REGULARITY AND MORSE INDEX OF THE SOLUTIONS TO CRITICAL QUASILINEAR ELLIPTIC SYSTEMS

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ABSTRACT. Regularity results and critical group estimates are studied for critical (p, r)-systems. Multiplicity results of solutions for a critical potential quasilinear system are also proved using Morse theory.

MSC2010: 35J92, 35J50, 58E05.

KEYWORDS: Critical Quasilinear Elliptic Systems, Regularity, Morse theory, Critical Groups.

1. INTRODUCTION

Let us consider the following (p, r)-system

$$\begin{cases} -\mathrm{div}\left((\alpha + |\nabla u|^{p-2})\nabla u\right) = \mathrm{D}_u F(x, u, v) & x \in \Omega \\ -\mathrm{div}\left((\alpha + |\nabla v|^{r-2})\nabla v\right) = \mathrm{D}_v F(x, u, v) & x \in \Omega \\ u = v = 0 & x \in \partial\Omega \end{cases} \tag{1.1}$$

where Ω is a smooth bounded domain of $\mathbb{R}^N$, p, r are real numbers larger than 2, $\alpha \geq 0$ and $N \geq \max\{p^2, r^2\}$. We assume that $F(x, \cdot, \cdot) \in C^2(\mathbb{R} \times \mathbb{R})$ a.e. $x \in \Omega$, $F(\cdot, u, v) \in C^1(\overline{\Omega})$ and that there exists $c > 0$ such that

$$|\mathrm{D}_{uu} F(x, u, v)| \leq c \left(|u|^{p^*-2} + |v|^{r^* \frac{p^*-2}{p^*}} + 1\right), \tag{1.2}$$

$$|\mathrm{D}_{uv} F(x, u, v)|,\ |\mathrm{D}_{vu} F(x, u, v)| \leq c \left(|u|^{p^*-1-\frac{p^*}{r^*}} + |v|^{r^*-1-\frac{r^*}{p^*}} + 1\right), \tag{1.3}$$

$$|\mathrm{D}_{vv} F(x, u, v)| \leq c \left(|u|^{p^* \frac{r^*-2}{r^*}} + |v|^{r^*-2} + 1\right), \tag{1.4}$$

where p^* and r^* denote respectively the conjugate Sobolev exponents of p and r ($s^* = sN/(N - s), s < N$). It is not restrictive to suppose also that $F(x, 0, 0) = 0$ for each $x \in \Omega$.

We notice that (1.2), (1.3), (1.4) imply that there exists $C > 0$, such that

$$|\mathrm{D}_u F(x, u, v)| \leq C \left(|u|^{p^*-1} + |v|^{r^* \frac{p^*-1}{p^*}} + 1\right), \tag{1.5}$$

The authors are partially supported by the Spanish-Italian Acción Integrada HI2008.0106/ Azione Integrata Italia-Spagna IT09L719F1.

The first and forth authors are partially supported by M.I.U.R., research project PRIN 09 *Variational and topological methods in the study of nonlinear phenomena.* The first author is also supported by GNAMPA research project 2012.

The second and third authors are partially supported by MICINN Ministerio de Ciencia e Innovación (Spain) MTM2009-10878, MTM-2012-31799 and Junta de Andalucia FQM-116 and FQM-194.

$$|\mathrm{D}_v F(x,u,v)| \le C\left(|u|^{p^*\frac{r^*-1}{r^*}} + |v|^{r^*-1} + 1\right) \tag{1.6}$$

and

$$|F(x,u,v)| \le C(|u|^{p^*} + |u| + |v|^{r^*} + |v|). \tag{1.7}$$

For example, one can consider $F(x,u,v) = \frac{1}{p}|u|^p + \frac{1}{r}|v|^r + \frac{2}{\gamma+\beta}|u|^\gamma|v|^\beta$ where $\gamma > 2$ and $\beta > 2$ satisfy $\frac{\gamma}{p^*} + \frac{\beta}{r^*} = 1$.

Systems involving quasilinear operators of p-laplacian type model some phenomena in non-Newtonian mechanics, nonlinear elasticity and glaciology, combustion theory, population biology; see [16, 20, 22, 23]. Existence, nonexistence and regularity results for such quasilinear elliptic systems are obtained by various authors, see for instance [5, 6, 15, 17, 32].

In the present paper we are interested to develop Morse index estimates for the weak solutions of the (p,r)-system (1.1).

Let us define the space $X := W_0^{1,p}(\Omega) \times W_0^{1,r}(\Omega)$ endowed with the norm

$$\|z\| = \|u\|_{1,p} + \|v\|_{1,r}$$

where $z = (u,v) \in X$. In what follows we shall denote respectively by $\|\cdot\|_s$ and $\|\cdot\|_{1,s}$ the usual norms in $L^s(\Omega)$ and $W_0^{1,s}(\Omega)$.

Weak solutions of problem (1.1) correspond to critical points of the Euler functional J_α on X defined by setting

$$\begin{aligned} J_\alpha(z) = J_\alpha(u,v) =& \frac{1}{p}\int_\Omega \left(\alpha + |\nabla u(x)|^2\right)^{\frac{p}{2}}\, dx + \frac{1}{r}\int_\Omega \left(\alpha + |\nabla v(x)|^2\right)^{\frac{r}{2}}\, dx \\ &- \int_\Omega F(x,u(x),v(x))\, dx,\ z=(u,v)\in X. \end{aligned}$$

By assumptions (1.2), (1.3) and (1.4), it follows that J_α is C^2 on X and for any $z_0 = (u_0,v_0) \in X$, $z = (u,v) \in X$ it results

$$\begin{aligned} \langle J'_\alpha(z_0), z\rangle =& \int_\Omega (\alpha + |\nabla u_0|^2)^{\frac{p-2}{2}} \nabla z_0 \nabla u + \int_\Omega (\alpha + |\nabla v_0|^2)^{\frac{r-2}{2}} \nabla v_0 \nabla v \\ &- \int_\Omega (\mathrm{D}_u F(x,u_0,v_0)u + \mathrm{D}_v F(x,u_0,v_0)v)dx. \end{aligned}$$

Moreover for any $z_1 = (u_1,v_1) \in X$, $z_2 = (u_2,v_2) \in X$, we have

$$\begin{aligned} \langle J''_\alpha(z_0)z_1, z_2\rangle = & \int_\Omega \Big((\alpha + |\nabla u_0|^2)^{(p-2)/2}(\nabla u_1|\nabla u_2) \\ & + (p-2)(\alpha + |\nabla u_0|^2)^{(p-4)/2}(\nabla u_0|\nabla u_1)(\nabla z_0|\nabla u_2)\Big)\, dx \\ & + \int_\Omega \Big((\alpha + |\nabla v_0|^2)^{(r-2)/2}(\nabla v_1|\nabla v_2) \\ & + (r-2)(\alpha + |\nabla v_0|^2)^{(r-4)/2}(\nabla v_0|\nabla v_1)(\nabla v_0|\nabla v_2)\Big)\, dx \\ & - \int_\Omega \big(D^2_{uu}F(x,u_0,v_0)u_1u_2 + D^2_{vv}F(x,u_0,v_0)v_1v_2 \\ & + D^2_{uv}F(x,u_0,v_0)u_1v_2 + D^2_{vu}F(x,u_0,v_0)u_2v_1\big)\, dx. \end{aligned}$$

We stress that every critical point $z_0 \in X$ of J_α is degenerate in the usual sense, since, in general, X is not isomorphic to its dual space, so that the classical Morse lemma does not hold. Moreover $J''_\alpha(z_0)$ is not even a Fredholm operator, so we can not use the Gromoll-Meyer splitting theorem and the perturbation results in [24].

In the literature some authors have introduced the definition of a weakly nondegenerate critical point, which coincides with the classical one in a Hilbert space (see [7, 30]). This notion of weakly nondegeneracy is stronger than the mere injectivity of the second derivative at the critical point in a Banach (not Hilbert) space and, in particular, it requires that the critical point is isolated.

Recently, in [10, 11] (see also [8]), for the scalar case and $\alpha > 0$, the authors have proved that the injectivity of the second derivative in a critical point is enough for developing Morse index estimates. The fact that the mere injectivity of the second derivative at the critical point could sometimes be the reasonable notion of nondegeneracy, was conjectured by Smale in an unpublished article, as mentioned by Uhlenbeck in [30].

In the framework of the (p, r)-systems, we begin by giving the following definition:

- *a critical point z_0 of J_α is said to be nondegenerate if $J''_\alpha(z_0) : X \to X^*$ is injective.*

The aim of this paper is to develop a local Morse theory for the functional J_α, extending the results of [11] to the quasilinear critical system (1.1). The strategy is to perform a finite dimensional reduction, introducing an auxiliary Hilbert space, where the linearized operator at the critical point becomes a Fredholm operator. This approach strongly relies on the C^1-regularity up to the boundary of the weak solutions of (1.1).

As far as we know, there are no C^1-regularity results for critical (p, r)-systems. For subcritical quasilinear systems, we quote [5, 6, 32]. In the scalar (critical) case, this regularity was proved in [18], but, due to the mixed critical growth of D_uF, D_vF, the extension to critical systems it is not straightforward, at least for the cases $p = r \neq 2$ or $p \neq r$. In section 2 we establish the following regularity result:

Theorem 1.1. *Assume that $z = (u, v) \in X$ is a solution of (1.1) with $\alpha \geq 0$ and F satisfying (1.5) and (1.6). Then $u, v \in C^{1,\eta}(\overline{\Omega})$, for some $0 < \eta < 1$. Moreover, for every bounded set $A \subset X$, there exists a positive constant K, independent of α, such that $\|u\|_{C^{1,\eta}(\overline{\Omega})} \leq K$ and $\|v\|_{C^{1,\eta}(\overline{\Omega})} \leq K$, for every solution $(u, v) \in A$.*

A local Morse theory for the functional J_α requires a result correlating the Morse index with the critical groups of the functional at a critical point. We recall the definition of these concepts from Morse theory.

Definition 1.2. The Morse index $m(J_\alpha, z_0)$ of J_α at a critical point z_0 is the supremum of the dimensions of the subspaces where $J''_\alpha(z_0)$ is negative definite. The large Morse index $m^*(J_\alpha, z_0)$ is the sum of $m(J_\alpha, z_0)$ and the dimension of the kernel of $J''_\alpha(z_0)$.

Definition 1.3. The q-th critical group of J_α at a critical point z_0, denoted by $C_q(J_\alpha, z_0)$ is given by

$$C_q(J_\alpha, z_0) = H^q(J^c_\alpha \cap U, (J^c_\alpha \setminus \{u\}) \cap U), \quad q = 0, 1, 2, \ldots,$$

where U is a neighborhood of z_0, $c = J_\alpha(z_0)$, $J^c_\alpha = \{z \in X : J_\alpha(z) \leq c\}$ and $H^q(A, B)$ stands for the q-th Alexander-Spanier cohomology group of the pair (A, B) with coefficients in a field $\mathbb{K}$.

The critical group computations for J_α rely on some local compactness at every level set. Due to the critical exponents involved in (1.1), we are led to investigate the local Palais-Smale condition around the critical points. We devote section 3 to the study of the local properties of the functional. In section 4 we perform a finite dimensional reduction and we introduce a reduction map φ. In section 5 we establish that φ is C^2. The proof looks very technical, since we can not apply the Implicit Function Theorem directly.

Finally in section 5 we derive the following main result:

Theorem 1.4. *Let $\alpha > 0$ and z_0 be a critical point of the functional J_α such that $J''_\alpha(z_0)$ is injective from X to X^* (nondegenerate). Then $m(J_\alpha, z_0)$ is finite and*

$$C_j(J_\alpha, z_0) \cong \mathbb{K}, \quad \text{if } j = m(J_\alpha, z_0),$$

$$C_j(J_\alpha, z_0) = \{0\}, \quad \text{if } j \neq m(J_\alpha, z_0).$$

Moreover, if $J''_\alpha(z_0)$ is not injective, then $m(J_\alpha, z_0)$ and $m^*(J_\alpha, z_0)$ are finite and

$$C_j(J_\alpha, z_0) = \{0\}$$

for any $j \leq m(J_\alpha, z_0) - 1$ and for any $j \geq m^*(J_\alpha, z_0) + 1$.

We remark that the first part of Theorem 1.4 extends a classical result in Hilbert spaces for nondegenerate critical points (cf. Theorem 4.1 in [7]), showing that the critical groups of J_α in z_0 depend only on its Morse index. Conversely the second part establishes a finiteness result for critical groups when the critical point is degenerate.

In the last section, we apply the critical group estimates in Theorem 1.4 to obtain multiplicity results of solutions for the potential p-laplacian system

$$\begin{cases} -\Delta_p u = \lambda |u|^{q-2}u + 2\frac{\gamma}{p^*}|u|^{\gamma-2}u|v|^\beta & \text{in } \Omega \\ -\Delta_p v = \mu |v|^{q-2}v + 2\frac{\beta}{p^*}|u|^{\gamma}|v|^{\beta-2}v & \text{in } \Omega \\ u > 0, \ v > 0 & \text{in } \Omega \\ u = v = 0 & \text{on } \partial\Omega \end{cases} \tag{1.8}$$

where Ω is a smooth domain, $N \geq p^2$, $2 < p \leq q < p^*$, $\lambda > 0$, $\mu > 0$ and $\gamma > 2$, $\beta > 2$ satisfy $\gamma + \beta = p^*$.

Using an entrance map and a barycenter map, we can correlate the topological properties of the domain with the number of solutions, counted with their multiplicities, of the p-laplacian system (1.8), for small values of λ, μ. However, it is not clear what multiplicity means in the Banach setting X. Indeed, in our setting we cannot apply the Marino-Prodi perturbation result in [24], based on the infinite dimensional version of Sard's Lemma, due to Smale. Nevertheless, we overcome the lack of Fredholm properties of the second derivative of the Euler functional thanks to the C^2-regularity of the reduction map φ. Indeed we apply the finite dimensional Sard's Lemma to the reduction map φ and we are able to give an interpretation of the multiplicity in terms of distinct solutions of approximating differential systems. Hence, we derive a quantitative result for the p-laplacian system (1.8).

In order to state it, we recall the following topological notion.

Definition 1.5. For any $A, B \subset \mathbb{R}^N$, we denote by $\mathcal{P}_t(A,B)$ the Poincaré polynomial of the topological pair (A,B), namely

$$\mathcal{P}_t(A,B) = \sum_{q=0}^{+\infty} \dim H^q(A,B)\, t^q.$$

The Poincaré polynomial of A, denoted by $\mathcal{P}_t(A)$, is the Poincaré polynomial of the topological pair $(A,\emptyset)$.

Theorem 1.6. *Assume that $\partial\Omega$ satisfies the interior sphere condition and that $2 < p \le q < p^*$, $N \ge p^2$ and $\gamma > 2$, $\beta > 2$ with $\gamma + \beta = p^*$. There exist $\lambda^* > 0, \mu^* > 0$ such that, for any $\lambda \in (0,\lambda^*)$ and $\mu \in (0,\mu^*)$, either the system (1.8) has at least $\mathcal{P}_1(\Omega)$ distinct solutions or, if not, for any sequence (α_n), with $\alpha_n > 0$, $\alpha_n \to 0$, there exist two sequences (f_n) and (g_n) in $C^1(\overline{\Omega})$, such that $\|f_n\|_{C^1(\overline{\Omega})} \to 0$, $\|g_n\|_{C^1(\overline{\Omega})} \to 0$ and problem*

(1.9)
$$\begin{cases} -\mathrm{div}\big((\alpha_n + |\nabla u|^2)^{(p-2)/2}\nabla u\big) = \lambda|u|^{q-2}u + 2\frac{\gamma}{p^*}|u|^{\gamma-2}u|v|^{\beta} + f_n & \text{in } \Omega\\ -\mathrm{div}\big((\alpha_n + |\nabla v|^2)^{(p-2)/2}\nabla v\big) = \mu|v|^{q-2}v + 2\frac{\beta}{p^*}|u|^{\gamma}|v|^{\beta-2}v + g_n & \text{in } \Omega\\ u > 0,\ v > 0 & \text{in } \Omega\\ u = v = 0 & \text{on } \partial\Omega \end{cases}$$

has at least $\mathcal{P}_1(\Omega)$ distinct solutions, for n large enough.

An existence result for a p-laplacian system involving homogeneous nonlinearities with critical Sobolev exponent degrees has been proved in [15]. A multiplicity result has been studied in [17], using Ljusternik–Schnirelman category.

We remark that Morse theory yields better results than Ljusternik–Schnirelman theory for topologically rich domains. For example if Ω is obtained by an open contractible domain cutting off k holes, we derive that the number of solutions of (1.8) is affected by k, even if the category of Ω is 2 (see [4]). For the scalar case, a multiplicity result of positive solutions has been proved in [13] via Morse theory.

2. Regularity

In this section we prove the $C^{1,\eta}$-regularity up to the boundary for X-solutions of (1.1). This result has been proved for the scalar case in [18], however the extension to systems is not straightforward due to the coupling of u with v. Here, inspired by [18], we prove Theorem 1.1.

Proof of Theorem 1.1. First we prove that $u, v \in L^t(\Omega)$ for every $t > 1$.

