THE PRINCIPAL EIGENVALUE FOR A CLASS OF SINGULAR QUASILINEAR ELLIPTIC OPERATORS AND APPLICATIONS

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Abstract. We characterize the principal eigenvalue associated to the singular quasilinear elliptic operator $-\Delta u - \mu(x)\frac{|\nabla u|^q}{u^{q-1}}$ in a bounded smooth domain $\Omega \subset \mathbb{R}^N$ with zero Dirichlet boundary conditions. Here, $1 < q \leq 2$ and $0 \leq \mu \in L^\infty(\Omega)$. As applications we derive some existence of solutions results (as well as uniqueness, nonexistence and homogenization results) to a problem whose model is

$$\begin{cases} -\Delta u = \lambda u + \mu(x)\dfrac{|\nabla u|^q}{|u|^{q-1}} + f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\lambda \in \mathbb{R}$ and $f \in L^p(\Omega)$ for some $p > \frac{N}{2}$.

1. Introduction

We consider a bounded domain $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ with $C^{1,1}$ boundary and study the quasilinear elliptic problem:

$$(P_\lambda) \qquad \begin{cases} -\text{div}(m(x)\nabla u) = \lambda u + \mu(x)\dfrac{|\nabla u|^q}{|u|^{q-1}} + f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\lambda \in \mathbb{R}$, $0 \leq \mu \in L^\infty(\Omega)$, $f \in L^p(\Omega)$ with $p > \frac{N}{2}$, $1 < q \leq 2$ and $0 < \eta \leq m \in L^\infty(\Omega) \cap W^{1,\infty}_{\text{loc}}(\Omega)$. We say that a solution to problem (P_λ) is a function $u \in H^1_0(\Omega) \cap L^\infty(\Omega)$ such that $\mu(x)\frac{|\nabla u|^q}{|u|^{q-1}} \in L^1(\{|u| > 0\})$ and

$$\int_\Omega m(x)\nabla u \nabla \phi = \lambda \int_\Omega u\phi + \int_{\{|u|>0\}} \mu(x)\frac{|\nabla u|^q}{|u|^{q-1}}\phi + \int_\Omega f(x)\phi,$$

for every $\phi \in H^1_0(\Omega) \cap L^\infty(\Omega)$.

The aim of this note is to summarize the known results, obtained in [6] and [7], concerning the existence, uniqueness, homogenization and nonexistence of solution to problem (P_λ) (which improve, in some sense, those contained in [2] for $q = 2$). In these mentioned papers, it is shown that the validity of such results depends on the existence of a principal eigenvalue for the eigenvalue problem

$$(E_\lambda) \qquad \begin{cases} -\text{div}(m(x)\nabla u) = \lambda u + \mu(x)\dfrac{|\nabla u|^q}{|u|^{q-1}} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

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Inspired by [3], the principal eigenvalue can be characterized by

$$\lambda^* = \sup\left\{ \lambda \in \mathbb{R} \;\middle|\; \begin{array}{c} \text{there exists a supersolution } v \text{ to } (E_\lambda) \\ \text{such that } v \geq c \text{ in } \Omega \text{ for some } c > 0 \end{array} \right\}, \tag{1.1}$$

where the precise meaning of supersolution used in (1.1) is specified in the next section.

2. Principal eigenvalue

We say that $v \in H^1(\Omega) \cap L^\infty(\Omega)$ is a *supersolution* to (E_λ) if $v > 0$ a.e. in Ω, $\frac{|\nabla v|^q}{v^{q-1}} \in L^1_{\mathrm{loc}}(\Omega)$ and the following inequality holds

$$\int_\Omega m(x)\nabla v \nabla\phi \geq \lambda \int_\Omega v\phi + \int_\Omega \mu(x)\frac{|\nabla v|^q}{v^{q-1}}\phi \qquad \forall \phi \in H^1_0(\Omega)\cap L^\infty(\Omega), \quad \phi \geq 0\,. \tag{2.1}$$

Analogously it is defined de concept of supersolution for (P_λ) and, with the reverse inequality, the concept of subsolution. Moreover, we say that

- v satisfies condition (v_1) if $v \geq c$ in Ω for some $c > 0$.
- v satisfies condition (v_2) if $v - c \in H^1_0(\Omega)$ for some $c > 0$.
- v satisfies condition (v_3) if, for some $\gamma_0 < 1$, $v^\gamma \in H^1(\Omega)$ for every $\gamma > \gamma_0$.

Thus, in order to summarize the main properties and characterizations of λ^*, we define for $i = 1, 2, 3$,

$$I_i = \left\{ \lambda \in \mathbb{R} \;\middle|\; \begin{array}{c} \text{there exists a supersolution } v \text{ to } (E_\lambda) \\ \text{such that } v \text{ satisfies } (v_i) \end{array} \right\}.$$

Proposition 2.1. *Assume that* $1 < q \leq 2$, $0 \leq \mu \in L^\infty(\Omega)$ *and* $0 < \eta \leq m \in L^\infty(\Omega)$*. Then the sets* I_1, I_2 *and* I_3 *are nonempty intervals which are unbounded from below, so* $\lambda^* = \sup I_1$ *is well defined. Moreover,* $I_1 = I_2$ *and* $\lambda^* = \sup I_3$*. In addition,* $0 < \lambda^* \leq \lambda_1(m) \equiv \inf_{w \in H^1_0(\Omega)\setminus\{0\}} \dfrac{\int_\Omega m(x)|\nabla w|^2}{\int_\Omega w^2}$.

