EXISTENCE AND REGULARIZING EFFECT OF DEGENERATE LOWER ORDER TERMS IN ELLIPTIC EQUATIONS BEYOND THE HARDY CONSTANT

DAVID ARCOYA, ALEXIS MOLINO, AND LOURDES MORENO-MÉRIDA

Dedicated with deep admiration and friendship to Ireneo Peral for his seventieth birthday.

Abstract. In this paper we study the regularizing effect of lower order terms in elliptic problems involving a Hardy potential. Concretely, our model problem is

$$-\Delta u + h(x)|u|^{p-1}u = \lambda \frac{u}{|x|^2} + f(x) \quad \text{in } \Omega,$$

with Dirichlet conditions on $\partial\Omega$, where $p > 1$ and $f \in L^m(\Omega; hdx)$ with $m \geq \frac{p+1}{p}$. We prove that there is a solution of the above problem even for $\lambda \geq \mathcal{H} = \frac{(N-2)^2}{4}$ and $0 \leq h \in L^1(\Omega)$ which could be vanished in a subset of Ω. Moreover, we show that all the solutions are in $L^{pm}(\Omega; hdx)$. These results improve and generalize the case $h(x) \equiv h_0$ treated in [9] and recently in [2].

1. Introduction

For a bounded domain $\Omega \subset \mathbb{R}^N$ $(N > 2)$ with smooth boundary $\partial\Omega$ and $0 \in \Omega$, we consider the following problem

$$(1) \qquad \begin{cases} -\Delta u + h(x)|u|^{p-1}u = \lambda \dfrac{u}{|x|^2} + f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

being $\lambda > 0$, $p > 1$, $0 \leq h \in L^1_{loc}(\Omega)$ and $f \in L_h^{\frac{p+1}{p}}(\Omega)$, (i.e. $|f|^{\frac{p+1}{p}} h \in L^1(\Omega)$).

If $h \equiv 0$, it is proved in [7] the existence of a solution for every $f \in W^{-1,2}(\Omega)$ when $\lambda < \mathcal{H} = \frac{(N-2)^2}{4}$ ($\mathcal{H}$ is called the Hardy constant). From this pioneering paper the case $h \equiv 0$ has been studied by many authors. When $\lambda = 0$ (i.e. no Hardy potential appears in (1)), it was proved in [3, 5] that the lower order term $h(x)|u|^{p-1}u$ has a regularizing effect. More recently, it is proved in [2, 9]

University of Granada, 18071, Granada, Spain
E-mail addresses: darcoya@ugr.es, alexis.molino@gmail.com, lumore87@gmail.com.
Supported by "Ministerio de Economía y Competitividad" MINECO-FEDER (Spain) grant MTM2015-68210-P and "Consejería de Economía, Innovación, Ciencia y Empleo, Junta de Andalucía" FQM-116.

that if $h(x) \equiv h_0 > 0$, then the lower order term has a regularizing effect: there exists a solution belonging to $W_0^{1,2}(\Omega) \cap L^{pm}(\Omega)$ for every $\lambda \geq 0$ provided that $\frac{p+1}{p} \leq m < \frac{N}{2}\frac{p-1}{p}$. The solution is obtained as limit of solutions of a sequence of suitable approximate problems. In particular the $L^{pm}(\Omega)$-regularity of the solution is only obtained for this specific solution obtained by approximation. We remark explicitly that the assumption that $h(x)$ is uniformly away from zero is essential in these papers.

Our first goal is to deal with the existence of solutions for $\lambda \geq \mathcal{H}$ and terms h which can vanish in a subset of Ω. Indeed, in Section 2 we handle functions $h(x)$ that can be zero in a neighbourhood $\Omega_\delta = \{x \in \Omega : \text{ dist } (x, \partial\Omega) < \delta\}$ of $\partial\Omega$. First we prove in Theorem 2.1-a) that if

$$\int_{\Omega\setminus\Omega_\delta} |x|^{\frac{2(p+1)}{1-p}} h(x)^{\frac{2}{1-p}} < \infty\,, \tag{2}$$

then there exists a solution u of (1) for every $\lambda \leq \Lambda(\delta)$, where $\Lambda(\delta) \to \infty$ as $\delta \to 0$. Observe that in the particular case that $h(x) \equiv a > 0$, the above condition is satisfied provided that $p > 2^* - 1$. Hence, our result contains also the existence result of [2, 9] when $m = \frac{p+1}{p}$ (see Corollary 2.3). The case that h is zero in Ω_δ is also considered in Corollary 2.5.

