GELFAND TYPE PROBLEM FOR SINGULAR QUADRATIC QUASILINEAR EQUATIONS

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Abstract. In this paper, we study the existence of positive solutions for the quasilinear elliptic singular problem

$$\begin{cases} -\Delta u + c\,\frac{|\nabla u|^2}{u^\gamma} = \lambda\, f(u)\,, & \text{in } \Omega, \\ u = 0\,, & \text{on } \partial\Omega, \end{cases}$$

where $c, \lambda > 0$, $\gamma \in (0,1)$, f is strictly increasing and derivable in $[0,\infty)$ with $f(0) > 0$. We show that there exists $\lambda^* > 0$ such that $(0, \lambda^*]$ is the maximal set of values such there exists solution. In addition, we prove that for $\lambda < \lambda^*$ there exists minimal and bounded solutions. Moreover, we give sufficient conditions for existence and regularity of solutions for $\lambda = \lambda^*$.

1. introduction

Gelfand-type problems constitute one of the most studied fields of semilinear elliptic equations and it has been considered since the very earliest stages of development of the theory of Partial Differential Equations. There are several reasons for this interest, foremost among them are the wide applications to physical models (we refer to [17, 19, 20, 22] and references therein) and the open problems relating to the existence and boundedness of solutions which still remain unsolved. We recall that a Gelfand-type problem aims to study the following semilinear elliptic equation

$$\begin{cases} -\Delta u = \lambda\, f(u)\,, & \text{in } \Omega, \\ u \geq 0\,, & \text{in } \Omega, \\ u = 0\,, & \text{on } \partial\Omega, \end{cases} \tag{G_λ}$$

where Ω is a smooth bounded, open subset of $\mathbb{R}^N$ $(N \geq 1)$, $\lambda \geq 0$ and the nonlinearity term satisfies

$$f \text{ is } \mathcal{C}^1[0,\infty), \text{ positive, increasing and convex such that } f(0) > 0. \tag{F}$$

Research supported by MEC-FEDER (Spain) MTM2012- 31799 and Junta de Andalucía FQM-116.

Typical examples for f are the power-like $(1+u)^p$ with $p>1$ and the exponential e^u. If a solution u of (G_λ) belongs to $L^\infty(\Omega)$ it is said that it is *regular* and *minimal* if $u \le v$ being v any other solution of (G_λ).

M.G. Crandall and P.H. Rabinowitz in [18] (see also F. Mignot and J.P. Puel [23]) proved, under the hypothesis f is superlinear at infinity (i.e. $\frac{f(s)}{s} \to \infty$), the following result

Proposition 1.1. *[Crandall-Rabinowitz, 1973]* [18] *There exists a positive number λ^* called the extremal parameter such that*

- *If $\lambda < \lambda^*$ the problem (G_λ) admits a minimal bounded solution w_λ.*
- *If $\lambda > \lambda^*$ the problem (G_λ) admits no solution.*

Even more, they showed that the sequence of minimal solutions $\{w_\lambda\}$ of (G_λ) is increasing. Furthermore, the minimal solutions are stable, namely they satisfy the following condition

$$\int_\Omega \left(|\nabla \xi|^2 - \lambda f'(w_\lambda)\xi^2\right) \ge 0, \qquad \forall \xi \in \mathcal{C}_c^\infty(\Omega).$$

An important role is played by the stability condition in order to prove the existence and regularity of $u^* := \lim_{\lambda \to \lambda^*} w_\lambda$, called *extremal solution*. In particular, it has been used to achieve optimal results of regularity of extremal solution depending on the dimension N. Special mention should be made of the exponential case $f(s) = e^s$, obtaining regularity for $N < 10$ as well as the power-like $f(s) = (1+s)^p$ for $N < 4 + 2(1-1/p) + 4\sqrt{1-1/p}$ (see [18]).

In [14] H. Brezis and J.L. Vázquez proved that u^* is a weak solution of (G_{λ^*}). But, as far as regularity of u^* is concerned , for general nonlinearities f satisfying (F), a few results are obtained. More specifically, assuming the superlinearity of f, G. Nedev proved the boundedness of extremal solutions for dimension $N \le 3$ ([24]) and S. Villegas in [26] for $N = 4$. See also X. Cabré et al. in [15, 16] for convex domains Ω.

On the other hand, quasilinear Dirichlet problems having lower order terms with quadratic growth with respect to the gradient whose simplest model is the following boundary value problem

$$\begin{cases} -\Delta u + H(x,u)\,|\nabla u|^2 = f_0(x)\,, & \text{in } \Omega, \\ u > 0\,, & \text{in } \Omega, \\ u = 0\,, & \text{on } \partial\Omega, \end{cases} \tag{Q}$$

have also been extensively studied. A simple motivation relies in the fact that they arise naturally in Calculus of Variations. For example, the Euler-Lagrange equation of the functional

$$I(u) = \frac{1}{2}\int_\Omega a(x,u)|\nabla u|^2 - \int_\Omega f_0(x)u,$$

is formally

$$-\mathrm{div}\,(a(x,u)\nabla u) + \frac{1}{2}a'_u(x,u)|\nabla u|^2 = f_0(x),$$

wich contains a quadratic gradient term.

In the 1980s, L. Boccardo, F. Murat and J.P. Puel discussed, among other important aspects, the case $H(x,s) = g(s)$ continuous in $[0,\infty)$, giving a huge literature since then (see [11, 12] and references therein). It can be observed in the previous example of Calculus of Variations that if we consider functions with unbounded derivative in zero, for instance $a(x,u) = 1 + |u|^\delta$ with $\delta \in (0,1)$, it shows that the Euler-Lagrange equation associated should have a singularity in the quadratic term. In recent years, the case $H(x,s)$ with a singularity at $s = 0$ has been studied by D. Arcoya et al. ([1, 2, 3, 6]) and some applications are described by this kind of equations, see for instance [7, 8, 21].

The goal of this work is to bring together the two areas above, that is, a Gelfand-type problem with a singularity in the gradient term. To be more specifically, we propose to study the existence and regularity of positive solutions for the following problem

$$\begin{cases} -\Delta u + g(u)\,|\nabla u|^2 = \lambda\, f(u)\,, & \text{in } \Omega,\\ u > 0\,, & \text{in } \Omega,\\ u = 0\,, & \text{on } \partial\Omega, \end{cases} \tag{P_λ}$$

were Ω is a smooth bounded and open subset of $\mathbb{R}^N$ ($N \geq 3$), $\lambda > 0$, f strictly increasing, derivable in $[0,\infty)$ with $f(0) > 0$ and respect to g a nontrivial and positive function that either is continuous in $[0,\infty)$ or it is continuous in $(0,\infty)$, decreasing and integrable in a neighborhood of zero. Typical example is $g(s) = \frac{1}{s^\gamma}$ with $\gamma \in (0,1)$.