Proof. We include here the main steps in the proof, further detail may be found in [6] in the case $m(x) = 1$.

First we observe that, from the concept of supersolution it is easily deduced that $(-\infty, \lambda] \subset I_i$ whenever $\lambda \in I_i$. Moreover, taking $v = 1$ as a supersolution to (E_0) we derive that $(-\infty, 0] \subset I_i$. In particular, I_i is an interval unbounded from below.

Step 1. $I_1 = I_2$. Observe that, since $(-\infty, 0] \subset I_1 \cap I_2$ then it is enough to prove that $I_1 \cap (0, +\infty) = I_2 \cap (0, +\infty)$. Assume that $0 < \lambda \in I_2 \cap (0, +\infty)$. Hence, there exist $v \in H^1(\Omega) \cap L^\infty(\Omega)$ and $c > 0$ with $v > 0$ in Ω, $v - c \in H^1_0(\Omega)$, and

$$-\mathrm{div}(m(x)\nabla(v - c)) = -\mathrm{div}(m(x)\nabla v) \geq \lambda v + \mu(x)\frac{|\nabla v|^q}{v^{q-1}} \geq 0 \text{ in } \Omega.$$

Therefore, the maximum principle yields to $v \geq c$ in Ω, and so $\lambda \in I_1 \cap (0, +\infty)$.

Conversely if $0 < \lambda \in I_1$ and $-\mathrm{div}(m(x)\nabla v) \geq \lambda v + \mu(x)\frac{|\nabla v|^q}{v^{q-1}}$ in Ω for some $v \in H^1(\Omega) \cap L^\infty(\Omega)$ with $v \geq c > 0$ then $v - c$ is a non-negative supersolution to the problem (without singularity)

$$\begin{cases} -\mathrm{div}(m(x)\nabla u) = \lambda u + \mu(x)\dfrac{|\nabla u|^q}{(|u| + c)^{q-1}} + \lambda c & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.2}$$

Since the trivial function it is a subsolution there exists a solution $u \in H^1_0(\Omega) \cap L^\infty(\Omega)$ to (2.2) (see [5, Théorème 3.1]) with $0 \leq u \leq v - c$ in Ω. Thus, $u + c$ is a supersolution to (E_λ) that satisfies (v_2) and therefore $\lambda \in I_2$.

Step 2. $\lambda^* = \sup I_3$. First we observe the trivial inclusion $I_1 \subset I_3$. Thus in order to prove Step 2 we are going to show that $I_3 - \epsilon \subset I_1$ for every $\epsilon > 0$ small enough. Indeed, assume that $\lambda \in I_3$, i.e. there exist $u \in H^1(\Omega) \cap L^\infty(\Omega)$ and $\widetilde{\gamma} \in (0,1)$ satisfying

$$u > 0 \text{ in } \Omega, \quad -\mathrm{div}(m(x)\nabla u) \geq \lambda u + \mu(x)\frac{|\nabla u|^q}{u^{q-1}} \text{ in } \Omega, \quad u^\gamma \in H^1(\Omega) \quad \forall \gamma > \widetilde{\gamma}.$$

Then we show that $\lambda - \epsilon \in I_1$. In fact we prove that a supersolution to $(E_{\lambda-\epsilon})$ is $v = \varepsilon(\varphi_1^\gamma + 1) + u^\gamma$ where ε is a small enough positive constant, $\gamma \in \left(\max\left\{\frac{1}{2}, \widetilde{\gamma}, \frac{\lambda-\epsilon}{\lambda}\right\}, 1\right)$ and $\varphi_1 > 0$ is the the principal positive and normalized eigenfunction associated to $\lambda_1(m)$, that is,

$$\begin{cases} -\mathrm{div}(m(x)\nabla\varphi_1) = \lambda_1(m)\varphi_1 & \text{in } \Omega, \\ \varphi_1 = 0 & \text{on } \partial\Omega. \end{cases}$$

Observe that, since $\gamma > \frac{1}{2}$ it is easy to deduce that $\varphi_1^\gamma \in H_0^1(\Omega)$. Indeed, take $(\varphi_1 + \delta)^{2\gamma-1} - \delta^{2\gamma-1}$ as test function in the equation satisfied by φ_1 and use Fatou lemma as $\delta \to 0$.

Thus, since $\gamma > \widetilde{\gamma}$, we have $v \in H^1(\Omega) \cap L^\infty(\Omega)$ and, clearly, $v \geq \varepsilon$ in Ω and only remains to prove that v is a supersolution to $(E_{\lambda-\epsilon})$.

