Radial solutions for a Gelfand type quasilinear elliptic problem with quadratic gradient terms

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Abstract. In this paper, for a positive parameter λ, we study the existence of solutions of quasilinear elliptic problems whose model is

$$\begin{cases} -\Delta u + \dfrac{\mu(x)}{1+u}|\nabla u|^2 = \lambda(1+u)^p & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $0 < m \leq \mu(x) \leq M$ and $p > 1$. A particular emphasis will be placed on the existence of radial solution when the domain is the unit ball and $\mu(x) \equiv 1$.

1. Introduction

For an open and bounded set Ω in $\mathbb{R}^N$ $(N \geq 3)$ and $\lambda > 0$ we confront a quasilinear elliptic differential operator having lower order terms with quadratic growth in the gradient ([**5**]) with a Gelfand type nonlinearity. More precisely, we consider the boundary value problem

$$\begin{cases} -\Delta u + h(x,u)|\nabla u|^2 = \lambda f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where, from now on, f is a continuously derivable, strictly increasing, nonnegative function in $[0, +\infty)$ with $f(0) > 0$, $h(x, \cdot)$ is continuously derivable and nonnegative for a.e. $x \in \Omega$ and $h(\cdot, s)$ is measurable for every $s \geq 0$.

We say that $0 < u \in H_0^1(\Omega)$ is solution of (1.1) if $h(x,u)|\nabla u|^2, f(u) \in L^1(\Omega)$ and

$$\int_\Omega \nabla u \nabla \phi + \int_\Omega h(x,u)|\nabla u|^2 \phi = \lambda \int_\Omega f(u)\phi,$$

for every $\phi \in H_0^1(\Omega) \cap L^\infty(\Omega)$.

The aim of this note is twofold. First, we summarize the results about the existence of a solution of (1.1) proved in [**1**]. Indeed, we state that the maximal set of λ for which the problem has at least one positive solution is an interval $(0, \lambda^*]$,

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with $\lambda^* > 0$, and that there exists a minimal regular positive solution u_λ for every $\lambda \in (0, \lambda^*)$. Moreover, under additional technical assumptions, $u^* = \lim_{\lambda \to \lambda^*} u_\lambda$ is an *extremal solution*. We also proved, under suitable conditions depending on h, f and the dimension N, that for $\lambda = \lambda^*$ there exists a minimal regular positive solution.

The other goal of this work is to study the sharpness of the above restrictions on the dimension N. By this reason, we present here new results which provide a complete description of the set of radial solution when Ω is a ball. In order to do that we use the so-called Emden transform (see [**9, 13, 14**]) to reduce our radial quasilinear problem to a nonlinear system for which the phase plane study can be accomplished.

2. The general problem

In order to study the problem (1.1), we need to impose suitable conditions on the function h. Specifically, we suppose that there exists a nonnegative C^1-function g and positive constants $0 < m < M$ such that for every $x \in \Omega$, $s \geq 0$,

$$mg(s) \leq h(x, s) \leq Mg(s). \tag{2.1}$$

In addition, we assume that there exists $\theta > 0$ and $0 < \eta \leq 1$ such that a.e. $x \in \Omega$

$$0 \leq \theta \frac{\partial}{\partial s}\left(h(x,s) - mg(s)\right) + \left(h(x,s) - mg(s)\right)\left[(1+\theta)mg(s) - h(x,s)\right] \tag{2.2}$$

and

$$0 \geq (1-\eta) \frac{\partial}{\partial s}\left(h(x,s) - Mg(s)\right) + \left(h(x,s) - Mg(s)\right)\left[h(x,s) - \eta Mg(s)\right]. \tag{2.3}$$

Observe that (2.2) and (2.3) are trivially satisfied in the case $m = M$.