Most recently in [5] D. Arcoya et al. solved problem (P_λ) in the case g continuous in $[0,\infty)$. Consequently, in the just mentioned paper the authors proved analogous results to that of semilinear elliptic problem (G_λ). They established that the maximal set of λ for which the problem (P_λ) has at least one solution is a closed interval $[0,\lambda^*]$, with $\lambda^* > 0$, and there exists a minimal regular solution for every $\lambda \in [0,\lambda^*)$ (compare Proposition 1.1). They also proved, under suitable conditions, that for $\lambda = \lambda^*$ there exists a minimal regular solution. Even more, they characterized minimal solutions as those solutions satisfying a stability condition. Motivated by this paper, our intention in the current work is to address this matter and provide statements that apply to the quasilinear problem having a singularity in the quadratic gradient term. To make our discussion more precise, under suitable hypotheses (see below hypotheses (H1)-(H4)) we prove in Theorem 2.9 a similar version of Crandall-Rabinowitz result (Proposition 1.1) for problem (P_λ). Moreover, assuming that

$$\lim_{s\to\infty} \frac{s(f'(s)-g(s)f(s))}{f(s)} = \alpha \in (1,\infty]$$

then u^* is a stable solution of (P_{λ^*}) (Theorem 3.6 and Corollary 3.7). We suggest that the reader refers to [14] and compare this condition with $\lim_{s\to\infty} \frac{sf'(s)}{f(s)} = \alpha \in (1,\infty]$. We recall, following the definition introduced by D. Arcoya et al. [5], that a stable solution in the literature of elliptic equations with quadratic growth in the gradient is a positive solution satisfying

$$\int_\Omega |\nabla\phi|^2 \geq \lambda \int_\Omega (f'(u) - g(u)f(u))\phi^2$$

for every $\phi \in W_0^{1,2}(\Omega)$. Stability condition plays an important role in the process to determine when the extremal solutions are regular, we give sufficient conditions in Theorem 4.1. Finally, under the extra condition $f'(s) - g(s)f(s)$ is strictly increasing, we prove that stable solutions are minimal (Theorem 3.8). We would like to point out that, unlike the work of D. Arcoya et al., we use this extra condition exclusively for this last result.

The rest of this paper proceeds as follows: in Section 2, it is shown the existence of bounded minimal solutions for (P_λ) up to a given value λ^*. In addition, we prove that sequence of minimal solutions is increasing respect to λ. In Section 3, we deal with the stability and the issue of the circumstances under which u^* is a stable solution. Also, we establish the relation between minimal and stable solution. Finally, in Section 4 we proceed with the study of regularity of extremal solution and some examples are stated.

Notation. We denote by $|\Omega|$ the Lebesgue measure of $\Omega \subset \mathbb{R}^N$ and by 2^* the critical Sobolev exponent $2N/(N-2)$, $N > 2$. For every $s \in \mathbb{R}$ we consider $s^+ = \max\{s,0\}$, $s^- = \min\{s,0\}$ and the functions $G(s) = \int_0^s g(t)dt$, $\psi(s) = \int_0^s e^{-G(t)}dt$.

2. Existence of bounded minimal solutions

This section is devoted to the study of solutions of problem (P_λ). As in the semilinear case, it is expected that there exists an interval of values of λ such that there is at least one solution. Even more, we prove that there exists a parameter $\lambda^* > 0$ such that the problem has a minimal solution u_λ which is bounded if $0 < \lambda < \lambda^*$ and no solution for $\lambda > \lambda^*$.

We recall that a function $0 < u \in W_0^{1,2}(\Omega)$ is a (weak) solution of (P_λ) if $g(u)|\nabla u|^2$, $f(u) \in L^1(\Omega)$ and it satisfies

$$\int_\Omega \nabla u \nabla \phi + \int_\Omega g(u)\,|\nabla u|^2\,\phi = \int_\Omega \lambda\, f(u)\,\phi, \tag{2.1}$$

for all test function $\phi \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$. As usual, supersolution (respectively subsolution) is defined analogously by replacing the equality "$=$" by the inequality "$\geq$", (resp. $\leq$), for positive test function.

We are interested in the case of functions g which are singular at zero, as a model case $g(s) = \dfrac{1}{s^\gamma}$, $\gamma \in (0,1)$. In this way, the function g will be required to satisfy the following hypotheses

$$\limsup_{s\to\infty} g(s) < \infty \tag{H1}$$

$$f'(s) - g(s)f(s) > 0 \text{ and non-singular}\,(s \geq 0) \tag{H2}$$

$$e^{-G(s)} \in L^1(1,\infty) \tag{H3}$$

$$\forall C > 0,\ \exists \tilde{C} > 0 \,:\, g(Cs) \leq \tilde{C} g(s), \quad \forall s < 1 \tag{H4}$$

Remark 2.1. We want to point out that the hypothesis (H2), which involves function f, in particular it implies that the function $f(s)e^{-G(s)}$ is increasing for $s \geq 0$. Moreover, model case satisfies hypotheses (H1), (H3), (H4) and (H2) taking for instance functions of kind $f(s) = h(s)e^{\frac{s^{1-\gamma}}{1-\gamma}}$, with $h(s)$ increasing and $h(0) > 0$, which also implies that $f(s)$ is concave in a neighborhood of zero. Another interesting case is $g(s) = \frac{1}{log(1+s^\gamma)}$ with $\gamma \in (0,1)$. Additionally, we would like to highlight that functions $g(s) = c$ ($c > 0$) are also considered.

One of the main keys to study problems with singularities in the quadratic gradient term is to treat with test functions with compact support. For this reason it is appropriate to enunciate the following result, which ensures that solutions have a convenient estimate from below in compact sets.

Proposition 2.2. *For every compactly contained open subset $\omega \subset \Omega$ (i.e., $\omega \subset\subset \Omega$) there exists a constant $c_\omega > 0$ such that $u(x) \geq c_\omega$ a. e. $x \in \omega$ for every $u \in W_0^{1,2}(\Omega)$ supersolution of problem (P_λ).*

Proof. To prove it we follow closely [4, Proposition 2.4]. By the fact that $\lambda f(s) \geq \lambda f(0) \neq 0$ for every $s \geq 0$ then every supersolution $u \in W_0^{1,2}(\Omega)$ of (P_λ) is a supersolution of problem

$$\begin{cases} -\Delta w + g(w)\,|\nabla w|^2 = \lambda\, f(0)\,, & \text{in } \Omega, \\ w > 0\,, & \text{in } \Omega, \\ w = 0\,, & \text{on } \partial\Omega. \end{cases} \qquad (P_0)$$

The problem (P_0) has a solution w_0 in $W_0^{1,2}(\Omega) \cap \mathcal{C}(\Omega)$ (see [9, Theorem 3.1]), in particular, since w_0 is continuous, it follows that for every compactly contained subset $\omega \subset \Omega$ there exists $\min_{\overline{\omega}} w_0 = c_\omega > 0$. Now by comparison principle due to [6, Theorem 2.7] we obtain that $u(x) \geq w_0(x) \geq c_\omega$ a.e. $x \in \omega$. $\square$

Lemma 2.3. *If g satisfies (H1), (H2) and (H3), then there exists $\bar{\lambda}$ such that (P_λ) admits no solution for $\lambda > \bar{\lambda}$.*

Proof. Let $u \in W_0^{1,2}(\Omega)$ be a solution of (P_λ) and let ϕ_1 be the positive eigenfunction associated to λ_1, the first positive eigenvalue of the Laplacian operator $-\Delta$ with zero Dirichlet boundary conditions. We take $\varphi_n = e^{-G(u)}\tilde{\phi}_n$, $n \in \mathbb{N}$, where $0 \leq \tilde{\phi}_n \in \mathcal{C}_c^\infty(\Omega)$ such that $\tilde{\phi}_n \to \phi_1$ in $W_0^{1,2}(\Omega)$. Since $\varphi_n \in L^\infty(\Omega)$ and $|\nabla\varphi_n| \leq e^{-G(u)}g(u)\tilde{\phi}_n|\nabla u| + e^{-G(u)}|\nabla\tilde{\phi}_n| \in L^2(\Omega)$ (by Proposition 2.2 and hypothesis (H1)), the function φ_n belongs to $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ and we can take it as test function in (2.1) to have

$$\int_\Omega e^{-G(u)}\nabla u \nabla\tilde{\phi}_n \geq \lambda \int_\Omega f(u)e^{-G(u)}\tilde{\phi}_n,$$

taking limits as n tends to ∞, we get

$$\int_\Omega e^{-G(u)}\nabla u \nabla\phi_1 = \lambda \int_\Omega f(u)e^{-G(u)}\phi_1.$$