Let $\phi \in H_0^1(\Omega) \cap L^\infty(\Omega)$ be such that $\phi \geq 0$ in Ω and has compact support. Direct computations yield to

$$\begin{aligned} &\int_\Omega \left(-m(x)\nabla v\nabla\phi + (\lambda-\epsilon)v\phi + \mu(x)\frac{|\nabla v|^q}{v^{q-1}}\phi\right) \leq -(\gamma\lambda - (\lambda-\epsilon))\int_\Omega u^\gamma\phi + \\ &\varepsilon\int_\Omega\left(-\gamma(1-\gamma)m(x)\frac{|\nabla\varphi_1|^2}{\varphi_1^{2-\gamma}} + ((\lambda-\epsilon) - \gamma\lambda_1(m))\varphi_1^\gamma + (\lambda-\epsilon) + \|\mu\|_{L^\infty(\Omega)}C_1\frac{|\nabla\varphi_1|^q}{\varphi_1^{q(1-\gamma)}}\right)\phi. \end{aligned} \tag{2.3}$$

Using Hopf lemma we can assure that $|\nabla\varphi_1|$ is bounded away from zero in a small neighborhood Ω_δ of the boundary. Using also that $\gamma > \frac{\lambda-\epsilon}{\lambda}$ and $q(1-\gamma) < 2-\gamma$, we choose δ sufficiently small and independent of ϵ, such that, in Ω_δ

$$\Psi(x) \equiv -\gamma(1-\gamma)m(x)\frac{|\nabla\varphi_1|^2}{\varphi_1^{2-\gamma}} + ((\lambda-\epsilon) - \gamma\lambda_1(m))\varphi_1^\gamma + (\lambda-\epsilon) + \|\mu\|_{L^\infty(\Omega)}C_1\frac{|\nabla\varphi_1|^q}{\varphi_1^{q(1-\gamma)}} \leq 0. \tag{2.4}$$

Consequently, we take ϵ small enough in order to have in $\Omega \setminus \Omega_\delta$

$$\epsilon\Psi(x) \leq \epsilon C_3 \leq (\gamma\lambda - (\lambda-\epsilon))\inf_{\Omega\setminus\Omega_\delta}(u^\gamma) \leq (\gamma\lambda - (\lambda-\epsilon))u^\gamma. \tag{2.5}$$

Gathering (2.3), (2.4) and (2.5) together we conclude that

$$\int_\Omega m(x)\nabla v\nabla\phi \geq (\lambda-\epsilon)\int_\Omega v\phi + \int_\Omega \mu(x)\frac{|\nabla v|^q}{\cdot}v^{q-1}\phi.$$

Step 3. $\lambda^* > 0$. First we choose $c > 0$ large enough and $\delta > 0$ small enough in order to assure (using [9, Theorem 3.4]) the existence of solution $0 \leq u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ to

$$\begin{cases} -\mathrm{div}(m(x)\nabla u) = \frac{\mu(x)}{c^{q-1}}|\nabla u|^q + \delta & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.6}$$

Then, for some $\lambda > 0$ small, $v = u + c \in H^1(\Omega) \cap L^\infty(\Omega)$ is a supersolution to (E_λ) which satisfies (v_1). Indeed,

$$-\mathrm{div}(m(x)\nabla v) = -\mathrm{div}(m(x)\nabla u) \geq \mu(x)\frac{|\nabla v|^q}{v^{q-1}} + \lambda v + (\delta - \lambda\|v\|_{L^\infty(\Omega)}).$$

Step 4 $\lambda^* \le \lambda_1(m)$. Assume that $0 < \lambda \in I_2$. From Step 1, there exists $\psi \ge 0$ solution to (2.2) for some $c > 0$. Taking φ_1 as test function in (2.2) we have

$$\lambda_1(m) \int_\Omega \varphi_1 \psi = \int_\Omega m(x) \nabla \varphi_1 \nabla \psi = \lambda \int_\Omega \psi \varphi_1 + \int_\Omega \mu(x) \frac{|\nabla \psi|^q}{(\psi + c)^{q-1}} \varphi_1 + \lambda c \int_\Omega \varphi_1.$$

In particular, $(\lambda_1(m) - \lambda) \int_\Omega \psi \varphi_1 > 0$ and $\lambda < \lambda_1(m)$. □

Now we characterize λ^* as the unique possible value of the parameter λ for which (E_λ) may admit a positive solution.

Proposition 2.2. *Assume that* $1 < q \le 2$, $0 \le \mu \in L^\infty(\Omega)$, *with* $\|\mu\|_{L^\infty(\Omega)} < 1$ *if* $q = 2$, *and* $0 < \eta \le m \in L^\infty(\Omega) \cap W^{1,\infty}_{loc}(\Omega)$. *If there exists a positive solution to* (E_λ), *then* $\lambda = \lambda^*$.

Proof. Arguing by contradiction, if there exists a positive solution u to (E_λ) for some $\lambda > \lambda^*$ then, in particular, it is a supersolution and taking $(u+\varepsilon)^{2\gamma-1} - \varepsilon^{2\gamma-1}$ as test function and using Fatou lemma it is possible to prove that $u^\gamma \in H^1(\Omega)$ for every $\gamma > \frac{1}{2}$ if $q < 2$ or $\gamma > \frac{1+\|\mu\|_{L^\infty(\Omega)}}{2}$ if $q = 2$. This implies that $\lambda \in I_3$ and using Proposition 2.1 we have that

$$\lambda \le \sup I_3 = \lambda^* < \lambda.$$

On the other hand, if there exists a positive solution u to (E_λ) for some $\lambda < \lambda^*$, then tu is also a solution for every $t > 0$ and, using the characterization of λ^* given in Proposition 2.1, we have that (E_λ) admits a positive supersolution v. In the case $\mu \equiv 0$ this is a contradiction, since the comparison principle assures then that $tu \le v$ for every $t > 0$, which is not possible.

