NON EXISTENCE RESULT OF NONTRIVIAL SOLUTIONS TO THE EQUATION $-\Delta u = f(u)$

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ABSTRACT. In this paper we prove the nonexistence of nontrivial solution to

$$\begin{cases} -\Delta u = f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

being $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ a bounded domain and f locally Lispchitz with non-positive primitive. As a consequence, we discuss the long-time behavior of solutions to the so-called sine-Gordon equation.

1. INTRODUCTION

In this paper we are interested in nonexistence results of nontrivial solutions for semilinear elliptic differential equations. Specifically, given $f : \mathbb{R} \to \mathbb{R}$ any locally Lipschitz function and $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ a bounded domain with smooth boundary, we obtain neccesary conditions to Ω and f for the nonexistence of nontrivial solutions for the Dirichlet problem

$$(P) \qquad \begin{cases} -\Delta u = f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Along this note, a *classical solution* to (P) (*solution* from now on) will be a function $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}^{1,\alpha}(\overline{\Omega})$, for some $\alpha \in (0,1)$, satisfying (P) pointwise. Observe that, by regularity results, every bounded weak solution is a solution to this problem (see e.g. Struwe (2008)).

When studying any kind of problem involving differential equations, it is always useful to know necessary conditions for the existence of solution. In this way, it follows immediately that a necessary condition for the existence of a solution u to (P) is that u must satisfy the identity

$$(1) \qquad \int_\Omega |\nabla u|^2 = \int_\Omega f(u)u.$$

As a consequence, a straightforward nonexistence result for problem (P) states that if

$$(2) \qquad f(s)s \leq 0, \quad \textit{for all } s \in \mathbb{R},$$

MSC: 35J05, 35J15, 35J25. *Keywords:* Elliptic Equation, Dirichlet Problem, Nonexistence.

there exists no nontrivial solution to (P). In addition, the well-known Pohožaev identity (Pohožaev (1965)) yields a sort of generalization of this simple result. To be more precise, every solution u to (P) must satisfy the following identity:

$$\frac{1}{2}\int_{\partial\Omega}|\nabla u(x)|^2\, x\cdot\nu(x)dx+\frac{N-2}{2}\int_{\Omega}|\nabla u(x)|^2dx=N\int_{\Omega}F(u(x))dx, \tag{3}$$

where $F(s)=\int_0^s f(t)dt$ and ν denotes the unit vector normal to $\partial\Omega$ pointing outwards. Observe that if Ω is starshaped with respect to 0 (i.e., $x\cdot\nu(x)>0$ on $\partial\Omega$) and $N\geq 3$, the left hand side of (3) is non-negative. Therefore, if Ω is starshaped, $N\geq 3$ and

$$F(s)\leq 0, \quad \textit{for all } s\in\mathbb{R}, \tag{4}$$

there exists no nontrivial solution to (P). Note that (4) implies that $f(0)=0$. Consequently, the trivial solution is always a solution.

Condition $sf(s)\leq 0$ clearly guarantees $F(s)\leq 0$, but not conversely. In this way, a natural question is whether the condition Ω is starshaped is essential for the nonexistence of nontrivial solution to (P), for any bounded domain Ω and f satisfying (4).

A similar situation arises when one analyzes the well-known supercritical case result, also derived from (3). In fact, if $f(s)=\lambda|s|^{p-2}s$, for $\lambda>0$ and $p\geq 2^*$, there exists no nontrivial solution to (P) provided $N\geq 3$ and Ω is starshaped. However, there are examples of non-starshaped domains for which, surprisingly, there exist nontrivial solutions for $p\geq 2^*$. For instance, positive solutions have been found when the domain is an annulus (see the seminal paper Kazdan and Warner (1975) and references therein) or for domains with small holes (del Pino et al. (2002)).