For every $\gamma \ge 2$ and $t \ge 0$ we define

$$h_{k,\gamma}(s) = \begin{cases} s|s|^{\gamma-1} & |s| \le k,\\ \gamma k^{\gamma-1}s + \mathrm{sign}(s)(1-\gamma)k^{\gamma} & |s| > k,\end{cases}$$
$$\Phi_{k,t,\gamma}(s) = \int_0^s \left|h'_{k,\gamma}(r)\right|^{\frac{t}{\gamma}}\, dr.$$

Observe that $h_{k,\gamma}$ and $\Phi_{k,t,\gamma}$ are C^1-functions with bounded derivative. Thus $\Phi_{k,t,\gamma}(u) \in W_0^{1,p}(\Omega)$ and $\Phi_{k,t,\gamma}(v) \in W_0^{1,r}(\Omega)$. Moreover, for every $t \ge \gamma$, there exists a positive constant C independent on k, such that

$$|s|^{\frac{t}{\gamma}-1}|\Phi_{k,t,\gamma}(s)| \le C|h_{k,\gamma}(s)|^{\frac{t}{\gamma}} \text{ and } |\Phi_{k,t,\gamma}(s)| \le C|h_{k,\gamma}(s)|^{\frac{1}{\gamma}(1+t\frac{\gamma-1}{\gamma})}. \tag{2.1}$$

We choose $\Phi_{k,\gamma p,\gamma}(u)$ as test function in the first equation of (1.1), for some $\gamma > 1$. Using Sobolev inequality, we obtain

$$\begin{aligned}\mathcal{S}_p\left(\int_\Omega |h_{k,\gamma}(u)|^{p^*}\right)^{\frac{p}{p^*}} &\le \int_\Omega |\nabla h_{k,\gamma}(u)|^p = \int_\Omega |\nabla u|^p |h'_{k,\gamma}(u)|^p \\ &\le \int_\Omega (\alpha + |\nabla u|^{p-2})|\nabla u|^2 |h'_{k,\gamma}(u)|^p = \int_\Omega (\alpha + |\nabla u|^{p-2})\nabla u \cdot \nabla \Phi_{k,\gamma p,\gamma}(u) \\ &= \int_\Omega D_u F(x,u,v)\Phi_{k,\gamma p,\gamma}(u)\end{aligned}$$

where $\mathcal{S}_p$ is the best Sobolev constant of the embedding of $W_0^{1,p}(\Omega)$ into $L^{p^*}(\Omega)$ defined by

$$\mathcal{S}_p = \inf_{u\in W_0^{1,p}(\Omega)\setminus\{0\}} \frac{\|u\|_{1,p}^p}{\left(\int_\Omega |u|^{p^*}dx\right)^{p/p^*}}.$$

Thus, using (1.5), we have

$$\mathcal{S}_p\left(\int_\Omega |h_{k,\gamma}(u)|^{p^*}\right)^{\frac{p}{p^*}} \le c\int_\Omega (|u|^{p^*-1}+1)|\Phi_{k,\gamma p,\gamma}(u)| + c\int_\Omega |v|^{r^*\frac{p^*-1}{p^*}}|\Phi_{k,\gamma p,\gamma}(u)|. \tag{2.2}$$

For a fixed $\sigma > 1$, to be determined later, we denote $\Omega_{\sigma,w} = \{x \in \Omega : |w(x)| > \sigma\}$. Therefore, using (2.1) and, if necessary, redefining the positive constant C independent on k, we get

$$\begin{aligned}\int_\Omega (|u|^{p^*-1}+1)|\Phi_{k,\gamma p,\gamma}(u)| \le& (\sigma^{p^*-1}+1)\int_\Omega |\Phi_{k,\gamma p,\gamma}(u)| \\ &+ C\int_{\Omega_{\sigma,u}} |u|^{p^*-p}|u|^{p-1}|\Phi_{k,\gamma p,\gamma}(u)| \\ \le& C\int_\Omega |h_{k,\gamma}(u)|^{\frac{1}{\gamma}(1+(\gamma-1)p)} + C\int_{\Omega_{\sigma,u}} |u|^{p^*-p}|h_{k,\gamma}(u)|^p.\end{aligned}$$

Using Hölder inequality we deduce

$$\begin{aligned}\int_\Omega (|u|^{p^*-1}+1)\Phi_{k,\gamma p,\gamma}(u) \le& C\left(\int_\Omega |h_{k,\gamma}(u)|^{p^*}\right)^{\frac{p}{p^*}\frac{\gamma p+1-p}{\gamma p}} \\ &+ C\|u^{p^*-p}\|_{L^{\frac{N}{p}}(\Omega_{\sigma,u})}\left(\int_\Omega |h_{k,\gamma}(u)|^{p^*}\right)^{\frac{p}{p^*}}.\end{aligned} \tag{2.3}$$

We deal with the second integral in (2.2) and similarly, using (2.1) and that $\Phi_{k,t,\gamma}$ is non decreasing, we yield

$$\begin{aligned}
\int_\Omega |v|^{r^*\frac{p^*-1}{p^*}}|\Phi_{k,\gamma p,\gamma}(u)| \le{} & C\left(\int_\Omega |h_{k,\gamma}(u)|^{p^*}\right)^{\frac{p}{p^*}\frac{\gamma p+1-p}{\gamma p}} \\
& + C\int_{\Omega_{\sigma,v}} |v|^{\frac{r^*}{p^*}(p^*-p)}|v|^{\frac{r^*}{p^*}(p-1)}|\Phi_{k,\gamma p,\gamma}(u)| \\
\le{} & C\left(\int_\Omega |h_{k,\gamma}(u)|^{p^*}\right)^{\frac{p}{p^*}\frac{\gamma p+1-p}{\gamma p}} + C\int_{\Omega_{\sigma,v}\cap\{|v|^{\frac{r^*}{p^*}}\le|u|\}} |v|^{\frac{r^*}{p^*}(p^*-p)}|h_{k,\gamma}(u)|^p \\
& + C\int_{\Omega_{\sigma,v}\cap\{|v|^{\frac{r^*}{p^*}}\ge|u|\}} |v|^{\frac{r^*}{p^*}(p^*-p)}\left(h_{k,\gamma}(|v|^{\frac{r^*}{p^*}})\right)^p \\
\le{} & C\left(\int_\Omega |h_{k,\gamma}(u)|^{p^*}\right)^{\frac{p}{p^*}\frac{\gamma p+1-p}{\gamma p}} \\
& + C\|(|v|^{\frac{r^*}{p^*}})^{p^*-p}\|_{L^{\frac{N}{p}}(\Omega_{\sigma,v})}\left(\int_\Omega |h_{k,\gamma}(u)|^{p^*}\right)^{\frac{p}{p^*}} \\
& + C\|(|v|^{\frac{r^*}{p^*}})^{p^*-p}\|_{L^{\frac{N}{p}}(\Omega_{\sigma,v})}\left(\int_\Omega \left(h_{k,\gamma}(|v|^{\frac{r^*}{p^*}})\right)^{p^*}\right)^{\frac{p}{p^*}}. \qquad (2.4)
\end{aligned}$$

Observe that there exists a positive constant C, independent on k, such that

$$\left|h_{k,\gamma}(|v|^{\frac{r^*}{p^*}})\right|^{p^*} \le C\left|h_{k^{\frac{p^*}{r^*}},\gamma}(|v|)\right|^{r^*}. \qquad (2.5)$$

Taking into account (2.3), (2.4), (2.5), and using Young inequality ($\frac{\gamma p+1-p}{\gamma p} < 1$), we obtain from (2.2) that

$$\begin{aligned}
\frac{\mathcal{S}_p}{2}\left(\int_\Omega |h_{k,\gamma}(u)|^{p^*}\right)^{\frac{p}{p^*}} \le{} & C + C\|(|v|^{\frac{r^*}{p^*}})^{p^*-p}\|_{L^{\frac{N}{p}}(\Omega_{\sigma,v})}\left(\int_\Omega |h_{k^{\frac{p^*}{r^*}},\gamma}(v)|^{r^*}\right)^{\frac{p}{p^*}} \\
& + C\left(\|u^{p^*-p}\|_{L^{\frac{N}{p}}(\Omega_{\sigma,u})} + \|(|v|^{\frac{r^*}{p^*}})^{p^*-p}\|_{L^{\frac{N}{p}}(\Omega_{\sigma,v})}\right)\left(\int_\Omega |h_{k,\gamma}(u)|^{p^*}\right)^{\frac{p}{p^*}}.
\end{aligned}$$

Choosing σ such that $\|u^{p^*-p}\|_{L^{\frac{N}{p}}(\Omega_\sigma)}$, $\|(|v|^{\frac{r^*}{p^*}})^{p^*-p}\|_{L^{\frac{N}{p}}(\Omega_{\sigma,v})} \le \frac{\mathcal{S}_p}{8C}$ we deduce

$$\frac{\mathcal{S}_p}{4}\left(\int_\Omega |h_{k,\gamma}(u)|^{p^*}\right)^{\frac{p}{p^*}} \le C + C\|(|v|^{\frac{r^*}{p^*}})^{p^*-p}\|_{L^{\frac{N}{p}}(\Omega_{\sigma,v})}\left(\int_\Omega |h_{k^{\frac{p^*}{r^*}},\gamma}(v)|^{r^*}\right)^{\frac{p}{p^*}}.$$

In particular, we get

$$\int_\Omega |h_{k,\gamma}(u)|^{p^*} \le C + C\|(|v|^{\frac{r^*}{p^*}})^{p^*-p}\|^{\frac{p^*}{p}}_{L^{\frac{N}{p}}(\Omega_{\sigma,v})}\int_\Omega |h_{k^{\frac{p^*}{r^*}},\gamma}(v)|^{r^*}. \qquad (2.6)$$

Analogously, we choose $\Phi_{k,\gamma r,\gamma}(v)$ as test function in the second equation of (1.1) and it follows that

$$\int_\Omega |h_{k,\gamma}(v)|^{r^*} \le C + C\|(|u|^{\frac{p^*}{r^*}})^{r^*-r}\|^{\frac{r^*}{r}}_{L^{\frac{N}{r}}(\Omega_{\sigma,u})}\int_\Omega |h_{k^{\frac{r^*}{p^*}},\gamma}(u)|^{p^*}. \qquad (2.7)$$

Using (2.7) for $k^{\frac{p^*}{r^*}}$ in (2.6) we have

$$\int_\Omega |h_{k,\gamma}(u)|^{p^*} \le C + C\|(|v|^{\frac{r^*}{p^*}})^{p^*-p}\|_{L^{\frac{N}{p}}(\Omega_{\sigma,v})}^{\frac{p^*}{p}} \int_\Omega |h_{k^{\frac{p^*}{r^*}},\gamma}(v)|^{r^*} \le C$$
$$+ C\|(|v|^{\frac{r^*}{p^*}})^{p^*-p}\|_{L^{\frac{N}{p}}(\Omega_{\sigma,v})}^{\frac{p^*}{p}} \left(C + C\|(|u|^{\frac{p^*}{r^*}})^{r^*-r}\|_{L^{\frac{N}{r}}(\Omega_{\sigma,u})}^{\frac{r^*}{r}} \int_\Omega |h_{(k^{\frac{p^*}{r^*}})^{\frac{r^*}{p^*}},\gamma}(u)|^{p^*} \right).$$

Taking σ in order to get $\|(|v|^{\frac{r^*}{p^*}})^{p^*-p}\|_{L^{\frac{N}{p}}(\Omega_{\sigma,v})}^{\frac{p^*}{p}}$ and $\|(|u|^{\frac{p^*}{r^*}})^{r^*-r}\|_{L^{\frac{N}{r}}(\Omega_{\sigma,u})}^{\frac{r^*}{r}}$ small enough we deduce

$$\int_\Omega |h_{k,\gamma}(u)|^{p^*} \le C.$$

Reasoning analogously we also have $\int_\Omega |h_{k,\gamma}(v)|^{r^*} \le C$.

Thus, we can use Fatou Lemma and passing to the limit for $k \to +\infty$ we get

$$\int_\Omega |u|^{\gamma p^*}, \int_\Omega |v|^{\gamma r^*} \le C. \tag{2.8}$$

Since γ is any arbitrary number with $\gamma \ge 1$ we have that $u, v \in L^\eta(\Omega)$ for any $\eta \ge 1$. In particular, using (1.5) and (1.6) we derive that $\mathrm{D}_u F(x,u,v)$, $\mathrm{D}_u F(x,u,v) \in L^\tau(\Omega)$ for some $\tau > \max\{N/p, N/r\}$ and

$$\|\mathrm{D}_u F(x,u,v)\|_\tau, \|\mathrm{D}_v F(x,u,v)\|_\tau \le C_1,$$

for some constant C_1 depending continuously on Ω, p, r and $\|u\|_{p^*}, \|v\|_{r^*}$. Thus, using Stampacchia method it follows that $u, v \in L^\infty(\Omega)$ and

$$\|u\|_\infty, \|v\|_\infty \le C_2,$$

for some constant C_2 depending continuously on Ω, p, r and the $L^\tau(\Omega)$ norm of $\mathrm{D}_u F(x,u,v)$ and $\mathrm{D}_v F(x,u,v)$. Finally, using the regularity result in [28, 29] we deduce that $u, v \in C^{1,\eta}(\overline{\Omega})$, for some $0 < \eta < 1$ and

$$\|u\|_{C^{1,\eta}(\overline{\Omega})}, \|v\|_{C^{1,\eta}(\overline{\Omega})} \le C_3,$$

for some constant C_3 depending continuously on Ω, p, r, $\|u\|_\infty$ and $\|v\|_\infty$. Moreover C_1, and consequently C_2 and C_3, may be chosen independent on (u, v) in a bounded set $A \subset X$. Thus, for every bounded set $A \subset X$ there exists a positive constant K such that $\|u\|_{C^{1,\eta}(\overline{\Omega})} \le K$ and $\|v\|_{C^{1,\eta}(\overline{\Omega})} \le K$, for every solution $(u, v) \in A$. □

Remark 2.1. We point out that the above proof still works for the quasilinear problem

$$\begin{cases} -\mathrm{div}\left((\alpha + |\nabla u|^{p-2})\nabla u\right) = G_1(x,u,v) + g_1(x) & x \in \Omega \\ -\mathrm{div}\left((\alpha + |\nabla v|^{r-2})\nabla v\right) = G_2(x,u,v) + g_2(x) & x \in \Omega \\ u = v = 0 & x \in \partial\Omega \end{cases}$$

for any functions G_1, $G_2 \in C(\overline{\Omega} \times \mathbb{R} \times \mathbb{R})$, satisfying

$$|G_1(x,u,v)| \le C\left(|u|^{p^*-1} + |v|^{r^*\frac{p^*-1}{p^*}} + 1\right),$$
$$|G_2(x,u,v)| \le C\left(|u|^{p^*\frac{r^*-1}{r^*}} + |v|^{r^*-1} + 1\right),$$

and for any bounded functions $g_1, g_2 \in C(\overline{\Omega})$. Moreover, for any bounded set $A \subset C(\overline{\Omega})$, the constant K may be chosen independent on $g_1, g_2 \in A$.

3. Local properties of J_α

In this section we prove that J_α is locally weakly lower semicontinuous and locally satisfies $(P.S.)$ condition. In what follows, it will be useful the following inequality obtained in [14].

Lemma 3.1. *For any $\alpha \geq 0$, $p \geq 2$ and $x, y \in \mathbb{R}^N$*

$$(\alpha + |y|^2)^{p/2} \geq (\alpha + |x|^2)^{p/2} + p(\alpha + |x|^2)^{(p-2)/2}(x|y-x) + \frac{|y-x|^p}{2^{p-1}-1}.$$

□

Lemma 3.2. *Assume that $G(x,u,v) \in C^1(\overline{\Omega} \times \mathbb{R} \times \mathbb{R})$ satisfies* (1.5) *and* (1.6). *Let $z_n = (u_n, v_n)$ be a bounded sequence in X which weakly converges to $z = (u,v)$ in X, then, up to subsequences,*

$$\int_\Omega (G(x,u_n,v_n) - G(x,u,v))\,dx = \int_\Omega G(x, u_n - u, v_n - v)dx + o(1).$$

Proof. From Fubini Theorem we have

$$\begin{aligned}\int_\Omega (G(x,u_n,v_n) &- G(x,u_n-u,v_n-v))dx \\ &= \int_\Omega \left(\int_0^1 \frac{d}{dt} G(x, u_n + (t-1)u, v_n + (t-1)v)dt\right) dx \\ &= \int_0^1 \left(\int_\Omega \mathrm{D}_u G(x, u_n + (t-1)u, v_n + (t-1)v)u dx\right) dt \\ &\quad + \int_0^1 \left(\int_\Omega \mathrm{D}_v G(x, u_n + (t-1)u, v_n + (t-1)v)v dx\right) dt.\end{aligned}$$

As $u_n(x) \to u(x)$ and $v_n(x) \to v(x)$ a.e. in Ω, u_n and v_n are bounded respectively in $L^{p^*}(\Omega)$ and $L^{r^*}(\Omega)$, taking account of (1.5) and (1.6), by Lebesgue Theorem, up to subsequences we get

$$\begin{aligned}\lim_{n\to\infty} &\int_\Omega (G(x,u_n,v_n) - G(x,u_n-u,v_n-v))dx \\ &= \int_0^1 \left(\int_\Omega \mathrm{D}_u G(x,tu,tv)u + \mathrm{D}_v G(x,tu,tv)v dx\right) dt = \int_\Omega G(x,u,v)dx.\end{aligned}$$

□

Proposition 3.3. *There exists $R > 0$ such that, for any fixed $\alpha \geq 0$ and any $z_0 \in X$, the functional J_α is weakly lower semicontinuous in $\overline{B}_R(z_0) = \{w \in X \mid \|w - z_0\| \leq R\}$.*

Proof. Let $R > 0$, $z_0 \in X$ and z_n a sequence in X such that $\|z_n - z_0\| \leq R$. Assume that z_n weakly converges to $z = (u,v)$. We will show that $J_\alpha(z) \leq \liminf_n J_\alpha(z_n)$, if R is chosen small enough. First we observe that there is a subsequence $z'_n = (u'_n, v'_n) = z_{k_n}$ of z_n such that $\lim_n J_\alpha(z'_n) = \liminf_n J_\alpha(z_n)$. Using Lemma 3.1 and Lemma 3.2 (with $G = F$) we deduce that, for a suitable subsequence $z''_n =$