On the one hand, let ψ be given by $\psi(s) = \int_0^s e^{-G(t)}dt$, then $e^{-G(u)}\nabla u = \nabla\psi(u)$ and $\psi(u) \in W_0^{1,2}(\Omega)$ since $\psi(s)$ is a Lipschitz function, and on the other hand by hypothesis (H2) $f(s)e^{-G(s)} \geq f(0)$, we obtain

$$\int_\Omega \nabla\phi_1\nabla\psi(u) \geq \lambda f(0)\int_\Omega \phi_1.$$

Taking into account $\psi(s) \leq c_1$ by hypothesis (H3) and integrability of g near to zero,

$$\int_\Omega \nabla\phi_1\nabla\psi(u) \geq \frac{\lambda f(0)}{c_1}\int_\Omega \phi_1\psi(u).$$

Lastly, using that ϕ_1 is the eingefunction associated to λ_1, we conclude the proof taking $\bar{\lambda} \geq \dfrac{\lambda_1 c_1}{f(0)}$. □

Remark 2.4. Even more, there exists $\bar{\lambda}$ such that (P_λ) admits no supersolution for $\lambda > \bar{\lambda}$. Indeed, the proof is similar starting with u a supersolution in place of a solution of (P_λ).

We will consider $\mathcal{I}$ the set of values of $\lambda > 0$ such that there exists a solution of (P_λ). By the previous lemma $\mathcal{I} \subset (0, \bar{\lambda}]$. In order to prove the main result of this section let $\Phi(s)$ be a positive function given by

$$\Phi(s) = \psi^{-1}\left(\frac{\lambda}{\mu}\psi(s)\right), \quad 0 < \lambda < \mu. \tag{2.2}$$

We give some properties of function $\Phi(s)$.

Lemma 2.5. *Let $\Phi(s)$ be a positive function defined by (2.2). Then, following properties are satisfied:*

(1) $0 \leq \Phi(s) \leq s$.
(2) *If (H3) is satisfied then Φ is bounded.*
(3) $0 < \Phi'(s) \leq \dfrac{\lambda}{\mu}$.
(4) $\Phi''(s) = \Phi'(s)\left[g\left(\Phi(s)\right)\Phi'(s) - g(s)\right]$.

Proof.

(1) Clearly $\Phi(s) \geq 0$. On the other hand, since $\dfrac{\lambda}{\mu}\psi(s) \leq \psi(s)$ and ψ^{-1} is increasing then
$$\Phi(s) = \psi^{-1}\left(\frac{\lambda}{\mu}\psi(s)\right) \leq \psi^{-1}\left(\psi(s)\right) = s.$$
(2) Since $\psi(\infty) < \infty$ and $\dfrac{\lambda}{\mu} < 1$ we get the result.
(3) An easy computation shows that
$$\Phi'(s) = \frac{\lambda}{\mu}\frac{e^{-G(s)}}{e^{-G(\Phi(s))}} = \frac{\lambda}{\mu}e^{G(\Phi(s))-G(s)} \leq \frac{\lambda}{\mu}.$$
using in the last inequality that G is increasing and $\Phi(s) \leq s$. Consequently, Φ is strictly increasing.
(4) We may now compute the second derivative to conclude that
$$\Phi''(s) = \left(\frac{\lambda}{\mu}e^{G(\Phi(s))-G(s)}\right)' = \frac{\lambda}{\mu}e^{G(\Phi(s))-G(s)}\left(g(\Phi(s))\Phi'(s) - g(s)\right).$$
□

Proposition 2.6. *If g satisfies hypothesis (H1)-(H4) and u is a solution of (P_μ) ($\mu > 0$) then, for every fixed $\lambda < \mu$, $\Phi(u)$ is a bounded supersolution of (P_λ).*

Proof. $\psi(s)$ is well-defined since g is continuous in $(0, \infty)$ and integrable near to zero. Furthermore, by hypothesis (H3) it is bounded, therefore $\Phi(u)$ is bounded using property (1) from Lemma 2.5. By the other hand, taking into account

$$|\nabla\Phi(u)| = \Phi'(u)|\nabla u| \leq \frac{\lambda}{\mu}|\nabla u| \in L^2(\Omega),$$

and $\Phi(u) = 0$ on $\partial\Omega$, it therefore follows that $\Phi(u) \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$. Moreover, we claim that functions $f(\Phi(u))$ and $g(\Phi(u))|\nabla\Phi(u)|^2$ are in $L^1(\Omega)$. Indeed, since f is continuous and Φ is bounded we deduce that $f(\Phi(u)) \in L^1(\Omega)$. Now we prove that $g(\Phi(u))|\nabla\Phi(u)|^2 \in L^1(\Omega)$, to this end, we define the subset of Ω_ε as $\{x \in \Omega : u(x) < \varepsilon\}$ where $0 < \varepsilon < 1$ is such that $g(s)$ is decreasing in $(0, \varepsilon)$. On one side, if $u \geq \varepsilon$ then $\Phi(u) \geq \Phi(\varepsilon)$ since Φ is increasing, in addition of $\Phi(u)$ is bounded and g is continuous gives $g(\Phi(u)) \leq C$ a.e. $x \in \Omega \backslash \Omega_\varepsilon$ and from the fact that $\Phi(u) \in W_0^{1,2}(\Omega)$ we obtain that $g(\Phi(u))|\nabla\Phi(u)|^2 \in L^1(\Omega \setminus \Omega_\varepsilon)$.

On the other side, again by property (1) from Lemma 2.5 we obtain $0 < \Phi(s) \leq \varepsilon$, $s \in (0, \varepsilon)$ and since

$$\lim_{s \to 0^+} \frac{\Phi(s)}{s} = \lim_{s \to 0^+} \Phi'(s) = \frac{\lambda}{\mu},$$

let $C_\varepsilon > 0$ be the infimum of $\frac{\Phi(s)}{s}$ for $s \in (0, \varepsilon)$, namely, $\Phi(s) \geq C_\varepsilon s$ $\forall s \in (0, \varepsilon)$. Now, by the fact that $g(s)$ is decreasing in $(0, \varepsilon)$ and $\Phi(s), C_\varepsilon s \in (0, \varepsilon)$ then $g(\Phi(s)) \leq g(sC_\varepsilon)$ in $(0, \varepsilon)$. Taking also into account the hypothesis (H4) there exists $\tilde{C}_\varepsilon > 0$ such that $g(sC_\varepsilon) \leq \tilde{C}_\varepsilon g(s)$ and

$$g(\Phi(u))|\nabla\Phi(u)|^2 \leq \tilde{C}_\varepsilon \left(\frac{\lambda}{\mu}\right)^2 g(u)|\nabla u|^2 \in L^1(\Omega_\varepsilon),$$

proving the claim. As a result, up to now $\Phi(u) \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ and $f(\Phi(u)), g(\Phi(u))|\nabla\Phi(u)|^2 \in L^1(\Omega)$. To conclude the proof we verify that $\Phi(u)$ is a supersolution of (P_λ), i.e.,

$$\int_\Omega \nabla\Phi(u)\nabla\varphi + \int_\Omega g(\Phi(u))\,|\nabla\Phi(u)|^2\,\varphi \geq \int_\Omega \lambda\, f(\Phi(u))\,\varphi,$$

for all $0 \leq \varphi \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$. For every fixed $0 \leq \varphi \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ let $\{\varphi_n\}_{n\in\mathbb{N}}$ be positive functions in $\mathcal{C}_c^\infty(\Omega)$ such that $\varphi_n \to \varphi$

in $W_0^{1,2}(\Omega)$. Then $\phi_n = \Phi'(u)\varphi_n \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$, indeed, since $\Phi'(u) \leq \frac{\lambda}{\mu}$ then $\phi_n \in L^\infty(\Omega)$ and by property (4) from Lemma 2.5

$$|\nabla \phi_n|^2 \leq \left(\frac{\lambda}{\mu}\right)^2 \left(|\nabla \varphi_n|^2 + (g(\Phi(u))|\nabla \Phi(u)| + g(u)|\nabla u|)^2 \varphi_n^2\right),$$

and the fact that $u(x) \geq c_{\omega_n}$ for a. e. $x \in \omega_n$, where $\omega_n = \text{supp } \varphi_n$, in addition to hypothesis (H1) we obtain that $g(u), g(\Phi(u)) \in L^\infty(\omega_n)$ and $|\nabla \phi_n|^2 \in L^1(\Omega)$.