Nevertheless, much less is known about the influence of the geometry of Ω in the existence of solution to problem (P) in the case $F(s)\leq 0$ and the literature contains only partial nonexistence results. Observe that for functions f globally Lipschitz, with $L-$Lipschitz constant, it follows that $|f(s)|\leq L|s|$. Thus, applying Poincaré inequality in (1), we obtain

$$\lambda_1\int_{\Omega}u^2\leq\int_{\Omega}|\nabla u|^2=\int_{\Omega}f(u)u\leq L\int_{\Omega}u^2.$$

Therefore, this simple computation gives the nonexistence of nontrivial solutions as long as $L<\lambda_1$, being λ_1 the first eingenvalue for the Laplacian operator with zero Dirichlet boundary conditions. In this line, in Ricceri (2008) and Fan (2009) the authors prove the nonexistence of nontrivial solution provided that $N\geq 2$ and $L<3\lambda_1$ or $L\leq 3\lambda_1$, respectively, without assuming any geometric condition on $\partial\Omega$. Recently, in Goubet and Ricceri (2019), the nonexistence of nontrivial solutions is shown if either $\partial\Omega$ has non-negative mean curvature or Ω is an annulus, also for functions f globally Lipschitz and $N\geq 2$. On the other hand, in Clément and Sweers (1987) (see also Dancer and Schmitt (1987)), a condition similar to $F(s)\leq 0$ for positive solutions is imposed. Specifically, the result states as follows.

Theorem 1.1 (Clément and Sweers (1987)). *Let f be a $\mathcal{C}^1$ function. Suppose that there are two numbers $s_1 < s_2$, with $s_2 > 0$, such that $f(s_1) = f(s_2) = 0$ and $f > 0$ in (s_1, s_2). In additon, assume that there is $\overline{s} \in [0, s_2)$ such that $F(\overline{s}) \geq F(s_2)$. Then, there is no positive solution u of (P) satisfying* $\max u \in (s_1, s_2)$.

In the present note, inspired by the above result, we prove that there is no nontrivial solution to problem (P) (not necessarily positive) provided $\int_0^s f(t)dt \leq 0$, being f a locally Lispchitz function (Theorem 2.1). Here, no additional hypotheses on Ω and N are required. This exposes the unexpected fact that there is no geometric assumption on Ω that gives a nontrivial solution. As a consequence, solution to the sine-Gordon equation (11) tends to 0 as $t \to \infty$ (Proposition 3.1).

2. MAIN RESULT

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^N$ ($N \geq 1$) a bounded domain with smooth boundary and $f : \mathbb{R} \to \mathbb{R}$ a locally Lipschitz function satisfying the condition $F(s) = \int_0^s f(t)dt \leq 0$ for all $s \in \mathbb{R}$. Then, $u \equiv 0$ is the unique solution to* (P).

Proof. Clearly, zero is a solution. We argue by contradiction and assume that there exists a nontrivial solution u to (P). First of all, notice that $-u$ is a solution to

$$\begin{cases} -\Delta u = -f(-u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Since the function $-f(-s)$ satisfies the hypotheses of the theorem, there is no loss of generality in assuming that $u_\infty := \max_{x\in\overline{\Omega}} u(x) > 0$. On the other hand, since f is locally Lipschitz and the value of $f(s)$ for $s > u_\infty$ is irrelevant, we can also assume that f is globally Lipschitz, with Lipschitz constant $L > 0$, and that $\lim_{s\to+\infty} f(s) = -\infty$.

It is easy to check that $f(u_\infty) > 0$. Indeed, arguing by contradiction, assume that $f(u_\infty) \leq 0$. Then,

$$-\Delta u_\infty + Lu_\infty \geq f(u_\infty) + Lu_\infty \quad \text{in } \Omega. \tag{5}$$

Moreover,

$$-\Delta u + Lu = f(u) + Lu \quad \text{in } \Omega. \tag{6}$$

Subtracting (6) from (5), and using that $f(s) + Ls$ is non-decreasing, we obtain

$$-\Delta(u_\infty - u) + L(u_\infty - u) \geq f(u_\infty) + Lu_\infty - f(u) - Lu \geq 0 \quad \text{in } \Omega.$$

Since $u_\infty > u$ on $\partial\Omega$, the strong maximum principle implies that $u_\infty > u$ in Ω, which is a contradiction.

Thus, the fact that $f(u_\infty) > 0$ implies that there are $s_1, s_2 > 0$ such that $s_1 < u_\infty < s_2$ and

$$f(s) > 0 \quad \forall s \in (s_1, s_2). \tag{7}$$

Moreover, since $F(s) \leq 0$ and $\lim_{s\to+\infty} f(s) = -\infty$, we can choose respectively s_1 and s_2 such that $f(s_1) = f(s_2) = 0$. Further, we can assume that $F(s_2) < 0$ since, otherwise (i.e., if $F(s_2) = 0$), we can modify f to another L-Lipschitz function f^* such that $f(s) > f^*(s) > 0$ for $s \in (u_\infty, s_2)$ and $f = f^*$ elsewhere. In this way, u is still a solution to (P), but now $F(s_2) < 0$.