For the proof of Theorem 2.1-a) we take advantage of the variational nature of (1) by finding its solution as a critical point of the associated Euler $\mathcal{C}^1$-functional I_λ (see (4) below). Indeed, we show that I_λ is coercive and bounded from below. By using the Variational Principle of Ekeland we also prove that a suitable minimizing sequence of this functional is weakly convergent to a critical point $u \in W_0^{1,2}(\Omega) \cap L_h^{p+1}(\Omega)$ of I_λ, i.e., a solution of (1).

In addition, in Theorem 2.1-b) we also prove that if we strengthen the condition (2) by assuming that there exists $\bar{s} \in (2, p+1)$ such that

$$\int_{\Omega\setminus\Omega_\delta} |x|^{\frac{2\bar{s}}{2-\bar{s}}} h(x)^{\frac{2\bar{s}}{(p+1)(2-\bar{s})}} < \infty, \tag{3}$$

then I_λ is weakly lower semicontinuous (see Remark 2.2-iv) for a comparation with [7, Theorem 3.4]) and thus u is a minimum of the functional I_λ. We also use this additional variational characterization of this found solution to obtain the existence of a non-zero solution of the problem (1) when $f \equiv 0$ (see Corollary 2.6) and improve the corresponding existence results of [10, 11] (see Remark 2.7).

We devote the section 3 to study the regularity of every solution of (1). Specifically we prove in Theorem 3.1 that if $f \in L_h^m(\Omega)$ with $m \geq \frac{p+1}{p}$ and $|x|^{\frac{2pm}{1-p}} h^{1-\frac{pm}{p-1}} \in L^1(\Omega)$, then every solution u of (1) verifies $u \in L_h^{pm}(\Omega)$ improving the previously mentioned regularity result of [2, 9] for solutions which are only obtained as limit of solutions of approximate problems (see Remark 3.4-ii)).

2. Coercivity and existence of solutions

For $0 \le h \in L^1_{loc}(\Omega)$ let $L^{p+1}_h(\Omega)$ be the linear space of all measurable functions in Ω such that $|f|^{p+1}h \in L^1(\Omega)$. It can be equiped with the seminorm

$$|u|_{L^{p+1}_h(\Omega)} = \left(\int_\Omega |u|^{p+1}\, h\right)^{\frac{1}{p+1}}, \ \forall u \in L^{p+1}_h(\Omega),$$

which is a norm in the particular case that $h(x) > 0$ a.e. $x \in \Omega$.

We consider the reflexive space

$$E = W^{1,2}_0(\Omega) \cap L^{p+1}_h(\Omega)$$

endowed with the norm

$$\|u\|_E = \|\nabla u\|_{L^2(\Omega)} + |u|_{L^{p+1}_h(\Omega)}.$$

Observe that every function $f \in L_h^{\frac{p+1}{p}}(\Omega)$ has associated a functional φ_f in the dual space E^* (of E) given by

$$\langle \varphi_f, g\rangle = \int_\Omega fgh, \ \forall g \in L^{p+1}_h(\Omega).$$

Hence, we understand that a solution of (1) is just a critical point of the $\mathcal{C}^1$-functional I_λ defined in E by setting

$$I_\lambda(u) = \int_\Omega \frac{|\nabla u|^2}{2} + \frac{1}{p+1}\int_\Omega |u|^{p+1}\, h - \frac{\lambda}{2}\int_\Omega \frac{u^2}{|x|^2} - \int_\Omega f\, u\, h, \ \forall u \in E; \tag{4}$$

i.e. a function $u \in E$ satisfying

$$\int_\Omega \nabla u \nabla v + \int_\Omega |u|^{p-1} u\, v\, h - \lambda \int_\Omega \frac{u}{|x|^2} v - \int_\Omega f(x)\, v\, h = 0, \ \forall v \in E.$$