When $\mu \not\equiv 0$ we conclude the proof in a similar way once we generalize the comparison principle which in addition requires to prove stronger regularity of solutions (see Theorem 2.4 and Lemma 2.3 below, proved in [6]). □

In the next lemma, proved with the regularity theory developed by Ladyzenskaya and Ural'tseva in [10], we resume the main regularity properties of solutions to (P_λ) and, in particular, to (E_λ). Here we replace the $C^{1,1}$ regularity of $\partial\Omega$ by a less restrictive hypothesis.

Lemma 2.3. *Let* $1 < q \le 2$, $0 \le \mu \in L^\infty(\Omega)$, *with* $\|\mu\|_{L^\infty(\Omega)} < 1$ *if* $q = 2$, $0 < \eta \le m \in L^\infty(\Omega) \cap W^{1,\infty}_{loc}(\Omega)$, $0 \lneq f \in L^p(\Omega)$ *with* $p > \frac{N}{2}$, *and let* $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ *be a solution to* (P_λ) *for some* $\lambda \in \mathbb{R}$. *Assume also that there exist* $r_0, \theta_0 > 0$ *such that, if* $x \in \partial\Omega$ *and* $0 < r < r_0$, *then*

$$|\Omega_r| \le (1 - \theta_0)|B_r(x)|$$

for every connected component Ω_r *of* $\Omega \cap B_r(x)$, *where* $B_r(x)$ *denotes the ball centered at* x *with radius* r. *Then* $u \in C^{0,\alpha}(\overline{\Omega}) \cap W^{1,2p}_{loc}(\Omega)$ *for some* $\alpha \in (0,1)$.

Now we state the main comparison principle that we have obtained in this context

Theorem 2.4. *Let* $1 < q \le 2$, $\lambda \in \mathbb{R}$, $0 < \eta \le \mu \in L^\infty(\Omega)$, $0 \le h \in L^1_{loc}(\Omega)$, $0 < \eta \le m \in L^\infty_{loc}(\Omega)$, *and assume that* $u, v \in C(\Omega) \cap W^{1,N}_{loc}(\Omega)$ *are such that* $u, v > 0$ *in* Ω *and satisfy*

$$\limsup_{x \to x_0} \frac{u(x)}{v(x)} \le 1 \quad \forall x_0 \in \partial\Omega, \tag{2.7}$$

$$\int_\Omega m(x) \nabla u \cdot \nabla \phi \le \lambda \int_\Omega u\phi + \int_\Omega \mu(x) \frac{|\nabla u|^q}{u^{q-1}} \phi + \int_\Omega h(x)\phi, \tag{2.8}$$

and

$$\int_\Omega m(x)\nabla v\cdot\nabla\phi \geq \lambda\int_\Omega v\phi + \int_\Omega \mu(x)\frac{|\nabla v|^q}{v^{q-1}}\phi + \int_\Omega h(x)\phi, \tag{2.9}$$

for all $0\leq\phi\in H_0^1(\Omega)\cap L^\infty(\Omega)$ *with compact support. Then* $u\leq v$ *in* Ω.

Proof. The result is obtained arguing as in [1, Lemma 2.2] (see also the references therein) in the equations satisfied by $u_1=\log(u)$ and $v_1=\log(v)$ and taking into account, using (2.7), that, for every $k>0$, the function $(u_1-v_1-k)^+$ has compact support in Ω. □

3. Applications

3.1. **Existence of solution.** In order to avoid the singularity we consider a sequence of approximating nonsingular problems.

$$(Q_n)\qquad \begin{cases} -\mathrm{div}(m(x)\nabla u_n) = \lambda u_n + \mu(x)g_n(u_n)|\nabla u_n|^q + f_n(x) & \text{in } \Omega,\\ u_n = 0 & \text{on } \partial\Omega,\end{cases}$$

where $f_n(x)=\max\{-n,\min\{f(x),n\}\}$ and $g_n(s)=\dfrac{1}{|s+\frac{1}{n}|^{q-1}}$ when $0\lneq f$, otherwise

$$g_n(s) \overset{\text{def}}{=} \begin{cases} \dfrac{1}{|s|^{q-1}} & |s|\geq \dfrac{1}{n},\\ |s|n^q & |s|\leq\dfrac{1}{n}.\end{cases}$$

The role of λ^*, the principal eigenvalue of (E_λ), is that it allows to prove an a priori estimate, for $\lambda<\lambda^*$, in the $L^\infty(\Omega)$ norm of this sequence of approximating solutions. Thanks to this estimate one can pass to the limit and prove the existence of solution to (P_λ).

The main result in [7] for the existence of solution, which includes the case where f may change sign, is the following (see also [6] for positive data).

Theorem 3.1. *Assume that* $1<q\leq 2$, $f\in L^p(\Omega)$ *for some* $p>\frac{N}{2}$, $0\leq\mu\in L^\infty(\Omega)$ *and* $0<\eta\leq m\in L^\infty(\Omega)\cap W^{1,\infty}_{loc}(\Omega)$. *If* $q=2$, *assume additionally that* $f\gneq 0$ *and* $\|\mu\|_{L^\infty(\Omega)}<1$. *Then there exists at least a solution to problem* (P_λ) *for every* $\lambda<\lambda^*$.