Therefore, taking ϕ_n as a test function in problem (P_μ)

$$\int_\Omega \nabla u \left(\Phi''(u)\nabla u\, \varphi_n + \Phi'(u)\nabla \varphi_n\right) + \int_\Omega g(u)|\nabla u|^2 \Phi'(u)\varphi_n =$$
$$\mu \int_\Omega f(u)\Phi'(u)\varphi_n \geq \lambda \int_\Omega f(\Phi(u))\varphi_n,$$

using in the last inequality that $\mu f(u)\Phi'(u) = \lambda f(\Phi(u)) \dfrac{e^{-G(u)} f(u)}{e^{-G(\Phi(u))} f(\Phi(u))}$ and hypothesis (H2).

Lastly, adding and subtracting $|\nabla \Phi(u)|^2 g(\Phi(u))\varphi_n$ together with the fact that the term $\frac{\Phi''(u)}{\Phi'(u)} + g(u) - \Phi'(u)g(\Phi(u))$ is equal to zero, we have for all $n \in \mathbb{N}$

$$\int_\Omega \nabla \Phi(u)\nabla \varphi_n + \int_\Omega g(\Phi(u))\, |\nabla \Phi(u)|^2\, \varphi_n \geq \int_\Omega \lambda\, f(\Phi(u))\, \varphi_n,$$

since $|\nabla \Phi(u)|^2$, $g(\Phi(u))\, |\nabla \Phi(u)|^2$, $f(\Phi(u)) \in L^1(\Omega)$ and $\varphi_n \to \varphi$ in $W_0^{1,2}(\Omega)$, we take the limit when n tends to ∞ and we conclude the proof. $\square$

Remark 2.7. Contrary to others works on this topic, this super-solution depends on the quadratic gradient term $g(s)$, and not on the nonlinearity term $f(s)$ (compare [5] and [13]). This allows us to deal with functions f less restrictive, for instance, in [5] the authors impose $f'(s) - g(s)f(s)$ is an increasing function, conversely this condition is not required in this section, in fact no-convex functions such as $f(s) = e^{G(s)}e^{(s+\delta)^\delta}$ with δ small enough are allowed, being $f'(s) - g(s)f(s)$ decreasing near to zero.

This result will prove to be extremely useful in the following theorem which ensures that set $\mathcal{I}$ is an interval.

Theorem 2.8. *Assume that g satisfies hypotheses (H1)-(H4) and fix $\mu \in \mathcal{I}$, then for every $\lambda \in (0, \mu)$ there exists a bounded minimal solution of (P_λ).*

Proof. First we prove that there exists a bounded solution. To prove it we use a standard monotone iteration argument: let w_0 the bounded solution of problem (P_0) in the proof of Proposition 2.2, we point out that w_0 is unique due to [6, Theorem 2.9]. For every $n \geq 1$ we define the recurrent sequence $\{w_n\}$ by

$$\begin{cases} -\Delta w_n + g(w_n)\,|\nabla w_n|^2 = \lambda\, f(w_{n-1})\,, & \text{in } \Omega, \\ w_n > 0\,, & \text{in } \Omega, \\ w_n = 0\,, & \text{on } \partial\Omega. \end{cases} \qquad (P_n)$$

The sequence $\{w_n\}$ is well defined by [9] and [6], even more, the sequence is increasing, to check that it suffices to prove that $w_0 \leq w_1$. Indeed, taking in account that $0 < w_0$ and f is increasing we obtain $\lambda f(0) \leq \lambda f(w_0)$ and by comparison principle, which is due to [6], it follows that $w_0 \leq w_1$ and by induction argument $0 < w_0 \leq w_1 \leq \cdots \leq w_n$, for all $n \geq 1$. By the fact that $\Phi(u)$, defined by (2.2), is a supersolution of problem (P_0), with a similar argument we prove that $w_n \leq \Phi(u)$ for every $n \in \mathbb{N}$.

Since $\Phi(u) \in L^\infty(\Omega)$, the sequence $\{w_n(x)\}$ is increasing and bounded by $\Phi(u)(x)$ for a. e. $x \in \Omega$. Let $w_\lambda(x)$ be the limit almost every where in Ω (i. e., $w_\lambda(x) := \lim_{n\to\infty} w_n(x)$ a. e. $x \in \Omega$). We claim that $w_\lambda \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$. Indeed, clearly $w_\lambda \in L^\infty(\Omega)$ since $w_\lambda \leq \Phi(u) \in L^\infty(\Omega)$. Moreover, as $w_n \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ we can take it as a test function in problem (P_n)

$$\int_\Omega |\nabla w_n|^2 + \int_\Omega g(w_n)|\nabla w_n|^2 w_n = \lambda \int_\Omega f(w_{n-1}) w_n,$$

dropping the positive term $g(w_n)|\nabla w_n|^2 w_n$, since $w_{n-1} \leq w_n \leq \Phi(u)$ and f is increasing it follows that

$$\int_\Omega |\nabla w_n|^2 \leq \lambda \int_\Omega f(\Phi(u))\Phi(u) \leq \lambda f(\|\Phi(u)\|_\infty)\|\Phi(u)\|_\infty |\Omega|.$$

That is, $\{w_n\}$ is uniformly bounded in $W_0^{1,2}(\Omega)$ and, up to a subsequence, there exists $\tilde{w}$ such that w_n converges weakly to $\tilde{w}$ in $W_0^{1,2}(\Omega)$ and $w_n(x) \to \tilde{w}(x)$ a. e. $x \in \Omega$, by the unicity of the limit $w_\lambda = \tilde{w} \in W_0^{1,2}(\Omega)$ and we conclude the claim.

We now verify that w_λ is solution of (P_λ). In order to prove it we define the operator $K : W_0^{1,2}(\Omega) \to W_0^{1,2}(\Omega)$ by $K[v]$ as the unique solution of problem

$$\begin{cases} -\Delta u + g(u)\,|\nabla u|^2 = v^+ + \lambda\, f(0)\,, & \text{in } \Omega, \\ u > 0\,, & \text{in } \Omega, \\ u = 0\,, & \text{on } \partial\Omega, \end{cases}$$

K is well defined (see [9] and [6]), even more, due to [4, Proposition 2.5] K is a compact operator. We remark that with this notation w_n is solution of (P_n) if and only if $w_n = K[\lambda(f(w_{n-1}) - f(0))]$. Now taking limits and considering that w_n converges weakly to w_λ in $W_0^{1,2}(\Omega)$ we obtain that $w_\lambda = K[\lambda(f(w_\lambda) - f(0))]$, that is, w_λ is a solution of (P_λ).