On the other hand, for every $\delta \ge 0$, we define the set

$$\Omega_\delta = \{x \in \Omega : \ \text{dist}\,(x, \partial\Omega) < \delta\}.$$

Observe that $\Omega_0 = \emptyset$ and that clearly there exists $\delta_0 > 0$ such that for every $\delta \in [0, \delta_0]$ the boundary $\partial\Omega_\delta$ of Ω_δ is smooth and $0 \notin \bar\Omega_\delta$, where $\bar\Omega_\delta$ denotes the clousure of Ω_δ. We point out that in the sequel the positive constant δ will be always assumed to be smaller than δ_0.

Our first goal is to study the existence of solutions for the problem (1) with functions h that can vanish in Ω_δ. Concretely, we are going to prove the following existence theorem.

Theorem 2.1. *Assume that $p > 1$, $f \in L_h^{\frac{p+1}{p}}(\Omega)$ and that there exists $\delta \ge 0$ such that $\partial\Omega_\delta$ is smooth, $0 \notin \bar\Omega_\delta$ and $h > 0$ a.e. in $\Omega \setminus \Omega_\delta$.*

a) *If condition* (2) *holds true, then there exists* $\Lambda(\delta)$ *such that* (1) *has a solution* $u \in E$ *for every* $\lambda \le \Lambda(\delta)$. *In addition,* $\Lambda(\delta) \to \infty$ *as* $\delta \to 0$.
b) *If, in addition, there exists* $\bar{s} \in (2, p+1)$ *such that condition* (3) *holds true, then* u *is a minimum of functional* I_λ *given by* (4).

Remarks 2.2.

i) As it has been previously observed, every function $f \in L_h^{\frac{p+1}{p}}(\Omega)$ can be considered as an element of the dual space E^* of E. We will see in the proof that for the above existence result the hypothesis $f \in L_h^{\frac{p+1}{p}}(\Omega)$ can be relaxed to $f \in E^*$.
ii) Observe that condition (2) is equivalent to $\frac{1}{|x| h^{\frac{1}{p+1}}} \in L^{\frac{2(p+1)}{p-1}}(\Omega)$, while condition (3) means that $\frac{1}{|x| h^{\frac{1}{p+1}}} \in L^{\frac{2\bar{s}}{\bar{s}-2}}(\Omega)$. Observe that if $2 < \bar{s} < p+1$, then $2 < \frac{2(p+1)}{p-1} < \frac{2\bar{s}}{\bar{s}-2}$ and it follows that (3) implies (2).
iii) Moreover, (3) is clearly satisfied in the case in which $h(x)$ is a Hardy potential term of order $p+1$ on the left hand of equation (1), i.e. $h(x) = 1/|x|^{p+1}$. Indeed, in this context condition, (3) holds true due to the boundedness of the domain Ω.
iv) In the case $h \equiv 0$, the part b) of the above theorem has to be compared with [7, Theorem 3.4] where the authors proved the existence of a minimum of the functional by using an argument that do not require the weak lower semicontinuity of the functional I_λ leaving this semicontinuity as an open problem. As for us, we prove that the hypothesis (3) implies that I_λ is w.l.s.c.

Proof. a) By (2), using the Hölder inequality with exponent $\dfrac{p+1}{2}$, we obtain for every $u \in E$

$$\int_\Omega \frac{u^2}{|x|^2} = \int_{\Omega_\delta} \frac{u^2}{|x|^2} + \int_{\Omega\setminus\Omega_\delta} \frac{u^2}{|x|^2} = \int_{\Omega_\delta} \frac{u^2}{|x|^2} + \int_{\Omega\setminus\Omega_\delta} \frac{u^2 h(x)^{\frac{2}{p+1}}}{h(x)^{\frac{2}{p+1}} |x|^2}$$
$$\le \frac{1}{\rho(\delta)^2} \int_{\Omega_\delta} u^2 + C_1 \left(\int_{\Omega\setminus\Omega_\delta} |u|^{p+1}\, h \right)^{\frac{2}{p+1}},$$

where $\rho(\delta) := \mathrm{dist}(0, \Omega_\delta) > 0$.