$(u_n'', v_n'') = z'_{h_n}$ of z'_n,

$$\begin{aligned} J_\alpha(u_n'', v_n'') - J_\alpha(u,v) &= \frac{1}{p}\int_\Omega (\alpha + |\nabla u_n''|^2)^{\frac{p}{2}} - \frac{1}{p}\int_\Omega (\alpha + |\nabla u|^2)^{\frac{p}{2}} \\ &\quad + \frac{1}{r}\int_\Omega (\alpha + |\nabla v_n''|^2)^{\frac{r}{2}} - \frac{1}{r}\int_\Omega (\alpha + |\nabla v|^2)^{\frac{r}{2}} \\ &\quad - \int_\Omega [F(x, u_n'', v_n'') - F(x,u,v)]\, dx \\ &\geq \int_\Omega (\alpha + |\nabla u|^2)^{\frac{p-2}{2}} (\nabla u | \nabla u_n'' - \nabla u) + \frac{1}{(2^{p-1}-1)p}\int_\Omega |\nabla u_n'' - \nabla u|^p \\ &\quad + \int_\Omega (\alpha + |\nabla v|^2)^{\frac{r-2}{2}} (\nabla v | \nabla v_n'' - \nabla v) + \frac{1}{(2^{r-1}-1)r}\int_\Omega |\nabla v_n'' - \nabla v|^r \\ &\quad - \int_\Omega F(x, u_n'' - u, v_n'' - v) + o(1). \end{aligned}$$

Taking into account (1.7) we know that

$$\int_\Omega F(x, u_n'' - u, v_n'' - v) \leq c\left(\|u_n'' - u\|_{p^*}^{p^*} + \|v_n'' - v\|_{r^*}^{r^*}\right) + o(1).$$

Thus, from Sobolev inequality it follows that

$$\begin{aligned} J_\alpha(u_n'', v_n'') - J_\alpha(u,v) \geq\, & o(1) + \int_\Omega (\alpha + |\nabla u|^2)^{\frac{p-2}{2}} (\nabla u | \nabla u_n'' - \nabla u) \\ & + \|u_n'' - u\|_{1,p}^p \left(\frac{1}{(2^{p-1}-1)p} - c\|u_n'' - u\|_{1,p}^{p^*-p}\right) \\ & + \int_\Omega (\alpha + |\nabla v|^2)^{\frac{r-2}{2}} (\nabla v | \nabla v_n'' - \nabla v) \\ & + \|v_n'' - v\|_{1,r}^r \left(\frac{1}{(2^{r-1}-1)r} - c\|v_n'' - v\|_{1,r}^{r^*-r}\right). \end{aligned}$$

On the other hand, if $R > 0$ is small enough, considering that

$$\|u_n - u\|_{1,p} + \|v_n - v\|_{1,r} = \|z_n - z\| \leq \|z_n - z_0\| + \|z_0 - z\| < 2R,$$

it follows that

$$c\|u_n'' - u\|_{1,p}^{p^*-p} < \frac{1}{(2^{p-1}-1)p},$$

$$c\|v_n'' - v\|_{1,r}^{r^*-r} < \frac{1}{(2^{r-1}-1)r},$$

$$\int_\Omega (\alpha + |\nabla u|^2)^{\frac{p-2}{2}} (\nabla u | \nabla u_n'' - \nabla u) \to 0,$$

and

$$\int_\Omega (\alpha + |\nabla v|^2)^{\frac{r-2}{2}} (\nabla v | \nabla v_n'' - \nabla v) \to 0$$

which implies

$$\liminf_{n\to+\infty} J_\alpha(z_n) = \lim_{n\to+\infty} J_\alpha(z_n'') \geq J_\alpha(z).$$

Thus, the functional J_α is weakly lower semicontinuous in $\overline{B}_R(z_0)$. □

In order to prove a local Palais-Smale condition for J_α, we extend a technical result proved in [25] about the weak convergence of Palais-Smale sequences.

Lemma 3.4. *Let $z_n = (u_n, v_n)$ be a sequence in X weakly convergent to $z = (u, v) \in X$. Assume that $J'_\alpha(z_n) \to 0$, then $D_i u_n \to D_i u$ and $D_i v_n \to D_i v$ a.e. in Ω for any $i = 1, \ldots, N$. Moreover, the sequence $(\alpha + |\nabla u_n|^2)^{\frac{p-2}{2}} D_i u_n$ weakly converges to $(\alpha + |\nabla u|^2)^{\frac{p-2}{2}} D_i u$ in $L^{p/(p-2)}(\Omega)$ and $(\alpha + |\nabla v_n|^2)^{\frac{r-2}{2}} D_i v_n$ weakly converges to $(\alpha + |\nabla v|^2)^{\frac{r-2}{2}} D_i v$ in $L^{r/(r-2)}(\Omega)$ for any $i = 1, \ldots, N$.*

Proof. Observe that $u_n \to u$ weakly in $W_0^{1,p}(\Omega)$, $u_n \to u$ strongly in $L^\eta(\Omega)$ with $\eta < p^*$ and $u_n(x) \to u(x)$ a.e. in Ω. In the same way, $v_n \to v$ weakly in $W_0^{1,r}(\Omega)$, $v_n \to v$ strongly in $L^\eta(\Omega)$ with $\eta < r^*$ and $v_n(x) \to v(x)$ a.e. in Ω.

We extend u_n and v_n to $\mathbb{R}^N$ and we assume that for some $R > 0$ supp $u_n \subset B_R(0)$ and supp $v_n \subset B_R(0)$. In particular, the sequences $\{|\nabla u_n|^p\}$ and $\{|\nabla v_n|^r\}$ are tight. Thus, using [27], there exist bounded nonnegative measures μ, ν, τ, υ on $\mathbb{R}^N$ such that

$$|\nabla u_n|^p \to \mu \quad \text{weakly and } |u_n|^{p^*} \to \nu \quad \text{tightly} \tag{3.1}$$

and

$$|\nabla v_n|^r \to \tau \quad \text{weakly and } |v_n|^{r^*} \to \upsilon \quad \text{tightly.} \tag{3.2}$$

Lemma 1.1 of [27] states the existence of at most a countable set I and of the sequences $\{x_i\}_{i\in I} \subset \mathbb{R}^N$, $\{\mu_i\}_{i\in I}, \{\nu_i\}_{i\in I} \subset (0, +\infty)$ such that $\mu_i \geq \mathcal{S}_p \nu_i^{\frac{p}{p^*}}$ for every $i \in I$ and

$$\nu = |u|^{p^*} + \sum_{i\in I} \nu_i \delta_{x_i} \tag{3.3}$$

and

$$\mu \geq |\nabla u|^p + \sum_{i\in I} \mu_i \delta_{x_i}, \tag{3.4}$$

where δ_{x_i} denotes a Dirac measure. Similarly, there exist $I' \subset \mathbb{N}$, $\{y_i\}_{i\in I'} \subset \mathbb{R}^N$, $\{\tau_i\}_{i\in I'}, \{\upsilon_i\}_{i\in I'} \subset (0, \infty)$ such that $\tau_i \geq \mathcal{S}_r \upsilon_i^{\frac{r}{r^*}}$ and

$$\upsilon = |v|^{r^*} + \sum_{i\in I'} \upsilon_i \delta_{y_i} \tag{3.5}$$

and

$$\tau \geq |\nabla v|^r + \sum_{i\in I'} \tau_i \delta_{y_i}. \tag{3.6}$$

Now, we show that I and I' are finite. Indeed, we consider $\varphi \in C_0^\infty(B_1(0))$, $0 \leq \varphi(z) \leq 1$, and $\varphi(w) \equiv 1$ for $|w| \leq 1/2$. For a given $\varepsilon > 0$ we denote

$$\varphi_{\varepsilon,n} = \left(\varphi\left(\frac{w - x_i}{\epsilon}\right) u_n, \varphi\left(\frac{w - y_j}{\epsilon}\right) v_n \right),$$

where $x_i \in \{x_i\}_{i\in I}$ and $y_j \in \{y_i\}_{i\in I'}$. Since z_n is bounded in X and $J'_\alpha(z_n) \to 0$ then $\langle J'_\alpha(z_n), \varphi_{\varepsilon,n}\rangle = o(1)\|\varphi_{\varepsilon,n}\| = o(1)$ and we deduce

$$\begin{aligned} &\int_{\mathbb{R}^N} (\alpha + |\nabla u_n|^2)^{\frac{p-2}{2}} \left(|\nabla u_n|^2 \varphi\left(\frac{w - x_i}{\varepsilon}\right) + \nabla u_n \nabla \varphi\left(\frac{w - x_i}{\varepsilon}\right) u_n \right) \\ &+ \int_{\mathbb{R}^N} (\alpha + |\nabla v_n|^2)^{\frac{r-2}{2}} \left(|\nabla v_n|^2 \varphi\left(\frac{w - y_j}{\varepsilon}\right) + \nabla v_n \nabla \varphi\left(\frac{w - y_j}{\varepsilon}\right) v_n \right) \\ &= \int_{\mathbb{R}^N} (\mathrm{D}_u F(x, u_n, v_n), \mathrm{D}_v F(x, u_n, v_n)) \cdot \varphi_{\varepsilon,n} + o(1). \end{aligned} \tag{3.7}$$

Observe that, using Young inequality, we obtain, for an arbitrary $\delta > 0$,

$$\begin{aligned}\int_{\mathbb{R}^N} |\alpha + |\nabla u_n|^2|^{\frac{p-2}{2}} |\nabla u_n| \left|\nabla\varphi\left(\frac{w-x_i}{\varepsilon}\right)\right| |u_n| \\ &\le \int_{\mathbb{R}^N} |\alpha + |\nabla u_n|^2|^{\frac{p-1}{2}} \left|\nabla\varphi\left(\frac{w-x_i}{\varepsilon}\right)\right| |u_n| \\ &\le \delta \int_\Omega |\alpha + |\nabla u_n|^2|^{\frac{p}{2}} + C_\delta \int_{\mathbb{R}^N} \left|u_n \nabla\varphi\left(\frac{w-x_i}{\varepsilon}\right)\right|^p . \\ &\le \delta \int_\Omega |\alpha + |\nabla u_n|^2|^{\frac{p}{2}} + C_\delta \int_{B_\varepsilon(x_i)} |u_n|^p \\ &\le C\delta + C_\delta \int_{B_\varepsilon(x_i)} |u_n|^p.\end{aligned}$$

Similarly, we get

$$\int_{\mathbb{R}^N} |\alpha + |\nabla v_n|^2|^{\frac{r-2}{2}} |\nabla v_n| \left|\nabla\varphi\left(\frac{w-y_j}{\varepsilon}\right)\right| |v_n| \le C\delta + C_\delta \int_{B_\varepsilon(y_j)} |v_n|^r.$$

Thus, taking into account (1.5), (1.6), (3.1), (3.2) and taking limit in (3.7) for $n \to +\infty$, we obtain

$$\begin{aligned}\mu(B_{\frac{\epsilon}{2}(x_i)}) + \tau(B_{\frac{\epsilon}{2}(y_j)}) &\le \int_{B_\epsilon(x_i)} \varphi\left(\frac{w-x_i}{\varepsilon}\right) d\mu + \int_{B_\epsilon(y_j)} \varphi\left(\frac{w-y_j}{\varepsilon}\right) d\tau \\ &= \lim_{n\to\infty} \left(\int_{\mathbb{R}^N} |\nabla u_n|^p \varphi\left(\frac{w-x_i}{\varepsilon}\right) + \int_{\mathbb{R}^N} |\nabla v_n|^r \varphi\left(\frac{w-y_j}{\varepsilon}\right)\right) \\ &\le \limsup_{n\to\infty} \left(\int_{\mathbb{R}^N} (\alpha + |\nabla u_n|^2)^{\frac{p-2}{2}} |\nabla u_n|^2 \varphi\left(\frac{w-x_i}{\varepsilon}\right)\right. \\ &\quad \left. + \int_{\mathbb{R}^N} (\alpha + |\nabla v_n|^2)^{\frac{r-2}{2}} |\nabla v_n|^2 \varphi\left(\frac{w-y_j}{\varepsilon}\right)\right) \\ &\le C\delta + C_\delta \int_{B_\varepsilon(x_i)} |u|^p + C_\delta \int_{B_\varepsilon(y_j)} |v|^r \\ &\quad + C \limsup_{n\to\infty} \left(\int_{B_\epsilon(x_i)} \left(|u_n|^{p^*-1} + |v_n|^{r^* \frac{p^*-1}{p^*}} + 1\right) u_n\right. \\ &\quad \left. + \int_{B_\epsilon(y_j)} \left(|v_n|^{r^*-1} + |u_n|^{p^* \frac{r^*-1}{r^*}} + 1\right) v_n\right).\end{aligned}$$

Therefore, using Young inequality, (3.1), (3.2), (3.3) and (3.5), we have for ε small enough and $x_i \notin \{y_j\}_{j\in I'}, y_j \notin \{x_i\}_{i\in I}$, that

$$\begin{aligned}\mu(B_{\frac{\epsilon}{2}(x_i)}) + \tau(B_{\frac{\epsilon}{2}(y_j)}) &\le C\delta + C_\delta \int_{B_\varepsilon(x_i)} |u|^p + C_\delta \int_{B_\varepsilon(y_j)} |v|^r + C \int_{B_\epsilon(x_i)} |u|^{p^*} \\ &\quad + C\nu_i + C \int_{B_\epsilon(x_i)} |v|^{r^*} + C \int_{B_\epsilon(y_j)} |v|^{r^*} + C\upsilon_j + C \int_{B_\epsilon(y_j)} |u|^{p^*}.\end{aligned}$$

Passing to the limit as $\varepsilon \to 0$ in the previous inequality, using that $\mu(B_{\frac{\varepsilon}{2}(x_i)}) \ge \mu_i \ge \mathcal{S}_p {\nu_i}^{\frac{p}{p^*}}$ and $\tau(B_{\frac{\epsilon}{2}(y_j)}) \ge \tau_j \ge \mathcal{S}_r {\upsilon_j}^{\frac{r}{r^*}}$ we deduce

$$\mathcal{S}_p {\nu_i}^{\frac{p}{p^*}} + \mathcal{S}_r {\upsilon_j}^{\frac{r}{r^*}} \le C(\delta + \nu_i + \upsilon_j).$$

This implies that $\nu_i, \upsilon_j > c$ (observe that $\lim_{(s,t)\to(0,0)} \frac{s+t}{s^{\frac{p}{p^*}}+t^{\frac{r}{r^*}}} = 0$). Since ν and υ are finite measures, the above inequality, (3.3) and (3.5) imply that the sets $\{x_i\}_{i\in I} \setminus \{y_j\}_{j\in I'}$ and $\{y_j\}_{j\in I'} \setminus \{x_i\}_{i\in I}$ are finite.

Analogously for $x_i \in \{x_i\}_{i\in I} \cap \{y_j\}_{j\in I'}$ ($x_i = y_j$ for some $j \in I'$), we have

$$\mathcal{S}_p {\nu_i}^{\frac{p}{p^*}} + \mathcal{S}_r {\upsilon_j}^{\frac{r}{r^*}} \le C(\delta + 2(\nu_i + \upsilon_j)).$$

Reasoning as before we also conclude that $\{x_i\}_{i\in I} \cap \{y_j\}_{j\in I'}$ is finite which implies that I and I' are finite.