Idea of the proof. **Step 1.** First we deduce, by means of the subsolution and supersolution method in [5], the existence of $u_n\in H_0^1(\Omega)\cap L^\infty(\Omega)$ solution to (Q_n). Here, the supersolution for (Q_n) for any $\lambda<\lambda^*$ is given by a positive multiple of the supersolution for (E_λ) for any $\bar\lambda\in(\lambda,\lambda^*)\subset I_1$. The subsolution is either a negative multiple of the same function or the zero function when $0\lneq f$.

Step 2. Arguing as in [10, Theorem 1.1] at Section 4 (p. 249-251) we deduce that $u_n\in C^{0,\alpha}(\overline{\Omega})$ for some $\alpha\in(0,1)$ (see also [6, Appendix]).

Step 3. $\{u_n\}$ is uniformly bounded from below. This is deduced from the maximum principle. Indeed, $u_n\geq z$ with $z\in H_0^1(\Omega)\cap L^\infty(\Omega)$ and $-\mathrm{div}(m(x)\nabla z)=\lambda z-|f(x)|$ in Ω. Moreover, if $0\lneq f$ we have that $u_n\geq w$ with $-\mathrm{div}(m(x)\nabla w)=\lambda z+f_1(x)$ and the strong maximum principle assures that $\{u_n\}$ is uniformly bounded away from zero in compactly embedded subsets of Ω.

Step 4 $\{u_n\}$ is bounded in $L^\infty(\Omega)$ and the proof finishes when $0\lneq f$. Indeed, we argue by contradiction and take $\|u_n\|_{L^\infty(\Omega)}\to\infty$ (up to a subsequence). Then, we have that the function $z_n\equiv\frac{u_n}{\|u_n\|_{L^\infty(\Omega)}}\in$

$H_0^1(\Omega) \cap L^\infty(\Omega)$ satisfies, for every n, that

$$\begin{cases} -\mathrm{div}(m(x) z_n) = \lambda z_n + \mu(x) \dfrac{|\nabla z_n|^q}{\left(z_n + \frac{1}{n\|u_n\|_{L^\infty(\Omega)}}\right)^{q-1}} + \dfrac{T_n(f(x))}{\|u_n\|_{L^\infty(\Omega)}} & \text{in } \Omega, \\ z_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.1}$$

Taking z_n as test function and using that $\|z_n\|_{L^\infty(\Omega)} = 1$ we obtain that $\{z_n\}$ is bounded in $H_0^1(\Omega)$ and we deduce that there exists $0 \le z \in H_0^1(\Omega) \cap L^\infty(\Omega)$ such that, passing to a subsequence, $z_n \rightharpoonup z$ weakly in $H_0^1(\Omega)$ and $z_n \to z$ uniformly in $\overline{\Omega}$ (due to the compact embedding of $C^{0,\alpha}(\overline{\Omega})$ in $C^0(\overline{\Omega})$ and the uniform bound in $C^{0,\alpha}(\overline{\Omega})$ that regularity yields from the $L^\infty(\Omega)$ bound of z_n). In particular, $\|z\|_{L^\infty(\Omega)} = 1$ and as a consequence $z \gneqq 0$ in Ω. Moreover, using weak limits $\displaystyle\int_\Omega m(x)\nabla z \nabla\phi - \lambda \int_\Omega z\phi \ge 0$, and the strong maximum principle ($\lambda < \lambda^* \le \lambda_1(m)$) leads to the facts that $z > 0$ in Ω and $\frac{|\nabla z|^q}{z^{q-1}} \in L^1_{\mathrm{loc}}(\Omega)$. Furthermore, the uniform convergence implies that z_n satisfies $z_n \ge c_\omega > 0$, $\forall \omega \subset\subset \Omega$, $\forall n \in \mathbb{N}$. This implies that $\{-\Delta z_n\}_{n\in\mathbb{N}}$ is bounded in $L^1_{\mathrm{loc}}(\Omega)$, that combined with the H^1 bound implies (see [4]) that

$$\nabla z_n \to \nabla z \ \text{ strongly in } L^r(\Omega)^N \text{ for any } r < 2.$$

The local lower bound and the convergence of the gradients will allow us to pass to the limit in (3.1). In this respect, the case $q = 2$ is special since in principle we do not have strong convergence of the gradients in $L^2(\Omega)^N$. Nevertheless, Fatou lemma can be applied to prove that z is both a subsolution and a supersolution to (E_λ), and here the assumptions $\|\mu\|_{L^\infty(\Omega)} < 1$ and $f \gneqq 0$ are essential. In either case, we deduce that z is a solution to problem (E_λ), which is a contradiction with Proposition 2.2 since $\lambda < \lambda^*$.

The contradiction confirms that $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $L^\infty(\Omega)$ and arguing as for the sequence $\{z_n\}$ we conclude the proof of the result by passing to the limit in (Q_n).

Step 5 $\{u_n\}$ is bounded in $L^\infty(\Omega)$ for changing sign f.