Moreover, since $u = 0$ in $\partial\Omega$ and $\partial\Omega \subset \partial\Omega_\delta$ we can use a Poincaré inequality in Ω_δ (see e.g. [8], [12, Section 4.6] see also [1, Section 8]) to assert that

$$\int_{\Omega_\delta} u^2 \le C(\delta) \int_{\Omega_\delta} |\nabla u|^2$$

with the positive constant $C(\delta)$ satisfying

$$C(\delta) = C_2\sqrt{\frac{|\Omega_\delta|}{C_{1,2}(\partial\Omega)}} \to 0, \qquad \text{as } \delta \to 0, \tag{5}$$

where $C_{1,2}(\partial\Omega)$ denotes the capacity of $\partial\Omega$.

Hence, the functional I_λ given by (4) satisfies for every $u \in E$ that

$$\begin{aligned} I_\lambda(u) \geq \int_\Omega \frac{|\nabla u|^2}{2} + \frac{1}{p+1}\int_\Omega |u|^{p+1}\, h - \frac{\lambda C(\delta)}{\rho(\delta)^2}\int_{\Omega_\delta} \frac{|\nabla u|^2}{2} \\ - \frac{\lambda\, C_1}{2}\left(\int_{\Omega\setminus\Omega_\delta} |u|^{p+1}\, h\right)^{\frac{2}{p+1}} - \int_\Omega f\, u\, h \\ \geq \left(1 - \frac{\lambda C(\delta)}{\rho(\delta)^2}\right)\int_\Omega \frac{|\nabla u|^2}{2} + \frac{1}{p+1}\int_\Omega |u|^{p+1}\, h - \frac{\lambda\, C_1}{2}\left(\int_\Omega |u|^{p+1}\, h\right)^{\frac{2}{p+1}} \\ - \|f\|_{E^*}\|u\|_E. \end{aligned}$$

Thus, since $\frac{2}{p+1} < 1$, we obtain that I_λ is coercive and bounded from below provided that

$$\lambda \leq \Lambda(\delta) := \frac{\rho(\delta)^2}{C(\delta)}.$$

As a consequence, by the *Variational Principle of Ekeland* [6], there is a bounded minimizing sequence $\{u_n\} \subset E$ such that

$$I_\lambda(u_n) \to \inf_E I_\lambda \tag{6}$$

and $I'_\lambda(u_n) \to 0$ in E^*, i.e., there exists a sequence of positive numbers $\{\varepsilon_n\}$ converging to zero such that

$$\left|\int_\Omega \nabla u_n \nabla v + \int_\Omega |u_n|^{p-1} u_n\, v\, h - \lambda\int_\Omega \frac{u_n}{|x|^2} v - \int_\Omega f(x)\, v\, h\right| \leq \varepsilon_n \|v\|_E,\ \forall v \in E. \tag{7}$$

We are going to pass to the limit in this inequality as n tends to infinity. The boundedness of $\{u_n\}$ in E implies that, up to a subsequence, we have the weak convergence of u_n in E to some $u \in E$. In particular, up to a subsequence, we can assume that

(A) $u_n \rightharpoonup u$ in $W_0^{1,2}(\Omega)$,
(B) $u_n h^{\frac{1}{p+1}} \rightharpoonup u h^{\frac{1}{p+1}}$ in $L^{p+1}(\Omega)$,
(C) $u_n \to u$ in $L^q(\Omega)$ $(1 \leq q < 2^*)$,
(D) $u_n(x) \to u(x)$ a.e. in Ω,
(E) $\exists\, g \in L^q(\Omega)$ $(1 \leq q < 2^*)$ such that $|u_n(x)| \leq g(x)$.