In [**1**] we show the existence of a parameter λ^* such that (1.1) has a solution if $\lambda < \lambda^*$ and no solution provided that $\lambda > \lambda^*$. Specifically, we prove:

Theorem 2.1. *Assume that h satisfies hypotheses* (2.1), (2.2) *and that $f'(s) - h(x,s)f(s)$ is an increasing function in s for every $x \in \Omega$. If $\frac{1}{f(s)} \in L^1(0, +\infty)$ and there exists a positive constant c such that*

$$\liminf_{s \to +\infty} \frac{f(s) e^{-M \int_0^s g(t)dt}}{\int_0^s e^{-M \int_0^r g(t)dt} dr} > 0, \tag{2.4}$$

and

$$\left| \frac{f'(s)}{f^2(s)} \right| \leq c(1 + \sqrt{g(s)}), \ \forall s \geq 0, \tag{2.5}$$

then there exists $\lambda^* \in (0, +\infty)$ such that (1.1) *admits a bounded minimal positive solution u_λ for every $\lambda \in (0, \lambda^*)$ and no positive solution for $\lambda > \lambda^*$. Moreover, if we also assume that the function g given by* (2.1) *is bounded, condition* (2.3) *holds and*

$$\lim_{s \to +\infty} \frac{s^2 (f'(s) - Mg(s)f(s)) e^{m \int_0^s g(t)dt}}{f(s) \int_0^s e^{m \int_0^t g(r)dr} dt} = \rho > \frac{1}{\eta},$$

then $\lim_{\lambda \to \lambda^*} u_\lambda(x) = u^*(x)$ almost everywhere in Ω, $u^* \in H_0^1(\Omega)$ and u^* is a solution of the problem (1.1) *with $\lambda = \lambda^*$.*

REMARK 2.2. We point out that the monotonicity condition imposed to the function $f'(\cdot) - h(x,\cdot)f(\cdot)$ substitutes the role of the usually required convexity of the function f in the study of Gelfand semilinear problems.

IDEA OF THE PROOF. The hypothesis (2.4) implies the non existence of solution of (1.1) for $\lambda >> 0$. This and the method of lower and upper solutions [**6**] imply that the set Λ of λ's for which (1.1) has a solution is a bounded interval. In addition, by the condition (2.5), we prove that if $\lambda_0 \in \Lambda$, then for every $\lambda < \lambda_0$, there exists a bounded upper solution of (1.1). On the other hand, condition (2.2) implies that the quasilinear operator $\Delta u + h(x,u)|\nabla u|^2$ satisfies the comparison principle. We remark explicitly that, in contrast with the comparison principle in [**2, 4**], we do not require neither that h is independent on $x \in \Omega$, nor the (increasing) monotonicity of h with respect to s. As a consequence, we establish the existence of a minimal solution u_λ of (1.1) provided that $0 < \lambda < \lambda^* := \sup \Lambda$.

To show the existence of extremal solution u^*, we have to verify mainly the uniform (for $\lambda \in (0,\lambda^*)$) boundedness of u_λ in $H_0^1(\Omega)$. The main difficulty for this is to establish that, similarly to the (variational) Gelfand semilinear problems, the minimal solutions u_λ of the (nonvariational) quasilinear problems (1.1) satisfy a suitable *stability condition*, namely,

$$\frac{1}{\eta}\int_\Omega |\nabla\phi|^2 \geq \lambda \int_\Omega [f'(u_\lambda) - Mg(u_\lambda)f(u_\lambda)]\phi^2, \quad \forall \phi \in H_0^1(\Omega), \tag{2.6}$$

where η is given by (2.3). □

In general, the extremal solution u^* given by the preceding theorem is not necessarily bounded. Using again the stability condition (2.6), we prove in [**1**] sufficient conditions on f and the dimension N to assure that u^* is *regular*, i.e. bounded. In particular, as a consequence, we obtain the following results for the model problem

$$\begin{cases} -\Delta u + \dfrac{\mu(x)}{1+u}|\nabla u|^2 = \lambda(1+u)^p & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.7}$$