Although we can not argue as in Step 4, we can use Step 4 in order to prove that u_n is uniformly bounded from above, which in addition to Step 3 implies that $\{u_n\}$ is bounded in $L^\infty(\Omega)$. Observe that, this uniform bound from above is trivial if the open set

$$\omega_n = \{x \in \Omega : u_n(x) > 0\}$$

is empty. Otherwise, since $u_n \in C^{0,\alpha}(\omega_n)$, then we deduce that $u_n \in W^{1,N}_{\mathrm{loc}}(\omega_n)$ and u_n is a subsolution to the problem

$$\begin{cases} -\mathrm{div}(m(x)\nabla\zeta) = \lambda\zeta + \mu(x)\dfrac{|\nabla\zeta|^q}{\zeta^{q-1}} + |f(x)| + 1 & \text{in } \omega_n, \\ \zeta > 0 & \text{in } \omega_n, \\ \zeta = 0 & \text{on } \partial\omega_n. \end{cases} \tag{3.2}$$

On the other hand, from Step 4, there exists a solution v to

$$\begin{cases} -\mathrm{div}(m(x)\nabla v) = \lambda v + \mu(x)\dfrac{|\nabla v|^q}{v^{q-1}} + |f(x)| + 1 & \text{in } \Omega, \\ v > 0 & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

Moreover, $v \in C(\overline{\Omega}) \cap W^{1,N}_{\mathrm{loc}}(\Omega)$ reasoning as before. Then, v is a supersolution to (3.2) and applying Theorem 2.4 we deduce that $u_n \le v \le \|v\|_{L^\infty(\Omega)}$ in ω_n and as a consequence $u_n \le \|v\|_{L^\infty(\Omega)}$ in Ω.

Step 6. Passing to the limit for general data f and $1 < q < 2$. Arguing as in Step 4, we can deduce that there exists $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ such that $u_n \to u$ strongly in $H_0^1(\Omega)$ and in $L^r(\Omega)$ for every $r \in [1, \infty)$. Moreover, given $\phi \in H_0^1(\Omega) \cap L^\infty(\Omega)$,

$$\lim_{n\to\infty} \int_\Omega \mu(x) g_n(u_n) |\nabla u_n|^q \phi = \lim_{n\to\infty} \left(\int_\Omega m(x) \nabla u_n \cdot \nabla \phi - \lambda \int_\Omega u_n \phi - \int_\Omega f_n(x)\phi \right) = \int_\Omega m(x)\nabla u \cdot \nabla\phi - \lambda \int_\Omega u\phi - \int_\Omega f(x)\phi.$$

The main difficulty is to prove that $\lim_{n\to\infty} \int_\Omega \mu(x) g_n(u_n)|\nabla u_n|^q \phi = \int_{\{|u|>0\}} \mu(x) \frac{|\nabla u|^q}{|u|^{q-1}} \phi$.

In order to do that we choose a convenient decreasing sequence of positive real numbers $\delta_m \to 0$ and prove, using that $|u_n|^{\frac{1-\varepsilon}{q}}$ is bounded in $H_0^1(\Omega)$, that

$$\lim_{m\to\infty} \left(\lim_{n\to\infty} \int_{\{|u_n| \le \delta_m\}} \mu(x) g_n(u_n) |\nabla u_n|^q \phi \right) = 0.$$

On the other hand, using that $|u_n|^{\frac{1}{q}}$ is bounded in $H_0^1(\Omega)$ and Lebesgue Theorem

$$\lim_{m\to\infty} \left(\lim_{n\to\infty} \int_{\{|u_n|>\delta_m\}} \mu(x) g_n(u_n)|\nabla u_n|^q \phi \right) = \lim_{m\to\infty} \left(\int_{\{|u|>\delta_m\}} \mu(x) \frac{|\nabla u|^q}{|u|^{q-1}} \phi \right) = \int_{\{|u|>0\}} \mu(x) \frac{|\nabla u|^q}{|u|^{q-1}} \phi.$$

This concludes the proof that u is a solution to (P_λ). □

3.2. **Nonexistence of solution to** (P_λ)**.** As in the proof of Proposition 2.2, when $0 \lneq f$, the main role of the principal eigenvalue λ^* for the nonexistence of positive solution is due to the characterization as $\lambda^* = \sup I_3$, since existence of positive solution for some λ implies $\lambda \in I_3$. As a consequence, no solution exists for $\lambda > \lambda^*$.

Proposition 3.2. *Assume that* $1 < q \le 2$, $0 \lneq f \in L^p(\Omega)$ *for some* $p > \frac{N}{2}$, $0 \le \mu \in L^\infty(\Omega)$ *and* $0 < \eta \le m \in L^\infty(\Omega) \cap W^{1,\infty}_{loc}(\Omega)$. *If* $q = 2$, *assume additionally that* $\|\mu\|_{L^\infty(\Omega)} < 1$. *Then, there is no solution to* (P_λ) *for any* $\lambda > \lambda^*$.