Our next claim is that the interval $\mathcal{I}$ is not empty. Indeed, we proceed to show that there exists $\tilde{\lambda} \in \mathcal{I}$. In order to get this, we fix $k > 0$ and we consider $\tilde{u} \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$, $\|\tilde{u}\|_\infty \leq \tilde{c}$, the unique solution of problem

$$\begin{cases} -\Delta u + g(u)\,|\nabla u|^2 = k\,, & \text{in } \Omega, \\ u > 0\,, & \text{in } \Omega, \\ u = 0\,, & \text{on } \partial\Omega, \end{cases}$$

we take $\tilde{\lambda} \in (0, \delta)$, where $0 < \delta \leq \dfrac{k}{f(\tilde{c})}$, to obtain for all $\varphi \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$

$$\int_\Omega \nabla\tilde{u}\nabla\varphi + \int_\Omega g(\tilde{u})|\nabla\tilde{u}|^2\varphi = \int_\Omega k\varphi \geq \int_\Omega \delta f(\tilde{c})\varphi \geq \tilde{\lambda}\int_\Omega f(\tilde{u})\varphi,$$

that $\tilde{u}$ is a bounded supersolution of $(P_{\tilde{\lambda}})$. We now apply the standard monotone iteration argument again, with the bounded supersolution $\Phi(u)$ replaced by $\tilde{u}$, to obtain $u_{\tilde{\lambda}}$ a bounded solution of problem $(P_{\tilde{\lambda}})$ and finally that $\mathcal{I} \neq \emptyset$.

Note that we have actually proved that if $\mu \in \mathcal{I}$ then $(0, \mu] \subset \mathcal{I}$, even more, for every $\lambda \in (0, \mu)$ there exists a bounded solution of (P_λ). The proof is completed by showing that solutions w_λ are minimal, indeed, let v_λ be a solution of problem (P_λ), by a similar argument of comparison principle and by induction in n we have $w_n \leq v_\lambda$ for all $n \in \mathbb{N}$ as $w_\lambda(x) := \lim_{n\to\infty} w_n(x)$ a. e. $x \in \Omega$ thus $w_\lambda \leq v_\lambda$.

□

Theorem 2.8 and Lemma 2.3 may be summarized by formulating our main result of this section

Theorem 2.9. *Assume that g satisfies hypotheses (H1)-(H4). Then there exists $\lambda^* \in (0, \bar{\lambda}]$ such that there is a bounded minimal solution of (P_λ) for every $\lambda < \lambda^*$ and no solution for $\lambda > \lambda^*$.*

Remark 2.10. We note that if $\lambda_1 \leq \lambda_2 < \lambda^*$, taking w_{λ_2} as a supersolution of problem (P_{λ_1}) and arguing as the proof of Theorem 2.8 we obtain $w_{\lambda_1} \leq w_{\lambda_2}$. That is, the family of functions $\{w_\lambda\}_{\lambda\in\mathcal{I}}$ are increasing.

Remark 2.11. It is worth pointing out that for every fixed arbitrary $\mu \in \mathcal{I}$ sufficiently small and u a solution of (P_μ), it follows that $\Phi(u) = \psi^{-1}\left(\frac{\lambda}{\mu}\psi(u)\right)$ tends to zero as $\lambda \to 0$. Hence, for every $\varepsilon > 0$ there exists $\eta(\varepsilon) > 0$ such that $w_\nu(x) < \varepsilon$ for every $0 < \nu < \eta$.

3. STABILITY AND EXTREMAL SOLUTIONS

As we have stated at the Remark 2.10, the mapping $\lambda \to u_\lambda$ is increasing in $(0, \lambda^*)$, a.e. $x \in \Omega$. This allows one define $u^* := \lim_{\lambda \to \lambda^*} u_\lambda$ and we call u^* the extremal solution of problem (P_λ). In [13] and [5] the authors proved that u^* is a weak solution for the semilinear and quasilinear problem, respectively. In order to prove the same effect for the singular quadratic quasilinear case we give a property of the minimal solutions, its stability.

Definition 3.1. Let u be a solution of (P_λ), we say that u is stable if $f'(u) - g(u)f(u) \in L^1_{loc}(\Omega)$ and

$$\int_\Omega |\nabla \phi|^2 \geq \lambda \int_\Omega (f'(u) - g(u)f(u))\phi^2 \tag{3.1}$$

holds for every $\phi \in \mathcal{C}_c^\infty(\Omega)$.

Since $f'(u) - g(u)f(u) > 0$ it follows that, by a standard approximation argument and Fatou Lemma, one can take $\phi \in W_0^{1,2}(\Omega)$ in the above definition.

The following result may be proved in much the same way as [5, Lemma 3.7].

Lemma 3.2. *Minimal bounded solutions of (P_λ) are stable.*

Our next goal is to prove that stability condition (3.1) (and under extra condition) allows us to ensure that minimal bounded solutions are uniformly bounded in $W_0^{1,2}(\Omega)$. For that purpose we give the following technnical lemma.

Lemma 3.3. *Let f and g be two positive continuous functions in $(0, \infty)$ with f increasing and sastifying the condition*

$$\lim_{s \to \infty} \frac{s(f'(s) - g(s)f(s))}{f(s)} > 0.$$

Then, for every positive $\delta < \alpha$, there exits a positive constant $C(\delta)$ (depending only on δ) such that $f(s)s \leq \frac{1}{\delta}s^2(f'(s) - g(s)f(s)) + C(\delta)$ for all $s \geq 0$.

Proof. By definition of limit: for all $\varepsilon > 0$ there exists $s_0(\varepsilon)$ depends to ε such that

$$\left|\frac{s(f'(s) - g(s)f(s))}{f(s)} - \alpha\right| < \varepsilon, \ \ \forall s \geq s_0(\varepsilon),$$

choosing $\varepsilon = \alpha - \delta$ and multiplying by s we obtain that there exists $s_0(\delta)$ such that

$$s^2(f'(s) - g(s)f(s)) \geq \delta s f(s), \ \ \forall s \geq s_0(\delta),$$

By the other hand, since f is increasing, $f(s)s < f(s_0(\delta))s_0(\delta)$ for all $s < s_0(\delta)$. Hence taking $C(\delta) = \frac{f(s_0(\delta))s_0(\delta)}{\delta}$ we conclude the proof. □

Proposition 3.4. *Let $\{w_\lambda\}$ be a sequence of minimal bounded solutions of problem (P_λ) such that f and g satisfy the condition*

$$\lim_{s\to\infty} \frac{s(f'(s) - g(s)f(s))}{f(s)} = \alpha \in (1, \infty]. \tag{3.2}$$

Then, the sequence is uniformly bounded in $W_0^{1,2}(\Omega)$.

Proof. Let w_λ be the minimal bounded solution of (P_λ) taken as a test function in (2.1) and dropped the positive term $g(w_\lambda)|\nabla w_\lambda|^2 w_\lambda$ we obtain

$$\int_\Omega |\nabla w_\lambda|^2 \leq \lambda \int_\Omega f(w_\lambda)w_\lambda.$$

In addition, by Lemma 3.3

$$\int_\Omega |\nabla w_\lambda|^2 \leq \frac{\lambda}{\delta} \int_\Omega \left(f'(w_\lambda) - g(w_\lambda)f(w_\lambda)\right) w_\lambda^2 + C_1,$$

with $C_1 = \lambda^* C(\delta)|\Omega|$.

While on the other hand, by Lemma 3.2 w_λ satisfies stability condition, hence choosing $\phi = w_\lambda$ in (3.1)

$$\int_\Omega |\nabla w_\lambda|^2 \geq \lambda \int_\Omega \left(f'(w_\lambda) - g(w_\lambda)f(w_\lambda)\right) w_\lambda^2.$$

Finally, by combining the last two inequalities and taking $\delta > 1$ the proposition follows. □

Remark 3.5. We note that above proof also involves the boundedness of $\int_\Omega f(w_\lambda)w_\lambda$ for $\lambda \in (0, \lambda^*)$.

The remainder of this section will be devoted to the proof of our main result, namely the extremal solution u^* is a solution of problem (P_{λ^*}).