We denote by w_i, $i = 1, 2, \dots, k$ the elements of the finite set $\{x_i\}_{i\in I} \cup \{y_j\}_{j\in I'}$. Let $\epsilon_0 > 0$ fixed small enough such that $B_{\epsilon_0}(w_i) \cap B_{\epsilon_0}(w_j) = \emptyset$ for $i \ne j$, $1 \le i, j \le k$, and $\cup_{j=1}^k B_{\epsilon_0}(w_j) \subset B_{\frac{1}{2\epsilon_0}}(0)$. For every $0 < \epsilon \le \epsilon_0$, we define

$$A_\epsilon = B_{\frac{1}{2\epsilon_0}}(0) \setminus \cup_{j=1}^k B_\epsilon(w_j)$$

and

$$\psi_\epsilon = \varphi(\epsilon x) - \sum_{j=1}^k \varphi\left(\frac{x - w_j}{\epsilon}\right)$$

where $\varphi \in C_0^\infty(B_1(0))$, $\varphi(x) = 1$ for $|x| \le \frac{1}{2}$. Thus, $0 \le \psi_\epsilon \le 1$, supp $\psi_\epsilon \subset B_{\frac{1}{\epsilon}}(0)$ and

$$\psi_\epsilon(x) = \begin{cases} 0 & \text{if } x \in \cup_{j=1}^k B_{\frac{\epsilon}{2}}(w_j), \\ 1 & \text{if } x \in A_\epsilon. \end{cases} \tag{3.8}$$

Since $\{z_n\}$ is bounded in X, in view of supp z_n, it follows from the continuity of the embeddings $W_0^{1,p}(\mathbb{R}^N) \hookrightarrow L^{p^*}(\mathbb{R}^N)$, $W_0^{1,r}(\mathbb{R}^N) \hookrightarrow L^{r^*}(\mathbb{R}^N)$ and Holder's inequality that $\{\psi_\epsilon z_n\}$ is bounded in X. Therefore, we derive

$$\langle J'_\alpha(z_n), \psi_\epsilon z_n \rangle = o(1)\,(\|\psi_\epsilon z_n\|) = o(1),$$

namely

$$\begin{aligned} o(1) = &\int_{\mathbb{R}^N} (\alpha + |\nabla u_n|^2)^{\frac{p-2}{2}} \nabla u_n \left(u_n \nabla\psi_\epsilon + \psi_\epsilon \nabla u_n\right) \\ &+ \int_{\mathbb{R}^N} (\alpha + |\nabla v_n|^2)^{\frac{r-2}{2}} \nabla v_n \left(v_n \nabla\psi_\epsilon + \psi_\epsilon \nabla v_n\right) \\ &- \int_{\mathbb{R}^N} \left[\mathrm{D}_u F(x, u_n, v_n)\psi_\epsilon u_n + \mathrm{D}_v F(x, u_n, v_n)\psi_\epsilon v_n\right]. \end{aligned} \tag{3.9}$$

Similarly, we get

$$\begin{aligned} o(1) = &\int_{\mathbb{R}^N} (\alpha + |\nabla u_n|^2)^{\frac{p-2}{2}} \nabla u_n \left(u \nabla\psi_\epsilon + \psi_\epsilon \nabla u\right) \\ &+ \int_{\mathbb{R}^N} (\alpha + |\nabla v_n|^2)^{\frac{r-2}{2}} \nabla v_n \left(v \nabla\psi_\epsilon + \psi_\epsilon \nabla v\right) \\ &- \int_{\mathbb{R}^N} \left[\mathrm{D}_u F(x, u_n, v_n)\psi_\epsilon u + \mathrm{D}_v F(x, u_n, v_n)\psi_\epsilon v\right]. \end{aligned} \tag{3.10}$$

We claim now that

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N} \mathrm{D}_u F(x, u_n, v_n)\psi_\epsilon u_n = \int_{\mathbb{R}^N} \mathrm{D}_u F(x, u, v)\psi_\epsilon u. \tag{3.11}$$

Indeed, using (1.5) and Lemma 3.2 successively with $G(x,u,v) = D_u F(x,u,v) u\psi_\epsilon$ and $G(x,u,v) = \left(|u|^{p^*} + |v|^{r^*}\right)\psi_\epsilon$, we deduce

$$\begin{aligned}\left|\int_{\mathbb{R}^N} \mathrm{D}_u F(x,u_n,v_n)\psi_\epsilon u_n - \int_{\mathbb{R}^N} \mathrm{D}_u F(x,u,v)\psi_\epsilon u\right| &= o(1) \\ &\quad + \left|\int_{\mathbb{R}^N} \mathrm{D}_u F(x,u_n-u,v_n-v)(u_n-u)\psi_\epsilon\right| \\ &\le o(1) + \int_{\mathbb{R}^N} |u_n-u|^{p^*}\psi_\epsilon + \int_{\mathbb{R}^N} |v_n-v|^{r^*}\psi_\epsilon \\ &= o(1) + \int_{\mathbb{R}^N} |u_n|^{p^*}\psi_\epsilon - \int_{\mathbb{R}^N} |u|^{p^*}\psi_\epsilon + \int_{\mathbb{R}^N} |v_n|^{r^*}\psi_\epsilon - \int_{\mathbb{R}^N} |v|^{r^*}\psi_\epsilon.\end{aligned}$$

Taking into account (3.3), (3.5) and (3.8) we have

$$\left|\int_{\mathbb{R}^N} \mathrm{D}_u F(x,u_n,v_n)\psi_\epsilon u_n - \int_{\mathbb{R}^N} \mathrm{D}_u F(x,u,v)\psi_\epsilon u\right| = o(1),$$

and the claim is proved. Similarly, we observe

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N} \mathrm{D}_v F(x,u_n,v_n)\psi_\epsilon v_n = \int_{\mathbb{R}^N} \mathrm{D}_u F(x,u,v)\psi_\epsilon v. \tag{3.12}$$

For an arbitrary $\delta > 0$, from definition of ψ_ϵ and Hölder inequality we get

$$\begin{aligned}\int_{\mathbb{R}^N} (\alpha + |\nabla u_n|^2)^{\frac{p-2}{2}} \nabla u_n \cdot u_n \nabla\psi_\epsilon + \int_{\mathbb{R}^N} (\alpha + |\nabla v_n|^2)^{\frac{r-2}{2}} \nabla v_n \cdot v_n \nabla\psi_\epsilon \\ \le \delta \int_{\mathbb{R}^N} \left((\alpha + |\nabla u_n|^2)^{\frac{p-2}{2}} |\nabla u_n|\right)^{\frac{p}{p-1}} + c_3 \int_{\mathbb{R}^N} |u_n|^p |\nabla\psi_\epsilon|^p \\ + \delta \int_{\mathbb{R}^N} \left((\alpha + |\nabla v_n|^2)^{\frac{r-2}{2}} |\nabla v_n|\right)^{\frac{r}{r-1}} + c_3 \int_{\mathbb{R}^N} |v_n|^r |\nabla\psi_\epsilon|^r \\ \le C\delta + C \int_{\mathbb{R}^N} |u|^p |\nabla\psi_\epsilon|^p + C \int_{\mathbb{R}^N} |v|^r |\nabla\psi_\epsilon|^r \\ \le C\delta + C\epsilon^p \|\nabla\varphi\|_\infty^p \|u\|_p^p + Ck\epsilon^{N-p} \|\nabla\varphi\|_N^p \|u\|_{p^*}^p + C\epsilon^r \|\nabla\varphi\|_\infty^r \|u\|_r^r \\ + Ck\epsilon^{N-r} \|\nabla\varphi\|_N^r \|u\|_{r^*}^r \le C\delta + o_\epsilon(1).\end{aligned} \tag{3.13}$$

Using (3.13) in (3.9) and arguing similarly with (3.10) we derive

$$\begin{aligned}C\delta + o_\epsilon(1) \ge \limsup_{n\to+\infty} \Bigg(\int_{\mathbb{R}^N} \left[(\alpha + |\nabla u_n|^2)^{\frac{p-2}{2}} |\nabla u_n|^2 - \mathrm{D}_u F(x,u_n,v_n) u_n\right] \psi_\epsilon \\ + \int_{\mathbb{R}^N} \left[(\alpha + |\nabla v_n|^2)^{\frac{r-2}{2}} |\nabla v_n|^2 - \mathrm{D}_v F(x,u_n,v_n) v_n\right] \psi_\epsilon \Bigg).\end{aligned} \tag{3.14}$$

$$\begin{aligned}C\delta + o_\epsilon(1) \ge \limsup_{n\to+\infty} \Bigg(\int_{\mathbb{R}^N} \left[(\alpha + |\nabla u_n|^2)^{\frac{p-2}{2}} \nabla u_n \nabla u - \mathrm{D}_u F(x,u_n,v_n) u\right] \psi_\epsilon \\ + \int_{\mathbb{R}^N} \left[(\alpha + |\nabla v_n|^2)^{\frac{r-2}{2}} \nabla v_n \nabla v - \mathrm{D}_v F(x,u_n,v_n) v\right] \psi_\epsilon \Bigg).\end{aligned} \tag{3.15}$$

We note that from Clarkson inequality (Lemma 3.1) it follows that

$$0 \le \sum_{i=1}^{N} \left((\alpha+|\nabla u_n|^2)^{\frac{p-2}{2}}\mathrm{D}_i u_n - (\alpha+|\nabla u|^2)^{\frac{p-2}{2}}\mathrm{D}_i u\right)(\mathrm{D}_i u_n - \mathrm{D}_i u) \equiv \sum_{i=1}^{N} V_{1,in}$$

$$0 \le \sum_{i=1}^{N} \left((\alpha+|\nabla v_n|^2)^{\frac{r-2}{2}}\mathrm{D}_i v_n - (\alpha+|\nabla v|^2)^{\frac{r-2}{2}}\mathrm{D}_i v\right)(\mathrm{D}_i v_n - \mathrm{D}_i v) \equiv \sum_{i=1}^{N} V_{2,in}.$$

Therefore, we have

$$\begin{aligned}
0 &\le \int_{A_\epsilon}\sum_{i=1}^{N} V_{1,in} + \int_{A_\epsilon}\sum_{i=1}^{N} V_{2,in} \le \int_{\mathbb{R}^N}\sum_{i=1}^{N}(V_{1,in}\psi_\epsilon + V_{2,in}\psi_\epsilon)\\
&= \int_{\mathbb{R}^N}\left[(\alpha+|\nabla u_n|^2)^{\frac{p-2}{2}}|\nabla u_n|^2 - (\alpha+|\nabla u_n|^2)^{\frac{p-2}{2}}\nabla u_n\nabla u\right]\psi_\epsilon\\
&\quad + \int_{\mathbb{R}^N}\left[(\alpha+|\nabla u|^2)^{\frac{p-2}{2}}|\nabla u|^2 - (\alpha+|\nabla u|^2)^{\frac{p-2}{2}}\nabla u\nabla u_n\right]\psi_\epsilon\\
&\quad + \int_{\mathbb{R}^N}\left[(\alpha+|\nabla v_n|^2)^{\frac{r-2}{2}}|\nabla v_n|^2 - (\alpha+|\nabla v_n|^2)^{\frac{r-2}{2}}\nabla v_n\nabla v\right]\psi_\epsilon\\
&\quad + \int_{\mathbb{R}^N}\left[(\alpha+|\nabla v|^2)^{\frac{r-2}{2}}|\nabla v|^2 - (\alpha+|\nabla v|^2)^{\frac{r-2}{2}}\nabla v\nabla v_n\right]\psi_\epsilon\\
&\quad - \int_{\mathbb{R}^N}[\mathrm{D}_u F(x,u_n,v_n)u_n - \mathrm{D}_u F(x,u_n,v_n)u]\,\psi_\epsilon\\
&\quad - \int_{\mathbb{R}^N}[\mathrm{D}_v F(x,u_n,v_n)v_n - \mathrm{D}_v F(x,u_n,v_n)v]\,\psi_\epsilon.
\end{aligned}$$

We conclude from (3.11), (3.12), (3.14), (3.15), and the weak convergence of z_n to z that

$$\limsup_{n\to+\infty}\left(\int_{A_\epsilon}\sum_{i=1}^{N} V_{1,in} + \int_{A_\epsilon}\sum_{i=1}^{N} V_{2,in}\right) \le 2C\delta + o_\epsilon(1).$$

Thus, we get

$$\lim_{n\to+\infty}\int_{A_\epsilon}\left((\alpha+|\nabla u_n|^2)^{\frac{p-2}{2}}\nabla u_n - (\alpha+|\nabla u|^2)^{\frac{p-2}{2}}\nabla u\right)(\nabla u_n - \nabla u) = 0$$

and

$$\lim_{n\to+\infty}\int_{A_\epsilon}\left((\alpha+|\nabla v_n|^2)^{\frac{r-2}{2}}\nabla v_n - (\alpha+|\nabla v|^2)^{\frac{r-2}{2}}\nabla v\right)(\nabla v_n - \nabla v) = 0.$$

Since ϵ is arbitrarily small, we deduce

$$\lim_{n\to+\infty}\int_{\Omega}\left((\alpha+|\nabla u_n|^2)^{\frac{p-2}{2}}\nabla u_n - (\alpha+|\nabla u|^2)^{\frac{p-2}{2}}\nabla u\right)(\nabla u_n - \nabla u) = 0$$

and

$$\lim_{n\to+\infty}\int_{\Omega}\left((\alpha+|\nabla v_n|^2)^{\frac{r-2}{2}}\nabla v_n - (\alpha+|\nabla v|^2)^{\frac{r-2}{2}}\nabla v\right)(\nabla v_n - \nabla v) = 0.$$

It follows that for any $i = 1,\dots,N$, $D_i u_n \to D_i u$ and $D_i v_n \to D_i v$ a.e. in Ω, as $n \to +\infty$. □

Proposition 3.5. *There exists $R > 0$ such that, for any fixed $\alpha \ge 0$ and any $z_0 \in X$, the functional J_α satisfies $(P.S.)$ condition on $\overline{B}_R(z_0) = \{w \in X \mid \|w - z_0\| \le R\}$.*

Proof. Let $R > 0$ and $z_n \in \overline{B}_R(z_0)$ be a sequence such that $J'_\alpha(z_n) \to 0$, then since z_n is bounded, up to subsequences, it weakly converges in X to some $\overline{z} = (\overline{u}, \overline{v}) \in \overline{B}_R(z)$. Therefore $u_n \to \overline{u}$ weakly in $W^{1,p}_0(\Omega)$, $u_n \to \overline{u}$ strongly in $L^t(\Omega)$ with $t < p^*$, $u_n(x) \to \overline{u}(x)$ a.e. in Ω, $v_n \to \overline{v}$ weakly in $W^{1,r}_0(\Omega)$, $v_n \to \overline{v}$ strongly in $L^t(\Omega)$ with $t < r^*$ and $v_n(x) \to \overline{v}(x)$ a.e. in Ω.

Using Lemma 3.2 and Lemma 3.4 we can show that, for every $z = (\varphi, \psi) \in X$

$$\begin{aligned}0 = \lim_{n\to+\infty} \langle J'(z_n), z\rangle =& \int_\Omega (\alpha + |\nabla \overline{u}|^2)^{\frac{p-2}{2}} \nabla \overline{u} \nabla \varphi + \int_\Omega (\alpha + |\nabla \overline{v}|^2)^{\frac{r-2}{2}} \nabla \overline{v} \nabla \psi \\ &- \int_\Omega \mathrm{D}_u F(x, \overline{u}, \overline{v})\varphi - \int_\Omega \mathrm{D}_v F(x, \overline{u}, \overline{v})\psi \\ =& \langle J'(\overline{z}), z\rangle.\end{aligned}$$

Moreover, the almost everywhere convergence of the gradients in Lemma 3.4 allows us to deduce

$$\begin{aligned}&\int_\Omega (\alpha + |\nabla u_n - \nabla \overline{u}|^2)^{\frac{p-2}{2}} |\nabla u_n - \nabla \overline{u}|^2 \\ &= \int_\Omega (\alpha + |\nabla u_n|^2)^{\frac{p-2}{2}} |\nabla u_n|^2 - \int_\Omega (\alpha + |\nabla \overline{u}|^2)^{\frac{p-2}{2}} |\nabla \overline{u}|^2 + o(1)\end{aligned}$$

and

$$\begin{aligned}&\int_\Omega (\alpha + |\nabla v_n - \nabla \overline{v}|^2)^{\frac{r-2}{2}} |\nabla v_n - \nabla \overline{v}|^2 \\ &= \int_\Omega (\alpha + |\nabla v_n|^2)^{\frac{r-2}{2}} |\nabla v_n|^2 - \int_\Omega (\alpha + |\nabla \overline{v}|^2)^{\frac{r-2}{2}} |\nabla \overline{v}|^2 + o(1).\end{aligned}$$

Observe that, using Lemma 3.2, we have

$$\begin{aligned}&\int_\Omega \mathrm{D}_u F(x, u_n - \overline{u}, v_n - \overline{v})(u_n - \overline{u}) + \mathrm{D}_v F(x, u_n - \overline{u}, v_n - \overline{v})(v_n - \overline{v}) \\ &\qquad = \int_\Omega \mathrm{D}_u F(x, u_n, v_n) u_n + \mathrm{D}_v F(x, u_n, v_n) v_n \\ &\qquad\quad - \int_\Omega \mathrm{D}_u F(x, \overline{u}, \overline{v})\overline{u} + \mathrm{D}_v F(x, \overline{u}, \overline{v})\overline{v} + o(1).\end{aligned}$$

Thus, we get

$$\begin{aligned}o(1) =& \langle J'_\alpha(z_n), z_n\rangle - \langle J'_\alpha(\overline{z}), \overline{z}\rangle = o(1) + \int_\Omega (\alpha + |\nabla u_n - \nabla \overline{u}|^2)^{\frac{p-2}{2}} |\nabla u_n - \nabla \overline{u}|^2 \\ &+ \int_\Omega (\alpha + |\nabla v_n - \nabla \overline{v}|^2)^{\frac{r-2}{2}} |\nabla v_n - \nabla \overline{v}|^2 \\ &+ \int_\Omega \mathrm{D}_u F(x, u_n - \overline{u}, v_n - \overline{v})(u_n - \overline{u}) + \mathrm{D}_v F(x, u_n - \overline{u}, v_n - \overline{v})(v_n - \overline{v}) \\ \geq& \, o(1) + \|u_n - \overline{u}\|^p_{1,p} + \|v_n - \overline{v}\|^r_{1,r} - c_1 \|u_n - \overline{u}\|^{p^*}_{L^{p^*}(\Omega)} - c_2 \|v_n - \overline{v}\|^{r^*}_{L^{r^*}(\Omega)} \\ \geq& \, o(1) + \|u_n - \overline{u}\|^p_{1,p} \left(1 - \mathcal{S}_p c_1 \|u_n - \overline{u}\|^{p^*-p}_{1,p}\right) \\ &+ \|v_n - \overline{v}\|^r_{1,r} \left(1 - \mathcal{S}_r c_2 \|v_n - \overline{v}\|^{r^*-r}_{1,r}\right).\end{aligned}$$

Choosing $R > 0$, small enough, we derive that z_n strongly converges to $\overline{z}$ in X. $\square$

Remark 3.6. For any $f, g \in C^1(\overline{\Omega})$ we can consider $J_{\alpha,f,g}(u,v) = J_\alpha(u,v) - \int_\Omega fu - \int_\Omega gv$. From the previous proof, it is clear that Proposition 3.5 still holds for $J_{\alpha,f,g}$, with the same R required by J_α. In particular $J_{\alpha,f,g}$ satisfies $(P.S.)$ on each ball $\overline{B}_R(z_0)$, where R is independent from $\alpha \geq 0$, $z_0 \in X$ and $f,\ g \in C^1(\overline{\Omega})$.