Obviously, by (A),

$$\lim_{n\to\infty}\int_\Omega \nabla u_n \nabla v = \int_\Omega \nabla u \nabla v, \ \forall v \in W_0^{1,2}(\Omega)$$

and by (B) the sequence $|u_n|^{p-1}u_n$ is bounded in $L_h^{p+1}(\Omega)$ and due to almost every convergence (D), it follows that $|u_n|^{p-1}u_n \rightharpoonup |u|^{p-1}u$ in $L^{p+1}(\Omega; hdx)$. Hence, by (E), Lebesgue dominated convergence theorem implies that

$$\lim_{n\to\infty}\int_\Omega |u_n|^{p-1}u_n\, v\, h = \int_\Omega |u|^{p-1}u\, v\, h, \quad \forall v \in L^{p+1}(\Omega).$$

In order to get the convergence of the term with Hardy potential, i.e., $\displaystyle\int_\Omega \frac{u_n}{|x|^2}v$, we point out that for each $v \in W_0^{1,2}(\Omega)$ the operator $T_v : W_0^{1,2}(\Omega) \to \mathbb{R}$ defined as

$$T_v(u) = \int_\Omega \frac{u}{|x|^2}\, v, \ \forall v \in W_0^{1,2}(\Omega)$$

is linear and continuous since (by using Hölder and Hardy inequalities)

$$|T_v(u)| \leq \left(\int_\Omega \left(\frac{u}{|x|}\right)^2\right)^{1/2} \left(\int_\Omega \left(\frac{v}{|x|}\right)^2\right)^{1/2} \leq \mathcal{H}\|u\|_{W_0^{1,2}(\Omega)}\|v\|_{W_0^{1,2}(\Omega)}$$

for every $v \in W_0^{1,2}(\Omega)$, ($\mathcal{H}$ is the Hardy constant).

In particular, since T_v has finite range, it is also compact and hence $T_v(u_n)$ strongly converges to $T_v(u)$, i.e.

$$\lim_{n\to\infty}\int_\Omega \frac{u_n(x)}{|x|^2}v(x) = \int_\Omega \frac{u(x)}{|x|^2}v(x).$$

In conclusion, taking limits in (7) we obtain that $u \in E$ is a solution of problem (1) for $\lambda < \Lambda(\delta)$.

In addition, since $\rho(\delta) \to \mathrm{dist}(0, \partial\Omega) > 0$ as $\delta \to 0$, then (5) implies that $\Lambda(\delta) \to \infty$ as $\delta \to 0$.

b) As it has been seen in the proof of the part a), for every $\lambda \leq \Lambda(\delta)$ the functional I_λ is bounded from below and coercive. Thus, in order to deduce that I_λ attains its minimum, it suffices to show that it is weak lower semicontinuous. Assume hence that $\{u_n\}$ is a sequence weakly convergent in E. As before, up to a subsequence, we can assume that $\{u_n\}$ verifies the convergences (A)-(E). In addition, we note that the boundedness of $u_n h^{\frac{1}{p+1}}$ in $L^{p+1}(\Omega)$ and the a.e. convergence (D) of u_n imply the strong convergence of $u_n h^{\frac{1}{p+1}}$ in $L^s(\Omega)$ for every $1 \leq s < p+1$. As a consequence, there exists $G \in L^s(\Omega)$ such that (again up to a subsequence) $|u_n(x)h^{\frac{1}{p+1}}(x)| \leq G(x)$, for all $n \in \mathbb{N}$.

We claim that

$$\lim_{n\to\infty}\int_\Omega \frac{u_n(x)^2}{|x|^2} = \int_\Omega \frac{u(x)^2}{|x|^2}. \tag{8}$$

Indeed, if we consider the function $g \in L^2(\Omega)$ given in (E) with $q = 2$ which satisfies that $|u_n(x)| \leq g(x)$ for every $n \in \mathbb{N}$ and almost everywhere for $x \in \Omega$ then

$$\frac{u_n^2(x)}{|x|^2} \leq H(x) \text{ a.e. } x \in \Omega,$$

where the function H is defined in Ω as

$$H(x) = \begin{cases} \dfrac{g^2(x)}{|x|^2}, & \text{if } x \in \bar{\Omega}_\delta, \\[2ex] \dfrac{G^2(x)}{|x|^2 h(x)^{\frac{2}{p+1}}}, & \text{if } x \in \Omega\setminus\bar{\Omega}_\delta. \end{cases}$$