Now we will find a family of supersolutions to (P) which will lead to a contradiction by comparison with u. For this purpose, we follow the original reasoning in Clément and Sweers (1987), which in principle is performed for $f \in \mathcal{C}^1(\mathbb{R})$. Here we adapt the proof to our setting and check that it also works for Lipschitz functions.

Indeed, consider the following initial value problem

$$\begin{cases} -w''(r) = f(w(r)), & \forall r > 0, \\ w(0) = s_2, \\ w'(0) = -\sqrt{-F(s_2)}. \end{cases}$$

Since f is Lipschitz there is a unique solution $w \in \mathcal{C}^2([0,+\infty))$. Multiplying the equation by $w'(r)$ and integrating, we obtain

$$\begin{aligned} (w'(r))^2 &= -F(s_2) + 2\int_{w(r)}^{s_2} f(s)ds \\ &= F(s_2) - 2F(w(r)). \end{aligned} \tag{8}$$

Thus, using (7) we get that

$$(w'(r))^2 > 0 \text{ for } w(r) \in [s_1, s_2]. \tag{9}$$

Now, since $w(0) = s_2$ and $w'(0) < 0$, we deduce easily that $w(r) \in (s_1, s_2)$ for all $r > 0$ small enough. We claim now that there exists $r_0 > 0$ such that $w(r_0) = s_1$. Indeed, assume by contradiction that $w(r) > s_1$ for all $r > 0$. Then, by (9) we have that w is decreasing in $(0,+\infty)$. Hence, there exists $s_3 \in [s_1, s_2)$ such that $\lim_{r\to+\infty} w(s) = s_3$. But this is impossible since $w''(r) = -f(w(r)) < 0$ for all $r > 0$, i.e. w is concave.

In consequence, since $w(r_0) = s_1$ and $w'(r_0) < 0$, we deduce that $\inf_{r\geq 0} w(r) < s_1$. Moreover, it is easy to show that $\inf_{r\geq 0} w(r) > 0$. Indeed, assuming otherwise, there exists a sequence $\{r_n\} \subset [0,+\infty)$ such that $\lim_{n\to\infty} w(r_n) = 0$. Then, for n large enough, we deduce from (8) that $(w'(r_n))^2 < \frac{F(s_2)}{2} < 0$, a contradiction.

Thus, we have proved that

$$0 < \inf w < s_1. \tag{10}$$

Next, we define

$$W(r) = \begin{cases} s_2, & r \in (-\infty, 0], \\ \min\{w(r), s_2\}, & r \in (0, \infty). \end{cases}$$

Since we can assume that $f(s) < 0$ for $s > s_2$, it follows that w is convex if $w(r) > s_2$. This implies that, if $w(r_2) = s_2$ for some $r_2 > 0$, then $W(r) = s_2$

for all $r \geq r_2$. Otherwise, $w(r) < s_2$ for all $r > 0$, so $W(r) = w(r)$ for all $r > 0$.

For every $t \in \mathbb{R}$, consider the family of parametric functions $v_t(x) = W(x_1 - t)$ for all $x = (x_1, ..., x_N) \in \mathbb{R}^N$. We will prove now that $u(x) \leq v_t(x)$ for all $x \in \overline{\Omega}$ and for all $t \in \mathbb{R}$ using the sweeping principle by Serrin. Indeed, let

$$U = \{t \in \mathbb{R} : u(x) \leq v_t(x) \text{ for all } x \in \overline{\Omega}\}.$$

Note that $v_t = s_2$ for t large enough, and $u < s_2$ in $\overline{\Omega}$, so U is nonempty. Notice also that W is a globally Lipschitz function, so the function $t \mapsto v_t(x)$ is continuous uniformly in x. In particular, U is closed.