4. The finite dimensional reduction

In this section we perform a finite dimensional reduction which will be useful in order to compute the critical groups for J_α at a critical point. We fix $\alpha > 0$ and $z_0 = (u_0, v_0) \in X$ a critical point of J_α. By Theorem 1.1 we have that $u_0, v_0 \in C^{1,\eta}(\overline{\Omega})$ for some $\eta \in (0,1)$. Now we can introduce a Hilbert space H_0, depending on the critical point z_0, in which X is embedded, so that a suitable splitting can be obtained. Precisely, let H_0 be the closure of $C_0^\infty(\Omega) \times C_0^\infty(\Omega)$ under the scalar product

$$\begin{aligned}(z_1, z_2)_0 = &\int_\Omega \Big((\alpha + |\nabla u_0|^2)^{(p-2)/2} (\nabla u_1 | \nabla u_2) \\ &+ (p-2)(\alpha + |\nabla u_0|^2)^{(p-4)/2} (\nabla u_0|\nabla u_1)(\nabla u_0|\nabla u_2) \Big)\, dx \\ &+ \int_\Omega \Big((\alpha + |\nabla v_0|^2)^{(r-2)/2} (\nabla v_1 | \nabla v_2) \\ &+ (r-2)(\alpha + |\nabla v_0|^2)^{(r-4)/2} (\nabla v_0|\nabla v_1)(\nabla v_0|\nabla v_2) \Big)\, dx,\end{aligned}$$

for any $z_1 = (u_1, v_1), z_2 = (u_2, v_2) \in H_0$.

As $u_0, v_0 \in C^{1,\eta}(\overline{\Omega})$, the norm $\|\cdot\|_0$ induced by $(\cdot,\cdot)_0$ is equivalent to the usual norm $\|\cdot\|_{1,2} + \|\cdot\|_{1,2}$ of $H_0^1(\Omega) \times H_0^1(\Omega)$. Hence H_0 is isomorphic to $H_0^1(\Omega) \times H_0^1(\Omega)$ and the embedding $W_0^{1,p}(\Omega) \times W_0^{1,r}(\Omega) \hookrightarrow H_0$ is continuous.
Denoting by $\langle \cdot, \cdot \rangle : H_0^* \times H_0 \to \mathbb{R}$ the duality pairing in H_0, $J''_\alpha(z_0)$ can be extended to the operator $L_0 : H_0 \to H_0^*$ defined by setting

$$\langle L_0 z_1, z_2 \rangle = (z_1, z_2)_0 \ - \langle K z_1, z_2 \rangle$$

for any $z_1 = (u_1, v_1), z_2 = (u_2, v_2) \in H_0$, where

$$\begin{aligned}\langle K z_1, z_2 \rangle = &\int_\Omega \big(D^2_{uu} F(x, u_0, v_0) u_1 u_2 + D^2_{vv} F(x, u_0, v_0) v_1 v_2 \\ &+ D^2_{uv} F(x, u_0, v_0) u_1 v_2 + D^2_{vu} F(x, u_0, v_0) u_2 v_1 \big)\, dx.\end{aligned}$$

Lemma 4.1. *L_0 is a compact perturbation of the Riesz isomorphism from H_0 to H_0^*. In particular, L_0 is a Fredholm operator with index zero.*

Proof. In order to prove the assertion it is sufficient to show that K is a compact operator from H_0 to H_0^*. Let $z_n = (u_n, v_n)$ be a bounded sequence in H_0. Then there exists $z = (u, v) \in H_0$ such that, up to a subsequence, u_n converges weakly to u in $H_0^1(\Omega)$ and strongly in $L^2(\Omega)$, v_n converges weakly to v in $H_0^1(\Omega)$ and strongly in $L^2(\Omega)$. There is a constant $C > 0$ such that, for any $w = (w_1, w_2) \in H_0$,

$\|w\|_0 = 1$ we have

$$\left|\int_\Omega D^2_{uu}F(x,u_0,v_0)\,(u_n-u)w_1 + D^2_{vv}F(x,u_0,v_0)(v_n-v)w_2 \right.$$
$$\left. + \int_\Omega D^2_{uv}F(x,u_0,v_0)(u_n-u)w_2 + D^2_{vu}F(x,u_0,v_0)(v_n-v)w_1\right|$$
$$\le C\int_\Omega \left(|u_n-u|^2 + |v_n-v|^2\right)dx$$

which tends to zero as $n\to+\infty$, uniformly with respect to w. This implies that K is a compact operator. □

Now let us denote by $m(L_0)$ the maximal dimension of a subspace of H_0 on which L_0 is negative definite. Obviously $m(J_\alpha, z_0) \le m(L_0)$. Furthermore let us denote by $m^*(L_0)$ the sum of $m(L_0)$ and the dimension of the kernel of L_0. By Lemma 4.1 we conclude that $m(L_0)$ and $m^*(L_0)$ are finite.
Since L_0 is a Fredholm operator in H_0, we can consider the natural splitting

$$H_0 = H^- \oplus H^0 \oplus H^+$$

where $H^-,\ H^0,\ H^+$ are, respectively, the negative, null and positive spaces, according to the spectral decomposition of L_0 in $L^2(\Omega)\times L^2(\Omega)$. Therefore one can easily show that there exists $\gamma_0>0$ such that

$$\langle L_0 v, v\rangle \ \ge \gamma_0\|v\|_0^2 \quad \forall v\in H^+.$$

Moreover,

$$\langle L_0 v, w\rangle = 0 \qquad \forall\, v\in H^-\oplus H^0 \quad \forall\, w\in H^+ \tag{4.1}$$

and $m(L_0)$ and $m^*(L_0)$ are respectively the dimensions of H^- and $H^-\oplus H^0$. Since $u_0, v_0 \in C^{1,\eta}(\overline{\Omega})$, we derive that $H^-\oplus H^0 \subset X\cap \left(C^{1,\beta}(\overline{\Omega})\times C^{1,\beta}(\overline{\Omega})\right)$ for some $\beta\in(0,1)$ (see [19]).

Consequently, if we denote by $W = H^+\cap X$ and $V = H^-\oplus H^0$, we get the splitting $X = V\oplus W$ and

$$\langle J''_\alpha(z_0)w, w\rangle \ge \gamma_0\|w\|_0^2 \quad \forall w\in W, \tag{4.2}$$

so that $m(L_0) = m(J_\alpha, z_0)$ and $m^*(L_0) = m^*(J_\alpha, z_0)$.

The next proposition states a sort of local $(P.S.)$ condition for J_α in the direction of W.

Proposition 4.2. *There exists $R>0$ such that, for any fixed $\alpha>0$ and $z\in X$, if $z_m\in \overline{B}_R(z)$ and*

$$\sup_{w\in W\setminus\{0\}} \langle {J_\alpha}'(z_m), w\rangle/\|w\|_0 \ \to\ 0 \quad m\to+\infty, \tag{4.3}$$

then z_m has a convergent subsequence.

Proof. Observe that fixed $R>0$ and $\alpha>0$, up to a subsequence, z_m weakly converges to some $\overline{z}\in\overline{B}_R(z)$ and, as in the proof of Proposition 3.5, for any $\phi\in X$,

$$\langle J'_\alpha(z_m), \phi\rangle \ \to\ \langle J'_\alpha(\overline{z}), \phi\rangle. \tag{4.4}$$

Now let $\{(e_{1,1}, e_{1,2}), \dots, (e_{m^*,1}, e_{m^*,2})\}$ be an $L^2(\Omega)\times L^2(\Omega)$-orthogonal basis of V, where $m^* = m^*(J_\alpha, z_0)$.

For any $\phi = (\phi_1, \phi_2) \in X$ we denote

$$P_V(\phi) = \left(\sum_{i=1}^{m^*} \left(\int_\Omega e_{i,1}\phi_1 \, dx \right) e_{i,1}, \sum_{i=1}^{m^*} \left(\int_\Omega e_{i,2}\phi_2 \, dx \right) e_{i,2} \right).$$

It is clear that $z - P_V(z) \in W$, for any $z \in X$. Moreover $P_V(z_m)$ strongly converges to $P_V(\overline{z})$. Exploiting (4.3) and (4.4) we get

$$\begin{aligned} \langle J'_\alpha(z_m), z_m\rangle - \langle J'_\alpha(\overline{z}), \overline{z}\rangle =& \langle J'_\alpha(z_m), z_m - \overline{z}\rangle\rangle + \langle J'_\alpha(z_m), \overline{z}\rangle\rangle - \langle J'_\alpha(\overline{z}), \overline{z}\rangle \\ =& \langle J'_\alpha(z_m), P_V(z_m) - P_V(\overline{z})\rangle\rangle \\ &+ \langle J'_\alpha(z_m), z_m - \overline{z} - P_V(z_m - \overline{z})\rangle\rangle + o(1) = o(1). \end{aligned}$$

The thesis follows now arguing as in the proof of Proposition 3.5. □

The following lemma, found in [21], allow us to study some kind of local convexity in the direction of W.

Lemma 4.3. *Let $I : L^p(\Omega, \mathbb{R}^k) \times L^q(\Omega, \mathbb{R}^m) \to]-\infty, +\infty]$ be a functional of the form*

$$I(u, v) = \int_\Omega \phi(x, u, v) \, dx$$

where $\phi(x, u, v)$ is a nonnegative, continuous function and $\phi(x, u, \cdot)$ is convex. Then I is lower semicontinuous with respect to the strong convergence of the component u in L^p and with respect to the weak convergence of the component v in L^q. □

In what follows, we denote respectively by $\|\cdot\|_{C^1}$ and $\|\cdot\|_{C^{1,\beta}}$ the usual norm of $C^1(\overline{\Omega}) \times C^1(\overline{\Omega})$ and $C^{1,\beta}(\overline{\Omega}) \times C^{1,\beta}(\overline{\Omega})$.

Lemma 4.4. *For any $K > 0$ there exist $R_0 > 0$ and $C > 0$ such that, for any $z = (z_1, z_2) \in C^1(\overline{\Omega}) \times C^1(\overline{\Omega})$, with $\|z\|_{C^1} \le K$ and $\|z - z_0\| < R_0$, we have*

$$\langle J''_\alpha(z)w, w\rangle \ge C\|w\|_0^2 \quad \forall w \in W. \tag{4.5}$$

Proof. By contradiction, there exist $K > 0$ and two sequences $z_n = (u_n, v_n) \in C^1(\overline{\Omega}) \times C^1(\overline{\Omega})$ and $w_n = (w_{n,1}, w_{n,2}) \in W$ such that $\|w_n\|_0 = 1$, $\|z_n\|_{C^1} \le K$, $\|z_n - z_0\| = \|u_n - u_0\|_{1,p} + \|v_n - v_0\|_{1,r} \to 0$ and

$$\langle J''_\alpha(z_n)w_n, w_n\rangle < \frac{1}{n}. \tag{4.6}$$

Firstly we observe that, for a suitable positive constant C_0, we have

$$\begin{aligned} \langle J''_\alpha(z_n)w_n, w_n\rangle \ge \quad & C_0\|w_n\|_0^2 - \int_\Omega \big(D^2_{uu}F(x, u_n, v_n)w_{n,1}^2 + D^2_{vv}F(x, u_n, v_n)w_{n,2}^2 \\ & + 2D^2_{uv}F(x, u_n, v_n)w_{n,1}w_{n,2}\big). \end{aligned}$$

Since w_n is bounded in H_0, it converges to some $\bar{w}$ weakly in H_0 and strongly in $L^2(\Omega) \times L^2(\Omega)$. Moreover, using that $F(x, \cdot, \cdot)$ is C^2, we derive

$$\begin{aligned} &\lim_{n\to\infty} \int_\Omega \big(D^2_{uu}F(x, u_n, v_n)w_{n,1}^2 + D^2_{vv}F(x, u_n, v_n)w_{n,2}^2 + 2D^2_{uv}F(x, u_n, v_n)w_{n,1}w_{n,2}\big) \\ &\quad = \int_\Omega \big(D^2_{uu}F(x, u_0, v_0)\overline{w}_1^2 + D^2_{vv}F(x, u_0, v_0)\overline{w}_2^2 + 2D^2_{uv}F(x, u_0, v_0)\overline{w}_1\overline{w}_2\big). \end{aligned}$$

This in particular implies that $\overline{w} \neq 0$ since in other case we should have $C_0 \leq 0$. By Lemma 4.3 we have

$$\begin{aligned}
&\int_\Omega \Big((\alpha + |\nabla u_0|^2)^{(p-2)/2} |\nabla \overline{w}_1|^2 + (p-2)(\alpha + |\nabla u_0|^2)^{(p-4)/2} (\nabla u_0 | \nabla \overline{w}_1)^2 \Big)\, dx \\
&+ \int_\Omega \Big((\alpha + |\nabla v_0|^2)^{(r-2)/2} |\nabla \overline{w}_2|^2 + (r-2)(\alpha + |\nabla v_0|^2)^{(r-4)/2} (\nabla v_0 | \nabla \overline{w}_2)^2 \Big)\, dx \\
&\leq \liminf_{n\to\infty} \int_\Omega \Big((\alpha + |\nabla u_n|^2)^{(p-2)/2} |\nabla w_{n,1}|^2 \\
&\qquad\qquad + (p-2)(\alpha + |\nabla u_n|^2)^{(p-4)/2} (\nabla u_n | \nabla w_{n,1})^2 \Big)\, dx \\
&\quad + \liminf_{n\to\infty} \int_\Omega \Big((\alpha + |\nabla v_n|^2)^{(r-2)/2} |\nabla w_{n,2}|^2 \\
&\qquad\qquad + (r-2)(\alpha + |\nabla v_n|^2)^{(r-4)/2} (\nabla v_n | \nabla w_{n,2})^2 \Big)\, dx\,.
\end{aligned}$$

This inequality jointly with (4.2) and (4.6) implies $\gamma_0 \|\overline{w}\|_0^2 \leq 0$, which is a contradiction because $\bar{w} \neq 0$. □

Following the same arguments of Lemma 4.5 in [10] and Lemma 4.2 in [11], we yield that z_0 is a strict minimum point in the direction of W.

Lemma 4.5. *There exist $\delta > 0$ and $\mu' > 0$ such that, for any $w = (w_1, w_2) \in W$ with $\|w\| \leq \delta$, we have*

$$J_\alpha(z_0 + w) - J_\alpha(z_0) \geq \mu' \left(\|w_1\|^p + \|w_2\|^r \right).$$

In particular, for any $\delta' \in (0, \delta)$, there is a constant $k_{\delta'} > 0$ such that

$$J_\alpha(z_0 + w) - J_\alpha(z_0) \geq k_{\delta'} \qquad \forall\, w \in W,\ \|w\| = \delta'.$$

□

We can now prove the following result which is crucial for developing a finite dimensional reduction.

Proposition 4.6. *There exist $r_0 > 0$ and $\rho \in (0, r_0)$ such that for each v in $V \cap \overline{B}_\rho(0)$, there exists one and only one $\overline{w} \in W \cap B_{r_0}(0) \cap \big(C^1(\overline{\Omega}) \times C^1(\overline{\Omega})\big)$ such that for any $w \in W \cap \overline{B}_{r_0}(0)$ we have*

$$J_\alpha(z_0 + v + \overline{w}) \leq J_\alpha(z_0 + v + w).$$

Moreover, $\overline{w}$ is the only element of $W \cap \overline{B}_{r_0}(0)$ such that

$$\langle J'_\alpha(z_0 + v + \overline{w}), w \rangle = 0, \qquad \forall w \in W$$

and

$$S(r_0, \rho) = \big\{ z_0 + v + w \mid v \in V \cap \overline{B}_\rho(0),\ w \in W \cap \overline{B}_{r_0}(0) \big\} \subset B_R(z_0) \tag{4.7}$$

where R is defined by Proposition 3.3 and Proposition 4.2.

Proof. Let us denote again $m^* = m^*(J_\alpha, z_0)$ and $e_i = (e_{i,1}, e_{i,2})$, $i = 1, \dots m^*$, an $L^2(\Omega) \times L^2(\Omega)$ - orthogonal basis of $V \subset C^1(\overline{\Omega}) \times C^1(\overline{\Omega})$.

Let $\delta > 0$ be defined by Lemma 4.5 and $z = (u, v) \in B_\delta(z_0)$ be a solution of $\langle J'_\alpha(z), w \rangle = 0$ for any $w \in W$. Denoting by $f_{z,1}(x) = \sum_{i=1}^{m^*} \langle J'_\alpha(z), e_i \rangle e_{i,1}(x)$, and

$f_{z,2}(x) = \sum_{i=1}^{m^*} \langle J'_\alpha(z), e_i\rangle e_{i,2}(x)$, z solves the system

$$\begin{cases} -\mathrm{div}\,\big((\alpha + |\nabla u|^{p-2})\nabla u\big) = \mathrm{D}_u F(x,u,v) + f_{z,1}(x) & x \in \Omega \\ -\mathrm{div}\,\big((\alpha + |\nabla v|^{r-2})\nabla v\big) = \mathrm{D}_v F(x,u,v) + f_{z,2}(x) & x \in \Omega \\ u = v = 0 & x \in \partial\Omega. \end{cases}$$

There exists M_1 depending just on δ such that $f_{z,i} \in C^1(\overline{\Omega})$ and $\|f_{z,i}\|_{C^1(\overline{\Omega})} \le M_1$. Reasoning as in Theorem 1.1 (see Remark 2.1), $z \in C^1(\overline{\Omega}) \times C^1(\overline{\Omega})$ and $\|z\|_{C^1} \le M$, where $M > 0$ depends on δ. Now by Lemma 4.4 in correspondence of M there exist $R_0 \in (0,\delta)$ and $C > 0$ such that (4.5) holds.

We fix $r_0 \in \big(0, \min\big\{\frac{R}{2}, \frac{R_0}{2}\big\}\big)$ so that (4.7), Proposition 3.3 and Proposition 4.2 hold in $S(r_0,\rho)$, for any $\rho \in (0, r_0)$. In particular, for any fixed $v \in V \cap B_{r_0}(0)$, there is a minimum point $\bar w \in W \cap \overline{B}_{r_0}(0)$ of the function $w \in W \cap \overline{B}_{r_0}(0) \mapsto J_\alpha(z_0+v+w)$.