By (D) we also have the convergence of $\frac{u_n(x)^2}{|x|^2}$ to $\frac{u(x)^2}{|x|^2}$ for almost every $x \in \Omega$. Therefore, by the dominated convergence theorem, the claim will be proved if we show that $H \in L^1(\Omega)$. For this purpose, observe that taking into account that $0 \notin \bar{\Omega}_\delta$, we deduce that $\frac{g^2(x)}{|x|^2} \in L^1(\bar{\Omega}_\delta)$, i.e., $H \in L^1(\bar{\Omega}_\delta)$. To prove the integrability in $\Omega\setminus\bar{\Omega}_\delta$, we use the Hölder inequality with exponent $\frac{s}{2} > 1$ to obtain

$$\int_{\Omega\setminus\Omega_\delta} \frac{G^2(x)}{|x|^2 h(x)^{\frac{2}{p+1}}} \leq \left(\int_{\Omega\setminus\Omega_\delta} \frac{1}{|x|^{\frac{2s}{s-2}} h(x)^{\frac{2s}{(s-2)(p+1)}}}\right)^{\frac{s-2}{s}} \left(\int_{\Omega\setminus\Omega_\delta} G(x)^s\right)^{\frac{2}{s}}.$$

The last two integral terms are finite due to hypothesis (3) and that $G \in L^s(\Omega)$. Consequently, we also have $H \in L^1(\Omega\setminus\bar{\Omega}_\delta)$ and the claim is proved.

By the other hand, the result of [4, Theorem 2.1] implies that (up to a subsequence) $\nabla u_n \to \nabla u$ strongly in $(L^q(\Omega))^N$ $(1 < q < 2)$ and in particular (up to a subsequence) it converges almost everywhere in Ω. Then, applying the Fatou lemma we have

$$\liminf_{n\to\infty}\left(\int_\Omega \frac{|\nabla u_n|^2}{2} + \frac{1}{p+1}\int_\Omega |u_n|^{p+1}\, h\right) \geq \int_\Omega \frac{|\nabla u|^2}{2} + \frac{1}{p+1}\int_\Omega |u|^{p+1}\, h \tag{9}$$

Summarizing (8) and (9) we obtain

$$\liminf_{n\to\infty} I_\lambda(u_n) \geq I_\lambda(u),$$

i.e. the functional I_λ is w.l.s.c. and the proof is concluded. □

If we take $\delta = 0$, then $\Omega_\delta = \emptyset$ and by observing that $\int_\Omega |x|^{\frac{2(p+1)}{1-p}} < \infty$ provided that $p > 2^* - 1$, we derive from Theorem 2.1 the following consequence for the case that h is a positive constant in all Ω.

Corollary 2.3. *Assume $p > 2^* - 1$, $f \in L^{\frac{p+1}{p}}(\Omega)$ and $h(x) \equiv h_0 > 0$ in Ω. There exists $u \in E$, solution of problem (1) for every $\lambda \in \mathbb{R}$.*

Remark 2.4. In particular, we recover the existence result of [2, 9]: there exists a solution in $E = W_0^{1,2}(\Omega) \cap L^{p+1}(\Omega)$.

A simple case in which h vanishes in Ω_δ is the following one.

Corollary 2.5. *Let $p > 2^* - 1$, $0 < \delta \leq \delta_0$, $f \in L^{\frac{p+1}{p}}(\Omega \setminus \Omega_\delta)$ and $h \equiv h_0\, \chi_{\Omega\setminus\Omega_\delta}$ for some $h_0 > 0$. Then, there is a solution of (1) in E for $\lambda \leq \Lambda(\delta)$.*

If $\mathcal{H} < \lambda$ then it is possible to choose $w \in W_0^{1,2}(\Omega)$ such that

$$\int_\Omega |\nabla w|^2 - \lambda \int_\Omega \frac{w^2}{|x|^2} < 0.$$

and since $p > 1$, we deduce in the case $f \equiv 0$ that $\inf_E I_\lambda \leq I_\lambda(tw) < 0 = I_\lambda(0)$ provided that t is close to zero. This allows to conclude this section by showing a simple consequence of the additional information that the solution u given in Theorem 2.1 is a minimum of I_λ.