Theorem 3.6. *Under the hypotheses (H1)-(H4) and condition* (3.2)*, $w_\lambda(x)$ converges to $u^*(x)$ a. e. $x \in \Omega$, a solution of (P_{λ^*}).*

Proof. Thanks to Proposition 3.4 there exists $C_1 > 0$ independent of λ such that $\|w_\lambda\|_{W_0^{1,2}(\Omega)} \leq C_1$ for all $\lambda \in (0, \lambda^*)$. Therefore, up to a subsequence, w_λ converges to u^* weakly in $H_0^1(\Omega)$ ($w_\lambda \rightharpoonup u^*$), strongly in $L^s(\Omega)$ ($1 \leq s < 2^*$) and almost everywhere in Ω,

$$w_\lambda(x) \longrightarrow u^*(x), \qquad \text{a.e. } x \in \Omega. \tag{3.3}$$

It should be noted that, as $w_\lambda(x)$ is increasing, the whole sequence converges almost everywhere to $u^*(x) > 0$.

Now we prove that u^* is a solution of (P_{λ^*}), i. e. $g(u^*)|\nabla u^*|^2$, $f(u^*) \in L^1(\Omega)$ and satisfies (2.1). First we claim that $f(w_\lambda)$ is uniformly bounded in $L^1(\Omega)$, indeed fixed $\rho > 0$ then $f(s) \leq f(\rho) + \frac{1}{\rho} f(s)s$ for every $s \geq 0$, thus

$$\int_\Omega f(w_\lambda) \leq f(\rho)|\Omega| + \frac{1}{\rho}\int_\Omega f(w_\lambda)w_\lambda$$

and by Remark 3.5, the last expression is bounded, proving the claim. Therefore the boundedness of $f(w_\lambda)$ in $L^1(\Omega)$ combined with the fact that $f(w_\lambda)$ is increasing, the monotone convergence theorem implies that $f(u^*) \in L^1(\Omega)$.

Concerning the term $g(u^*)\,|\nabla u^*|^2$, taking $\varphi = \frac{T_\varepsilon(w_\lambda)}{\varepsilon}$ as test function in (2.1), where $T_\varepsilon(s) := \min\{s, \varepsilon\}$, thereby $\frac{T_\varepsilon(w_\lambda)}{\varepsilon} \leq 1$ and $\nabla T_\varepsilon(w_\lambda) = \nabla w_\lambda \cdot \chi_{\{w_\lambda \leq \varepsilon\}}$, we get

$$\int_\Omega |\nabla w_\lambda|^2 \cdot \chi_{\{w_\lambda \leq \varepsilon\}} + \int_\Omega g(w_\lambda)|\nabla w_\lambda|^2 \frac{T_\varepsilon(w_\lambda)}{\varepsilon} \leq \lambda^* \int_\Omega f(w_\lambda).$$

Dropping the positive term $|\nabla w_\lambda|^2 \cdot \chi_{\{w_\lambda \leq \varepsilon\}}$ and taking into account the boundedness of $f(w_\lambda)$ in $L^1(\Omega)$ we obtain that there exists a positive constant C_2 such that

$$\int_\Omega g(w_\lambda)|\nabla w_\lambda|^2 \frac{T_\varepsilon(w_\lambda)}{\varepsilon} \leq C_2.$$

Taking the limit as $\varepsilon \to 0$ and having in mind that $\lim\limits_{\varepsilon \to 0} \frac{T_\varepsilon(w_\lambda)}{\varepsilon} = 1$, we get from the Lebesgue dominated convergence theorem

$$\int_\Omega g(w_\lambda)|\nabla w_\lambda|^2 \leq C_2,$$

for every $\lambda \in (0, \lambda^*)$. Now, the result of [10, Theorem 2.1] yields that (up to a subsequence) $\nabla w_\lambda \to \nabla u^*$ converges strongly in $(L^q(\Omega))^N$

$(1 < q < 2)$, particularly it converges almost everywhere in Ω. Then we have, by Fatou lemma, $g(u^*)|\nabla u^*|^2 \in L^1(\Omega)$.

To close, following closely [9], we proceed to show that u^* satisfies the equation (2.1). Since $\phi = \phi^+ + \phi^-$, it is enough to prove it for every nonegative function $\phi \in H_0^1(\Omega) \cap L^\infty(\Omega)$. Furthermore, by density, it is sufficient to prove it when $0 \leq \phi \in H_0^1(\Omega) \cap C_c(\Omega)$. First we claim that u^* is a subsolution. Indeed, from

$$\int_\Omega g(w_\lambda)\,|\nabla w_\lambda|^2\,\phi = \lambda \int_\Omega f(w_\lambda)\phi - \int_\Omega \nabla w_\lambda \nabla \phi,$$

we apply the Fatou lemma on the left side. In regards to the right-hand side, since w_λ converges weakly to u^* in $W_0^{1,2}(\Omega)$ and the boundedness of $f(w_\lambda)$ in $L^1(\Omega)$ we take limits and the claim is proved.

On the other hand, our next claim is that u^* is a supersolution. Choosing $\varphi = e^{G(T_k(u^*))-G(w_\lambda)}\,\phi$ as a test function we obtain

$$\int_\Omega e^{G(T_k(u^*))-G(w_\lambda)}\nabla w_\lambda \nabla \phi + \int_\Omega e^{G(T_k(u^*))-G(w_\lambda)} g(T_k(u^*))\nabla T_k(u^*)\nabla w_\lambda \phi$$
$$= \lambda \int_\Omega f(w_\lambda) e^{G(T_k(u^*))-G(w_\lambda)}\,\phi.$$

Since w_λ converges weakly to u^* and by the strong convergence of $e^{G(T_k(u^*))-G(w_\lambda)}$ to $e^{G(T_k(u^*))-G(u^*)}$, hence taking limits as λ tends to λ^* and again by Fatou lemma on the right side it follows that

$$\int_\Omega e^{G(T_k(u^*))-G(u^*)}\nabla u^* \nabla \phi + \int_\Omega e^{G(T_k(u^*))-G(u^*)} g(T_k(u^*))\nabla T_k(u^*)\nabla u^* \phi$$
$$\geq \lambda^* \int_\Omega f(u^*) e^{G(T_k(u^*))-G(u^*)}\,\phi.$$

Finally, according to ϕ has compact support, there exists a positive constant such that $u^* \geq w_\lambda \geq C_\phi$, that is, $g(u^*)$ is bounded in $supp\,\phi$. We pass to the limit as $k \to \infty$ and by dominated convergence theorem yield the desired another inequality for compact support functions. Using density argument we finish the proof.

□

Corollary 3.7. *Under the hypotheses of Theorem 3.6 the extremal solution u^* is stable.*

Proof. Since w_λ is stable, it follows that

$$\int_\Omega |\nabla \phi|^2 \geq \lambda \int_\Omega (f'(w_\lambda) - g(w_\lambda)f(w_\lambda))\phi^2,$$

letting $\lambda \to \lambda^*$ and by Fatou lemma imply that u^* satisfies condition (3.1). Theorem 3.6 now shows that u^* is stable. $\square$

We have been working under the assumption that $f'(s) - g(s)f(s)$ is not necessarily increasing. In the remainder of this section we assume $f'(s) - g(s)f(s)$ to be increasing.

Theorem 3.8. *Assume the hypotheses (H1)-(H4) hold and $f'(s) - g(s)f(s)$ is strictly increasing. Then every stable solution of problem (P_λ) is minimal.*

Proof. Let u be a minimal solution of (P_λ) and suppose, contrary to our claim, that there exists $v \in W_0^{1,2}(\Omega)$ a solution of (P_λ) and $\mathcal{O} \subset \Omega$ ($|\mathcal{O}| \neq 0$) such that $v < u$ in $\mathcal{O}$.