COROLLARY 2.1. *If $1 < m \leq \mu(x) \leq M < p$, then there exists $\lambda^* > 0$ such that (2.7) has a minimal regular positive solution u_λ for every $\lambda < \lambda^*$ and no positive solution for every $\lambda > \lambda^*$. Moreover, if $(p-M)(m+1) > \frac{M-1}{m-1}$ then $u^* = \lim_{\lambda\to\lambda^*} u_\lambda$ is an extremal solution. Even more, u^* is regular provided that*

$$3 \leq N < 4\frac{(p-M)(m-1)}{(p-1)(M-1)} + 2 + \frac{4(m-1)}{M-1}\sqrt{\frac{p-M}{p-1}}.$$

COROLLARY 2.2. *If $\mu(x) \equiv c \in [0,p)$ with $p > 1$, then there exists $\lambda^* > 0$ such that (2.7) has a minimal regular positive solution u_λ for every $\lambda < \lambda^*$ and no positive solution for every $\lambda > \lambda^*$. Moreover, if $(p-c)(c+1) > 1$ then $u^* = \lim_{\lambda\to\lambda^*} u_\lambda$ is an extremal solution. Even more, u^* is regular provided that*

$$3 \leq N < 4\frac{p-c}{p-1} + 2 + 4\sqrt{\frac{p-c}{p-1}}.$$

REMARK 2.3. Observe that (2.7) reduces to a semilinear problem when $\mu(x) \equiv 0$. Also, in the particular case $\mu(x) \equiv 1$, $p = 2$, the change $w = \ln(1+u)$ relates the problem (2.7) with the classical Gelfand problem

$$\begin{cases} -\Delta w = \lambda e^w & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.8}$$

This kind of semilinear problems appears in a vast amount of works, among others [**3, 8, 13, 14**], being motivated by different applications (study of an isothermal gas in gravitational equilibrium [**7**], thermal self-ignition in combustion theory [**10**], temperature distribution in an object heated by a uniform electric current [**11, 12**], etc.).

3. Radial solutions for the Gelfand quasilinear problem

From now on we assume that $\Omega = B$, the unit ball in $\mathbb{R}^N$ $(N \geq 3)$, $\mu(x) \equiv 1$ and $p = 2$. Thus problem (2.7) reduces to

$$\begin{cases} -\Delta u + \dfrac{|\nabla u|^2}{1+u} = \lambda(1+u)^2 & \text{in } B \\ u = 0 & \text{on } \partial B. \end{cases} \tag{3.1}$$

Observe that in general solutions of (3.1) are not necessarily non negative and in order to completely describe the set of radial solutions, we say that $u \in H_0^1(B)$ is a radial solution of (3.1) if $u(x)$ depends only on the modulus $|x|$ of x, meas $\{x \in B : u(x) = -1\} = 0$, $\frac{|\nabla u|^2}{1+u} \in L^1(B)$ and

$$\int_B \nabla u \nabla \varphi + \int_B \frac{|\nabla u|^2}{1+u}\varphi = \lambda \int_B (1+u)^2 \varphi,$$

for every $\varphi \in H_0^1(B) \cap L^\infty(B)$.

THEOREM 3.1. *For every $\lambda > 0$, problem (3.1) admits infinitely many negative radially increasing solutions with $u(0) = -1$. Moreover, there exist infinitely many bounded and sign-changing radial solutions with $u(0) = -1$. Even more, $\lambda_* = \sup\{\lambda \in \mathbb{R}^+ : (3.1)$ admits positive radial solution$\}$ is a finite number satisfying the following:*

(1) *If $N \geq 10$, then $\lambda_* = 2(N-2)$ and (3.1) has a unique positive radial regular solution for every $\lambda \in (0, \lambda_*)$.*
(2) *If $2 < N < 10$, then $2(N-2) < \lambda_*$ and there exists $0 < \delta < 2(N-2)$ such that*
   (a) *For every $\lambda \in (0, \delta)$ there exists a unique positive regular radial solution.*
   (b) *For every $\lambda \in [\delta, 2(N-2)) \cup (2(N-2), \lambda_*)$ we have finite multiplicity of positive regular radial solutions.*
   (c) *For $\lambda = 2(N-2)$ problem (3.2) has infinitely many positive regular radial solutions and for $5 \leq N < 10$ a unique positive singular solution.*
   (d) *For $\lambda = \lambda_*$ there exists a unique positive regular radial solution.*

REMARK 3.2. Observe that the classical Gelfand semilinear problem (2.8) has neither negative nor sign-changing solutions (see Remark 3.7). In contrast, due to its quasilinear nature, we prove the existence of infinitely many negative radial solutions of (3.1) for every $\lambda > 0$.