We claim that there exists $\rho \in (0, r_0)$ such that for any $v \in V \cap \overline{B}_\rho(0)$

$$\inf\{J_\alpha(z_0 + v + w) \,|\, w \in W, \|w\| = r_0\} > J_\alpha(z_0 + v). \tag{4.8}$$

Indeed, arguing by contradiction, we assume that there exist $w_n \in W$ and $v_n \in V$ such that $\|w_n\| = r_0$, $\|v_n\| \to 0$ and

$$J_\alpha(z_0 + v_n + w_n) \le J_\alpha(z_0 + v_n) + o(1). \tag{4.9}$$

Moreover, $J_\alpha(z_0 + v_n + w_n) - J_\alpha(z_0 + w_n) = \langle J'_\alpha(z_0 + \beta_n v_n + w_n), v_n\rangle$, where $\beta_n \in (0,1)$, so that

$$J_\alpha(z_0 + v_n + w_n) = J_\alpha(z_0 + w_n) + o(1)$$

which combined with (4.9) and Lemma 4.5 gives

$$\begin{aligned} 0 <& k_{r_0} \le J_\alpha(z_0 + w_n) - J_\alpha(z_0) = J_\alpha(z_0 + v_n + w_n) - J_\alpha(z_0) + o(1) \\ \le& J_\alpha(z_0 + v_n) - J_\alpha(z_0) + o(1) = o(1). \end{aligned}$$

This is a contradiction and the claim is proved. Consequently, by (4.8), we have that for any $v \in V \cap \overline{B}_\rho(0)$ the minimum point $\bar w$ belongs to $W \cap B_{r_0}(0)$ and then it solves

$$\langle J'_\alpha(z_0 + v + \bar w), w\rangle = 0 \quad \forall\, w \in W. \tag{4.10}$$

Therefore $\bar w \in C^1(\overline{\Omega}) \times C^1(\overline{\Omega})$ and $\|z_0 + v + \bar w\|_{C^1} \le M$. We can also recognize that $\bar w$ is unique. In fact, if we suppose that there exist $w_1 \ne w_2 \in W \cap \overline{B}_{r_0}(0)$ which solves (4.10), for any $t \in [0,1]$ we have $\|v + w_1 + t(w_2 - w_1)\|_0 \le 2r_0 < R_0$, $\|z_0 + v + w_1 + t(w_2 - w_1)\|_{C^1} \le M$ and, applying Lemma 4.4,

$$\begin{aligned} 0 =& \langle J'_\alpha(z_0 + v + w_2) - J'_\alpha(z_0 + v + w_1), w_2 - w_1\rangle \\ =& \int_0^1 \langle J''_\alpha(z_0 + v + w_1 + t(w_2 - w_1))(w_2 - w_1), w_2 - w_1\rangle\, dt > 0. \end{aligned}$$

The contradiction allow us to conclude the uniqueness of $\bar w$. □

Remark 4.7. The previous result still holds replacing V with a new subspace $\bar V$ and W with $\bar W$, when

- $X = \bar V \oplus \bar W$
- $\bar W \subset W$
- $\bar V \subset C^1(\overline{\Omega}) \times C^1(\overline{\Omega})$ is finite dimensional
- $\bar V$ and $\bar W$ are orthogonal in $L^2(\Omega) \times L^2(\Omega)$.
- J_α satisfies $(P.S.)$ in the direction of $\bar W$.

5. The critical groups estimates

By applying Proposition 4.6, we can define the map

$$\psi : V \cap \overline{B}_\rho(0) \to W \cap B_{r_0}(0)$$

where $\psi(v)$ is the unique minimum point of the function $w \in W \cap \overline{B}_{r_0}(0) \mapsto J_\alpha(z_0 + v + w)$. Furthermore for any $v \in V \cap \overline{B}_\rho(0)$, $\psi(v)$ is the only point in $W \cap \overline{B}_{r_0}(0)$ such that $\langle J'_\alpha(z_0 + v + \psi(v)), \theta\rangle = 0$, for any $\theta \in W$.

Remark 5.1. By Theorem 1.1 and Lemma 4.4 (reasoning as in the proof of Proposition 4.6), there exist $R_0,\ M,\ \mu > 0$ such that

- if $z \in B_{R_0}(z_0)$ and $\langle J'_\alpha(z), \theta\rangle = 0$ for any $\theta \in W$, then $z \in C^{1,\beta}(\overline{\Omega}) \times C^{1,\beta}(\overline{\Omega})$ and $\|z\|_{C^{1,\beta}} \le M$, with $\beta \in]0, 1[$;
- setting $\tilde{K} = \left\{z \in B_{R_0}(z_0) \cap (C^{1,\beta}(\overline{\Omega}) \times C^{1,\beta}(\overline{\Omega})) \mid \|z\|_{C^{1,\beta}} \le M\right\}$, there is $\mu > 0$ such that, if $z \in \tilde{K}$, then $\langle J''_\alpha(z)w, w\rangle \ge \mu \|w\|_0^2$ for any $w \in W$;
- $\tilde{K}$ is convex and $z_0 + v + \psi(v) \in \tilde{K}$, for any $v \in V \cap \overline{B}_\rho(0)$.

We begin to derive the following lemma.

Lemma 5.2. *The map $\psi : V \cap \overline{B}_\rho(0) \to W$ is Lipschitz continuous with respect to the norm $\|\cdot\|_0$ on W.*

Proof. Let $v, z \in V \cap \overline{B}_\rho(0)$. We evaluate

$$\begin{aligned} 0 &= \langle J'_\alpha(z_0 + v + \psi(v)), \psi(v) - \psi(z)\rangle - \langle J'_\alpha(z_0 + z + \psi(z)), \psi(v) - \psi(z)\rangle \\ &= \langle J''_\alpha(z_0 + tv + t\psi(v) + (1-t)z + (1-t)\psi(z))(\psi(v) - \psi(z)), v - z + \psi(v) - \psi(z)\rangle \end{aligned}$$

for a suitable $t \in (0, 1)$. By Remark 5.1 we have that $z_t = z_0 + tv + t\psi(v) + (1 - t)z + (1-t)\psi(z) \in \tilde{K}$, so that

$$\begin{aligned} \|\psi(v) - \psi(z)\|_0^2 &\le 1/\mu \langle J''_\alpha(z_t)(\psi(v) - \psi(z)), \psi(v) - \psi(z)\rangle \\ &= -1/\mu \langle J''_\alpha(z_t)(\psi(v) - \psi(z)), v - z\rangle \le K\|v - z\|\, \|\psi(v) - \psi(z)\|_0. \end{aligned}$$

Hence we have

$$\|\psi(v) - \psi(z)\|_0 \le K\|v - z\|$$

where K is a positive constant. □

Set $q_1(\xi) = \frac{1}{p}\left(\alpha + |\xi|^2\right)^{p/2}$ and $q_2(\xi) = \frac{1}{r}\left(\alpha + |\xi|^2\right)^{r/2}$, with $\xi = (\xi_1, \dots \xi_N) \in \mathbb{R}^N$. As $p > 2$, $r > 2$ it results that $q_1, q_2 \in C^2$ and $|q_1''(\xi)| \le C_1 + C_2|\xi|^{p-2}$ $|q_2''(\xi)| \le C_1 + C_2|\xi|^{r-2}$ where $q_i''(\xi)$ denotes the Hessian matrix with $i = 1, 2$. More precisely $q_1''(\xi) = \left(\alpha + |\xi|^2\right)^{(p-2)/2} I + (p-2)\left(\alpha + |\xi|^2\right)^{(p-4)/2} A_\xi$ and $q_2''(\xi) = \left(\alpha + |\xi|^2\right)^{(r-2)/2} I + (r-2)\left(\alpha + |\xi|^2\right)^{(r-4)/2} A_\xi$, where I is the identity matrix and $(A_\xi)_{ij} = \xi_i\xi_j$.

If $u = (u_1, u_2) \in \tilde{K}$, we can extend $J'_\alpha(u)$ to H_0 by defining $A_1(u) : H_0 \to \mathbb{R}$

$$\begin{aligned} \langle A_1(u), z\rangle = &\int_\Omega \left(\alpha + |\nabla u_1|^2\right)^{(p-2)/2} (\nabla u_1 | \nabla z_1)\, dx \\ &+ \int_\Omega \left(\alpha + |\nabla u_2|^2\right)^{(r-2)/2} (\nabla u_2 | \nabla z_2)\, dx \\ &- \int_\Omega (D_u F(x, u_1, u_2) z_1 + D_v F(x, u_1, u_2) z_2)\, dx \end{aligned}$$

for any $z = (z_1, z_2) \in H_0$. Analogously we can extend $J''_\alpha(u)$ by defining $A_2(u)$: $H_0 \times H_0 \to \mathbb{R}$

$$\begin{aligned}\langle A_2(u)z, \theta\rangle &= \int_\Omega ((q_1''(\nabla u_1)\nabla z_1|\nabla\theta_1) + (q_2''(\nabla u_2)\nabla z_2|\nabla\theta_2))\,dx \\ &- \int_\Omega (D^2_{uu}F(x,u_1,u_2)z_1\theta_1 + D^2_{uv}F(x,u_1,u_2)z_1\theta_2)dx \\ &- \int_\Omega (D^2_{vu}F(x,u_1,u_2)z_2\theta_1 + D^2_{vv}F(x,u_1,u_2)z_2\theta_2\big)\ dx\end{aligned}$$

for any $z = (z_1, z_2) \in H_0$, $\theta = (\theta_1, \theta_2) \in H_0$.

It is easy to see that $A_1(u)$ is linear, $A_2(u)$ is bilinear and symmetric, both are continuous and the following result holds.

Lemma 5.3. *It results that*

(1) *if $v \in V \cap \overline{B}_\rho(0)$ then $\langle A_1(z_0 + v + \psi(v)), z\rangle = 0$ for any $z \in H^+$;*
(2) *there is $\mu > 0$ such that $\langle A_2(u)z, z\rangle \geq \mu\|z\|_0^2$ for any $u \in \tilde{K}$ and $z \in H^+$;*
(3) *if $u_1, u_2 \in \tilde{K}$ and $z \in H_0$, the real function $g : (0,1) \to \mathbb{R}$ defined by $g(t) = \langle A_1(tu_1 + (1-t)u_2), z\rangle$ is C^1 and $g'(t) = \langle A_2(tu_1 + (1-t)u_2)z, u_1 - u_2\rangle$.*

In what follows, we prove directly that ψ is C^1 map with respect to the norm $\|\cdot\|_0$ on W. The same argument can be also performed for the scalar case. We also precise that in Lemma 2.2 of [12], the C^1 regularity of the map ψ is already stated for a quasilinear elliptic equation. However, even if the result is true, that proof does not work, since it relies on the introduction of a penalized functional, which is not C^2 on the Hilbert space (see, for instance, Proposition 2.8, Chapter 1 in [2]).

Theorem 5.4. *The map ψ is C^1 with respect to the $\|\cdot\|_0$ norm on W.*

Proof. We begin to prove that ψ is differentiable with respect to the $\|\cdot\|_0$ norm on W. Let us consider $\bar{v} \in V \cap \overline{B}_\rho(0)$. Setting $\bar{u} = z_0 + \bar{v} + \psi(\bar{v})$, by (2) of Lemma 5.3 we have that $L_{\bar{u}} : H^+ \to (H^+)^*$ defined by $\langle L_{\bar{u}}(z), \theta\rangle = \langle A_2(\bar{u})z, \theta\rangle$ is a linear and continuous isomorphism. Moreover, for any $h \in V$, $\langle A_2(\bar{u})\cdot, h\rangle$ belongs to $(H^+)^*$. We denote by $B_{\bar{u}}(h) = L_{\bar{u}}^{-1}(\langle A_2(\bar{u})\cdot, h\rangle)$, so that $B_{\bar{u}}(h)$ is the only element of H^+ verifying the equality

$$\langle A_2(\bar{u})z, B_{\bar{u}}(h)\rangle = \langle A_2(\bar{u})z, h\rangle \quad \forall z \in H^+. \tag{5.1}$$

It is obvious that $B_{\bar{u}} : V \to H^+$ is linear, moreover it is also continuous, as

$$\|B_{\bar{u}}(h)\|_0 \leq \|L_{\bar{u}}^{-1}\| \sup_{z\in H,^+ \|z\|_0=1} |\langle A_2(\bar{u})z, h\rangle| \leq C\|h\|. \tag{5.2}$$

If we show that

$$\lim_{h\to 0} \frac{\|\psi(\bar{v}+h) - \psi(\bar{v}) + B_{\bar{u}}(h)\|_0}{\|h\|} = 0$$

then the differentiability of ψ is proved, being $\psi'(\bar{v}) = -B_{\bar{u}}$.

Let us fix $h \in V$, $h \neq 0$ such that $\bar{v} + h \in V \cap B_\rho(0)$.
Denoting by $z_h = \psi(\bar{v}+h) - \psi(\bar{v}) + B_{\bar{u}}(h) \in H^+$, by Lemma 5.3 and (5.1) we have, for a suitable $t \in (0,1)$, that

$$\begin{aligned} 0 &= \langle A_1(z_0 + \bar{v} + h + \psi(\bar{v}+h)), z_h\rangle - \langle A_1(z_0 + \bar{v} + \psi(\bar{v})), z_h\rangle \\ &= \langle A_2(z_0 + \bar{v} + th + t\psi(\bar{v}+h) + (1-t)\psi(\bar{v}))z_h, h\rangle \\ &+\langle A_2(z_0 + \bar{v} + th + t\psi(\bar{v}+h) + (1-t)\psi(\bar{v}))z_h, z_h\rangle \\ &-\langle A_2(z_0 + \bar{v} + th + t\psi(\bar{v}+h) + (1-t)\psi(\bar{v}))z_h, B_{\bar{u}}(h)\rangle \\ &+\langle A_2(\bar{u})z_h, B_{\bar{u}}(h)\rangle - \langle A_2(\bar{u})z_h, h\rangle. \end{aligned} \tag{5.3}$$

In what follows we denote by $u_{th} = z_0 + \bar{v} + th + t\psi(\bar{v} + h) + (1-t)\psi(\bar{v})$ and by z^j the j-coordinate of the generic point $z \in H_0$.

For any $z, \theta \in H_0$

$$
\begin{aligned}
&\langle \big(A_2(u_{th}) - A_2(\bar{u})\big)z, \theta\rangle \\
\leq & \int_\Omega |\big(q_1''(\nabla u_{th}^1) - q_1''(\nabla \bar{u}^1)\big)\nabla z_1 \nabla \theta_1| + \int_\Omega \big|\big(q_2''(\nabla u_{th}^1) - q_2''(\nabla \bar{u}^1)\big)\nabla z_2 \nabla \theta_2\big| \\
+ & \int_\Omega |D_{uu}F(x, u_{th}) - D_{uu}F(x, \bar{u})||z_1\theta_1| + \int_\Omega |D_{uv}F(x, u_{th}) - D_{uu}F(x, \bar{u})||z_1\theta_2| \\
+ & \int_\Omega |D_{vu}F(x, u_{th}) - D_{uu}F(x, \bar{u})||z_2\theta_1| + \int_\Omega |D_{vv}F(x, u_{th}) - D_{uu}F(x, \bar{u})||z_2\theta_2|
\end{aligned}
$$

The Ascoli-Arzelà Theorem assures that $\lim_{h\to 0} \|u_{th} - \bar{u}\|_{C^1} = 0$, so the previous inequality gives

$$
|\langle \big(A_2(u_{th}) - A_2(\bar{u})\big)z, \theta\rangle| \leq o(h)\|z\|_0\|\theta\|_0 \qquad \forall z, \theta \in H_0 \tag{5.4}
$$

From Lemma 5.3, taking account of (5.3), (5.4) and (5.2), we get

$$
\begin{aligned}
&\mu\|z_h\|_0^2 \leq \langle A_2(u_{th})z_h, z_h\rangle \\
&= \langle \big(A_2(u_{th}) - A_2(\bar{u})\big)z_h, B_{\bar{u}}(h)\rangle + \langle \big(A_2(\bar{u}) - A_2(u_{th})\big)z_h, h\rangle \\
&\leq o(h)\,\|z_h\|_0\|h\|
\end{aligned}
$$

so that

$$
\lim_{h\to 0} \frac{\|z_h\|_0}{\|h\|} = 0.
$$

Now we recognize that ψ is C^1. We consider a sequence $(v_n) \subset V$ such that $v_n \to \bar{v}$ with $\bar{v} \in V$, as $n \to +\infty$. Let us denote $u_n = z_0 + v_n + \psi(v_n)$ and $L_{u_n} = A_2(u_n)_{|_{H^+}} : H^+ \to (H^+)^*$ the linear isomorphism. It results that

$$
\begin{aligned}
\|\psi'(v_n) - \psi'(\bar{v})\| &= \sup_{\|h\|=1} \|\psi'(v_n)h - \psi'(\bar{v})h\|_0 \\
&\leq \sup_{h\in V, \|h\|=1} \|L_{u_n}^{-1}(\langle A_2(u_n)\cdot, h\rangle) - L_{\bar{u}}^{-1}(\langle A_2(\bar{u})\cdot, h\rangle)\|_0 \\
&\leq \sup_{h\in V, \|h\|=1} \|L_{u_n}^{-1}(\langle A_2(u_n)\cdot, h\rangle) - L_{u_n}^{-1}(\langle A_2(\bar{u})\cdot, h\rangle)\|_0 \\
&\quad + \sup_{h\in V, \|h\|=1} \|L_{u_n}^{-1}(\langle A_2(\bar{u})\cdot, h\rangle) - L_{\bar{u}}^{-1}(\langle A_2(\bar{u})\cdot, h\rangle)\|_0 \\
&\leq \|L_{u_n}^{-1}\| \sup_{h\in V, \|h\|=1} \sup_{z\in W, \|w\|=1} |\langle A_2(u_n) - A_2(\bar{u})z, h\rangle| \\
&\quad + \|L_{u_n}^{-1} - L_{\bar{u}}^{-1}\| \sup_{h\in V, \|h\|=1} \sup_{z\in W, \|w\|=1} |\langle A_2(\bar{u})z, h\rangle|
\end{aligned}
$$

which tends to zero as $n \to +\infty$, as u_n^j tends to $\bar{u}^j$ and ∇u_n^j tends to $\nabla \bar{u}^j$ uniformly in $\overline{\Omega}$, as $n \to +\infty$ for any $j = 1, 2$. □

Remark 5.5. We notice that $\psi'(0) = 0$. In fact, for each $h \in V$, by (4.1) we have that $\langle A_2(z_0)\cdot, h\rangle = 0$ on H^+ and so, from the previous proof,

$$
\psi'(0)(h) = -L_{z_0}^{-1}\big(\langle A_2(z_0)\cdot, h\rangle\big) = 0.
$$

Lemma 5.6. *Let $H{=}V$ or $H{=}H^+$. The function $B_H : V\cap \overline{B}_\rho(0)\to H^*$ defined by*

$$\langle B_H(v), z\rangle = \langle A_1\left(z_0+v+\psi(v)\right), z\rangle \quad \forall v\in V\cap\overline{B}_\rho(0), z\in H,$$

is C^1 and

$$\langle B'_H(v)h, z\rangle = \langle A_2(z_0+v+\psi(v))(h+\psi'(v)h), z\rangle \tag{5.5}$$

for any $v\in V\cap\overline{B}_\rho(0)$, $h\in V$, $z\in H$.