On the one hand, choosing $e^{-G(u)}\phi$ ($\phi \in \mathcal{C}_c^\infty$) as a test function on the equation (2.1) satisfied by u

$$\int_\Omega e^{-G(u)} \nabla u \nabla \phi = \lambda \int_\Omega f(u) e^{-G(u)} \phi, \tag{3.4}$$

and by a standard approximation argument the above equation it is satisfied for every $\phi \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$.

Analogously, choosing $e^{-G(v)}\phi$ on the equation which it satisfied by v,

$$\int_\Omega e^{-G(v)} \nabla v \nabla \phi = \lambda \int_\Omega f(v) e^{-G(v)} \phi, \tag{3.5}$$

for every $\phi \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$. Now, subtracting (3.5) from (3.4) and writing $\psi(s)$ instead of $\int_0^s e^{-G(t)} dt$, this gives

$$\int_\Omega \nabla(\psi(u) - \psi(v)) \nabla \phi = \lambda \int_\Omega \left(f(u) e^{-G(u)} - f(v) e^{-G(v)} \right) \phi.$$

Taking $\phi = (\psi(u) - \psi(v))^+$ in the above equation, which it is zero in $\Omega \setminus \mathcal{O}$, since ψ is increasing and $v < u$ in $\mathcal{O}$. We have

$$\int_{\mathcal{O}} |\nabla(\psi(u) - \psi(v))|^2 = \lambda \int_{\mathcal{O}} \left(f(u) e^{-G(u)} - f(v) e^{-G(v)} \right) (\psi(u) - \psi(v)). \tag{3.6}$$

On the other hand, taking $\phi = (\psi(u) - \psi(v))^+$ on the stability condition (3.1) satisfied by u, it gives

$$\int_{\mathcal{O}} |\nabla(\psi(u) - \psi(v))^+|^2 \geq \lambda \int_{\mathcal{O}} (f'(u) - g(u) f(u)) \left[(\psi(u) - \psi(v))^+ \right]^2. \tag{3.7}$$

Now combining (3.6) with (3.7) yields

$$\int_{\mathcal{O}} \left[(f'(u) - g(u)f(u))z - \left(f(u)e^{-G(u)} - f(v)e^{-G(v)}\right)\right] z \leq 0, \tag{3.8}$$

here and subsequently, z denotes $\psi(u) - \psi(v)$. Note that $z > 0$ in $\mathcal{O}$. Our claim is that $(f'(u) - g(u)f(u))z - (f(u)e^{-G(u)} - f(v)e^{-G(v)}) > 0$, which leads to a contradiction with (3.8), therefore $z \leq 0$ and concluding that $u \leq v$ in $\mathcal{O}$. To prove the claim it is sufficient to show that $f'(u) - g(u)f(u) - \frac{f(u)e^{-G(u)} - f(v)e^{-G(v)}}{z}$ is positive. Thus, since by the Mean Value Theorem there exists $\tilde{u} \in [v, u]$, a. e. $x \in \mathcal{O}$, such that

$$\frac{f(u)e^{-G(u)} - f(v)e^{-G(v)}}{z} = \frac{f'(\tilde{u})e^{-G(\tilde{u})} - g(\tilde{u})f(\tilde{u})e^{-G(\tilde{u})}}{e^{-G(\tilde{u})}}$$

$$= f'(\tilde{u}) - g(\tilde{u})f(\tilde{u}),$$

hence, with the fact that $f'(s) - g(s)f(s)$ is strictly increasing and $\tilde{u} \leq u$ a. e. in $\mathcal{O}$, the claim is proved and the theorem follows. □

Corollary 3.9. *Under the hypotheses of Theorem 3.6. If in addition, $f'(s) - g(s)f(s)$ is strictly increasing. Then the extremal solution u^* is stable and minimal.*

Proof. Clearly, by Corollary 3.7 the extremal solution u^* given by Theorem 3.6 is stable and consequently, applying Theorem 3.8 we complete the proof. □

Corollary 3.10. *Under the assumptions of Theorem 3.8. If u is an stable and singular solution of (P_λ) then $\lambda = \lambda^*$.*

Proof. By Theorem 3.8 u is the minimal solution of (P_λ) and Theorem 2.8 assures that u is bounded for $\lambda \in (0, \lambda^*)$ which implies, since u is singular, that $\lambda = \lambda^*$. □

4. Regularity of extremal solutions

The *extremal solution* u^* may be bounded or singular. In [14] H. Brezis and J.L. Vázquez raised the question of determining the regularity of u^* depending on the dimension N, this problem led to the study of the regularity theory of stable solutions which many authors are interested ([24, 26, 15]). In this section, we will obtain, under suitable conditions depending on the dimension N, the regularity of extremal solutions for the quasilinear case with singularity in the quadratic gradient term.

In what follows, we write the nonlinearity term of (P_λ) as $e^{G(s)}h(s)$ instead of $f(s)$, where $h(0) > 0$ and h is a derivable function in $[0, \infty)$.

We note that with this notation hypothesis (H2) is equivalent to impose $h(s)$ is increasing. In this way, we replace problem (P_λ) by the following

$$\begin{cases} -\Delta u + g(u)\,|\nabla u|^2 = \lambda\, e^{G(u)} h(u)\,, & \text{in } \Omega, \\ u > 0\,, & \text{in } \Omega, \\ u = 0\,, & \text{on } \partial\Omega. \end{cases} \tag{Q_λ}$$

We can now formulate our main result of this section.

Theorem 4.1. *Under hypotheses (H1)-(H4) and*

$$\lim_{s\to\infty} \frac{sh'(s)}{h(s)} > 1 \tag{4.1}$$

The extremal solution of (Q_λ) given in Theorem 3.6 is bounded whenever

$$N < \frac{4 + 2(\tilde\mu + \tilde\alpha) + 4\sqrt{\tilde\mu + \tilde\alpha}}{1 + \tilde\alpha}, \tag{4.2}$$

$\tilde\alpha$ and $\tilde\mu$ being the following parameters

$$\tilde\alpha := \lim_{s\to\infty} \frac{g(s)h(s)}{h'(s)}, \quad \tilde\mu := \lim_{s\to\infty} \frac{h''(s)h(s)}{(h'(s))^2}. \tag{4.3}$$

Remark 4.2. Comparing the above theorem with [5, Theorem 4.7] we obtain similar results replacing $\tilde\alpha$ and $\tilde\mu$ by

$$\alpha = \frac{\tilde\alpha}{1 + \tilde\alpha}, \quad \mu = \frac{\tilde\alpha + \tilde\mu}{\tilde\alpha + 1}.$$

However, in addition to the singularity of function g, some hypotheses of [5, Theorem 4.7] such as $\alpha < 1$, $\frac{1}{f} \in L^1$, $\left|\frac{f'(s)}{f^2(s)}\right| \le c_2(1 + \sqrt{g(s)})$ or $f'(s) - g(s)f(s)$ is increasing, are not necessary. We wish to emphasize that last hypothesis allow us to deal with functions $f(s)$ no-convex.

Proof. Due to Stampachia Lemma ([25, Lemma 5.1]), what is left is to show that $e^{G(u^*)}h(u^*) \in L^\beta(\Omega)$ with $\beta > N/2$.