The proof of Theorem 3.1 requires several previous lemmas. First, observe that classical radial solutions of (3.1) are characterized by means of real functions $u : [0,1] \to \mathbb{R}$ satisfying $u_r'(0) = 0$ and

$$\begin{cases} -u_r'' - u_r' \left(\dfrac{N-1}{r} \right) + \dfrac{u_r'^2}{1+u} = \lambda(1+u)^2, & r \in (0,1) \\ u(1) = 0. \end{cases} \tag{3.2}$$

As usual (see [**7, 10, 11**]), we consider the so-called Emden transform $s = \log r$ to rewrite the quasilinear equation in (3.2) as

$$-u_s'' \frac{1}{e^{2s}} + u_s' \frac{1}{e^{2s}} - u_s' \frac{1}{e^s} \frac{(N-1)}{e^s} + \frac{(u_s')^2}{e^{2s}(1+u)} = \lambda(1+u)^2.$$

Using

$$\begin{cases} x = \dfrac{u_s'(s)}{1+u(s)} \\ y = -\lambda(1+u(s))e^{2s}, \end{cases}$$

we deduce that the radial solutions of (3.1) satisfy the following autonomous system of ordinary differential equations in the plane (x, y)

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} -(N-2)x + y \\ (x+2)y \end{pmatrix}. \tag{3.3}$$

The autonomous system (3.3) appears in [**14**] and its phase portrait is also studied in [**9**]. It has two critical points, $P_1 = (0,0)$ which is a saddle point and $P_2 = (-2, -2(N-2))$ which is either a stable focus for (3.3) if $2 < N \leq 9$ or a stable node if $N \geq 10$. In the following lemmas we give the main properties of its phase portrait.

Lemma 3.3 ([**9**]). *The orbit Γ_u corresponding to the unstable manifold of P_1 is bounded and it is contained in the third quadrant $x, y < 0$. Moreover, it corresponds to a curve joining the points P_1 and P_2 which is either the graph of a monotone function if $N \geq 10$ or a spiral if $3 \leq N \leq 9$. Even more, given a solution $(x(s), y(s))$ of (3.3) with Γ_u as associated orbit, there exists $s_1 < 0$ such that, for every $s < s_1$,*

$$0 > \frac{1}{\frac{1}{4N}e^{2s_1} - \frac{1}{4N}e^{2s} + \frac{e^{2s_1}}{y(s_1)}} > y(s)e^{-2s} > \frac{1}{\frac{1}{N}e^{2s_1} - \frac{1}{N}e^{2s} + \frac{e^{2s_1}}{y(s_1)}}. \tag{3.4}$$

□

Now we prove that there are only three bounded orbits in the semiplane $y \leq 0$, namely P_1, P_2 and Γ_u and any other orbit for $y < 0$ intersects both the third and fourth quadrants.

Lemma 3.4. *Let $x_0 \in \mathbb{R}$, $y_0 < 0$ and assume that $(x_0, y_0) \notin P_1 \cup P_2 \cup \Gamma_u$. Then the solution $(x(s), y(s))$ of (3.3) such that $(x(0), y(0)) = (x_0, y_0)$ is defined for every $s \in \mathbb{R}$ and verifies that*

$$\lim_{s \to -\infty} (x(s), y(s)) = (+\infty, 0) \quad \textit{and} \quad \lim_{s \to +\infty} (x(s), y(s)) = P_2.$$