Proof. Let us consider $v\in V\cap\overline{B}_\rho(0)$ and $h\in V$, such that $v+h\in V\cap\overline{B}_\rho(0)$. Denoting $\omega_h\equiv\psi(v+h)-\psi(v)$, we have, for a suitable $t\in(0,1)$,

$$\begin{aligned}
&\|B_H(v+h)-B_H(v)-\langle A_2(z_0+v+\psi(v))(h+\psi'(v)h),\cdot\rangle\| \\
&\quad=\sup_{z\in H,\|z\|=1}|\langle A_1(z_0+v+h+\psi(v+h)), z\rangle-\langle A_1(z_0+v+\psi(v)), z\rangle \\
&\qquad\qquad -\langle A_2(z_0+v+\psi(v))(h+\psi'(v)h), z\rangle| \\
&\quad=\sup_{z\in H,\|z\|=1}|\langle A_2(z_0+v+th+\psi(v)+t\omega_h)z, h\rangle \\
&\qquad\qquad +\langle A_2(z_0+v+th+\psi(v)+t\omega_h)z,\omega_h\rangle \\
&\qquad\qquad -\langle A_2(z_0+v+\psi(v))(h+\psi'(v)h), z\rangle| \\
&\quad\le\sup_{z\in H,\|z\|=1}|\langle(A_2(z_0+v+th+\psi(v)+t\omega_h)-A_2(z_0+v+\psi(v))z, h\rangle| \\
&\qquad\qquad +|\langle A_2(z_0+v+th+\psi(v)+t\omega_h)-A_2(z_0+v+\psi(v))\omega_h, z\rangle| \\
&\qquad\qquad +|\langle A_2(z_0+v+\psi(v))\,z,(\psi(v+h)-\psi(v)-\psi'(v)h)\rangle|
\end{aligned}$$

From the above inequality we immediately derive

$$\lim_{\|h\|\to 0}\frac{\|B_H(v+h)-B_H(v)-\langle A_2(z_0+v+\psi(v))(h+\psi'(v)h),\cdot\rangle\|}{\|h\|}=0.$$

In order to prove continuity of B'_H, let us consider a sequence $(v_n)\subset\big(V\cap\overline{B}_\rho(0)\big)$ such that $v_n\to\bar v$. Reasoning as in (5.4), we have that

$$\begin{aligned}
&|\langle B'_H(v_n)h, z\rangle-\langle B'_H(\bar v)h, z\rangle| \\
&=|\langle A_2(z_0+v_n+\psi(v_n))(h+\psi'(v_n)h), z\rangle-\langle A_2(z_0+\bar v+\psi(\bar v))(h+\psi'(\bar v)h), z\rangle| \\
&\le|\langle\big(A_2(z_0+v_n+\psi(v_n))-A_2(z_0+\bar v+\psi(\bar v))\big)h, z\rangle| \\
&+|\langle\big(A_2(z_0+v_n+\psi(v_n))-A_2(z_0+\bar v+\psi(\bar v))\big)\psi'(v_n)h, z\rangle| \\
&+|\langle A_2(z_0+\bar v+\psi(\bar v))\big)z,\psi'(v_n)h-\psi'(\bar v)h\rangle|\le o(n)\,\|h\|\,\|z\|.
\end{aligned}$$

Therefore

$$\lim_{n\to\infty}\|B'_H(v_n)-B'_H(\bar v)\|=0$$

$\square$

Proposition 5.7. *For any $v\in V\cap\overline{B}_\rho(0)$ and $h\in V$*

$$\psi'(v)h\in\big(C^1(\overline{\Omega})\times C^1(\overline{\Omega})\big)\cap H^+\subset W. \tag{5.6}$$

Proof. Using the notations of Lemma 5.6, where $H = H^+$, from (1) of Lemma 5.3 $B_{H^+} : V \cap \overline{B}_\rho(0) \to (H^+)^*$ is constantly equal to zero, so that (5.5) gives

$$\langle A_2(z_0 + v + \psi(v))(h + \psi'(v)h), z\rangle = 0, \quad \forall\, v \in V \cap \overline{B}_\rho(0),\ h \in V,\ z \in H^+. \tag{5.7}$$

Since $z_0 + v + \psi(v) \in C^{1,\beta}(\overline{\Omega}) \times C^{1,\beta}(\overline{\Omega})$, we derive that $h + \psi'(v)h$ belongs to $\big(C^{1,\eta}(\overline{\Omega}) \times C^{1,\eta}(\overline{\Omega})\big)$ for some $\eta \in (0, 1)$ (see [19]), so that, as $V \subset C^1(\overline{\Omega}) \times C^1(\overline{\Omega})$ and $\psi'(v)h \in H^+$, (5.6) is proved.

□

Now we can derive the following regularity result.

Theorem 5.8. *The map $\varphi : V \cap \overline{B}_\rho(0) \to \mathbb{R}$ defined by $\varphi(v) = J_\alpha(z_0 + v + \psi(v))$ is C^2 and, for any $v \in V \cap \overline{B}_\rho(0)$ and $z, h \in V$*

$$\langle \varphi'(v), z\rangle = \langle J'_\alpha(z_0 + v + \psi(v)), z\rangle \tag{5.8}$$

$$\langle \varphi''(v)h, z\rangle = \langle J''_\alpha(z_0 + v + \psi(v))(h + \psi'(v)h), z\rangle \tag{5.9}$$

(5.10) *$\varphi''(v)$ is an isomorphism if and only if $J''_\alpha(z_0 + v + \psi(v))$ is injective.*

Proof. Fix $v \in V \cap \overline{B}_\rho(0)$, $h \in V$, such that $v + h \in V \cap \overline{B}_\rho(0)$. We have, for suitable $t, s, \tau \in (0, 1)$ and denoting $\omega_h \equiv \psi(v + h) - \psi(v)$,

$$\begin{aligned}
&\varphi(v + h) - \varphi(v) - \langle J'_\alpha(z_0 + v + \psi(v)), h\rangle \\
&\quad= \langle J'_\alpha(z_0 + v + th + \psi(v) + t(\psi(v + h) - \psi(v))) - J'_\alpha(z_0 + v + \psi(v)), h\rangle \\
&\qquad + \langle J'_\alpha(z_0 + v + th + \psi(v) + t\omega_h) - J'_\alpha(z_0 + v + \psi(v)), \omega_h\rangle \\
&\quad= t\langle A_2(z_0 + v + sth + \psi(v) + st\omega_h)h, h\rangle \\
&\qquad + t\langle A_2(z_0 + v + sth + \psi(v) + st\omega_h)\omega_h, h\rangle \\
&\qquad + t\langle A_2(z_0 + v + \tau th + \psi(v) + \tau t\omega_h)\omega_h, h\rangle \\
&\qquad + t\langle A_2(z_0 + v + \tau th + \psi(v) + t\tau\omega_h)\omega_h, \omega_h\rangle.
\end{aligned}$$

We infer that

$$\begin{aligned}
&\frac{|\varphi(v + h) - \varphi(v) - \langle J'_\alpha(z_0 + v + \psi(v)), h\rangle|}{\|h\|} \\
&\le \frac{|\langle A_2(z_0 + v + \psi(v) + sth + st\omega_h)h, h\rangle|}{\|h\|} \\
&+ \frac{|\langle A_2(z_0 + v + \psi(v) + sth + st\omega_h)\omega_h, h\rangle|}{\|h\|} \\
&+ \frac{|\langle A_2(z_0 + v + \psi(v) + \tau th + \tau t\omega_h)\omega_h, h\rangle|}{\|h\|} \\
&+ \frac{|\langle A_2(z_0 + v + \psi(v) + t\tau h + t\tau\omega_h)\omega_h, \omega_h\rangle|}{\|h\|}
\end{aligned}$$

which tends to zero as $\|h\| \to 0$, so that (5.8) is proved.

Moreover, by Lemma 5.6, where $H = V$, we immediately see that $\varphi' = B_V$, φ is C^2 and

$$\langle \varphi''(v)h, z\rangle = \langle A_2(z_0 + v + \psi(v))(h + \psi'(v)h), z\rangle \quad \forall\, h, z \in V. \tag{5.11}$$

Proposition 5.7 assures that any $h + \psi'(v)h \in X$ which, together with (5.11), gives (5.9).

In order to prove (5.10) we fix $v \in V \cap \overline{B}_\rho(0)$ and suppose that $\varphi''(v)$ is an isomorphism. By way of contradiction, if $J''_\alpha(z_0 + v + \psi(v))$ is not injective, there exists $\bar{z} \in X \setminus \{0\}$ such that

$$\langle J''_\alpha(z_0 + v + \psi(v))z, \bar{z}\rangle = 0, \quad \forall\, z \in X.$$

Writing $\bar{z} = \bar{v} + \bar{w}$, where $\bar{v} \in V$ and $\bar{w} \in W$, by (5.7) and (5.9) we infer

$$\begin{aligned}\langle \varphi''(v)h, \bar{v}\rangle &= \langle J''_\alpha(z_0 + v + \psi(v))(h + \psi'(v)h), \bar{v}\rangle \\ &= \langle J''_\alpha(z_0 + v + \psi(v))(h + \psi'(v)h), \bar{z}\rangle = 0, \qquad \forall\, h \in V\end{aligned}$$

so that $\bar{v} = 0$ and $\bar{z} \in W$. By (2) of Lemma 5.3, $\bar{z} = 0$ which is a contradiction.

On the other side, if $J''_\alpha(z_0 + v + \psi(v))$ is injective but $\varphi''(v)$ is not, there is $\bar{v} \in V \setminus \{0\}$ such that

$$\langle J''_\alpha(z_0 + v + \psi(v))(\bar{v} + \psi'(v)\bar{v}), h\rangle = \langle \varphi''(v)\bar{v}, h\rangle = 0, \qquad \forall\, h \in V$$

which, by (5.7), gives

$$\langle J''_\alpha(z_0 + v + \psi(v))(\bar{v} + \psi'(v)\bar{v}), z\rangle = 0, \qquad \forall\, z \in H_0.$$

As $J''_\alpha(z_0 + v + \psi(v))$ is injective, this means that $\bar{v} + \psi'(v)\bar{v} = 0$, so also $\bar{v} = 0$ which is again a contradiction. □

Corollary 5.9. *Any nondegenerate critical point is isolated.*

Proof. If z_0 is nondegenerate, $J''_\alpha(z_0)$ is injective and (5.10) assures that $\varphi''(0)$ is an isomorphism. As V is finite dimensional, this implies that 0 is an isolated critical point for φ and, by (5.8), z_0 is an isolated critical point for J_α. □

The critical group computations may now be established.

Proof of Theorem 1.4. Taking account of Proposition 4.2 and, as in [14], using a pseudo-gradient flow we can derive that

$$C_j(J_\alpha, z_0) \simeq C_j(\varphi, 0). \tag{5.12}$$

In the non-degenerate case, that is, $J''_\alpha(z_0)$ is injective, we have $H^0 = \{0\}$ and there is a suitable constant $\mu > 0$ such that

$$\langle J''_\alpha(z_0)v, v\rangle \le -\mu\|v\|^2, \quad \text{for any } v \in V.$$

As a consequence, z_0 is a local isolated maximum of J_α along V, thus 0 is a local isolated maximum of φ in $V \cap \overline{B}_{\rho'}$ and by (5.12) the assert comes (see [7, Example 1, page 33]).

In the degenerate case, by (5.12), it is clear that $C_j(J_\alpha, z_0) = \{0\}$ if $j > \dim V$. Moreover, by Corollary 6.4 proved by Lancelotti in [26], we have $C_j(J_\alpha, z_0) = \{0\}$ for any $j < m(J_\alpha, z_0)$. □

6. Applications

In this section we apply the critical group estimates in Theorem 1.4 to obtain multiplicity results of solutions for the potential p-laplacian system (1.8).

Let $I_{\lambda,\mu} : X \to \mathbb{R}$ be the Euler functional associated to (1.8) defined by

$$I_{\lambda,\mu}(z) = \frac{1}{p}\int_\Omega (|\nabla u|^p + |\nabla v|^p)\,dx - \frac{1}{q}\int_\Omega (\lambda(u^+)^q + \mu(v^+)^q)\,dx - \frac{2}{p^*}\int_\Omega (u^+)^\gamma (v^+)^\beta\,dx.$$

Since $2 < p \le q < p^*$, $\gamma > 2$, $\beta > 2$, $\gamma + \beta = p^*$, the functional $I_{\lambda,\mu}$ is of class C^2 on X. We remark that if $p > 2$ and $1 < \gamma \le 2$, $1 < \beta \le 2$, the Euler functional associated to the potential system is of class C^1, but not C^2.

Moreover we consider the Nehari manifold

$$M_{\lambda,\mu} = \{z \in X \setminus \{0\} \mid \langle I'_{\lambda,\mu}(z), z\rangle = 0\}$$

and the related number

$$c_{\lambda,\mu} = \inf\{I_{\lambda,\mu}(z) \ : \ z \in M_{\lambda,\mu}\}.$$

Lemma 2.2 in [17] assures that *Palais-Smale* condition holds under a level depending on $\tilde{S}$ where

$$\tilde{S} = \inf_{u,v \in W_0^{1,p}(\Omega)\setminus\{0\}} \frac{\|u\|_{1,p}^p + \|v\|_{1,p}^p}{\left(\int_\Omega |u|^\gamma |v|^\beta dx\right)^{p/p^*}}.$$

By [17] (see also [1]), it is known that $\tilde{S} = [(\gamma/\beta)^{\beta/p^*} + (\beta/\gamma)^{\gamma/p^*}]\mathcal{S}_p$. Since $\mathcal{S}_p$ is independent of Ω and it is achieved if and only if $\Omega = \mathbb{R}^N$, it follows that $\tilde{S}$ is independent of Ω.

Taking into account Lemma 2.3 in [17], we have the following result.

Lemma 6.1. *Assume that $N \ge p^2$. The functional $I_{\lambda,\mu}$ satisfies $(P.S.)_c$ condition for all $c \in \left(0, \frac{2}{N}\left(\frac{\tilde{S}}{2}\right)^{N/p}\right)$. Moreover $c_{\lambda,\mu} \in \left(0, \frac{2}{N}\left(\frac{\tilde{S}}{2}\right)^{N/p}\right)$ for any $\lambda > 0$ and $\mu > 0$ if $q > p$ and for any $\lambda \in (0, \lambda^*)$ and $\mu \in (0, \mu^*)$ if $q = p$ with $\lambda^*, \mu^* > 0$ small enough.* □

Now, we consider the space $W^{1,p}_{0,rad}(B_r) = \{u \in W_0^{1,p}(B_r) \mid u(x) = u(|x|)\}$ and $E_r = W_0^{1,p}(B_r) \times W_0^{1,p}(B_r)$. We set

$$\begin{aligned} I_{\lambda,\mu,r}(z) =& \frac{1}{p}\int_{B_r} (|\nabla u|^p + |\nabla v|^p)\,dx \ - \frac{1}{q}\int_{B_r} (\lambda(u^+)^q + \mu(v^+)^q)\,dx \\ &- \frac{2}{p^*}\int_{B_r} (u^+)^\gamma (v^+)^\beta\,dx. \end{aligned}$$

Define the Nehari manifold

$$M^r_{\lambda,\mu} = \{z \in E_r \setminus \{0\} \mid \langle I'_{\lambda,\mu,r}(z), z\rangle = 0\}$$

and set $m^r_{\lambda,\mu} = \inf\{I_{\lambda,\mu,r}(z) \,|\, z \in M_{\lambda,\mu,r}\}$.

Arguing as in [13, 17], if we fix $r > 0$ small enough, we can construct an entrance and a barycenter map and we derive the following topological result:

Lemma 6.2. *There exist $\lambda^*, \mu^* > 0$ such that if $\lambda \in (0, \lambda^*)$, $\mu \in (0, \mu^*)$ and $a \in (0, c_{\lambda,\mu})$ we have*

$$\mathcal{P}_t(I_{\lambda,\mu}^{m^r_{\lambda,\mu}}, I^a_{\lambda,\mu}) = t(\mathcal{P}_t(\Omega) + \mathcal{Z}_{\lambda,\mu}(t)),$$

where $\mathcal{Z}_{\lambda,\mu}(t)$ is a polynomial with nonnegative integer coefficients. □

As usual, we denote by $\mathcal{P}_t(f,u)$ the Morse polynomial of f in a critical point u, that is

$$\mathcal{P}_t(f,u) = \sum_{q=0}^{+\infty} \dim\, C_q(f,u)t^k$$

and the *multiplicity* of u by the number $\mathcal{P}_1(f,u)$.

The following result correlates the topological properties of the domain and the number of solutions of system (1.8), counted with their multiplicities.

