.

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