Proof. Assume that $(x(s), y(s))$ is defined in an interval (α, β) with $-\infty \leq \alpha < 0 < \beta \leq +\infty$. Let us denote by $\Gamma^+ = \{(x(s), y(s)) : s > 0\}$ and $\Gamma^- = \{(x(s), y(s)) : s \leq 0\}$. In [**9**] it is proved that Γ^+ is bounded, being P_2 its unique accumulation point. In particular, $\beta = +\infty$. Moreover, it is proved that Γ^-

intersects the fourth quadrant and we can assume (with no loss of generality) that Γ^- is contained in the fourth quadrant. By (3.3) we also have

$$e^{(N-2)s}x(s) = x_0 + \int_0^s e^{(N-2)t}y(t)dt \leq x_0 + y_0 s, \quad y(s) = y_0 e^{\int_0^s (x(t)+2)dt},$$

for $s < 0$. Thus, $\Gamma^- \subset (x_0, (x_0 + y_0\alpha)e^{-(N-2)\alpha}) \times (y_0, 0)$, that is, Γ^- is unbounded if and only if $\alpha = -\infty$. Using that x, y are decreasing we have that $\lim_{s\to-\infty} x(s) \in (x_0, +\infty]$ and $\lim_{s\to-\infty} y(s) \in (y_0, 0]$. Therefore, necessarily $\lim_{s\to-\infty} x(s) = +\infty$. In addition, if $\lim_{s\to-\infty} y(s) \neq 0$ then, using (3.3), $\lim_{s\to-\infty} y'(s) = -\infty$ which contradicts that $\lim_{s\to-\infty} y(s)$ is a real number. We conclude that $\lim_{s\to-\infty}(x(s), y(s)) = (+\infty, 0)$. □

We prove the relation between solutions of (3.3) and radial solutions of (3.5).

LEMMA 3.5. *Let $(x(s), y(s))$ be a solution of (3.3) with $(x(0), y(0)) = (x_0, y_0) \in \mathbb{R} \times (-\infty, 0)$ and $v(r) = \frac{y(\ln r)}{y_0 r^2} - 1$. Then, $u(z) = v(|z|)$ is a classical solution of*

$$-\Delta u + \frac{|\nabla u|^2}{1+u} = \lambda(1+u)^2 \quad \text{in } B \setminus \{0\}. \tag{3.5}$$

with $\lambda = -y_0$. If we also assume that $\lim_{r\to0^+} r^{N-1}v(r) = \lim_{r\to0^+} r^{N-1}v'(r) = 0$ and $(1+u)^2, |\nabla u|, \frac{|\nabla u|^2}{1+u} \in L^1(B)$, then u is a solution of (3.1) in the sense of distributions. In particular, if in addition $u \in H_0^1(B)$, then it is a weak solution of (3.1).

PROOF. A direct computation shows that v is a classical solution of (3.2) with $\lambda = -y_0$ in the interval $r \in (0, 1]$, i.e., u solves (3.5) with $\lambda = -y_0$. Assume now that $\lim_{r\to0^+} r^{N-1}v(r) = \lim_{r\to0^+} r^{N-1}v'(r) = 0$ and $(1+u)^2, |\nabla u|, \frac{|\nabla u|^2}{1+u} \in L^1(B)$. Taking into account that $\lim_{r\to0^+} r^{N-1}v(r) = 0$ and $u, |\nabla u| \in L^1(\Omega)$, u is weakly differentiable in B. In order to show that $u(z)$ is a solution in the sense of distributions for (3.1) in B, with $\lambda = -y_0$, we multiply by a test function ϕ in the space $C_0^\infty(\Omega)$ of the C^∞ functions with compact support in B and integrating by parts in $B \setminus B_\rho$ we have that

$$\int_{B\setminus B_\rho} \nabla u \nabla \phi - \int_{\partial B_\rho} \frac{\partial u}{\partial n}\phi + \int_{B\setminus B_\rho} \frac{|\nabla u|^2}{1+u}\phi = \lambda \int_{B\setminus B_\rho} (1+u)^2\phi.$$