Theorem 6.3. *Assume that $N \geq p^2$, $2 < p \leq q < p^*$, $\gamma > 2$, $\beta > 2$ with $\gamma + \beta = p^*$. There exist $\lambda^* > 0$ and $\mu^* > 0$ such that, for any $\lambda \in (0,\lambda^*)$ and $\mu \in (0,\mu^*)$, the system* (1.8) *has at least $\mathcal{P}_1(\Omega)$ positive solutions, possibly counted with their multiplicities.*

Proof. Let $\lambda \in (0,\lambda^*)$ and $\mu \in (0,\mu^*)$ be such that Lemma 6.1 and Lemma 6.2 hold. For simplicity we drop the dependence of I, m^r and other objects from λ and μ in the rest of this proof. Let us denote by K the set of critical points of I in $I^{-1}(a, m^r)$ for some $0 < a < c_{\lambda,\mu}$. If, for any $\rho \in (0,r)$, m^ρ is a critical value for I, then K is an infinite set, otherwise m^r (or $m^{r'}$ where $r' \in (0,r)$) is a regular value. In the second case, by Theorem 4.3 in [7] and Lemma 6.2 we have that

$$\begin{aligned}\sum_{q=0}^{+\infty} a_q t^q =& \mathcal{P}_t(I^{m^r}, I^a) + (1+t)Q(t)\\ =& t(\mathcal{P}_t(\Omega) + \mathcal{Z}_{\lambda,\mu}(t)) + (1+t)Q(t)\end{aligned}$$

where $a_q = \sum_{z\in K} \dim C_q(I,z)$. So it is proved that, in any case, K has at least $\mathcal{P}_1(\Omega)$ elements, possibly counted with their multiplicities. It remains to prove that if $z = (u,v) \in K$, then $u > 0$ and $v > 0$. As $\langle I'(z),(u^-,0)\rangle = \langle I'(z),(0,v^-)\rangle = 0$, then $u \geq 0$ and $u \geq 0$ in Ω, while the strict positivity is assured by [31]. □

We need a deep insight into the notion of multiplicity. In order to do that, we recall an abstract theorem, proved in [9] (see also [3] and [7]).

Theorem 6.4. *Let A be an open subset of a Banach space X. Let f be a C^1-functional on A and $u \in A$ be an isolated critical point of f. Assume that there exists an open neighborhood U of u such that $\overline{U} \subset A$, u is the only critical point of f in $\overline{U}$ and f satisfies the Palais–Smale condition in $\overline{U}$.*
Then there exists $\bar{\mu} > 0$ such that, for any $g \in C^1(A,\mathbb{R})$ with

- $\|f-g\|_{C^1(A)} < \bar{\mu}$,
- *g satisfies the Palais–Smale condition in $\overline{U}$,*
- *g has a finite number $\{u_1,u_2,\ldots,u_m\}$ of critical points in U,*

we have

$$\sum_{j=1}^{m} \mathcal{P}_t(g,u_j) = \mathcal{P}_t(f,u) + (1+t)Q(t),$$

where $Q(t)$ is a formal series with coefficients in $\mathbb{N} \cup \{+\infty\}$. □

Proof of Theorem 1.6. Let $\lambda^*, \mu^* > 0$ be defined by Theorem 6.3 and $(\lambda,\mu) \in (0,\lambda^*) \times (0,\mu^*)$. By Theorem 6.3, problem (1.8) has at least $\mathcal{P}_1(\Omega)$ solutions, possibly counted with their multiplicities. If (1.8) has less than $\mathcal{P}_1(\Omega)$ distinct solutions, this means, in particular, that $I_{\lambda,\mu}$ has a finite number of critical points $\tilde{z}_1, \dots \tilde{z}_h$, having multiplicities $\tilde{m}_1, \dots \tilde{m}_h$, where $1 \le h < \mathcal{P}_1(\Omega)$ and

$$\sum_{j=1}^{h} \tilde{m}_j \ge \mathcal{P}_1(\Omega). \tag{6.1}$$

Let $R > 0$ be defined in Proposition 3.5. As $\tilde{z}_1, \dots \tilde{z}_h$ are isolated, let $\gamma_j \in (0, R)$ be such that $I_{\lambda,\mu}$ has not critical points other than $\tilde{z}_j$ in $\overline{B_{\gamma_j}(\tilde{z}_j)}$. Denoting by $U_j = B_{\gamma_j/2}(\tilde{z}_j)$, let $\bar{\mu}_j$ be defined by Theorem 6.4 relatively to U_j and $B_{\gamma_j}(\tilde{z}_j)$. Moreover we call $A = \bigcup_{j=1}^{h} B_{\gamma_j}(\tilde{z}_j)$ and $\bar{\mu} = \min\{\bar{\mu}_1, \dots \bar{\mu}_h\}$.

Let α_n be a sequence such that $\alpha_n > 0$, $\alpha_n \to 0$, and denote by

$$\begin{aligned} T_{\alpha_n}(z) = T_{\alpha_n}(u, v) = & \frac{1}{p} \int_{\Omega} \left(\alpha_n + |\nabla u(x)|^2\right)^{\frac{p}{2}} + \left(\alpha_n + |\nabla v(x)|^2\right)^{\frac{p}{2}} \, dx \\ & - \frac{1}{q} \int_{B_r} (\lambda (u^+)^q + \mu (v^+)^q) \, dx - \frac{2}{p^*} \int_{B_r} (u^+)^{\gamma} (v^+)^{\beta} \, dx. \end{aligned}$$

so that $T_{\alpha_n} \in C^2(X, \mathbf{R})$ and $\|I_{\lambda,\mu} - T_{\alpha_n}\|_{C^1(A)} < \bar{\mu}$, if n is sufficiently large.

If T_{α_n} has at least $\mathcal{P}_1(\Omega)$ critical points in A, we choose $f_n = g_n = 0$, otherwise we denote by $\bar{z}_1, \dots \bar{z}_k$ the critical points of T_{α_n} in A, and by $\bar{m}_1, \dots \bar{m}_k$ their multiplicities. For simplicity, we omit the dependence of $\bar{z}_1, \dots \bar{z}_k$ (and their related objects) from n. By Theorem 6.4 and (6.1) we have

$$\sum_{i=1}^{k} \bar{m}_i \ge \sum_{j=1}^{h} \tilde{m}_j \ge \mathcal{P}_1(\Omega) \tag{6.2}$$

where $h \le k < \mathcal{P}_1(\Omega)$, so $\bar{m}_i \ge 2$ for some $i \in \{1, \dots k\}$.

Now, repeating for any $i = 1, \dots, k$ the same procedure introduced in Section 4, we get k splittings $X = V_i \oplus W_i$. In particular, setting $V = V_1 + V_2 + \cdots + V_k$ and $W = \bigcap_{i=1}^{k} W_i$, there are $r > 0$ and $\rho \in (0, r)$ such that

(1) $X = V \oplus W$;
(2) $V \subset C^1(\overline{\Omega})$ is finite dimensional;
(3) V and W are orthogonal in $L^2(\Omega) \times L^2(\Omega)$;
(4) for any $M > 0$ there exist $r_0 > 0$ and $C > 0$ such that if $z \in C^1(\overline{\Omega})$, $\|z\|_{C^1(\overline{\Omega})} \le M$ and $\|z - \bar{z}_i\| < r_0$ for some $i \in \{1, \dots, k\}$, then

$$\langle T''_{\alpha_n}(z)w, w\rangle \ge C\|w\|_0^2, \quad \forall w \in W;$$

(5) for any $i \in \{1, \dots k\}$ and $v \in V \cap B_\rho(0)$ there exists one and only one $\bar{w}_i = \bar{w}_i(v) \in W \cap B_r(0)$ such that

$$\langle T'_{\alpha_n}(\bar{z}_i + v + \bar{w}_i), w\rangle = 0, \quad \forall w \in W. \tag{6.3}$$

Moreover, denoting by $\bar{U}_i = \bar{z}_i + (V \cap B_\rho(0)) + (W \cap B_r(0))$, for any $i \in \{1, \dots, k\}$ $\bar{U}_i \subset A$ and $\bar{U}_{i_1} \cap \bar{U}_{i_2} = \emptyset$ if $i_1 \ne i_2$. In fact all these properties are consequences of Lemma 4.4, Lemma 4.6 and Remark 4.7, choosing r and ρ small enough.

For any $i = 1, \dots k$, let us introduce the maps

$$\psi_i : V \cap B_\rho(0) \to W \cap B_r(0) \text{ and } \varphi_i = V \cap B_\rho(0) \to \mathbb{R}$$

where $\psi_i(v)$ is the only element $\bar{w}_i \in W \cap B_r(0)$ satisfying (6.3) and $\varphi_i(v) = T_{\alpha_n}(\bar{z}_i + v + \psi_i(v))$.

By Theorem 5.8, φ_i is a C^2-map and, for any $v \in V \cap B_\rho(0)$, $z \in V$

$$\langle {\varphi_i}'(v), z\rangle = \langle T'_{\alpha_n}\left(\bar z_i + v + \psi_i(v)\right), z\rangle.$$

Furthermore we have that ${\varphi_i}''(v)$ is an isomorphism if and only if is injective $T''_{\alpha_n}\left(\bar z_i + v + \psi_i(v)\right)$.

Let $e_i = (e_{i,1}, e_{i,2})$, $i = 1, \dots, l$, be an $L^2(\Omega) \times L^2(\Omega)$ - orthogonal basis of V, where $l = \dim V$. For any $v' \in V'$ we introduce

$$f_{v'}(x) = \sum_{i=1}^{l} \langle v', e_i\rangle e_{i,1}(x) \text{ and } g_{v'}(x) = \sum_{i=1}^{l} \langle v', e_i\rangle e_{i,2}(x).$$

It is clear that $f_{v'}$, $g_{v'} \in C^1(\overline{\Omega})$,

$$\lim_{\|v'\|_{V'} \to 0} \|f_{v'}\|_{C^1(\overline{\Omega})} = 0 \text{ and } \lim_{\|v'\|_{V'} \to 0} \|g_{v'}\|_{C^1(\overline{\Omega})} = 0. \tag{6.4}$$

We also define $L_{v'} : X \to \mathbb{R}$ by

$$L_{v'}(z) = L_{v'}(u, v) = \int_\Omega f_{v'} u + \int_\Omega g_{v'} v\,.$$

By definition

$$L_{v'}(w) = 0, \quad \forall\, w \in W \qquad \text{and} \qquad L_{v'}(v) = \langle v', v\rangle_{V'}, \quad \forall\, v \in V. \tag{6.5}$$

Let $J_{\alpha,v'} = T_\alpha - L_{v'}$, so that $J_{\alpha,v'} \in C^2(X)$,

$$\langle J'_{\alpha,v'}(\bar z), z\rangle = \langle T'_\alpha(\bar z), z\rangle - L_{v'}(z) \text{ and } J''_{\alpha,v'}(\bar z) = T''_\alpha(\bar z), \qquad \forall\, \bar z, z \in X. \tag{6.6}$$

Moreover, if B is a bounded subset of X,

$$\lim_{\|v'\| \to 0} \|J_{\alpha,v'} - T_\alpha\|_{C^1(B)} = 0.$$

By Sard's Lemma, for any $\varepsilon > 0$, $\hat v' \in V'$ and $i = 1, \dots, k$, there is $\bar v' \in V'$ such that $\|\bar v'\|_{V'} < \varepsilon$ and, if $\varphi'_i(v) = \hat v' + \bar v'$, then $\varphi''_i(v)$ is an isomorphism. Moreover, there is $\beta > 0$, depending on $\hat v' + \bar v'$, such that if $v' \in V'$, $\|v'\|_{V'} \le \beta$ and $\varphi'_i(v) = \hat v' + \bar v' + v'$, then $\varphi''_i(v)$ is an isomorphism.

Let μ_i be defined by Theorem 6.4 relatively to T_{α_n}, $\bar z_i$, A and $\bar U_i$, for any $i = 1, \dots, k$, and $\mu = \min\{\mu_1, \dots, \mu_k\}$. We fix $\varepsilon' > 0$ such that $\|L_{v'}\|_{C^1(A)} < \mu$, if $v' \in V'$ and $\|v'\|_{V'} < \varepsilon'$.

In correspondence of $\varepsilon_1 = \min\{\varepsilon'/k, 1/n\}$, there are $v'_1 \in V'$ and $\beta_1 > 0$ such that $\|v'_1\|_{V'} < \varepsilon_1$ and, if $v' \in V'$, $\|v'\|_{V'} \le \beta_1$ and $\varphi'_1(v) = v'_1 + v'$, then $\varphi''_1(v)$ is an isomorphism.

In this way, for any $i = 2, \dots, k$, we define recursively $\varepsilon_i = \min\{\varepsilon_{i-1}, \frac{\beta_{i-1}}{k-i+1}\}$, and there are $v'_i \in V'$ and $\beta_i > 0$ such that $\|v'_i\| < \varepsilon_i$ and, if $v' \in V'$, $\|v'\| \le \beta_i$ and $\varphi'_i(v) = v'_1 + v'_2 + \cdots + v'_i + v'$, then $\varphi''_i(v)$ is an isomorphism.

So it is sufficient to choose $f_n = f_{\bar v'_n}$ and $g_n = g_{\bar v'_n}$, where $\bar v'_n = \sum_{i=1}^k v'_i$.

In fact, we see that, as $\|\bar v'_n\|_{V'} \le k/n$, then $\lim\limits_{n\to\infty} \|\bar v'_n\|_{V'} = 0$, thus by (6.4) $\lim\limits_{n\to\infty} \|f_n\|_{C^1(\overline{\Omega})} = 0$ and $\lim\limits_{n\to\infty} \|g_n\|_{C^1(\overline{\Omega})} = 0$, as required.

Moreover, the solutions of

$$(P_n)\quad \begin{cases} -\mathrm{div}\big((\alpha_n + |\nabla u|^2)^{(p-2)/2}\nabla u\big) = \lambda |u|^{q-2}u + \frac{2\gamma}{p^*}|u|^{\gamma-2}u|v|^\beta + f_n & x \in \Omega \\ -\mathrm{div}\big((\alpha_n + |\nabla v|^2)^{(p-2)/2}\nabla v\big) = \mu |v|^{q-2}v + \frac{2\beta}{p^*}|u|^{\gamma}|v|^{\beta-2}v + g_n & x \in \Omega \\ u = v = 0 & x \in \partial\Omega \end{cases}$$

are critical points of $J_n = J_{\alpha_n, \bar v'_n} = T_{\alpha_n} - L_{\bar v'_n}$.

It is crucial to see that any critical point $\bar{z}$ of J_n belonging to $\bar{U} = \bigcup_{i=1}^k \bar{U}_i$ is nondegenerate. In fact we can write $\bar{z} = \bar{z}_i + \bar{v} + \bar{w}$ where $i \in \{1, \dots, k\}$, $(\bar{v}, \bar{w}) \in V \times W$ so, by (6.5) and (6.6),

$$\langle T'_{\alpha_n}(\bar{z}), w \rangle = \langle J'_n(\bar{z}), w \rangle + L_{\bar{v}'_n}(w) = 0, \qquad \forall\, w \in W,$$

so that $\bar{w} = \psi_i(\bar{v})$.

Moreover $\varphi'_i(\bar{v}) = \bar{v}'_n = v'_1 + v'_2 + \cdots + v'_i + (v'_{i+1} + \cdots + v'_k)$, where $\|v'_{i+1} + \cdots + v'_k\|_{V'} < (k-i)\varepsilon_{i+1} \leq \beta_i$.

So by construction $\varphi''_i(\bar{v})$ is an isomorphism and $J''_n(\bar{z}) = T''_{\alpha_n}(\bar{z})$ is injective.

In particular, Theorem 1.4 assures that the multiplicity of any critical point of J_n in $\bar{U}$ is 1.

Let us denote by k_n the number of critical points of J_n in $\bar{U}$, if it is finite, otherwise (P_n) has infinite (hence more than $\mathcal{P}_1(\Omega)$) distinct solutions.

As $\|\bar{v}'_n\|_{V'} < \varepsilon'$, it is $\|L_{\bar{v}'_n}\|_{C^1(A)} < \mu$, and in particular $\|J_n - T_{\alpha_n}\|_{C^1(A)} < \mu_i$ for any $i = 1, \dots, k$. By Remark 3.6 $J_n = J_{\alpha_n, f_{\bar{v}'_n}, g_{\bar{v}'_n}}$ satisfies $(P.S.)$ in A, so by Theorem 6.4 and (6.2) we infer that $k_n \geq \mathcal{P}_1(\Omega)$, hence in any case (P_n) has at least $\mathcal{P}_1(\Omega)$ distinct solutions. It remains to prove that these solutions are positive. If $z_n = (u_n, v_n) \in \bar{U}$ is a critical point of J_n, as $\bar{U} \subset A$, there is $j \in \{1, \dots, h\}$ such that $z_n \in B_{\gamma_j}(\tilde{z}_j)$. By construction, for any $k \geq n$ there is at least a critical point z_k of J_k such that $z_k = (u_k, v_k) \in B_{\gamma_j}(\tilde{z}_j)$. Theorem 1.1, see also Remark 2.1, assures that any $u_k, v_k \in C^{1,\eta}(\overline{\Omega})$ and the sequences $\{u_k\}_{k \geq n}$, $\{v_k\}_{k \geq n}$ are both bounded in $C^{1,\eta}(\overline{\Omega})$, so we infer that $z_k \to \tilde{z}_j = (\tilde{u}_j, \tilde{v}_j)$ in $C^1(\overline{\Omega})$-norm. By [31], we know that $\tilde{u}_j,\ \tilde{v}_j > 0$ in Ω and $\frac{\partial \tilde{u}_j}{\partial \nu}(x_0) \geq \xi > 0$ uniformly with respect to $x_0 \in \partial\Omega$, where ν is the interior normal of x_0. Consequently also $u_n > 0$, $v_n > 0$, if n is sufficiently large. □

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