Corollary 2.6. *If $p > 1$, the function h satisfies (3) with $h > 0$ a.e. in $\Omega \setminus \Omega_\delta$ and $\mathcal{H} < \lambda \leq \Lambda(\delta)$, then the problem*

$$\begin{cases} -\Delta u + h(x)|u|^{p-1}u = \lambda \dfrac{u}{|x|^2} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{10}$$

has at least one nonzero solution.

Remark 2.7. As usual by considering instead of I_λ the functional J_λ given by

$$J_\lambda(u) = \int_\Omega \frac{|\nabla u|^2}{2} + \frac{1}{p+1}\int_\Omega |u|^{p+1}\, h - \frac{\lambda}{2}\int_\Omega \frac{(u^+)^2}{|x|^2}, \quad u \in E,$$

it is possible to deduce the existence of a positive solution of the problem (10). Therefore we improve the corresponding existence result of [11] where it is required additionally that h is a continuous and positive function in $\bar{\Omega}$ and of [10], where the case $h(x) = 1/|x|^\beta$ with $\beta < 2$ is studied. (Observe that in both cases considered in those papers, $\Lambda(\delta) = \infty$ in the above corollary).

3. Regularity of the solutions

In this section, for the reader's convenience we assume that $h \in L^1(\Omega)$. In this case, by Hölder inequality, it is easy to verify that $L_h^r(\Omega) \subset L_h^s(\Omega)$ for every $r \geq s \geq 1$. Next, we give a sufficient condition on the function h for which if we strength the condition $f \in L_h^{\frac{p+1}{p}}(\Omega)$ by assuming that $f \in L_h^m(\Omega)$ with $m \geq \frac{p+1}{p}$, then the solution (given by Theorem 2.1) $u \in W_0^{1,2}(\Omega) \cap L_h^{p+1}(\Omega)$ of (1) is more regular: it belongs also to $L_h^{pm}(\Omega)$.

Theorem 3.1. *Assume that $h \in L^1(\Omega)$ with $h(x) > 0$ a.e. in Ω and that there exists $m \geq \frac{p+1}{p}$ such that*

i) $f \in L_h^m(\Omega)$,

ii) $|x|^{\frac{2pm}{1-p}} h^{1-\frac{pm}{p-1}} \in L^1(\Omega)$.

If u is a solution of (1), then $u \in L_h^{pm}(\Omega)$.

Remark 3.2. If instead of assuming that $h \in L^1(\Omega)$ we only assume that $h \in L^1_{loc}(\Omega)$, then the above hypothesis i) should be replaced by $f \in L_h^{\frac{p+1}{p}}(\Omega) \cap L_h^m(\Omega)$.

Proof. For every $k > 0$, we define the auxiliary function $T_k : \mathbb{R} \to \mathbb{R}$ as usual

$$T_k(s) = \begin{cases} k, & s > k, \\ s, & |s| \leq k, \\ -k, & s < -k. \end{cases}$$

Let $u \in E$ be a solution of (1). Since $m \geq (p+1)/p$, we have $\gamma := pm - 1 - p > 0$ and we can choose $|T_k(u)|^\gamma T_k(u)$ as a test function in problem (1) to obtain, by dropping the positive term coming from the principal part, that

$$\int_\Omega h|u|^p |T_k(u)|^{\gamma+1} \leq \lambda \int_\Omega \frac{|u||T_k(u)|^{\gamma+1}}{|x|^2} + \int_\Omega f|T_k(u)|^{\gamma+1} h. \tag{11}$$

Next, we estimate each term of the above inequality. In order to do it, we define

$$F_k(u) := |u|^{p-\delta} |T_k(u)|^{1+\gamma+\delta} h\,,$$

where

$$\delta = \frac{(1+\gamma)(p-1)}{\gamma+2} = \frac{p(m-1)(m-1)}{pm-p+1} \in (0, p-1).$$