Observe that, since $(1+u)^2, |\nabla u|, \frac{|\nabla u|^2}{1+u} \in L^1(B)$, then

$$\int_{B\setminus B_\rho} (1+u)^2\phi \to \int_B (1+u)^2\phi, \quad \int_{B\setminus B_\rho} \frac{|\nabla u|^2}{1+u}\phi \to \int_B \frac{|\nabla u|^2}{1+u}\phi$$

$$\text{and} \quad \int_{B\setminus B_\rho} \nabla u \nabla \phi \to \int_B \nabla u \nabla \phi,$$

as $\rho \to 0$. Furthermore, since $\lim_{\rho\to0^+} \rho^{N-1}v'(\rho) = 0$, if we denote by w_N the measure of the unity ball B, we have

$$\left| \int_{\partial B_\rho} \frac{\partial u}{\partial n}\phi \right| \leq N w_N \|\phi\|_\infty \rho^{N-1}|v'(\rho)| \overset{(\rho\to0)}{\longrightarrow} 0.$$

Therefore we obtain that

$$\int_B \nabla u \nabla \phi + \int_B \frac{|\nabla u|^2}{1+u}\phi = \lambda \int_B (1+u)^2 \phi,$$

i.e., $u(z)$ is a solution in the sense of distributions for (3.1) in B, with $\lambda = -y_0$. □

Now, we analyze how we can pass from solutions of (3.3) to radial solutions of (3.1).

LEMMA 3.6. *Let $(x(s), y(s))$ be a solution of (3.3), with $(x(0), y(0)) = (x_0, y_0) \in \mathbb{R} \times (-\infty, 0)$ and associated orbit Γ. If we denote by $v(r) = \frac{y(\ln r)}{y_0 r^2} - 1$, $\lambda = -y_0$ and $u(z) = v(|z|)$, then:*

(1) *If $N \geq 5$ and $\Gamma = \{P_2\}$ then $u(z) = \frac{1}{|z|^2} - 1$, is a singular solution, in the sense of distributions, of (3.1) for $\lambda = 2(N-2)$. Moreover, u is a singular weak solution for every $N \geq 7$.*
(2) *If $\Gamma = \Gamma_u$ then $u(z) = \frac{y(\ln |z|)}{y_0 |z|^2} - 1$ is a positive and bounded, radially decreasing solution of (3.1) for $\lambda = -y_0$.*
(3) *If Γ^- is contained in the fourth quadrant then $u(z) = \frac{y(\ln |z|)}{y_0 |z|^2} - 1$ is a negative, radially increasing solution of (3.1) with $\lambda = -y_0$ and moreover,*

$$\lim_{|z| \to 0} u(z) = -1, \quad \lim_{|z| \to 1} \frac{\partial u(z)}{\partial n} = x_0 \geq 0.$$

PROOF. (1) In this case $(x(s), y(s)) = (-2, -2(N-2)) = (x_0, y_0)$ and thus $v(r) = 1/r^2 - 1$, for every $r \in (0, 1]$ and $u(z) = 1/|z|^2 - 1$, for every $z \in B \setminus \{0\}$. Observing that $N \geq 5$ implies that $\lim_{r \to 0^+} r^{N-1} v(r) = \lim_{r \to 0^+} r^{N-1} v'(r) = 0$, and using Lemma 3.5, we conclude the proof of this case. Moreover, since $u \in W_0^{1,p}(B)$ for every $p < \frac{N}{3}$ we have that u is a singular weak solution for every $N \geq 7$.

(2) First we observe that $v(1) = \frac{y(\ln 1)}{y_0} - 1 = 0$ and $v'(r) = \frac{x(\ln r) y(\ln r)}{y_0 r^3} < 0$, since Γ^- contained in the third quadrant. Thus u is positive and radially decreasing. Moreover, using (3.4), we deduce the existence of $s_1 < 0$ such that, for every $|z| < e^{s_1}$,

$$u(z) < \frac{1}{\frac{1}{N} e^{2s_1} - \frac{1}{N}|z|^2 + \frac{e^{2s_1}}{y(s_1)}} - 1.$$

In particular, u is bounded.