By (4.2) we fix

$$\beta \in \left(\frac{N}{2}, \frac{2 + (\tilde\mu + \tilde\alpha) + 2\sqrt{\tilde\mu + \tilde\alpha}}{1 + \tilde\alpha}\right), \tag{4.4}$$

and let us considerer the following positive differentiable function

$$\phi(s) = \sqrt{\frac{h(s)^\beta \left(e^{G(s)}\right)^{\beta - 1}}{h'(s)}}, \quad s \ge R,$$

such that $\phi(0)=0$ and $\phi\in\mathcal{C}^1[0,R]$. For $\lambda<\lambda^*$ let u_λ be the bounded minimal solution of (Q_λ) given by Theorem 2.8 which, under the assumptions of Theorem 3.6 with condition (3.2) replaced by condition (4.1), converges to $u^*(x)$ a. e. $x\in\Omega$. In addition to Lemma 3.2, u_λ satisfies the stability condition, in this way, taking $\phi(u_\lambda)$ in (3.1) (clearly $\phi(u_\lambda)\in W_0^{1,2}(\Omega)$ since u_λ is bounded) we obtain

(4.5)
$$\int_\Omega(\phi'(u_\lambda))^2|\nabla u_\lambda|^2\geq\lambda\int_{\Omega_R}e^{G(u_\lambda)}h'(u_\lambda)\phi^2(u_\lambda)+\lambda\int_\Omega e^{\beta G(u_\lambda)}h^\beta(u_\lambda)$$
$$-\lambda\int_{\Omega_R}e^{\beta G(u_\lambda)}h^\beta(u_\lambda)$$

where $\Omega_R=\{x\in\Omega:u_\lambda(x)<R\}$. Computing, we have

$$\phi'(s)=\frac{\phi(s)}{2}\left(\beta\frac{h'(s)}{h(s)}+(\beta-1)g(s)-\frac{h''(s)}{h'(s)}\right).\tag{4.6}$$

While on the other hand, we define

$$\zeta(s):=e^{-G(s)}\int_0^s(\phi'(t))^2e^{G(t)}dt,$$

since u_λ is bounded if follows that $\zeta(u_\lambda)\in L^\infty(\Omega)$, and applying L'Hôpital rule we obtain

$$\lim_{s\to0}\frac{\int_0^s(\phi'(t))^2e^{G(t)}dt}{s}=(\phi'(0))^2<\infty$$

since $\phi\in\mathcal{C}^1[0,R]$. Thus

$$\zeta'(0)=\lim_{s\to0}\frac{\zeta(s)}{s}$$

and therefore $\zeta'(u_\lambda)\in L^\infty(\Omega)$ and $\zeta(u_\lambda)\in W_0^{1,2}(\Omega)$. Futhermore, using (4.3) and L'Hopital rule we get

$$\lim_{s\to\infty}\frac{\zeta(s)}{\left(e^{G(s)}h(s)\right)^{\beta-1}}=\lim_{s\to\infty}\frac{\int_0^s(\phi'(t))^2e^{G(t)}dt}{e^{\beta G(s)}h(s)^{\beta-1}}$$

$$=\lim_{s\to\infty}\frac{(\phi'(s))^2e^{(1-\beta)G(s)}}{h(s)^{\beta-2}(\beta g(s)h(s)+(\beta-1)h'(s))}$$

$$=\lim_{s\to\infty}\frac{h^2(s)\left(\beta\frac{h'(s)}{h(s)}+(\beta-1)g(s)-\frac{h''(s)}{h'(s)}\right)^2}{4h'(s)\left(\beta g(s)h(s)+(\beta-1)h'(s)\right)}=\frac{(\beta+(\beta-1)\tilde\alpha-\tilde\mu)^2}{4(\tilde\alpha\beta+\beta-1)},$$

which is less than 1 due to (4.4). Thereby, there exist $\gamma < 1$ and $K > 0$ such that

$$\zeta(s) \leq \gamma \left(e^{G(s)}h(s)\right)^{\beta-1} + K, \ \ s \geq R.$$

In this way, choosing $\zeta(u_\lambda)$ as a test function in (2.1) we obtain

$$\int_\Omega (\phi'(u_\lambda))^2 |\nabla u_\lambda|^2 = \lambda \int_\Omega e^{G(u_\lambda)} h(u_\lambda)\zeta(u_\lambda)$$

$$\leq \gamma\lambda \int_\Omega e^{\beta G(u_\lambda)} h^\beta(u_\lambda) + K\lambda \int_\Omega e^{G(u_\lambda)} h(u_\lambda).$$

Combining this last inequality with (4.5) (and dropping the positive term $e^{G(u_\lambda)}h'(u_\lambda)\phi^2(u_\lambda)$) we can assert that

$$(1-\gamma)\lambda \int_\Omega e^{\beta G(u_\lambda)} h^\beta(u_\lambda) \leq \lambda \int_{\Omega_R} e^{\beta G(u_\lambda)} h^\beta(u_\lambda) + K\lambda \int_\Omega e^{G(u_\lambda)} h(u_\lambda),$$

and taking into account that h is increasing (hypothesis (H2)) together with the Lebesgue dominated convergence theorem we deduce that

$$\int_\Omega e^{\beta G(u_\lambda)} h^\beta(u_\lambda) \leq \frac{f(R)^\beta |\Omega|}{\lambda^*(1-\gamma)} + \frac{K}{1-\gamma} \int_\Omega e^{G(u^*)} h(u^*)$$

and $e^{G(u^*)}h(u^*) \in L^1(\Omega)$ since u^* is a solution of (P_{λ^*}) (Theorem 3.6). Finally we conclude, from the Fatou Lemma applied on the left-hand side of the above inequality, that $e^{G(u^*)}h(u^*) \in L^\beta(\Omega)$ with $\beta > N/2$ which is the desired conclusion. □

We now give few examples, according to the different types of function g.

Example 1. Let us consider the problem

$$\begin{cases} -\Delta u + c\,|\nabla u|^2 = \lambda\, e^u\,, & \text{in } \Omega, \\ u > 0\,, & \text{in } \Omega, \\ u = 0\,, & \text{on } \partial\Omega, \end{cases} \qquad (P_\lambda)$$

with $c < 1$. By Theorem 2.9, since $g(s) = c$ satisfies hypotheses (H1)-(H4), there exists $\lambda^* > 0$ such that there is a bounded minimal solution for every $\lambda < \lambda^*$ and no solution for $\lambda > \lambda^*$. Moreover, there exists u^* solution for $\lambda = \lambda^*$ (Theorem 3.6) and it is stable and minimal (Corollary 3.9). Furthermore, since $\tilde{\alpha} = \frac{c}{1-c}$ and $\tilde{\mu} = 1$, it follows from Theorem 4.1 that u^* is bounded provided that

$$N < 4(1-c) + 2 + 4\sqrt{1-c}.$$

We remark that letting $c \to 0$ we obtain the regularity of extremal solution for the well known semilinear elliptic equation (G_λ) in the exponential case, i.e., $N < 10$

Example 2. In the singularity case $g(s) = \frac{c}{s^\gamma}$ with $0 < \gamma < 1$, a relevant example would be the case $f(s)$ no-convex. Thus, if we take as $h(s) = e^{(s+\delta)^{1-\gamma}}$ with δ small enough then $f'(s) - g(s)f(s)$ is no increasing (see Remark 2.7). Therefore, Theorem 2.9 ensures that there exist $\lambda^* > 0$ and bounded minimal solutions for $\lambda < \lambda^*$, and no solutions for $\lambda > \lambda^*$. Even more, since condition 3.2 is satisfied, u^* is a stable solution for $\lambda = \lambda^*$ (Theorem 3.6 and Corollary 3.7) and not necessarily minimal. In addition, since $\tilde{\alpha} = \frac{c}{1-\gamma}$ and $\tilde{\mu} = 1$, due to Theorem 4.1 we obtain for

$$N < \frac{6(1-\gamma) + 2c + 4\sqrt{(c+1-\gamma)(1-\gamma)}}{c+1-\gamma},$$

the regularity of the extremal solution. We would like to stress that letting $c \to 0$ we have $N < 10$.

Acknowledgement

The author thanks Dr. J. Carmona Tapia for his guidance and all his comments on this note.

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