Let us now take $t \in U$. Observe that $v_t \in W^{1,\infty}(\Omega)$ and $-\Delta v_t \geq f(v_t)$ in Ω (in the weak sense). Then, since $s \mapsto f(s) + Ls$ is non-decreasing and $u \leq v_t$ in $\overline{\Omega}$, we have that $-\Delta(v_t - u) + L(v_t - u) \geq 0$ in Ω. Notice that

$$u(x) = 0 < \inf w \leq v_t(x) \quad \forall x \in \partial\Omega,$$

so $v_t \not\equiv u$. Then, the strong maximum principle implies that $u(x) < v_t(x)$ for all $x \in \overline{\Omega}$. Therefore, the uniform continuity of $s \mapsto v_s$ implies that there exists $T > 0$, independent of x, such that $u(x) < v_s(x)$ for all $x \in \overline{\Omega}$ and for all $s \in (t - T, t + T)$. That is to say, $(t - T, t + T) \subset U$, so U is open. In conclusion, $U = \mathbb{R}$, and thus, $u \leq v_t$ for all $t \in \mathbb{R}$. In consequence,

$$u(x) \leq \inf_{t \in \mathbb{R}} v_t(x) = \inf_{r>0} w(r) < s_1, \quad \forall x \in \Omega,$$

which is a contradiction with the fact that $u_\infty \in (s_1, s_2)$. □

3. Sine-Gordon equation

In this section we study the asymptotic behavior, as $t \to \infty$, of global and bounded solutions of the second order evolution problem

$$\text{(11)} \qquad \begin{cases} u_{tt} + \alpha u_t - \Delta u + \beta \sin u = 0 & \text{in } \mathbb{R}^+ \times \Omega, \\ u = 0 & \text{on } \mathbb{R}^+ \times \partial\Omega, \\ u(0, x) = u_0(x) & x \text{ in } \Omega, \\ u_t(0, x) = u_1(x) & x \text{ in } \Omega, \end{cases}$$

also called the sine-Gordon equation. Here, $\alpha, \beta > 0$, the unknown is a scalar function $u(t, x)$ which maps $[0, \infty) \times \Omega$ into $\mathbb{R}$, u_t and u_{tt} denote the first and second derivatives of u with respect to the variable t and the initial data (u_0, u_1) belongs to $H_0^1(\Omega) \times L^2(\Omega)$. In physics the sine-Gordon equation is used to model, for instance, the dynamics of a Josephson junction driven by a current source. It is well known the existence and uniqueness of solution $u \in \mathcal{C}([0, T], H_0^1(\Omega))$ and $u_t \in \mathcal{C}([0, T], L^2(\Omega))$ for any $T > 0$ (Temam, 1997, IV. Theorem 2.1.).

Proposition 3.1. *Let $N \geq 1$, Ω be a bounded smooth domain of $\mathbb{R}^N$ and u be the solution to (11) with initial data (u_0, u_1) belonging to the energy space $H_0^1(\Omega) \times L^2(\Omega)$. Then*

$$\lim_{t \to \infty} \|u_t(t, \cdot)\|_{L^2(\Omega)} = \lim_{t \to \infty} \|u(t, \cdot)\|_{H_0^1(\Omega)} = 0.$$

Proof. Directly applying (Haraux and Jendoubi, 1999, Th. 1.2) we obtain that

$$\lim_{t\to\infty} \|u_t(t,\cdot)\|_{L^2(\Omega)} = \lim_{t\to\infty} \|u(t,\cdot) - \varphi(\cdot)\|_{H_0^1(\Omega)} = 0,$$

where $\varphi \in H^2(\Omega) \cap H_0^1(\Omega)$ is a solution of

$$-\Delta\varphi = -\beta \sin\varphi,$$

see also (Haraux and Jendoubi, 1999, Ex. 4.1.1).

On the other hand, since $F(s) = \beta(\cos s - 1)$ is non-positive for all $s \in \mathbb{R}$, it follows by Theorem 2.1 that $\varphi \equiv 0$ and the proof is finished. □

Remark 3.2. Proposition 3.1 improves the recent result by Goubet (2019), which arrives at the same asymptotic behavior of the solution but under the restrictions of the dimension ($N \geq 2$) and, either the domain Ω has non-negative mean curvature (Goubet, 2019, Th. 2.1), or Ω is an annulus of $\mathbb{R}^N$ (Goubet, 2019, Prop. 2.1).

Acknowledgements

First and second author are supported by PGC2018-096422-B-I00 (MCIU / AEI / FEDER, UE) and by Junta de Andalucía FQM-116 (Spain). First author is also supported by Programa de Contratos Predoctorales del Plan Propio de la Universidad de Granada.

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