Proof of Proposition 3.2. As commented above, the main difficulty lies on the proof of the existence of $\frac{1}{2} \le \gamma(q) < 1$ such that $u^\gamma \in H^1(\Omega)$ for every $\gamma > \gamma(q)$ and for every u solution to (P_λ), which implies that $\lambda \in I_3$ and thus $\lambda \le \lambda^*$. □

3.3. **Uniqueness.** The comparison principle given in Theorem 2.4 guarantees uniqueness of $C(\Omega) \cap W^{1,N}_{\text{loc}}(\Omega)$ positive solution to (P_λ) when any pair of possible solutions satisfy (2.7). In [6] we derive another comparison result to improve the uniqueness result for (P_λ).

Theorem 3.3. *Let* $1 < q \le 2$, $\lambda \in \mathbb{R}$, $0 \le \mu \in L^\infty(\Omega)$, $0 \le h \in L^1_{loc}(\Omega)$, $0 < \eta \le m \in L^\infty_{loc}(\Omega)$. *Assume that* $u, v \in C(\Omega) \cap W^{1,N}_{loc}(\Omega)$, *with* $u, v > 0$ *in* Ω, *and satisfy* (2.8) *and* (2.9) *respectively. Assume also that, for all* $\varepsilon > 0$,

$$\limsup_{x\to x_0} \left(\frac{u(x)}{v(x)+\varepsilon} \right) \le 1 \quad \forall x_0 \in \partial\Omega. \tag{3.3}$$

Furthermore, if $\lambda > 0$, *assume also that* h *is locally bounded away from zero and* $\lambda < \lambda^*$. *Then,* $u \le v$ *in* Ω.

Theorem 3.4. *Assume that* $1 < q \le 2$, $0 \lneq f \in L^p(\Omega)$ *with* $p > \frac{N}{2}$, $0 \le \mu \in L^\infty(\Omega)$, *with* $\|\mu\|_{L^\infty(\Omega)} < 1$ *if* $q = 2$, *and* $0 < \eta \le m \in L^\infty(\Omega) \cap W^{1,\infty}_{loc}(\Omega)$. *Then* (P_λ) *has a unique solution if either* $\lambda \le 0$, *or* f *is locally bounded away from zero and* $\lambda < \lambda^*$.

Proof. We observe that if u, v are two solutions to (P_λ), then Lemma 2.3 implies that $u, v \in C(\overline{\Omega}) \cap W^{1,N}_{\mathrm{loc}}(\Omega)$. In particular, using the continuity up to the boundary of u, v and the fact that $u(x_0) = 0$ for any $x_0 \in \partial\Omega$, we have that u, v satisfy (3.3) for any $\varepsilon > 0$. Moreover, they obviously satisfy (2.8) and (2.9) respectively. Therefore, Theorem 3.3 implies that $u \le v$ in Ω. The reverse inequality follows by interchanging the roles of u and v. □

3.4. **Bifurcation.** As in the semilinear case ($\mu \equiv 0$) we prove that the principal eigenvalue λ^* is in fact a bifurcation point from infinity for (P_λ) when $f \gneq 0$. This, in addition, is useful to get that (E_{λ^*}) admits solution.

Theorem 3.5. *Assume that* $1 < q \le 2$, $0 \lneq f \in L^p(\Omega)$ *with* $p > \frac{N}{2}$, $0 \le \mu \in L^\infty(\Omega)$, *with* $\|\mu\|_{L^\infty(\Omega)} < 1$ *if* $q = 2$, *and* $0 < \eta \le m \in L^\infty(\Omega) \cap W^{1,\infty}_{loc}(\Omega)$. *Then,* λ^* *is the unique possible bifurcation point from infinity of* (P_λ). *Moreover, if* f *is locally bounded away from zero, then the set*

$$\Sigma := \{(\lambda, u_\lambda) \in \mathbb{R} \times C(\overline{\Omega}) : u_\lambda \text{ is a solution to } (P_\lambda)\}$$

is a continuum. In this case, the continuum is unbounded and bifurcates from infinity at λ^* *to the left whenever* (P_λ) *has no solution for* $\lambda = \lambda^*$.

Remark 3.6. *Observe that there are conditions on* f *such that there are no solutions to* (P_{λ^*}). *For instance,* $f \ge c$ *for some* $c > 0$. *See* [6] *for more details.*

Proof. For the first part we observe that, as before, if $\bar{\lambda} \in \mathbb{R}$ is a bifurcation point from infinity then the normalized sequence converges to a solution to $(E_{\bar{\lambda}})$ which implies that $\lambda = \lambda^*$.

For the existence of the continuum we observe that, when f is locally bounded away from zero, one has uniqueness of solution for all $\lambda < \lambda^*$. Then, we can define a map $\lambda \mapsto u_\lambda$, where u_λ is the unique solution to problem (P_λ) for all $\lambda < \lambda^*$. The proof that this map is continuous is deduced by deriving an L^∞ estimate and passing to the limit, as in Theorem 3.1.