Using that $|T_k(s)| \leq |s|$ for all $s \in \mathbb{R}$, we deduce that

$$|u|^p |T_k(u)|^{\gamma+1} h = F_k(u)|T_k(u)|^{-\delta}/|u|^{-\delta} \geq F_k(u)$$

and thus

$$\int_\Omega h|u|^p |T_k(u)|^{\gamma+1} \geq \int_\Omega F_k(u), \tag{12}$$

On the other hand, using Hölder inequality with exponent $p - \delta > 1$ and that $1 + \delta + \gamma = (1+\gamma)(p-\delta)$, we get

$$
\begin{aligned}
\lambda\int_\Omega \frac{|u||T_k(u)|^{\gamma+1}}{|x|^2} &= \lambda\left(\int_\Omega |x|^{\frac{2pm}{1-p}}h^{1-\frac{pm}{p-1}}\right)^{\frac{1}{(p-\delta)'}}\left(\int_\Omega F_k(u)\right)^{\frac{1}{p-\delta}}\\
&\le C_1\left(\int_\Omega F_k(u)\right)^{\frac{1}{p-\delta}}, \qquad (13)
\end{aligned}
$$

where the last inequality is a consequence of hypothesis ii).

In addition, using Hölder with exponent m and taking into account that

$$\frac{(\gamma+1)m}{m-1} = pm = \gamma+1+p$$

we obtain by i)

$$
\begin{aligned}
\int_\Omega f|T_k(u)|^{\gamma+1}\,h &= \int_\Omega f\,h^{\frac{1}{m}}|T_k(u)|^{\gamma+1}\,h^{\frac{m-1}{m}}\\
&\le \left(\int_\Omega |f|^m\,h\right)^{\frac{1}{m}}\left(\int_\Omega |T_k(u)|^{\frac{(1+\gamma)m}{m-1}}\,h\right)^{\frac{m-1}{m}} \le C_2\left(\int_\Omega F_k(u)\right)^{\frac{m-1}{m}}. \qquad (14)
\end{aligned}
$$

In conclusion, substituting (12), (13) and (14) into (11), we deduce that

$$\int_\Omega F_k(u) \le C_1\left(\int_\Omega F_k(u)\right)^{\frac{1}{p-\delta}} + C_2\left(\int_\Omega F_k(u)\right)^{\frac{m-1}{m}}. \qquad (15)$$

Since $\frac{1}{p-\delta}$ and $\frac{m-1}{m}$ are less than 1, (15) implies the existence of $k_0>0$ and $C_3>0$ (independent of k and u) such that

$$\int_\Omega |u|^{p-\delta}|T_k(u)|^{1+\gamma+\delta}h = \int_\Omega F_k(u) \le C_3, \qquad \text{for all } k\ge k_0.$$

Fatou's lemma when k tends to ∞ and the fact that $\gamma+1+p=pm$ implies that

$$\int_\Omega |u|^{pm}h(x)dx = \int_\Omega |u|^{p+1+\gamma}h \le C_3$$

as we desired.

□

A particular interesting case is when the function h can be compared with a Hardy potential of different order.

Corollary 3.3. *Assume that $f\in L^m_h(\Omega)$ for $m\ge \frac{p+1}{p}$, and that there exist $\mu>0$ and $\beta\ge 0$ such that the function $h\in L^1(\Omega)$ satisfies*

$$h(x)\ge \frac{\mu}{|x|^\beta},\quad a.e.\ x\in\Omega.$$

If u is a solution of (1), then $u \in L^{pm}\left(\Omega; \frac{dx}{|x|^\beta}\right)$ for every

$$m \in \begin{cases} \left[\frac{p+1}{p}, \frac{(N-\beta)(p-1)}{(2-\beta)p}\right), & \text{if } \beta \in [0,2), \\ \left[\frac{p+1}{p}, \infty\right), & \text{if } \beta \geq 2. \end{cases}$$

Remarks 3.4.

i) The integrability of h implies that necessarily $\beta < N$.

ii) Observe that if $\beta \in [0,2)$, then the interval $[\frac{p+1}{p}, \frac{(N-\beta)(p-1)}{(2-\beta)p}]$ of the possibles values of m is not empty (i.e., $\frac{p+1}{p} < \frac{(N-\beta)(p-1)}{(2-\beta)p}$) if and only if h satisfies condition (2).

iii) We note that in the particular case $\beta = 0$ the regularity result is proved in [2] only for a solution obtained as limit of solutions of a sequence of suitable approximate problems, but not for every solution as in the previous result.

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