Let us prove that $u(z) = \frac{y(\ln |z|)}{y_0 |z|^2} - 1$ is a solution of (3.1) for $\lambda = -y_0$. Observe that, since $N \geq 3$, we have that $\lim_{r \to 0^+} r^{N-1} v(r) = \lim_{r \to 0^+} r^{N-1} v'(r) = 0$. On the other hand, note that

$$|\nabla u(z)| = \left| \frac{x(\log |z|) y(\log |z|)}{y_0 |z|^3} \right| = \left| \frac{x(\log |z|)}{|z|} \right| (1 + u(z)),$$

and that we can assume $0 > x(s) \geq -2$ for every $s < 0$. Thus, for some positive constant c,

$$\int_B |\nabla u|^2 \leq c \int_B \frac{x^2(\ln |z|)}{|z|^2} dz \leq 4c \int_B \frac{1}{|z|^2} dz < +\infty,$$

i.e., $|\nabla u| \in L^2(B)$, which, being u positive and bounded, implies that $(1+u)^2$, $|\nabla u|$, $\frac{|\nabla u|^2}{1+u} \in L^1(B)$. The proof of the case is concluded by using the Lemma 3.5.

(3) Since Γ^- is contained in the fourth quadrant, we deduce that $v'(r) = \frac{x(\ln r)y(\ln r)}{y_0 r^3} > 0$. Thus, u is negative and radially increasing, which implies the existence of $\lim_{|z|\to 0} u(z)$. Moreover, integrating the second equation in (3.3), we have

$$0 \le 1 + u(z) = \frac{y(\ln|z|)}{y_0|z|^2} = e^{-\int_{\ln|z|}^0 x(t)dt}$$

for every $0 < |z| < 1$. Then, using Lemma 3.4, we obtain that $\lim_{|z|\to 0} u(z) = -1$. Even more, since $\frac{v'(r)}{1+v(r)} = \frac{x(\ln r)}{r}$, we derive that $\lim_{r\to 1} v'(r) = x_0$, that is, $\lim_{|z|\to 1} \frac{\partial u(z)}{\partial n} = x_0$.

On the other hand, by (3.3), $x'(s) + (N-2)x(s) \ge y_0$ for $s < 0$. In particular,

$$\left(e^{(N-2)s}x(s)\right)' \ge e^{(N-2)s}y_0.$$

Integrating in the interval $(s, 0)$,

$$e^{(N-2)s}x(s) \le x_0 - \frac{y_0}{N-2}(1 - e^{(N-2)s}) \le x_0 - \frac{y_0}{N-2}.$$

In consequence, $r^{N-2}x(\ln r)$ is bounded for $r \to 0^+$ and, using that $\lim_{r\to 0^+} u(r) = -1$, we get $(N \ge 3)$

$$\lim_{r\to 0^+} r^{N-1}v'(r) = \lim_{r\to 0^+} r^{N-2}x(\ln r)(1 + v(r)) = 0,$$

and

$$\lim_{r\to 0^+} r^{N-1}v(r) = \lim_{r\to 0^+} r^{N-3}y(\ln r) = 0.$$

Now we also prove that $\frac{|\nabla u|^2}{1+u} \in L^1(B)$. Indeed, observe that for some positive constant c

$$\begin{aligned}\int_B \frac{|\nabla u|^2}{1+u} &= \int_B \left|\frac{x(\log|z|)}{|z|}\right|^2 (1+u(z)) = \int_B \frac{x^2(\ln|z|)}{|z|^2}\frac{y(\ln|z|)}{y_0|z|^2}dz \\ &= \int_0^1 \frac{x^2(\ln r)}{r^2}\frac{y(\ln r)}{y_0 r^2}cr^{N-1}dr = c\int_{-\infty}^0 x^2(s)\frac{y(s)}{y_0e^{2s}}e^{(N-2)s}ds.\end{aligned}$$