Finally, for the global behavior of Σ we observe that, if for $\lambda_n \to \lambda^*$ the sequence $\{u_{\lambda_n}\}_{n\in\mathbb{N}}$ is bounded in $L^\infty(\Omega)$, then we can pass to the limit in (P_{λ_n}) to find a solution to (P_{λ^*}), which contradicts the assumption. □

Corollary 3.7. *Assume that* $1 < q \le 2$, $0 \le \mu \in L^\infty(\Omega)$, *with* $\|\mu\|_{L^\infty(\Omega)} < 1$ *if* $q = 2$, *and* $0 < \eta \le m \in L^\infty(\Omega) \cap W^{1,\infty}_{loc}(\Omega)$. *Then* (E_{λ^*}) *admits solution.*

Proof. We may choose $\lambda_n \to \lambda^*$ such that $\|u_{\lambda_n}\|_{L^\infty(\Omega)} \to \infty$, where u_{λ_n} denotes, for any n, the unique solution to the problem

$$\begin{cases} -\Delta u = \lambda_n u + \mu(x)\dfrac{|\nabla u|^q}{u^{q-1}} + 1 & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Then we prove that the normalized sequence converges to is a solution to (E_{λ^*}). □

3.5. **Homogenization.** Here we state the homogenization result obtained in [7]. Following [8], we consider for every $\varepsilon > 0$ a finite number, $n(\varepsilon) \in \mathbb{N}$, of closed subsets $T_i^\varepsilon \subset \mathbb{R}^N$, $1 \le i \le n(\varepsilon)$, which are the holes. Let us denote $D^\varepsilon = \mathbb{R}^N \setminus \bigcup_{i=1}^{n(\varepsilon)} T_i^\varepsilon$. The domain Ω^ε is defined by removing the holes T_i^ε from Ω, that is

$$\Omega^\varepsilon = \Omega - \bigcup_{i=1}^{n(\varepsilon)} T_i^\varepsilon = \Omega \cap D^\varepsilon.$$

We assume that the sequence of domains Ω^ε is such that there exist a sequence of functions $\{w^\varepsilon\}$ and $\sigma \in H^{-1}(\Omega)$ such that

$$w^\varepsilon \in H^1(\Omega) \cap L^\infty(\Omega), \tag{3.4}$$

$$0 \le w^\varepsilon \le 1 \text{ a.e. } x \in \Omega, \tag{3.5}$$

$$w^\varepsilon \phi \in H_0^1(\Omega^\varepsilon) \cap L^\infty(\Omega^\varepsilon) \ \forall \phi \in H_0^1(\Omega) \cap L^\infty(\Omega), \tag{3.6}$$

$$w^\varepsilon \rightharpoonup 1 \text{ weakly in } H^1(\Omega), \tag{3.7}$$

and given $z^\varepsilon, \phi, z \in H^1(\Omega) \cap L^\infty(\Omega)$ such that $z^\varepsilon \phi \in H_0^1(\Omega^\varepsilon) \cap L^\infty(\Omega^\varepsilon)$ and z^ε weakly converges in $H^1(\Omega)$ to z, the following holds

$$\int_\Omega m(x) \nabla w^\varepsilon \cdot \nabla (z^\varepsilon \phi) \to \langle \sigma, z\phi \rangle_{H^{-1}(\Omega), H_0^1(\Omega)}. \tag{3.8}$$

For a function $u^\varepsilon \in H_0^1(\Omega^\varepsilon)$, we denote by $\widetilde{u^\varepsilon} \in H_0^1(\Omega)$ the extension of u by zero in $\Omega \setminus \Omega^\varepsilon$.

Theorem 3.8. *Assume that the sequence of perforated domains Ω^ε satisfies* (3.4), (3.5), (3.6), (3.7) *and* (3.8). *Suppose also that $1 < q < 2$, $f \in L^p(\Omega)$ for some $p > \frac{N}{2}$, $0 \le \mu \in L^\infty(\Omega)$, $0 < \eta \le m \in L^\infty(\Omega) \cap W_{loc}^{1,\infty}(\Omega)$, $\lambda < \lambda^*$ and that both Ω and D^ε satisfy the regularity condition of the domain in Lemma 2.3, where $D^\varepsilon = \mathbb{R}^N \setminus \bigcup_{i=1}^{n(\varepsilon)} T_i^\varepsilon$. Then, there exists a sequence $\{u^\varepsilon\}$ of solutions to problem*

$$\begin{cases} -\operatorname{div}(m(x)\nabla u^\varepsilon) = \lambda u^\varepsilon + \mu(x)) \dfrac{|\nabla u^\varepsilon|^q}{|u^\varepsilon|^{q-1}} + f(x) & \text{in } \Omega^\varepsilon, \\ u^\varepsilon = 0 & \text{on } \partial\Omega^\varepsilon, \end{cases}$$

such that $\{\widetilde{u^\varepsilon}\}$ is bounded in $L^\infty(\Omega)$ and $\widetilde{u^\varepsilon}$ weakly converges in $H_0^1(\Omega)$ to a solution u to

$$\begin{cases} -\operatorname{div}(m(x)\nabla u) + \sigma u = \lambda u + \mu(x) \dfrac{|\nabla u|^q}{|u|^{q-1}} + f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

in the sense that $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$, $\frac{|\nabla u|^q}{|u|^{q-1}} \in L^1(\{|u| > 0\})$ and

$$\int_\Omega m(x)\nabla u \cdot \nabla \phi + \langle \sigma, u\phi \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} = \lambda \int_\Omega u\phi + \int_{\{|u|>0\}} \mu(x) \frac{|\nabla u|^q}{|u|^{q-1}} \phi + \int_\Omega f(x)\phi$$

for all $\phi \in H_0^1(\Omega) \cap L^\infty(\Omega)$.

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