Integrating the second equation in (3.3) we deduce that $\frac{y(s)}{y(0)e^{2s}} = e^{-\int_s^0 x(t)dt}$, for every $s < 0$. Moreover, by the first equation in (3.3) we also have

$$(e^{(N-2)s}x(s))' = e^{(N-2)s}y(s).$$

Thus, integrating by parts and using that $x(0) = 0$ and $\lim_{s\to 0} x(s) = +\infty$, we deduce that

$$\begin{aligned}\int_B \frac{|\nabla u|^2}{1+u} &= -cx(0) + c\lim_{s\to-\infty} e^{-\int_s^0 x(t)dt}x(s)e^{(N-2)s} + c\int_{-\infty}^0 \frac{y(s)}{y(0)e^{2s}}y(s)e^{(N-2)s}ds \\ &= c\int_{-\infty}^0 \frac{y(s)}{y(0)e^{2s}}y(s)e^{(N-2)s}ds.\end{aligned}$$

Moreover,

$$\frac{y^2(s)e^{(N-4)s}}{e^{Ns}} = \left(\frac{y(s)}{e^{2s}}\right)^2 = \left(y_0e^{-\int_s^0 x(t)dt}\right)^2 \overset{(s\to-\infty)}{\longrightarrow} 0.$$

Therefore, taking into account that e^{Ns} is integrable in $(-\infty, 0)$ we conclude that $\int_B \frac{|\nabla u|^2}{1+u} < +\infty$ and thus the proof by using Lemma 3.5. $\square$

Remark 3.7. • If we consider the radial solutions of the Dirichlet boundary value problem for the semilinear equation $-\Delta u = \lambda e^u$ with zero boundary condition, by the change $y(s) = -\lambda e^{u(s)} e^{2s}$ and $x(s) = u'(s)$ we achieve the identical phase diagram. In this case, by the maximum principle, there is no negative solution. Therefore, if Γ^- is contained in the fourth quadrant, then $u(z) = \ln\left(\frac{y(\ln|z|)}{-y_0|z|^2}\right)$ is a classical solution in $B \setminus \{0\}$ which can not be extended to a solution in B.

• Taking into account Lemmas 3.4 and 3.5, the above lemma shows that bounded radial positive solutions of (3.1) correspond with solutions of (3.3) with initial data in Γ_u. Moreover, the unique unbounded radial positive solution correspond to $\frac{1}{|z|^2} - 1$ for $\lambda = 2(N-2)$ and $N \geq 5$. Even more, since Γ_u is bounded, (3.1) has no positive radial solution for every λ greater than the infimum of the projection of Γ_u in the axis of y.

• In the case of item (2) or item (3) of the above lemma and with the same notation, we observe that if $y_0 \in (-\infty, 0)$, then for every $x_0 \in \mathbb{R}$ there exists a solution $(x(s, x_0), y(s, x_0))$ of (3.3) such that $(x(0, x_0), y(0, x_0)) = (x_0, y_0)$. Thus $u(z, x_0) = \frac{y(\ln z, x_0)}{y_0|z|^2} - 1$ is a radial solution of (3.1) for $\lambda = -y_0$ with $\frac{\partial u}{\partial n}(z, x_0) = x_0$ for $|z| = 1$. In particular, problem (3.1) for $\lambda = -y_0$ has infinitely many radial solutions, which are:

- positive and bounded provided that $(x_0, y_0) \in \Gamma_u$,
- negative and bounded if $x_0 \geq 0$ or
- sign-changing solutions for $x_0 < 0$ and $(x_0, y_0) \notin \Gamma_u \cup P_2$.

Proof of Theorem 3.1. The first part is deduced from Lemma 3.6, see Remark 3.7.

(1) The trajectory Γ_u joining P_1 and P_2 is the unstable manifold for the node P_1. By Lemmas 3.3 and 3.4, Γ_u is a monotone curve contained in the region $-2 < x < 0$, $-2(N-2) < y < 0$, then for each line $y = -\lambda$ we have a unique point of intersection and, therefore, a unique regular radial solution for each $\lambda \in (0, 2(N-2))$.

(2) In this case, by Lemmas 3.3 and 3.4, Γ_u has a spiral shape. Thus the number of intersection points of the manifold Γ_u with the straight line $y = -\lambda$ is

- a unique point for λ small enough, case (a), or $\lambda = \lambda_*$, case (d).
- a finite number of points in the case (b).
- infinite points in the case (c). $\square$

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